$$\lambda_k(H) = \inf_{\substack{V \subset F_+ \\ \dim V = k}} \sup_{\substack{x \in V \oplus F_- \\ \|x\|=1}} \langle x, Hx \rangle$$

variational principle

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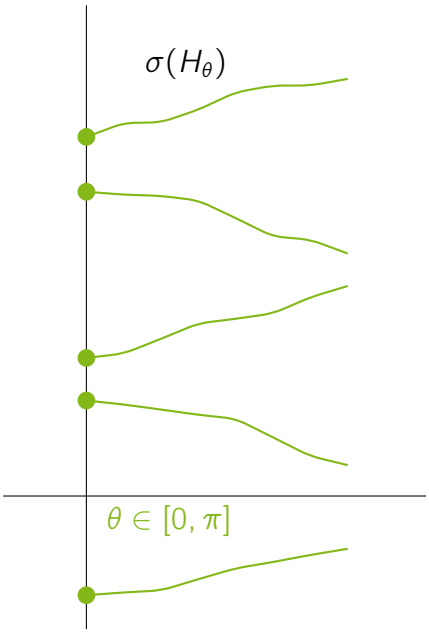
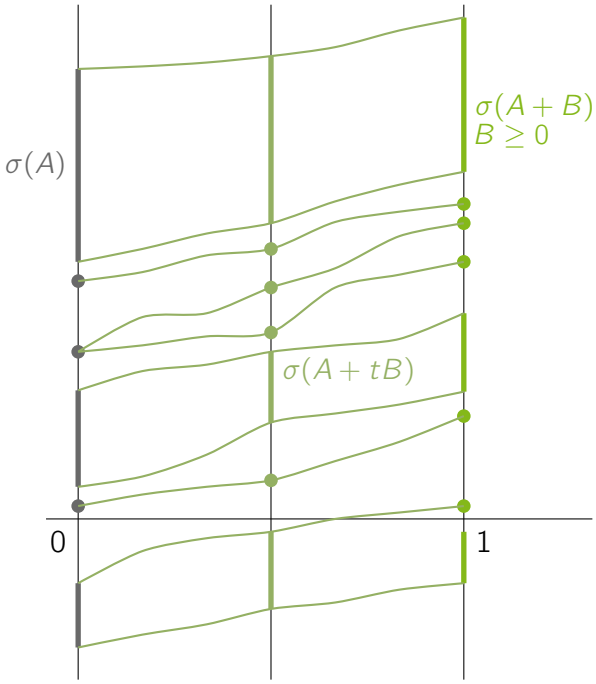
essential spectrum

$$\sigma(H) = \bigcup_n \bigcup_{\theta \in [0,2\pi)^d} E_n(H_\theta)$$

Dirac operator

$$H = -i\alpha \cdot \nabla + \beta + V$$





Organizers

Albrecht Seelmann and **Ivan Veselić**

Support

Technical Support: Alexander Dicke

Zoom Support: Alexander Dicke, Max Kämper

Meeting information

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Zoom

About

Workshop on Perturbation of Spectral Bands and Gaps

TU Dortmund
26^{th} – 30^{th} July 2021

technische universität dortmund

This event will bring together mathematicians interested and working in various types of problems where spectral bands and spectral gaps play a role. This includes

- perturbation theory for eigenvalues, essential spectrum, spectral subspaces of (self-adjoint) operators, in particular of Dirac, Schrödinger, and more general elliptic second order differential operators,
- variational principles for eigenvalues,
- periodic operators, generalised Fourier transform, direct integral decomposition,
- periodic partial differential equations, in particular Schrödinger operators, Floquet-Bloch theory, dispersion relations,
- perturbation of periodic models, in particular random ones,
- homogenisation of elliptic problems.

