technische universität
dortmund

On a minimax principle in spectral gaps

Albrecht Seelmann

March 2021

Croatian German meeting on analysis and mathematical physics

arXiv:2008.02489

Motivation

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\mathbf{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\mathbf{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;
- $\nu > 0$, $v_* \geq 0$.

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\boldsymbol{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;
- $\nu > 0$, $\upsilon_* \geq 0$.

On $\mathcal{H} = L^2(\Omega)^n \oplus L^2(\Omega)$ consider the self-adjoint extension B_S of the (non-closed) block operator matrix

$$\begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix} \quad \text{on} \quad (H^2(\Omega) \cap H_0^1(\Omega))^n \oplus H^1(\Omega).$$

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\boldsymbol{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;
- $\nu > 0$, $\upsilon_* \geq 0$.

On $\mathcal{H} = L^2(\Omega)^n \oplus L^2(\Omega)$ consider the self-adjoint extension B_S of the (non-closed) block operator matrix

$$\begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix} \quad \text{on} \quad (H^2(\Omega) \cap H_0^1(\Omega))^n \oplus H^1(\Omega).$$

B_S *Stokes operator*.

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\mathbf{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;
- $\nu > 0$, $v_* \geq 0$.

On $\mathcal{H} = L^2(\Omega)^n \oplus L^2(\Omega)$ consider the self-adjoint extension B_S of the (non-closed) block operator matrix

$$\begin{pmatrix} -\nu\mathbf{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix} \quad \text{on} \quad (H^2(\Omega) \cap H_0^1(\Omega))^n \oplus H^1(\Omega).$$

B_S *Stokes operator*.

By Grubb, Geymonat 1977; Faierman, Fries, Mennicken, Möller 2000:

$$\sigma_{\text{ess}}(B_S) = \left\{-\frac{v_*^2}{\nu}, -\frac{v_*^2}{2\nu}\right\} \subset (-\infty, 0).$$

Motivation

Let

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$, bounded C^2-domain;
- $\mathbf{\Delta} = \Delta \cdot I_{\mathbb{C}^n}$ vector-valued Dirichlet Laplacian on Ω;
- $\nu > 0$, $v_* \geq 0$.

On $\mathcal{H} = L^2(\Omega)^n \oplus L^2(\Omega)$ consider the self-adjoint extension B_S of the (non-closed) block operator matrix

$$\begin{pmatrix} -\nu\mathbf{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix} \quad \text{on} \quad (H^2(\Omega) \cap H_0^1(\Omega))^n \oplus H^1(\Omega).$$

B_S *Stokes operator*.

By Grubb, Geymonat 1977; Faierman, Fries, Mennicken, Möller 2000:

$$\sigma_{\mathrm{ess}}(B_S) = \Big\{ -\frac{v_*^2}{\nu}, -\frac{v_*^2}{2\nu} \Big\} \subset (-\infty, 0).$$

In particular: $\sigma(B_S) \cap (0, \infty)$ discrete.

Let

- E_{B_S}: projection-valued spectral measure for B_S;

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\mathrm{Ran}\,\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset\operatorname{Dom}[\mathfrak{b}_S]\cap\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))\\ \dim\mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M}\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset\operatorname{Dom}[\mathfrak{b}_S]\cap\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))\\ \dim\mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M}\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset \operatorname{Dom}[\mathfrak{b}_S]\cap \operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))\\ \dim\mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M}\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Goal: Find representation of λ_k that does not make use of $\mathsf{E}_{B_S}((0,\infty))$.

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset\operatorname{Dom}[\mathfrak{b}_S]\cap\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))\\ \dim\mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M}\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Goal: Find representation of λ_k that does not make use of $\mathsf{E}_{B_S}((0,\infty))$.

Several existing abstract approaches:

- Griesemer, Lewis, Siedentop '99;

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M} \subset \operatorname{Dom}[\mathfrak{b}_S] \cap \operatorname{Ran}\mathsf{E}_{B_S}((0,\infty)) \\ \dim \mathfrak{M} = k}} \ \sup_{\substack{f \in \mathfrak{M} \\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Goal: Find representation of λ_k that does not make use of $\mathsf{E}_{B_S}((0,\infty))$.

