GAMM-workshop in UQ, TU Dortmund

Characterization of fluctuations in stochastic homogenization

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Random medium ...

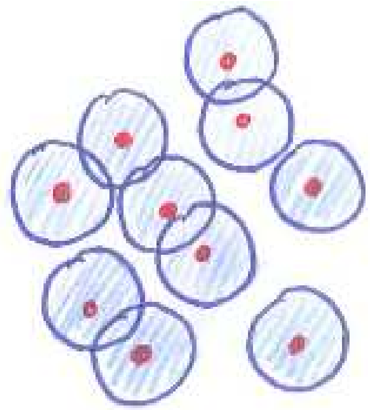
symmetric coefficient field $a = a(x)$ on d-dimensional space

$\lambda|\xi|^2 \le \xi \cdot a(x)\xi \le |\xi|^2$ for all points x and vectors ξ

$\rightsquigarrow$ uniformly elliptic operator $-\nabla \cdot a\nabla u$

Ensemble $\langle\cdot\rangle$ of such coefficient fields a

Example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a =$ id on union, $a = \lambda$id on complement,



Stationarity: a and $a(y+\cdot)$ have same distribution under $\langle\cdot\rangle$

... = elliptic operator with random stationary coefficient field

Plan for talk

1) Error in Representative Volume Element (RVE) Method:
Scaling of random and systematic contribution
in terms of RVE-size

2) Fluctuations of macroscopic observables:
leading-order pathwise characterization
RVE method for extraction

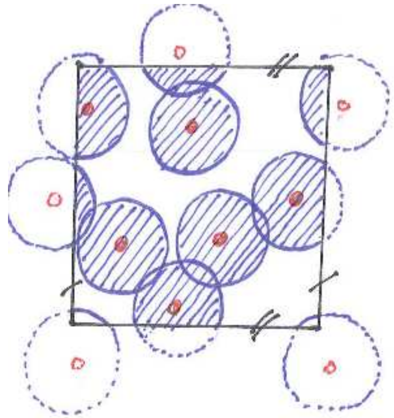
Representative Volume Element method

to extract effective tensor $\bar{a}$:

Scaling of random and systematic error in RVE size

Gloria, Neukamm

Goal: Extract effective behavior $\bar{a}$ from $\langle\cdot\rangle$ ...



Recall example of ensemble $\langle\cdot\rangle$:
points Poisson distributed with density 1,
union of balls of radius $\frac{1}{4}$ around points,
$a = \mathrm{id}$ on union, $a = \lambda\,\mathrm{id}$ on complement,

ensemble $\langle\cdot\rangle$ $\rightsquigarrow$ effective conductivity $\bar{a}$

$$\left\{ \begin{array}{c} \text{density of points } 1 \\ \text{radius of inclusions } \frac{1}{4} \\ \text{conductivity in pores } \lambda \end{array} \right\} \rightsquigarrow \bar{a} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix} = \bar{\lambda}\,\mathrm{id}$$

3 numbers $\rightsquigarrow$ 1 number

... via Representative Volume Element (RVE)

Representative Volume Element method

Introduce artificial period L

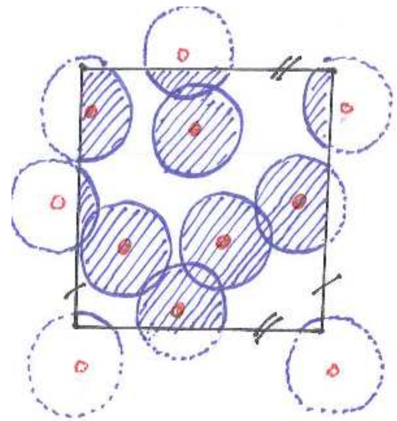
Periodized ensemble $\langle\cdot\rangle_L$

points Poisson distributed with density 1,

on d-dimensional torus $[0,L)^d$

union of balls of radius $\frac{1}{4}$ around points,

$a = \mathrm{id}$ on union, $a = \lambda\,\mathrm{id}$ on complement,



Given coordinate direction $i = 1, \cdots, d$ seek L-periodic φ_i with

$$-\nabla\cdot a(e_i + \nabla\varphi_i) = 0 \quad \text{on } [0,L)^d.$$

Spatial average $\fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$ of flux $a(e_i + \nabla\varphi_i)$

as approximation to $\bar{a}e_i$ for $L \gg 1$;

φ_i is approximate corrector, e_i unit vector in i-th coordinate direction

Solving d elliptic equations $-\nabla \cdot a(e_i + \nabla\varphi_i) = 0$ ...