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List of Abstracts

Global extrema of dispersion relation of tight-binding models

Gregory Berkolaiko, Texas A&M University (TAMU) (Slides, Recording)

Tight-binding approximation is frequently used in physics to analyze wave propagation through a periodic medium. Its Floquet–Bloch transform is a compact graph with a parameter-dependent operator defined on it. The graph of the eigenvalues as functions of parameters is called the dispersion relation. Extrema (minima and maxima) of the dispersion relation give rise to band edges: endpoints of intervals supporting continuous spectrum and therefore allowing wave propagation. Locating the extrema can be difficult in general; there are examples where extrema occur away from the set of parameters with special symmetry.

In this talk we will show that a large family of tight-binding models have a curious property: there is a local condition akin to the second derivative test that detects if a critical point is a global (sic!) extremum. With some additional assumptions (time-reversal invariance and dimension 3 or less), we show that any local extremum of a given sheet of the dispersion relation is in fact the global extremum.

Based on a joint project with Yaiza Canzani, Graham Cox, Jeremy Marzuola.

Improve convergence by creating a gap, is this a good idea?

Lyonell Boulton, Heriot-Watt University (Slides, Recording)

In this talk we argue on the pros and cons of the following strategy for computing the smallest discrete eigenvalue of an elliptic operator. To improve the convergence rate, map the problem so the original eigenvalue becomes an embeded eigenvalue in a spectral gap. This might sound counter-intuitive at first, but we will show a concrete realisation of this idea by considering the Laplacian on a 2D domain with fairly general fractal boundary. The talk is based on work in collaboration with Lehel Banjai.

Low lying spectral gaps induced by slowly varying magnetic fields

Horia Cornean, Aalborg University (AAU) (Recording)

Consider a periodic Schrödinger operator in two dimensions, perturbed by a weak magnetic field whose intensity slowly varies around a positive mean. We show in great generality that the bottom of the spectrum of the corresponding magnetic Schrödinger operator develops spectral minibands separated by gaps, reminding of a Landau-level structure. The talk will summarize results from JFA (2016) http://dx.doi.org/10.1016/j.jfa.2017.04.002 and JST (2019) https://doi.org/10.4171/jst/274 obtained in collaboration with B. Helffer and R. Purice.

Diagonalization of indefinite saddle point forms and applications in fluid dynamics

Luka Grubišić, University of Zagreb

We study the block diagonalization (by a direct rotation) of sign-indefinite symmetric sesquilinear forms as well as of the associated operators. We are interested in examples of the operators which need not be semi-bounded, neither from below nor from above. In the semi-bounded case, we refine the obtained results and, as an example, revisit the block Stokes operator from fluid dynamics. We also discuss the ramifications of these results for enabling a geometric interpretation of the generalized Reynolds number (in fluid dynamics) introduced by Ladyzhenskaya. Time permitting, we will also discuss the use of this theory for the design of efficient numerical schemes for similarly structured problems in the theory of elasticity.

Estimates of spectral bands and spectral gaps

Evgeny Korotyaev, Saint Petersburg State University and HSE University (Slides, Recording)

In the first part of the talk I describe estimates of bands and spectral gaps for Schrödinger operators with periodic potentials on periodic graphs. In particular I describe new two-sided estimates total bandwidth for Schrödinger operators in terms of geometric parameters of graphs and potentials. Note that these estimates are sharp. It means that these estimates become identities for specific graphs and potentials. In the second part I consider the continuous case. Here I describe known results about estimates of bands and spectral gaps for Schrödinger operators with periodic potentials on the real line.

Generic properties of dispersion relations for discrete periodic operators

Peter Kuchment, Texas A&M University (TAMU) (Recording)

An old problem in mathematical physics deals with the structure of the dispersion relation of the Schrödinger operator $-\Delta + V(x)$ in $\mathbb{R}^n$ with periodic potential near the edges of the spectrum, i.e., near extrema of the dispersion relation. A well-known and widely believed conjecture says that generically (with respect to perturbations of the periodic potential), the extrema are attained by a single branch of the dispersion relation, are isolated, and have non-degenerate Hessian (i.e., dispersion relations are graphs of Morse functions). The important notion of effective masses in solid state physics, as well as the Liouville property, Green's function asymptotics, and so on hinge upon this property. It is natural to try to look at discrete problems (the tight binding model), where the dispersion relation is an algebraic, rather than analytic, variety. Alas, counterexamples exist even for simple 2D-periodic two-atomic structures, where the generic non-degeneracy fails.