Several existing abstract approaches:

- Griesemer, Lewis, Siedentop ’99;
- Dolbeault, Esteban, Séré 2000; Morozov, D. Müller 2015;

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset\operatorname{Dom}[\mathfrak{b}_S]\cap\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))\\ \dim\mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M}\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Goal: Find representation of λ_k that does not make use of $\mathsf{E}_{B_S}((0,\infty))$.

Several existing abstract approaches:

- Griesemer, Lewis, Siedentop '99;
- Dolbeault, Esteban, Séré 2000; Morozov, D. Müller 2015;
- Langer, Strauss 2016.

Let

- E_{B_S}: projection-valued spectral measure for B_S;
- $\lambda_k = \lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))})$, $k \in \mathbb{N}$: k-th positive eigenvalue of B_S (counting multiplicities);
- $\mathfrak{b}_S$: form associated with B_S.

Then

$$\lambda_k = \inf_{\substack{\mathfrak{M}\subset \operatorname{Dom}[\mathfrak{b}_S]\cap \operatorname{Ran}\mathsf{E}_{B_S}((0,\infty)) \\ \dim \mathfrak{M}=k}} \sup_{\substack{f\in\mathfrak{M} \\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

Goal: Find representation of λ_k that does not make use of $\mathsf{E}_{B_S}((0,\infty))$.

Several existing abstract approaches:

- Griesemer, Lewis, Siedentop '99;
- Dolbeault, Esteban, Séré 2000; Morozov, D. Müller 2015;
- Langer, Strauss 2016.

An abstract result

An abstract result

Let

- A, B self-adjoint with $\mathrm{Dom}(|A|^{1/2}) = \mathrm{Dom}(|B|^{1/2})$;

An abstract result

Let

- A, B self-adjoint with $\mathrm{Dom}(|A|^{1/2}) = \mathrm{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_{\bullet}((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;

An abstract result

Let

- A, B self-adjoint with $\mathrm{Dom}(|A|^{1/2}) = \mathrm{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \mathrm{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;

An abstract result

Let

- A, B self-adjoint with $\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \operatorname{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;
- $\mathfrak{b}[x, y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y \rangle$ form associated to B.

An abstract result

Let

- A, B self-adjoint with $\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \operatorname{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;
- $\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle$ form associated to B.

Theorem (Griesemer, Lewis, Siedentop '99)

If

- $\mathfrak{b}[x,x] \le 0 \quad \forall x \in \mathfrak{D}_-$

An abstract result

Let

- A, B self-adjoint with $\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \operatorname{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;
- $\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle$ form associated to B.

Theorem (Griesemer, Lewis, Siedentop '99)

If

- $\mathfrak{b}[x,x] \le 0 \quad \forall x \in \mathfrak{D}_-$, and
- $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$,

An abstract result

Let

- A, B self-adjoint with $\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \operatorname{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;
- $\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle$ form associated to B.

Theorem (Griesemer, Lewis, Siedentop '99)

If

- $\mathfrak{b}[x,x] \le 0 \quad \forall x \in \mathfrak{D}_-$, and
- $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$,

then

$$\lambda_k(B|_{\operatorname{Ran} P_{B,+}}) = \inf_{\substack{\mathfrak{M}_+ \subset \mathfrak{D}_+ \\ \dim \mathfrak{M}_+ = k}} \sup_{\substack{x \in \mathfrak{M}_+ \oplus \mathfrak{D}_- \\ \|x\|=1}} \mathfrak{b}[x,x]$$

for $k \le \dim \operatorname{Ran} P_{A,+}$.

An abstract result

Let

- A, B self-adjoint with $\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|B|^{1/2})$;
- $P_{\bullet,+} := \mathsf{E}_\bullet((0,\infty))$, $P_{\bullet,-} := I - P_{\bullet,+}$, $\bullet \in \{A, B\}$;
- $\mathfrak{D}_\pm := \operatorname{Dom}(|A|^{1/2}) \cap \operatorname{Ran} P_{A,\pm}$;
- $\mathfrak{b}[x,y] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}y\rangle$ form associated to B.

