Quantum dynamics for random models: why, what and how

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1. The Anderson model ▶ WHY?

2. Quantum dynamics ▶ WHAT?

3. Random models ▶ WHAT?

4. Methods ▶ HOW?

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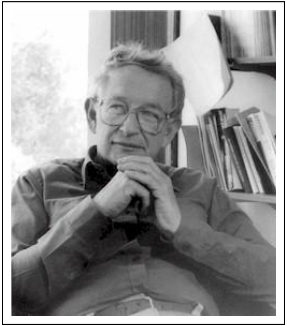


In the celebrated article [4] "Absence of Diffusion in Certain Random Lattices". *Phys. Rev.* **109** (5): 1492 –1505 from 1958, P.W. Anderson proposed a mathematical model to explain the phase transition from insulator to metal in disordered solids.

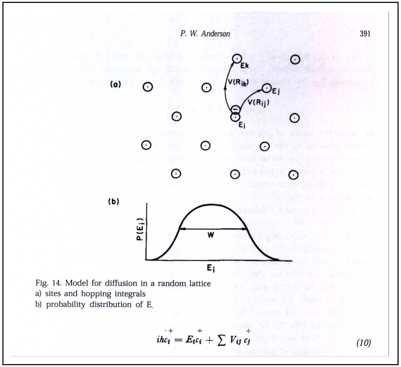


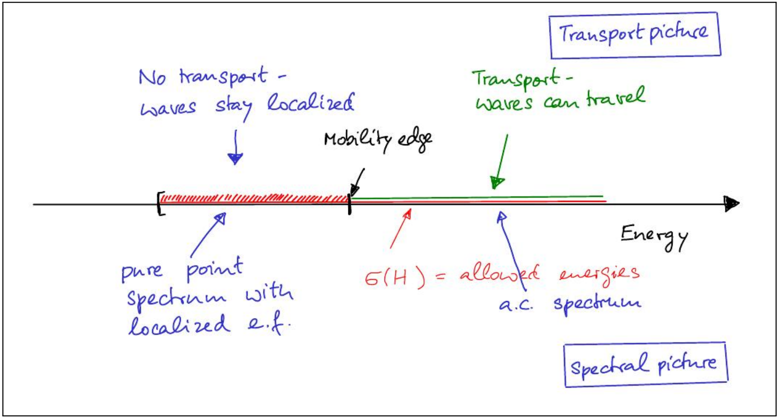
Fig. 14. Model for diffusion in a random lattice
a) sites and hopping integrals
b) probability distribution of E_i

$$i\hbar \dot{c}_i^{+} = E_i c_i^{+} + \sum V_{ij} c_j^{+} \qquad (10)$$

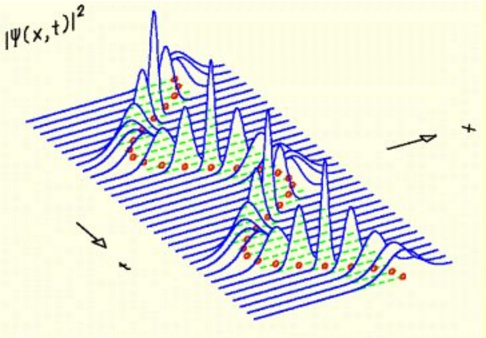
from Anderson's Nobel lecture, [3]: The original Anderson model

- Random model for solids: Hopping term plus random onsite potential. In our terminology:
  $$H_\omega \psi(x) = -\Delta_{disc}\psi(x) + V_\omega(x)\psi(x),$$
  defines a random operator. The ingredients:
  x - site on a "lattice"
  Δ_{disc} - Hopping terms
  V_ω - random potential
- Need probability distributions not averages: no real atom is an average atom

Anderson proved: nontransport theorem, **localization**, and suggested transition, now called Anderson transition.



$$\dot{\psi}(t) = -i(-\Delta + V)\psi(t), \psi(0) = \psi_0,$$



Easy: the solution of the Schrödinger equation is

$$\psi(t) = e^{-iHt}\psi_0.$$

But: what does it look like?
To this end one studies the spectral resolution of H.

