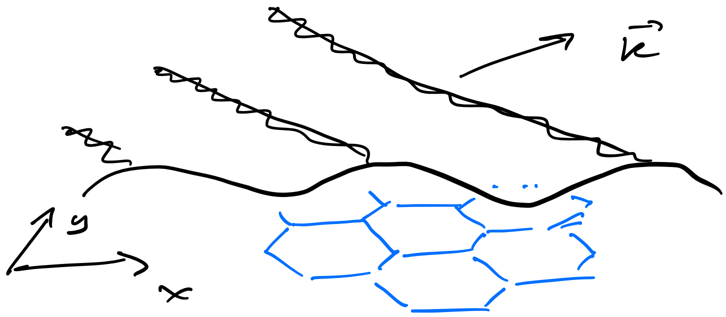
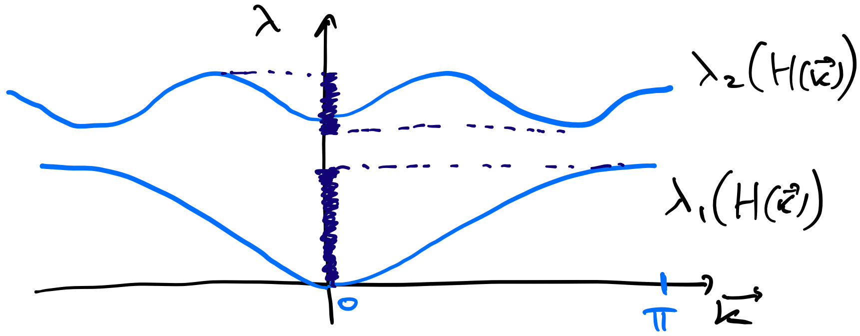
Joint Work with Y. Canzani, G. Cox, J. Marzuola



$$A(\vec{x},t) = \hat{f}_{\vec{k}}(\vec{x})\, e^{i(\vec{x}\cdot\vec{k} - et)}$$

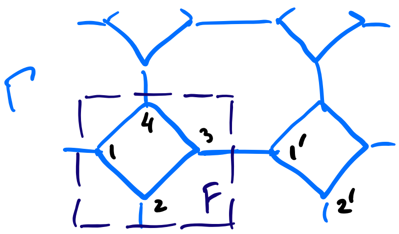
periodic $\hat{f}_{\vec{k}}(\vec{x})$

"Wave should be compatible with microstructure"



Band - Gap Structure

Consider $\mathbb{Z}^d$-periodic (discrete) graph Γ



$$(Hf)_u = \sum_{v\sim u} H_{u,v} f_v \qquad H_{v,u} = \overline{H_{u,v}}$$

$$H_{gu,gv} = H_{u,v} \quad \forall g \in \mathbb{Z}^d$$

Floquet-Bloch: Expand H into direct sum over irreps of the translation group $\mathbb{Z}^d$

$$\mathbb{Z}^d \ni \vec{n} = (n_1, n_2 .. n_d) \mapsto e^{i\vec{k}\cdot\vec{n}} \qquad \vec{k} \in T^d$$

$$H = \int_{T^d}^{\oplus} H(\vec{k})\, d\vec{k} \Rightarrow \sigma(H) = \bigcup_{\vec{k}\in T^d} \sigma(H(\vec{k}))$$

$$T^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$$

Example:

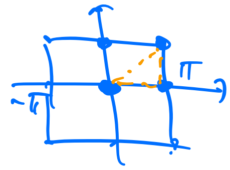
$$(Hf)_3 = -f_2 - f_4 - f_{1'} + (3+q_3) f_3$$

$$(H(\vec{k})f)_3 = -f_2 - f_4 - e^{ik_1} f_1 + (3+q_3) f_3$$

$$\ldots \quad H(\vec{k}) = \begin{pmatrix} 3+q_1 & -1 & -e^{-ik_1} & -1 \\ -1 & 3+q_2 & -1 & -e^{-ik_2} \\ -e^{ik_1} & -1 & 3+q_3 & -1 \\ -1 & -e^{ik_2} & -1 & 3+q_4 \end{pmatrix}$$

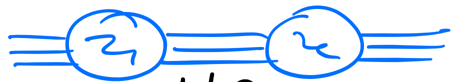
4×4
$\vec{k}$-dep.

