Friedrichs Extension and Min–Max Principle for Operators with a Gap

joint work with J. P. Solovej and S. Tokus

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Overview

- The Friedrichs extension of a semibounded operator and its variational principle
- A similar result for a gapped operator
- Applying the result to the Coulomb–Dirac operator
- Applying the result to Dirac operators on manifolds with boundary
- Main proof idea and comparison

The Friedrichs extension of semibounded operators

Let A be a symmetric operator on a domain $D(A) \subset \mathcal{H}$. In Mathematical Physics it is often of importance to find a (distinguished) self-adjoint extension of A.

Theorem (Friedrichs 1934)

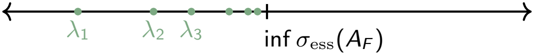
Let $A : D(A) \to \mathcal{H}$ be symmetric and bounded from below

$$\lambda_1 = \inf_{z \in D(A)} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|_{\mathcal{H}}^2} > -\infty \,.$$

Then there exists a distinguished self-adjoint extension A_F such that for $k \geq 1$

$$\lambda_k = \inf_{\substack{V \subset D(A) \\ \dim V = k}} \sup_{z \in V} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|_{\mathcal{H}}^2}$$

are all the eigenvalues of A_F below the essential spectrum of A_F.



Note that we only need $D(A)$ to compute the eigenvalues of A_F. The proof is based on the fact that a closed quadratic form corresponds to a self-adjoint operator.

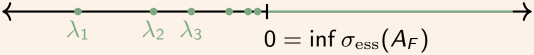
The Friedrichs extension of a Schrödinger operator

An application is the Schrödinger operator

$$A = -\Delta + V \quad \text{on} \quad \mathcal{C}_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)\,.$$

Example: Coulomb potential $V = -\frac{\nu}{|x|}$

By Hardy's inequality/Sobolev inequality $\lambda_1 > -\infty$ for any $\nu \in \mathbb{R}$.



$$\lambda_k = \inf_{\substack{V \subset \mathcal{C}_0^\infty(\mathbb{R}^3) \\ \dim V = k}} \sup_{z \in V} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|_{\mathcal{H}}^2}\,.$$

The main result for operators with a gap

Operators with gaps

Friedrichs' construction is not applicable to operators that only have a finite gap

$$\left\| \left(A - \frac{\lambda_0 + \lambda_1}{2} \right) z \right\|_{\mathcal{H}} \geq \frac{\lambda_1 - \lambda_0}{2} \|z\|_{\mathcal{H}} , \quad \forall z \in D(A) .$$

Example: Dirac operator

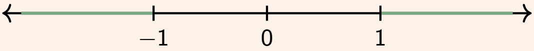
On $\mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4)$ we consider

$$A = \begin{pmatrix} 1 & \sigma \cdot \mathrm{i}\nabla \\ \sigma \cdot \mathrm{i}\nabla & -1 \end{pmatrix} .$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The operator is only essentially self-adjoint and satisfies a gap condition with $\lambda_0 = -1, \lambda_1 = 1$.



Operators with a gap

Friedrichs' construction is not applicable to operators that only have a finite gap

$$\left\| \left(A - \frac{\lambda_0 + \lambda_1}{2} \right) z \right\|_{\mathcal{H}} \geq \frac{\lambda_1 - \lambda_0}{2} \|z\|_{\mathcal{H}}, \quad \forall z \in D(A).$$

Example: Coulomb–Dirac operator

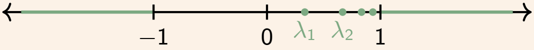
On $\mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^4) \subset L^2(\mathbb{R}^3;\mathbb{C}^4)$ we consider

$$A = \begin{pmatrix} 1 - \frac{\nu}{|x|} & \sigma \cdot \mathrm{i}\nabla \\ \sigma \cdot \mathrm{i}\nabla & -1 - \frac{\nu}{|x|} \end{pmatrix}.$$

Here $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator is only essentially self-adjoint for $\nu \leq \sqrt{3}/2$ but satisfies a gap condition with $\lambda_0 = -1, \lambda_1 = 0$ for $\nu \leq 1$.



In a similar way to the semibounded case, one would like to solve the following problems.

