Integrability and approximation of solutions to flow equations with conductivity given by Lévy random fields

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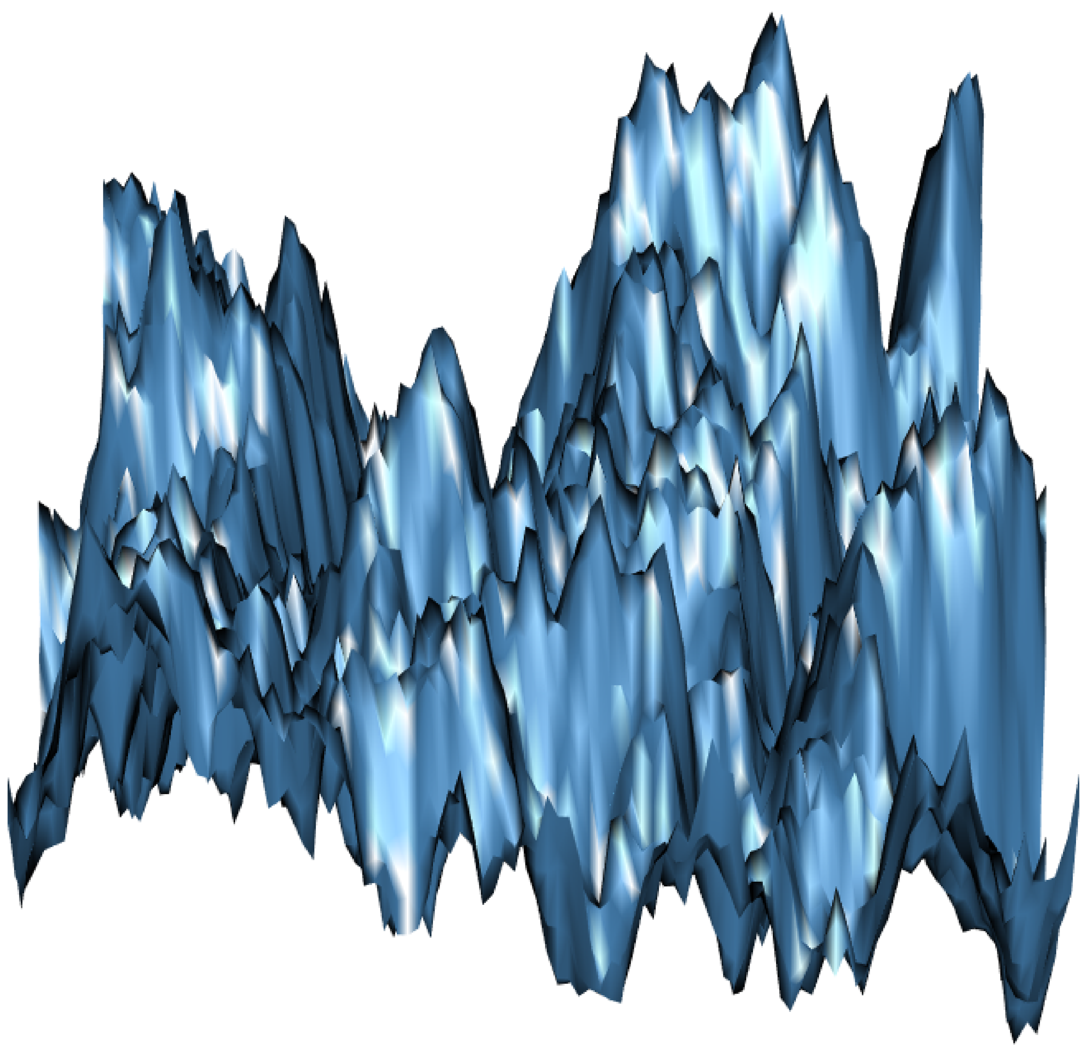
Introduction

Darcy's flow equation models the transport of pollutants in groundwater and oil recovery processes through conductive mediums (e.g. sand). This equation on $D \subset \mathbb{R}^d$ is given by

