Optimal Reliability for Ceramic Structures

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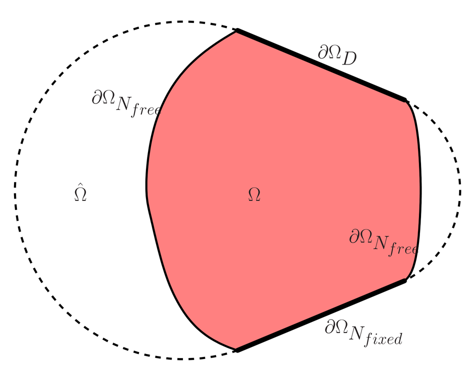
Introduction

Ceramic is a material frequently used in industry because of its favorable properties. Common approaches in shape optimization for ceramic structures aim to minimize the tensile stress acting on the component, as it is the main driver for failure. In contrast to this, we follow a more natural approach by minimizing the component's probability of failure under a given tensile load.
Since the fundamental work of Weibull, the probabilistic description of the strength of ceramics is standard and has been widely applied. Here, for the first time, the resulting failure probabilities are used as objective functionals in PDE constrained shape optimization.

Problem

Let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a domain with Lipschitz boundary $\partial\Omega$. It is assumed to be filled with ceramic material. Ω represents the ceramic component in its initial, force free state. Furthermore, we assume that the boundary $\partial\Omega$ can be divided into three different parts

$$\partial\Omega = \overline{\partial\Omega}_D \cup \overline{\partial\Omega}_{N_{fixed}} \cup \overline{\partial\Omega}_{N_{free}}. \tag{1}$$



Forces may act on the object with the shape given by Ω.
The volume force is represented by a function $f \in L^2(\Omega, \mathbb{R}^d)$,
The surface force by a function $g \in L^2(\partial\Omega_N, \mathbb{R}^d)$.
The linear elasticity PDE

$$B(u, v) = L(v) \, \forall v \in H_0^1(\Omega, \mathbb{R}^d), \tag{2}$$

$$B(u, v) = \int_\Omega \sigma(u) : \varepsilon(v) \, dx \tag{3}$$

$$L(v) = \int_\Omega f \cdot v \, dx + \int_{\partial\Omega_N} g \cdot v \, dA \tag{4}$$

must hold.

Adjoint Euqation

After discretising the objective functional via finite elements we want to calculate the shape gradient.

$$\frac{dJ(X,U)}{dX} = \frac{\partial J(X,U)}{\partial X} + \frac{\partial J(X,U)}{\partial U}\frac{\partial U}{\partial X}. \tag{5}$$

The calculation of $\frac{\partial U(X)}{\partial X}$ is very costly, consider the corresponding Lagrange function instead.

$$\mathcal{L}(X, U, \Lambda) := J(X, U) - \Lambda^T(B(X)U - F(X)). \tag{6}$$

Hence,

$$\frac{\partial J(X,U)}{\partial X} + \Lambda\left[\frac{\partial F(X)}{\partial X} - \frac{\partial B(X)}{\partial X}U\right] = 0$$

$$B^T(X)\Lambda = \frac{\partial J(X,U)}{\partial U} \tag{7}$$

$$B(X)U = F(X)$$

gives the first order conditions to the shape minimization problem of the probability of failure.

References

[1] M. Bolten, H. Gottschalk, and S. Schmitz. Minimal failure probability for ceramic design via shape control. *J. Optim. Theory Appl.*, pages 983–1001, 2015.

[2] E. Weibull. A statistical theory of the strength of materials. *Ingeniörsvetenskapsakedemiens Handlingar*, 151:1–45, 1939.

[3] O. Kallenberg. *Random Measures*. Akademie-Verlag, Berlin, 1983.