Theorem (Griesemer, Lewis, Siedentop '99)

If

- $\mathfrak{b}[x,x] \le 0 \quad \forall x \in \mathfrak{D}_-$, and
- $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$, (Note: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \subset \mathfrak{D}_+$ automatic)

then

$$\lambda_k(B|_{\operatorname{Ran} P_{B,+}}) = \inf_{\substack{\mathfrak{M}_+ \subset \mathfrak{D}_+ \\ \dim \mathfrak{M}_+ = k}} \ \sup_{\substack{x \in \mathfrak{M}_+ \oplus \mathfrak{D}_- \\ \|x\|=1}} \mathfrak{b}[x,x]$$

for $k \le \dim \operatorname{Ran} P_{A,+}$.

Two projections

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X \colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I + X^*X)^{-1}$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I+X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I+X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I+X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Essentially due to Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2020:

Lemma

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I+X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Essentially due to Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2020:

Lemma

If $(I+XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$, *then* $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I + X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Essentially due to Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2020:

Lemma

If $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$, then $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Note: If A is lower semibounded, then $\mathfrak{D}_- = \operatorname{Ran} P_{A,-}$.

Two projections

Suppose that $\|P_{A,+} - P_{B,+}\| < 1$.

Then: $\exists!\, X\colon \operatorname{Ran} P_{A,+} \to \operatorname{Ran} P_{A,-}$ bounded linear with

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Thus, $P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,+}} = (I+X^*X)^{-1}$.
In particular: $\operatorname{Ran}(P_{A,+}P_{B,+}|_{\mathfrak{D}_+}) \supset \mathfrak{D}_+$ iff $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Essentially due to Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2020:

Lemma

If $(I+XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$, then $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Note: If A is lower semibounded, then $\mathfrak{D}_- = \operatorname{Ran} P_{A,-}$.

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$.

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\mathrm{Ran}\, P_{A,+}})f$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$.   To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_-$

Proof

Let $(I+XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I+XX^*)^{-1}Xf = X(I+X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_- \quad \Rightarrow h := (I+XX^*)Xf \in \mathfrak{D}_-$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I+XX^*)^{-1}Xf = X(I+X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_-$ $\quad \Rightarrow h := (I+XX^*)Xf \in \mathfrak{D}_-$

Now:

$$(I+X^*X)f = f + X^*Xf$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$. Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_- \quad \Rightarrow h := (I + XX^*)Xf \in \mathfrak{D}_-$

Now:

$$(I + X^*X)f = f + X^*Xf = f + X^*(I + XX^*)^{-1}h$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_- \quad \Rightarrow h := (I + XX^*)Xf \in \mathfrak{D}_-$

Now:

$$(I + X^*X)f = f + X^*Xf = f + X^*(I + XX^*)^{-1}h = f + (I + X^*X)^{-1}X^*h$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I + X^*X)^{-1} & (I + X^*X)^{-1}X^* \\ X(I + X^*X)^{-1} & X(I + X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\operatorname{Ran} P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_- \quad \Rightarrow h := (I + XX^*)Xf \in \mathfrak{D}_-$

Now:

$$\begin{aligned}(I + X^*X)f &= f + X^*Xf = f + X^*(I + XX^*)^{-1}h = f + (I + X^*X)^{-1}X^*h \\ &= f + (P_{A,+}P_{B,+}|_{\operatorname{Ran} P_{A,-}})h\end{aligned}$$

Proof

Let $(I + XX^*)\mathfrak{D}_- \subset \mathfrak{D}_-$. To show: $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.
Recall

$$P_{B,+} = \begin{pmatrix} (I+X^*X)^{-1} & (I+X^*X)^{-1}X^* \\ X(I+X^*X)^{-1} & X(I+X^*X)^{-1}X^* \end{pmatrix}.$$

Let $f \in \mathfrak{D}_+$.Then

$$(I + XX^*)^{-1}Xf = X(I + X^*X)^{-1}f = (P_{A,-}P_{B,+}|_{\mathrm{Ran}\, P_{A,+}})f \in \mathfrak{D}_-$$

$\Rightarrow Xf \in \mathfrak{D}_- \quad \Rightarrow h := (I + XX^*)Xf \in \mathfrak{D}_-$

Now:

$$\begin{aligned}(I + X^*X)f &= f + X^*Xf = f + X^*(I + XX^*)^{-1}h = f + (I + X^*X)^{-1}X^*h \\ &= f + (P_{A,+}P_{B,+}|_{\mathrm{Ran}\, P_{A,-}})h \in \mathfrak{D}_+. \quad q.e.d.\end{aligned}$$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \, \mathrm{d}x.$$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \,\mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form.