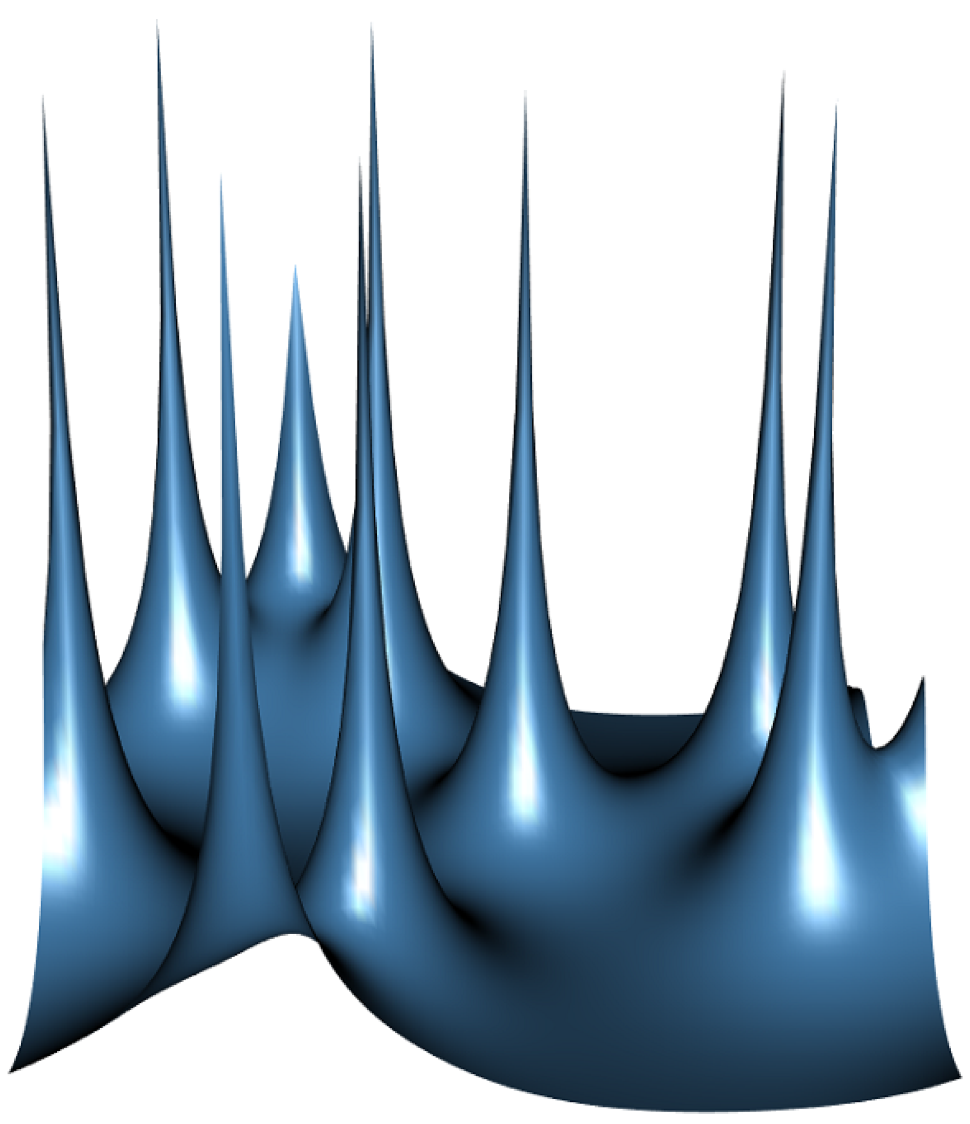
$$\begin{aligned} -\nabla \cdot (k(x)\nabla p(x)) &= f(x), && \text{in } D \\ p(x) &= g_D(x), && \text{on } \Gamma_D \\ -k(x)\nabla p(x) n &= g_N(x), && \text{on } \Gamma_D \end{aligned}$$

where D is a Lipschitz domain, $p : D \to \mathbb{R}$ is called the potential (or the pressure) and the volumetric flow is described by an vector field $k(x)\nabla p(x) : D \to \mathbb{R}^d$ where $k : D \to \mathbb{R}$ denotes the conductivity of the medium. The randomness originate from the unknown properties of the conductive medium. Therefore we modeled the coefficient function k as a transformed random field and investigated the solution p on its integrability and approximability.

Gaussian field



Smoothed compound Poisson noise



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Integrability

We investigated the integrability of the Sobolev norm of the weak solution p by using the following estimation which can be derived with help from the trace theorem and the Poincaré-inequality out of the weak form of Darcy's flow equation

$$\|p\|_{H^1(D)} \leq \frac{1 + C_p^2}{\inf_{x\in D} k(x)} (\|f\|_{L^2(D)} + \|g_D\|_{H^1(D)} + \|g_N\|_{H^1(D)})$$

1) For conductivity's given as a Gaussian Field we additionaly used the following estimation from Talagrand which under certain assumptions estimates the extreme values of an Gaussian Field Z.

$$P(\sup_{x\in D} Z(x) \geq z) \leq \left(\frac{KAz}{\sqrt{v}\sigma^2}\right)^v \phi\left(\frac{z}{\sigma}\right), \tag{1}$$

where $\phi(z) := \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\frac{s^2}{\sigma}}\, ds$. The result of this approach is given by:

Result 1 *Let $k(x) = T(Z(x))$ with $T(z) \geq Re^{-\rho|z|^h}$, for $0 < h < 2$ and $R, \rho > 0$. Assume that $Z(x)$, $x \in D$ is an continuous Gaussian process and suppose that (1) holds. Then $\|p\|_{H^1(D)} \in L^n(\Omega)$ for all $n \in \mathbb{N}$, where p is the weak solution to Darcy's flow equation.*

Note that if $h = 2$ the integrability still holds for $n < \frac{1}{2\sigma^2\rho}$.

