Shape optimization for quadratic functionals and states with random right-hand sides

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Overview

- Problem statement
- Deterministic reformulation
- Shape calculus
- Low-rank approximation
- Numerical results

Problem statement

- Consider an elliptic state equation with random right-hand side, for example, the equations of linear elasticity with random forcing:

$$\begin{aligned} -\operatorname{div}\left[\mathbf{A}e\big(\mathbf{u}(\omega)\big)\right] &= \mathbf{f}(\omega) && \text{in } D,\\ \mathbf{A}e\big(\mathbf{u}(\omega)\big)\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^{\text{free}},\\ \mathbf{A}e\big(\mathbf{u}(\omega)\big)\mathbf{n} &= \mathbf{g}(\omega) && \text{on } \Gamma_N^{\text{fix}},\\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D. \end{aligned}$$

  where $e(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}})/2$ stands for the linearized strain tensor and $\mathbf{A}$ is given by

$$\mathbf{A}\mathbf{B} = 2\mu\mathbf{B} + \lambda \operatorname{tr}(\mathbf{B})\mathbf{I} \text{ for all } \mathbf{B} \in \mathbb{R}^{d\times d}$$

  with the Lamé coefficients λ and μ satisfying $\mu > 0$ and $\lambda + 2\mu/d > 0$.

- Consider a quadratic shape functional, for example, the compliance of shapes:

$$\begin{aligned} \mathcal{C}(D,\omega) &= \int_D \mathbf{A}e\big(\mathbf{u}(\mathbf{x},\omega)\big) : e\big(\mathbf{u}(\mathbf{x},\omega)\big)\,\mathrm{d}\mathbf{x}\\ &= \int_D \langle \mathbf{f}(\omega), \mathbf{u}(\omega)\rangle\,\mathrm{d}\mathbf{x} + \int_{\Gamma_N^{\text{fix}}} \langle \mathbf{g}(\mathbf{x},\omega), \mathbf{u}(\mathbf{x},\omega)\rangle\,\mathrm{d}\sigma_{\mathbf{x}}, \end{aligned}$$

- We aim at minimizing the expectation $\mathbb{E}[\mathcal{C}(D,\omega)]$ of the quadratic shape functional.

Related work

References.

P.D. Dunning and H.A. Kim.
Robust topology optimization. Minimization of expected and variance of compliance.
AIAA Journal, 51(11):2656–2664, 2013.

S. Conti, H. Held, M. Pach, M. Rumpf, and R. Schultz.
Shape optimization under uncertainty. A stochastic programming approach.
SIAM J. Optim., 19(4):1610–1632, 2009.

G. Allaire and C. Dapogny.
A deterministic approximation method in shape optimization under random uncertainties.
SMAI J. Comput. Math., 1:83–143, 2015.

Statistical quantities

- Expectation or mean:

$$\mathbb{E}[v](\mathbf{x}) := \int_\Omega v(\mathbf{x},\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

- Correlation:

$$\mathrm{Cor}[v](\mathbf{x},\mathbf{y}) := \int_\Omega v(\mathbf{x},\omega)v(\mathbf{y},\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[v(\mathbf{x})v(\mathbf{y})]$$

- Covariance:

$$\begin{aligned}\mathrm{Cov}[v](\mathbf{x},\mathbf{y}) &:= \int_\Omega \big(v(\mathbf{x},\omega)-\mathbb{E}[v](\mathbf{x})\big)\big(v(\mathbf{y},\omega)-\mathbb{E}[v](\mathbf{y})\big)\,\mathrm{d}\mathbb{P}(\omega)\\ &= \mathrm{Cor}[v](\mathbf{x},\mathbf{y}) - \mathbb{E}[v](\mathbf{x})\mathbb{E}[v](\mathbf{y})\end{aligned}$$

- Variance:

$$\begin{aligned}\mathrm{Var}[v](\mathbf{x}) &:= \int_\Omega \big(v(\mathbf{x},\omega)-\mathbb{E}[v](\mathbf{x})\big)^2\,\mathrm{d}\mathbb{P}(\omega)\\ &= \mathrm{Cor}[v](\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}} - \mathbb{E}[v]^2(\mathbf{x}) = \mathrm{Cov}[v](\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\end{aligned}$$

- k-th moment:

$$\mathcal{M}[v](\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_k) := \int_\Omega v(\mathbf{x}_1,\omega)v(\mathbf{x}_2,\omega)\cdots v(\mathbf{x}_k,\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

PDEs with random right-hand side

Random boundary value problem:

$$-\operatorname{div}\left[\alpha\nabla u(\omega)\right] = f(\omega) \text{ in } D, \quad u(\omega) = 0 \text{ on } \partial D$$

$\longrightarrow$ the random solution depends linearly on the random input parameter

Theorem (Schwab/Todor [2003]): It holds

$$-\operatorname{div}\left[\alpha\nabla\mathbb{E}[u]\right] = \mathbb{E}[f] \text{ in } D, \quad \mathbb{E}[u] = \mathbb{E}[g] \text{ on } \partial D$$

and

$$\begin{aligned}(\operatorname{div}\otimes\operatorname{div})\left[(\alpha\otimes\alpha)(\nabla\otimes\nabla)\operatorname{Cor}[u]\right] &= \operatorname{Cor}[f] && \text{in } D\times D,\\ \operatorname{Cor}[u] &= 0 && \text{on } \partial(D\times D).\end{aligned}$$

Numerical solution of the correlation equation:

- sparse grid approximation by the combination technique

  H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *J. Comput. Phys.*, 252:128–141, 2013.