Operators with a gap

Friedrichs' construction is not applicable to symmetric operators with finite gap

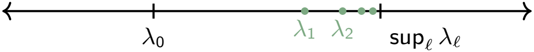
$$\left\| \left(A - \frac{\lambda_0 + \lambda_1}{2} \right) z \right\|_{\mathcal{H}} \geq \frac{\lambda_1 - \lambda_0}{2} \|z\|_{\mathcal{H}}, \quad \forall z \in D(A).$$

Problem 1

Define a distinguished self-adjoint extension A_F of A that preserves the gap.

Problem 2

Provide a simple variational principle that allows to compute the eigenvalues of A_F, ideally only from the symmetric operator A.



There are results establishing distinguished extensions[1] and variational principles[2] for (essentially) self-adjoint operators but no general results establishing a connection.

[1] Krein 1947; Esteban & Loss 2008

[2] Griesemer & Siedentop 1999; Dolbeault, Esteban & Séré 2000; Kraus, Langer & Tretter 2004; Morozov & Müller 2015

Operators with gaps: Problem 1

Problem 1

Define a distinguished self-adjoint extension A_F of A that preserves the gap.

For the Coulomb–Dirac operator:

- For $\nu < 1$ a distinguished extension with both finite kinetic and finite potential energy was found by Schmincke, Nenciu and Wüst.
- For $\nu = 1$ a distinguished extension was defined by Esteban and Loss[3] by using a stronger gap condition.

In the general case:

- Existence of extensions with gap was proved by Krein.[4]
- Esteban and Loss[5] constructed a distinguished gap-preserving self-adjoint extension using stronger gap conditions.

[3] Esteban & Loss 2006

[4] Krein 1947

[5] Esteban & Loss 2008

Operators with gaps: Problem 2

Problem 2

Provide a simple variational principle that allows to compute the eigenvalues of A_F, ideally only from the symmetric operator A.

For the Coulomb–Dirac operator:

- Talman as well as Datta and Deviah suggested to split the optimisation in the variational principle

$$\lambda_1 = \inf_{\psi \in F_+} \sup_{\varphi \in F_-} \frac{\left\langle \binom{\psi}{\varphi}, A\binom{\psi}{\varphi} \right\rangle_2}{\|\psi\|_2^2 + \|\varphi\|_2^2} .$$

- For $\nu \leq 1$ Esteban, Lewin and Séré[6] proved this for the distinguished extension with $F_\pm = \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$.

In the general case:

- There are results establishing variational principles[7] for (essentially) self-adjoint operators.

Only for the Coulomb–Dirac operator do we know a connection between Problem 1 and 2.

[6] Esteban, Lewin & Séré 2019

[7] Griesemer & Siedentop 1999; Dolbeault, Esteban & Séré 2000; Kraus, Langer & Tretter 2004; Morozov & Müller 2015

Operators with gaps: Problem 2

Problem 2

Provide a simple variational principle that allows to compute the eigenvalues of A_F, ideally only from the symmetric operator A.

For the Coulomb–Dirac operator:

- Talman as well as Datta and Deviah suggested to split the optimisation in the variational principle

$$\lambda_k = \inf_{\substack{V \subset F_+ \\ \dim V = k}} \sup_{\binom{\psi}{\varphi} \in (V \times F_-) \setminus \{0\}} \frac{\langle \binom{\psi}{\varphi}, A \binom{\psi}{\varphi} \rangle_2}{\|\psi\|_2^2 + \|\varphi\|_2^2}$$

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The main result

Theorem (LS, Solovej and Tokus, 2020)

Let A be a densely defined symmetric operator on $D(A) \subset \mathcal{H}$ such that:

1. **Orthogonal decomposition:** *There are orthogonal projections $\Lambda_\pm$ on $\mathcal{H}$ such that*

$$\mathcal{H} = \Lambda_+\mathcal{H} \oplus \Lambda_-\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \,, \quad \text{and} \quad F_\pm := \Lambda_\pm D(A) \subset D(A)\,.$$

