On the Implementation of a Generalized Hyperinterpolation Process

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Summary

Generalized hyperinterpolation on the sphere is a constructive and uniformly convergent approximation method. Even in the case of discretized Newman-Shapiro operators, where the kernel is known in formula, the degree of complexity is high, but it can be reduced considerably by truncation if product Gauß quadratures are used. To avoid unnecessary many evaluations, it is essential that the quadrature is changed off the equator. This work is a completed version of [4].

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1 Introduction

Generalized hyperinterpolation on the unit sphere S^{r-1} in $\mathbb{R}^r$, $r \in \mathbb{N}\setminus\{1\}$, is a constructive approximation method, which arises from hyperinterpolation by summation, [2], [3]. Not too surprising in a typical multivariate problem, the order of complexity is large. But in a particular case, the numerical expense can be reduced considerably by a particular technique. This is the topic of what follows.

By ω we denote the standard measure on S^{r-1}, and put $\lambda := \frac{r-2}{2}$. For $\mu \in \mathbb{N}_0$ and $\nu := \lfloor \frac{\mu}{2} \rfloor$, the Newman–Shapiro operator L_μ is defined by

$$(L_\mu F)(x) := \int_{S^{r-1}} F(t) K_\mu(tx)\, d\omega(t) \tag{1.1}$$

for $F \in C(S^{r-1})$ and $x \in S^{r-1}$, where

$$K_\mu(\eta) = \left[\gamma_{\nu+1} \cdot \frac{C^\lambda_{\nu+1}(\eta)}{\eta - \eta_{\nu+1}}\right]^2 \tag{1.2}$$

is a positive polynomial of degree 2ν, and where the constant $\gamma_{\nu+1}$ is chosen such that

$$\int_{S^{r-1}} K_\mu(tx)\, d\omega(t) \;=\; 1 \tag{1.3}$$

holds for $x \in S^{r-1}$. In (1.2), $\eta_{\nu+1}$ denotes the greatest zero of the Gegenbauer polynomial $C^\lambda_{\nu+1}(\eta)$. Note that the value of the constant $\gamma_{\nu+1}$ is well-known, ([3], p. 239). Newman and Shapiro used these operators in order to prove Jackson's inequality on the sphere, [1], but actually they even allow an error estimate by the modulus of continuity of the second order, ([3], Th. 6.36). Moreover, if the integral (1.1) is evaluated by means of a positive quadrature with nodes $t_1, \ldots, t_M$ and weights $A_1, \ldots, A_M$, which is exact for all polynomials of degree 2μ, then the positive linear polynomial operators $\hat{L}_\mu$, defined by

$$(\hat{L}_\mu F)(x) \;:=\; \sum_{j=1}^{M} A_j F(t_j) K_\mu(t_j x) \tag{1.4}$$

for $F \in C(S^{r-1})$ and $x \in S^{r-1}$, arise and form a generalized hyperinterpolation process, which is convergent in the uniform norm $\|\cdot\|_\infty$ on S^{r-1}, ([3], Th. 6.34). Note that, as in interpolation, the approximant $\hat{L}_\mu F$ is defined by the function values $F(t_1), \ldots, F(t_M)$ only, which we assume to be available in the storage.

Moreover, for all sufficiently smooth functions, the error law

$$\|F - \hat{L}_\mu F\|_\infty \;=\; \mathcal{O}(\mu^{-2})$$

is valid for $\mu \to \infty$. This is the best possible approximation order attainable for a general function by positive polynomial operators, and caused our particular interest in the discretized Newman–Shapiro operators (1.4).

Convergence still holds if the quadratures are exact only of degree $\mu + 1$, see proof of ([3], Th. 6.34). However, they should use not too many nodes and yet provide high exactness. It is advisable to store the real numbers

$$a_j(F) \;:=\; A_j F(t_j),\; j = 1, \ldots, M,$$

in advance. A good choice are product Gauß quadratures, see [5] or [3], e.g. But even in their case, the cost of a standard evaluation of (1.4) at a single point x are at least of the order μ^r. By means of the fast Fourier transform, they can be reduced to the order $\mu^{r-1} log\, \mu$, though at the price

of much organisation work. This tool need not be given up in what follows, but the issue must be to avoid kernel evaluations in advance where ever this is possible. This the more, if the graph of $\hat{L}_\mu F$ is to be made visable, say by the help of a 50×50–grid, which requires 2500 operator evaluations by an immense amount of arithmetical operations.

However, the kernel function $K_\mu(\eta)$ is very steep near to the point $\eta = 1$. In other words, it is quickly decreasing off this point. Therefore it suffices in the calculation of (1.4) to take into account only those nodes which belong to a given neighbourhood $U_\mu(x)$ of x, i.e. to consider, instead of $\hat{L}_\mu$, the operator $\hat{M}_\mu$ defined by

$$(\hat{M}_\mu F)(x) := \sum_{t_j \in U_\mu(x)} a_j(F) \cdot K_\mu(t_j x) \tag{1.5}$$

for F and x as above. We call $\hat{M}_\mu$ a truncated generalized hyperinterpolation operator, and proved in [3] that under proper assumptions the approximation order is not distroyed, while the cost are reduced considerably – provided the calculation of the nodes, which belong to the neighbourhood, is not too expensive. However, just this is a serious problem. We show how it can be mastered in case of product Gauß quadratures, where the nodes are known explicitly. But foremost we discuss the important question how high degree kernels can be evaluated efficiently and stably at all.