- low-rank approximation by the pivoted Cholesky decomposition

  H. Harbrecht, M. Peters, and R. Schneider. On the low-rank approximation by the pivoted Cholesky decomposition. *Appl. Numer. Math.*, 62:428–440, 2012.

- adaptive low-rank approximation by means of $\mathcal{H}$-matrices

  J. Dölz, H. Harbrecht, and C. Schwab. Covariance regularity and $\mathcal{H}$-matrix approximation for rough random fields. *Numer. Math.*, 135(4):1045–1071, 2017.

Deterministic reformulation of the shape functional

Theorem. The expectation of the quadratic shape functional can be rewritten by

$$\mathbb{E}[\mathcal{C}(D,\omega)] = \int_D \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\,\mathrm{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\,\mathrm{d}\mathbf{x},$$

where

$$(\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}}) : \left[H^1_{\Gamma_D}(D)\right]^d \otimes \left[H^1_{\Gamma_D}(D)\right]^d \to L^2(D) \otimes L^2(D)$$

is the linear operator induced from the bilinear mapping

$$\mathbf{u}\mathbf{v}^{\mathsf{T}} \mapsto \mathbf{A}e(\mathbf{u}) : e(\mathbf{v}).$$

Proof. The assertion follows from

$$\begin{aligned}
\mathbb{E}[\mathcal{C}(D,\omega)] &= \int_\Omega \int_D \mathbf{A}e\big(\mathbf{u}(\mathbf{x},\omega)\big) : e\big(\mathbf{u}(\mathbf{x},\omega)\big)\,\mathrm{d}\mathbf{x} \\
&= \int_D \left[(\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}}) \int_\Omega \mathbf{u}(\mathbf{x},\omega)\mathbf{u}(\mathbf{y},\omega)^{\mathsf{T}}\,\mathrm{d}\mathbb{P}(\omega)\right]\Bigg|_{\mathbf{x}=\mathbf{y}} \mathrm{d}\mathbf{x} \\
&= \int_D \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\,\mathrm{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

□

How to compute the correlation?

Theorem. The two-point correlation function $\operatorname{Cor}[\mathbf{u}] \in [H^1_{\Gamma_D}(D)]^d \otimes [H^1_{\Gamma_D}(D)]^d$ is the unique solution to the following tensor-product boundary value problem:

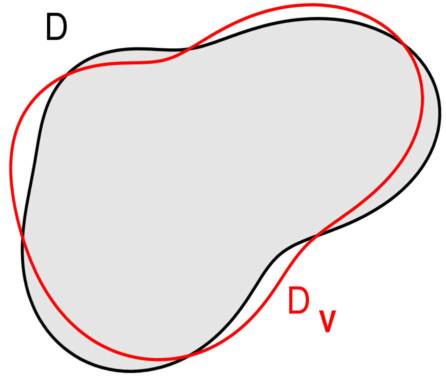
$$\begin{aligned}
(\operatorname{div}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})\big[(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}]\big] &= \operatorname{Cor}[\mathbf{f}] && \text{in } D \times D,\\
(\operatorname{div}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{I}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } D \times \Gamma_N^{\text{fix}\cup\text{free}},\\
(\mathbf{I}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_N^{\text{fix}\cup\text{free}} \times D,\\
(\operatorname{div}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } D \times \Gamma_D,\\
(\mathbf{I}_{\mathbf{x}} \otimes \operatorname{div}_{\mathbf{y}})(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } \Gamma_D \times D,\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \big(\Gamma_N^{\text{fix}\cup\text{free}} \times \Gamma_N^{\text{fix}\cup\text{free}}\big)\\
&&& \quad \setminus (\Gamma_N^{\text{fix}} \times \Gamma_N^{\text{fix}}),\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \operatorname{Cor}[\mathbf{g}] && \text{on } \Gamma_N^{\text{fix}} \times \Gamma_N^{\text{fix}},\\
(\mathbf{A}e_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{n}_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_N^{\text{fix}\cup\text{free}} \times \Gamma_D,\\
(\mathbf{I}_{\mathbf{x}} \otimes \mathbf{A}e_{\mathbf{y}})\operatorname{Cor}[\mathbf{u}](\mathbf{I}_{\mathbf{x}} \otimes \mathbf{n}_{\mathbf{y}}) &= \mathbf{0} && \text{on } \Gamma_D \times \Gamma_N^{\text{fix}\cup\text{free}},\\
\operatorname{Cor}[\mathbf{u}] &= \mathbf{0} && \text{on } \Gamma_D \times \Gamma_D.
\end{aligned}$$

Proof. The assertion follows by tensorizing the state equation and the exploiting the linearity when taking the expectation. □

Computing the shape gradient

Domain perturbation.

$$D_{\mathbf{V}} = (I+\mathbf{V})(D), \quad \mathbf{V} \in W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d), \quad \|\mathbf{V}\|_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)} \leq \frac{1}{2}.$$



Definition (Shape derivative). A functional $J(D)$ of the domain is shape differentiable at D if the underlying mapping $\mathbf{V} \mapsto J(D_{\mathbf{V}})$ which maps $W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ into $\mathbb{R}$ is Fréchet-differentiable at $\mathbf{V}=\mathbf{0}$. The related Fréchet derivative $\mathbf{V} \mapsto J'(D)[\mathbf{V}]$ at D satisfies the following asymptotic expansion in the vicinity of $\mathbf{V}=\mathbf{0}$:

$$J(D_{\mathbf{V}}) = J(D) + J'(D)[\mathbf{V}] + o(\mathbf{V}), \text{ where } \frac{\|o(\mathbf{V})\|}{\|\mathbf{V}\|_{W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d)}} \xrightarrow{\mathbf{V}\to\mathbf{0}} 0.$$

