Asymptotic behaviour of the heat equation in twisted waveguides

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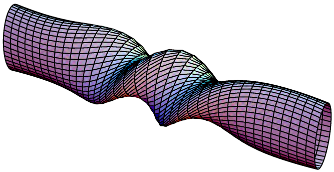
Layout

Introduction

- Straight waveguide $\Omega_0 := \mathbb{R} \times \omega$, $\omega \in \mathbb{R}^2$ is non-circular cross-section
- Twisted waveguide Ω_θ, ω is rotating with respect to non-constant angle $\theta(x_1)$.



The Hamiltonian of a particle moving inside is descibed by the Dirichlet Laplacian $-\Delta_D^{\Omega_\theta} : L^2(\Omega_\theta) \to L^2(\Omega_\theta)$.
It is associated with the quadratic form $\psi \mapsto \|\nabla\psi\|^2$ with the domain $\mathcal{D}(\Omega_\theta) := H_0^1(\Omega_\theta)$.

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Spectral stability

$-\Delta_D^{\Omega_\theta}$ and $-\Delta_D^{\Omega_0}$ have the same spectrum as a set:

$$\sigma(-\Delta_D^{\Omega_\theta}) = \sigma_{\text{ess}}(-\Delta_D^{\Omega_\theta}) = [E_1, \infty).$$

- E_1 is the threshold energy of $-\Delta_D^\omega$
- Difference: existence of the Hardy inequality [Ekholm, Kovařík, Krejčiřík 2008]

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Hardy-type inequality

$$-\Delta_D^{\Omega_\theta} - E_1 \geq \rho, \qquad \rho \text{ is a positive function.}$$

- Operator $-\Delta_D^{\Omega_\theta} - E_1$ is subcritical, $-\Delta_D^{\Omega_0} - E_1$ is critical

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Heat equation on Ω_θ

$$u_t(x,t) - \Delta u(x,t) = 0$$

subject to Dirichlet boundary conditions on $\partial\Omega_\theta$.

From the semigroup theory: $\forall u_0 \in L^2(\Omega_\theta)$ exists uniquely determined generalized solution of the heat equation in the form

$$u(x,t) = e^{\Delta_D^{\Omega_\theta} t} u_0(x),$$

where $e^{\Delta_D^{\Omega_\theta} t} : L^2(\Omega_\theta) \to L^2(\Omega_\theta)$ is the semigroup operator associated with the Laplacian $-\Delta_D^{\Omega_\theta}$.
It follows from the spectral mapping theorem that

$$\|e^{\Delta_D^{\Omega_\theta} t}\|_{L^2(\Omega_\theta)\to L^2(\Omega_\theta)} = e^{-E_1 t}.$$

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Decay rate

We are interested in an additional (polynomial) decay of the semigroup, which will follow if we restrict the class of initial data to the weighted space $L^2(\Omega_\theta, K)$, where $K(x) := e^{x_1^2/4}$. Let us define the decay rate:

$$\gamma(\Omega_\theta) := \sup\left\{\gamma \mid \exists C_\gamma > 0, \forall t \geq 0, \left\|e^{(\Delta_\theta + E_1)t}\right\|_K \leq C_\gamma (1+t)^{-\gamma}\right\},$$

where $\|.\|_K : L^2(\Omega_\theta, K) \to L^2(\Omega_\theta)$.

Theorem

We have $\gamma(\Omega_\theta) = 1/4$ if Ω_θ is untwisted, while $\gamma(\Omega_\theta) = 3/4$ if Ω_θ is twisted.

[D. Krejčiřík and E. Zuazua, 2011]

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Self-similarity transformation

Motivation: The Dirichlet Laplacian converges in the norm resolvent sense to one dimensional Schrödinger operator whose potential holds the information about twisting [Šediváková, Krejčiřík, 2011].
One-dimensional heat equation:

$$u_t - u_{xx} + V(x)u = 0, \quad \text{with } V = C_\omega \dot{\theta}^2.$$

Self-similarity transformation

$$u(x,t) = (t+1)^{-1/4} w(y,s),$$
$$y := (t+1)^{-1/2} x, \quad s := \ln(t+1).$$

Using self-similarity transformation we arrive at:

$$w_s - \frac{1}{2} y w_y - \frac{1}{4} w - w_{yy} + e^s V(e^{s/2} y) w = 0.$$

Finally, obeying

$$z(y,s) := e^{y^2/8} w(y,s),$$

we get expression in $(y,s) \in \Omega_0 \times (0,\infty)$:

$$z_s - z_{yy} + \frac{y^2}{16} z + e^s V(e^{s/2} y) z = 0$$

This is a parabolic equation with time-dependent coefficients. However, this form is advantageous because of the compactness of the resolvent. Self-similarity transformation is unitary: $\|u(t)\| = \|w(s)\|$. According to theorem, we expect

$$V = 0,\ \|u(t)\| \sim t^{-1/4}, \qquad V \geq 0,\ \|u(t)\| \sim t^{-3/4}.$$

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Exact solution for $V = 0$

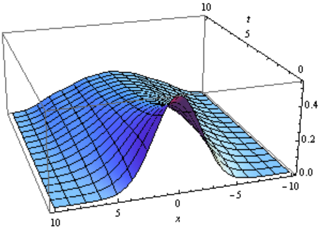
Semigroup operator of the heat equation in an integral operator

$$u(x,t) = \int_{\mathbb{R}} G(x,y,t)u_0(y)dy,$$

where the heat kernel is defined as

$$G(x,y,t) = \frac{e^{-\frac{|x-y|^2}{4t}}}{\sqrt{4\pi t}}.$$

We are able to find numerical solution (Wolfram Mathematica 7.0).



Expansion to oscillator basis

The first method uses the expansion to the eigenbasis of the harmonic oscillator:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(x).$$

ψ satisfies the Helmoltz equality $-\psi'' + \frac{x^2}{16}\psi = \lambda\psi$, we know explicitly the eigenvalues and -vectors from quantum mechanics:

$$\lambda_n = \frac{1}{2}\left(n + \frac{1}{2}\right), \qquad \psi_n(x) = \mathcal{N}_n H_n\left(\frac{x}{2}\right) e^{-\frac{x^2}{8}}.$$

Plugging into the heat equation:

$$a(t) = e^{-Mt}a(0), \qquad \text{where } M_{mn} = \lambda_n\delta_{mn} - \left\langle \psi_m, \frac{x^2}{16}\psi_n \right\rangle.$$

Self-similarity solution

The solution of self-similarity-transformed equation $z_s - z_{yy} + \frac{y^2}{16} z = 0$ may be found again as expansion to harmonic oscillator eigenbasis. Then

$$a(t) = e^{-Mt} a(0), \qquad \text{where } M_{mn} = \lambda_n \delta_{mn}.$$

Finally, we apply backward self-similarity transformation.
The decay rate function is defined

$$q(t) := -\frac{\ln \|u(t)\|}{\ln(1+t)}.$$

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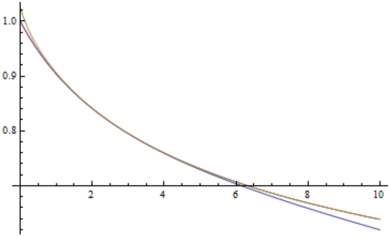
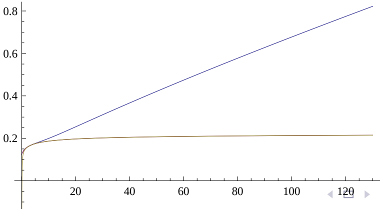
Comparison

Time evolution of the norm $\|u\|$:



Decay rate $q(t)$:



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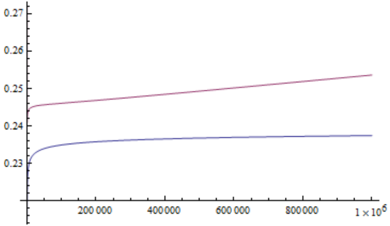
The twisted waveguide

The potential increases the decay rate about $1/2$. Analytically, the mathematical background lies in scalling which is more singular than the Dirac's delta interaction and thus leads to the Dirichlet condition at the origin.
This problem is already explicitely solvable: the first eigenvalue is $3/4$, which coincides with the second eigenvalue of the harmonic oscillator without Dirichlet condition. Decay rate $q(t)$:



Conclusion

- Aim: support the data given by Krejčiřík and Zuazua; show that the decay rate posseses increase about $1/2$ in the twisted waveguide in comparison to the untwisted case.
- Possible extensions: Computation for non-approximated 3D waveguides.

Admiration of the audience

THANK YOU FOR YOUR ATTENTION