2. **Gap condition:**

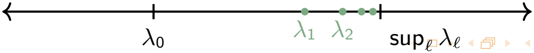
$$\sup_{y_- \in F_- \setminus \{0\}} \frac{\langle y_-, Ay_- \rangle_{\mathcal{H}}}{\|y_-\|_{\mathcal{H}}^2} =: \lambda_0 < \lambda_1 := \inf_{x_+ \in F_+ \setminus \{0\}} \sup_{y_- \in F_-} \frac{\langle x_+ + y_-, A(x_+ + y_-) \rangle_{\mathcal{H}}}{\|x_+ + y_-\|_{\mathcal{H}}^2}\,.$$

3. *The operator $\Lambda_- A|_{F_-} : F_- \to \mathcal{H}_-$ is essentially self-adjoint.*

Then there exists a self-adjoint extension A_F of A such that for $k \geq 1$ the numbers

$$\lambda_k := \inf_{\substack{V \subset F_+ \\ \dim V = k}} \sup_{z \in (V \oplus F_-) \setminus \{0\}} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|_{\mathcal{H}}^2}$$

are the eigenvalues of A_F in the set $(\lambda_0, \sup_{\ell \geq 1} \lambda_\ell)$.



Application to Coulomb–Dirac operators

Application to the Coulomb–Dirac operator: 1st condition

1st condition of the theorem

1. **Orthogonal decomposition:** There are orthogonal projections $\Lambda_\pm$ on $\mathcal{H}$ such that

$$\mathcal{H} = \Lambda_+\mathcal{H} \oplus \Lambda_-\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \,, \quad \text{and} \quad F_\pm := \Lambda_\pm D(A) \subset D(A)\,.$$

This allows us to write the operator as a block-matrix operator

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

For the Coulomb–Dirac operator on $\mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^4) \subset L^2(\mathbb{R}^3;\mathbb{C}^4)$ we consider

$$A = \begin{pmatrix} 1 - \frac{\nu}{|x|} & \sigma \cdot \mathrm{i}\nabla \\ \sigma \cdot \mathrm{i}\nabla & -1 - \frac{\nu}{|x|} \end{pmatrix}.$$

We can choose the Talman projections

$$\Lambda_+ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \quad \Lambda_- \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \quad \Lambda_+\mathcal{C}_0^\infty(\mathbb{R}^3,\mathbb{C}^4) = \mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^2) \times \{0\}\,.$$

Application to the Coulomb–Dirac operator: 3rd condition

3rd condition of the theorem

3. The operator $A_{--} : F_- \to \mathcal{H}_-$ is essentially self-adjoint.

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

For the Coulomb–Dirac operator on $\mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \subset L^2(\mathbb{R}^3; \mathbb{C}^4)$ we consider

$$A = \begin{pmatrix} 1 - \frac{\nu}{|x|} & \sigma \cdot \mathrm{i}\nabla \\ \sigma \cdot \mathrm{i}\nabla & -1 - \frac{\nu}{|x|} \end{pmatrix}.$$

If we choose the Talman projections then $A_{--} = -1 - \frac{\nu}{|x|}|_{\mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^2)}$ is essentially self-adjoint.

Application to the Coulomb–Dirac operator: 2nd condition

2nd condition of the theorem

2. **Gap condition:**

$$\sup_{y_- \in F_- \setminus \{0\}} \frac{\langle y_-, Ay_- \rangle_{\mathcal{H}}}{\|y_-\|_{\mathcal{H}}^2} =: \lambda_0 < \lambda_1 := \inf_{x_+ \in F_+ \setminus \{0\}} \sup_{y_- \in F_-} \frac{\langle x_+ + y_-, A(x_+ + y_-) \rangle_{\mathcal{H}}}{\|x_+ + y_-\|_{\mathcal{H}}^2} .$$

It is possible to show that this condition implies

$$\left\| \left(A - \frac{\lambda_0 + \lambda_1}{2} \right) z \right\|_{\mathcal{H}} \geq \frac{\lambda_1 - \lambda_0}{2} \|z\|_{\mathcal{H}} , \quad \forall z \in D(A) .$$

