Eigenvalue asymptotics for polynomially compact pseudodifferential operators and applications

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27 July 2021

Pseudodifferential operators

$$(\mathfrak{A}\mathbf{u})(x) = \mathcal{F}^{-1}_{\xi\to x}\mathfrak{a}(x,\xi)\widehat{\mathbf{u}}(\xi) =$$
$$(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi}\mathfrak{a}(x,\xi)\mathbf{u}(y)dyd\xi.$$

$\mathfrak{a}(x,\xi)$ -symbol. Classical PsDO: symbol is polyhomogeneous, $\mathfrak{a}(x,\xi) \sim \mathfrak{a}_s(x,\xi) + \mathfrak{a}_{s-1}(x,\xi) + ..., \ \xi \to \infty$
$\mathfrak{a}_\rho(x,\xi)$ is positively homogeneous order ρ in ξ; $\mathfrak{a}_s(x,\xi)$ principal symbol. We are interested in the vector case: $\mathbf{u} = (u_1, \dots, u_N)^\top$, $\mathfrak{a}(x,\xi)$ is $N \times N$ matrix.

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Passage from $\mathbb{R}^d$ to a compact d-dimensional manifold Γ: localization. The principal symbol $\mathfrak{a}_s(x,\xi)$ is an invariant object on $T^*(\Gamma)$, lower order symbols, not invariant.

Spectral theory in the self-adjoint case:

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$s < 0$, compact operator, eigenvalues tend to 0, a version of Weyl law (Birman-Solomyak):

$$\boldsymbol{\lambda}_k^{\pm} \sim \pm C_{\pm}(\mathfrak{a}_s) k^{d/s};$$

$$C_{\pm}(\mathfrak{a}_s) = c(d,s)\left[\int_{S^*(\Gamma)} \operatorname{tr}\left([\mathfrak{a}_s(x,\xi)]_{\pm}^{-d/s}\right) dx d\xi\right]^{-s/d}$$

$s = 0$?

Zero order operators

$s = 0$; the principal symbol zero order homogeneous. Present interest in various physical models. One of the first: D.Yafaev (2005), scattering with magnetic field, now Colin de Verdiere, Zworski,... forced interior waves in fluids. Essential spectrum: the union of spectra of the principal symbol. Discrete spectrum??

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Polynomially compact zero order operators.

$\mathfrak{A}$: there exists a polynomial $\mathbf{p}(\boldsymbol{\lambda})$: $\mathfrak{B}_0 = \mathbf{p}(\mathfrak{A})$ is compact – a pseudodifferential operator with principal matrix symbol $\mathbf{p}(\mathfrak{a}_0(x,\xi))$. Compact $\rightrightarrows$ the principal symbol must vanish. Eigenvalues of the matrix $\mathfrak{a}_0(x,\xi)$ must coincide with zeros of the polynomial $\mathbf{p}(\boldsymbol{\lambda})$. By continuity, these eigenvalues must be constant; denoted by ω_j. The principal symbol is a matrix with eigenvalues independent of x, ξ. We choose the polynomial $\mathbf{p}(\boldsymbol{\lambda})$ of smallest possible degree. The zeros ω_j are simple. Let γ_j, $0 \le j \le N$. Denote by $n_\pm(\pm\tau + \omega_j; T)$ the number of eigenvalues of an operator T in the interval $(\gamma_{j-1}, -\tau + \omega_j)$, resp., in $\tau + \omega_j$, these are intervals on both sides of ω_j.

$$\ldots\ldots\gamma_{j-1}\overset{n_-(\omega_j-\tau)}{\ldots\ldots\ldots\ldots\ldots}[\omega_j-\tau]\ldots\ldots\ldots\omega_j\ldots\ldots\ldots\ldots[\omega_j+\tau]\overset{n_+(\omega_j+\tau)}{\ldots\ldots\ldots\ldots\ldots}\gamma_j\ldots\ldots$$

We study: estimates and asymptotics as $\tau \to 0$.

