Improve convergence by creating a gap, is this a good idea?

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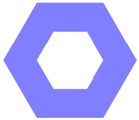
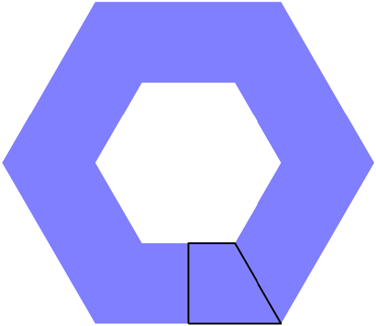
Main motto:

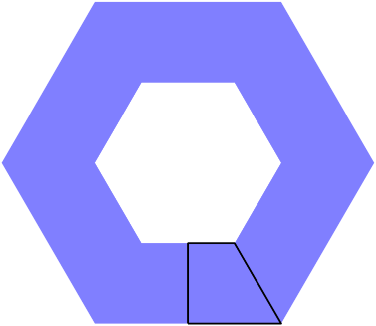
brute force does not work

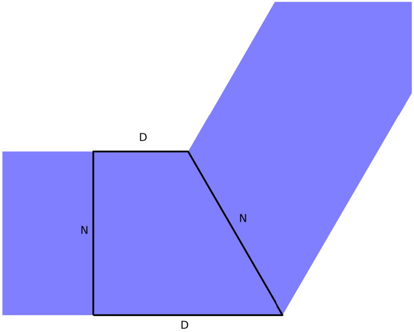
Problem.
Let $\Sigma \subset \mathbb{R}^2$ be a bounded domain. Compute the first eigenvalue of the Dirichlet Laplacian on Σ with as many correct digits as possible.
...say

$\Sigma =$ 

1- Beyond brute force





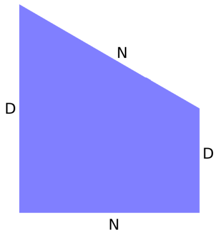


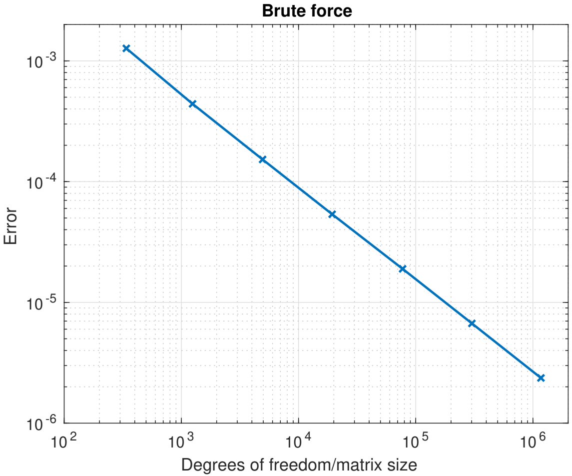
$$\Sigma = \bigcup \Omega_n$$

$$\begin{cases} -\Delta u = \omega^2 u & \text{in } \Omega_n \\ u = 0 & \text{on D} \\ \partial_n u = 0 & \text{on N} \end{cases}$$

... Matlab PDE Toolbox

does not go beyond single precision

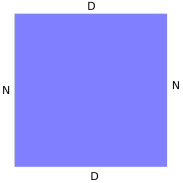




$\omega^2 \approx 49.814*$

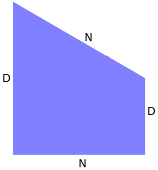
Conformal transplantation

crucial idea from [Banjai 2007] based on [Trefethen-Driscoll 2003]



$$f - \text{conformal}$$

$$\Xi \longrightarrow \Omega_n$$



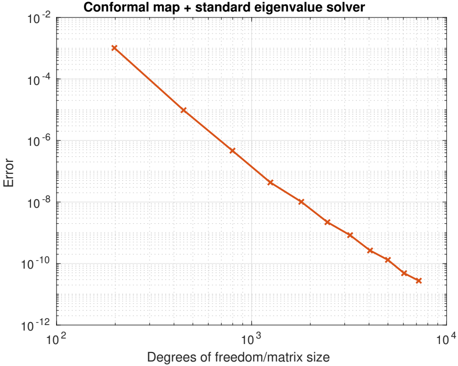
$$\begin{cases} -\Delta v = \omega^2 |f'|^2 v & \text{in } \Xi \\ v = 0 & \text{on D} \end{cases}$$

