Non-standard limits for a family of autoregressive stochastic sequences

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An autoregressive sequence and its invariant distribution

- Let $\alpha \in (0,1)$, $\beta \in (0,\infty)$ and let ξ be a random variable with $\mathbb{E}\xi = 0$ and $0 < \sigma^2 := \mathbb{E}\xi^2 < \infty$. Let $(\xi_n)_{n\in\mathbb{N}}$ be i.i.d. copies of ξ.
- Consider the autoregressive sequence

$$X_{t+1} = \alpha X_t + \beta \xi_{t+1}, \quad t \in \mathbb{N}_0, \quad \text{and} \quad X_0 = 0.$$

- Let X have the stationary (invariant) distribution of $(X_t)_{t\in\mathbb{N}_0}$, i.e.,

$$X \overset{d}{=} \alpha X + \beta \xi.$$

- Let $(\xi_j')_{j\in\mathbb{N}_0}$ be i.i.d. copies of ξ. Then,

$$X \overset{d}{=} \beta \sum_{j=0}^{\infty} \alpha^j \xi_j'.$$

- It follows that

$$\operatorname{Var} X = \beta^2 \sum_{j=0}^{\infty} \alpha^{2j} \sigma^2 = \frac{\beta^2 \sigma^2}{1-\alpha^2}.$$

A family of autoregressive sequences

Let $(\alpha_m)_{m\in\mathbb{N}}$ and $(\beta_m)_{m\in\mathbb{N}}$ be sequences and $a \in (0,\infty)$ such that

$$1-\alpha_m \sim \frac{a}{m} \quad \text{and} \quad \beta_m \sim \frac{1}{\sqrt{m}} \quad \text{as} \quad m \to \infty.$$

Let $X^{(m)}$ be the invariant distribution of $(X_t^{(m)})_{t\in\mathbb{N}_0}$ such that

$$X_{t+1}^{(m)} = \alpha_m X_t^{(m)} + \beta_m \xi_{t+1}, \quad t \in \mathbb{N}_0, \quad \text{and} \quad X_0^{(m)} = 0,$$

i.e.,

$$X^{(m)} \stackrel{d}{=} \alpha_m X^{(m)} + \beta_m \xi.$$

Then,

$$X^{(m)} \stackrel{d}{=} \beta_m \sum_{j=0}^{\infty} \alpha_m^j \xi_j' \quad \text{and} \quad \operatorname{Var} X^{(m)} = \frac{\beta_m^2 \sigma^2}{1-\alpha_m^2} \sim \frac{\sigma^2}{m} \frac{1}{\frac{2a}{m}} = \frac{\sigma^2}{2a} \text{ as } m \to \infty$$

and

$$X^{(m)} \xrightarrow{d} N(0, \sigma^2/(2a)) \quad \text{as} \quad m \to \infty.$$

Heavy traffic scaling

The scaling

$$1-\alpha_m \sim \frac{a}{m} \quad \text{and} \quad \beta_m \sim \frac{1}{\sqrt{m}} \quad \text{as} \quad m \to \infty$$

is called heavy-traffic scaling.

If we apply the Euler-Maruyama method with step width $1/m$ to the SDE

$$dZ(s) = -aZ(s)ds + dB(s), \quad s \geq 0, \quad with \quad Z(0) = 0,$$

where $(B(s))_{s\geq 0}$ is a Brownian motion, we obtain

$$\widehat{Z}((k+1)/m) - \widehat{Z}(k/m) = -\frac{a}{m}\widehat{Z}(k/m) + (B((k+1)/m) - B(k/m))$$

or

$$\widehat{Z}((k+1)/m) = \left(1-\frac{a}{m}\right)\widehat{Z}(k/m) + \frac{1}{\sqrt{m}}\xi_{k+1} \quad \text{with} \quad \xi_{k+1} \stackrel{d}{=} N(0,1).$$