We establish first the following natural dichotomy: the non-degeneracy of either fails or holds in the complement of a proper algebraic subset of the parameters. Thus, a random choice of a point in the parameter space gives the correct answer "with probability one." This is a simple result in the discrete case, but still unproven in the continuous case.

Noticing that the known counterexample has only two free parameters, one can suspect that this might be too tight for the genericity to hold. We consider the maximal $\mathbb{Z}^2$-periodic two-atomic nearest-cell interaction graph, which has nine edges per unit cell and the discrete "Laplace–Beltrami" operator with nine free parameters. Using methods from computational and combinatorial algebraic geometry we prove the genericity conjecture for this graph. We show three different approaches to the genericity, which might be suitable in various situations. We also prove in this case that adding more parameters indeed cannot destroy the genericity result. This allows us to list all "bad" periodic subgraphs of the one we consider and discover that in all these cases the genericity fails for "trivial" reasons only.

(Joint work with Ngoc T. Do and Frank Sottile)

Variational principles for operator functions

Matthias Langer, University of Strathclyde (Recording)

In this talk I will review variational principles for operator functions that can be used to characterise eigenvalues in gaps of the essential spectrum. Such variational principles can be used, for instance, to obtain information about eigenvalues under perturbations. In particular, I will consider variational principles with a finite index shift and triple variational principles; the latter somehow correspond to an infinite index shift and can be used for arbitrary gaps in the essential spectrum.

Critical magnetic field for 2d magnetic Dirac-Coulomb operators

Michael Loss, Georgia Institute of Technology (Recording)

This talk is about the two-dimensional Dirac-Coulomb operator in presence of an Aharonov-Bohm external magnetic potential. We characterize the highest intensity of the field for which the ground state energy lies in the gap of the continuous spectrum. This critical magnetic field is also the threshold value for which the operator is self-adjoint. The method relies on optimal constant of a Hardy inequality for the two-dimensional magnetic Pauli operator. This is joint work with Jean Dolbeault and Maria Esteban.

Dissipation, repetition or oscillation: calculating eigenvalues of perturbed periodic operators in spectral gaps

Marco Marletta, Cardiff University School of Mathematics (Recording)

In this talk I shall speak about work on numerical calculation of eigenvalues of perturbed periodic operators which lie in spectral gaps (in the selfadjoint case) or in the essential numerical range (in the non-selfadjoint case). The talk will compare dissipative barrier methods, both concrete and abstract; supercell methods; and methods based on Floquet theory. Parts of the work will be joint with collaborators including Salma Aljawi, Sabine Boegli, Rob Scheichl and Christiane Tretter, though I shall also mention some interesting single-author works by my former postdoc Michael Strauss and my current PhD student Alexei Stepanenko.

On the Bethe-Sommerfeld property and integrated density of states of periodic matrix operators

Sergey Morozov, Ludwig Maximilian University of Munich (LMU)

I will present some recent results concerning high energy spectral structure of periodic matrix operators. The main example is the Dirac operator with periodic potential of any nature. The first result is the Bethe-Sommerfeld property, i.e. the absence of spectral gaps for large values of the spectral parameter. The second one is the existence and structure of the asymptotics of the integrated density of states for large energies. The talk is based on joint work with J. Lagace, L. Parnovski, B. Pfirsch, and R. Shterenberg.