2 Evaluation of the Approximant

2.1 Reduction of the Kernel Degree

It follows from (1.2) that the evaluation of the kernel $K_\mu(\eta)$ at a point η requires two multiplications after the evaluation of the polynomial

$$A_\nu(\eta) := \frac{C^\lambda_{\nu+1}(\eta)}{\eta - \eta_{\nu+1}}. \tag{2.1}$$

Near to $\eta_{\nu+1}$, and this is the interesting area, the division by $\eta - \eta_{\nu+1}$ is numerically outmost problematic – and avoidable. To prove this we use the formulae

$$C^\lambda_{2\kappa}(\eta) = a^\lambda_\kappa \cdot P^{(\lambda-\frac{1}{2},-\frac{1}{2})}_\kappa(2\eta^2 - 1), \tag{2.2}$$

$$C^\lambda_{2\kappa+1}(\eta) = b^\lambda_\kappa \cdot \eta \cdot P^{(\lambda-\frac{1}{2},\frac{1}{2})}_\kappa(2\eta^2 - 1), \tag{2.3}$$

which hold with proper coefficients a_κ^λ and b_κ^λ, see ([6], p.178).

Now let us assume first that $\nu = 2\kappa$ is even, and that ζ_κ is the greatest zero of the Jacobi polynomial $P_\kappa^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta)$. Substituting

$$\zeta = 2\eta^2 - 1, \quad \zeta_\kappa = 2\eta_\kappa^2 - 1,$$

we get from (2.1) and (2.3)

$$A_\nu(\eta) \;=\; b_\kappa^\lambda \cdot \eta \cdot (\eta + \eta_{\nu+1}) \cdot \frac{P_\kappa^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta)}{\zeta - \zeta_\kappa}, \tag{2.4}$$

where ζ_κ is the greatest zero of $P_\kappa^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta)$. We calculate ζ_κ in advance, and get $\eta_{\nu+1}$ by putting

$$\eta_{\nu+1} \;:=\; \sqrt{\frac{1+\zeta_\kappa}{2}}. \tag{2.5}$$

Finally we use the Formula of Christoffel–Darboux, which in our case takes the form

$$\frac{P_\kappa^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta)}{\zeta - \zeta_\kappa} \;=\; \sum_{k=0}^{\kappa-1} c_k^\lambda \cdot P_k^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta_\kappa) \cdot P_k^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta) \tag{2.6}$$

with well-known coefficients c_κ^λ, [6]. We write this in the form

$$\frac{P_\kappa^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta)}{\zeta - \zeta_\kappa} \;=\; \sum_{k=0}^{\kappa-1} \gamma_k^\lambda \cdot P_k^{(\lambda-\frac{1}{2},\frac{1}{2})}(\zeta), \tag{2.7}$$

and assume that the coefficients, which fortunately are all positive, are evaluated and stored also in advance. Now it is easy to evaluate (2.7) by means of the recurrence relation of the Jacobi polynomial in a Clenshaw-type algorithm, which is remarkably stable.

Summarizing we state that the problematic evaluation of the kernel, which has the degree $\mu = 4\kappa$, has been reduced to an unproblematic evaluation of a polynomial of degree $\kappa - 1$ by means of 6 arithmetical operations.

The case $\nu = 2\kappa + 1$ can be treated likewise by means of (2.1) and of (2.2).

2.2 Truncation

In what follows let $r = 3$. We want to define $\hat{L}_\mu$ and likewise $\hat{M}_\mu$ by means of a product Gauß quadrature which is exact of degree $2m+1$, where $m \in \mathbb{N}_0$ is allowed to be different from μ, but is assumed to satisfy $2m \geq \mu$. Such a quadrature is described by the formula

$$\int_{S^2} F(x)\, d\omega(x) = \sum_{j=1}^{2m+2} \sum_{k=1}^{m+1} \tfrac{\pi}{m+1} \cdot c_k \cdot F\Big(\cos \tfrac{j-\frac{1}{2}}{m+1}\pi \cdot \sin\psi_k, \sin \tfrac{j-\frac{1}{2}}{m+1}\pi \cdot \sin\psi_k, \cos\psi_k \Big)$$

to hold for all polynomials F of degree $2m+1$, where the points $\xi_k = \cos\,\psi_k$ are the zeros of the Legendre polynomial $C_{m+1}^{\frac{1}{2}}$, put into the order

$$-1 < \xi_{m+1} < \ldots < \xi_1 < 1, \tag{2.8}$$

while the c_k are the corresponding interpolatory weights, see [5] or ([3], p.211 ff.), e.g. Both are assumed to be calculated in advance at full precision – a serious problem itself.

We introduce the map $x : \mathbb{R} \times [0,\pi] \to S^2$ by

$$x(\phi,\psi) := (\cos\phi \sin\psi, \sin\phi \sin\psi, \cos\psi)', \tag{2.9}$$

and define the nodes

$$t_{j,k} := x(\phi_j, \psi_k) \in S^2 \tag{2.10}$$

by the help of the angles

$$\phi_j := \tfrac{j-\frac{1}{2}}{m+1}\pi \tag{2.11}$$

for $j \in \mathbb{Z}$ and $k \in \{1,\ldots,m+1\}$. Note that the nodes are periodic with respect to j with period $2\mu+2$. This will make the notation easier in what follows. Note also that our quadrature can now be written in the form

$$\int_{S^2} F(x)\, d\omega(x) \;=\; \sum_{j=1}^{2m+2} \sum_{k=1}^{m+1} \tfrac{\pi}{m+1} \cdot c_k \cdot F(t_{j,k}). \tag{2.12}$$

It is uniquely determined by its node system

$$T_3^m := \Big\{ t_{j,k} \,|\, j = 1,\ldots,2m+2,\; k = 1,\ldots,m+1 \Big\}, \tag{2.13}$$

where the subscript indicates that it is generated from the (x_1,x_2)–plane by a product with respect to the x_3–axis. Likewise we define the node systems

T_1^m and T_2^m with respect to cyclically permutated axes.

