Uniform Existence of the IDS on metric Cayley graphs

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Amenable Groups

Let $\mathcal{G}$ be a group, $\mathcal{S} \subseteq \mathcal{G}$ finite set of generators.
Let $\mathcal{F}$ denote the set of all finite subsets of $\mathcal{G}$.
$\mathcal{G}$ *amenable* $:\Longleftrightarrow$ there exists a *Følner sequence* $(Q_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$, i.e.,

$$\lim_{n\to\infty} \frac{|\mathcal{S}Q_n \setminus Q_n|}{|Q_n|} = 0.$$

$Q \in \mathcal{F}$ *symmetrically tiles* $\mathcal{G}$ with grid $T \subseteq \mathcal{G}$ $:\Longleftrightarrow$ $T = T^{-1}$, and

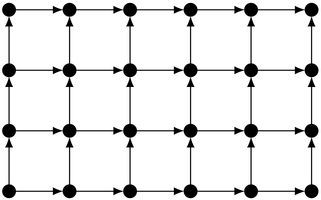
$$\mathcal{G} = \bigcup_{g\in T} Qg.$$

Cayley graphs as metric graphs

Let $\Gamma(\mathcal{G}, \mathcal{S}) = (\mathcal{V}, \mathcal{E}, \gamma)$ with

- vertex set $\mathcal{V} = \mathcal{G}$,
- edge set $\mathcal{E}$
- $\gamma = (\gamma_0, \gamma_1) : \mathcal{E} \to \mathcal{V}^{\{0,1\}}$ encoding starting and end vertices.

Edge e from v to w, if there is $s \in \mathcal{S}$: $w = sv$. Edge $e \sim (0,1)$.



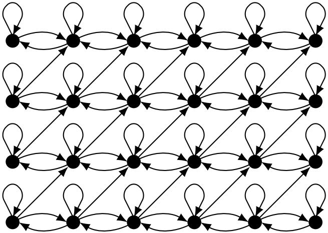


Figure: $\Gamma(\mathbb{Z}^2, \{(1,0),(0,1)\})$ and $\Gamma(\mathbb{Z}^2, \{(0,0),(1,1),(1,0),(-1,0)\})$

Schrödinger Operators

Hilbert space:

$$\mathcal{H}_\Gamma := \bigoplus_{e\in\mathcal{E}} L_2(0,1).$$

Let $\mathcal{B} \subseteq L_\infty(0,1)$ be finite and $V_e \in \mathcal{B}$ $(e \in \mathcal{E})$.
Maximal operator:

$$D(\hat{H}) := \bigoplus_{e\in\mathcal{E}} W_2^2(0,1),$$

$$(\hat{H}f)_e := -f_e'' + V_e f_e \quad (e \in \mathcal{E}).$$

For selfadjointness: boundary conditions.

Local boundary conditions

For $v \in \mathcal{V}$:

$$\mathcal{E}_{v,j} := \{e \in \mathcal{E};\ \gamma_j(e) = v\} \quad (j \in \{0,1\}),$$
$$\mathcal{E}_v := (\mathcal{E}_{v,0} \times \{0\}) \cup (\mathcal{E}_{v,1} \times \{1\}),$$

edges connected with v.
Then, for $f \in D(\hat{H})$: $\operatorname{tr}_v f, \operatorname{str}_v f' \in \mathbb{K}^{\mathcal{E}_v}$:

$$\operatorname{tr}_v f(e,j) := f_e(j) \quad ((e,j) \in \mathcal{E}_v),$$
$$\operatorname{str}_v f'(e,j) := (-1)^j f_e'(j) \quad ((e,j) \in \mathcal{E}_v).$$

Lagrangian subspace $U_v \subseteq \mathbb{K}^{\mathcal{E}_v} \oplus \mathbb{K}^{\mathcal{E}_v}$ with $\dim U_v = |\mathcal{E}_v|$ and

$$\left(f_1' \,\middle|\, f_2\right) - \left(f_1 \,\middle|\, f_2'\right) = 0 \quad ((f_1, f_1'), (f_2, f_2') \in U_v).$$

Selfadjoint realizations

$$D(H) = \left\{ f \in D(\hat{H});\ (\operatorname{tr}_v f, \operatorname{str}_v f')_{v\in\mathcal{V}} \in (U_v)_{v\in\mathcal{V}} \right\},$$
$$(Hf)_e = (\hat{H}f)_e = -f_e'' + V_e f_e \quad (e \in \mathcal{E}).$$