Lifting of spectral band edges of Schrödinger operators on unbounded domains

Ivica Nakić, University of Zagreb (Recording)

In the talk I will present lower and upper Lipschitz bounds on the function parametrizing locally a chosen edge of the essential spectrum of a Schrödinger operator in dependence of a coupling constant, together with analogous estimates for eigenvalues, possibly in gaps of the essential spectrum. These results are based on two theorems: a quantitative, scale-free unique continuation estimate for functions in a spectral subspace of a Schrödinger operator on a bounded or unbounded domain and a perturbation and lifting estimate for edges of the essential spectrum of a self-adjoint operator under a semidefinite perturbation. The talk is based on a joint work with Albrecht Seelmann, Matthias Täufer, Martin Tautenhahn and Ivan Veselić.

Bethe-Sommerfeld property of multidimensional Schrödinger operators with periodic and almost-periodic potentials

Leonid Parnovski, University College London (UCL)

We say that an elliptic operator satisfies Bethe-Sommerfeld property if its spectrum has no high-energy spectral gaps (i.e., any sufficiently large value of energy is inside its spectrum). I will give a survey on establishing this property for multidimensional periodic and almost-periodic Schrödinger operators, both scalar and vector-valued.

Gap localization of TE-modes by arbitrarily weak defects

Michael Plum, Karlsruhe Institute of Technology (KIT) (Slides, Recording)

We consider the propagation of TE-modes in photonic crystal waveguides. The waveguide is created by introducing a linear defect into a periodic background medium. Both the periodic background problem and the perturbed problem are modelled by a divergence type equation. A feature of our analysis is that we allow discontinuities in the coefficients of the operator, which is required in many photonic crystal models. It is shown that arbitrarily weak perturbations introduce spectrum into the spectral gaps of the background operator.

(joint work with B.M. Brown, V. Hoang, M. Radosz, and I. Wood)

Eigenvalue distribution of polynomially compact pseudodifferential operators

Grigori Rozenblum, Chalmers University of Technology, The Euler International Mathematical Institute, and N.-T. University Sirius (Slides, Recording)

We consider a polynomially compact self-adjoint zero order pseudodifferential operator **A** acting on vector-functions on a closed smooth manifold. The essential spectrum of **A** consists of several points ω_j and, possibly, eigenvalues forming sequences converging to these points. In particular, these eigenvalues may lie in the gaps of the essential spectrum. A method is proposed to study the asymptotics of these eigenvalues as they approach the points ω_j. The motivating example is the Neumann-Poincare (the double layer potential) operator in $3D$ elasticity; we apply the general approach and obtain information on the behavior of its eigenvalues.

Effective gaps in continuous Floquet Hamiltonians

Amir Sagiv, Columbia University (Recording)

Applying time-periodic forcing is a common technique to effectively change materials properties. A well-known example is the transformation of graphene from a conductor to an insulator in the presence of time-periodic magnetic potential. Can this property be derived in the continuous time-periodic Schrödinger model? We first show that the dynamics of a certain type of wave-packets can be approximated by a homogenized time-dependent Dirac equation. The Monodromy of this Dirac equation is then shown to have a spectral gap property. Our main result is that this property "carries back" to the Schrödinger equation in the form of an "effective gap". This latter notion is a new physically-relevant relaxation of a spectral gap, one which quantitatively characterizes parts of the spectrum and their relation to the spectrum of the unforced model. Based on a joint work with Michael I. Weinstein.

Friedrichs extension and min-max principle for operators with a gap

Lukas Schimmer, Mittag-Leffler Institute of the Royal Swedish Academy of Sciences

In many cases the physically relevant self-adjoint extension of a lower semibounded, symmetric operator is the Friedrichs extension. This extension preserves the lower bound, and its eigenvalues below the essential spectrum can be computed in terms of a variational principle that only depends on the domain of the symmetric operator. This makes its spectrum especially accessible to numerical methods.

In my talk I will present a generalisation of the Friedrichs extension to the setting of a symmetric operator satisfying a gap condition. This extension remains gapped, and its eigenvalues above the gap are again given by a variational principle that involves only the domain of the symmetric operator.