We have not yet decided which of these quadratures is to be used in the definition of the approximants $\hat{L}_\mu$ and $\hat{M}_\mu$. We decide to use all three together, in dependence of the location of x, and this for the following reason. The node density, say of T_3^m, near to the poles $\pm e_3$ is much larger than at the equator $x_3 = 0$. Therefore the number of nodes, which are located, for instance, in a neighbourhood of e_3, causes much more evaluation work than if the neighbourhood belongs to an equator point. For this reason we use T_3^m in the definition of the approximant only if x belongs to the collar

$$-\frac{1}{\sqrt{3}} \le x_3 \le \frac{1}{\sqrt{3}}. \tag{2.14}$$

For T_1^m and T_2^m similar holds. To be more precise, under the condition (2.14), (1.5) takes the form

$$\left(\hat{M}_\mu^m F\right)(x) = \left(\hat{M}_{\mu,3}^m F\right)(x) := \sum_{t_{j,k} \in U_\mu(x)} a_{j,k}(F) \cdot K_\mu(t_{j,k} x), \tag{2.15}$$

with

$$a_{j,k}(F) := \tfrac{\pi}{m+1} \cdot c_k \cdot F(t_{j,k}),$$

see (2.12) and (2.13), and we change and complete the definition in the remaining cases by putting

$$\left(\hat{M}_\mu^m F\right)(x) := \begin{cases} \left(\hat{M}_{\mu,3}^m F\right)(x) & for\ x_3^2 \le \frac{1}{3}, \\ \left(\hat{M}_{\mu,1}^m F\right)(x) & for\ x_3^2 > \frac{1}{3}\ and\ x_1^2 \le \frac{1}{3}, \\ \left(\hat{M}_{\mu,2}^m F\right)(x) & for\ x_3^2 > \frac{1}{3},\ x_1^2 > \frac{1}{3},\ and\ x_2^2 < \frac{1}{3}, \end{cases}$$

where $\hat{M}_{\mu,1}^m$ and $\hat{M}_{\mu,2}^m$ are defined by means of T_1^m or T_2^m, respectively.

After that we consider without restriction of generality the case (2.14), (2.15), where $(\hat{M}_\mu^m F)(x)$ is defined by the help of T_3^m. We assume that all values $a_{j,k}(F)$, at least all which are really needed, are calculated and stored in advance. Moreover we assume that a truncation number $\kappa \in I\!N_0$ is given which satisfies $2\kappa < m$ and

$$-\xi_{m+1-\kappa} = \xi_{1+\kappa} > \frac{1}{\sqrt{3}}. \tag{2.16}$$

Such a number exists if m is sufficiently large in comparison to κ.

After that let $x = x(\phi, \psi) \in S^2$ be given and satisfying (2.14). We want to determine the neighbourhood of x such that $Q_\kappa(x) := U_\mu(x) \cap T_3^m$ takes the form

$$Q_\kappa(x) := \left\{ t_{j,k} \,|\, j \in \{p-\kappa, \ldots, p+\kappa\},\, k \in \{q-\kappa, \ldots, q+\kappa\} \right\} \tag{2.17}$$

with numbers $p \in \{1, \ldots, 2m+2\}$ and $q \in \{1+\kappa, \ldots, m+1-\kappa\}$. The center $t_{p,q}$ of $Q_\kappa(x)$ should be located as near to x, as possible, and its evaluation should be rather easy.

In order to realize these requirements, we put

$$p := \left\lfloor \frac{m+1}{\pi}\, \phi \right\rfloor + 1, \tag{2.18}$$

such that

$$\phi_p - \frac{\pi}{2(m+1)} \leq \phi < \phi_p + \frac{\pi}{2(m+1)} \tag{2.19}$$

holds, while the calculation of q takes two steps.

First we define $\bar{q} \in \{1+\kappa, \ldots, m-\kappa\}$ such that

$$\psi_{\bar{q}} \leq \psi < \psi_{\bar{q}+1}$$

holds. Such a number exists because of the assumption (2.16). Moreover, following ([6], p. 183), we get the inequalities

$$\frac{\bar{q} - \frac{1}{2}}{m + \frac{3}{2}}\, \pi \;\leq\; \psi_{\bar{q}} \;\leq\; \psi \;<\; \psi_{\bar{q}+1} \;\leq\; \frac{\bar{q}+1}{m+\frac{3}{2}}\, \pi. \tag{2.20}$$

Now we calculate the number

$$s \;:=\; \left\lfloor \frac{m+\frac{3}{2}}{\pi}\, \psi \right\rfloor. \tag{2.21}$$

¿From (2.20) we obtain $s - 1 < \bar{q} \leq s + 1$, and hence $\bar{q} \in \{s, s+1\}$, and applying the cosine-function to (2.20) we find that either

$$\xi_{s+1} < x_3 \leq \xi_s$$

or

$$\xi_{s+2} < x_3 \leq \xi_{s+1}$$

must hold. Therefore we get $\bar{q}$ by the statement

$$if\ \xi_{s+1} < x_3\ then\ \bar{q} := s\ else\ \bar{q} := s+1. \tag{2.22}$$

Moreover, it follows from the definition of $\bar{q}$ that

$$\xi_{\bar{q}+1} < x_3 \leq \xi_{\bar{q}}$$

holds, where one of the points is located nearest to x_3, say ξ_q. We calculate q finally by the statement

$$if\ (\xi_{\bar{q}} + \xi_{\bar{q}+1})/2 \leq x_3\ then\ q := \bar{q}\ else\ q := \bar{q}+1. \tag{2.23}$$

As a summary we state, that we could determine the point $t_{p,q} \in T_3^m$ under the restriction (2.14) by the simple statements (2.18) and (2.21) – (2.23), and this such that

$$|\phi - \phi_p| \leq \frac{\pi}{2(m+1)}$$

and

$$|x_3 - \xi_q| \leq \frac{\xi_{\bar{q}} - \xi_{\bar{q}+1}}{2}$$

is valid. In this sense $t_{p,q}$ is located nearest to x. Finally we remark that (2.15) takes now the form

$$\left(\hat{M}_\mu^{m,\kappa} F\right)(x) = \left(\hat{M}_{\mu,3}^{m,\kappa} F\right)(x) := \sum_{t_{j,k} \in Q_\kappa(x)} a_{j,k}(F) \cdot K_\mu(t_{j,k} x), \tag{2.24}$$

If x does not belong to the collar $x_3^2 \leq \frac{1}{3}$, then we have either $x_1^2 \leq \frac{1}{3}$ or $x_2^2 < \frac{1}{3}$, and the definition of $(\hat{M}_\mu^{m,\kappa} F)(x)$ is completed by a similar construction, where T_1^m or T_2^m, respectively, takes the part of T_3^m.