- If ψ_0 is an eigenvector of H with eigenvalue $E_0 \Rightarrow$

$$\psi(t) = e^{-iE_0 t}\psi_0.$$

  One speaks of a **bound state**.
- If the spectral measure $\rho^H_{\psi_0}$ of ψ_0 w.r.t H is continuous,

$$\Rightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T}^{T} \|\chi_B \psi(t)\|^2 dt = 0 \text{ for compact } B.$$

  One speaks of a **scattering state**.

- Atomic models are described by a negative potential V that decays rapidely enough near infinity:one gets bound states for negative energies and scattering states for positive energies.
- Solid states with perfect order are described by a periodic $V \Rightarrow H = -\Delta + V$ has only scattering states.
- P.W. Anderson proposed a new paradigm for disordered solids in dimension ≥ 3:

Once translated into the language of spectral theory there is a transition from a **localized phase** that exhibits pure point spectrum
(= only bound states = no transport)
to a
delocalized phase with absolutely continuous spectrum
(= scattering states = transport)
The dynamical properties of solutions to the Schrödinger equation

$$\dot{\psi}(t) = -i(-\Delta + V)\psi(t), \psi(0) = \psi_0,$$

depend drastically on the energy of ψ_0!

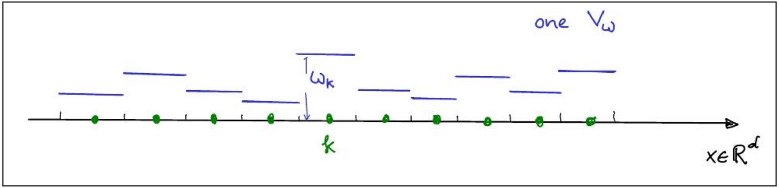
However, for genuine random models, there is no rigorous proof for the existence of a transition or even of the appearance of spectral components other than pure point, so far.
This is quite a strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed one can prove the opposite spectral behavior only.

$$H(\omega) = -\Delta + V_\omega \text{ in } L^2(\mathbb{R}^d)$$

E.g., $\Omega = \mathbb{R}^{\mathbb{Z}^d}, \mathbb{P} = \mu^{\mathbb{Z}^d}$ a probability space describing independent, identically distributed coupling constants

$$V_\omega = \sum_{k \in \mathbb{Z}^d} \omega_k \cdot v(\cdot - k)$$

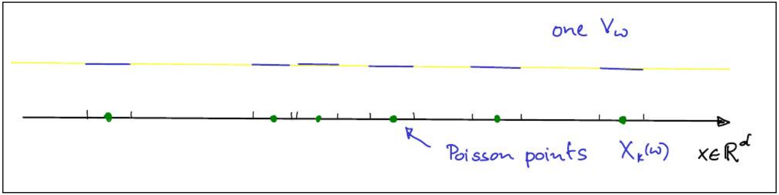
sometimes also called **alloy type models**. Localization has been proven for such models under additional technical hypotheses on μ and v.



$$H(\omega) = -\Delta + V_\omega \text{ in } L^2(\mathbb{R}^d)$$

with impurities located according to a Poisson process

$$V_\omega = \sum_{k \in \mathbb{N}} v(\cdot - X_k(\omega))$$



Localization was shown for such models in [8]

Here the operators are given by

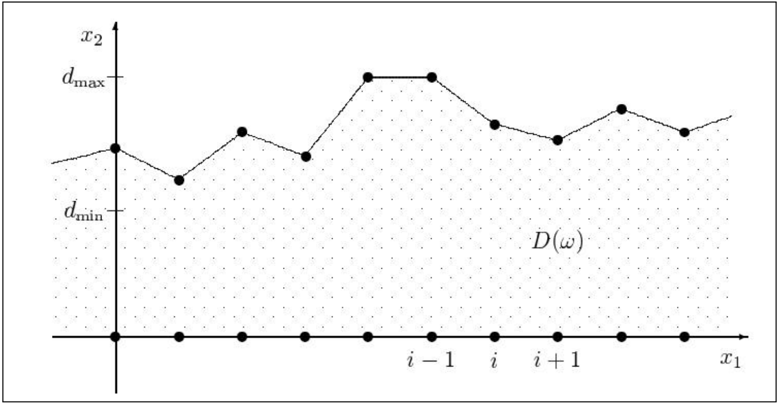
$$H(\omega) = -\nabla \mathbf{a}_\omega \nabla \text{ in } L^2(\mathbb{R}^d)$$

