Optimal Reliability for Metal Components under Cyclic Loading

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Metal devices are exposed to strong forces like friction, tension and rotation causing stress states that influence the reliability of the component significantly. Using the PDE of linear elasticity and a material theoretic approach the deterministic lifetime of the component can be calculated. But, since it is impossible to predict exactly when and where damage will happen the lifetime model becomes more realistic when a stochastic approach based on Poisson-Point-Processes is integrated. Last but not least, the shape of the component itself affects reliability.

The resulting objective functional $J(\Omega, \sigma(u_\Omega))$ determines the failure probability depending on the shape Ω and the stress tensor $\sigma(u_\Omega)$. We use shape calculus methods to minimize this functional in order to find shapes with optimal reliability.

State-Equation:

- **shape/device**: $\Omega \subset \mathbb{R}^3$ (compact, class C^4)
- **linear elasticity equation**[2]:

$$\begin{aligned} -\text{div}(\sigma(u)) &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega_D \\ \sigma(u)\cdot\nu &= g \quad \text{on } \partial\Omega_N, \end{aligned} \qquad (1)$$

$$\partial\Omega_D \cap \partial\Omega_N = \emptyset,\ \partial\Omega_D \cup \partial\Omega_N = \partial\Omega$$

displacement: $u = u_\Omega$

stress: $\sigma(u) = \lambda \text{tr}(Du)I + \mu(Du + Du^\top)$

Origins of Failure

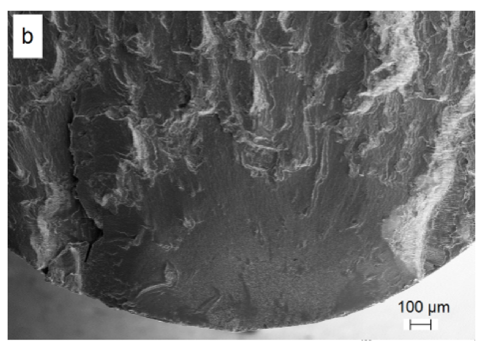


Figure 1: Metal Surface

- intrusions & extrusions cause stress peaks at the surface
- cyclic loadings lead to cracks (damage mechanism: low cycle fatigue (LCF))

From Stress to Deterministic Lifetime

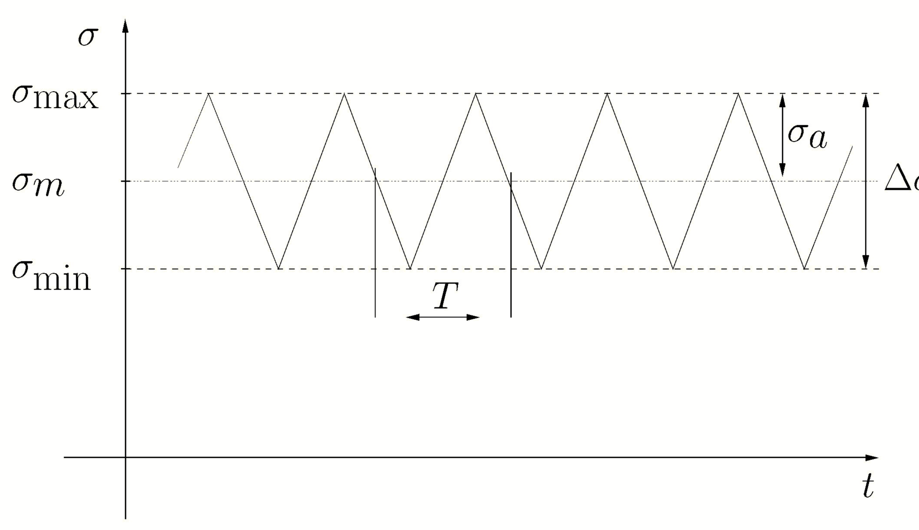


Figure 2: Load Cycles with Stress Amplitude

Use a material theoretic approach to calculate durability of the component

Outcome:

$$N_{det} : \mathbb{R}^{3\times3} \to \mathbb{R}^+ \cup \infty,\ N_{det}(0) = \infty$$

maps the stress tensor $\sigma(u(x))$ to the **deterministic number of load cycles to crack initiation** at the surface point x