Survival Probabilities from Linear Fracture Mechanics

Find full deduction of the functional here [1]. Optimize the reliability for a ceramic body Ω by minimizing its probability of failure under one given tensile load σ_n. From its production process, in ceramic material small flaws arise, which under load may become the initial point of a rupture. Describe these flaws by

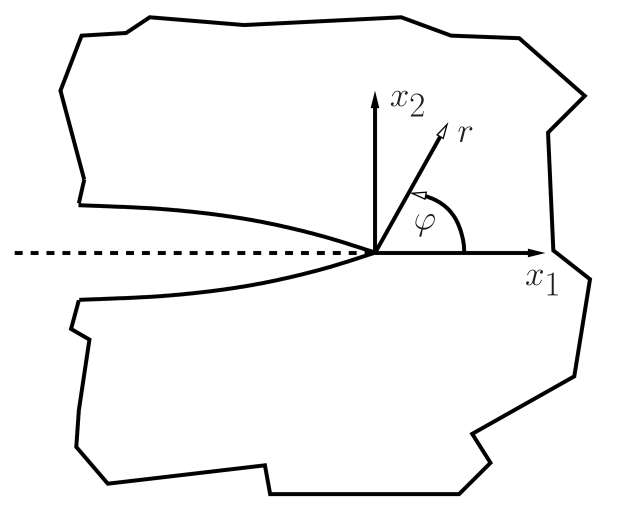
- their location $x \in \Omega$,
- their orientation $\mu \in S^{d-1}$,
- their radius $a \in \mathbb{R}$.

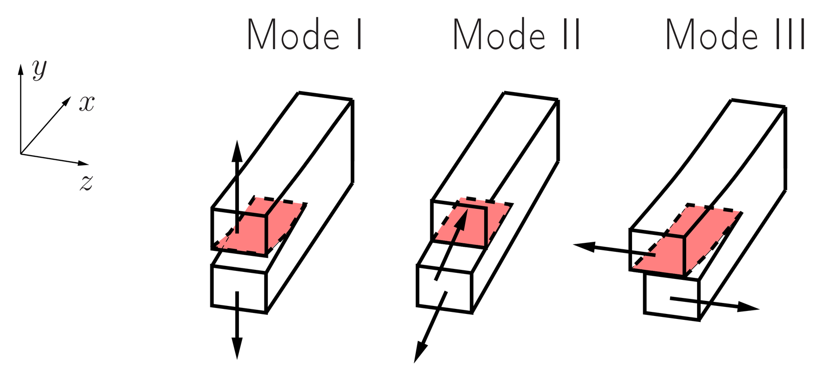
Crack configuration space $\mathcal{C} := (\bar{\Omega} \times (0, \infty) \times S^{d-1})$.
Stress field close to the tip of the crack is given by

$$\sigma = \frac{1}{\sqrt{2\pi r}}\{K_I\tilde{\sigma}^I(\phi) + K_{II}\tilde{\sigma}^{II}(\phi) + K_{III}\tilde{\sigma}^{III}(\phi)\}$$

+regular terms.

$K_I := \frac{2}{\pi}\sigma_n\sqrt{\pi a}$ is most relevant for this case, with tensile load $\sigma_n := (n \cdot \sigma(Du)n)^+$.





Ω fails, if K_I exceeds a critical value K_{I_C}.

$$A_C := A_C(\Omega, Du) = \{(x, a, \mu) \in \mathcal{C} : K_I(a, \sigma_n(x)) > K_{I_C}\}$$

Assume that the distribution of cracks in different parts of the component is statistically independent, conclude with [3] that the random number of cracks $N(A)$, $A \subseteq \mathcal{C}$ is Poisson distributed, i.e. $N(A)$ is a Poisson point process. $\mathbb{P}(N(A) = n) = e^{-\nu(A)}\frac{\nu(A)^n}{n!} \sim Po(\nu(A))$, with intensity measure $\nu : \mathcal{C} \to \mathbb{R}$.
The component fails if $N(A_C) \geq 1$. With this, the survival probability is given by

$$p_s(\Omega|Du) = P(N(A_c(\Omega, Du)) = 0) = \exp\{-\nu(A_c(\Omega, Du))\}.$$

This gives our objective functional

$$J(\Omega, Du) := \nu(A_c(\Omega, Du)) = \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\int_\Omega\int_{S^{d-1}}\left(\frac{\sigma_n}{\sigma_0}\right)^m dn dx.$$

With some remodeling, we see that $p_s(\Omega|Du)$ follows a Weibull distribution. This corresponds to the statistical strength of brittle materials as described in [2].

Computation & Validation

We calculate the shape gradient using the adjoint formalism (7).
We consider a simple example in $d = 2$:

- Set the parameters E and ν to those of Aluminum oxide (Al_2O_3) ceramics.
- Set $m = 10$, which is a reasonable value and still leads to tractable numerics.
- The test object is a bent rod of length 0.6m and height 0.1m

We obtain a first optimization process,

- using geometric mesh morphing,
- and small step sizes
- project the volume gradient from the shape gradient
- $X_{new} = X_{old} - \alpha\left(\frac{dJ}{dX} - \frac{\partial \text{Vol}}{\partial X}\right)$.