[8] Dolbeault, Esteban, Loss & Vega 2004

Application to the Coulomb–Dirac operator: 2nd condition

2nd condition of the theorem

2. **Gap condition:**

$$\sup_{y_-\in F_-\setminus\{0\}} \frac{\langle y_-, Ay_-\rangle_{\mathcal{H}}}{\|y_-\|^2_{\mathcal{H}}} =: \lambda_0 < \lambda_1 := \inf_{x_+\in F_+\setminus\{0\}} \sup_{y_-\in F_-} \frac{\langle x_+ + y_-, A(x_+ + y_-)\rangle_{\mathcal{H}}}{\|x_+ + y_-\|^2_{\mathcal{H}}} .$$

It is equivalent to $\lambda_0 < \infty$ and

$$q_E(x_+, x_+) := \langle x_+, (A_{++} - E)x_+\rangle_{\mathcal{H}} - \langle A_{-+}x_+, (A_{--} - E)^{-1} A_{-+}x_+\rangle_{\mathcal{H}} \geq 0$$

for some $\lambda_0 < E$ (and then for all $\lambda_0 < E \leq \lambda_1$).

For the Coulomb–Dirac operator in the case of Talman projections this is a consequence of the Dirac–Hardy inequality[8]

$$q_0(\psi,\psi) = \int_{\mathbb{R}^3} \frac{|\sigma\cdot\nabla\psi(x)|^2}{1+\frac{1}{|x|}}\,\mathrm{d}x + \int_{\mathbb{R}^3} \left(1 - \frac{1}{|x|}\right)|\psi(x)|^2\,\mathrm{d}x \geq 0\,.$$

[8] Dolbeault, Esteban, Loss & Vega 2004

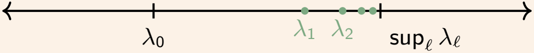
Application to the Coulomb–Dirac operator: Result

Result of the theorem

Then there exists a self-adjoint extension A_F of A such that for $k \geq 1$ the numbers

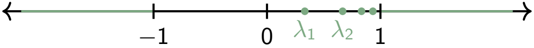
$$\lambda_k := \inf_{\substack{V \subset F_+ \\ \dim V = k}} \sup_{z \in (V \oplus F_-) \setminus \{0\}} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|_{\mathcal{H}}^2}$$

are the eigenvalues of A_F in the set $(\lambda_0, \sup_{\ell \geq 1} \lambda_\ell)$ and $\sup_\ell \lambda_\ell$.



For the Coulomb–Dirac operator we obtain the well-known distinguished self-adjoint extension[9] and the variational principle of Esteban, Lewin and Séré[10]

$$\lambda_k = \inf_{\substack{V \subset C_0^\infty(\mathbb{R}^3;\mathbb{C}^2) \\ \dim V = k}} \sup_{\binom{\psi}{\varphi} \in (V \times C_0^\infty(\mathbb{R}^3;\mathbb{C}^2)) \setminus \{0\}} \frac{\langle \binom{\psi}{\varphi}, A \binom{\psi}{\varphi} \rangle_2}{\|\psi\|_2^2 + \|\varphi\|_2^2} .$$



[9] Schmincke 1972; Nenciu 1976; Wüst 1975; Esteban & Loss 2007

[10] Esteban, Lewin & Séré 2019

Application to the Coulomb–Dirac operator: Result

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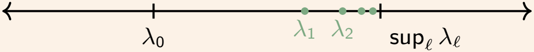
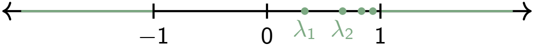
are the eigenvalues of A_F in the set $(\lambda_0, \sup_{\ell \geq 1} \lambda_\ell)$ and $\sup_\ell \lambda_\ell$.



If $\Lambda_\pm$ are the projections onto the positive/negative spectral subspace of the free Dirac operator, we obtain the same self-adjoint extension and the variational principle[11]

$$\lambda_k = \inf_{\substack{V \subset \Lambda_+ \mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^4) \\ \dim V = k}} \sup_{\psi \in (V \oplus \Lambda_- \mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^4)) \setminus \{0\}} \frac{\langle \psi, A\psi \rangle_2}{\|\psi\|_2} .$$



[11] Esteban, Lewin & Séré 2019

Application to Dirac operators on manifolds with boundary

Application to Dirac-type operators (product case)

We consider the Dirac operator on a finite cylinder $(0,1)\times\mathbb{S}^1$.