$$\begin{cases} -\Delta u = \omega^2 u & \text{in } \Omega_n \\ u = 0 & \text{on D} \\ \partial_n u = 0 & \text{on N} \end{cases}$$

$$v = u \circ f$$

... almost double precision

$\omega^2 \approx 49.81459823*$ with standard eigenvalue solver



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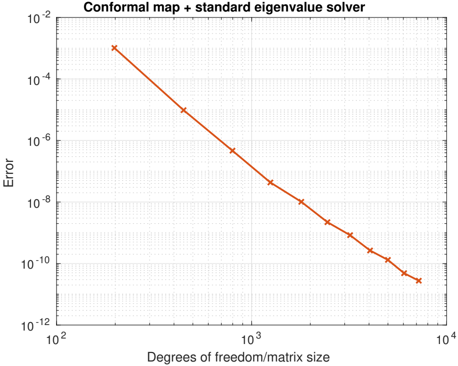
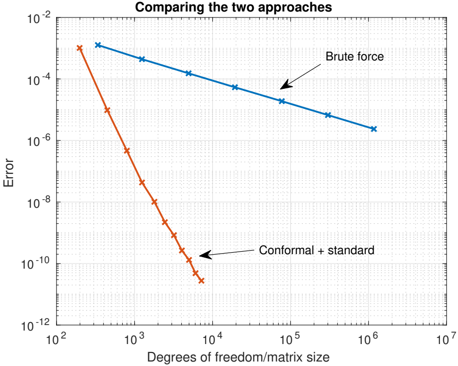
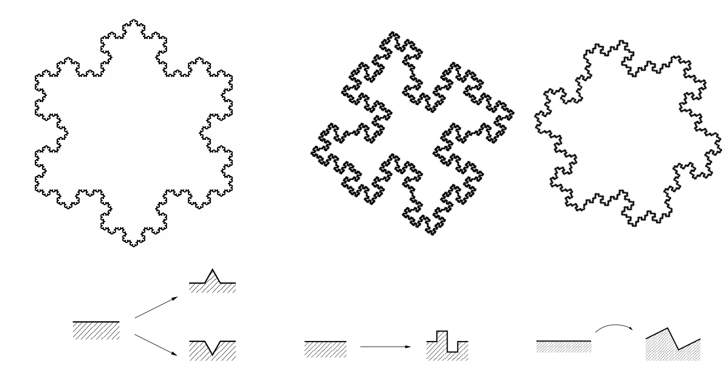
$\omega^2 \approx 49.81459823*$ with standard eigenvalue solver





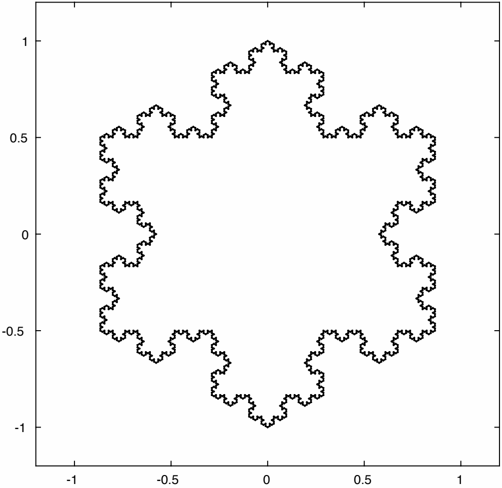
2- Concrete challenge

What about $\partial\Sigma$ one of the following?



Specific conditions on what is allowed will come later.

For example, we know $\omega^2 = 13.166^2_0*$

$\partial\Sigma \quad = $ 

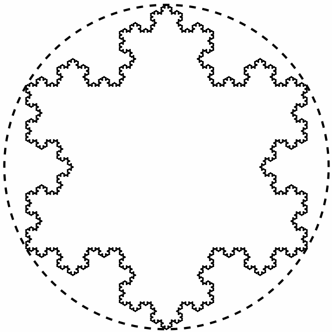
See [*Journal of Fractal Geometry* **8** (2021) 153–188]

3- The method

Main steps

1. *Domain monotonicity.*
   Enclose Σ (tightly) between two nested families of polygons $\mathsf{T}_j \subset \mathsf{T}_{j+1} \subset \Sigma \subset \mathsf{H}_{j+1} \subset \mathsf{H}_j$. Levels $\Sigma_j \in \{\mathsf{T}_j, \mathsf{H}_j\}$.
2. *Conformal transplantation.*
   Find a simple polygon Σ_0 to transplant Σ_j into, then get conformal $f : \Sigma_0 \longrightarrow \Sigma_j$ via Schwarz-Christoffel formula.
3. *Open a gap.*
   Write the (singular) eigenvalue problem in Σ_0 as a system of order 1.
4. *Compute.*
   Compute the eigenvalue approximation by means of your favorite "gaps" method.

