Sharp geometric condition for null-controllability of the heat equation on the whole space

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The controlled heat equation

Let $T > 0$. Let Ω be an open set in $\mathbb{R}^d$, $\omega \subset \Omega$, and Δ be the Laplacian on Ω. The *controlled heat equation* on Ω is defined as

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = \chi_\omega v(t,x) & \text{on } [0,T] \times \Omega \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

The system is called *null-controllable* in time $T > 0$ if for every $u_0 \in L^2(\Omega)$ there exists a *control function* $v \in L^2([0,T] \times \omega)$ such that the unique solution $u(x,t)$ of (1) satisfies $u(T,\cdot) \equiv 0$.

If the system is null-controllable, the *control cost* is the quantity

$$C_T := \sup_{\|u_0\|_{L^2(\Omega)}=1} \inf\{\|v\|_{L^2([0,T]\times\omega)} \mid u \text{ solution of (1) with r.h.s. } \chi_\omega \cdot v \text{ and } u(T,\cdot) = 0\}.$$

Known null-controllability results

If $\Omega \subset \mathbb{R}^d$ is a bounded domain:

- any open subset $\omega \subset \Omega$ is a control set.
- recently proved that any measurable ω of positive measure is a control set.

If $\Omega = \mathbb{R}^d$:

- A sufficient condition for null-controllability is
$$\exists\, R, r > 0 : \ \forall\, x \in \mathbb{R}^d \ \exists\, z \in \mathbb{R}^d \ \text{ s. t. } |z - x| < R \text{ and } B_r(z) \subset \omega. \tag{2}$$
- A necessary condition for null-controllability is
$$\exists\, R > 0 \text{ such that } \forall\, x \in \mathbb{R}^d : \quad B_R(x) \cap \omega \neq \emptyset. \tag{3}$$

Our questions:

(1) What is a sharp condition for null-controllability in $\mathbb{R}^d$?
(2) Is it possible to estimate explicitely the control cost?

Sharp condition for null-controllability

Theorem 1: Let $\Omega = \mathbb{R}^d$. The followings are equivalent.
(i) ω is a thick set.
(ii) The heat equation in (1) is null-controllable in time $T > 0$.

Let $S \subset \mathbb{R}^d$. The set S is *thick* if it is measurable and there exist $\delta > 0$ and $a \in \mathbb{R}^d$, $a_j > 0$ for all $j = 1, \dots, d$ such that

$$|S \cap (x + [0,a_1] \times \ldots \times [0,a_d])| \geq \delta \prod_{j=1}^{d} a_j \qquad \forall\, x \in \mathbb{R}^d.$$

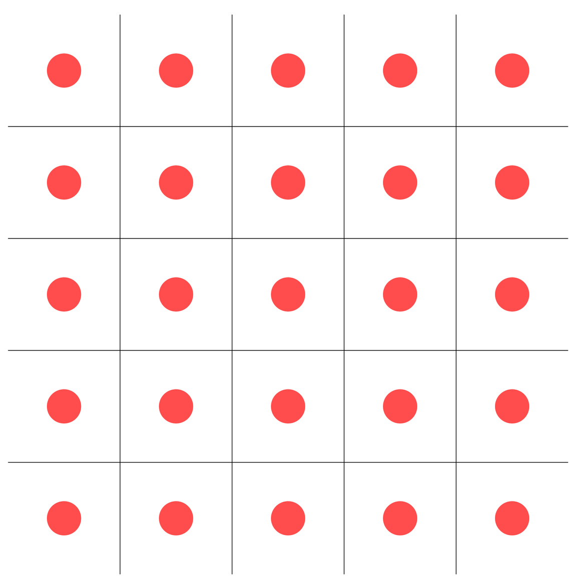

Figure 1: A periodic arrangement of balls in $\mathbb{R}^2$ is a thick set.