Theorem. The functional $\mathbb{E}[J(D,\omega)]$ is shape differentiable at any shape $D \in \mathcal{U}_{ad}$ and its derivative reads

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{V}}\mathbb{E}[\mathcal{C}(D,\omega)] = \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V},\mathbf{n}\rangle \big((\mathbf{A}e_{\mathbf{x}} : e_{\mathbf{y}})\,\mathrm{Cor}[\mathbf{u}]\big)(\mathbf{x},\mathbf{y})\big|_{\mathbf{x}=\mathbf{y}}\,\mathrm{d}\sigma_{\mathbf{x}}.$$

A further example

Problem. Minimize the L^2-tracking type functional

$$J(D,\omega) = \frac{1}{2}\int_B |u(\mathbf{x},\omega) - u_0(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x},$$

where the state $u(\omega)$ is given by

$$-\Delta u(\omega) = f(\omega) \text{ in } D, \quad u(\omega) = 0 \text{ in } \partial D.$$

Reformulation. The expectation of the functional $J(D,\omega)$ can be rewritten as

$$\mathbb{E}[J(D,\omega)] = \frac{1}{2}\int_B \Big(\mathrm{Cor}[u](\mathbf{x},\mathbf{x}) - 2u_0(\mathbf{x})\mathbb{E}[u](\mathbf{x}) + u_0^2(\mathbf{x})\Big)\,\mathrm{d}\mathbf{x},$$

where

$$-\Delta\mathbb{E}[u] = \mathbb{E}[f] \text{ in } D, \quad \mathbb{E}[u] = 0 \text{ on } \partial D,$$

and

$$\begin{aligned} (\Delta_{\mathbf{x}} \otimes \Delta_{\mathbf{y}})\,\mathrm{Cor}[u] &= \mathrm{Cor}[f] && \text{in } D \times D, \\ \mathrm{Cor}[u] &= 0 && \text{in } \partial(D \times D). \end{aligned}$$

A further example

Shape gradient. The gradient is given by

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{V}} J(D,\omega) = -\int_{\partial D} \langle \mathbf{V}, \mathbf{n} \rangle \frac{\partial p}{\partial \mathbf{n}}(\omega) \frac{\partial u}{\partial \mathbf{n}}(\omega) \, \mathrm{d}\sigma_{\mathbf{x}},$$

where the adjoint state $p(\omega) \in H_0^1(D)$ satisfies the boundary value problem

$$-\Delta p(\omega) = -\chi_B \big(u(\omega) - u_0\big) \text{ in } D, \quad p(\omega) = 0 \text{ on } \partial D.$$

Reformulation. The shape gradient of the expected shape functional is given by

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{V}} \mathbb{E}[J(D,\omega)] = -\int_{\partial D} \langle \mathbf{V}, \mathbf{n} \rangle \left[\left(\frac{\partial}{\partial \mathbf{n}} \otimes \frac{\partial}{\partial \mathbf{n}} \right) \mathrm{Cor}[p,u](\mathbf{x},\mathbf{y}) \right] \Bigg|_{\mathbf{x}=\mathbf{y}} \mathrm{d}\sigma_{\mathbf{x}}.$$

Here, the correlation function $\mathrm{Cor}[p,u] \in H_0^1(D) \otimes H_0^1(D)$ can be calculated by solving the boundary value problem

$$\begin{aligned} -(\Delta_{\mathbf{x}} \otimes \mathbf{I}_{\mathbf{y}}) \, \mathrm{Cor}[p,u] &= -(\chi_B \otimes \mathbf{I}_{\mathbf{y}}) \big(\mathrm{Cor}[u] - u_0 \otimes \mathbb{E}[u] \big) && \text{in } D \times D, \\ \mathrm{Cor}[p,u] &= 0 && \text{on } \partial(D \times D). \end{aligned}$$

Low-rank approximation

- **Approximation of the input correlation.** Assume low-rank approximations

$$\operatorname{Cor}[\mathbf{f}] \approx \sum_i \mathbf{f}_i \mathbf{f}_i^\intercal, \quad \operatorname{Cor}[\mathbf{g}] \approx \sum_j \mathbf{g}_j \mathbf{g}_j^\intercal.$$

  Such expansions can efficiently be computed by e.g. a pivoted Cholesky decomposition.
- **Approximation of the shape functional.** The shape functional is simply given by

$$\mathbb{E}[\mathcal{C}(D,\omega)] = \int_D \sum_{i,j} \mathbf{A}e(\mathbf{u}_{i,j}) : e(\mathbf{u}_{i,j}) \, \mathrm{d}\mathbf{x},$$

  where

$$\begin{aligned} -\operatorname{div}\left[\mathbf{A}e(\mathbf{u}_{i,j})\right] &= \mathbf{f}_i && \text{in } D,\\ \mathbf{A}e(\mathbf{u}_{i,j})\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_N^{\text{free}},\\ \mathbf{A}e(\mathbf{u}_{i,j})\mathbf{n} &= \mathbf{g}_j && \text{on } \Gamma_N^{\text{fix}},\\ \mathbf{u}_{i,j} &= \mathbf{0} && \text{on } \Gamma_D. \end{aligned}$$

- **Approximation of the shape gradient.** The shape gradient is given by

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{V}} \mathbb{E}[J(D,\omega)] = \int_{\Gamma_N^{\text{free}}} \langle \mathbf{V}, \mathbf{n} \rangle \sum_{i,j} \mathbf{A}e(\mathbf{u}_{i,j}) : e(\mathbf{u}_{i,j}) \, \mathrm{d}\sigma_{\mathbf{x}}.$$

- **Alternative approach.** A direct discretization of $\operatorname{Cor}[\mathbf{u}]$ in a sparse grid space is possible as well.

