A Multi-Level Monte Carlo Method for Stresses along Paths

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- **Equations of linear elasticity** describe stress problems in the upper earth's crust.
- Stochastic finite element methods model propagation of measurement uncertainty.
- We estimate the **expected stress along a path** by **multi-level Monte Carlo finite elements**.
- Error of the expected stress along a path converges **linearly** with respect to mesh width.

Linear elasticity problem with stochastic data

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $(\Theta, \mathscr{F}, \mathbb{P})$ a probability space. Find $u(x,\theta) : \Omega \times \Theta \to \mathbb{R}^3$, such that $\mathbb{P}$-almost surely

$$\begin{aligned} -\nabla \cdot \sigma(x,\theta) &= f(x,\theta) && \text{in } \Omega, \\ u_i(x,\theta) &= u_{D,i}(x) && \text{on } \Gamma_D^i \quad \text{for } i = 1,2,3, \\ (\sigma(x,\theta)n(x))_i &= g_i(x) && \text{on } \Gamma_N^i \quad \text{for } i = 1,2,3. \end{aligned}$$

where σ is the Cauchy stress tensor given by Hooke's law $\sigma = \mathbb{C}\epsilon(u)$, ϵ is the linearized strain tensor and $\mathbb{C}$ is the elasticity tensor.

Multi-level Monte Carlo finite element method (MLMCFEM)

- Linear finite elements on regular triangulation.
- Nested sequence of hierarchical finite element spaces by halving the mesh width: $V_1 \subset V_2 \subset \cdots \subset V_L$.
- Path $\mathscr{P}$ is a one-dimensional submanifold of domain Ω.
- Finite element solution $u^l \in V_l \to$ Restriction to path: $u^{l,\mathscr{P}}$.
- **Definition 1**: Construct unique gradient $\nabla^M u^l(\theta)$ by choosing componentwise the maximum of the gradient $\nabla u^l(\theta)$ of the neighboring tetrahedrons for faces, edges and nodes.
  $\to$ Restriction to path: $\nabla u^{l,\mathscr{P}}(\theta) = \nabla^M u^l(\theta)|_{\mathscr{P}}$.
- The expectation of the restricted stress on the finest level is equal to the sum of differences in expectation between restricted stresses on consecutive levels [1, 2]:

$$\mathbb{E}[\sigma^{L,\mathscr{P}}] = \sum_{l=1}^{L} \mathbb{E}[\sigma^{l,\mathscr{P}} - \sigma^{l-1,\mathscr{P}}]$$

  with $\sigma^{0,\mathscr{P}} = 0$, $\sigma^{l,\mathscr{P}} = \mathbb{C}\epsilon(u^{l,\mathscr{P}})$ for $l = 1, \dots, L$.
- MLMC estimator: Estimate each of these expectations independently by MC estimator:

$$E^L[\sigma^{\mathscr{P}}] = \sum_{l=1}^{L} E_{N_l}[\sigma^{l,\mathscr{P}} - \sigma^{l-1,\mathscr{P}}]$$

  where

$$E_{N_l}[\sigma^{l,\mathscr{P}} - \sigma^{l-1,\mathscr{P}}] = \frac{1}{N_l}\sum_{n=1}^{N_l}(\sigma_n^{l,\mathscr{P}} - \sigma_n^{l-1,\mathscr{P}})$$

  with $\sigma_n^{l,\mathscr{P}}$ i.i.d. random variables with the same probability distribution as $\sigma^{l,\mathscr{P}}$.

Convergence analysis [3]

- **Assumption 1**: $u(\theta) \in H^3(\Omega)$
  $\Rightarrow$ Sobolev's embedding theorem: There exists a continuous representative for the gradient.
- **Assumption 2**: There exists $C > 0$, independent of u and h_l, such that $\mathbb{P}$-almost surely

$$\|\nabla u(\theta) - \nabla u^l(\theta)\|_{L^\infty(\Omega)} \le C h_l |u(\theta)|_{W^{2,\infty}(\Omega)}.$$