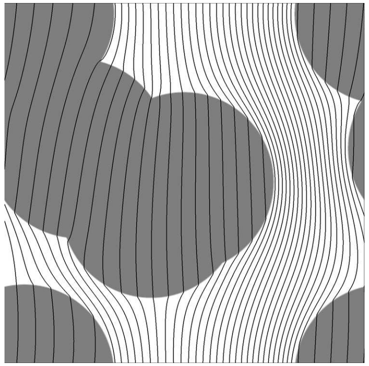
direction e_1

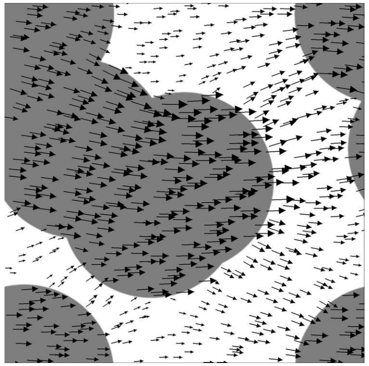
potential

$x_1 + \varphi_1$

flux

$a(e_1 + \nabla\varphi_1)$





average flux

$\fint a(e_1 + \nabla\varphi_1)$

$= \begin{pmatrix} 0.49641 \\ -0.02137 \end{pmatrix}$

$\approx \bar{a}e_1$

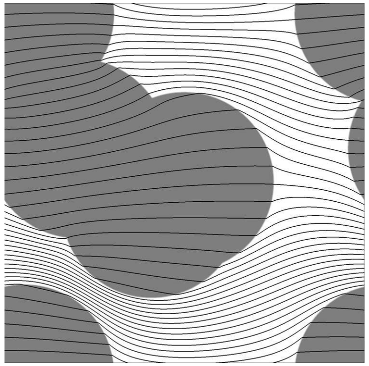
direction e_2

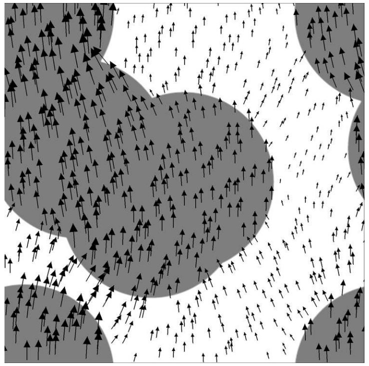
potential

$x_2 + \varphi_2$

flux

$a(e_2 + \nabla\varphi_1)$





average flux

$\fint a(e_2 + \nabla\varphi_2)$

$= \begin{pmatrix} -0.02137 \\ 0.53240 \end{pmatrix}$

$\approx \bar{a}e_2$

simulations by R. Kriemann (MPI)

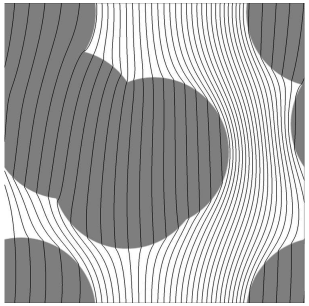
... gives approximation $\bar{a}_L$

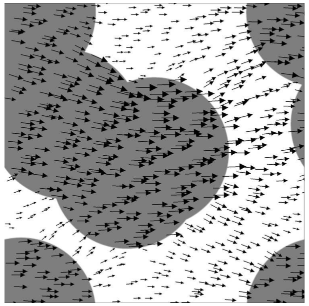
Random error: approx. $\bar{a}_L$ depends on realization

realization 1

potential,

current



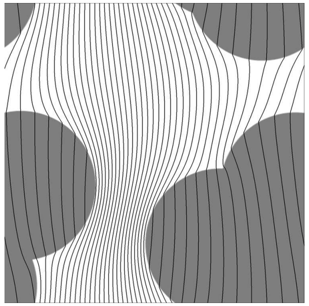


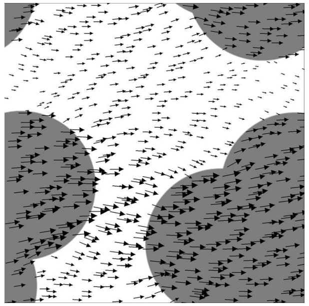
$$\bar{a}_L = \begin{pmatrix} 0.49641 & -0.02137 \\ -0.02137 & 0.53240 \end{pmatrix}$$

realization 2

potential,

current



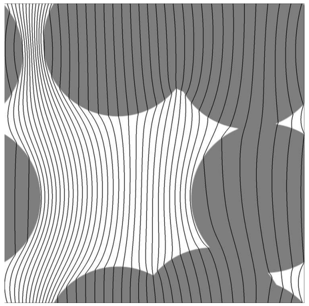


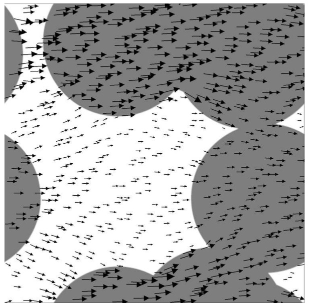
$$\bar{a}_L = \begin{pmatrix} 0.45101 & 0.01104 \\ 0.01104 & 0.45682 \end{pmatrix}$$