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \, \mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form. $(\mathfrak{a} \sim A := -\nu\boldsymbol{\Delta} \oplus 0)$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \, \mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form. $(\mathfrak{a} \sim A := -\nu\boldsymbol{\Delta} \oplus 0)$
$\mathfrak{D}_+ = H_0^1(\Omega)^n \oplus 0, \quad \mathfrak{D}_- = 0 \oplus L^2(\Omega).$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \, \mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form. $(\mathfrak{a} \sim A := -\nu\boldsymbol{\Delta} \oplus 0)$
$\mathfrak{D}_+ = H_0^1(\Omega)^n \oplus 0, \quad \mathfrak{D}_- = 0 \oplus L^2(\Omega)$.
On $\operatorname{Dom}[\mathfrak{v}] := \operatorname{Dom}[\mathfrak{a}]$ consider the symmetric form

$$\mathfrak{v}[v \oplus q, u \oplus p] = -\upsilon_* \langle \operatorname{div} v, p \rangle_{L^2(\Omega)} - \upsilon_* \langle q, \operatorname{div} u \rangle_{L^2(\Omega)}.$$

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \,\mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form. $(\mathfrak{a} \sim A := -\nu\boldsymbol{\Delta} \oplus 0)$
$\mathfrak{D}_+ = H_0^1(\Omega)^n \oplus 0, \quad \mathfrak{D}_- = 0 \oplus L^2(\Omega)$.
On $\operatorname{Dom}[\mathfrak{v}] := \operatorname{Dom}[\mathfrak{a}]$ consider the symmetric form

$$\mathfrak{v}[v \oplus q, u \oplus p] = -v_* \langle \operatorname{div} v, p \rangle_{L^2(\Omega)} - v_* \langle q, \operatorname{div} u \rangle_{L^2(\Omega)}.$$

It is known that $\mathfrak{v}$ is infinitesimally $\mathfrak{a}$-bounded

The Stokes operator revisited

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

On $\operatorname{Dom}[\mathfrak{a}] := H_0^1(\Omega)^n \oplus L^2(\Omega)$ consider

$$\mathfrak{a}[v \oplus q, u \oplus p] = \nu \sum_{j=1}^{n} \int_\Omega \langle \partial_j v(x), \partial_j u(x) \rangle_{\mathbb{C}^n} \, \mathrm{d}x.$$

$\mathfrak{a}$ is a closed, densely defined, and nonnegative form. $(\mathfrak{a} \sim A := -\nu\boldsymbol{\Delta} \oplus 0)$
$\mathfrak{D}_+ = H_0^1(\Omega)^n \oplus 0, \quad \mathfrak{D}_- = 0 \oplus L^2(\Omega)$.
On $\operatorname{Dom}[\mathfrak{v}] := \operatorname{Dom}[\mathfrak{a}]$ consider the symmetric form

$$\mathfrak{v}[v \oplus q, u \oplus p] = -\upsilon_* \langle \operatorname{div} v, p \rangle_{L^2(\Omega)} - \upsilon_* \langle q, \operatorname{div} u \rangle_{L^2(\Omega)}.$$

It is known that $\mathfrak{v}$ is infinitesimally $\mathfrak{a}$-bounded, so there is a unique lower semibounded self-adjoint operator B_S associated with $\mathfrak{b}_S := \mathfrak{a} + \mathfrak{v}$.

$$B_S \supset \begin{pmatrix} -\nu \boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

$$B_S \supset \begin{pmatrix} -\nu\mathbf{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \le 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

$$B_S \supset \begin{pmatrix} -\nu\mathbf{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f]$$