$x \in (0,1)$

$y \in \mathbb{S}^1$ $\qquad A = \mathrm{i}\sigma_2(\partial_x + \mathrm{i}\sigma_3\partial_y)$

The operator is symmetric on $\mathcal{C}_0^1((0,1); D(\mathrm{i}\sigma_3\partial_y)) \subset L^2((0,1); L^2(\mathbb{S}^1)) \cong L^2((0,1)\times\mathbb{S}^1)$.

Choice of Λ_+:

The eigenvalues of $\mathrm{i}\sigma_3\partial_y$ are $k\in\mathbb{Z}$ with eigenfunctions

$$\begin{pmatrix} \mathrm{e}^{-\mathrm{i}ky} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}ky} \end{pmatrix}$$

Let Λ_+ be the orthogonal projection onto

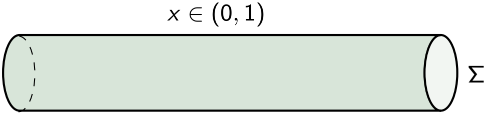
$$\mathrm{span}\left\{ \begin{pmatrix} \mathrm{e}^{-\mathrm{i}ky} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}ky} \end{pmatrix} \,:\, k>0 \right\} \oplus \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

i.e. Λ_+ maps

$$\sum_{k\in\mathbb{Z}} u_k(x)\begin{pmatrix} \mathrm{e}^{-\mathrm{i}ky} \\ 0 \end{pmatrix} + \sum_{k\in\mathbb{Z}} v_k(x)\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}ky} \end{pmatrix} \mapsto \sum_{k>0} u_k(x)\begin{pmatrix} \mathrm{e}^{-\mathrm{i}ky} \\ 0 \end{pmatrix} + \sum_{k>0} v_k(x)\begin{pmatrix} 0 \\ \mathrm{e}^{\mathrm{i}ky} \end{pmatrix} + u_0(x)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Application to Dirac-type operators (product case)

More generally, we consider a Dirac-type operator on a cylinder $(0,1) \times \Sigma$



$$A = \sigma(\partial_x + B)$$

The operator A is symmetric on $\mathcal{C}_0^1((0,1); D(B)) \subset L^2((0,1); \mathcal{K})$ where $\mathcal{K} = L^2(\Sigma)$.

Properties of $B : D(B) \subset \mathcal{K} \to \mathcal{K}$:

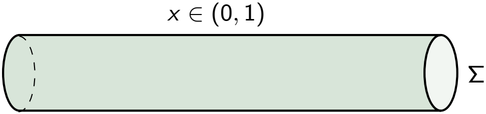
- is independent of x, is self-adjoint, has discrete spectrum (Ψ_k, ℓ_k) with $k \in \mathbb{Z}$
- $\dim \ker B < \infty$, $\ker B = \mathcal{N}_+ \oplus \mathcal{N}_-$.

Properties of $\sigma : \mathcal{K} \to \mathcal{K}$:

- is independent of x, is an automorphism,
- $\sigma^2 = -\mathbb{I}$, $\sigma^* = -\sigma$,
- $\{B, \sigma\} = B\sigma + \sigma B = 0$,
- $\sigma(\mathcal{N}_-) = \mathcal{N}_+$.

Application to Dirac-type operators (product case)

More generally, we consider a Dirac-type operator on a cylinder $(0,1) \times \Sigma$



$$A = \sigma(\partial_x + B)$$

The operator A is symmetric on $\mathcal{C}_0^1((0,1); D(B)) \subset L^2((0,1); \mathcal{K})$ where $\mathcal{K} = L^2(\Sigma)$.