realization 3

potential,

current

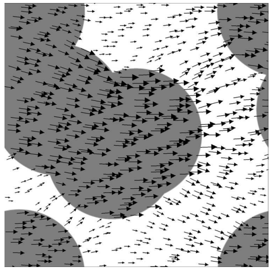




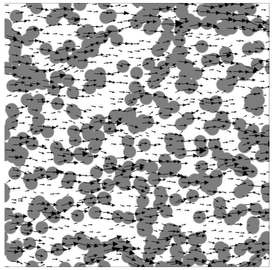
$$\bar{a}_L = \begin{pmatrix} 0.56213 & 0.00857 \\ 0.00857 & 0.60043 \end{pmatrix}$$

... and thus fluctuates is random

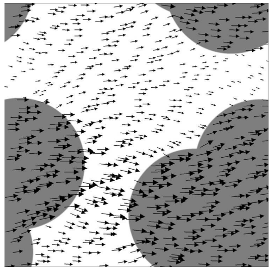
Fluctuations of $\bar{a}_L$ decrease with increasing L



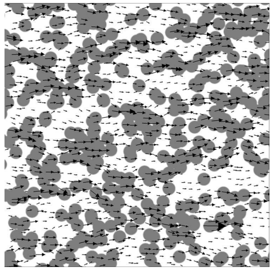
$$\bar{a}_L = \begin{pmatrix} 0.50 & -0.02 \\ -0.02 & 0.53 \end{pmatrix}$$



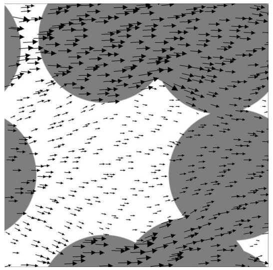
$$\bar{a}_L = \begin{pmatrix} 0.518 & 0.004 \\ 0.004 & 0.511 \end{pmatrix}$$



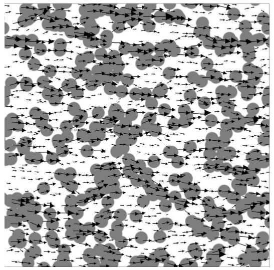
$$\bar{a}_L = \begin{pmatrix} 0.45 & 0.01 \\ 0.01 & 0.46 \end{pmatrix}$$



$$\bar{a}_L = \begin{pmatrix} 0.532 & 0.005 \\ 0.005 & 0.523 \end{pmatrix}$$



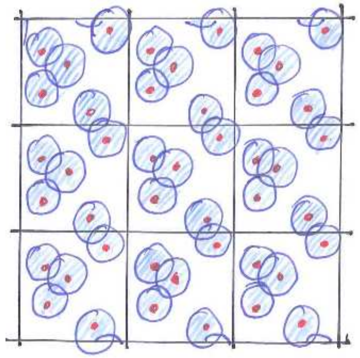
$$\bar{a}_L = \begin{pmatrix} 0.56 & 0.01 \\ 0.01 & 0.60 \end{pmatrix}$$



$$\bar{a}_L = \begin{pmatrix} 0.515 & -0.001 \\ -0.001 & 0.521 \end{pmatrix}$$

... scaling of variance $\mathrm{var}(\bar{a}_L)$ **in** L**?**

Systematic error, decreases with increasing L

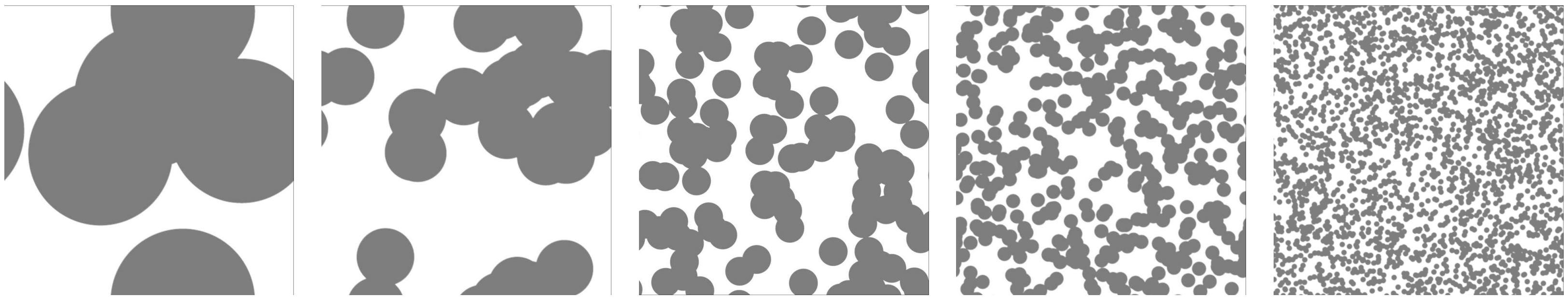


Also expectation $\langle \bar{a}_L \rangle_L$ depends on L
since from $\langle \cdot \rangle$ to $\langle \cdot \rangle_L$ statistics are altered
by artificial long-range correlations