Theorem

$\mathfrak{B}_0 = \mathbf{p}(\mathfrak{A})$ *is a pseudodifferential operator of order* -1. *Its eigenvalues are* $\mathbf{p}(\boldsymbol{\lambda})$, *where* $\boldsymbol{\lambda}$ *are eigenvalues of* $\mathfrak{A}$. *The counting function of* $\mathfrak{B}_0$ *is therefore* $n_{\pm}(\tau; \mathfrak{B}_0) \leq C\tau^{-d}, \tau \to 0$. *Therefore, for each of* ω_j,

$$n_{\pm}(\pm\tau + \omega_j; \mathfrak{A}) = O(\tau^{-d}).$$

We obtained the estimate for eigenvalues near all of ω_j. Even B-S asymptotic formula for the UNION of the sets of $\mathbf{p}(\boldsymbol{\lambda}_k)$. We need to separate them, find the asymptotics for $n_\pm(\pm\tau+\omega_j,\omega_j;\mathfrak{A})$, $\tau\to 0$.

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How to find the symbol (of order -1) of $\mathfrak{B}_\nu = \mathbf{p}_\nu(\mathfrak{A}) = \prod_{j\neq\nu}(\mathfrak{A}-\omega_j)^2(\mathfrak{A}-\omega_\nu)$? We must know the principal (order 0) and subprincipal (order -1) symbols of $\mathfrak{A}$. Then the product rule for 2 operators:

$$\mathfrak{A}\mathfrak{C} \sim \mathfrak{a}_0(x,\xi)\mathfrak{c}_0(x,\xi) + \frac{1}{i}\operatorname{grad}_\xi \mathfrak{a}_0 \operatorname{grad}_x \mathfrak{c}_0 +$$
$$\mathfrak{a}_{-1}(x,\xi)\mathfrak{c}_0(x,\xi) + \mathfrak{a}_0(x,\xi)\mathfrak{c}_{-1}(x,\xi).$$

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we need to use it many times. Lots of calculations. But the structure is clear. The result:

Theorem

The principal (order -1) symbol $\mathfrak{b}_{-1,\nu}$ of $\mathfrak{B}_\nu$ contains lots of terms of two types: products of:

1. *several copies of $\mathfrak{a}_0(x,\xi)$ and one copy of $\mathfrak{a}_{-1}(x,\xi)$, or*
2. *several copies of $\mathfrak{a}_0(x,\xi)$, one copy of $\operatorname{grad}_\xi \mathfrak{a}_0(x,\xi)$, and one copy of $\operatorname{grad}_x \mathfrak{a}_0(\xi)$.*

Eigenvalue asymptotics

As soon as the symbol $\mathfrak{b}_{\nu,-1}(x,\xi)$ is found, we can use the B-S asymptotic formula for the order -1 operator $\mathfrak{B}_\nu$

$$n_\pm(\omega_\nu \pm_\tau, \mathfrak{B}_\nu) \sim C\tau^{-d} \int_{S^*G} \mathrm{Tr}\,(\mathfrak{b}_{\nu,-1}(x,\xi)^d_\pm) dx d\xi, \tau \to 0.$$

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This is OK if the integrand is nonzero. If $\mathfrak{b}_{\nu,-1}(x,\xi)$ is identically zero, formula BS does not work

Degeneracy

If $\mathfrak{b}_{\nu,-1}(x, \xi) \equiv 0$, this means that $\mathfrak{B}_\nu$ is an operator of order less than -1 and the eigenvalues of $\mathfrak{A}$ converge to zero faster. It is the degenerate case. How to handle?

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A different polynomial: $p_{\nu,2}(\lambda) = \prod_{j\neq\nu}(\lambda-\omega_j)^3(\lambda-o_\nu)$.
Eigenvalues converging to $\omega_j, j \neq \nu$ are transformed to the ones converging to 0 as $k^{-3/d}$. Only the eigenvalues converging to ω_ν remain. The asymptotics of $n_\pm(\omega_n \pm \tau)$ is found by the BS formula for PsDO of order -2.

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Application: The Lamé system (homogeneous and isotropic material)

$$\mathcal{L}\mathbf{u} \equiv \mathcal{L}_{\mu,\lambda}\mathbf{u} \equiv \operatorname{div}(\mu \operatorname{grad} \mathbf{u}) + \operatorname{grad}((\lambda+\mu)\operatorname{div}\mathbf{u}) = 0, \qquad (0.1)$$
$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{D}, \mathbf{u} = (u_1, u_2, u_3)^\top,$$

where λ, μ are the Lamé constants.

