

Homogenization of the Periodic Schrödinger-type Equations

Tatiana Suslina

St. Petersburg State University

Workshop on Perturbation
of Spectral Bands and Gaps
Dortmund

Statement of the problem

Let Γ be a lattice in \mathbb{R}^d , let Ω be the cell of Γ .

Statement of the problem

Let Γ be a lattice in \mathbb{R}^d , let Ω be the cell of Γ . Let $\tilde{\Gamma}$ be the dual lattice. By $\tilde{\Omega}$ we denote the central Brillouin zone of $\tilde{\Gamma}$.

Statement of the problem

Let Γ be a lattice in \mathbb{R}^d , let Ω be the cell of Γ . Let $\tilde{\Gamma}$ be the dual lattice. By $\tilde{\Omega}$ we denote the central Brillouin zone of $\tilde{\Gamma}$.

Example: $\Gamma = \mathbb{Z}^d$, $\Omega = [0, 1)^d$, $\tilde{\Gamma} = (2\pi\mathbb{Z})^d$, $\tilde{\Omega} = (-\pi, \pi)^d$.

Statement of the problem

Let Γ be a lattice in \mathbb{R}^d , let Ω be the cell of Γ . Let $\tilde{\Gamma}$ be the dual lattice. By $\tilde{\Omega}$ we denote the central Brillouin zone of $\tilde{\Gamma}$.

Example: $\Gamma = \mathbb{Z}^d$, $\Omega = [0, 1)^d$, $\tilde{\Gamma} = (2\pi\mathbb{Z})^d$, $\tilde{\Omega} = (-\pi, \pi)^d$.

Let $\varepsilon > 0$ be a parameter. We use the notation

$$f^\varepsilon(\mathbf{x}) := f\left(\frac{\mathbf{x}}{\varepsilon}\right).$$

Statement of the problem

Main object

By A_ε we denote the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}).$$

Statement of the problem

Main object

By A_ε we denote the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}).$$

Here $g(\mathbf{x})$ is a Γ -periodic $(m \times m)$ -matrix-valued function such that

$$c' \mathbf{1}_m \leq g(\mathbf{x}) \leq c'' \mathbf{1}_m, \quad 0 < c' \leq c'' < \infty;$$

Statement of the problem

Main object

By A_ε we denote the operator in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ given by

$$A_\varepsilon = b(\mathbf{D})^* g^\varepsilon(\mathbf{x}) b(\mathbf{D}).$$

Here $g(\mathbf{x})$ is a Γ -periodic $(m \times m)$ -matrix-valued function such that

$$c' \mathbf{1}_m \leq g(\mathbf{x}) \leq c'' \mathbf{1}_m, \quad 0 < c' \leq c'' < \infty;$$

$b(\mathbf{D}) = \sum_{j=1}^d b_j D_j$ is a first order $(m \times n)$ -matrix DO. We assume that $m \geq n$ and that the symbol $b(\boldsymbol{\xi}) = \sum_{j=1}^d b_j \xi_j$ has maximal rank:

$$\text{rank } b(\boldsymbol{\xi}) = n, \quad 0 \neq \boldsymbol{\xi} \in \mathbb{R}^d.$$

Statement of the problem

The precise definition of A_ε is given in terms of the quadratic form

$$a_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u} \rangle dx, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Statement of the problem

The precise definition of A_ε is given in terms of the quadratic form

$$a_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u} \rangle dx, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

We have

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 dx \leq a_\varepsilon[\mathbf{u}, \mathbf{u}] \leq c_1 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 dx, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Statement of the problem

The precise definition of A_ε is given in terms of the quadratic form

$$a_\varepsilon[\mathbf{u}, \mathbf{u}] = \int_{\mathbb{R}^d} \langle g^\varepsilon(\mathbf{x}) b(\mathbf{D})\mathbf{u}, b(\mathbf{D})\mathbf{u} \rangle dx, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

We have

$$c_0 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 dx \leq a_\varepsilon[\mathbf{u}, \mathbf{u}] \leq c_1 \int_{\mathbb{R}^d} |\mathbf{D}\mathbf{u}|^2 dx, \quad \mathbf{u} \in H^1(\mathbb{R}^d; \mathbb{C}^n).$$

Example:

$$A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla = \mathbf{D}^* g^\varepsilon(\mathbf{x})\mathbf{D}.$$

In this case, we have $n = 1$ and $m = d$.

Statement of the problem

Problem

The problem is to study the behavior of the operator exponential

$$e^{-i\tau A_\varepsilon}, \quad \tau \in \mathbb{R},$$

for small ε .