Step 1. Domain monotonicity



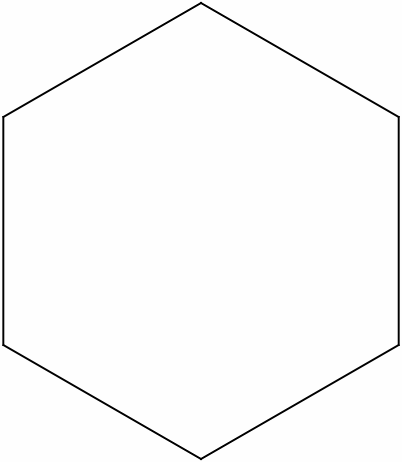
$\Rightarrow \omega_\Sigma^2 \geq j_{0,1}^2 = 5.7831*$

$\Sigma \subset \mathsf{D} = \{x^2 + y^2 < 1\}$

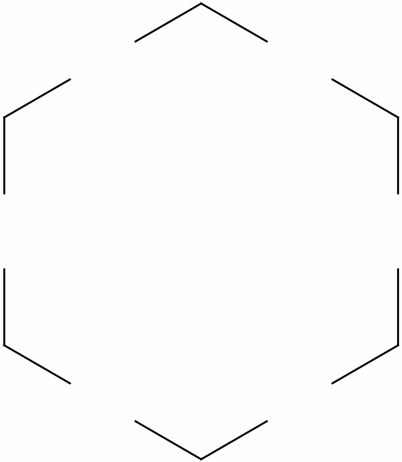
compare: $5.7831*$ vs $13.1160*$

... and $13.1162*$ vs $\frac{16\pi^2}{9} = 17.5459*$

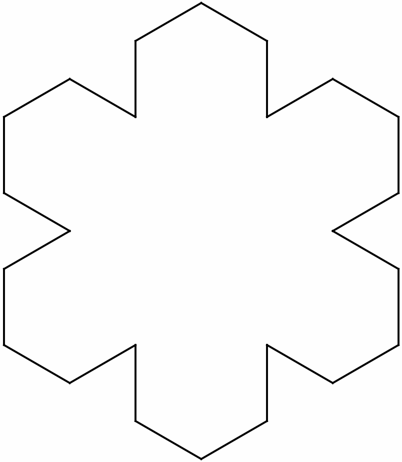
H_0



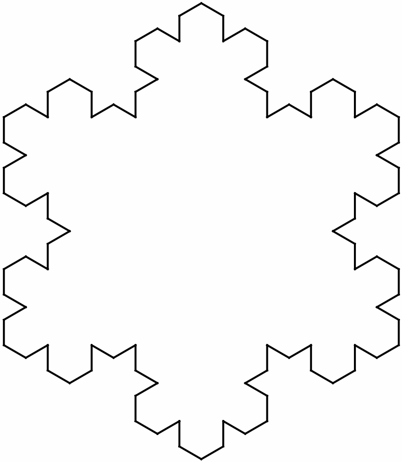
H_0 to H_1



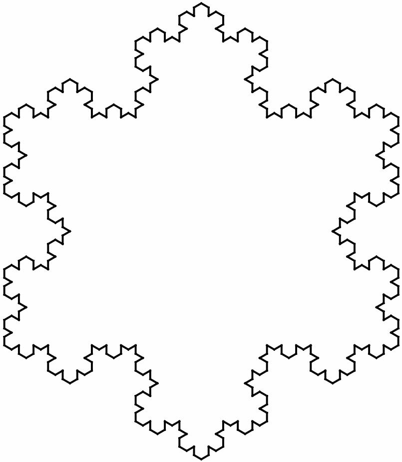
H_1



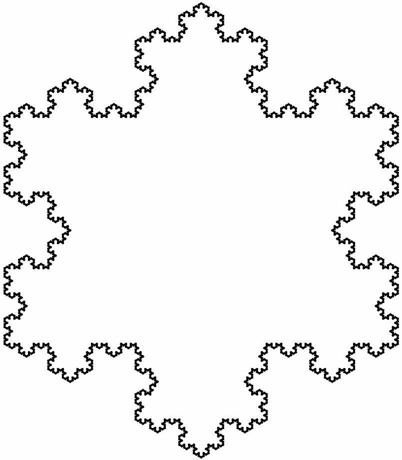
H_2



H_3



H_4

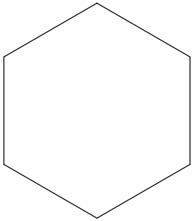


$$\mathsf{T}_j \subset \Sigma \subset \mathsf{H}_j \quad \Rightarrow \quad \omega^2_{\mathsf{H}_j} \leq \omega^2_{\Sigma} \leq \omega^2_{\mathsf{T}_j}$$

For the snowflake we computed

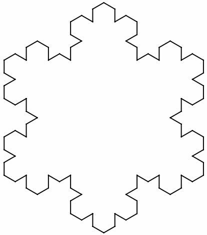
$$13.1160{*} < \omega^2_{\mathsf{H}_{10}} \quad \text{and} \quad \omega^2_{\mathsf{T}_{10}} < 13.1162{*}$$

Step 2. Conformal transplantation



f − conformal

$$\Sigma_0 \longrightarrow \Sigma_j$$



$$\begin{cases} -\Delta v = \omega^2 |f'|^2 v & \text{in } \Sigma_0 \\ v = 0 & \text{on } \partial\Sigma_0 \end{cases}$$

$$\begin{cases} -\Delta u = \omega^2 u & \text{in } \Sigma_j \\ u = 0 & \text{on } \partial\Sigma_j \end{cases}$$

$$v = u \circ f$$

Use Schwarz-Christoffel formula as follows

$g_j : \mathsf{D} \longrightarrow \Sigma_j$ conformal map

$f(z) \equiv f_j(z) = g_j(g_0^{-1}(z))$ for