Dirichlet boundary conditions:

$$U_v^D = \{0\}^{\mathcal{E}_v} \otimes \mathbb{K}^{\mathcal{E}_v} \quad (v \in \mathcal{V}).$$

Dirichlet Laplacian:

$$D(-\Delta_D) = \bigoplus_{e\in\mathcal{E}} W_{2,0}^1 \cap W_2^2(0,1),$$
$$(-\Delta_D f)_e = -f_e'' \quad (e \in \mathcal{E}).$$

Randomization

$\mathcal{G}$ acts on Γ: for $e \in \mathcal{E}$ and $g \in \mathcal{G}$ exists unique edge $e \circ g$:

$$\gamma(e \circ g) = (\gamma_0(e)g^{-1}, \gamma_1(e)g^{-1}).$$

Probability space $(\Omega, \mathcal{A}, \mathbb{P})$, ergodic and measure preserving action $\alpha : \mathcal{G} \times \Omega \to \Omega$.
Let $\mathcal{B} \subseteq L_\infty(0,1)$ finite and $\mathcal{U}$ finite set of local boundary conditions at id.
Random potential: $V : \Omega \to \prod_{e\in\mathcal{E}} \mathcal{B}$,

$$V(\alpha_g(\omega))_{e\circ g} = V(\omega)_e \quad (g \in \mathcal{G}, e \in \mathcal{E}).$$

Random boundary conditions: $U : \Omega \to \prod_{v\in\mathcal{V}} \mathcal{U}$,

$$U(\alpha_g(\omega))_v = U(\omega)_{vg} \quad (g \in \mathcal{G}, v \in \mathcal{V}).$$

Random Schrödinger Operators

$(H_\omega)_{\omega\in\Omega}$ on $\mathcal{H}_\Gamma$:

$$D(H_\omega) := \left\{ f \in D(\hat{H});\ (\operatorname{tr}_v f, \operatorname{str}_v f')_{v\in\mathcal{V}} \in U(\omega) \right\},$$
$$(H_\omega f)_e := -f_e'' + V(\omega)_e f_e \quad (e \in \mathcal{E}).$$

H_ω selfadjoint ($\omega \in \Omega$), there is $C \geq 0$: $H_\omega \geq -C$ ($\omega \in \Omega$). (H_ω) ergodic random Schrödinger operator.

finite subgraphs

For $Q \in \mathcal{F}$: $\Gamma_Q = (\mathcal{V}_Q, \mathcal{E}_Q, \gamma_Q)$ defined by:

$$\mathcal{E}_Q := \bigcup_{v \in Q} \mathcal{E}_{v,0}, \quad \mathcal{V}_Q := Q \cup \mathcal{S}Q, \quad \gamma_Q := \gamma|_{\mathcal{E}_Q}.$$

inner vertices $\mathcal{V}_Q^i$ and boundary vertices $\mathcal{V}_Q^\partial$:

$$\mathcal{V}_Q^i := \left\{v \in \mathcal{V}_Q;\ \mathcal{E}_{v,0} \cup \mathcal{E}_{v,1} \subseteq \mathcal{E}_Q\right\}, \quad \mathcal{V}_Q^\partial := \mathcal{V}_Q \setminus \mathcal{V}_Q^i,$$

inner edges $\mathcal{E}_Q^i$ and boundary edges $\mathcal{E}_Q^\partial$:

$$\mathcal{E}_Q^i := \left\{e \in \mathcal{E}_Q;\ \gamma_0(e), \gamma_1(e) \in \mathcal{V}_Q^i\right\}, \quad \mathcal{E}_Q^\partial := \mathcal{E}_Q \setminus \mathcal{E}_Q^i.$$

Restricition to finite subgraphs

Restriction H_ω^Q of H_ω to Γ_Q on

$$\mathcal{H}_{\Gamma_Q} = \bigoplus_{e\in\mathcal{E}_Q} L^2(0,1)$$

by

$$D(H_\omega^Q) := \Big\{ f \in \bigoplus_{e\in\mathcal{E}_Q} W^{2,2}(0,1);\ (\operatorname{tr}_v f, \operatorname{str}_v f') \in U(\omega)_v \quad (v \in \mathcal{V}_Q^i),$$
$$(\operatorname{tr}_v f, \operatorname{str}_v f') \in U_v^D \quad (v \in \mathcal{V}_Q^\partial) \Big\},$$
$$(H_\omega^Q f)_e := -f_e'' + V(\omega)_e f_e \quad (e \in \mathcal{E}_Q).$$

H_ω^Q selfadjoint, semibounded from below, purely discrete spectrum.
Eigenvalue sequence $(\lambda_n(H_\omega^Q))_{n\in\mathbb{N}}$.