Statement of the problem

Problem

The problem is to study the behavior of the operator exponential

$$e^{-i\tau A_\varepsilon}, \quad \tau \in \mathbb{R},$$

for small ε . The results can be applied to the Cauchy problem

$$\begin{aligned} i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x}, \tau) &= (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \tau \in \mathbb{R}; \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) &= \phi(\mathbf{x}). \end{aligned}$$

Statement of the problem

Problem

The problem is to study the behavior of the operator exponential

$$e^{-i\tau A_\varepsilon}, \quad \tau \in \mathbb{R},$$

for small ε . The results can be applied to the Cauchy problem

$$\begin{aligned} i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x}, \tau) &= (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \tau \in \mathbb{R}; \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) &= \phi(\mathbf{x}). \end{aligned}$$

The solution can be represented as

$$\mathbf{u}_\varepsilon(\cdot, \tau) = e^{-i\tau A_\varepsilon} \phi.$$

The effective operator

We show that, in some sense,

$$e^{-i\tau A_\varepsilon} \sim e^{-i\tau A^0}$$

for small ε .

The effective operator

We show that, in some sense,

$$e^{-i\tau A_\varepsilon} \sim e^{-i\tau A^0}$$

for small ε . Here A^0 is the *effective operator* with constant coefficients:

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}).$$

The effective operator

We show that, in some sense,

$$e^{-i\tau A_\varepsilon} \sim e^{-i\tau A^0}$$

for small ε . Here A^0 is the *effective operator* with constant coefficients:

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}).$$

Definition of the effective matrix g^0 :

Let $\Lambda(\mathbf{x})$ be the $(n \times m)$ -matrix-valued Γ -periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_{\Omega} \Lambda(\mathbf{x}) \, d\mathbf{x} = 0.$$

The effective operator

We show that, in some sense,

$$e^{-i\tau A_\varepsilon} \sim e^{-i\tau A^0}$$

for small ε . Here A^0 is the *effective operator* with constant coefficients:

$$A^0 = b(\mathbf{D})^* g^0 b(\mathbf{D}).$$

Definition of the effective matrix g^0 :

Let $\Lambda(\mathbf{x})$ be the $(n \times m)$ -matrix-valued Γ -periodic solution of the problem

$$b(\mathbf{D})^* g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m) = 0, \quad \int_{\Omega} \Lambda(\mathbf{x}) \, d\mathbf{x} = 0.$$

Then g^0 is an $(m \times m)$ -matrix given by

$$g^0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(\mathbf{x}) \, d\mathbf{x}, \quad \tilde{g}(\mathbf{x}) = g(\mathbf{x})(b(\mathbf{D})\Lambda(\mathbf{x}) + \mathbf{1}_m).$$

- In 2001, **Birman** and **Suslina** proved that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon \quad (1)$$

by the *operator-theoretic* (spectral) *approach*.

- In 2001, **Birman** and **Suslina** proved that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon \quad (1)$$

by the *operator-theoretic* (spectral) *approach*. In 2005, a more accurate approximation was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^2. \quad (2)$$

- In 2001, **Birman** and **Suslina** proved that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon \quad (1)$$

by the *operator-theoretic* (spectral) *approach*. In 2005, a more accurate approximation was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^2. \quad (2)$$

In 2006, approximation in the energy norm was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon. \quad (3)$$

- In 2001, **Birman** and **Suslina** proved that

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon \quad (1)$$

by the *operator-theoretic* (spectral) *approach*. In 2005, a more accurate approximation was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon^2. \quad (2)$$

In 2006, approximation in the energy norm was obtained:

$$\|(A_\varepsilon + I)^{-1} - (A^0 + I)^{-1} - \varepsilon K_1(\varepsilon)\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C\varepsilon. \quad (3)$$

Here a *corrector* $K_1(\varepsilon)$ is given by

$$K_1(\varepsilon) = \Lambda^\varepsilon \Pi_\varepsilon b(\mathbf{D})(A^0 + I)^{-1}, \quad (\Pi_\varepsilon \mathbf{u})(\mathbf{x}) = (2\pi)^{-d/2} \int_{\tilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x}, \boldsymbol{\xi} \rangle} \hat{\mathbf{u}}(\boldsymbol{\xi}) d\boldsymbol{\xi},$$

and $K(\varepsilon)$ has a more complicated structure:

$$K(\varepsilon) = K_1(\varepsilon) + K_1(\varepsilon)^* + K_3.$$

- In 2004, **Suslina** proved that

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon, \quad \tau > 0. \quad (4)$$

- In 2004, **Suslina** proved that

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon, \quad \tau > 0. \quad (4)$$

A more accurate approximation was found by **Vasilevskaya** in 2009:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon^2. \quad (5)$$

In 2010, approximation in the energy norm was obtained by **Suslina**:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K_1(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(\tau) \varepsilon. \quad (6)$$

- In 2004, **Suslina** proved that

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon, \quad \tau > 0. \quad (4)$$

A more accurate approximation was found by **Vasilevskaya** in 2009:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon^2. \quad (5)$$

In 2010, approximation in the energy norm was obtained by **Suslina**:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K_1(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(\tau) \varepsilon. \quad (6)$$

Here $K(\varepsilon, \tau)$ and $K_1(\varepsilon, \tau)$ are appropriate correctors. Estimates (1)–(6) are called *operator error estimates* in homogenization theory.