I will discuss how the result can be applied to Dirac operators with Coulomb potential (where we recover the well-known distinguished self-adjoint extension and its variational principle), as well as to Dirac operators on manifolds with boundary (where we recover the Atiyah–Patodi–Singer boundary condition).

This talk is based on joint work with Jan Philip Solovej and Sabiha Tokus.

Dirac-Coulomb operators: looking for the optimal charge distribution

Éric Séré, Paris Dauphine University (Slides, Recording)

This talk is based on joint work with M.J. Esteban and M. Lewin. Consider an electron moving in the attractive Coulomb potential generated by a positive finite measure representing an external charge density. If the total charge is fixed, it is well known that the lowest eigenvalue of the corresponding Schrödinger operator is minimized when the measure is a delta. We investigate the conjecture that the same holds for the relativistic Dirac-Coulomb operator. First we give conditions ensuring that this operator has a natural self-adjoint realisation and that its eigenvalues are given by min-max formulas. Then we define a critical charge such that, if the total charge is fixed below it, then there exists a measure minimising the first eigenvalue of the Dirac-Coulomb operator. We find that this optimal measure concentrates on a compact set of Lebesgue measure zero. The last property is proved using a new unique continuation principle for Dirac operators.

Homogenization of the periodic Schrödinger-type equations

Tatiana Suslina, Saint Petersburg State University (Slides, Recording)

In $L_2(\mathbb{R}^d;\mathbb{C}^n)$, we consider a selfadjoint matrix strongly elliptic second order differential operator A_ε. It is assumed that the coefficients of A_ε are periodic and depend on $\mathbf{x}/\varepsilon$, where $\varepsilon > 0$ is the small parameter. We study the behavior of the operator exponential $e^{-iA_\varepsilon \tau}$ for small ε and $\tau \in \mathbb{R}$. The results are applied to study the behavior of the solution $\mathbf{u}_\varepsilon$ of the Cauchy problem for the Schrödinger-type equation $i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x},\tau) = (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x},\tau)$ with the initial data from a special class. For a fixed $\tau \in \mathbb{R}$, the solution converges in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ to the solution of the homogenized problem, as $\varepsilon \to 0$; the error is $O(\varepsilon)$. We find approximation of the solution $\mathbf{u}_\varepsilon(\cdot,\tau)$ in $L_2(\mathbb{R}^d;\mathbb{C}^n)$ with an error $O(\varepsilon^2)$ and also approximation of $\mathbf{u}_\varepsilon(\cdot,\tau)$ in $H^1(\mathbb{R}^d;\mathbb{C}^n)$ with an error $O(\varepsilon)$. These approximations involve some correctors.

The research was supported by the Russian Science Foundation, grant no. 17-11-01069.

Band edge localization beyond regular Floquet eigenvalues

Matthias Täufer, University of Hagen

We prove that localization near band edges of multi-dimensional ergodic random Schrödinger operators with periodic background potential in $L^2(\mathbb{R}^d)$ is universal. By this, we mean that localization in its strongest dynamical form holds without extra assumptions on the random variables and independently of regularity or degeneracy of the Floquet eigenvalues of the background operator. The main novelty is an initial scale estimate the proof of which avoids Floquet theory altogether and uses instead an interplay between quantitative unique continuation and large deviation estimates. This talk is based on joint work with Albrecht Seelmann.

Generic simplicity for spectral edges with applications to homogenization theory

Vivek Tewary, TIFR Centre for Applicable Mathematics (Slides, Recording)

Motivated by the Floquet-Bloch wave method in homogenization, we consider perturbation theory for interior spectral edges of second order periodic elliptic operators. Interior spectral edges can be made simple by perturbations in the coefficients. The results suggest an interplay between simplicity and finiteness of points at which a spectral edge is attained.