$$g_j(\xi) = A_j + C_j \int^{\xi} \prod_{k=1}^{n(j)} (1 - \zeta/\xi_k)^{\alpha_k - 1}\, d\zeta$$

$z_k \in \partial\Sigma_0$ pre-vertices

$$z_k = g_0(\xi_k) \qquad f_j(z_k) = g_j(\xi_k) = w_k$$

$w_k \in \partial\Sigma_j$ all the corners

$$\text{for } \mathsf{H}_j \text{ interior angles are } \pi\alpha_k = \begin{cases} 2\pi/3 \\ 5\pi/3 \end{cases}$$

$$n(j) = 4^j 6 = \underbrace{6 + 4(4^j - 1)}_{\alpha_j = 2/3} + \underbrace{2(4^j - 1)}_{\alpha_j = 5/3}$$

Step 3. Open a gap

In $\Omega \equiv \Sigma_j$

$$\overbrace{\begin{bmatrix} 0 & i\,\mathrm{div} \\ & \\ i\,\mathrm{grad} & 0 \end{bmatrix}}^{\mathcal{G}} : \overbrace{\begin{matrix} H_0^1(\Omega) \\ \oplus \\ H(\mathrm{div},\Omega)^2 \end{matrix}}^{\mathrm{dom}\,\mathcal{G}} \longrightarrow \overbrace{\begin{matrix} L^2(\Omega) \\ \oplus \\ L^2(\Omega)^2 \end{matrix}}^{L^2(\Omega)^3}$$

$$\begin{bmatrix} 0 & i\,\mathrm{div} \\ i\,\mathrm{grad} & 0 \end{bmatrix} \begin{bmatrix} u \\ \underline{\sigma} \end{bmatrix} = \omega \begin{bmatrix} u \\ \underline{\sigma} \end{bmatrix} \qquad \Longleftrightarrow \qquad \omega^2 u = \underbrace{-\,\mathrm{div}\,\mathrm{grad}}_{-\Delta : H_0^1 \cap H^2 \longrightarrow L^2} \; u$$

$$\omega_\Omega = \min\left[\mathrm{Spec}\,\mathcal{G} \cap (0,\infty)\right]$$

Note

$$\mathrm{Spec}_{\mathrm{ess}}\,\mathcal{G} = \{0\} \qquad \text{and} \qquad \mathrm{Spec}\,\mathcal{G} = -\,\mathrm{Spec}\,\mathcal{G}$$