- In 2004, **Suslina** proved that

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon, \quad \tau > 0. \quad (4)$$

A more accurate approximation was found by **Vasilevskaya** in 2009:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \varepsilon^2. \quad (5)$$

In 2010, approximation in the energy norm was obtained by **Suslina**:

$$\left\| e^{-A_\varepsilon \tau} - e^{-A^0 \tau} - \varepsilon K_1(\varepsilon, \tau) \right\|_{L_2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(\tau) \varepsilon. \quad (6)$$

Here $K(\varepsilon, \tau)$ and $K_1(\varepsilon, \tau)$ are appropriate correctors. Estimates (1)–(6) are called *operator error estimates* in homogenization theory.

- A different approach to operator error estimates (the *shift method*) was suggested by **Zhikov** and **Pastukhova** in 2005. See a survey by Zhikov and Pastukhova (Russian Math. Surveys, 2016).

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

- In 2008, **Birman** and **Suslina** proved that

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (7)$$

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

- In 2008, **Birman** and **Suslina** proved that

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (7)$$

- In 2017, **Suslina** confirmed that in the general case this result is sharp with respect to the norm type.

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

- In 2008, **Birman** and **Suslina** proved that

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (7)$$

- In 2017, **Suslina** confirmed that in the general case this result is sharp with respect to the norm type.
- In 2021, it was shown by **Dorodnyi** that in the general case this result is sharp with respect to the dependence on τ .

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

- In 2008, **Birman** and **Suslina** proved that

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (7)$$

- In 2017, **Suslina** confirmed that in the general case this result is sharp with respect to the norm type.
- In 2021, it was shown by **Dorodnyi** that in the general case this result is sharp with respect to the dependence on τ .
- Under some additional assumptions, estimate (7) was improved:

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2}\varepsilon. \quad (8)$$

The situation with homogenization of *Schrödinger-type* equations and *hyperbolic* equations is different from the elliptic and parabolic cases.

- In 2008, **Birman** and **Suslina** proved that

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (7)$$

- In 2017, **Suslina** confirmed that in the general case this result is sharp with respect to the norm type.
- In 2021, it was shown by **Dorodnyi** that in the general case this result is sharp with respect to the dependence on τ .
- Under some additional assumptions, estimate (7) was improved:

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2}\varepsilon. \quad (8)$$

- Similar results were obtained for $\cos(\tau A_\varepsilon^{1/2})$ and $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$ by **Birman** and **Suslina**, **Meshkova**, **Dorodnyi** and **Suslina**.

Main questions

Problem

Is it possible to approximate the operator $e^{-i\tau A_\varepsilon}$ in the $(H^s \rightarrow L_2)$ -norm with error $O(\varepsilon^2)$ and in the $(H^s \rightarrow H^1)$ -norm with error $O(\varepsilon)$?

Main questions

Problem

Is it possible to approximate the operator $e^{-i\tau A_\varepsilon}$ in the $(H^s \rightarrow L_2)$ -norm with error $O(\varepsilon^2)$ and in the $(H^s \rightarrow H^1)$ -norm with error $O(\varepsilon)$?

Results: *We have obtained such approximations not for the exponential $e^{-i\tau A_\varepsilon}$, but for the operator $e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)$.*

Main questions

Problem

Is it possible to approximate the operator $e^{-i\tau A_\varepsilon}$ in the $(H^s \rightarrow L_2)$ -norm with error $O(\varepsilon^2)$ and in the $(H^s \rightarrow H^1)$ -norm with error $O(\varepsilon)$?

Results: We have obtained such approximations not for the exponential $e^{-i\tau A_\varepsilon}$, but for the operator $e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)$.

The results with corrector can be applied to the Cauchy problem

$$\begin{aligned} i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x}, \tau) &= (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \tau \in \mathbb{R}; \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) &= \psi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}, \end{aligned} \tag{9}$$

with **initial data from a special class**:

$$\psi_\varepsilon := \phi + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon \phi, \quad \phi \in H^s(\mathbb{R}^d; \mathbb{C}^n) \text{ with a suitable } s. \tag{10}$$

Main questions

Problem

Is it possible to approximate the operator $e^{-i\tau A_\varepsilon}$ in the $(H^s \rightarrow L_2)$ -norm with error $O(\varepsilon^2)$ and in the $(H^s \rightarrow H^1)$ -norm with error $O(\varepsilon)$?

Results: We have obtained such approximations not for the exponential $e^{-i\tau A_\varepsilon}$, but for the operator $e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)$.