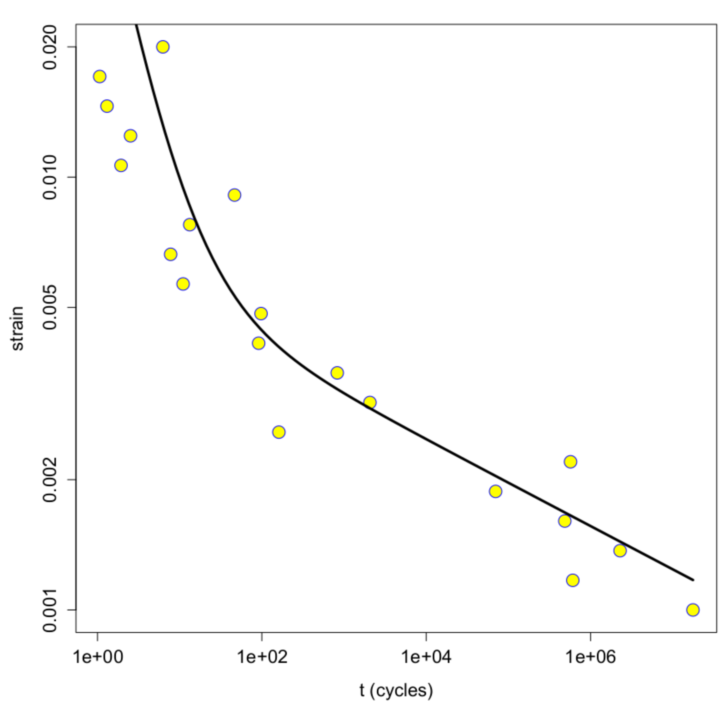


Figure 3: Relation Between Stress and Load Cycles to Failure

Probabilistic Reliability Model

Combine N_{det} with **stochastic-processes** for a more realistic model:

- $\mathcal{C} = [0,\infty) \times \partial\Omega$ possible **times and places of crack initiation**
- **simplicity**: cracks should be distinguishable
- **nonatomicness:** there is no time & location in which a crack will start with probability larger than zero
- **independent increments:** cracks in disjoint areas appear independently from each other

$\to$ Poisson-Point-Process γ on $\mathcal{C}$

Random failure time:

$$\mathcal{Z}_\gamma = \inf\{t > 0 \,|\, \gamma([0,t] \times \partial\Omega) > 0\}$$

Weibull Model for LCF

- Define the Radon measure ϱ by

$$\varrho([0,t] \times \partial\Omega) = \left\| \frac{t}{N_{det}(\sigma(u))} \right\|^m_{L^m(\partial\Omega)}$$

- γ associated PPP [4] with

$$P(\gamma(B) = n) = \exp(-\varrho(B))\frac{\varrho(B)^n}{n!} B \in \mathcal{B}_b(\mathcal{C})$$

- $\mathcal{Z} = \mathcal{Z}_\gamma$ is Weibull distributed

Probability of survival beyond time t

$$P(\mathcal{Z} > t) = P(\varrho([0,t] \times \partial\Omega) = 0) = e^{-\left\| \frac{t}{N_{det}(\sigma(u))} \right\|^m_{L^m(\partial\Omega)}}$$