In Σ_0

$$-\Delta v = \omega^2 |f'|^2 v \quad \Longleftrightarrow \quad \omega^2 \neq 0 \qquad \mathcal{G}\begin{bmatrix} v \\ \underline{s} \end{bmatrix} = \omega F^2 \begin{bmatrix} v \\ \underline{s} \end{bmatrix}$$

$$F = \begin{bmatrix} |f'| & 0 \\ 0 & I \end{bmatrix}$$

Formally

$$\underbrace{F^{-1}\mathcal{G}F^{-1}}_{\mathcal{T}} \begin{bmatrix} w \\ \underline{t} \end{bmatrix} = \omega \begin{bmatrix} w \\ \underline{t} \end{bmatrix} \qquad \text{for} \qquad \begin{bmatrix} w \\ \underline{t} \end{bmatrix} = F \begin{bmatrix} v \\ \underline{s} \end{bmatrix}$$

... One little problem, $|f'|$ may have lots of "nasty" singularities and zeros.

Step 4. Only rough details

Explicit computations for level $j = 0, \ldots, 10$ snowflake.

j	$(\omega^2_{\mathsf{H}_j})^{\text{upper}}_{\text{lower}}$	$(\omega^2_{\mathsf{T}_j})^{\text{upper}}_{\text{lower}}$
0	7.15533^{94}_{83}	17.5459633^{80}
1	11.78144^{39}_{19}	13.402^{73}_{24}
2	12.51986^{87}_{72}	13.268^{86}_{56}
3	12.89778^{23}_{06}	13.170^{77}_{69}
4	13.03710^{92}_{57}	13.1357^{54}_{33}
5	13.0876^{93}_{89}	13.123^{22}_{19}
6	13.10593^{82}_{52}	13.118^{71}_{68}
7	13.112^{51}_{49}	13.1170^{93}_{73}
8	13.1148^{63}_{59}	13.116^{52}_{49}
9	13.11570^{99}_{73}	13.116^{31}_{28}
10	13.11601^{20}_{15}	13.11622^{76}_{10}

Software:
own C_{++} code + Driscoll's Schwarz-Christoffel Toolbox for Matlab + Comsol LiveLink

4- Details

Assumptions on Σ

$\mathsf{T}_j \subset \mathsf{T}_{j+1} \subset \Sigma \subset \mathsf{H}_{j+1} \subset \mathsf{H}_j$

1. All polygons are open and simply connected
2. $\forall \varepsilon > 0$ there exists $k \in \mathbb{N}$

$$\{z \in \mathsf{H}_j \,:\, \mathrm{dist}(z, \partial \mathsf{H}_j) \geq \varepsilon\} \subset \mathsf{T}_j \qquad \forall j \geq k$$

   Hence $\Sigma = \bigcup \mathsf{T}_j = \mathrm{int} \bigcap \mathsf{H}_j$

3. For suitable self-similar Jordan curves F_n

$$\partial \Sigma = \bigcup_{n=1}^{N} F_n$$

4. The inner angles are uniformly away from multiples of 2π

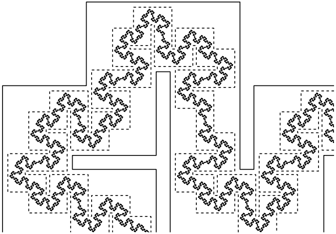
This covers classical domains, but demands careful attention. The above[1]

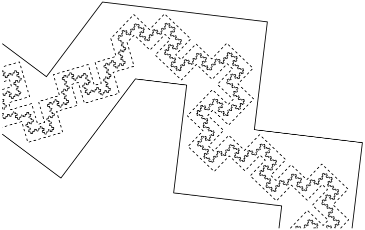
$$\Rightarrow \quad \omega_{\mathsf{T}_j}^2 - \omega_{\mathsf{H}_j}^2 < C\varepsilon^{1/2} \qquad \forall j \geq k.$$

[1] Relies on estimate of Pang [*Bulletin LMS* **29** (1997) 720–730]