Band edges = minima / maxima of $\lambda_n(H(\vec{k}))$ over $\mathbb{T}^d$



If $H = \overline{H}$, $\vec{k} = (0, 0, \dots)$, $(\pi, 0, \dots)$, $(0, \pi, \dots)$, $\dots$ are critical points (when smooth)

Is it enough to check those?

*  $\mathbb{Z}^1$-periodic – YES
* Harrison, Kuchment, Sobolev, Winn: $\mathbb{Z}^d$, $d \geq 2$ NO
* Exner, Kuchment, Winn: 

  even $\mathbb{Z}^1$ + multiple links NO

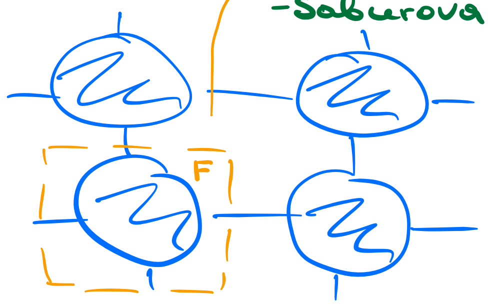
Good News:

Assumptions:

1. $\mathbb{Z}^d$-periodic

($\exists F$) with one crossing per generator

choice of F Korotyaev-Saburova



2. Eigenvalue $\lambda^\circ = \lambda_n(H(\vec{k}^\circ))$ is simple eigenfunction f° is $\neq 0$ on at least one end of each crossing edge

Proposition: Let $\lambda(k) := \lambda_n(H(k))$

Under Assumption 2 at $k = k^\circ$,

$$\nabla \lambda(k^\circ) = B^* f^\circ \qquad \text{Hess}\, \lambda(k^\circ) = 2 \,\mathrm{Re}\, W$$

$$B := D\left(H(k) f^\circ\right)\Big|_{k=k^\circ}, \qquad \Omega = \frac{1}{2} \text{Hess} \left\langle f^\circ, H(k) f^\circ \right\rangle \Big|_{k=k^\circ}$$

$$W = \Omega - B^* \left(H(k^\circ) - \lambda^\circ\right)^+ B.$$

Second derivative test: $\nabla\lambda = 0$, $\mathrm{Re}\, W > 0 \Rightarrow$ Loc. Min

Thm I: Under Assumptions 1 & 2

$\nabla\lambda(k^\circ) = 0$, $W \geq 0 \Rightarrow \lambda(k^\circ)$ is Global Min.

Thm II: Under Assumptions 1 & 2 & $H = \bar{H}$,

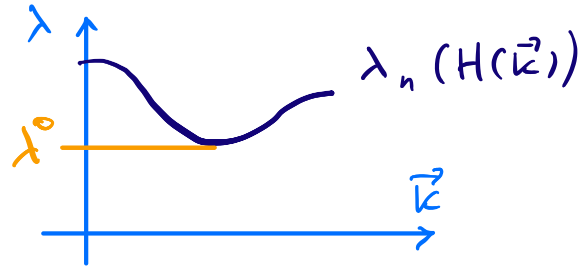
if λ has local extremum at $k = k^\circ$ and either

* $\vec{k}^\circ$ is "corner" ($\in \{0, \pi\}^d$)
* $d \leq 2$
* $d = 3$ and loc. min is non-degenerate.

Then $\lambda(k^\circ)$ is the global extremum.

Remark: For $d = 4$ we have an example of a local-not-global min (not "corner").