Spectral inclusion for operators with gaps

Christiane Tretter, University of Bern

Analytical information about the spectra and resolvents of non-selfadjoint operators is of great importance for numerical analysis and applications. However, even for perturbations of selfadjoint operators there are only a few classical results. In this talk relatively bounded, not necessarily symmetric perturbations of selfadjoint operators with spectral gaps are considered. We present new spectral inclusion results and various modifications e.g. for gaps of the essential spectrum or for infinitely many gaps, and some applications.

(Joint work with Jean-Claude Cuenin, Loughborough University, UK)

Perturbations of selfadjoint spectra

Krešimir Veselić, University of Hagen (Slides)

We give estimates for the movement of different kinds of selfadjoint spectrum (discrete, contionuous) under relativly bounded perturbations, in particular in gaps of the essential spectrum. The results will be illustrated on Dirac and Schrödinger operators.

Timetable

Monday, July 26

09:50	**Opening**	
Chair	**Ivan Veselić**	
10:00	**Christiane Tretter** Bern	Spectral inclusion for operators with gaps
11:00	**Matthias Langer** Glasgow	Variational principles for operator functions
Chair	**Werner Kirsch**	
14:00	**Lukas Schimmer** Djursholm	Friedrichs extension and min-max principle for operators with a gap
15:00	**Krešimir Veselić** Hagen	Perturbations of selfadjoint spectra
16:00	**Luka Grubišić** Zagreb	Diagonalization of indefinite saddle point forms and applications in fluid dynamics

Tuesday, July 27

Chair	**Albrecht Seelmann**	
10:00	**Grigori Rozenblum** Gothenburg/Saint Petersburg/Sochi	Eigenvalue distribution of polynomially compact pseudodifferential operators
11:00	**Ivica Nakić** Zagreb	Lifting of spectral band edges of Schrödinger operators on unbounded domains
Chair	**Grigori Rozenblum**	
14:00	**Vivek Tewary** Bangalore	Generic simplicity for spectral edges with applications to homogenization theory
15:00	**Tatiana Suslina** Saint Petersburg	Homogenization of the periodic Schrödinger-type equations
16:00	**Matthias Täufer** Hagen	Band edge localization beyond regular Floquet eigenvalues

Wednesday, July 28

Chair	**Pavel Exner**	
15:00	**Lyonell Boulton** Edinburgh	Improve convergence by creating a gap, is this a good idea?
16:00	**Marco Marletta** Cardiff	Dissipation, repetition or oscillation: calculating eigenvalues of perturbed periodic operators in spectral gaps

Thursday, July 29

Chair	**Konstantin Pankrashkin**	
10:00	**Michael Plum** Karlsruhe	Gap localization of TE-modes by arbitrarily weak defects
11:00	**Horia Cornean** Aalborg	Low lying spectral gaps induced by slowly varying magnetic fields
Chair	**Michael Gruber**	
14:00	**Amir Sagiv** New York City	Effective gaps in continuous Floquet Hamiltonians
15:00	**Michael Loss** Atlanta	Critical magnetic field for 2d magnetic Dirac-Coulomb operators
16:00	**Peter Kuchment** College Station	Generic properties of dispersion relations for discrete periodic operators

Friday, July 30

Chair	**Tatiana Suslina**	
10:00	**Sergey Morozov** Munich	On the Bethe-Sommerfeld property and integrated density of states of periodic matrix operators
11:00	**Leonid Parnovski** London	Bethe-Sommerfeld property of multidimensional Schrödinger operators with periodic and almost-periodic potentials
Chair	**Norbert Peyerimhoff**	
14:00	**Evgeny Korotyaev** Saint Petersburg	Estimates of spectral bands and spectral gaps
15:00	**Éric Séré** Paris	Dirac-Coulomb operators: looking for the optimal charge distribution
16:00	**Gregory Berkolaiko** College Station	Global extrema of dispersion relation of tight-binding models