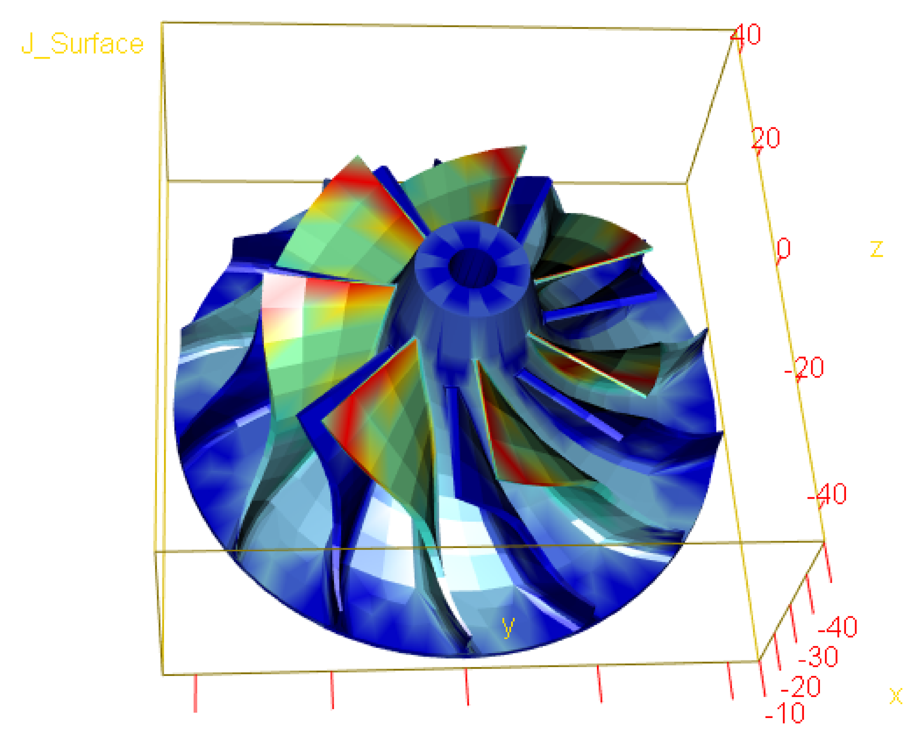


Figure 4: Failure Probabilities at the Surface of a Compressor Stage, Mohammed Saadi, BUW

Optimal Reliability [1, 3]

Task: Maximize the probability of survival

$$\begin{aligned} &\textbf{min} \quad \mathbf{J(\Omega, \sigma(u_\Omega))} := \left\| \frac{\mathbf{1}}{\mathbf{N_{det}(\sigma(u_\Omega))}} \right\|^{\mathbf{m}}_{\mathbf{L^m(\partial\Omega)}} \\ &\text{s.t.} \quad u(\Omega) \text{ solves (1)} \\ &\qquad \Omega \text{ is an admissible shape.} \end{aligned} \qquad (2)$$

J is called **LCF Reliability Functional**

Shape calculus: Setting

Idea: [5] Calculate the change of the objective functional when the shape Ω is changed along a **vector field** $V \in C_0^4(\mathbb{R}^3, \mathbb{R}^3)$, $\langle V, n\rangle = 0$ on $\partial\Omega^{ext}$

- $\Omega \subset \Omega^{ext} \subset \mathbb{R}^3$ initial shape (C_b^4-domain),
- $T_t[V]$ transformation along the vector field V at time t,
- $\Omega_t = T_t[V](\Omega)$ set transformed along V at time t.