Proof Idea: Given (λ°, f°), represent



$$H(\vec{k}) = S + P(\vec{k})$$

with $\quad P(\vec{k}) \geq 0 \quad \forall \vec{k}$

$\qquad P(\vec{k}_0) f^\circ = 0$

$\Rightarrow \quad S f^\circ = \lambda^\circ f^\circ$

Lateral Variation Principle $\Rightarrow \quad \lambda^\circ = \lambda_n(S)$

(gen. case : with P. Kuchment)

for simplicity rank $P = 2$

then $\quad P(k) = V(k)^* V(k) = \begin{pmatrix} | & | \end{pmatrix} \begin{pmatrix} — \\ — \end{pmatrix}$

rank2 $\Rightarrow \quad \lambda_n(S) \leq \lambda_n(H(k)) \leq \lambda_{n+2}(S)$

$\lambda^\circ \in \mathrm{spec}(S) \qquad \left(\lambda^\circ := \lambda_n(H(\vec{k}_0)) \right)$

If $\lambda^\circ = \lambda_n(S) \Rightarrow (\vec{k}_0, \lambda^\circ)$ is a local min

of $\lambda_n(S + V(k)^* V(k))$

It turns out that $\Leftarrow$

if $V(\vec{k})$ is "rich enough"

Rich enough: Family $V(\vec{k})$ transversal to

"boring perturbations" $\{ W : W f^\circ = 0 \}$

codim $= 2$

More generally

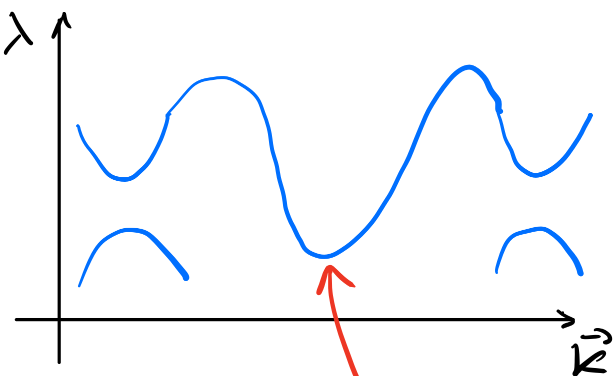
$$\lambda^0 = \lambda_{n+s}(S) \iff \lambda_n\left(S + V(k_0)^* V(k_0)\right)$$

is a C.P. of Morse index s

"proof":

$$\lambda'' \to \left\{ \begin{array}{l} \text{Schur complement} \\ \text{gen. DtN map} \\ \text{gen. Birman-Schwinger} \end{array} \right\} \to \text{spectral shift}$$

on the spectrum of H!