In all cases the calculation of $(\hat{M}_\mu^{m,\kappa} F)(x)$ is straight-forward and requiring $(2\kappa+1)^2$–times an evaluation of the kernel, i.e. of the univariate polynomial (2.7). By using the recurrence relation of the Jacobi polynomials, we can solve this task by $\approx \kappa^2 \cdot \mu$ arithmetical operations. By the fast Fourier transform, the cost could be reduced to $\approx \log \kappa \cdot \mu$ operations, but the numerical profit arising from formula (2.7) would get lost. Moreover, κ will be a small number in practice.

We finish this section by mentioning that it is possible to derive upper bounds to the error

$$\|F - \hat{M}_\mu^{m,\kappa} F\|_\infty$$

by the method presented in ([3], Section 6.10).

2.3 The Number of Nodes

Each of the systems T_3^m, T_1^m, and T_2^m contains $M(m) = 2(m+1)^2$ nodes, but not all of them are participating in the definition of $\hat{M}_\mu^{m,\kappa}$. We want to estimate the number of nodes which are really used. To this end we define the subsets

$$\begin{aligned} Z_3 &:= \{x \in S^2 \,|\, x_3^2 \leq \tfrac{1}{3}\}, \\ Z_1 &:= \{x \in S^2 \,|\, x_3^2 > \tfrac{1}{3},\ x_1^2 \leq \tfrac{1}{3}\}, \\ Z_2 &:= \{x \in S^2 \,|\, x_3^2 > \tfrac{1}{3},\ x_1^2 > \tfrac{1}{3},\ x_2^2 < \tfrac{1}{3}\}, \end{aligned}$$

and put

$$\begin{aligned} \hat{T}_j^m &:= T_j^m \cap Z_j, \\ n_j &:= card\,(\hat{T}_j^m) \end{aligned} \tag{2.25}$$

for $j = 1, 2, 3$. Obviously, for every $j \in \{1, 2, 3\}$ the n_j nodes of $\hat{T}_j^m$ participate, but there are at most $4(\kappa+1)(m+1)$ further nodes of T_j^m participating, namely those which are located on a parallel circle $x_j = \pm\xi_{n-1}, \ldots, \pm\xi_{n-1-\kappa}$, up to $2(m+1)$ in number on each circle, where n is the lowest number which satisfies

$$\xi_n \leq \frac{1}{\sqrt{3}}.$$

So the total number $N(m, \kappa)$ of nodes used in the definition of $\hat{M}_\mu^{m,\kappa}$ satisfies

$$N(m, \kappa) \leq n_1 + n_2 + n_3 + 12(\kappa + 1)(m + 1). \tag{2.26}$$

To get an upper bound of n_3, we count the parallel circles $x_3 = \xi_n, \ldots, \xi_{m+2-n}$ which are contained in Z_3. Their number is $m+3-2n$. Following again ([6], p. 183), we get

$$\arccos \frac{1}{\sqrt{3}} \leq \psi_n \leq \frac{n}{m + \frac{3}{2}} \cdot \pi,$$

and hence

$$n \geq (m + \tfrac{3}{2}) \cdot c \tag{2.27}$$

with the constant

$$c := \frac{1}{\pi} \arccos \frac{1}{\sqrt{3}}.$$

Moreover, each of these parallel circles contains exactly $2(m+1)$ nodes of T_3^m, so we get $n_3 = 2(m+3-2n)(m+1)$, and hence, in view of (2.27),

$$n_3 \le 2\Big(m+3-(2m+3)c\Big)\cdot(m+1). \tag{2.28}$$

Next we estimate n_1 by considering the orthogonal projection of S^2 onto the (x_2,x_3)–plane. Here the nodes of $\hat{T}_1^m$ occur on the circles $x_2^2+x_3^2 = 1-x_1^2$ with $x_1 \in \{\xi_n,\dots,\xi_{m+2-n}\}$, restricted to the domain $x_3^2 > \frac{1}{3}$. On each of these circle restrictions there are at most as many nodes, as angles $\phi_j,\ j \in \{1,\dots,2m+2\}$ exist which satisfy

$$|\sin\phi_j| > \frac{1}{\sqrt{3}}.$$

By symmetry, this is twice the number attainable under the restriction $j \in \{1,\dots,m+1\}$. In view of the symmetry $\phi_{m+2-j} = \pi - \phi_j$, we may assume that these are the angles

$$\phi_j,\ \ j \in \Big\{k,\dots,m+2-k\Big\},\ \ k \in \Big\{1,\dots,\lfloor\tfrac{m+2}{2}\rfloor\Big\}. \tag{2.29}$$

Obviously, their number is $m+3-2k$, where k satisfies $\sin\phi_k > \frac{1}{\sqrt{3}}$, which implies

$$k > \frac{1}{2} + (m+1)\cdot s,$$

where

$$s := \frac{1}{\pi}\arcsin\frac{1}{\sqrt{3}}.$$

Altogether this yields $n_1 \le 2(m+3-2n)(m+3-2k)$, and hence

$$n_1 \ \le\ 2\Big(m+3-(2m+3)c\Big)\cdot\Big(m+2-2(m+1)s\Big). \tag{2.30}$$

Likewise we estimate n_2 by considering the orthogonal projection of S^2 onto the (x_3,x_1)–plane. The nodes of $\hat{T}_2$ can now be counted by the number of parallel circles $x_3^2+x_1^2 = 1-x_2^2$, $x_2 \in \{\xi_n,\dots,\xi_{m+2-n}\}$, and the number of j–s which satisfy

$$|\sin\phi_j| > \frac{1}{\sqrt{3}} \quad \textit{and} \quad |\cos\phi_j| > \frac{1}{\sqrt{3}}.$$