Choice of Λ_+:

B has discrete spectrum

$$\{(\Psi_k, \ell_k) : \ k \in \mathbb{Z}\} \ .$$

Let Λ_+ be the orthogonal projection onto

$$\mathrm{span}\,\{\Psi_k : \ \ell_k > 0\} \oplus \mathcal{N}_+.$$

i.e. Λ_+ maps

$$\sum_{k \in \mathbb{Z}} u_k(x)\Psi_k \mapsto \sum_{k \in \mathbb{Z}: \ell_k > 0} u_k(x)\Psi_k + \sum_{k \in \mathbb{Z}: \Psi_k \in \mathcal{N}_+} u_k(x)\Psi_k \ .$$

Application to Dirac-type operators (product case): 1st condition

1st condition of the theorem

1. **Orthogonal decomposition:** There are orthogonal projections $\Lambda_\pm$ on $\mathcal{H}$ such that

$$\mathcal{H} = \Lambda_+\mathcal{H} \oplus \Lambda_-\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \,, \quad \text{and} \quad F_\pm := \Lambda_\pm D(A) \subset D(A)\,.$$

For a Dirac operator on a cylinder, it holds that

$$\Lambda_\pm \mathcal{C}_0^1((0,1); D(B)) \subset \mathcal{C}_0^1((0,1); D(B))\,.$$

Application to Dirac-type operators (product case): 3rd condition

3rd condition of the theorem

3. The operator $A_{--} : F_- \to \mathcal{H}_-$ is essentially self-adjoint.

For a Dirac operator on a cylinder, the diagonal entry A_{--} vanishes and is thus essentially self-adjoint

$$\Lambda_- \sigma(\partial_x + B)\Lambda_- = \sigma\Lambda_+(\partial_x + B)\Lambda_- = 0\,.$$

Application to Dirac-type operators (product case): 2nd condition

2nd condition of the theorem

2. **Gap condition:**

$$\sup_{y_-\in F_-\setminus\{0\}} \frac{\langle y_-, Ay_-\rangle_{\mathcal{H}}}{\|y_-\|_{\mathcal{H}}^2} =: \lambda_0 < \lambda_1 := \inf_{x_+\in F_+\setminus\{0\}} \sup_{y_-\in F_-} \frac{\langle x_+ + y_-, A(x_+ + y_-)\rangle_{\mathcal{H}}}{\|x_+ + y_-\|_{\mathcal{H}}^2} .$$

For a Dirac-type operator on a cylinder we have $\lambda_0 = 0$ and can compute explicitly that

$$\lambda_1 = \inf_{u\in C_0^1(0,1)\setminus\{0\}} \sup_{k>0} \left[\ell_k^2 + \frac{\|-\partial_x u\|_2^2}{\|u\|_2^2}\right]^{\frac{1}{2}} \geq \pi .$$

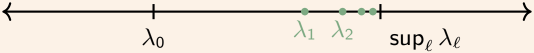
Application to Dirac-type operators (product case): Result

Result of the theorem

Then there exists a self-adjoint extension A_F of A such that for $k \geq 1$ the numbers

$$\lambda_k := \inf_{\substack{V \subset F_+ \\ \dim V = k}} \sup_{z \in (V \oplus F_-) \setminus \{0\}} \frac{\langle z, Az \rangle_{\mathcal{H}}}{\|z\|^2_{\mathcal{H}}}$$

are the eigenvalues of A_F in the set $(\lambda_0, \sup_{\ell \geq 1} \lambda_\ell)$ and $\sup_\ell \lambda_\ell$.



For a Dirac-type operator on a cylinder we obtain the following.

Theorem (LS, Solovej & Tokus, 2020)

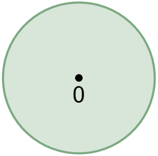
A_F is characterised by the Atiyah–Patodi–Singer boundary condition[a]*, i.e.*

$$D(A_F) = \left\{ \psi \in L^2((0,1); D(B)) \cap H^1((0,1); \mathcal{K}) : \Lambda_+ \psi|_{x=0} = \Lambda_+ \psi|_{x=1} = 0 \right\}.$$

[a] Atiyah, Patodi & Singer 1975

Application to Dirac-type operators (not product case)

We consider the Dirac operator on a disk



$|z| \leq 1$

$$A = \begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\overline{z}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}e^{-\mathrm{i}\varphi} \\ -\mathrm{i}e^{\mathrm{i}\varphi} & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}\sigma_3 \mathrm{i}\partial_\varphi \right)$$

Choice of Λ_+:

Let Λ_+ to be the orthogonal projection onto the positive eigenspace of $\sigma_3 \mathrm{i}\partial_\varphi$

$$\operatorname{span}\left\{ \begin{pmatrix} e^{-\mathrm{i}k\varphi} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^{\mathrm{i}k\varphi} \end{pmatrix} : k > 0 \right\}$$

then all the conditions of the main theorem are satisfied.