2) A smoothed compound Poison noise Z is given by

$$Z(x) = \sum_{i=1}^{N} S_i g(x, X_i),$$

where $g \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, N is Poisson distributed and $\{X_i\}_{i\in\mathbb{N}}$, $\{S_i\}_{i\in\mathbb{N}}$ are both families of i.i.d. random variables.

For a conductivity given by such a random field, the a-priori estimation of the sobolev norm we stated above and the Markov inequality suffices to obtain the integrability and we have the following statement:

Result 2 *Let p be the weak solution to Darcy's flow equation with transformed conductivity and let $k(x) = T(Z(x))$, where $Z(x), x \in D \subset \mathbb{R}$ is an smoothed compound Poisson noise.*

(i) If $T(z) \geq K(1 + |z|)^{-q}$, for some $K > 0$ and fixed $q \in \mathbb{N}$, and ν has finite moment of order $nq + 2$ for $n \in \mathbb{N}$, i.e. $\int_{\mathbb{R}\setminus\{0\}} |s|^{(nq+2)}\, d\nu(s) < \infty$. Then $\|p\|_{H^1(D)} \in L^n(\Omega)$.

(ii) If $T(z) \geq Re^{-\rho|z|^h}$, for $0 < h \leq 1$, $R, \rho > 0$ and the two sided Laplace transform of $\tilde{\nu}$ $M_{\tilde{\nu}}(\theta) = M_{S_1}(\theta) := \mathbb{E}[e^{-\theta S}]$ exists for all $\theta \in \mathbb{R}$. Then $\|p\|_{H^1(D)} \in L^n(\Omega)$.

Approximation

We used the Karhunen-Loéve expansion for random field to obtain approximated solutions which converge in the L^n norm. With this expansion which is given by

$$Z_{KL}(x, w) = \mu(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i(x) \hat{Z}_i(w),$$

we approximated the random coefficient function k. For the solution of the resulting PDE we derived:

$$\begin{aligned} \|\dot{p}_t^k\|_{H^1(D)} \leq C(C_p^2 + 1) \cdot \Big(& \frac{\sup_{x\in D} |\dot{T}(Z_1^k(x) + tZ_2^k(x))| \cdot |Z_2^k(x)|}{(\inf_{x\in D} |T(Z_1^k(x) + tZ_2^k(x))|)^2} \\ & + \frac{\sup_{x\in D} |\dot{T}(Z_1^k(x) + tZ_2^k(x))| \cdot |Z_2^k(x)| \cdot \|g\|_{H^1(D)}}{C \cdot (\inf_{x\in D} |T(Z_1^k(x) + tZ_2^k(x))|)} \Big), \end{aligned}$$

where $\dot{p}_t^k$ denotes the solution to Darcy's flow equation with conductivity $k(x) = T(Z_1^k(x) + tZ_2^k(x))$ and $Z_1^k(x) = \sum_{i=1}^k \sqrt{\lambda_i} e_i(x) \hat{Z}_i(w)$, $Z_2^k(x) = \sum_{i=k+1}^\infty \sqrt{\lambda_i} e_i(x) \hat{Z}_i(w)$. With this estimation we obtained the following results:

Result 3 *Let $k(x) = T(Z(x))$ be a transformed Gaussian random field which fulfills the conditions of the Karhunen-Loéve expansion and the conditions of (1). Assume that T is continuous differentiable with $T(z) \geq e^{-\rho|z|^h}$ and $|\dot{T}(z)| \leq f(|z|)e^{\rho|z|^h}$, where $\rho > 0$, $0 < h < 2$ and $f(z)$ is an monotonic increasing polynomial. Let p be the weak solution and p_t^k the approximated weak solution. Then*

$$\mathbb{E}[\|p - p_t^k\|_{H^1(D)}^n] \to 0, \quad \textit{as } k \to \infty,$$

for all $n \in \mathbb{N}$.

Result 4 *Let $k(x) = T(Z(x))$ be a transformed smoothed compound Poisson noise. Let p be the weak solution and p_t^k the approximated weak solution. Assume that one of the following assumptions holds:*

(i) T is continuous differentiable with $T(z) \geq K(1 + |z|)^{-q}$, for some $K > 0$ and fixed $q \in \mathbb{N}$, and ν has finite moments of order $2n(m + 2q) + 2$ for $n, m \in \mathbb{N}$. And $|\dot{T}(z)| \leq f(|z|)$, where $f(z)$ is an monotonic increasing polynomial of order m.

(ii) T is continuous differentiable with $T(z) \geq e^{-\rho|z|^h}$ and $|\dot{T}(z)| \leq f(|z|)e^{\rho|z|^h}$, where $\rho > 0$, $0 < h \leq 1$ and $f(z)$ is an monotonic increasing polynomial. Furthermore, the two sided Laplace transform of ν $M_\nu(\theta) = M_{S_1}(\theta) := \mathbb{E}[e^{-\theta S}]$ exists for all $\theta \in \mathbb{R}$.

Then

$$\mathbb{E}[\|p - p_t^k\|_{H^1(D)}^n] \to 0, \quad \textit{as } k \to \infty,$$

for all $n \in \mathbb{N}$.