Autoregressive sequences with restart mechanisms

For a sequence $(\gamma_m)_{m\in\mathbb{N}}$ and a bounded measurable set $A \subset \mathbb{R}$ with $0 \in \operatorname{int}(A)$ consider

$$Y_{t+1}^{(m)} = \alpha_m Y_t^{(m)} 1\{Y_t^{(m)} \notin \gamma_m A\} + \beta_m \xi_{t+1}, \quad t \in \mathbb{N}_0, \quad \text{and} \quad Y_0^{(m)} = 0.$$

Let $Y^{(m)}$ have the stationary distribution of $(Y_t^{(m)})_{t\in\mathbb{N}_0}$, i.e.,

$$Y^{(m)} \overset{d}{=} \alpha_m Y^{(m)} 1\{Y^{(m)} \notin \gamma_m A\} + \beta_m \xi.$$

Goal of this talk: What is the limiting distribution of $Y^{(m)}$ for $m \to \infty$?

Conjectures:

1) For $(\gamma_m)_{m\in\mathbb{N}}$ sufficiently small, $Y^{(m)} \xrightarrow{d} N(0, 1/(2a))$ as $m \to \infty$.
2) For $(\gamma_m)_{m\in\mathbb{N}}$ sufficiently large, $Y^{(m)} \xrightarrow{d} 0$ as $m \to \infty$.

Are there further regimes?

General framework

- ξ random vector in $\mathbb{R}^d$ with $\mathbb{E}\xi = 0$, $\mathbb{E}\|\xi\|^2 < \infty$ and $\Sigma := \mathbb{E}\xi\xi^T$ regular
- $(\alpha_m)_{m\in\mathbb{N}}$ sequence of positive random variables such that

$$1 - \mathbb{E}\alpha_m \sim \frac{a}{m} \quad \text{and} \quad 1 - \mathbb{E}\alpha_m^2 \sim \frac{2a}{m} \quad \text{as} \quad m \to \infty$$

- $(\beta_m)_{m\in\mathbb{N}}$ sequence of real-valued numbers with $\beta_m \sim \frac{1}{\sqrt{m}}$ as $m \to \infty$
- Measurable $A \subset \mathbb{R}^d$ with $B^d(0,\underline{r}) \subset A \subset B^d(0,\overline{r})$, $\underline{r},\overline{r} \in (0,\infty)$
- $Y^{(m)}$ random vector following the stationary distribution of

$$Y_{t+1}^{(m)} = \alpha_{m,t+1} Y_t^{(m)} \mathbb{1}\{Y_t^{(m)} \notin \gamma_m A\} + \beta_m \xi_{t+1}, \ t \in \mathbb{N}_0, \quad \text{and } Y_0^{(m)} = 0,$$

with $(\alpha_{m,t})_{t\in\mathbb{N}}$ i.i.d. copies of α_m and $(\xi_t)_{t\in\mathbb{N}}$ i.i.d. copies of ξ, i.e.,

$$Y^{(m)} \stackrel{d}{=} \alpha_m Y^{(m)} \mathbb{1}\{Y^{(m)} \notin \gamma_m A\} + \beta_m \xi.$$

Regenerative structure

The process $(Y_t^{(m)})_{t\in\mathbb{N}_0}$ is regenerative and coincides with the process

$$X_{t+1}^{(m)} = \alpha_{m,t+1} X_t^{(m)} + \beta_m \xi_{t+1}, \quad t \in \mathbb{N}_0, \quad \text{and} \quad X_0^{(m)} = 0,$$

until

$$\tau^{(m)} := \min\{t \geq 1 : X_t^{(m)} \in \gamma_m A\}.$$

Stationary distribution

Proposition: Foss/S. 2020

Let $m \in \mathbb{N}$ be such that $\mathbb{P}(\alpha_m \in [d/(d+1), 1)) > 0$ and $\mathbb{E}\alpha_m < 1$.