The results with corrector can be applied to the Cauchy problem

$$\begin{aligned} i\partial_\tau \mathbf{u}_\varepsilon(\mathbf{x}, \tau) &= (A_\varepsilon \mathbf{u}_\varepsilon)(\mathbf{x}, \tau), \quad \mathbf{x} \in \mathbb{R}^d, \tau \in \mathbb{R}; \\ \mathbf{u}_\varepsilon(\mathbf{x}, 0) &= \psi_\varepsilon(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}, \end{aligned} \quad (9)$$

with **initial data from a special class**:

$$\psi_\varepsilon := \phi + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon \phi, \quad \phi \in H^s(\mathbb{R}^d; \mathbb{C}^n) \text{ with a suitable } s. \quad (10)$$

Note that

$$(I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)^{-1} = I - \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon \Rightarrow \phi = \psi_\varepsilon - \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon \psi_\varepsilon.$$

Reduction 1: Scaling transformation

We apply the **operator-theoretic method** based on the scaling transformation, the Floquet–Bloch theory and analytic perturbation theory.

Reduction 1: Scaling transformation

We apply the **operator-theoretic method** based on the scaling transformation, the Floquet–Bloch theory and analytic perturbation theory. By the scaling transformation, the study of $e^{-i\tau A_\varepsilon}$ is reduced to the study of the operator

$$e^{-i\tau\varepsilon^{-2}A},$$

where $A = b(\mathbf{D})^*g(\mathbf{x})b(\mathbf{D})$.

Reduction 2: Direct integral expansion

Using the Floquet–Bloch theory, we expand the operator A in the direct integral:

$$A \sim \int_{\tilde{\Omega}} \oplus A(\mathbf{k}) d\mathbf{k}.$$

Reduction 2: Direct integral expansion

Using the Floquet–Bloch theory, we expand the operator A in the direct integral:

$$A \sim \int_{\tilde{\Omega}} \oplus A(\mathbf{k}) d\mathbf{k}.$$

The operator $A(\mathbf{k})$ acts in $L_2(\Omega; \mathbb{C}^n)$ and is given by

$$A(\mathbf{k}) = b(\mathbf{D} + \mathbf{k})^* g(\mathbf{x}) b(\mathbf{D} + \mathbf{k})$$

with periodic boundary conditions.

Reduction 2: Direct integral expansion

Using the Floquet–Bloch theory, we expand the operator A in the direct integral:

$$A \sim \int_{\tilde{\Omega}} \oplus A(\mathbf{k}) d\mathbf{k}.$$

The operator $A(\mathbf{k})$ acts in $L_2(\Omega; \mathbb{C}^n)$ and is given by

$$A(\mathbf{k}) = b(\mathbf{D} + \mathbf{k})^* g(\mathbf{x}) b(\mathbf{D} + \mathbf{k})$$

with periodic boundary conditions.

We have to approximate the operator

$$e^{-i\tau\varepsilon^{-2}A(\mathbf{k})}$$

uniformly in $\mathbf{k} \in \tilde{\Omega}$.

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete.

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ (for small \mathbf{k}) as a perturbed one.

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ (for small \mathbf{k}) as a perturbed one. We have

$$\mathfrak{N} := \text{Ker } A(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}.$$

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ (for small \mathbf{k}) as a perturbed one. We have

$$\mathfrak{N} := \text{Ker } A(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}.$$

Let P be the orthogonal projection onto \mathfrak{N} : $P\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x}) \, d\mathbf{x}$.

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ (for small \mathbf{k}) as a perturbed one. We have

$$\mathfrak{N} := \text{Ker } A(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}.$$

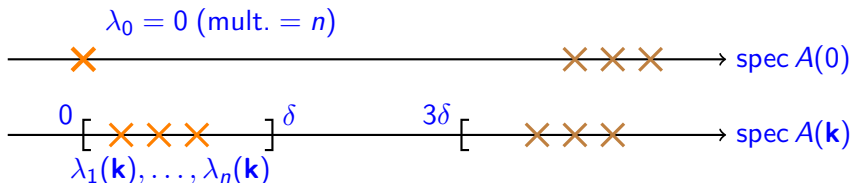
Let P be the orthogonal projection onto \mathfrak{N} : $P\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x}) d\mathbf{x}$. So, $\lambda_0 = 0$ is an n -multiple isolated eigenvalue of $A(0)$.