Quadric and Gosper-Peano islands

$\omega_{\mathsf{T}_j}^2 - \omega_{\mathsf{H}_j}^2 < C\sqrt{2\delta}\,\sin(\beta_0/2)^{-1/2}\ell_j^{1/2}$





$$\ell_j = \frac{1}{4^j}$$

$\beta_0 = \frac{3\pi}{2}$ $\quad \delta = \frac{1}{3}$

$$\ell_j = \frac{1}{5^{j/2}}$$

$\beta_0 = \frac{2\pi}{3}$ $\quad \frac{1}{5-\sqrt{2}} < \delta < \frac{1}{2}$

On the original domain

$\Omega = \Sigma_j$

$$\overbrace{\begin{bmatrix} 0 & i\,\mathrm{div}_x \\ i\,\mathrm{grad}_x & 0 \end{bmatrix}}^{\mathcal{G}_x} : \overbrace{\begin{matrix} \mathsf{H}^1_0(\Omega) \\ \times \\ \mathsf{H}(\mathrm{div},\Omega) \end{matrix}}^{\mathrm{D}(\mathcal{G}_x)} \longrightarrow \overbrace{\begin{matrix} \mathsf{L}^2(\Omega) \\ \times \\ \mathsf{L}^2(\Omega)^2 \end{matrix}}^{\mathsf{L}^2(\Omega)^3}$$

The densely defined operator $\mathcal{G}_x : \mathrm{D}(\mathcal{G}_x) \longrightarrow \mathsf{L}^2(\Omega)^3$ is selfadjoint, as the adjoint of the minimal operator $i\,\mathrm{grad}_x : \mathsf{H}^1_0(\Omega) \longrightarrow \mathsf{L}^2(\Omega)^2$ is the maximal operator $i\,\mathrm{div}_x : \mathsf{H}(\mathrm{div},\Omega) \longrightarrow \mathsf{L}^2(\Omega)$ and vice versa.
Denote the Dirichlet Laplacian by

$$-\Delta_x : \mathrm{D}(\Delta_x) \longrightarrow L^2(\Omega).$$

Via von Neumann's Theorem,

$$\mathrm{D}(\Delta_x) = \{u \in H^1_0(\Omega) : \mathrm{grad}\, u \in \mathsf{H}(\mathrm{div},\Omega)\} \subset L^2(\Omega).$$

The non-zero vector $\begin{bmatrix} u \\ \underline{s} \end{bmatrix} \in \mathrm{D}(\mathcal{G}_x)$ is an eigenfunction of $\mathcal{G}_x$ if and only if,

1. either $u \in \mathrm{D}(\Delta_x)$, $-\Delta_x u = \omega^2 u$ and $\underline{s} = \frac{\pm i}{|\omega|} \operatorname{grad}_x u$
2. or $u = 0$ and $\operatorname{div}_x \underline{s} = 0$.

Moreover, $\begin{bmatrix} u \\ \underline{s} \end{bmatrix}$ is associated to the eigenvalue $\pm\omega$ in the case 1 and to the eigenvalue 0 in the case 2.
Hence, the family

$$\mathcal{E} = \left\{ \begin{bmatrix} u_k \\ \pm \underline{s}_k \end{bmatrix} \right\}_{k \in \mathbb{N}} \cup \left\{ \begin{bmatrix} 0 \\ \underline{\sigma}_n \end{bmatrix} \right\}_{n \in \mathbb{N}}$$

where $\underline{s}_k = \frac{\pm i}{|\omega_k|} \operatorname{grad}_x u_k$ and we pick $\{\underline{\sigma}_n\}_{n=1}^{\infty} \subset H(\operatorname{div}, \Omega_j)$ an orthonormal basis of $\ker(\operatorname{div})$, is a complete family of eigenfunctions of $\mathcal{G}_x$.

On the transplanted domain

$\Omega = \Sigma_0$

$$\mathcal{D} := \left\{ \begin{bmatrix} \tilde{v} \\ \underline{t} \end{bmatrix} \in L^2(\Omega)^3 : |f'|^{-1}\tilde{v} \in H_0^1(\Omega),\ |f'|^{-1} \operatorname{div}_y \underline{t} \in L^2(\Omega) \right\}$$

and

$$\mathcal{T}_y = \begin{bmatrix} 0 & i|f'|^{-1} \operatorname{div}_y \\ i \operatorname{grad}_y |f'|^{-1} & 0 \end{bmatrix} : \mathcal{D} \longrightarrow L^2(\Omega_0)^3$$

is a densely defined symmetric operator.
We know that $\mathcal{T}_y$ has an orthonormal basis of eigenfunctions in its domain. The closure

$$\overline{\mathcal{T}_y} : \mathrm{D}(\overline{\mathcal{T}_y}) \longrightarrow L^2(\Omega_0)^3$$

is selfadjoint. Moreover,

$$\operatorname{spec}(\overline{\mathcal{T}_y}) = \operatorname{spec}(\mathcal{G}_x) = \{\pm w_k(\Omega_j), 0\}.$$