Material/ Shape/ Euler derivative

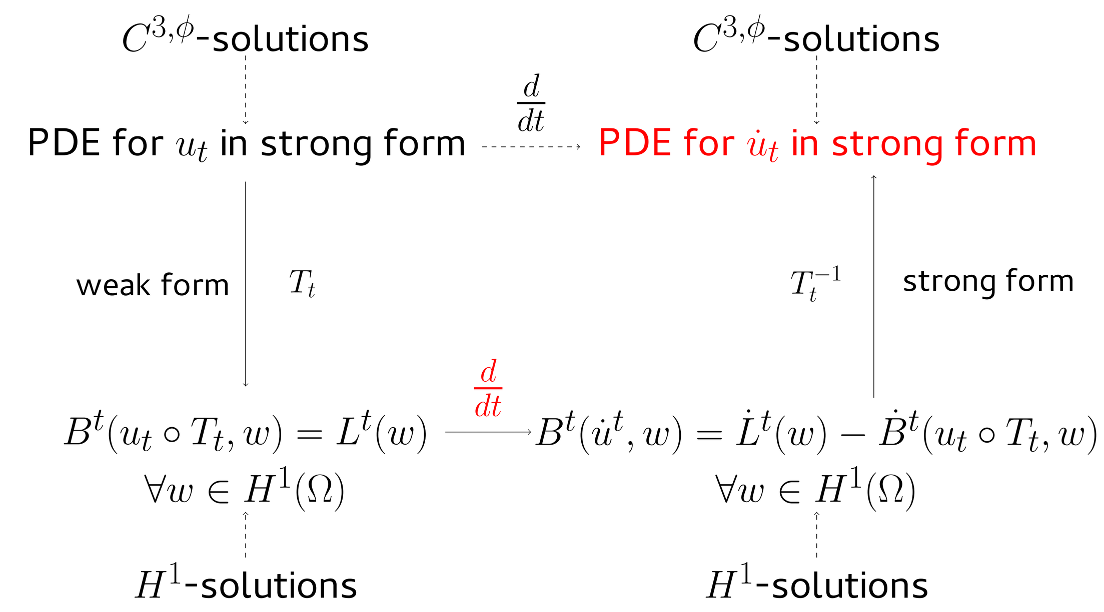
1) **material derivative**: $\dot{u}^t = \frac{d}{dt} u_t \circ T_t[V], \quad u_t = u_{\Omega_t}$

2) **shape derivative**: $u' = \dot{u} - \nabla u V, \quad \dot{u} = \dot{u}^0$

3) **Euler derivative**: $dJ(\Omega)[V] = \frac{d}{dt} J(\Omega_t)\Big|_{t=0}$

Differentiable Material Derivatives

- **need differentiable solutions and material/shape derivatives**
- direct differentiation w.r.t. Hölder topologies is very complicated
- Alternative: Use **weak formulations** and **regularity theory for elliptic PDE**



Hölder Material Derivatives: Linear Elasticity

Theorem: Let $f \in C^{1,\phi}(\overline{\Omega^{ext}}, \mathbb{R}^3)$, $g \in C^{2,\phi}(\overline{\Omega^{ext}}, \mathbb{R}^3)$, $\phi \in (0,1)$, u_t the unique $C^{3,\phi}$-solution of (1) on Ω_t and q_t the unique $C^{3,\phi}$-solution of

$$\begin{aligned} -\text{div}(\sigma(q)) &= Df \cdot V + \text{div}(V)f + f_{u_t} && \text{in } \Omega_t \\ q &= 0 && \text{on } \partial\Omega_{D,t} \\ \sigma(q)\cdot n_t &= DgV + g\,\text{div}_{\partial\Omega}(V) + gn_t && \text{on } \partial\Omega_{N,t} \end{aligned} \qquad (3)$$

Then

$$\dot{u}^t = q_t \circ T_t$$

is the **material derivative** of u_t w.r.t. the **strong $C^{3,\varphi}$-topology** for any $0 < \varphi < \phi$.

Eulerian Derivative: LCF Reliability

Theorem: The Euler derivative of the LCF reliability functional is given by

$$\begin{aligned} dJ(\Omega)[V] = \int_{\partial\Omega} &\langle D\mathcal{F}_{sur}(\sigma(u)), D\sigma(u) : n\rangle \langle V, n\rangle \\ &+ \kappa \mathcal{F}_{sur}(\sigma(u))\langle V, n\rangle \\ &+ \langle D\mathcal{F}_{sur}(\sigma(u)), \sigma(u')\rangle \, dA \end{aligned}$$

with $\mathcal{F}_{sur}(\sigma(u)) = N_{det}(\sigma(u))^{-m}$.

Necessary optimality condition:

$dJ(\Omega)[V] = 0$ for any admissible vector field V.

References

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[4] O. Kallenberg. *Random Measures*. Akademie-Verlag, Berlin, 1983.

[5] J. Sokolovski and J.-P. Zolesio. *Introduction to Shape Optimization - Shape Sensitivity Analysis*. Springer, Berlin Heidelberg, 1st edition, 1992.