Analytic perturbation theory

The operator $A(\mathbf{k})$ is an elliptic operator in a bounded domain; its spectrum is discrete. We consider $A(0)$ as an unperturbed operator and $A(\mathbf{k})$ (for small \mathbf{k}) as a perturbed one. We have

$$\mathfrak{N} := \text{Ker } A(0) = \{\mathbf{u} \in L_2(\Omega; \mathbb{C}^n) : \mathbf{u}(\mathbf{x}) = \mathbf{c} \in \mathbb{C}^n\}.$$

Let P be the orthogonal projection onto \mathfrak{N} : $P\mathbf{u} = |\Omega|^{-1} \int_{\Omega} \mathbf{u}(\mathbf{x}) d\mathbf{x}$. So, $\lambda_0 = 0$ is an n -multiple isolated eigenvalue of $A(0)$. Then for $|\mathbf{k}| \leq t_0$ the perturbed operator $A(\mathbf{k})$ has exactly n eigenvalues $\lambda_1(\mathbf{k}), \dots, \lambda_n(\mathbf{k})$ on $[0, \delta]$, while the interval $(\delta, 3\delta)$ is free of the spectrum.



Analytic perturbation theory

We put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and study the operator family

$$A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$$

by means of the *analytic perturbation theory* with respect to t .

Analytic perturbation theory

We put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and study the operator family

$$A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$$

by means of the *analytic perturbation theory* with respect to t .

By the Kato–Rellich theorem, for $t \leq t_0$ there exist real-analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of $A(t, \boldsymbol{\theta})$:

$$A(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}) = \lambda_l(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}), \quad l = 1, \dots, n,$$

and the set $\{\varphi_l(t, \boldsymbol{\theta})\}$ forms an orthonormal basis in the eigenspace of $A(t, \boldsymbol{\theta})$ corresponding to $[0, \delta]$.

Analytic perturbation theory

We put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and study the operator family

$$A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$$

by means of the *analytic perturbation theory* with respect to t .

By the Kato–Rellich theorem, for $t \leq t_0$ there exist real-analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of $A(t, \boldsymbol{\theta})$:

$$A(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}) = \lambda_l(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}), \quad l = 1, \dots, n,$$

and the set $\{\varphi_l(t, \boldsymbol{\theta})\}$ forms an orthonormal basis in the eigenspace of $A(t, \boldsymbol{\theta})$ corresponding to $[0, \delta]$. Then for small t we have

$$\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \nu_l(\boldsymbol{\theta})t^4 + \dots, \quad (11)$$

$$\varphi_l(t, \boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\varphi_l^{(1)}(\boldsymbol{\theta}) + \dots, \quad (12)$$

$l = 1, \dots, n$.

Analytic perturbation theory

We put $\mathbf{k} = t\boldsymbol{\theta}$, $t = |\mathbf{k}|$, $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, and study the operator family

$$A(\mathbf{k}) = A(t\boldsymbol{\theta}) =: A(t, \boldsymbol{\theta})$$

by means of the *analytic perturbation theory* with respect to t .

By the Kato–Rellich theorem, for $t \leq t_0$ there exist real-analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of $A(t, \boldsymbol{\theta})$:

$$A(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}) = \lambda_l(t, \boldsymbol{\theta})\varphi_l(t, \boldsymbol{\theta}), \quad l = 1, \dots, n,$$

and the set $\{\varphi_l(t, \boldsymbol{\theta})\}$ forms an orthonormal basis in the eigenspace of $A(t, \boldsymbol{\theta})$ corresponding to $[0, \delta]$. Then for small t we have

$$\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \nu_l(\boldsymbol{\theta})t^4 + \dots, \quad (11)$$

$$\varphi_l(t, \boldsymbol{\theta}) = \omega_l(\boldsymbol{\theta}) + t\varphi_l^{(1)}(\boldsymbol{\theta}) + \dots, \quad (12)$$

$l = 1, \dots, n$. Here $\gamma_l(\boldsymbol{\theta}) \geq c_* > 0$, and $\mu_l(\boldsymbol{\theta}), \nu_l(\boldsymbol{\theta}) \in \mathbb{R}$. The embryos $\omega_1(\boldsymbol{\theta}), \dots, \omega_n(\boldsymbol{\theta})$ form an orthonormal basis in \mathfrak{R} .

Analytic perturbation theory

Let $F(t, \theta)$ be the spectral projection of $A(t, \theta)$ for the interval $[0, \delta]$. We have the following *threshold approximations* for small t :

$$F(t, \theta) = P + tF_1(\theta) + O(t^2), \quad (13)$$

$$A(t, \theta)F(t, \theta) = t^2S(\theta)P + t^3K(\theta) + O(t^4). \quad (14)$$

Analytic perturbation theory

Let $F(t, \theta)$ be the spectral projection of $A(t, \theta)$ for the interval $[0, \delta]$. We have the following *threshold approximations* for small t :

$$F(t, \theta) = P + tF_1(\theta) + O(t^2), \quad (13)$$

$$A(t, \theta)F(t, \theta) = t^2S(\theta)P + t^3K(\theta) + O(t^4). \quad (14)$$

The operators $F_1(\theta)$, $S(\theta)$ and $K(\theta)$ can be described in the invariant terms, as well as in terms of the coefficients of power series expansions for $\lambda_I(t, \theta)$ and $\varphi_I(t, \theta)$.