$$\langle \bar{a}_L \rangle_L = \bar{\lambda}_L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because of symmetry of $\langle \cdot \rangle$ under rotation

$L = 2$	$L = 5$	$L = 10$	$L = 20$	$L = 50$



0.551	0.524	0.520	0.522	0.522

Scaling of both errors in L ...

Pick a according to $\langle\cdot\rangle_L$, **solve** for φ (period L),

compute spatial average $\bar{a}_L e_i := \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$

Take *random variable* $\bar{a}_L$ **as approximation to** $\bar{a}$

$\langle$**error**$^2\rangle_L$ = **random**^{2} + **systematic**^{2}:

$$\langle|\bar{a}_L - \bar{a}|^2\rangle_L = \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] + |\langle\bar{a}_L\rangle_L - \bar{a}|^2$$

Qualitative theory yields:

$$\lim_{L\uparrow\infty} \mathrm{var}_{\langle\cdot\rangle_L}[\bar{a}_L] = 0, \qquad \lim_{L\uparrow\infty} \langle\bar{a}_L\rangle_L = \bar{a}$$

... why rate is of interest?

Number of samples N vs. artificial period L

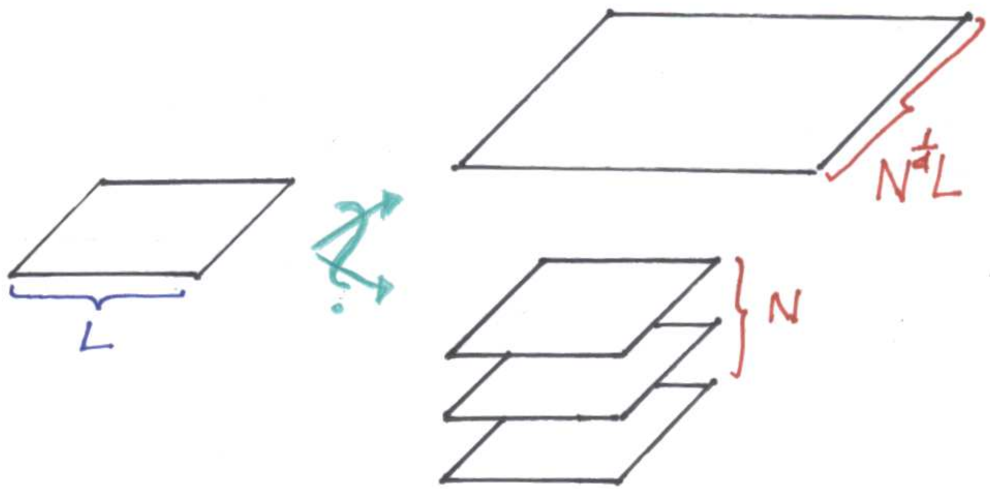
Take **N** samples, i. e. independent picks $a^{(1)}, \cdots, a^{(N)}$ from $\langle \cdot \rangle_L$.

Compute empirical mean $\frac{1}{N}\sum_{n=1}^{N} \fint_{[0,L)^d} a^{(n)}(e_i + \nabla\varphi_i^{(n)})$

$$\langle \textbf{total error}^2 \rangle_L = \frac{1}{\mathbf{N}} \textbf{random error}^2 + \textbf{systematic error}^2$$

L ↑ reduces systematic error and random error

N ↑ reduces only effect of **random error**



An optimal result

Let $\langle\cdot\rangle_L$ be ensemble of a's with period L,
with $\langle\cdot\rangle_L$ suitably coupled to $\langle\cdot\rangle$

For a with period L
solve $\nabla\cdot a(e_i+\nabla\varphi_i)=0$ for φ_i of period L.
Set $\bar{a}_L e_i = ⨍_{[0,L)^d} a(e_i+\nabla\varphi_i)$.