- **Assumption 3**: The elastic coefficients have $\mathbb{P}$-almost surely an upper bound almost everywhere on the path.
- Linear convergence of the restricted **semidiscretized stress**: Let $u \in L^2(\Theta, W^{2,\infty}(\Omega))$, then $\exists C > 0$, independent of u and h_l, such that

$$\|\sigma^{\mathscr{P}} - \sigma^{l,\mathscr{P}}\|_{L^2(\Theta, L^2(\mathscr{P}))} \le C h_l |u|_{L^2(\Theta, W^{2,\infty}(\Omega))}.$$

- Main aspects of the proof:
  - Error of the restricted stress can be estimated by the error of the restricted gradient with respect to $L^2(\mathscr{P})$ norm by means of Assumption 3.
  - Hölder's inequality: $L^2(\mathscr{P})$ norm can be estimated by $L^\infty(\mathscr{P})$ norm.
  - Assumption 1 and Definition 1 provide:

$$\begin{aligned} \|\nabla u^{\mathscr{P}}(\theta) - \nabla u^{l,\mathscr{P}}(\theta)\|_{L^\infty(\mathscr{P})} &= \sup_{x \in \mathscr{P}} |\nabla u^{\mathscr{P}}(\theta) - \nabla u^{l,\mathscr{P}}(\theta)| \\ &\le \sup_{x \in \Omega} |\nabla u(\theta) - \nabla^M u^l(\theta)| = \|\nabla u(\theta) - \nabla u^l(\theta)\|_{L^\infty(\Omega)}. \end{aligned}$$

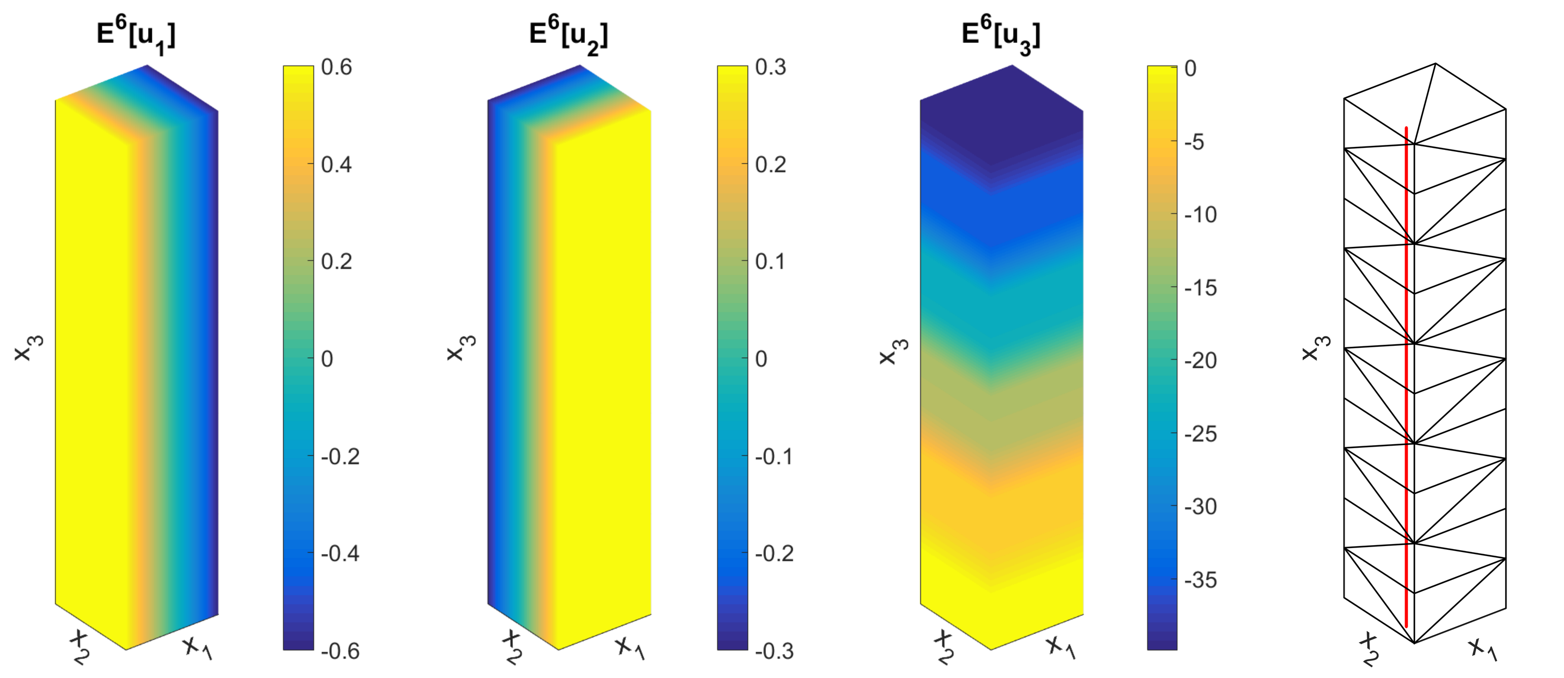
  - Assumption 2 can be applied.
- Convergence of the **MLMCFE error**: Let $u \in L^2(\Theta, W^{2,\infty}(\Omega))$, then $\exists C > 0$, independent of u and h_l, such that