Analytic perturbation theory

Let $F(t, \theta)$ be the spectral projection of $A(t, \theta)$ for the interval $[0, \delta]$. We have the following *threshold approximations* for small t :

$$F(t, \theta) = P + tF_1(\theta) + O(t^2), \quad (13)$$

$$A(t, \theta)F(t, \theta) = t^2S(\theta)P + t^3K(\theta) + O(t^4). \quad (14)$$

The operators $F_1(\theta)$, $S(\theta)$ and $K(\theta)$ can be described in the invariant terms, as well as in terms of the coefficients of power series expansions for $\lambda_I(t, \theta)$ and $\varphi_I(t, \theta)$. We have

$$F_1(\theta) = [\Lambda]b(\theta)P + ([\Lambda]b(\theta)P)^*.$$

Analytic perturbation theory

Let $F(t, \theta)$ be the spectral projection of $A(t, \theta)$ for the interval $[0, \delta]$. We have the following *threshold approximations* for small t :

$$F(t, \theta) = P + tF_1(\theta) + O(t^2), \quad (13)$$

$$A(t, \theta)F(t, \theta) = t^2S(\theta)P + t^3K(\theta) + O(t^4). \quad (14)$$

The operators $F_1(\theta)$, $S(\theta)$ and $K(\theta)$ can be described in the invariant terms, as well as in terms of the coefficients of power series expansions for $\lambda_l(t, \theta)$ and $\varphi_l(t, \theta)$. We have

$$F_1(\theta) = [\Lambda]b(\theta)P + ([\Lambda]b(\theta)P)^*.$$

The operator $S(\theta) = b(\theta)^*g^0b(\theta)$ is called the *spectral germ* of the operator family $A(t, \theta)$ at $t = 0$. The coefficients $\gamma_l(\theta)$ and the elements $\omega_l(\theta)$ are eigenvalues and eigenvectors of the spectral germ:

$$S(\theta)\omega_l(\theta) = \gamma_l(\theta)\omega_l(\theta), \quad l = 1, \dots, n.$$

Analytic perturbation theory

Next, we need to describe the operator $N(\theta) = PK(\theta)P$.

Analytic perturbation theory

Next, we need to describe the operator $N(\boldsymbol{\theta}) = PK(\boldsymbol{\theta})P$. In the invariant terms, we have

$$N(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta}) P,$$

$$L(\boldsymbol{\theta}) := |\Omega|^{-1} \int_{\Omega} (\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \tilde{g}(\mathbf{x}) + \tilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x})) dx.$$

Analytic perturbation theory

Next, we need to describe the operator $N(\boldsymbol{\theta}) = PK(\boldsymbol{\theta})P$. In the invariant terms, we have

$$N(\boldsymbol{\theta}) = b(\boldsymbol{\theta})^* L(\boldsymbol{\theta}) b(\boldsymbol{\theta}) P,$$

$$L(\boldsymbol{\theta}) := |\Omega|^{-1} \int_{\Omega} (\Lambda(\mathbf{x})^* b(\boldsymbol{\theta})^* \tilde{g}(\mathbf{x}) + \tilde{g}(\mathbf{x})^* b(\boldsymbol{\theta}) \Lambda(\mathbf{x})) d\mathbf{x}.$$

In terms of the coefficients,

$$N(\boldsymbol{\theta}) = N_0(\boldsymbol{\theta}) + N_*(\boldsymbol{\theta}),$$

$$N_0(\boldsymbol{\theta}) = \sum_{l=1}^n \mu_l(\boldsymbol{\theta}) (\cdot, \omega_l(\boldsymbol{\theta})) \omega_l(\boldsymbol{\theta}),$$

$$N_*(\boldsymbol{\theta}) = \sum_{l=1}^n \gamma_l(\boldsymbol{\theta}) ((\cdot, P\varphi_l^{(1)}(\boldsymbol{\theta})) \omega_l(\boldsymbol{\theta}) + (\cdot, \omega_l(\boldsymbol{\theta})) P\varphi_l^{(1)}(\boldsymbol{\theta})).$$

Theorem 1 [Birman and Suslina, 2008]

For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^3(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon.$$

Results with corrector

Theorem 2 [Suslina, 2021]

For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K(\varepsilon, \tau) \right\|_{H^6(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^2 \varepsilon^2.$$

Here the corrector $K(\varepsilon, \tau)$ is given by

$$K(\varepsilon, \tau) := [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon e^{-i\tau A^0} - i \int_0^\tau e^{-i(\tau-\rho)A^0} b(\mathbf{D})^* L(\mathbf{D}) b(\mathbf{D}) e^{-i\rho A^0} d\rho.$$

Theorem 3 [Suslina, 2021]

For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K_1(\varepsilon, \tau) \right\|_{H^1(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon.$$

Here the corrector $K_1(\varepsilon, \tau)$ is given by

$$K_1(\varepsilon, \tau) := [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon e^{-i\tau A^0}.$$

Results with corrector

Theorem 3 [Suslina, 2021]

For $\varepsilon > 0$ and $\tau \in \mathbb{R}$ we have

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K_1(\varepsilon, \tau) \right\|_{H^4(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon.$$

Here the corrector $K_1(\varepsilon, \tau)$ is given by

$$K_1(\varepsilon, \tau) := [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon e^{-i\tau A^0}.$$

Remark. 1) The correctors $K(\varepsilon, \tau)$ and $K_1(\varepsilon, \tau)$ are different. This agrees with the results for elliptic and parabolic equations.