Theorem [Gloria&O.'13, G.&Neukamm&O. Inventiones'15]

Random error$^2 = \mathrm{var}_{\langle\cdot\rangle_L}\big[\bar{a}_L\big] \;\le\; C(d,\lambda)L^{-d}$

Systematic error$^2 = \big|\langle\bar{a}_L\rangle_L - \bar{a}\big|^2 \;\le\; C(d,\lambda)L^{-2d}\ln^d L$

Gloria&Nolen '14: (random) error approximately Gaussian

Fischer '17: variance reduction

State of art in quantitative stochastic homogenization ...

Yurinskii '86 : suboptimal rates in L for mixing $\langle\cdot\rangle$

Naddaf & Spencer '98, & Conlon '00:
optimal rates for small contrast $1-\lambda \ll 1$,
for $\langle\cdot\rangle$ with spectral gap

Gloria & O. '11, & Neukamm '13, & Marahrens '13:
optimal rates for all $\lambda > 0$ for $\langle\cdot\rangle$ with spectral gap,
Logarithmic Sobolev (concentration of measure)

Armstrong & Smart '14, & Mourrat '14, & Kuusi '15,
Gloria & O. '15
optimal stochastic integrability for finite range $\langle\cdot\rangle$

Gaussianity of error:
Biskup & Salvi & Wolf '14, Rossignol '14, Nolen '14

... of linear equations in divergence form

Homogenization error on macroscopic observables

Characterization of leading-order variances via a pathwise characterization of leading-order fluctuations

Duerinckx, Gloria

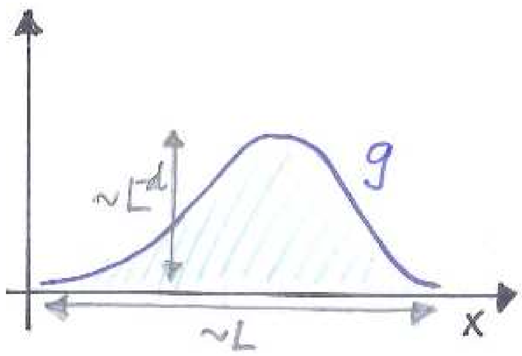
Macroscopic r. h. s. and observable ...

solution u of $\nabla \cdot a\nabla u = \nabla \cdot f$,

where r. h. s. $f(x) = \hat{f}(\frac{x}{L})$ deterministic

macroscopic observable $\int g \cdot \nabla u$,

where $g(x) = L^{-d}\hat{g}(\frac{x}{L})$ deterministic



Marahrens & O.'13: $\mathrm{var}(\int g \cdot \nabla u) = O(\frac{1}{L^d})$

Goal: Characterize limiting variance $\lim_{L\uparrow\infty} L^d \mathrm{var}(\int g \cdot \nabla u)$

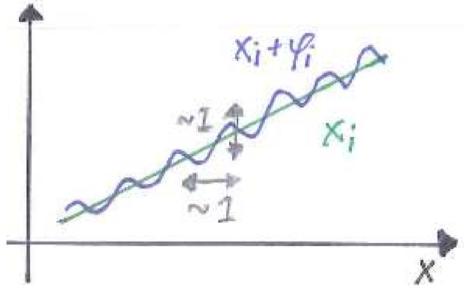
Naive approach via two-scale expansion

Goal: Characterize limiting variance $\lim_{L\uparrow\infty} L^d \mathrm{var}(\int g\cdot\nabla u)$

Corrector φ_i corrects affine x_i

such that $-\nabla\cdot a(e_i+\nabla\varphi_i)=0$

for coordinate direction $i=1,\cdots,d$

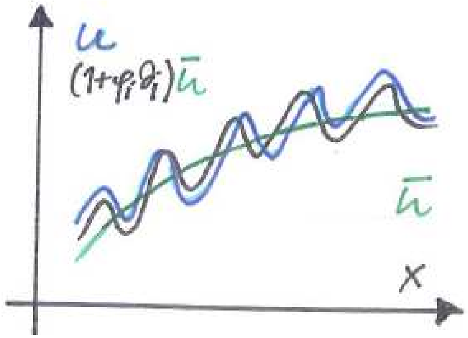


Solution $\bar u$ of homogenized equation

$\nabla\cdot(\bar a\nabla\bar u+f)=0$

Compare u to "two-scale expansion"

$(1+\varphi_i\partial_i)\bar u$ Einstein's summation rule



Naively expect $\mathrm{var}(\int g\cdot\nabla u)=\mathrm{var}(\int\nabla\cdot g\,u)\approx\mathrm{var}(\int\nabla\cdot g\,(1+\varphi_i\partial_i)\bar u)$

Hence study asymptotic covariance $\langle\varphi_i(x-y)\varphi_j(0)\rangle$

The subtle role of the two-scale expansion

Mourrat&O.'14: $\lim_{L\uparrow\infty} L^{d-2}\langle\varphi_i(L(\hat x-\hat y))\varphi_j(0)\rangle$ exists,
but $\neq$ a Green function $\bar G(\hat x-\hat y)$ (Gaussian free field)
Helffer-Sjöstrand, annealed Green's function bounds $\rightsquigarrow$ 4-tensor $\bar Q$