$$\|\mathbb{E}[\sigma^{\mathscr{P}}] - E^L[\sigma^{\mathscr{P}}]\|_{L^2(\Theta, L^2(\mathscr{P}))} \le C\Big(h_L + \sum_{l=1}^{L} \frac{h_l}{\sqrt{N_l}}\Big)|u|_{L^2(\Theta, W^{2,\infty}(\Omega))}.$$

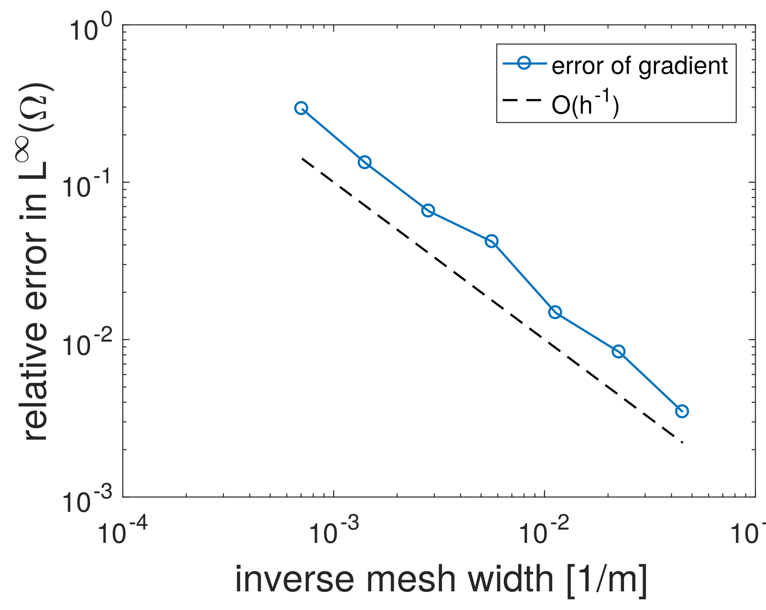
  - Proof according to [1] using the linear convergence of the semidiscretized stress.
- Linear convergence of the MLMCFE error with respect to mesh width h_L [1]:
  $\Rightarrow$ Number of samples $N_l = O(l^{2+2\epsilon}(h_l/h_L)^2) = O(l^{2+2\epsilon}2^{2(L-l)})$ for $\epsilon > 0$.