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$$\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{D}, \mathbf{u} = (u_1, u_2, u_3)^\top,$$

where λ, μ are the Lamé constants.
The fundamental solution $\mathcal{R}(\mathbf{x},\mathbf{y}) = [\mathcal{R}(\mathbf{x},\mathbf{y})]_{p,q=1,2,3}$ for the Lamé equations, *the Kelvin matrix*, known since long ago:

$$[\mathcal{R}(\mathbf{x},\mathbf{y})]_{p,q} = \lambda' \frac{\delta_{p,q}}{|\mathbf{x}-\mathbf{y}|} + \mu' \frac{(x_p - y_p)(x_q - y_q)}{|\mathbf{x}-\mathbf{y}|^3}, \qquad (0.2)$$

$$\lambda' = \frac{\lambda + 3\mu}{4\pi\mu(\lambda+2\mu)}, \ \mu' = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}, \ p, q = 1, 2, 3.$$

Coboundary (stress) operator

$$[T(\mathbf{x}, \partial_{\boldsymbol{\nu}})]_{p,q} = \lambda \nu_p \partial_q + \mu \nu_q \partial_p + \mu \delta_{p,q} \partial_{\boldsymbol{\nu}},$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x}) = (\nu_1, \nu_2, \nu_3)$ is the unit outward normal vector to Γ at the point $\mathbf{x}$ and $\partial_{\boldsymbol{\nu}(\mathbf{x})}$ is the directional derivative along $\boldsymbol{\nu}(\mathbf{x})$.

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The NP operator:

$$(\mathfrak{K}[\psi])(\mathbf{x}) = \int_\Gamma \mathcal{K}(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) dS(\mathbf{y}) \equiv \int_\Gamma T(\mathbf{y}, \partial_{\boldsymbol{\nu}(\mathbf{y})}) \mathcal{R}(\mathbf{x}, \mathbf{y})^\top \psi(\mathbf{y}) dS(\mathbf{y}), \tag{0.3}$$

dS is the natural surface measure on Γ and $T(\mathbf{y}, \partial_{\boldsymbol{\nu}(\mathbf{y})})$ denotes the coboundary operator at the point $\mathbf{y} \in \Gamma$. The operator is not compact!! (Singular integral operator).

The kernel of the integral operator:

$$[\mathcal{K}(\mathbf{x},\mathbf{y})]_{p,q} = \mathbf{a}\frac{\nu_p(\mathbf{y})(x_q - y_q) - \nu_q(\mathbf{y})(x_p - y_p)}{|\mathbf{x}-\mathbf{y}|^3} +$$

$$\left(\mathbf{b}\delta_{p,q} - \mathbf{c}\frac{(x_p - y_p)(x_q - y_q)}{|\mathbf{x}-\mathbf{y}|^2}\right)\sum_{l=1}^{3}\nu_l(\mathbf{y})\frac{x_l - y_l}{|\mathbf{x}-\mathbf{y}|^3} \equiv$$

$$\mathcal{K}_0(y, y-x) + \mathcal{K}_{-1}(y, y-x);\ p,q = 1,2,3,$$

where

$$\mathbf{a} = \frac{\mu}{2\pi(\lambda+2\mu)},\ \mathbf{b} = \frac{\lambda+\mu}{4\pi(\lambda+2\mu)},\ \mathbf{c} = \frac{3(\lambda+\mu)}{2\pi(\lambda+2\mu)}.$$

Spectrum was previously known for ONE case only, the sphere.
Three series of eigenvalues"

$$\boldsymbol{\lambda}_k^0(\mathfrak{K}) = \frac{3}{2(2k+1)} \sim \frac{3}{4k}, \tag{0.4}$$

$$\boldsymbol{\lambda}_k^-(\mathfrak{K}) = \frac{3\lambda - 2\mu(2k^2 - 2k - 3)}{2(\lambda + 2\mu)(4k^2 - 1)} \sim -\mathbb{k} + \frac{2\mu}{(\lambda + 2\mu)k},$$

$$\boldsymbol{\lambda}_k^+(\mathfrak{K}) = \frac{-3\lambda + 2\mu(2k^2 + 2k - 3)}{2(\lambda + 2\mu)(4k^2 - 1)}, \sim \mathbb{k} + \frac{2\mu}{(\lambda + 2\mu)k},$$

Asymptotics. Converge to the points of ess. spectrum from above.
Question: What happens for the general case?

