The Bloch spectrum of a quantum graph

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Quantum graphs

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Quantum graphs lie 'inbetween' manifolds and combinatorial graphs. They exhibit much of the richness of structure of manifolds but are still simple enough to inherit some of the ease of computations from combinatorial graphs.

Functions on quantum graphs

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This models a conservation of flow.
The space of functions is denoted by $\Lambda^0(G)$.

The Laplacian on a quantum graph

The standard Laplacian Δ_0 on a quantum graph acts as $-\frac{\partial^2}{\partial x^2}$ on each of the edges, seen as intervals.

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The spectrum of the Laplacian is the set of λ where the eigenvalue equation

$$\Delta_0 f = \lambda f$$

has a nontrivial solution.

An example of two isospectral quantum graphs

Two quantum graphs with the same spectrum are said to be isospectral.

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Two quantum graphs with the same spectrum are said to be isospectral.
There are examples of isospectral non-isomorphic quantum graphs, this one is from [vB01].

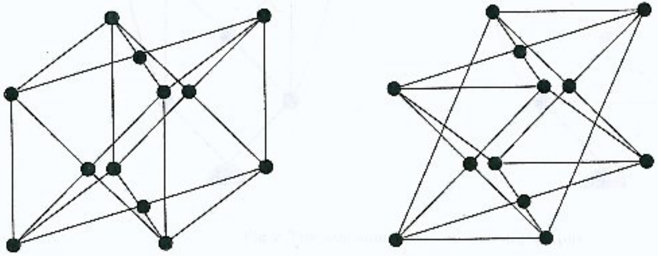

Fig.1 Two isospectral networks that are not isometric.

The questions

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What does the spectrum of an operator tell us about the quantum graph?

What additional information can we get if we look at multiple spectra at the same time?

Which properties of the quantum graph are spectrally determined? Which classes of quantum graphs are uniquely determined by their Bloch spectrum?

Differential forms

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Definition

A 1-form on a quantum graph is a 1-form on each edge with a suitable boundary condition at the vertices.
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The exterior derivative

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There is an exterior derivative $d : \Lambda^0(G) \to \Lambda^1(G)$ from functions to 1-forms.

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Definition

We define the following operators from functions to 1-forms

$$d_\alpha f := df + 2\pi i f \alpha$$

where α is a real 1-form.

The inner product and the adjoint

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Definition

The adjoint is characterised by

$$(d_\alpha f, \beta) = (f, d_\alpha^* \beta)$$

for all functions f and 1-forms β.

The operators

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where d_α^* is the adjoint via an edgewise hermitian inner product.

In the case where $\alpha = 0$ this is the standard Laplacian $\Delta_0 = -\frac{\partial^2}{\partial x^2}$. In general this is a second order differential operator with highest order part $-\frac{\partial^2}{\partial x^2}$.

The eigenvalue problem

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Proposition

The spectrum is real, infinite, bounded from below, with a single accumulation point at infinity. The multiplicity of each eigenvalue is finite.

deRham cohomology

Definition

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Proposition

For graphs homology and cohomology are isomorphic.

$$H^1_{dR}(G, \mathbb{R}) \cong \mathbb{R}^n \cong H_1(G, \mathbb{R})$$

The Bloch spectrum

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Let α and $\beta = \alpha + d\varphi$ be cohomologous 1-forms. Then Δ_α and Δ_β have the same spectrum.

To each element $[\alpha] \in H^1_{dR}(G)$ we associate the spectrum of the Schrödinger operator Δ_α.

Definition

We define the Bloch spectrum of a quantum graph to be the collection of spectra of G for the operators Δ_α for all $\alpha \in H^1_{dR}(G)$.

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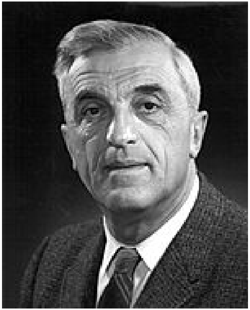

Figure: Felix Bloch

An example

We will compute the Bloch spectrum of the circle S^1. We can see the circle as a quantum graph with a single edge of length 2π.

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We have $H^1_{dR}(S^1) = \mathbb{R}$. A little more computation then shows that

$$Spec_\alpha(S^1) = \left\{ \left(k + \int_{S^1} \alpha \right)^2, k \in \mathbb{Z} \right\}$$

Contents

The main tool

Theorem

[KS99] The spectrum of each of the Δ_α determines the following exact trace formula.

$$\sum_n \delta_{\lambda-\lambda_n} = \frac{\mathcal{L}}{\pi} + (V - E - 1)\delta_\lambda + \frac{1}{\pi}\sum_{p\in PO} \mathcal{A}_p(\alpha)e^{i\lambda l_p}$$

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- *the first sum is over the eigenvalues including multiplicities*
- δ *is the Dirac-δ-distribution*
- $\mathcal{L}$ *denotes the length of the graph*
- *PO stands for periodic orbit*
- l_p *denotes the length of a PO*
- $\mathcal{A}_p(\alpha) = \tilde{l}_p \cos(\int_p \alpha) \prod_b \sigma_{bb'}$
- $\cos(\int_p \alpha)$ *is the phase factor or 'magnetic flux'*
- *the product incorporates the Kirchhoff boundary counditions*

Analysing the trace formula

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Corollary

The spectrum of a single Schrödinger operator determines the total edge length and the Euler characteristic of a quantum graph.

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Each homology class of POs has a shortest representative. We call this PO minimal.

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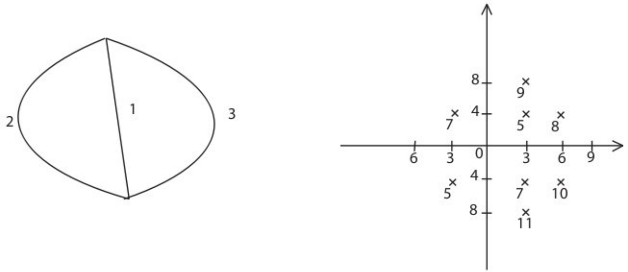
Theorem (R.)

The Bloch spectrum of G determines the length of a minimal PO of each element in $H_1(G, \mathbb{Z})$.

An example



A quantum graph and the lengths of some minimal POs.

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A graph is planar if and only if its cycle space has a simple basis.

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Remark

There are examples of a planar and a non planar quantum graph that are isospectral for the standard Laplacian [vB01].

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The dual of a graph is not necessarily unique. Any graph is a dual of the dual.

The Bloch spectrum determines a dual

Theorem (R.)

The Bloch spectrum determines a dual of a planar quantum graph.

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The Bloch spectrum identifies and uniquely determines
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Corollary (R.)

The Bloch spectrum identifies and uniquely determines 3*-connected planar quantum graphs.*

The theorem gives us the combinatorial graph, we still need to determine the edge lengths.

Thank you

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