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \mathfrak{a}[0 \oplus p, 0 \oplus p] + \mathfrak{v}[0 \oplus p, 0 \oplus p]$$

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \underbrace{\mathfrak{a}[0 \oplus p, 0 \oplus p]}_{=0} + \mathfrak{v}[0 \oplus p, 0 \oplus p]$$

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \underbrace{\mathfrak{a}[0 \oplus p, 0 \oplus p]}_{=0} + \underbrace{\mathfrak{v}[0 \oplus p, 0 \oplus p]}_{=0}$$

$$B_S \supset \begin{pmatrix} -\nu\mathbf{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \leq 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \underbrace{\mathfrak{a}[0 \oplus p, 0 \oplus p]}_{=0} + \underbrace{\mathfrak{v}[0 \oplus p, 0 \oplus p]}_{=0} \leq 0$$

$$B_S \supset \begin{pmatrix} -\nu \boldsymbol{\Delta} & \upsilon_* \operatorname{grad} \\ -\upsilon_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \le 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \underbrace{\mathfrak{a}[0 \oplus p, 0 \oplus p]}_{=0} + \underbrace{\mathfrak{v}[0 \oplus p, 0 \oplus p]}_{=0} \le 0$$

By Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2019, 2020:

$$\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| \le \frac{\sqrt{2}}{2} < 1$$

$$B_S \supset \begin{pmatrix} -\nu\boldsymbol{\Delta} & v_* \operatorname{grad} \\ -v_* \operatorname{div} & 0 \end{pmatrix}$$

Recall: Need to show $\mathfrak{b}_S[f,f] \le 0$ for all $f \in \mathfrak{D}_-$ and $\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| < 1$.

For $f = 0 \oplus p \in \mathfrak{D}_-$:

$$\mathfrak{b}_S[f,f] = \underbrace{\mathfrak{a}[0\oplus p, 0\oplus p]}_{=0} + \underbrace{\mathfrak{v}[0\oplus p, 0\oplus p]}_{=0} \le 0$$

By Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2019, 2020:

$$\|\mathsf{E}_A((0,\infty)) - \mathsf{E}_{B_S}((0,\infty))\| \le \frac{\sqrt{2}}{2} < 1 \quad \text{(à la Davis-Kahan } \tan 2\Theta \text{ theorem)}$$

Theorem

$$\lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))}) = \inf_{\substack{\mathfrak{M}_+\subset H_0^1(\Omega)^n\\ \dim\mathfrak{M}_+=k}} \sup_{\substack{u\oplus p\in\mathfrak{M}_+\oplus L^2(\Omega)\\ \|u\|^2_{L^2(\Omega)^n}+\|p\|^2_{L^2(\Omega)}=1}} \mathfrak{b}_S[u\oplus p, u\oplus p].$$

Theorem

$$\lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))}) = \inf_{\substack{\mathfrak{M}_+\subset H_0^1(\Omega)^n\\ \dim\mathfrak{M}_+=k}} \sup_{\substack{u\oplus p\in\mathfrak{M}_+\oplus L^2(\Omega)\\ \|u\|^2_{L^2(\Omega)^n}+\|p\|^2_{L^2(\Omega)}=1}} \mathfrak{b}_S[u\oplus p, u\oplus p].$$

Corollary

$$\nu\lambda_k(-\mathbf{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\mathbf{\Delta}) + \frac{v_*^2}{\nu}$$

Theorem

$$\lambda_k(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))}) = \inf_{\substack{\mathfrak{M}_+\subset H_0^1(\Omega)^n\\ \dim\mathfrak{M}_+=k}} \sup_{\substack{u\oplus p\in\mathfrak{M}_+\oplus L^2(\Omega)\\ \|u\|^2_{L^2(\Omega)^n}+\|p\|^2_{L^2(\Omega)}=1}} \mathfrak{b}_S[u\oplus p, u\oplus p].$$

Corollary

$$\nu\lambda_k(-\mathbf{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\mathbf{\Delta}) + \frac{v_*^2}{\nu}$$

Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2019 ($k=1$):

$$\nu\lambda_k(-\Delta) \le \lambda_1(B_S|_{\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))}) \le \nu\lambda_1(-\Delta) + v_*\|\operatorname{div} u_0\|_{L^2(\Omega)},$$