2) We see that the “expected” first order approximation $e^{-i\tau A^0} + \varepsilon K_1(\varepsilon, \tau)$ is close not to the operator $e^{-i\tau A_\varepsilon}$, but to $e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon)$.

Discussion

It is impossible to approximate the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$ in terms of the spectral characteristics at the bottom of the spectrum.

Discussion

It is impossible to approximate the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$ in terms of the spectral characteristics at the bottom of the spectrum. Indeed, after the scaling transformation and the direct integral expansion, this operator transforms to the operator family $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} [\Lambda] b(\mathbf{k}) P$ acting in $L_2(\Omega; \mathbb{C}^n)$.

Discussion

It is impossible to approximate the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$ in terms of the spectral characteristics at the bottom of the spectrum. Indeed, after the scaling transformation and the direct integral expansion, this operator transforms to the operator family $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} [\Lambda] b(\mathbf{k}) P$ acting in $L_2(\Omega; \mathbb{C}^n)$. We have $[\Lambda] = P^\perp [\Lambda]$, so, we deal with the operator $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} P^\perp$ which is close to $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} F(\mathbf{k})^\perp$ within the margin of error.

Discussion

It is impossible to approximate the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$ in terms of the spectral characteristics at the bottom of the spectrum. Indeed, after the scaling transformation and the direct integral expansion, this operator transforms to the operator family $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} [\Lambda] b(\mathbf{k}) P$ acting in $L_2(\Omega; \mathbb{C}^n)$. We have $[\Lambda] = P^\perp [\Lambda]$, so, we deal with the operator $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} P^\perp$ which is close to $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} F(\mathbf{k})^\perp$ within the margin of error. Here $F(\mathbf{k})^\perp$ is the spectral projection of $A(\mathbf{k})$ corresponding to the interval $[3\delta, \infty)$. Hence, this part of the spectrum is responsible for the behavior of the operator $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} F(\mathbf{k})^\perp$.

Discussion

It is impossible to approximate the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$ in terms of the spectral characteristics at the bottom of the spectrum. Indeed, after the scaling transformation and the direct integral expansion, this operator transforms to the operator family $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} [\Lambda] b(\mathbf{k}) P$ acting in $L_2(\Omega; \mathbb{C}^n)$. We have $[\Lambda] = P^\perp [\Lambda]$, so, we deal with the operator $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} P^\perp$ which is close to $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} F(\mathbf{k})^\perp$ within the margin of error. Here $F(\mathbf{k})^\perp$ is the spectral projection of $A(\mathbf{k})$ corresponding to the interval $[3\delta, \infty)$. Hence, this part of the spectrum is responsible for the behavior of the operator $e^{-i\tau \varepsilon^{-2} A(\mathbf{k})} F(\mathbf{k})^\perp$. Our conjecture is that all the spectral bands corresponding to the interval $[3\delta, \infty)$ will contribute to the desired approximation for the operator $\varepsilon e^{-i\tau A_\varepsilon} [\Lambda^\varepsilon] b(\mathbf{D}) \Pi_\varepsilon$.

Improvement of the results under additional assumptions

The results can be improved under the following condition.

Condition 1

Suppose that at least one of the following assumptions is satisfied:

1°. $N(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

2°. $N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (i. e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1, \dots, n$) and the number of different eigenvalues of $S(\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$.

Improvement of the results under additional assumptions

The results can be improved under the following condition.

Condition 1

Suppose that at least one of the following assumptions is satisfied:

1°. $N(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

2°. $N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (i. e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1, \dots, n$) and the number of different eigenvalues of $S(\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$.

Remark. If $A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$, where $g(\mathbf{x})$ has real entries, then $N(\boldsymbol{\theta}) \equiv 0$.

Improvement of the results under additional assumptions

The results can be improved under the following condition.

Condition 1

Suppose that at least one of the following assumptions is satisfied:

1°. $N(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$.

2°. $N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$ (i. e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1, \dots, n$) and the number of different eigenvalues of $S(\boldsymbol{\theta})$ does not depend on $\boldsymbol{\theta}$.