Gu&Mourrat'15: $\lim_{L\uparrow\infty} L^d\mathrm{var}(\int g\cdot\nabla u)$ exists,
but $\neq$ $\lim_{L\uparrow\infty} L^d\mathrm{var}(\int \nabla\cdot g\,(1+\varphi_i\partial_i)\bar u)$
Helffer-Sjöstrand $\rightsquigarrow$ same 4-tensor $\bar Q$, Gaussianity, heuristics
i. e. two-scale expansion cannot be applied naively

Duerinckx&Gloria&O.'16: Two-scale expansion
$\nabla u\approx\partial_i\bar u(e_i+\nabla\varphi_i)$ ok on level of "commutator"

$$\underbrace{a\nabla u}_{\text{flux}}-\bar a\underbrace{\nabla u}_{\text{field}}\approx\partial_i\bar u\Big(\underbrace{a(e_i+\nabla\varphi_i)-\bar a(e_i+\nabla\varphi_i)}_{=:\,\Xi_i}\Big)$$

Leading-order fluctuation of macro observables ...

$\Xi e_i = a(e_i+\nabla\varphi_i) - \bar{a}(e_i+\nabla\varphi_i)$ stationary tensor field

I) $a\nabla u$-$\bar{a}\nabla u \approx \Xi\nabla\bar{u}$ holds in quantitative sense of
$L^d\mathrm{var}\big(\int g\cdot(a\nabla u\text{-}\bar{a}\nabla u - \Xi\nabla\bar{u})\big) = O(L^{-2})$, which implies
$L^d\mathrm{var}\big(\int g\cdot\nabla u - \int\nabla\bar{v}\cdot\Xi\nabla\bar{u}\big) = O(L^{-2})$,
where $\bar{v}$ solves dual equation $\nabla\cdot(\bar{a}^*\nabla\bar{v}+g) = 0$

II) Ξ is local, ie $\Xi(a,x)$ depends little on $a(y)$ for $|y-x|\gg 1$,
thus $\Xi \approx$ tensorial white noise on large scales
more precisely, $L^d\big|\mathrm{var}(\int g\cdot\Xi f) - \int f\otimes g:\bar{Q}f\otimes g\big| = O(L^{-2})$
for four-tensor $\bar{Q}$ from Mourrat&O.

I)&II) $L^d\big|\mathrm{var}(\int g\cdot\nabla u) - \int\nabla\bar{v}\otimes\nabla\bar{u}:\bar{Q}\nabla\bar{v}\otimes\nabla\bar{u}\big| = O(L^{-2})$

... characterized via homogenization commutator

How to extract $\bar{Q}$ from $\langle\cdot\rangle$?

Homogenization commutator $\Xi e_i = a(e_i+\nabla\varphi_i)-\bar{a}(e_i+\nabla\varphi_i)$

$$L^d\mathrm{var}\Big(\int g\cdot\nabla u-\int\nabla\bar{v}\cdot\Xi\nabla\bar{u}\Big)=O(L^{-2}),\qquad \nabla\cdot(\bar{a}^*\nabla\bar{v}+g)=0$$

$$L^d\Big|\mathrm{var}\big(\int g\cdot\Xi f\big)-\int f\otimes g:\bar{Q}f\otimes g\Big|=O(L^{-2})$$

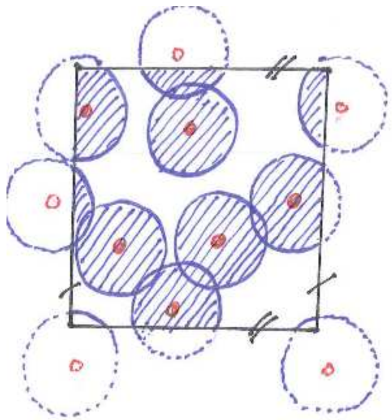
Duerinckx&Gloria&O.'17:

$$|L^d\mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L)-\bar{Q}|\le C(d,\lambda)L^{-d}\ln^d L\ ,$$

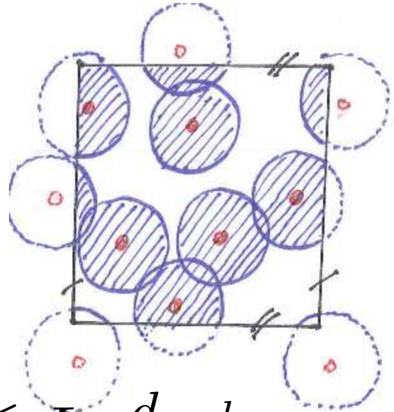
recall: $\langle\cdot\rangle_L$ ensemble of a's with period L,

solve $\nabla\cdot a(e_i+\nabla\varphi_i)=0$ for φ_i of period L,

Set $\bar{a}_L e_i=\fint_{[0,L)^d}a(e_i+\nabla\varphi_i)$.