where the leading order coefficient is a randomly chosen matrix, independently in different space regions, more precisely

$$\mathbf{a}_\omega = \mathbf{a}_{per} + \sum_{k \in \mathbb{Z}^d} \omega_k \cdot \chi(\cdot - k),$$

and $\omega = (\omega_k)_{k \in \mathbb{Z}^d} \in S^{\mathbb{Z}^d}$ is picked w.r.t. a product measure, see [6] and [12].

A model studied by Kleespies and P.S.



picture taken from [9]

Much more complicated are the models studied by Borisov and Veselic 2013, [5], where the curvature is random.
One main technical difficulty: missing monotonicity in the dependence on the random variables. In that respect these models share technical features with random displacement models, coming next:

$$H(\omega) = -\Delta + V_\omega \text{ in } L^2(\mathbb{R}^d)$$

with impurities randomly displaced from their lattice position

$$V_\omega = \sum_{k \in \mathbb{Z}^d} v(\cdot - k - \omega_k)$$

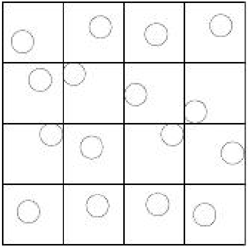
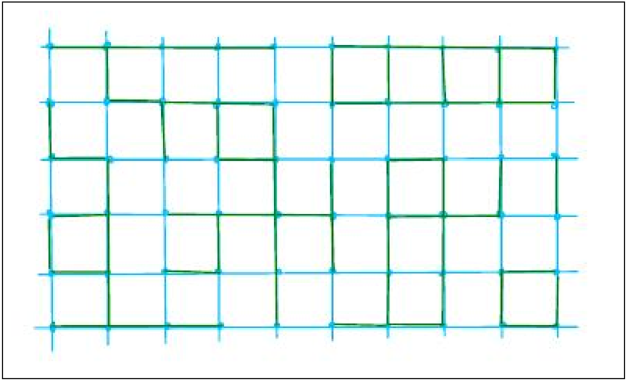


FIGURE 1. A typical configuration

... taken from [10].



Starting from a given graph (e.g., $\mathbb{Z}^2$) edges are deleted at random and independently; the random operator is the Laplacian on the resulting subgraph, see e.g. [11].

- New methods are required, traditional perturbation theory fails!
- Statistics of eigenvalues;
- decay properties of eigenfunctions;
- ... for finite volume restrictions of random operators

Two approaches:

- **Fractional moment method**: see Aizenman and Molchanov 1993, [1], for the start and the monograph [2] by Aizenman and Warzel for a recent account.
- **Multi–scale analysis**: see Fröhlich and Spencer 1983, [7], for the start and the monograph [13] for an extended proof.

Is an induction process proving “good decay properties” of resolvents on boxes C_L with high probability. With a little bit of oversimplification:

- Box of sidelength L decomposed into boxes of sidelength $\ell << L$.
- Small box is **good** if resolvent decays exponentially with rate $\gamma_\ell > 0$, that happens with probability p_ℓ.
- Decay for the large box is controlled by decay for “most” small boxes, giving a decay rate $\gamma_L < \gamma_\ell > 0$ with probability $p_L > p_\ell$.
- Use independence in the latter step!

Two main ingredients:

- **Nonresonance**, **Wegner estimate:** For all $E \in \mathbb{R}$,

$$\mathbb{P}\left(\operatorname{dist}(E, \sigma(H_\omega \restriction C_\ell)) \leq \epsilon\right) \leq C\ell^d \epsilon,$$

  where $H_\omega \restriction C_\ell$ denotes a selfadjoint restriction of the random operator to the box C_ℓ.
- **Initial length scale estimates** allow to start the induction procedure. For E in a certain energy regime, the probability that a box is good (resolvent at energy E) is high enough. Often proved in terms of **Lifshitz tail estimates**.

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