# Random Diffusion Equations with Lévy Coefficients

## Numerical Simulation of a Penetrating Spheres Model

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### Random Diffusion Equation

- $D = [0,1]^2$, $\partial_D = \{0,1\} \times [0,1]$, $\partial_N = [0,1] \times \{0,1\}$
- Consider diffusion equation describing e.g. Darcy flow in groundwater

$$\begin{aligned} -\nabla \cdot (k(x)\nabla p(x)) &= 0 && \text{in } D, \\ p(x) &= p_{\partial_D}(x) && \text{on } \partial_D, \\ -k(x)\nabla p(x) \cdot \mathbf{n} &= 0 && \text{on } \partial_N. \end{aligned}$$

- Conductivity $k$ is a random field.
- Quantity of interest: expected flux through the right boundary, i.e. $\mathcal{Q}(p) = \int_0^1 k(1, x_2)\nabla p(1, x_2) \cdot \mathbf{n}\, dx_2$.

### Objectives

- Estimate quantities of interest of solution $p$.
- e.g. flux across right boundary.
- Consider Lévy random fields as generalization of Gaussian fields.
- Numerical approximation by quadrature and Monte Carlo methods.

### Penetrating Spheres Model

- Conductivity $k(x) := k_0 + (k_1 - k_0)\Theta(\eta_S(x))$ describes a piecewise constant two phase-model, with $\Theta$ the Heavyside function.
- Deterministic jump sizes, i.e. $\nu = \lambda\delta_1$, fixed radius $r > 0$, smoothed random field defined as

$$\eta_S(x) := \sum_{i=1}^{N} \mathbb{1}_{B_r(Y_i)}(x).$$

- Classical construction of compound Poisson noise:
  - Given a Lévy measure $\nu$, compute intensity $\lambda = \nu(\mathbb{R} \setminus \{0\})$.
  - Random number of $N \sim Po(\lambda)$ particles.
  - Each particle has location $Y_i \sim \frac{1}{|D|}dx$ and jump distribution $S_i \sim \tilde{\nu} := \frac{\nu}{\lambda}$.
  - Obtain random field $\eta(x) = \sum_{i=1}^{N} S_i \delta_{Y_i}(x)$.

### Approximate Expectation of Quantity of Interest

- We want to compute

$$\mathbb{E}\mathcal{Q}(p(\eta_S)) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!} \int_{D^n} \mathcal{Q}(p(y_1, \ldots, y_n))\, dy_1 \ldots dy_n.$$

- Can truncate sum depending on intensity $\lambda$.
- Even for low intensity $\lambda = 1$, need to handle at least 10 particles, resulting in 20-dimensional integrals in 2D.
- Integrand does not depend smoothly on the random locations.

### Exploiting Symmetry in the Random Field

- Random field, PDE and solution do not change when permuting the particles.
- Parameter configurations generating the same geometry are put into equivalence classes.
- Evaluate only one representative per class and multiply the coefficient by the number of equivalent nodes.

| Particles | Nodes per Dimension | Full Grid | Unique Nodes | Rel. Reduction |
|---|---|---|---|---|
| 3 | 4 | 64 | 20 | 31.25% |
| | 9 | 729 | 165 | 22.63% |
| | 16 | 4096 | 816 | 19.92% |
| 5 | 4 | 1024 | 56 | 5.47% |
| | 9 | 59049 | 1287 | 2.18% |
| | 16 | 1048576 | 15504 | 1.48% |
| 7 | 4 | 16384 | 120 | 0.73% |
| | 9 | 4782969 | 6435 | 0.13% |
| | 16 | 268435456 | 170504 | 0.06% |

Table 1: Relative reduction of evaluations needed in a grid exploiting symmetries versus a full tensor product grid.

### Realizations of Penetrating Spheres Model

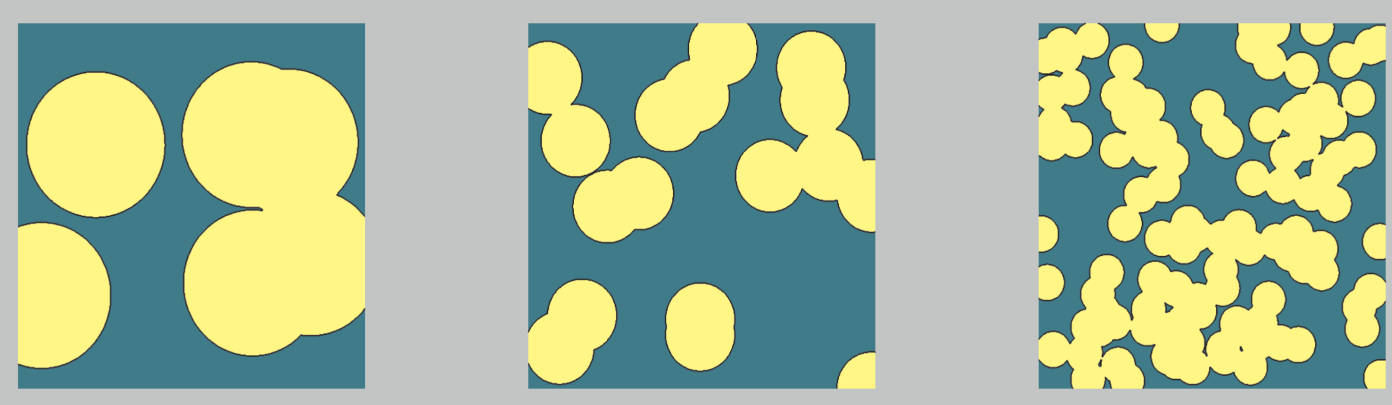

Figure 1: Realizations of random medium with different radii and intensities.