Implementation

- **Level-set approach to represent the domain under consideration.** Represent the shape $D \subset \mathbb{R}^d$ as the negative subdomain of a level set function

$$\phi : \mathbb{R}^d \to \mathbb{R} \text{ such that } \begin{cases} \phi(\mathbf{x}) < 0 & \text{if } \mathbf{x} \in D, \\ \phi(\mathbf{x}) = 0 & \text{if } \mathbf{x} \in \partial D, \\ \phi(\mathbf{x}) > 0 & \text{if } \mathbf{x} \in \mathbb{R}^d \setminus \overline{D}. \end{cases}$$

- **Motion of the domain.** The motion of the domain $D(t)$, induced by a velocity field $\mathbf{V}(t,\mathbf{x})$ with normal amplitude, translates in terms of an associated level set function $\phi(t,\cdot)$ as a Hamilton-Jacobi equation:

$$\frac{\partial \phi}{\partial t} + \langle \mathbf{V}, |\nabla \phi| \rangle = 0, \quad t \in (0,T),\ \mathbf{x} \in \mathbb{R}^d.$$

- In the present situation, $\mathbf{V}$ stems from the analytical formula for the shape derivative of the objective under consideration.
- **Ersatz material approach for computing the elastic displacement $\mathbf{u}$.** The boundary value problem problem is transferred to a problem on a box D_0 by filling the void part $D_0 \setminus \overline{D}$ with a very soft material, whose Hooke's law is $\varepsilon \mathbf{A}$ with $\varepsilon \ll 1$.
- **Low-rank approximation.** We use the pivoted Cholesky decomposition to compute the low-rank approximation.

First example

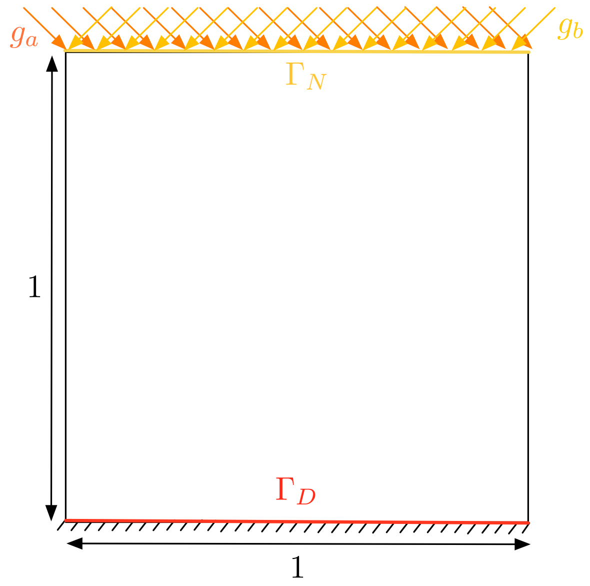
Problem. A bridge is clamped on its lower part two sets of loads $\mathbf{g}_a = (1, -1)$ and $\mathbf{g}_b = (-1, 1)$ are applied on its top, i.e.,

$$\mathbf{g}(\mathbf{x}, \omega) = \xi_1(\omega)\mathbf{g}_a(\mathbf{x}) + \xi_2(\omega)\mathbf{g}_b(\mathbf{x}).$$

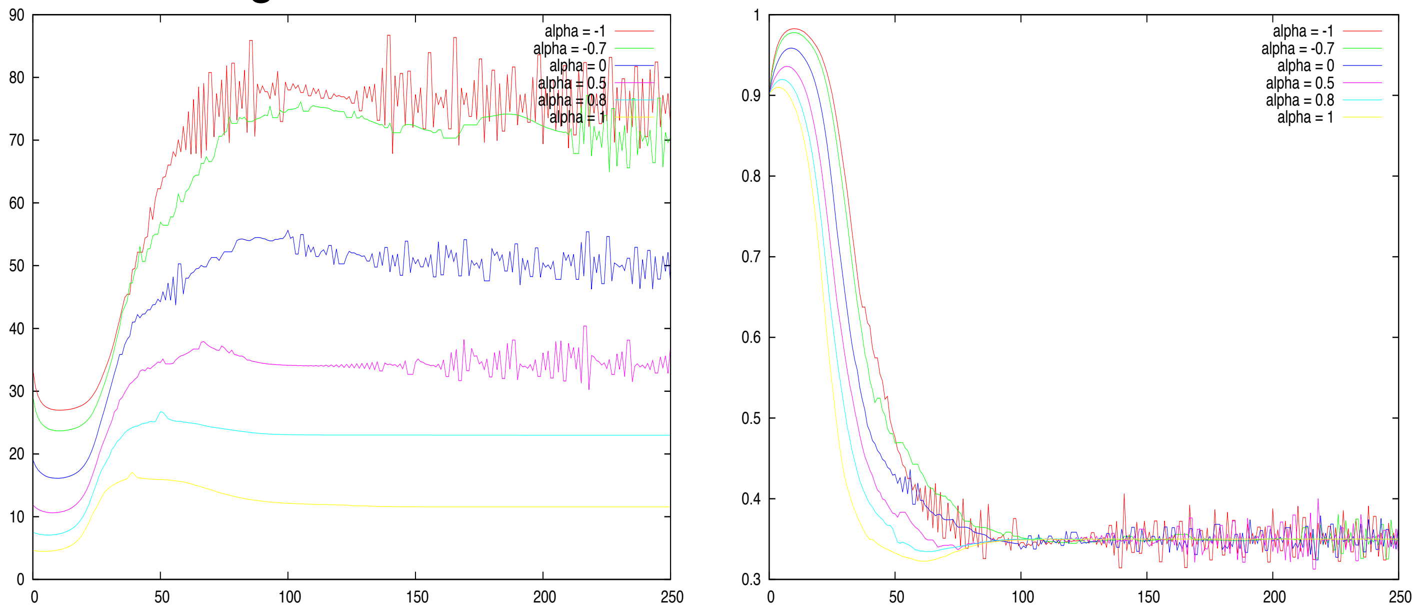
The choice $\mathbb{E}[\xi_i] = 0$, $\mathrm{Var}[\xi_i] = 1$, $\mathrm{Cor}[\xi_1, \xi_2] = \alpha$ implies