Theorem

$$\lambda_k(B_S|_{\operatorname{Ran} \mathsf{E}_{B_S}((0,\infty))}) = \inf_{\substack{\mathfrak{M}_+ \subset H_0^1(\Omega)^n \\ \dim \mathfrak{M}_+ = k}} \sup_{\substack{u \oplus p \in \mathfrak{M}_+ \oplus L^2(\Omega) \\ \|u\|^2_{L^2(\Omega)^n} + \|p\|^2_{L^2(\Omega)} = 1}} \mathfrak{b}_S[u \oplus p, u \oplus p].$$

Corollary

$$\nu\lambda_k(-\boldsymbol{\Delta}) \leq \lambda_k(B_S|\operatorname{Ran} \mathsf{E}_{B_S}((0,\infty))) \leq \nu\lambda_k(-\boldsymbol{\Delta}) + \frac{v_*^2}{\nu}$$

Grubišić, Kostrykin, Makarov, Schmitz, K. Veselić 2019 ($k = 1$):

$$\nu\lambda_k(-\Delta) \leq \lambda_1(B_S|_{\operatorname{Ran} \mathsf{E}_{B_S}((0,\infty))}) \leq \nu\lambda_1(-\Delta) + v_*\|\operatorname{div} u_0\|_{L^2(\Omega)},$$

where $u_0 \in (H^2(\Omega) \cap H_0^1(\Omega))^n$ normalized eigenfunction for $-\boldsymbol{\Delta}$ corresponding to $\lambda_1(-\Delta) = \lambda_1(-\boldsymbol{\Delta})$.

$$\nu\lambda_k(-\boldsymbol{\Delta}) \leq \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \leq \nu\lambda_k(-\boldsymbol{\Delta}) + \frac{v_*^2}{\nu}$$

$$\nu\lambda_k(-\boldsymbol{\Delta}) \leq \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \leq \nu\lambda_k(-\boldsymbol{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim \mathfrak{M}_+ = k$.

$$\nu\lambda_k(-\mathbf{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\mathbf{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim\mathfrak{M}_+ = k$.Then, $\mathfrak{v}[f,f] = 0$ for $f \in \mathfrak{D}_+$

$$\nu\lambda_k(-\boldsymbol{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\boldsymbol{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim\mathfrak{M}_+ = k$.Then, $\mathfrak{v}[f,f] = 0$ for $f \in \mathfrak{D}_+$, so

$$\sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{a}[f,f] = \sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{b}_S[f,f]$$

$$\nu\lambda_k(-\mathbf{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\mathbf{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim\mathfrak{M}_+ = k$.Then, $\mathfrak{v}[f,f] = 0$ for $f \in \mathfrak{D}_+$, so

$$\sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{a}[f,f] = \sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{b}_S[f,f] \le \sup_{\substack{f\in\mathfrak{M}_+\oplus\mathfrak{D}_-\\ \|f\|=1}} \mathfrak{b}_S[f,f].$$

$$\nu\lambda_k(-\boldsymbol{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\boldsymbol{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim\mathfrak{M}_+ = k$.Then, $\mathfrak{v}[f,f] = 0$ for $f \in \mathfrak{D}_+$, so

$$\sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{a}[f,f] = \sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{b}_S[f,f] \le \sup_{\substack{f\in\mathfrak{M}_+\oplus\mathfrak{D}_-\\ \|f\|=1}} \mathfrak{b}_S[f,f].$$

Now take infimum over $\mathfrak{M}_+$. □

$$\nu\lambda_k(-\mathbf{\Delta}) \le \lambda_k(B_S|\operatorname{Ran}\mathsf{E}_{B_S}((0,\infty))) \le \nu\lambda_k(-\mathbf{\Delta}) + \frac{v_*^2}{\nu}$$

Proof of 1st inequality.