In practise: Extract $\bar{Q}$ from RVE ...



Recall periodized ensemble $\langle\cdot\rangle_L$

$\bar{a}_L e_i = \fint_{[0,L)^d} a(e_i + \nabla\varphi_i)$

Previous result: $|\langle \bar{a}_L\rangle_L - \bar{a}|^2 \lesssim L^{-2d}\ln^d L$

Duerinckx&Gloria&O.'17: $|L^d \mathrm{var}_{\langle\cdot\rangle_L}(\bar{a}_L) - \bar{Q}|^2 \lesssim L^{-d}\ln^d L$

Hence get $\bar{a}$ and $\bar{Q}$ by same procedure:

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$

$$\Big\langle\Big|\frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)} - \bar{a}\Big|^2\Big\rangle_L \lesssim L^{-2d}\ln^d L,$$

$$\Big\langle\Big|\frac{L^d}{N-1}\sum_{m=1}^{N}\Big(\bar{a}_L^{(m)} - \frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)}\Big)^{\otimes 2} - \bar{Q}\Big|^2\Big\rangle_L \lesssim L^{-d}\ln^d L$$

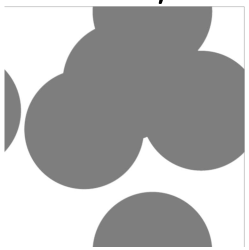
... at no further cost than $\bar{a}$

Back to numerical example

$N \sim L^{\frac{d}{2}}$ independent samples $\{a^{(n)}\}_{n=1,\cdots,N}$ from $\langle\cdot\rangle_L$,

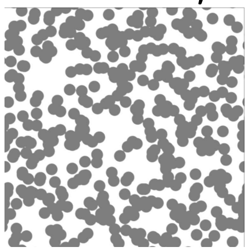
$$\Big\langle\Big|\frac{L^d}{N-1}\sum_{m=1}^{N}(\bar{a}_L^{(m)}-\frac{1}{N}\sum_{n=1}^{N}\bar{a}_L^{(n)})^{\otimes 2}-\bar{Q}\Big|^2\Big\rangle_L \lesssim L^{-d}\ln^d L$$

L=2, N=500



$$\bar{Q} = 10^{-2}\times\begin{pmatrix} 1.01 & 0.00 & 0.00 & 0.09 \\ 0.00 & 0.31 & 0.09 & 0.00 \\ 0.00 & 0.09 & 0.31 & 0.00 \\ 0.09 & 0.00 & 0.00 & 0.99 \end{pmatrix}$$

L=20, N=500



$$\bar{Q} = 10^{-2}\times\begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.23 \\ 0.00 & 0.56 & 0.23 & 0.00 \\ 0.00 & 0.23 & 0.56 & 0.00 \\ 0.23 & 0.00 & 0.00 & 1.01 \end{pmatrix}$$

Higher order comes naturally, i. e. 2nd order

2nd-order two-scale expansion: $u \approx (1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar{u}'$,
where $\bar{u}' := \bar{u} + \tilde{u}'$ with $\nabla\cdot(a\nabla\tilde{u}' + \bar{a}'_i\nabla\partial_i\bar{u}) = 0$
and tensor $\bar{a}'_i$ is 2nd-order homogenized coefficient.

k-the component of 2nd-order commutator:
$\Xi'_k[u] := e_k\cdot(a-\bar{a})\nabla u + \bar{a}^{*\prime}_k e_l \cdot \nabla\partial_l u$,
characterized by $\Xi_k[u] = \nabla^2 :$ something for a-harmonic u

Inject: $\Xi^{0\prime}[\bar{u}](x) := \Xi'\Big[(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})T'_x\bar{u}'\Big](x)$,
where $T'_x\bar{u}'$ is 2nd-order *Taylor polynomial* of $\bar{u}'$ at x

A relative error of $O(L^{-\frac{d}{2}})$

Recipe: Inject two-scale expansion $(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})\bar{u}'$
into commutator $\Xi'_k[u] := e_k\cdot(a-\bar{a})\nabla u+\bar{a}^{*\prime}_k e_l\cdot\nabla\partial_l u$,
in sense of $\Xi^{0\prime}[\bar{u}](x) := \Xi'\big[(1+\phi_i\partial_i+\phi'_{ij}\partial_{ij})T'_x\bar{u}'\big](x)$