Numerical example

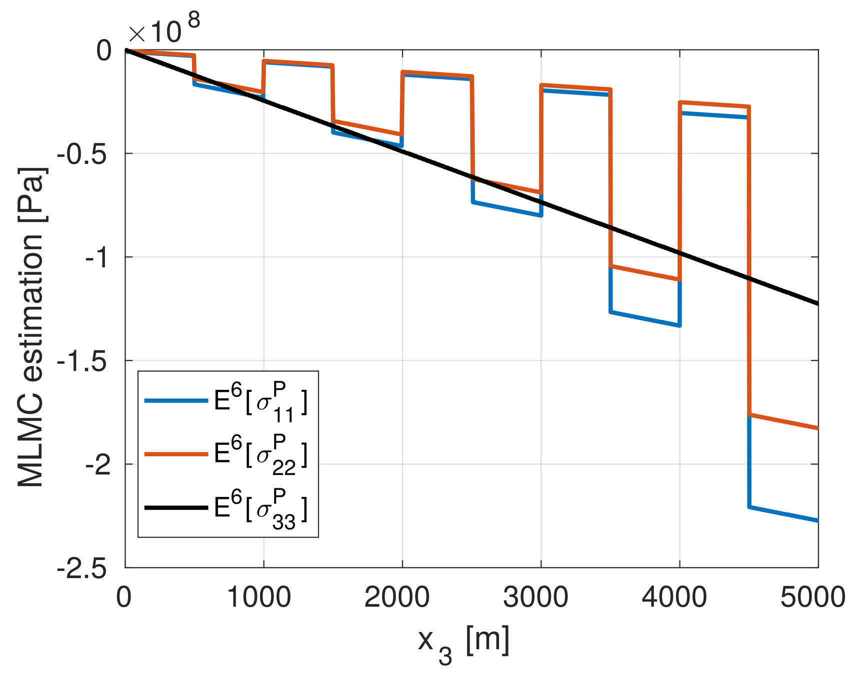
- $\Omega = (0,1000) \times (0,1000) \times (-5000, 0)$.
- 10 layers of different homogeneous and isotropic materials.
- Homogeneous Neumann and non-homogeneous Dirichlet boundary conditions.
- Path $\mathscr{P} = \{x \in \Omega : x_1 = 120, x_2 = 270\}$, displayed in red, intersects the tetrahedrons.

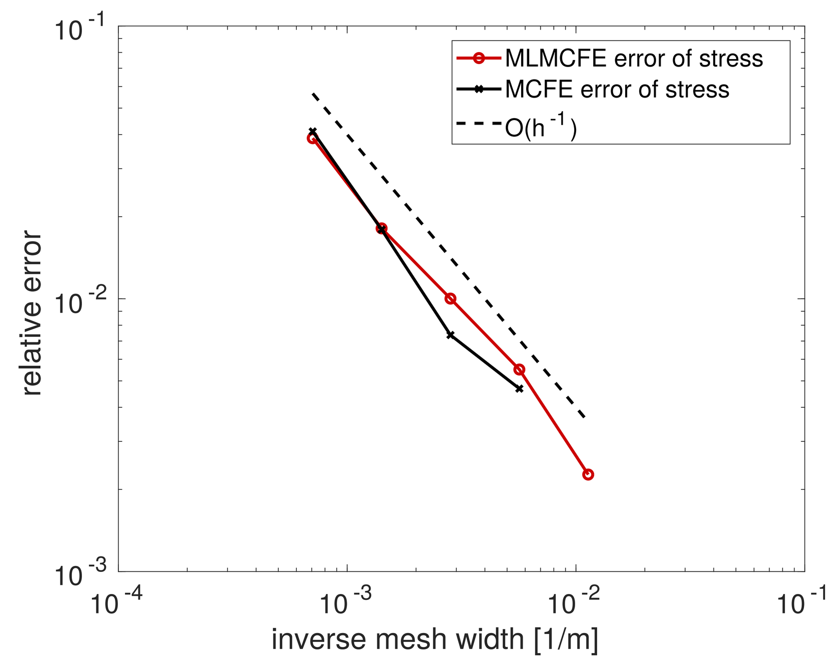


- Pointwise linear convergence of the gradient (Assumption 2) for the deterministic problem is confirmed.



- Error measured with respect to $L^2(\Theta, L^2(\mathscr{P}))$ norm.





Conclusion and outlook

- MLMCFE error of the stress along a path converges linearly in $L^2(\Theta, L^2(\mathscr{P}))$ norm with respect to mesh width.
- Numerical example: H^3 regularity of the solution is not fulfilled because of non-smooth elastic coefficients. However, we obtain a linear convergence rate.
- Next, we will investigate the H^3 regularity of the solution in subdomains, Assumption 2 and the condition $u \in L^2(\Theta, W^{2,\infty}(\Omega))$.
- Further, we will analyze the convergence assuming a lower spatial regularity of the solution.

References

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