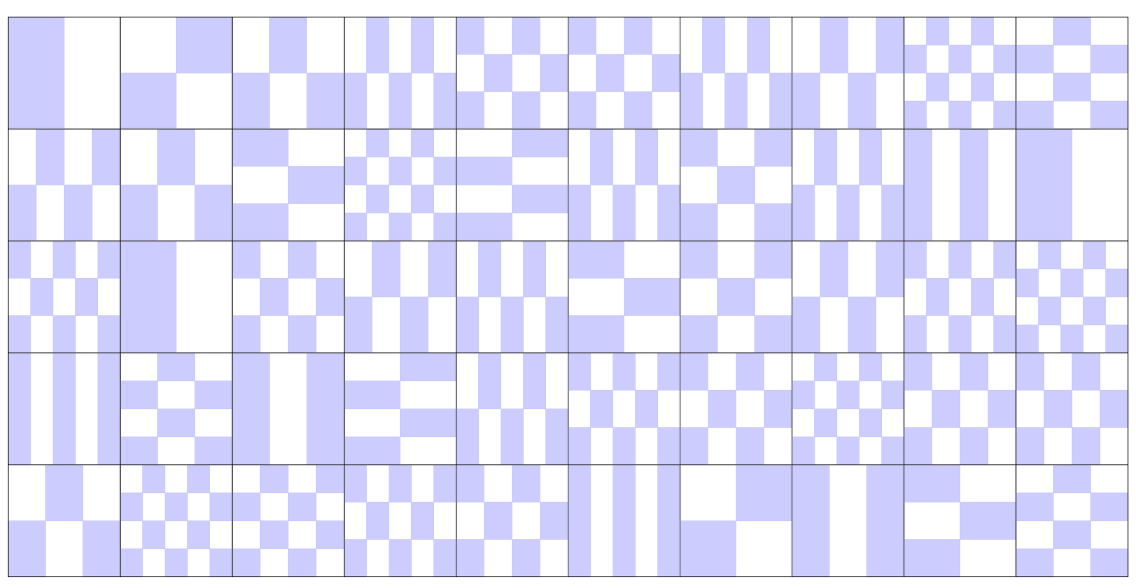

Figure 2: Example of a section of size 10×5 of a (γ, a)-thick set S in $\mathbb{R}^2$ where $a = (2,2)$ and $\gamma = 1/8$.

Comparison with previous results:

- (γ, a)-thickness is stronger than the sufficient condition in (2).
- The necessity condition (3) implies (γ, a)-thickness.

Methods of the proof

- Complex and Fourier Analysis results/techniques to derive spectral inequality.
- Observability estimate for the heat equation.
- Duality between observability and null-controllability to conclude.

Control cost estimate

Consider system (1) for $\Omega = \mathbb{R}^d$ and ω being (γ, a)-thick. Let $T > 0$, then

$$C_T \leq C_1^{1/2} \exp\left(\frac{C_1}{2T}\right), \quad \text{where} \quad C_1 = \left(\frac{K^d}{\gamma}\right)^{K(d+\|a\|_1)},$$

$\|a\|_1 = a_1 + \ldots + a_d$ and K is a universal constant.

- It is automatically given by the observability inequality.
- Explicit dependence on all possible parameters.

Control cost on finite cubes

Let $L > 0$ and

$$\Omega = \Lambda_L := \left[-\frac{L}{2}, \frac{L}{2}\right]^d, \qquad \omega = S \cap \Lambda_L,$$

where $S \subset \mathbb{R}^d$ is a (γ, a)-thick set with $a_j \geq L$ for all $j = 1, \dots, d$.

Theorem 2: Let $T > 0$, Λ_L and ω be as above, and consider system (1) where Δ is the Laplacian on Λ_L with Dirichlet, Neumann or periodic boundary conditions. Then, the controlled heat equation is null-controllable in time $T > 0$ and the control cost C_T satisfies

$$C_T \leq C_1^{1/2} \exp\left(\frac{C_1}{2T}\right), \quad \text{where} \quad C_1 = \left(\frac{K^d}{\gamma}\right)^{K(d+\|a\|_1)},$$

$\|a\|_1 = a_1 + \ldots + a_d$ and K is an universal constant.

- There is no dependence on L.
- The control cost is "of same type" of the control cost for $\mathbb{R}^d$.

Further questions

Control cost:

- Optimizing over a and γ.
  For example: What happens if $a \to 0$ and $\gamma \to 1$?
- Is the control cost estimate sharp in all parameters?

The similarity of the control cost for the problem on Λ_L and on $\mathbb{R}^d$ suggests we can

Compare control function and solutions for problems on Λ_L and $\mathbb{R}^d$:

- Approximate control problem on $\mathbb{R}^d$ by control problem on Λ_L:
  Given initial value $u_0 \in L^2(\mathbb{R}^d)$, a time $T > 0$, an error $\varepsilon > 0$, choose $L_0 > 0$ such that null-control functions on $\mathbb{R}^d$ and on Λ_L differ by at most ε if $L > L_0$.
  Or similar question for error of solution u.
- Approximate control problem on Λ_L by control problem on $\mathbb{R}^d$:
  Given $L > 0$ fixed, a time $T > 0$, and an error $\varepsilon > 0$, for which inital values $u_0 \in L^2(\mathbb{R}^d)$ is (minimal norm) null-control function for Λ_L ε-approximated by a null-control function on $\mathbb{R}^d$?

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