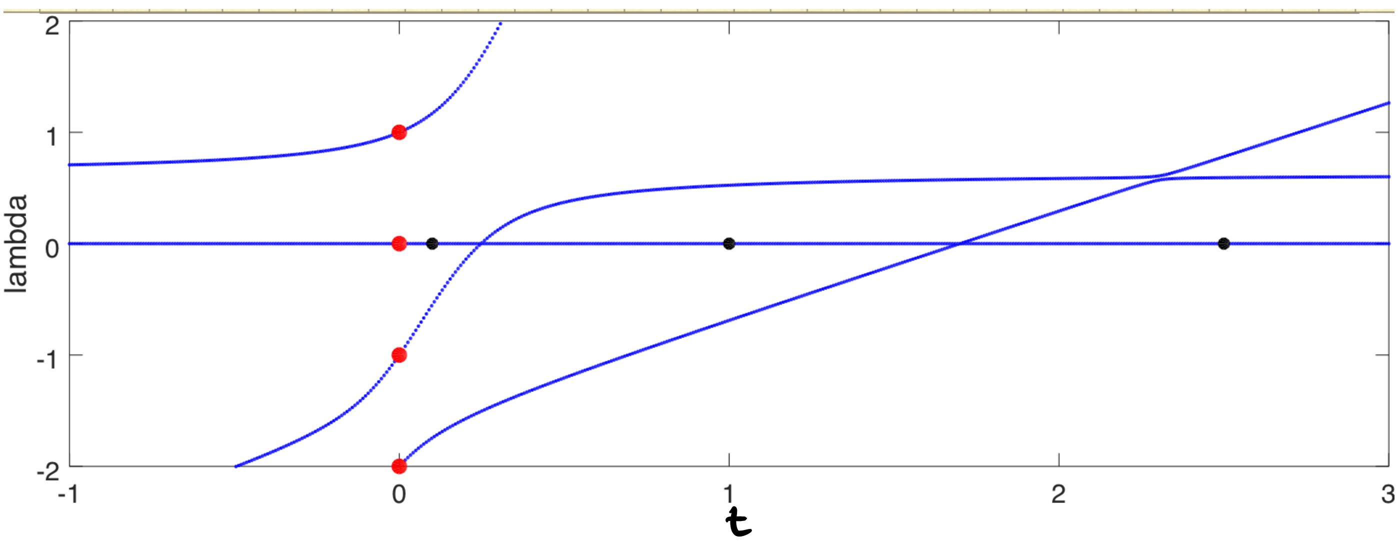


∴ Not possible!
(T&C apply)

Example :

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad V_0 = \begin{pmatrix} 0 & 1/2 & 1/2 & 3/2 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Consider
$$M_t = S + t\, V_0^* V_0$$

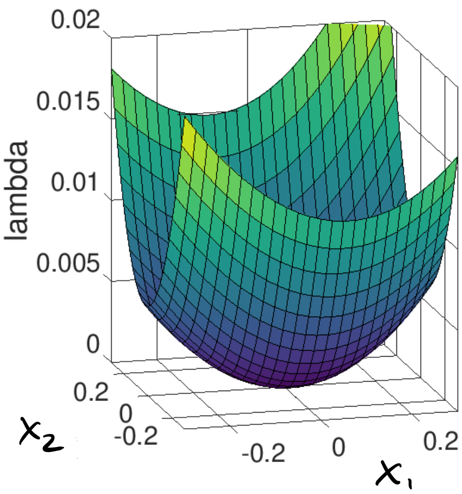


Fix t (say $t=1$)

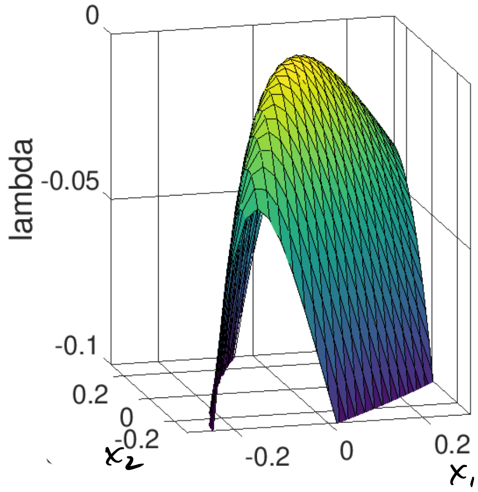
Consider
$$H(\vec{x}) = S + V(\vec{x})^* V(\vec{x})$$
where
$$(\vec{x}) = V_0 + x_1 V_1 + x_2 V_2$$
any V_1, V_2 s.t. $\{V_1 f, V_2 f\}$ lin. indep.

"transversality"

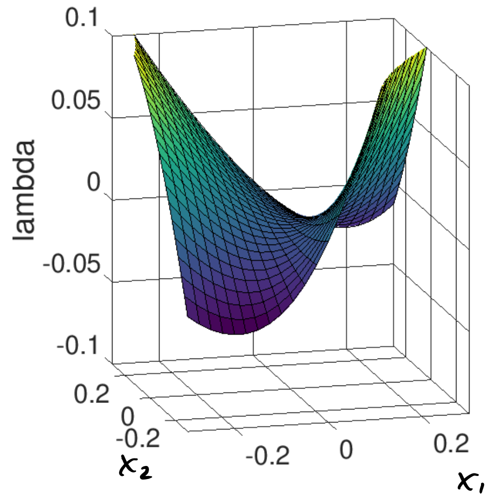
Plot $\lambda(\vec{x})$ — continuation of $\lambda=0$



$S + 0.1\, V^*(\vec{x})\, V(\vec{x})$



$S + 1 \cdot V^*(\vec{x})\, V(\vec{x})$



$S + 2.5\, V^*(\vec{x})\, V(\vec{x})$