$$\mathrm{Cor}[\mathbf{g}] = \mathbf{g}_a\mathbf{g}_a^{\mathsf{T}} + \mathbf{g}_b\mathbf{g}_b^{\mathsf{T}} + \alpha\left(\mathbf{g}_a\mathbf{g}_b^{\mathsf{T}} + \mathbf{g}_b\mathbf{g}_a^{\mathsf{T}}\right).$$

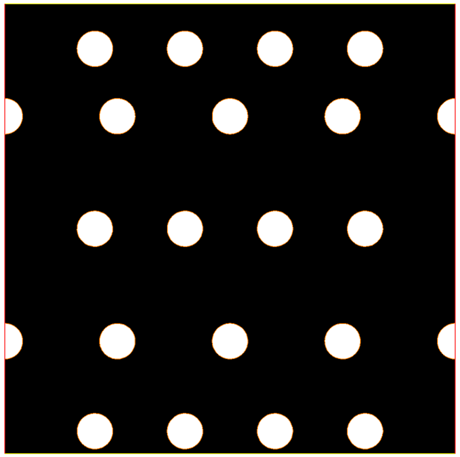
Sketch:



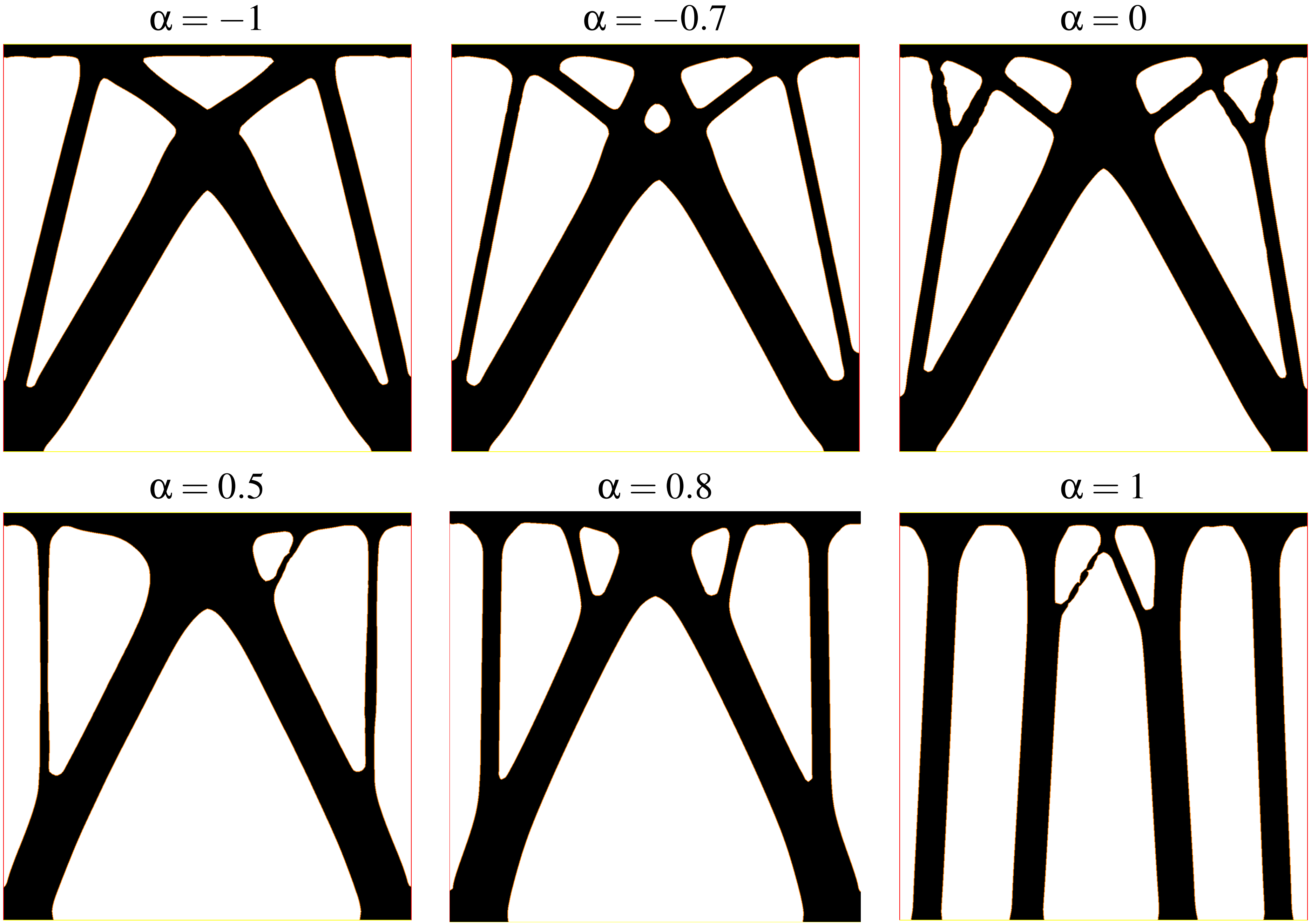
Convergence histories for the mean value and the volume:



Initial guess:



First example



Second example

Problem. A bridge is clamped on its lower part two sets of loads $\mathbf{g}^i = (g_1^i, g_2^i)$, $i = 1,2,3$, are applied on its top such that

$$\mathrm{Cor}[g_1^i](\mathbf{x},\mathbf{y}) = 10^5 h_i^+\left(\frac{x_1+y_1}{2}\right)e^{-10|x_1-y_1|},$$

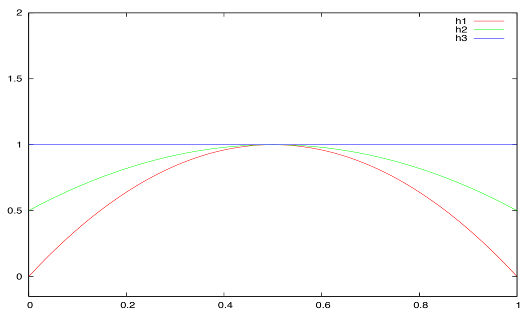
$$\mathrm{Cor}[g_2^i](\mathbf{x},\mathbf{y}) = 10^6 k_i^+\left(\frac{x_1+y_1}{2}\right)e^{-10|x_1-y_1|},$$

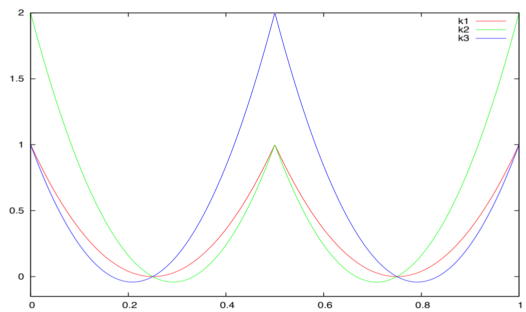
where

$$h_1(t) = 1-4\left(t-\frac{1}{2}\right)^2, \quad k_1(t) = \begin{cases}(4t-1)^2, & \text{if } t \le \frac{1}{2},\\ (4t-3)^2, & \text{else,}\end{cases}$$

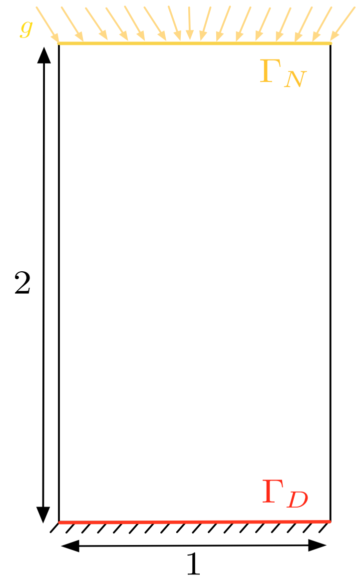
$$h_2(t) = 2t(1-t)+\frac{1}{2}, \quad k_2(t) = \begin{cases}(4t-1)(6t-2), & \text{if } t \le \frac{1}{2},\\ (4t-3)(6t-4), & \text{else,}\end{cases}$$

$$h_3(t) = 1, \quad k_3(t) = \begin{cases}(4t-1)(6t-1), & \text{if } t \le \frac{1}{2},\\ (4t-3)(6t-5), & \text{else.}\end{cases}$$

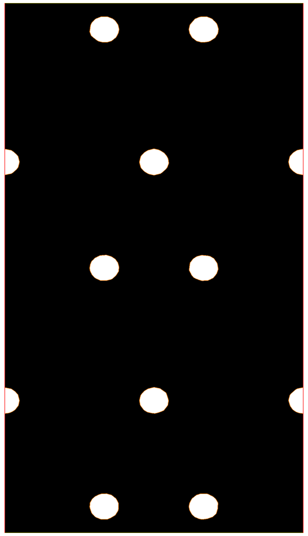




Sketch:

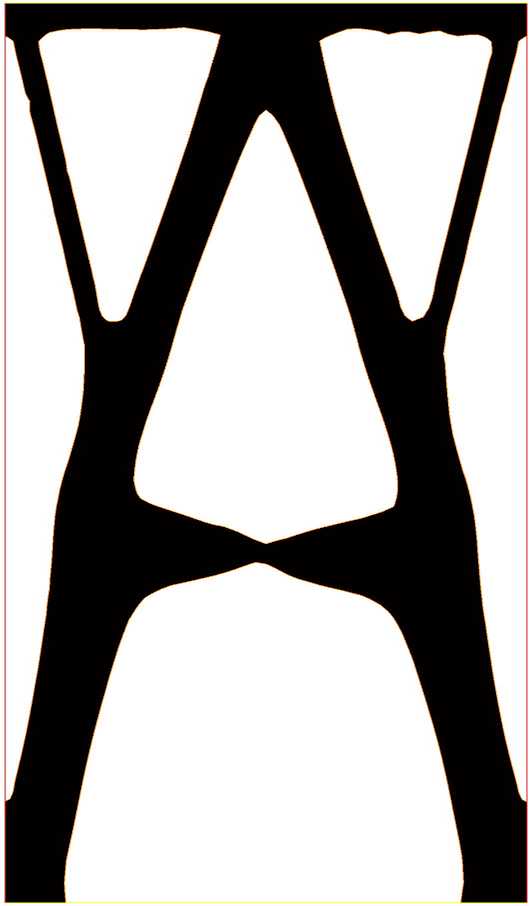


Initial guess:

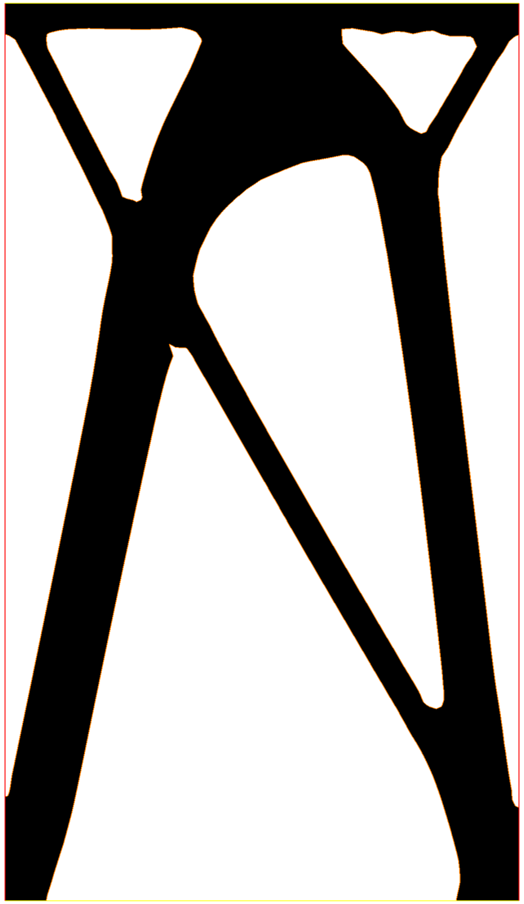


Second example

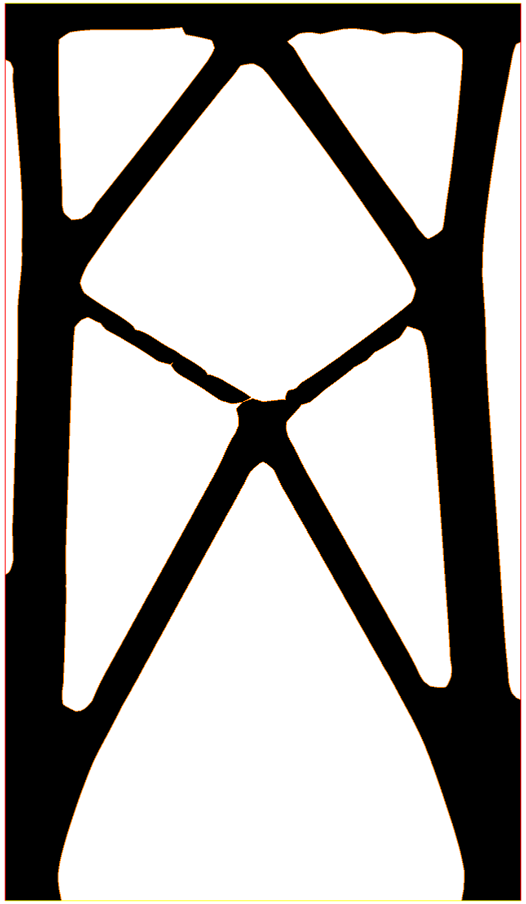
surface load $\mathbf{g}^1(\omega)$



surface load $\mathbf{g}^2(\omega)$



surface load $\mathbf{g}^3(\omega)$



About measurement noise in EIT

Problem. Minimize

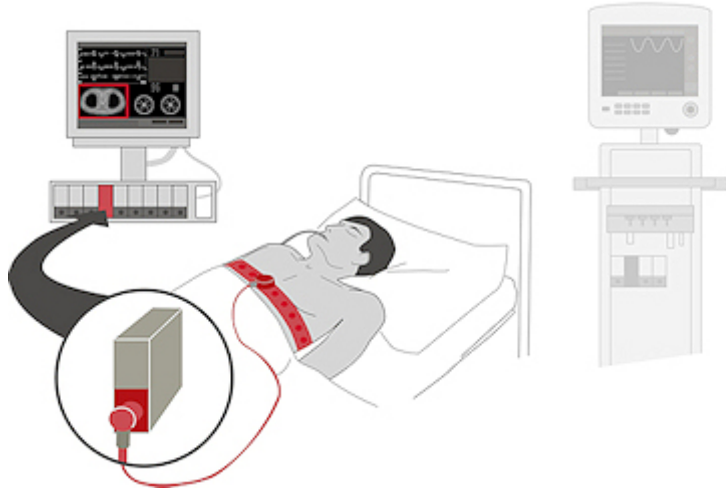
$$F(D) = (1-\alpha)\mathbb{E}\big[J(D,\omega)\big] + \alpha\sqrt{\mathrm{Var}\big[J(D,\omega)\big]} \to \inf,$$

where the random shape functional reads as

$$J(D,\omega) = \int_D \big\|\nabla\big(v(\omega)-w\big)\big\|^2 \mathrm{d}\mathbf{x} \to \inf$$

and the states read as

$$\begin{aligned} \Delta v(\omega) &= 0 & \Delta w &= 0 & &\text{in } D,\\ v(\omega) &= 0 & w &= 0 & &\text{on } \Gamma,\\ \frac{\partial v}{\partial \mathbf{n}}(\omega) &= g(\omega) & w &= f & &\text{on } \Sigma. \end{aligned}$$