### Product Quadrature

- Product trapezoidal on rule $D^n$
- Number of quadrature nodes grows fast despite exploitation of symmetry.
- Expensive since for each node there is a PDE to solve (with a grid respecting the geometry).
- Sparse grid quadrature not applicable due to insufficient smoothness.
- In 1D, exact solution and expectation available for few particles, rates of convergence deteriorate fast (order 2 for one particle, order 1 for two particles).
- Approach not efficient due to the curse of dimensionality.

### (Multilevel) Monte Carlo Quadrature

- Error behaves like $N^{-\frac{1}{2}}$, independent of dimension.
- Multilevel Monte Carlo (MLMC) yields same rate with smaller constant.
- Measure for the error: mean square error

$$MSE := \mathbb{E}(\hat{\mathcal{Q}}_h - \mathbb{E}\mathcal{Q})^2 = \mathbb{V}\mathcal{Q}_h + \mathbb{E}(\mathcal{Q}_h - \mathcal{Q})^2$$

- Cost for a given error $\epsilon^2$ with optimal choice of discetization parameter $h$ is $\mathcal{O}(\epsilon^{-3})$.
- Cost for MLMC with two levels has same rate with better constant.
- Fixing the coarsest grid and taking as many levels as necessary reduces cost to $\mathcal{O}(\epsilon^{-2})$.

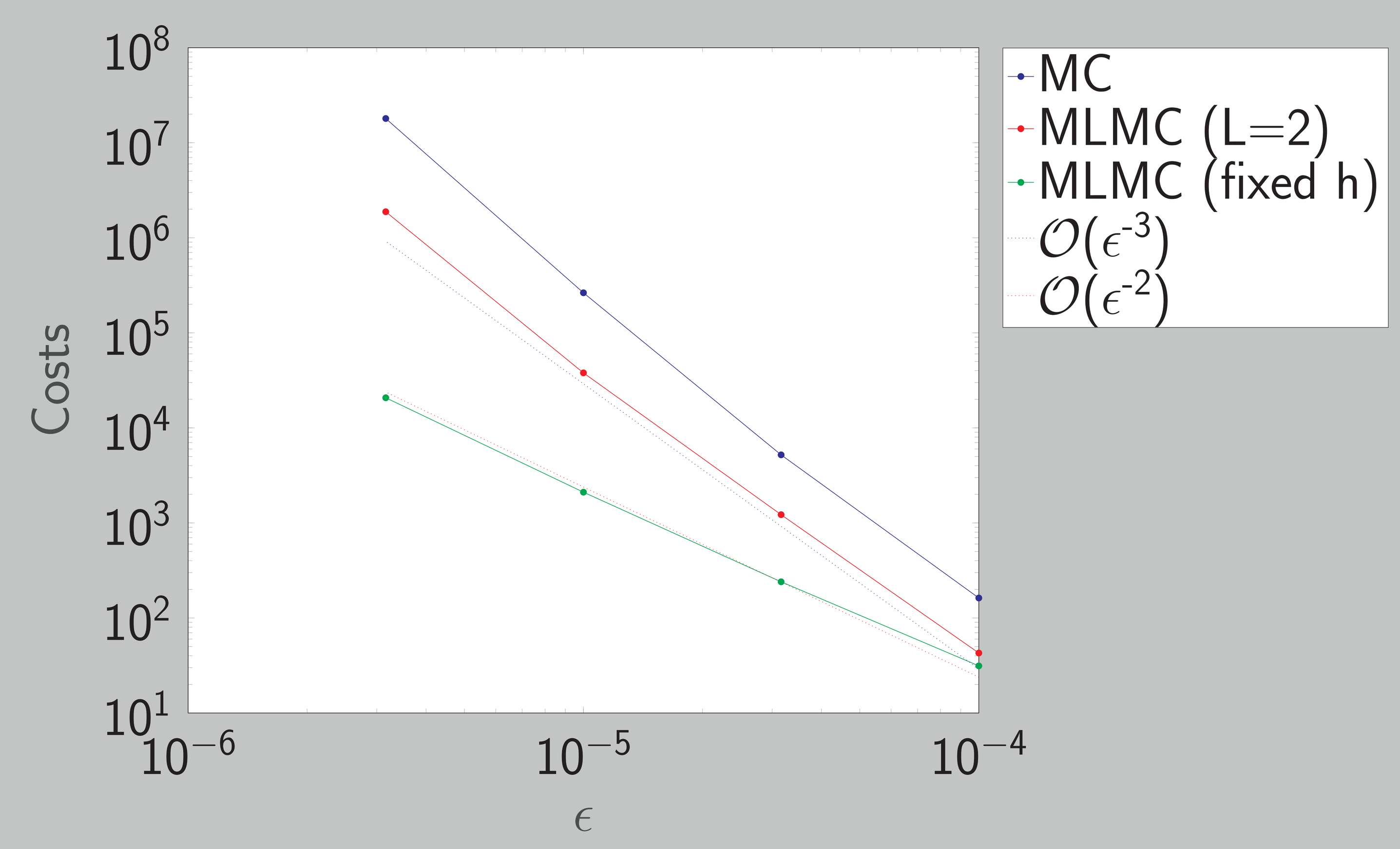


Figure 2: Costs of computation for a fixed square mean root error of $\epsilon$.

### Conclusions

- Standard product quadrature is too expensive and their rates drop too fast with dimension to be a suitable approach for this model.
- Insufficient smoothness for sparse quadrature techniques.
- MC/MLMC methods yield the expected results and are applicable for any number of particles.

### Future Work

- Try a more strongly smoothed random field and investige sparse grid methods on it.
- Use Karhunen-Loève or the Mercer expansion of smoothed field.
- Find other applications for Lévy distributions.


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