Let $\mathfrak{M}_+ \subset \mathfrak{D}_+$ with $\dim \mathfrak{M}_+ = k$.Then, $\mathfrak{v}[f,f] = 0$ for $f \in \mathfrak{D}_+$, so

$$\sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{a}[f,f] = \sup_{\substack{f\in\mathfrak{M}_+\\ \|f\|=1}} \mathfrak{b}_S[f,f] \le \sup_{\substack{f\in\mathfrak{M}_+\oplus\mathfrak{D}_-\\ \|f\|=1}} \mathfrak{b}_S[f,f].$$

Now take infimum over $\mathfrak{M}_+$. □

Proof of second inequality uses Young's inequality and $\nu\|\operatorname{div} u\|^2_{L^2(\Omega)} \le \mathfrak{a}[u\oplus 0, u\oplus 0]$ for all $u \in H_0^1(\Omega)^n$.

Off-diagonal operator perturbations

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1 (in particular, $\operatorname{Dom}(B) = \operatorname{Dom}(A)$).

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1 (in particular, $\operatorname{Dom}(B) = \operatorname{Dom}(A)$).

Proposition (Makarov, Schmitz, S. 2016; Tretter 2008)

$\|P_{A,+} - P_{B,+}\| \leq \sqrt{2}/2 < 1$

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1 (in particular, $\operatorname{Dom}(B) = \operatorname{Dom}(A)$).

Proposition (Makarov, Schmitz, S. 2016; Tretter 2008)

$\|P_{A,+} - P_{B,+}\| \leq \sqrt{2}/2 < 1$, *and* $Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$ *with* X *as before satisfies*

$$(I - Y)(A + V)(I - Y)^{-1} = A - YV$$

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1 (in particular, $\operatorname{Dom}(B) = \operatorname{Dom}(A)$).

Proposition (Makarov, Schmitz, S. 2016; Tretter 2008)

$\|P_{A,+} - P_{B,+}\| \leq \sqrt{2}/2 < 1$, *and* $Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$ *with* X *as before satisfies*

$$(I - Y)(A + V)(I - Y)^{-1} = A - YV = \begin{pmatrix} A_+ - X^*W^* & 0 \\ 0 & A_- + XW \end{pmatrix}.$$

Off-diagonal operator perturbations

Suppose that

$$B = A + V = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}$$

and that V is A-bounded with A-bound < 1 (in particular, $\operatorname{Dom}(B) = \operatorname{Dom}(A)$).

Proposition (Makarov, Schmitz, S. 2016; Tretter 2008)

$\|P_{A,+} - P_{B,+}\| \leq \sqrt{2}/2 < 1$, *and* $Y = \begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix}$ *with* X *as before satisfies*

$$(I - Y)(A + V)(I - Y)^{-1} = A - YV = \begin{pmatrix} A_+ - X^*W^* & 0 \\ 0 & A_- + XW \end{pmatrix}.$$

In particular, $(I - Y)\operatorname{Dom}(A) = \operatorname{Dom}(A)$.

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I - Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I - Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I + Y = 2I - (I - Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I - Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I + Y = 2I - (I - Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I - Y)(I + Y)$$

Consequence of Heinz inequality:

$$\mathrm{Dom}(|B|^{1/2}) = \mathrm{Dom}(|A|^{1/2}) \quad \text{and} \quad (I - Y)\,\mathrm{Dom}(|A|^{1/2}) = \mathrm{Dom}(|A|^{1/2}).$$

Now, also $I + Y = 2I - (I - Y)$ maps $\mathrm{Dom}(|A|^{1/2})$ into itself, and so does

$$(I - Y)(I + Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix} \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix}$$

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I+Y = 2I-(I-Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix} \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I+X^*X & 0 \\ 0 & I+XX^* \end{pmatrix}.$$

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I + Y = 2I - (I - Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix}\begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I + X^*X & 0 \\ 0 & I + XX^* \end{pmatrix}.$$

In particular, $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I+Y = 2I-(I-Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix}\begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I+X^*X & 0 \\ 0 & I+XX^* \end{pmatrix}.$$

In particular, $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Also

$$\langle x, Bx\rangle = \langle x, Ax\rangle \le 0 \quad \text{for} \quad x \in \operatorname{Ran} P_{A,-} \cap \operatorname{Dom}(A).$$

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I + Y = 2I - (I - Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix} \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I + X^*X & 0 \\ 0 & I + XX^* \end{pmatrix}.$$

In particular, $(I + X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Also

$$\langle x, Bx\rangle = \langle x, Ax\rangle \leq 0 \quad \text{for} \quad x \in \operatorname{Ran} P_{A,-} \cap \operatorname{Dom}(A).$$

By approximation: $\mathfrak{b}[x,x] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}x\rangle \leq 0$ for $x \in \mathfrak{D}_-$.