Remark. If $A_\varepsilon = -\operatorname{div} g^\varepsilon(\mathbf{x})\nabla$, where $g(\mathbf{x})$ has real entries, then $N(\boldsymbol{\theta}) \equiv 0$.

Theorem 4 [Dorodnyi, 2021]

Suppose that Condition 1 is satisfied. Then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$

$$\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2} \varepsilon. \quad (15)$$

Improvement of the results under additional assumptions

Theorem 5 [Suslina, 2021]

Suppose that Condition 1 is satisfied. Then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K(\varepsilon, \tau) \right\|_{H^4(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon^2.$$

Improvement of the results under additional assumptions

Theorem 5 [Suslina, 2021]

Suppose that Condition 1 is satisfied. Then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K(\varepsilon, \tau) \right\|_{H^4(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(1 + |\tau|) \varepsilon^2.$$

Theorem 6 [Suslina, 2021]

Suppose that Condition 1 is satisfied. Then for $\varepsilon > 0$ and $\tau \in \mathbb{R}$

$$\left\| e^{-i\tau A_\varepsilon} (I + \varepsilon \Lambda^\varepsilon b(\mathbf{D}) \Pi_\varepsilon) - e^{-i\tau A^0} - \varepsilon K_1(\varepsilon, \tau) \right\|_{H^3(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C(1 + |\tau|)^{1/2} \varepsilon.$$

Sharpness of the results

In the general case, all the results are sharp regarding the norm type and the dependence of estimates on τ .

Sharpness of the results

In the general case, all the results are sharp regarding the norm type and the dependence of estimates on τ .

Theorem 7 [Suslina, 2017; Dorodnyi, 2021]

Suppose that $N_0(\theta_0) \neq 0$ for some $\theta_0 \in \mathbb{S}^{d-1}$, i. e., $\mu_l(\theta_0) \neq 0$ for some l .

1) Let $\tau \neq 0$ and $s < 3$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate

$$\left\| e^{-i\tau A_\varepsilon} - e^{-i\tau A^0} \right\|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \quad (16)$$

holds for all sufficiently small ε .

2) Let $s \geq 3$. There does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \rightarrow \infty} C(\tau)/|\tau| = 0$ and estimate (16) holds for $\tau \in \mathbb{R}$ and sufficiently small ε .

Theorem 7 confirms that Theorem 1 is sharp.

Sharpness of the results

Theorem 8 [Suslina, 2021]

Suppose that $N_0(\theta_0) \neq 0$ for some $\theta_0 \in \mathbb{S}^{d-1}$, i. e., $\mu_l(\theta_0) \neq 0$ for some l . Then the results of Theorem 2 and 3 are sharp both regarding the norm type and regarding the dependence of estimates on τ .

Sharpness of the results

Finally, we show that the improved results (that are valid under Condition 1) are also sharp.

Theorem 9 [Dorodnyi, 2021]

Suppose that $N_0(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$, i. e., $\mu_l(\theta) \equiv 0$ for $l = 1, \dots, n$. Suppose that $\nu_j(\theta_0) \neq 0$ for some j and $\theta_0 \in \mathbb{S}^{d-1}$.

1) Let $\tau \neq 0$ and $s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate

$$\|e^{-i\tau A_\varepsilon} - e^{-i\tau A^0}\|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \quad (17)$$

holds for all sufficiently small ε .

2) Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \rightarrow \infty} C(\tau)/|\tau|^{1/2} = 0$ and the estimate (17) holds for $\tau \in \mathbb{R}$ and sufficiently small ε .

Theorem 9 shows that Theorem 4 is sharp.

Sharpness of the results

Theorem 10 [Suslina, 2021]

Suppose that $N_0(\boldsymbol{\theta}) = 0$ for any $\boldsymbol{\theta} \in \mathbb{S}^{d-1}$, i. e., $\mu_l(\boldsymbol{\theta}) \equiv 0$ for $l = 1, \dots, n$. Suppose that $\nu_j(\boldsymbol{\theta}_0) \neq 0$ for some j and $\boldsymbol{\theta}_0 \in \mathbb{S}^{d-1}$. Then the results of Theorems 5 and 6 are sharp both regarding the norm type and regarding the dependence of estimates on τ .

- M. Sh. Birman and T. A. Suslina, *Operator error estimates in the homogenization problem for nonstationary periodic equations*, Algebra i Analiz (St. Petersburg Math. J.) **20** (2008), no. 6.
- T. A. Suslina, *Spectral approach to homogenization of nonstationary Schrödinger-type equations with periodic coefficients*, J. Math. Anal. Appl. **446** (2017), no. 2.
- M. A. Dorodnyi, *Operator error estimates for homogenization of the nonstationary Schrödinger-type equations: Sharpness of the results*, Applicable Analysis (2021).
- T. A. Suslina, *Homogenization of the Schrödinger-type equations with periodic coefficients: Results with correctors*, in preparation.