This is fourtimes the number of the corresponding ϕ_j–s which are located in the interval $[0, \frac{\pi}{2})$. Assume these are exactly the angles

$$\phi_j, \;\; j \in \{k, \ldots, l\}, \tag{2.31}$$

where k is as above, while l satisfies $\cos\phi_l > \frac{1}{\sqrt{3}}$. Then their number is $l + 1 - k$, where

$$l < \frac{1}{2} + (m+1) \cdot c,$$

c as above. Altogether this yields $n_2 \leq 4(m + 3 - 2n)(l + 1 - k)$, and hence

$$n_2 \;\leq\; 4\Big(m + 3 - (2m+3)c\Big) \cdot \Big(1 + (m+1)(c-s)\Big). \tag{2.32}$$

Finally we obtain from (2.28),(2.30), and (2.32) the inequalities

$$\begin{aligned} n_1 + n_2 + n_3 &< 2\Big(m + 3 - (2m+3)c\Big) \cdot \Big(2m + 5 + 2(m+1)(c-2s)\Big) \\ &\lesssim 4(1-2c)(1+c-2s)(m+1)^2 \\ &< 0.72 \cdot M(m). \end{aligned} \tag{2.33}$$

Therefore the following corollary holds in view of (2.26).

Corollary.
Assume $\kappa = \kappa(m)$ satisfies $\kappa(m) = \mathrm{o}(m)$. Then the total number $N(m, \kappa)$ of nodes which occur in the definition of $\hat{M}_\mu^{m,\kappa}$ satisfies

$$N(m, \kappa) \lesssim 0.72 \cdot M(m)$$

for $m \to \infty$, where $M(m) = 2(m+1)^2$ is the number of nodes in a product Gauß quadrature which is exact of degree $2m + 1$.

The Corollary says that though we use three different product Gauß quadratures, instead of one, the number of function values which are used is reduced significantly. This effect is due to the minor node density near to the equator.

3 Numerical Examples

In ([3], Section 6.10) we discussed a numerical example in order to show that the approximation order is not increased essentially by modest truncation.

However, to get a reasonable processing time, we allowed the quadrature to depend on the point in calculation. This method is no more constructive, contrary to the method defined in Section 2.2 of this paper, which we investigate numerically in what follows for various parameter values. The kernel degree has the value $\mu = 160$ or 300, while the product Gauß quadratures are exact for all polynomials of degree to 161, 181, or 321, respectively.

To make figurative, how truncation works, we consider first the action of $\hat{L}_\mu$ on the unity function $F_0 = 1$. We use a product Gauß quadrature which is exact for all polynomials of degree $2m+1 = 161$, such that $L_\mu 1 = 1 = \hat{L}_\mu 1$ is valid. Its number of nodes is $M = 13122$. To obtain Figure 1, we ordered the nodes t_j by their distance from the test point $x = e_3$, and asked how much do the first n nodes contribute to the sum (1.4). In other words, Figure 1 shows the function f defined by

$$f(n) := \sum_{j=1}^{n} A_j K_{160}(t_j e_3) \to \left(\hat{L}_{160} 1\right)(e_3) = 1$$

for $n = 1, \ldots, M$, where we use a logarithmic scale in the n–direction.

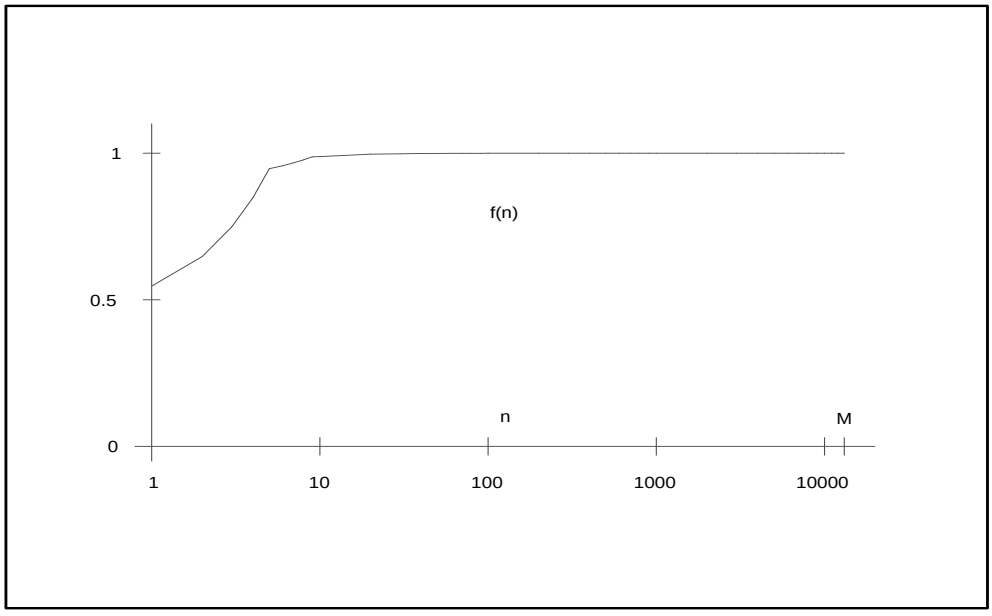

Figure 1. Contribution of the First n Points to Unity.

Obviously, the first 10 nodes are important, the next 40 increase the accuracy, but the remaining are more or less redundant. This result is covered

by our theory developped in ([3], Section 6.10).

To obtain the following numerical results, we used a Maple implementation of the operator $\hat{M}_{\mu}^{m,\kappa}$, as it is defined in Section 2.2 by the help of the node systems T_3^m, T_1^m, and T_2^m.