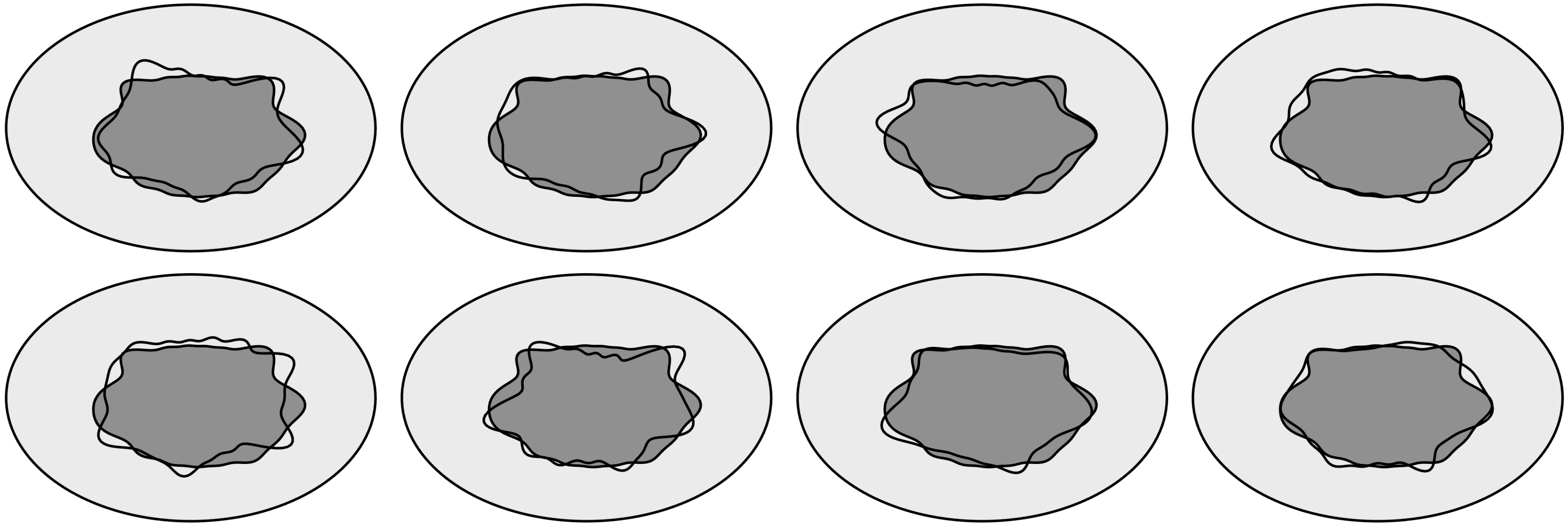
We assume that the Neumann data g are given as a Gaussian random field

$$g(\mathbf{x},\omega) = g_0(\mathbf{x}) + \sum_{i=1}^{M} g_i(\mathbf{x}) Y_i(\omega),$$

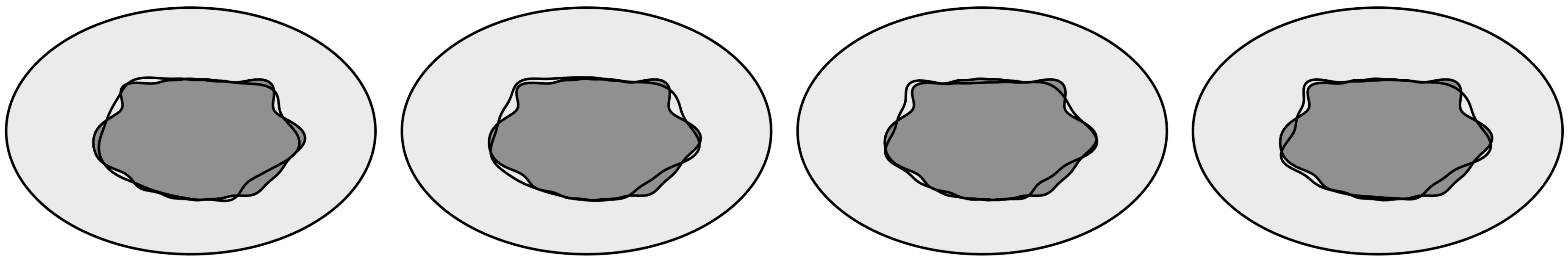
where the random variables are independent, satisfying $Y_i \sim \mathcal{N}(0,1)$.

Numerical results (5% noise, 10 samples)

Reconstructions for different realizations of the measurement:



Reconstructions for $\alpha = 0$, $\alpha = 0.5$, $\alpha = 0.75$, $\alpha = 0.875$



Conclusion

- We have shown that shape optimization of the expectation of a shape quadratic functional and a state with random right-hand sides is a deterministic problem.
- The numerical solution has been addressed without assuming any specific model for the randomness.
- Numerical results for the optimization of the compliance under random loadings have been presented.
- Our ideas can be extended to shape functionals containing polynomials of the state.
  $\rightsquigarrow$ involves higher-order moments
- Our ideas can be extended also to the variance of the shape functional and, hence, also to a combination of variance end expectation of the shape functional.
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References.

M. Dambrine, C. Dapogny, and H. Harbrecht.
Shape optimization for quadratic functionals and states with random right-hand sides.
SIAM J. Control Optim., 53(5):3081–3103, 2015.

M. Dambrine, H. Harbrecht, and B. Puig.
Incorporating knowledge on the measurement noise in electrical impedance tomography.
ESAIM Control Optim. Calc. Var., to appear.

Conclusion

- We have shown that shape optimization of the expectation of a shape quadratic functional and a state with random right-hand sides is a deterministic problem.
- The numerical solution has been addressed without assuming any specific model for the randomness.
- Numerical results for the optimization of the compliance under random loadings have been presented.
- Our ideas can be extended to shape functionals containing polynomials of the state.
  $\rightsquigarrow$ involves higher-order moments
- Our ideas can be extended also to the variance of the shape functional and, hence, also to a combination of variance end expectation of the shape functional.
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Thank you four your attention!