Theorem

A_F is characterised by the Atiyah–Patodi–Singer boundary condition[a] $\Lambda_+\psi|_{r=1} = 0$.

[a] Atiyah, Patodi & Singer 1975

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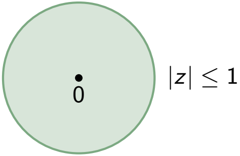

$$A = \begin{pmatrix} 0 & 2\partial_z \\ -2\partial_{\overline{z}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}e^{-\mathrm{i}\varphi} \\ -\mathrm{i}e^{\mathrm{i}\varphi} & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}\sigma_3 \mathrm{i}\partial_\varphi \right)$$

Choice of Λ_+:

Let Λ_+ to be the orthogonal projection onto the first component

$$\Lambda_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

then all the conditions of the main theorem are satisfied.

Theorem

A_F is characterised by the Zig-Zag boundary condition[a] $\Lambda_+ \psi|_{r=1} = 0$.

[a] Benguria, Fournais, Stockmeyer & Van Den Bosch 2017

Generalising the main theorem: Replacing the first condition

Theorem

The orthogonality condition can be replaced by

1. **Decomposition:** *There are bounded operators* $\Gamma_\pm$ *with*

$$\Gamma_+^*\Gamma_+ + \Gamma_-^*\Gamma_- = \mathbb{I} \quad \text{and} \quad \Gamma_\pm^*\Gamma_\pm D(A) \subset D(A)\,.$$

Adapting the conditions in a natural way, there exists a self-adjoint extension A_F *of* A *such that for* $k \geq 1$ *the numbers*

$$\lambda_k := \inf_{\substack{V \subset \Gamma_+ D(A) \\ \dim V = k}} \sup_{x_+ \in V \setminus \{0\}, y_- \in \Gamma_- D(A)} \frac{\langle \Gamma_+^* x_+ + \Gamma_-^* y_-, A(\Gamma_+^* x_+ + \Gamma_-^* y_-)\rangle_{\mathcal{H}}}{\|x_+\|_{\mathcal{H}}^2 + \|y_-\|_{\mathcal{H}}^2}$$

are the eigenvalues of A_F *in the set* $(\lambda_0, \sup_{\ell \geq 1} \lambda_\ell)$.

The main proof idea

The main proof idea

If A were a bounded self-adjoint operator ($F_\pm = \mathcal{H}_\pm$), then for any $E \notin \sigma(A_{--})$

$$\begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} - E\mathbb{I} = \begin{pmatrix} \mathbb{I} & -L_E^* \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} Q_E & 0 \\ 0 & -(B+E) \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -L_E & \mathbb{I} \end{pmatrix}$$

holds where

$$\begin{aligned} B &= -A_{--}\,, \\ L_E &= (B+E)^{-1} A_{-+}\,, \\ Q_E &= (A_{++} - E) + A_{+-}(B+E)^{-1} A_{-+}\,. \end{aligned}$$

The construction of B in the unbounded case

B is the unique self-adjoint extension of $-A_{--}$.

The main proof idea

If A were a bounded self-adjoint operator ($F_\pm = \mathcal{H}_\pm$), then for any $E \notin \sigma(A_{--})$

$$\begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} - E\mathbb{I} = \begin{pmatrix} \mathbb{I} & -L_E^* \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} Q_E & 0 \\ 0 & -(B+E) \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -L_E & \mathbb{I} \end{pmatrix}$$

holds where

$$B = -A_{--}\,,$$
$$L_E = (B+E)^{-1} A_{-+}\,,$$
$$Q_E = (A_{++} - E) + A_{+-}(B+E)^{-1} A_{-+}\,.$$

The construction of L_E in the unbounded case

L_E is the closure of $(B+E)^{-1}A_{-+}$.