Duerinckx&O.'18 ($d=3$):

$$L^d\mathrm{var}\Big(\int g\cdot\nabla u-\int\nabla\bar{v}'\cdot\Xi'[u]\Big)$$
$$+L^d\mathrm{var}\Big(\int g\cdot\Xi'[u]-\int g\cdot\Xi^{0\prime}[\bar{u}]\Big)\le C(d,\lambda)L^{-d}$$

where $\bar{v}'=\bar{v}+\tilde{v}'$ with $\nabla\cdot(\bar{a}^*\nabla\tilde{v}'+a^{*\prime}_k\nabla\partial_k\bar{v})=0$

Relies on stochastic estimates of ϕ'_{ij} (Gu, Bella&Fehrman&Fischer&O)

Helpful tool: Flux correctors ...

1st-order: $a e_i = \bar{a} e_i - a\nabla\phi_i + \nabla\sigma_i$, $\quad \sigma_i$ skew,

2nd-order: $(\phi_i a - \sigma_i) e_j = \bar{a}_i' e_j - a\nabla\phi_{ij}' + \nabla\cdot\sigma_{ij}'$

2nd-order two-scale expansion; for $\bar{a}$-harmonic $\bar{u}$:

$$\nabla\cdot a\nabla(1+\phi_i\partial_i+\phi_{ij}'\partial_{ij})\bar{u}' = \nabla\cdot\big((\phi_{ij}'a-\sigma_{ij}')\nabla\partial_{ij}\bar{u}' + \bar{a}_i'\nabla\partial_i\tilde{u}'\big),$$

2nd-order commutator; for a^*-harmonic u:

$$e_k\cdot(a-\bar{a})\nabla u + a_k^{*\prime} e_l\cdot\nabla\partial_l u = \partial_l\nabla\cdot\big((\phi_{kl}^{*}{}'a+\sigma_{kl}^{*}{}')\nabla u\big)$$

stochastic estimates: σ_i, σ_{ij}' like ϕ_i, ϕ_{ij} (Gloria&Neukamm&O.'13)

... bring residue in divergence-form

Helpful tool: Malliavin calculus and spectral gap

Spectral gap: $\mathrm{var}(F) \leq \langle \int |\frac{\partial F}{\partial a}|^2 dx \rangle$

random variable F = functional of a,

Malliavin-derivative = functional derivative

For 1st-order $F := \int g \cdot (a - \bar{a})(\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i))$

Crucial formula: δa infinitesimal perturbation of a

$$\delta\Big(e_j \cdot (a - \bar{a})\big(\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i)\big)\Big)$$

$$= \underbrace{(e_j + \nabla \phi_j^*)}_{O(1)} \cdot \delta a \Big(\underbrace{\nabla u - \partial_i \bar{u}(e_i + \nabla \phi_i)}_{O(L^{-1} \nabla \bar{u})}\Big)$$

$$- \underbrace{\nabla}_{\rightsquigarrow O(L^{-1})} \cdot \Big(\underbrace{(\phi_j^* a + \sigma_j^*)}_{O(1)} \underbrace{\nabla \delta u}_{O(\nabla \bar{u}\, \delta a)}\Big) + \partial_i \bar{u}\, \nabla \cdot \Big((\phi_j^* a + \sigma_j^*) \underbrace{\nabla \delta \phi_i}_{O(\delta a)}\Big)$$

$$- \nabla \cdot (\phi_j^* \delta a \nabla u) + \partial_i \bar{u}\, \nabla \cdot (\phi_j^* \delta a (e_i + \nabla \phi_i))$$

Credits

Gaussianity of various errors: Nolen'14 based on Stein/Chatterjee, Biskup&Salvi&Wolf'14, Rossignol'14, ...

Quartic tensor Q via Helffer-Sjöstrand and Mahrarens& O.'13: Mourrat&O'14, Gu&Mourrat'15

Heuristics of a path-wise approach w/o Ξ: Gu&Mourrat'15, based on variational approach by Armstrong&Smart '13

$\nabla\varphi = \bar{a}$-Helmholtz-projection of white noise: Armstrong&Mourrat&Kuusi'16, Gloria&O.'16 based on finite range rather than Spectral Gap

Summary

1) Error in Representative Volume Element (RVE) Method,
Scaling of random var$(\bar{a}_L)$
and systematic contribution $\langle \bar{a}_L \rangle - \bar{a}$
in terms of RVE-size L

2) Fluctuations of macroscopic observables $\int g \cdot \nabla u$,
leading-order pathwise characterization
via two-scale expansion used on level of commutator Ξ
in terms of fourth-order tensor $\bar{Q}$
extract from RVE at no additional cost
Natural higher-order versions