Reason, for

$$v_k = u_k \circ f, \qquad \tilde{v}_k = |f'| v_k,$$
$$\underline{t}_k = (\nabla_y f)^T \underline{s}_k \circ f \qquad \text{and} \qquad \underline{\tau}_n = (\nabla_y f)^T \underline{\sigma}_n \circ f,$$

the family

$$\tilde{\mathcal{E}} = \left\{ \begin{bmatrix} \tilde{v}_k \\ \pm \underline{t}_k \end{bmatrix}, \begin{bmatrix} 0 \\ \underline{\tau}_n \end{bmatrix} \right\}_{k,n \in \mathbb{N}} \subset \mathcal{D}$$

is complete in $L^2(\Omega)^3$ (not orthonormal).

A question

... This is how we constructed the transplanted self-adjoint operator, but out of curiosity...
$|f'|$ is singular and has zeros on $\partial\Omega$. If, say

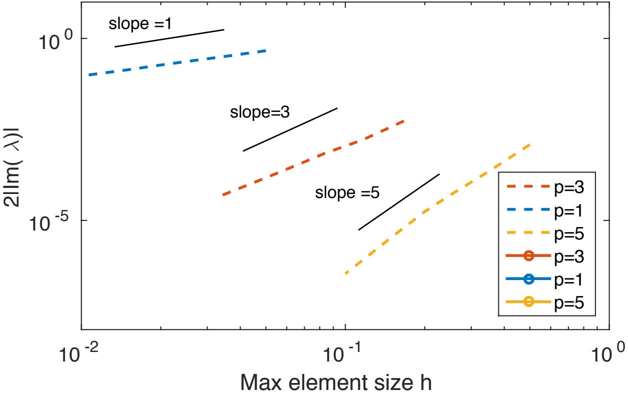
$$\int_{\Omega} |f'(z)|^{-2} dz < \infty$$

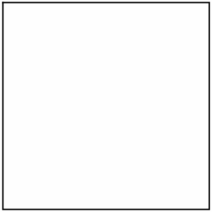
(this is connected with the angles in the vertices and pre-vertices),

Is $(\mathcal{T}, \mathcal{D})$ closed?

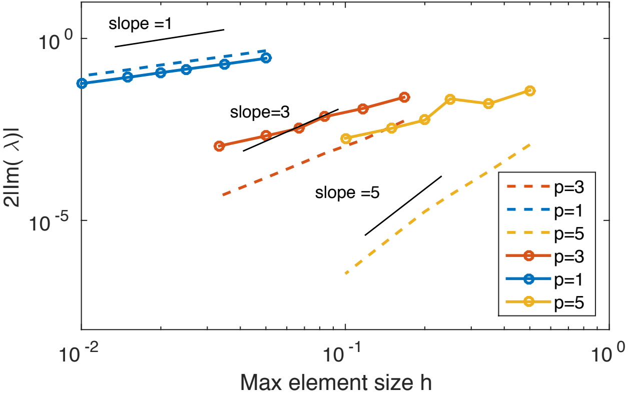
5- An argument in favour of “opening a gap”

Laplacian as a benchmark (take $F \equiv 1$ momentarilly)