The outcome is a procedure converging against the straightened rod, which is visualized in Figure 3.
We can also investigate the (discretised) failure probability of the component see Figure 2.
It is obvious that we actually decrease the probability of failure with the present procedure, which finally converges to the evident optimum.

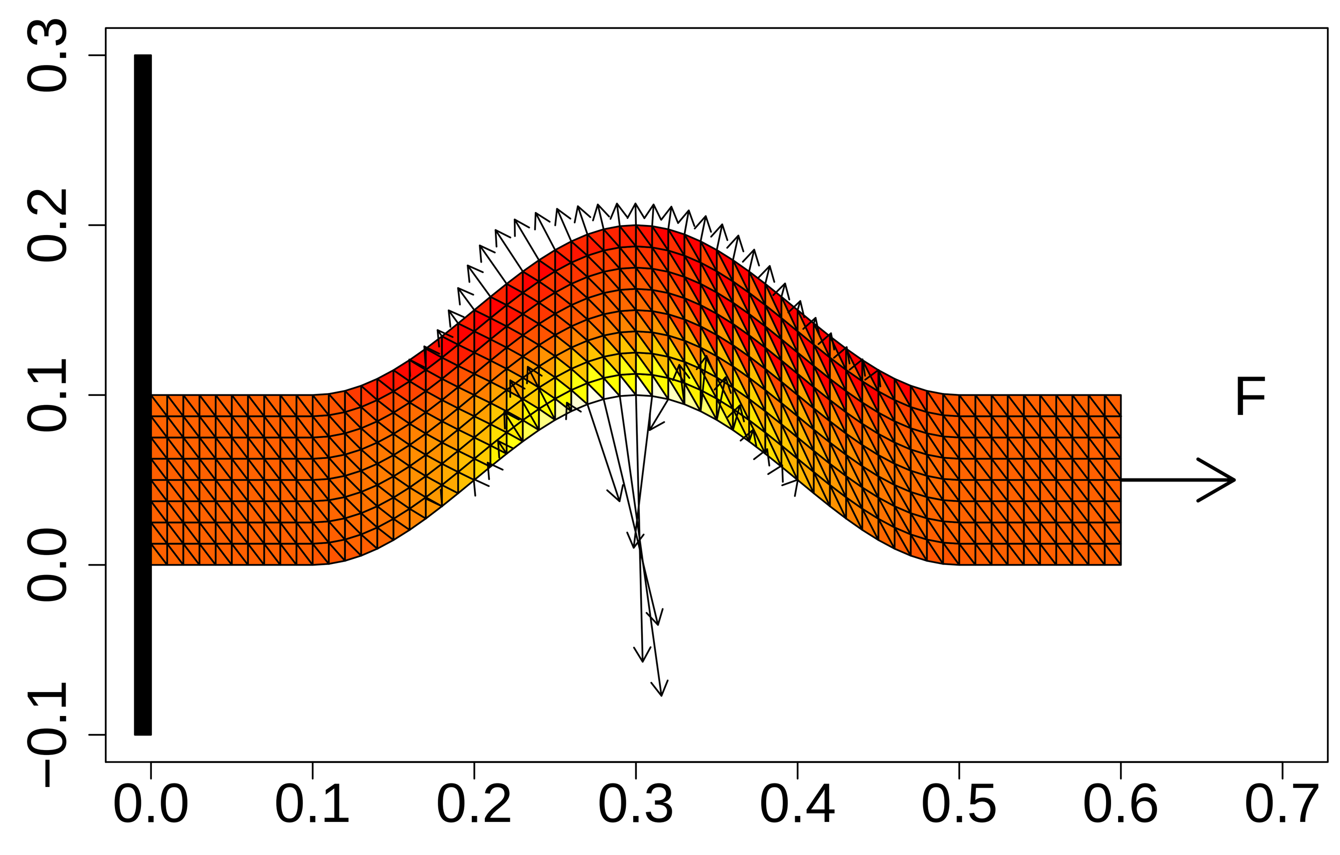


Figure 1: Visualization of the Gradient

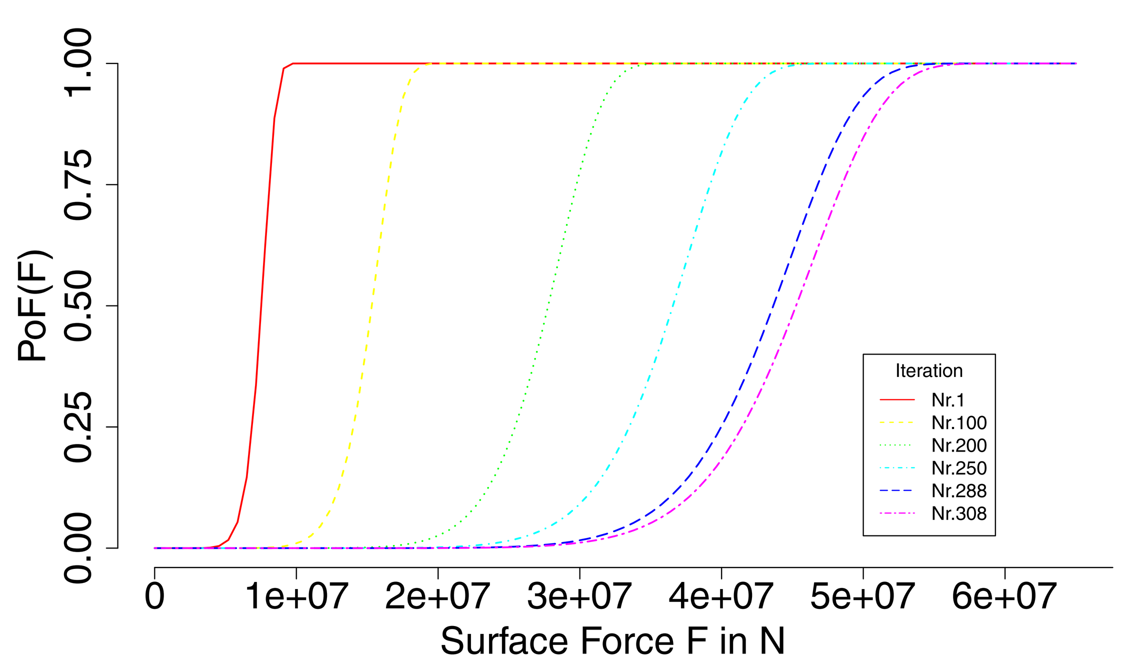


Figure 2: Distribution of Failure Probability

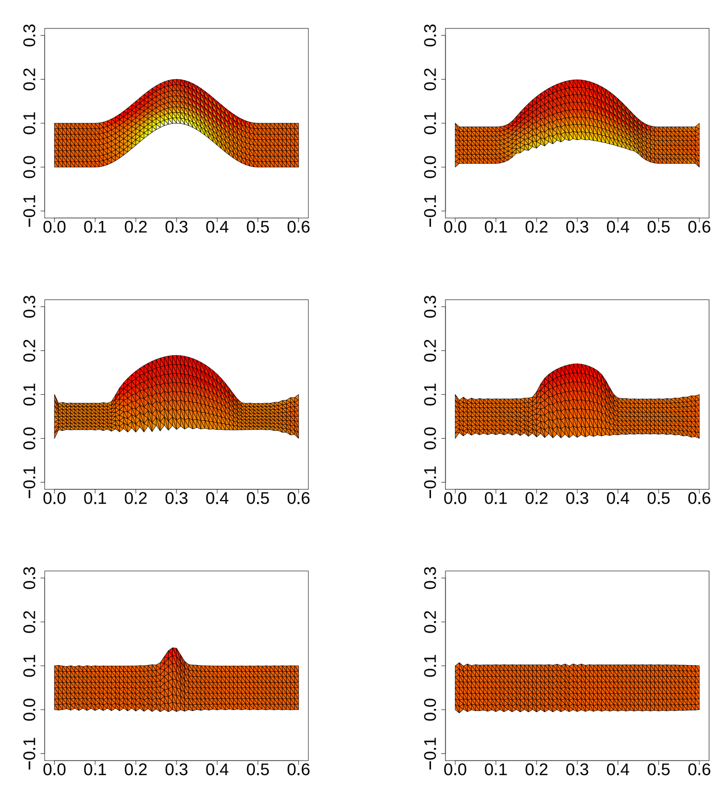


Figure 3: Visualization of the Opimization Process