Theorem (Agranovich et.al., 1999; Miyanishi, R., 2020) $\mathfrak{K}$ is a pseudodifferential operator of order 0 on Γ, with principal symbol

$$\mathfrak{k}_0(x,\xi) = \frac{i\Bbbk}{|\xi|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}, \Bbbk = \frac{\mu}{2(\lambda+2\mu)}.$$

The operator is polynomially compact, with polynomial $\mathbf{p}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}(\boldsymbol{\lambda}^2 - \Bbbk^2)$, the essential spectrum consists of 3 points $0; \pm\Bbbk$.

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The operator is polynomially compact, with polynomial $\mathbf{p}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}(\boldsymbol{\lambda}^2 - \Bbbk^2)$, the essential spectrum consists of 3 points $0; \pm\Bbbk$.
Question: Find information on eigenvalues.

To find the asymptotics of eigenvalues: 1. Find the subprincipal symbol of $\mathfrak{K}$. 2. Perform the calculations for the principal symbol of $\mathbf{p}_\nu(\mathfrak{K})$, $\nu = 1, 2, 3$ of degree 5. 3. Apply the general theorem on eigenvalue asymptotics.

In the general case– terrible calculations. Soft analysis to reduce the problem to an easier one. Recall: the structure of the symbol of $\mathfrak{M}_\nu = \mathbf{p}_\nu(\mathfrak{K})$. Many copies of the principal symbol and only one or 2 other entries. Then (by the general rule)- to find eigenvalues of these matrix symbols, in a symbolic form, integrate etc.

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The results of calculations are indistinct, obscure, and are impossible to understand. Therefore, we use a softer analysis, enabling us to obtain the properties of these symbols on the qualitative level, without hard calculations.

The analysis of the structure of the symbol $\mathfrak{m}_\nu(x, \xi)$ is performed on the basis of the general form of the symbol and its expression as integral operator. We have an integral operator, represent it as a pseudodifferential and find what the symbol can depend on.

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Theorem

The principal, order -1, symbol $\mathfrak{m}_\nu(x,\xi)$ is a linear function of the principal curvatures of the surface Γ at x, in some special co-ordinate system,

$$\mathfrak{m}_\nu(x,\xi) = \mathbf{k}_1(x)\mathbf{M}_{1,\nu}(\xi) + \mathbf{k}_2(x)\mathbf{M}_{2,\nu}(\xi),$$

where $\mathbf{M}_1, \mathbf{M}_2$ are some universal 3×3 matrices, depending only on the material constants λ, μ but not on the geometry of the body. Moreover, $\mathbf{M}_2(\xi_1,\xi_2) = V\mathbf{M}_1(\xi_2,\xi_1)V$, $V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So, it is only $\mathbf{M}_1(\xi)$ is needed!

More information about $\mathbf{M}(\xi)$ is obtained from the case of the sphere. $\mathbf{M}$ is a nonnegative matrix. But There is a task left for a harder analysis. Find the explicit expression for the matrix $\mathbf{M}_\nu(\xi)$ via the Lame constants. Here, since it does not depend on the surface, we may choose the surface in such way that the calculations are most simple.

More information about $\mathbf{M}(\xi)$ is obtained from the case of the sphere. $\mathbf{M}$ is a nonnegative matrix. But There is a task left for a harder analysis. Find the explicit expression for the matrix $\mathbf{M}_\nu(\xi)$ via the Lame constants. Here, since it does not depend on the surface, we may choose the surface in such way that the calculations are most simple. A cylinder. One of principal curvatures vanishes. Derivative in the axis direction vanishes, and some more terms vanish. $\mathbf{M}(\xi) = \mathbf{k}^{-1}\mathfrak{k}(\xi)$. But still a lot of calculations. There are symbols

$$\mathfrak{k}_0(\xi), \partial_{\xi_1}\mathfrak{k}_0(\xi), \partial x_1\mathfrak{k}_0(\xi), \mathfrak{k}_{-1}(\xi).$$

Multiply these matrices in proper order.