The main proof idea

If A were a bounded self-adjoint operator ($F_\pm = \mathcal{H}_\pm$), then for any $E \notin \sigma(A_{--})$

$$\begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix} - E\mathbb{I} = \begin{pmatrix} \mathbb{I} & -L_E^* \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} Q_E & 0 \\ 0 & -(B+E) \end{pmatrix} \begin{pmatrix} \mathbb{I} & 0 \\ -L_E & \mathbb{I} \end{pmatrix}$$

holds where

$$B = -A_{--}\,,$$
$$L_E = (B+E)^{-1} A_{-+}\,,$$
$$Q_E = (A_{++} - E) + A_{+-}(B+E)^{-1} A_{-+}\,,$$

The construction of Q_E in the unbounded case

We know that on F_+

$$q_E(x_+, x_+) = \langle x_+, (A_{++} - E)x_+ \rangle_{\mathcal{H}} + \langle A_{-+}x_+, (B+E)^{-1} A_{-+}x_+ \rangle_{\mathcal{H}} \geq 0\,.$$

- The form is not necessarily closable on $\mathcal{H}_+$.
- It is closable on $D(L_E)$ with the graph norm $\|\cdot\|_E$ of L_E.
- On the Hilbert space $(D(L_E), \|\cdot\|_E)$ the form is bounded from below for all $E > \lambda_0$.
- We define Q_E to be the unique self-adjoint operator associated to q_E on $D(L_E)$.
- λ_k is the unique number $\lambda > \lambda_0$ such that $\mu_k(Q_\lambda) = 0$.

The main proof idea: A bit more on Q_E

- The completion $\mathcal{F}_+$ of $F_+ = \Lambda_+ D(A)$ with respect to q_E in the above space is independent of E.
- By Riesz there exists an operator $\widehat{T_E} : (\mathcal{F}_+, q_E) \to (\mathcal{F}_+, q_E)'$ such that

$$[\widehat{T_E}(x_+)](y_+) = q_E(x_+, y_+) + K_E \langle x_+, y_+ \rangle_E$$

- Using the embedding $j : D(L_E) \to \mathcal{F}_+'$ the operator $Q_E = j^{-1}\widehat{T_E} - K_E$ is defined on

$$D(Q_E) = \left\{ x_+ \in \mathcal{F}_+ : \ \widehat{T_E} x_+ \in j(D(L_E)) \right\}$$

- Using the embedding $j : \mathcal{H}_+ \to \mathcal{F}_+'$ the operator $j^{-1}[\widehat{Q_E} + L_E'(B+E)(y_- - L_E x_+)]$ is well-defined on

$$\begin{aligned} D(A_F) = \{ x_+ + y_- \in \mathcal{F}_+ \oplus \mathcal{H}_- : y_- - L_E x_+ \in D(B) \\ \widehat{Q_E} + L_E'(B+E)(y_- - L_E x_+) \in j(\mathcal{H}_+) \} \end{aligned}$$

Here L_E' denotes the adjoint of $L_E : \mathcal{F}_+ \to \mathcal{H}_-$.

$$\begin{array}{ccccc} \mathcal{F}_+ & \xrightarrow{\mathbb{I}} & D(L_E) & \xrightarrow{\mathbb{I}} & \mathcal{H}_+ \\ \downarrow{\scriptstyle \widehat{T_E}} & & \downarrow & & \downarrow \\ \mathcal{F}_+' & \longleftarrow & D(L_E)' & \longleftarrow & \mathcal{H}_+' = \mathcal{H}_+ \end{array}$$

Comparison to known results

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Comparison to the self-adjoint extension of Esteban and Loss[12]:

- We do not require the off-diagonal operators to be self-adjoint.
- We do not require $A_{+-}(B+E)^{-1}A_{-+}$ to be well-defined.
- In our work the introduction of the Hilbert space $(D(L_E), \|\cdot\|_E)$ is necessary to guarantee that
  - q_E is closable,
  - the construction is independent of E,
  - the variational principle holds.

Comparison to the results of Esteban, Lewin and Séré for the Coulomb–Dirac operator[13]:

- In this special case q_E is closable on $\mathcal{H}_+$.
- In this special case the closure of q_E on $\mathcal{H}_+$ coincides with the closure of q_E on $(D(L_E), \|\cdot\|_E)$.
- We thus recover the same self-adjoint extension and variational principle.

[12] Esteban & Loss 2008

[13] Esteban, Lewin & Séré 2019

Thank you for your attention!