Eigenvalue counting function

Eigenvalue counting function: $n_\omega^Q : \mathbb{R} \to \mathbb{N}_0$,

$$n_\omega^Q(\lambda) := \left|\left\{n \in \mathbb{N};\ \lambda_n(H_\omega^Q) \le \lambda\right\}\right| = \operatorname{Tr} \mathbb{1}_{(-\infty,\lambda]}(H_\omega^Q).$$

Volume scaled version:

$$N_\omega^Q(\lambda) := \frac{1}{|\mathcal{E}_Q|} n_\omega^Q(\lambda) \quad (\lambda \in \mathbb{R}),$$

distribution function, unbounded.
For Dirichlet Laplacian:

$$N_D^Q(\lambda) := n_D(\lambda) \quad (\lambda \in \mathbb{R}),$$

n_D eigenvalue counting function of Dirichlet Laplacian on $L_2(0,1)$.

Spectral shift function

Define

$$\xi_\omega^Q(\lambda) := |\mathcal{E}_Q| \left(N_\omega^Q(\lambda) - N_D^Q(\lambda) \right) \quad (\lambda \in \mathbb{R}).$$

Then ξ_ω^Q bounded.

By Banach space valued ergodic theorem: $\left(\frac{\xi_\omega^{Q_j}}{\left|\mathcal{E}_{Q_j}\right|} \right)$ converges uniformly for a.a. $\omega \in \Omega$, where (Q_j) Følner sequence.

Uniform Existence of the IDS

Theorem

Let $(Q_l)_{l\in\mathbb{N}}$ be a Følner sequence in $\mathcal{G}$. Then there is $N : \mathbb{R} \to \mathbb{R}$ monotone increasing and right continuous (i. e. a distribution function), such that

$$\lim_{l\to\infty} \left\| N_\omega^{Q_l} - N \right\|_\infty = 0$$

for $\mathbb{P}$-a. a. $\omega \in \Omega$. In particular, $N_\omega^{Q_l} \to N$ pointwise for $\mathbb{P}$-a. a. $\omega \in \Omega$. Furthermore, for $\lambda \in \mathbb{R}$ and $Q \in \mathcal{F}$:

$$N(\lambda) = \frac{1}{|\mathcal{E}_Q|} \int_\Omega \operatorname{Tr}\left(\mathbb{1}_{\mathcal{E}_Q} \mathbb{1}_{(-\infty,\lambda]}(H_\omega)\right) \, d\mathbb{P}(\omega).$$

Note that $N(\lambda)$ does not depend on the choice of Q.

Related results

Theorem

There exist subsets $\Sigma, \Sigma_{pp}, \Sigma_{sc}, \Sigma_{ac}, \Sigma_{disc}, \Sigma_{ess} \subseteq \mathbb{R}$ *and* $\Omega' \subseteq \Omega$ *with* $\mathbb{P}(\Omega') = 1$ *such that* $\sigma(H_\omega) = \Sigma$ *and* $\sigma_\bullet(H_\omega) = \Sigma_\bullet$ *for all the spectral types* $\bullet \in \{pp, sc, ac, disc, ess\}$ *and all* $\omega \in \Omega'$.

Corollary

Σ *is the topological support of* μ, *the measure corresponding to* N.

Corollary

Let $D := \{f \in \mathcal{H}_\Gamma;\ \exists \mathcal{E}' \subseteq \mathcal{E} \text{ finite} : f_e = 0 \quad (e \in \mathcal{E} \setminus \mathcal{E}')\}$ *and*

$\Sigma_{comp} := \{\lambda \in \mathbb{R};\ \text{for } \mathbb{P}\text{-a. a. } \omega \in \Omega \text{ exists } f_\omega \in D(H_\omega) \cap D : H_\omega f_\omega = \lambda f_\omega\}$.

Then

$$\Sigma_{comp} = \{\lambda \in \mathbb{R};\ \mu(\{\lambda\}) > 0\}.$$

Thank you for your attention.