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I+Y = 2I-(I-Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix} \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I+X^*X & 0 \\ 0 & I+XX^* \end{pmatrix}.$$

In particular, $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Also

$$\langle x, Bx\rangle = \langle x, Ax\rangle \leq 0 \quad \text{for} \quad x \in \operatorname{Ran} P_{A,-} \cap \operatorname{Dom}(A).$$

By approximation: $\mathfrak{b}[x,x] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}x\rangle \leq 0$ for $x \in \mathfrak{D}_-$.

Thus

$$\lambda_k(B|_{\operatorname{Ran} P_{B,+}}) = \inf_{\substack{\mathfrak{M}_+ \subset \mathfrak{D}_+ \\ \dim \mathfrak{M}_+ = k}} \sup_{\substack{x \in \mathfrak{M}_+ \oplus \mathfrak{D}_- \\ \|f\|=1}} \mathfrak{b}[x,x].$$

Consequence of Heinz inequality:

$$\operatorname{Dom}(|B|^{1/2}) = \operatorname{Dom}(|A|^{1/2}) \quad \text{and} \quad (I-Y)\operatorname{Dom}(|A|^{1/2}) = \operatorname{Dom}(|A|^{1/2}).$$

Now, also $I+Y = 2I-(I-Y)$ maps $\operatorname{Dom}(|A|^{1/2})$ into itself, and so does

$$(I-Y)(I+Y) = \begin{pmatrix} I & X^* \\ -X & I \end{pmatrix} \begin{pmatrix} I & -X^* \\ X & I \end{pmatrix} = \begin{pmatrix} I+X^*X & 0 \\ 0 & I+XX^* \end{pmatrix}.$$

In particular, $(I+X^*X)\mathfrak{D}_+ \subset \mathfrak{D}_+$.

Also

$$\langle x, Bx\rangle = \langle x, Ax\rangle \leq 0 \quad \text{for} \quad x \in \operatorname{Ran} P_{A,-} \cap \operatorname{Dom}(A).$$

By approximation: $\mathfrak{b}[x,x] = \langle |B|^{1/2}x, \operatorname{sign}(B)|B|^{1/2}x\rangle \leq 0$ for $x \in \mathfrak{D}_-$.

Thus

$$\lambda_k(B|_{\operatorname{Ran} P_{B,+}}) = \inf_{\substack{\mathfrak{M}_+ \subset \mathfrak{D}_+ \\ \dim \mathfrak{M}_+ = k}} \sup_{\substack{x \in \mathfrak{M}_+ \oplus \mathfrak{D}_- \\ \|f\|=1}} \mathfrak{b}[x,x].$$

In particular, $\lambda_k(B|_{\operatorname{Ran} P_{B,+}}) \geq \lambda_k(A|_{\operatorname{Ran} P_{A,+}})$.

Outlook to general perturbations

Consider

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \; V = \begin{pmatrix} V_+ & W \\ W^* & V_- \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Outlook to general perturbations

Consider

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \; V = \begin{pmatrix} V_+ & W \\ W^* & V_- \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Write

$$B = A + V = \begin{pmatrix} A_+ + V_+ & 0 \\ 0 & A_- + V_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}$$

Outlook to general perturbations

Consider

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}, \; V = \begin{pmatrix} V_+ & W \\ W^* & V_- \end{pmatrix} \quad \text{w.r.t.} \quad \operatorname{Ran} P_{A,+} \oplus \operatorname{Ran} P_{A,-}.$$

Write

$$B = A + V = \begin{pmatrix} A_+ + V_+ & 0 \\ 0 & A_- + V_- \end{pmatrix} + \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix} =: \tilde{A} + \tilde{V},$$

and study $B = \tilde{A} + \tilde{V}$ in the off-diagonal framework.

Thank you for your attention!