Tabular 1 shows, again for $F_0 = 1$, the error caused by the truncated operators. This function stands for an arbitrary harmless function.

κ	node number	$\|1 - \hat{M}_{160}^{80,\kappa} 1\|_\infty \approx$
0	1	$7 \cdot 10^{-1}$
1	9	$3 \cdot 10^{-2}$
2	25	$4 \cdot 10^{-3}$
3	49	$1 \cdot 10^{-3}$
4	81	$5 \cdot 10^{-4}$
5	121	$3 \cdot 10^{-4}$

Tabular 1. Error of the Truncated Operator ($F_0 = 1$).
The product Gauß quadrature is exact of degree 161
and acts on M=13122 nodes.

In comparison to the cost, the gain of accuracy is great in the beginning, say from $\kappa = 0$ to $\kappa = 1$ or $\kappa = 2$, but a greater κ is just causing evaluation cost – in correspondence to our reasoning from above.

In the following we raise the parameters. For $m = 90$ (160), the product Gauß quadrature is exact for all polynomials of degree 181 (321), and using $M = 16252$ (52842) nodes. Moreover, in the representation of spherical functions, we use the parametrisation $x = x(u)$ of S^2 defined by

$$\begin{aligned} x_1 &= \sqrt{1 - \frac{u_1^2 + u_2^2}{4}} \cdot u_1, \\ x_2 &= \sqrt{1 - \frac{u_1^2 + u_2^2}{4}} \cdot u_2, \\ x_3 &= 1 - \frac{u_1^2 + u_2^2}{2} \end{aligned}$$

for $u_1^2 + u_2^2 \leq 4$. It is one-to-one, except for the margin $u_1^2 + u_2^2 = 4$, which corresponds to the 'south pole' $-e_3$. The map $u \mapsto x(u)$ is area preserving,

such that the originals of the nodes occur on the disk $u_1^2 + u_2^2 \leq 4$ with the same density as the nodes on the sphere S^2 themselves.

In this parametrisation, the next test function is given by

$$F_1(x(u)) \;=\; \frac{1}{1 + 8(u_1 - \frac{1}{2})^2 + 3(u_2 - \frac{3}{10})^2} \tag{3.1}$$

for $|u| \leq 2$. It is no more a polynomial, satisfies

$$0 \;\leq\; F_1(x(u)) \;\leq\; 1,$$

and represents an average function. Figure 2 shows this function, Figure 3a and Figure 3b error functions belonging to it for the truncation parameter $\kappa = 3$, both restricted to the northern hemisphere $S^2_+ := \{x \in S^2 \,|\, x_3 \geq 0\}$, which corresponds to the disk $u_1^2 + u_2^2 \leq 2$.

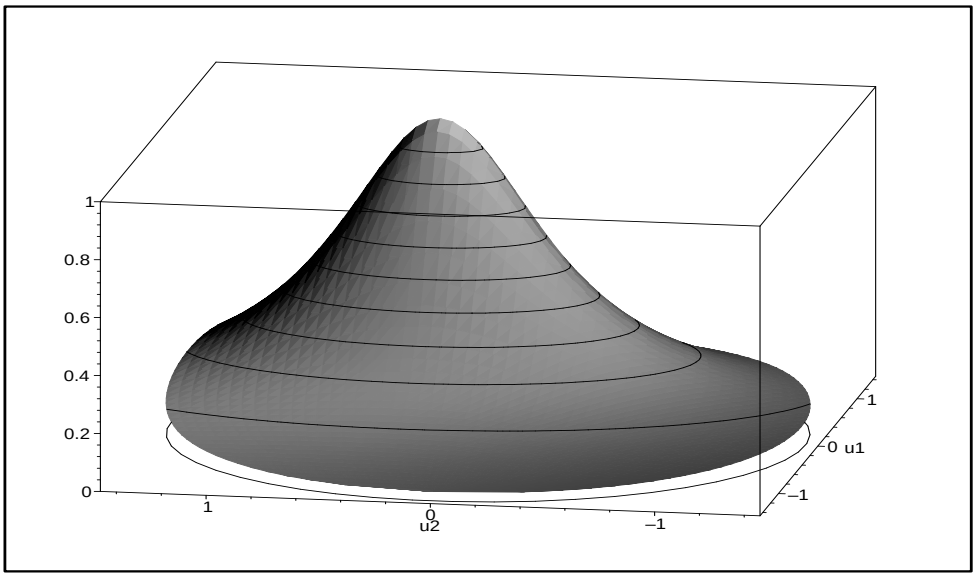

Figure 2. Original Function $F_1((x(u))$.

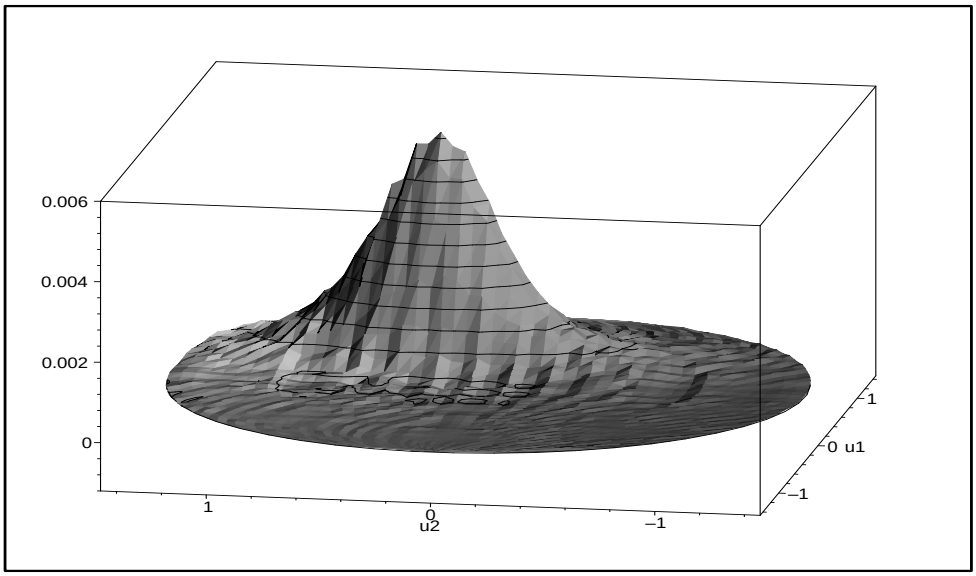

Figure 3a. Error Function belonging to Figure 2.
($\mu = 160,\ m = 90,\ \kappa = 3$)