Level 0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slope =5
10^0
10^{-5}
2|Im(λ)|
10^{-2}
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10^0
Max element size h
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p=1
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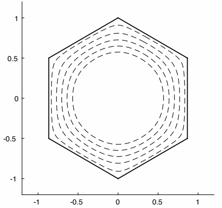
p=3
p=1
p=5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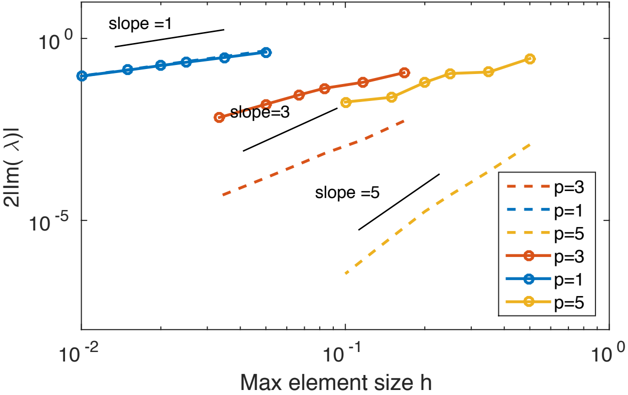
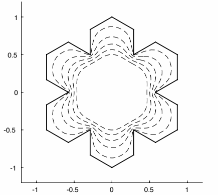
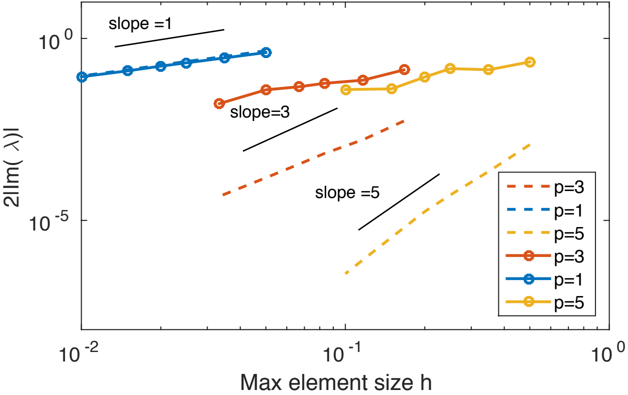
1

Level 1





Level 2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2|Im(λ)|
10^0
10^{-5}
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Max element size h
p=3
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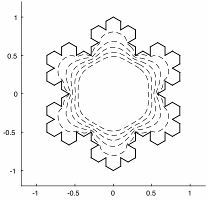
p=1
p=5


1
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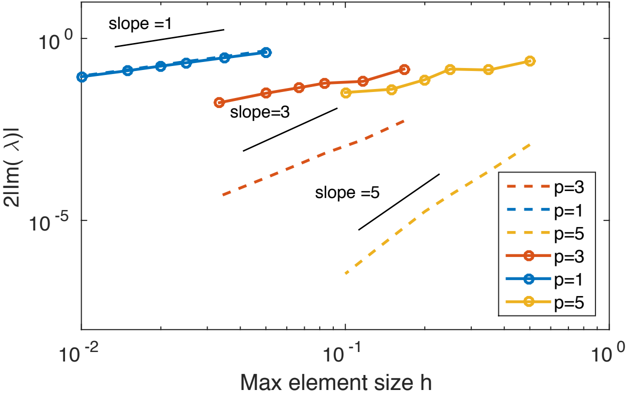
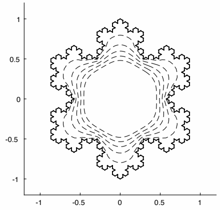
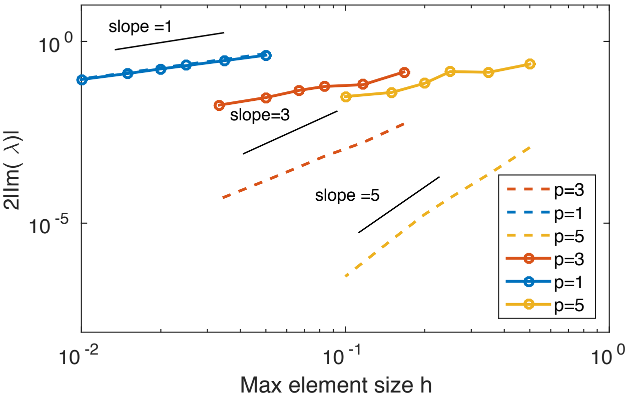
0
-0.5
-1
-1
-0.5
0
0.5
1

Level 3





Level 4



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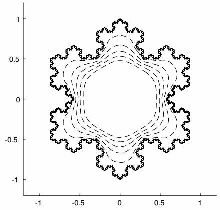
p=1
p=5



1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

Final message

- For numerical calculations it is better a rough operator on a simple region, than a simple operator on a rough region.
- If the rough operator has too singular coefficients, you do not need a too regular basis of approximation.
- Better to lower the order and deal with the gap in the spectrum with modern methods.

end of the presentation