1. 4 copies of $\mathfrak{k}_0(\xi)$ and one copy of $\mathfrak{k}_{-1}(\xi)$ in all possible orders,
2. 3 copies of $\mathfrak{k}_0(\xi)$, 1 copy of $\partial_{\xi_1}\mathfrak{k}_0(\xi)$, and 1 copy of $\partial_{x_1}\mathfrak{k}_0(\xi)$ in all possible orders.

And add them.

1. For any smooth bounded body there always exist infinitely many eigenvalues of the NP operator, approaching each point of the essential spectrum from above.
2. If there exists at least one point on Γ where the body is strictly concave, i.e., both principal curvatures are nonnegative, while at least one is positive, then there also exist infinitely many eigenvalues of the NP operator approaching the points of the essential spectrum from below.
3. The leading term in the asymptotics of eigenvalues approaching 0 does not depend on Lamé parameters.

Extensions. The case where the polynomially compact approach does not work – the nonhomogeneous elasticity. The principal symbol of the NP operator

$$\mathfrak{t}_0(x,\xi) = \frac{i\mathbb{k}(x)}{|\xi|} \begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}, \; \mathbb{k}(x) = \frac{\mu(x)}{2(\lambda(x) + 2\mu(x))}.$$

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The essential spectrum, of this operator consists of the point 0 ant two symmetric intervals $\pm(\min_{x\in\Gamma}\Bbbk(x), \max_{x\in\Gamma}\Bbbk(x))$. The question consists in finding the asymptotics of eigenvalues outside these intervals, converging to the endpoints.

$$.......-\Bbbk_+ \eqcirc\eqcirc\eqcirc\eqcirc\eqcirc\eqcirc\eqcirc -\Bbbk_-.......0.......\Bbbk_- \eqcirc\eqcirc\eqcirc\eqcirc\eqcirc\eqcirc\eqcirc \Bbbk_+.......$$

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If, say, $\Bbbk(x)$ has a nondegenerate minimum at x^-,
$\Bbbk(x) = \Bbbk_- + s|x - x^-|^2 + o(|x - x^-|^2)$.

$$..0.......\tau_0.....\bigstar....\bigstar....\bigstar..\Bbbk_- - \tau.....\Bbbk_- \circeq\circeq\circeq\circeq\circeq$$

To study eigenvalues: Local model using the Fourier transform.

Local model: $(\mathbf{H}_- u)(y) = \Bbbk_- - s\Delta u - p|y|^{-1}$, $y = (y_1, y_2) \in R^2$ in $L^2(\mathbb{R}^2)$, a Schrödinger operator with negative potential tending to zero at infinity slowly.

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A classical problem, studied since mid - XX century. The Weyl type asymptotics. Phase space volume:
$H(y,\eta)_- = \Bbbk_- + |\chi|^2 - |y|^{-1}$ symbol, The number of eigenvalues of $\mathbf{H}_-$ below $\Bbbk_- - \tau$ equals, asymptotically,

$$N(\mathbf{H}_-, \Bbbk_- - \tau) \sim C \int_{H_-(y,\eta) < \Bbbk_- - \tau} 1 dy d\eta.$$

Local model: $(\mathbf{H}_{-}u)(y) = \Bbbk_{-} - s\Delta u - p|y|^{-1}$, $y = (y_1, y_2) \in R^2$ in $L^2(\mathbb{R}^2)$, a Schrödinger operator with negative potential tending to zero at infinity slowly.
A classical problem, studied since mid - XX century. The Weyl type asymptotics. Phase space volume:
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$$N(\mathbf{H}_{-}, \Bbbk_{-} - \tau) \sim C \int_{H_{-}(y,\eta) < \Bbbk_{-} - \tau} 1 dy d\eta.$$

calculation gives $N(\mathbf{H}_{-}, \Bbbk_{-} - \tau) \sim C\tau^{-1}$, $\tau \to 0$, or that eigenvalues tend to $\Bbbk_{-}$ as $\Bbbk_{-} - \tau_j \sim j^{-1}$. It is in order twice as fast as for the homogeneous body.

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Thank you!!!