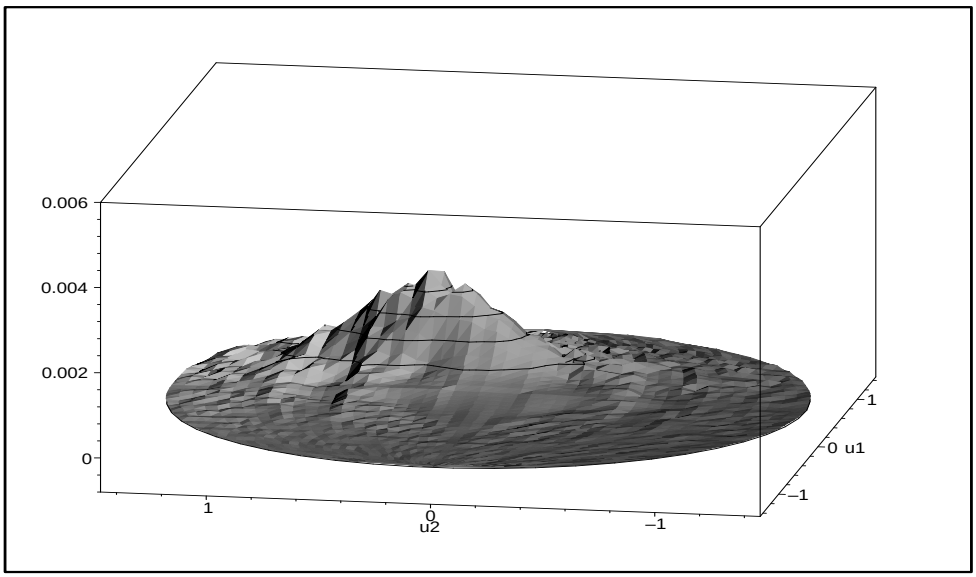

Figure 3b. Error Function belonging to Figure 2
($\mu = 300,\ m = 160,\ \kappa = 3$)

Finally we consider the test function defined by

$$F_2(x(u)) \;=\; \tfrac{1}{2}\left(1+\sin\left[5(u_1-u_2)(u_2-\tfrac{3}{5})\right]\right) \tag{3.2}$$

for $|u| \le 2$, which satisfies again

$$0 \;\le\; F_2(x(u)) \;\le\; 1,$$

but is no more so harmless. It is thought to simulate a geodetic situation, with mountains, valleys, and a saddle-point. We introduced it already as a test function in [3]. Note that it is far off being a polynomial.

Tabular 2 shows the error caused by truncation for several parameter values.

κ	node number	$\|F-\hat{M}_{160}^{90,\kappa}F\|_\infty \approx$	$\|F-\hat{M}_{160}^{160,\kappa}F\|_\infty \approx$	$\|F-\hat{M}_{300}^{160,\kappa}F\|_\infty \approx$
0	1	$7\cdot 10^{-1}$	$9\cdot 10^{-1}$	$8\cdot 10^{-1}$
1	9	$6\cdot 10^{-2}$	$3\cdot 10^{-1}$	$4\cdot 10^{-2}$
2	25	$3\cdot 10^{-2}$	$5\cdot 10^{-2}$	$1\cdot 10^{-2}$
3	49	$3\cdot 10^{-2}$	$3\cdot 10^{-2}$	$1\cdot 10^{-2}$
4	81	$3\cdot 10^{-2}$	$3\cdot 10^{-2}$	$1\cdot 10^{-2}$
5	121	$3\cdot 10^{-2}$	$3\cdot 10^{-2}$	$1\cdot 10^{-2}$

Tabular 2. Error of Truncated Approximation ($F = F_2$).

Obviously, in view of areas where the curvature of F is very large, it does not pay to use locally more than 25 or at most 49 nodes, or even to evaluate $(\hat{L}_\mu F)(x)$ at full precision, such that we may speak of κ–*saturation*.

Moreover, it is worthwhile to consider the general case $m = \mu$, where $\hat{L}_\mu$ is a hyperinterpolation operator in the strict sense, see ([3], Definition 6.7). In this case, the number of quadrature nodes is $M = 2(\mu+1)^2$, and the average of the weights equals $\frac{4\pi}{M}$. Therefore we get, by the arguments used in ([4], page 253, Remark 3), that the average–weighted kernel function satisfies

$$\frac{2\pi}{(\mu+1)^2}K_\mu(1) \sim \frac{1}{2j_{0,1}^2} \approx 0.0862 \approx \frac{1}{12},$$

where $j_{0,1}$ is the lowest positive zero of the Bessel function J_0. This says that the 11 or 12 nodes nearest to x, must contribute already the essential part

to the value of $(\hat{L}_\mu F)(x)$ – in accordance to our experimental results.

Figure 4 and Figure 5a and 5b show the original function and the corresponding error functions for $m = 90$ and $m = 160$, both restricted to S^2_+, again for the truncation parameter $\kappa = 3$.

Finally we want to give an impression of the processing time needed: A single evaluation $u \mapsto (\hat{M}^{90,3}_{160} F)(x(u))$, for instance, takes about 1.5 seconds on a simple personal computer (Pentium-S, CPU at 133 MHz). Accordingly, the calculation of a whole image, based on a usual 15×15–grid, requires about 6 minutes. However, the untruncated operator would need about 20 hours.

