On surjectivity of partial differential operators with a single characteristic direction and on Runge pairs for such operators

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Abstract:
We report on recent results concerning partial differential operators with constant coefficients $P(\partial)$ for which the characteristic set $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}$ of its symbol $P \in \mathbb{C}[X_1, \ldots, X_d]$ is a one-dimensional subspace of $\mathbb{R}^d$. Here P_m denotes the principal part of P, i.e. $P_m(\xi) := \sum_{|\alpha|=m} a_\alpha \xi^\alpha$ for $P(\xi) = \sum_{|\alpha|\leq m} a_\alpha \xi^\alpha$ with minimal $m \in \mathbb{N}_0$. Among others, this class of partial differential operators contains the time-dependent free Schrödinger operator as well as non-degenerate parabolic operators like the heat operator.

We characterize those open subsets X of $\mathbb{R}^d$ for which $P(\partial)$ is surjective on $C^\infty(X)$, or equivalently on $\mathscr{D}'_F(X)$, the space of distributions of finite order on X. Moreover, we give a sufficient geometrical/topological condition for pairs of open subsets $X_1 \subseteq X_2$ of $\mathbb{R}^d$ to be P-Runge pairs, which means that every smooth solution, resp. distributional solution, of the equation $P(\partial)u = 0$ in X_1 can be approximated by smooth solutions, resp. distributional solutions, of the same equation in X_2. This condition is in the spirit of Runge's Approximation Theorem from complex analysis which deals with the case when $P(\partial)$ is the Cauchy-Riemann operator.

Finally, we show that under the additional assumption of semi-ellipticity for such a differential operator surjectivity on $C^\infty(X)$ implies that its kernel $C^\infty_P(X) = \{f \in C^\infty(X); P(\partial)f = 0\}$ has the linear topological invariant (Ω) of Vogt and Wagner which plays a prominent role when dealing with the question of surjectivity of $P(\partial)$ on spaces of vector valued smooth functions. Via Grothendieck-Köthe duality this can be interpreted as an abstract version of another classical theorem from complex analysis, namely Hadamard's Three Circles Theorem.

References

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