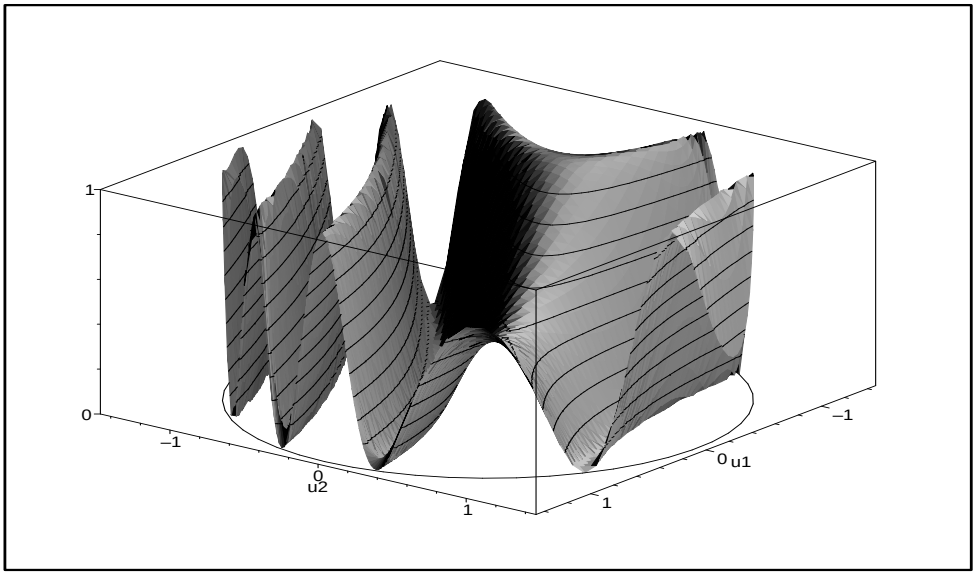

Figure 4. Original Function $F_2((x(u))$

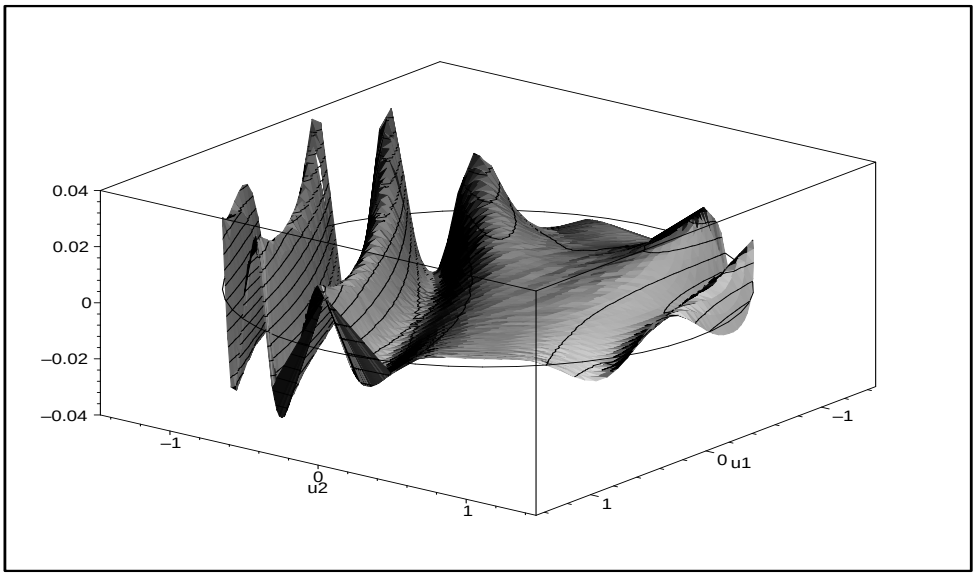

Figure 5a. Error Function belonging to Figure 4.
($\mu = 160,\ m = 90,\ \kappa = 3$)

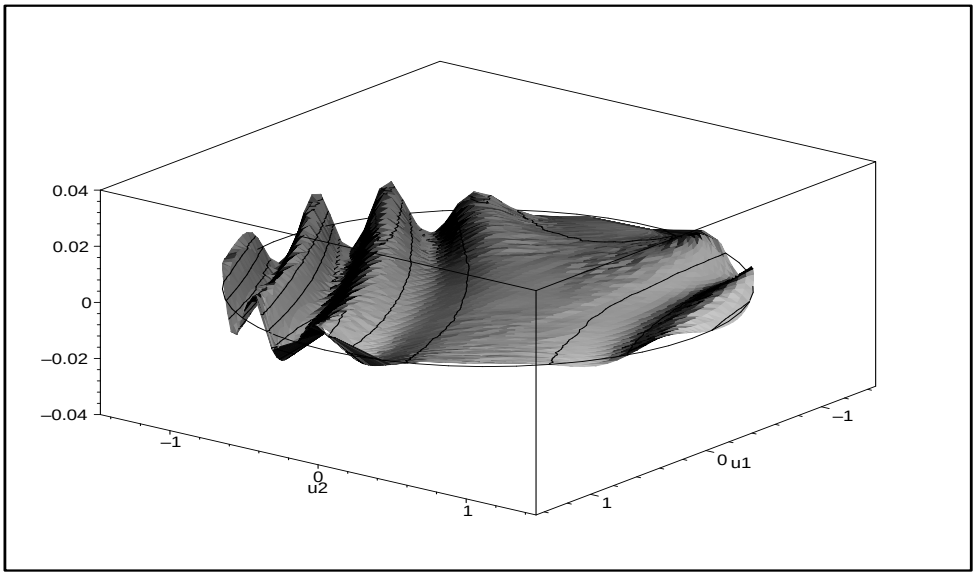

Figure 5b. Error Function belonging to Figure 4.
($\mu = 300,\ m = 160,\ \kappa = 3$)

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