(i) Then

$$\tau^{(m)} < \infty \quad \text{a.s.} \quad \text{and} \quad \mathbb{E}e^{c\tau^{(m)}} < \infty \quad \text{for some} \quad c = c(m) > 0.$$

(ii) The unique stationary distribution of $(Y_t^{(m)})_{t \in \mathbb{N}_0}$ is given by

$$\pi^{(m)}(\cdot) = \frac{1}{\mathbb{E}\tau^{(m)}} \mathbb{E} \sum_{j=1}^{\tau^{(m)}} 1\{X_j^{(m)} \in \cdot\} = \frac{1}{\mathbb{E}\tau^{(m)}} \mathbb{E} \sum_{j=1}^{\infty} 1\{X_j^{(m)} \in \cdot\} 1\{\tau^{(m)} \geq j\}$$

and $Y_t^{(m)}$ converge to it in the total variation norm as $t \to \infty$.

(iii) For any $c > 1$, if $\mathbb{E}\alpha_m^c < 1$ and $\mathbb{E}||\xi||^c < \infty$, then $\mathbb{E}||Y^{(m)}||^c < \infty$.

Limit theorem

Theorem: Foss/S. 2020

Suppose that

$$\mathbb{E}Y^{(m)} \to \mu \in \mathbb{R}^d \quad \text{and} \quad \mathbb{E}\tau^{(m)} \to \widehat{\tau} \in [1, \infty] \quad \text{as} \quad m \to \infty.$$

If $\widehat{\tau} \in [1, \infty)$, assume additionally that there exists a random variable τ such that $\tau^{(m)} \xrightarrow{\mathbb{P}} \tau$ as $m \to \infty$. Then

$$Y^{(m)} \xrightarrow{d} Y := B_1 \cdot Z \quad \text{as} \quad m \to \infty,$$

where B_1 and Z are independent,

$$p := \mathbb{P}(B_1 = 1) = 1 - \mathbb{P}(B_1 = 0) = \begin{cases} 1 - \mathbb{E}\tau/\widehat{\tau}, & \widehat{\tau} \in [1, \infty), \\ 1, & \widehat{\tau} = \infty, \end{cases}$$

and the d-dimensional random vector Z has an absolutely continuous distribution that is characterised by the following properties:

Limiting distribution

Theorem: continuation

i) The random vector Z has the characteristic function

$$\varphi_Z(u) = \left(1 + \mathrm{i}\frac{\sqrt{2a}\langle u,\mu\rangle}{p\sqrt{u^T\Sigma u}} \int_0^{\frac{\sqrt{u^T\Sigma u}}{\sqrt{2a}}} \exp\left(\frac{t^2}{2}\right) dt\right) \exp\left(-\frac{u^T\Sigma u}{4a}\right)$$

for $u \in \mathbb{R}^d$.

ii) For any $v \in \mathbb{R}^d$,

$$\langle v, Z\rangle \overset{d}{=} \frac{\sqrt{v^T\Sigma v}}{\sqrt{2a}} B_{2,v}|N|,$$

where N and $B_{2,v}$ are two independent random variables with N having the standard normal distribution and $B_{2,v}$ such that

$$\mathbb{P}(B_{2,v} = 1) = \frac{1}{2} + \frac{\sqrt{\pi a}\langle v,\mu\rangle}{2p\sqrt{v^T\Sigma v}} \text{ and } \mathbb{P}(B_{2,v} = -1) = \frac{1}{2} - \frac{\sqrt{\pi a}\langle v,\mu\rangle}{2p\sqrt{v^T\Sigma v}}.$$

Limiting distribution

Theorem: continuation

iii) The density of Z is given by

$$f_Z(x) = \frac{\sqrt{a}^d}{\sqrt{\det(\Sigma)}\sqrt{\pi}^d} \exp(-ax^T\Sigma^{-1}x) + \widetilde{f}_Z(x), \ x \in \mathbb{R}^d,$$

where, for odd dimensions $d = 1, 3, 5, \ldots$ and $x \in \mathbb{R}^d$,

$$\widetilde{f}_Z(x) = \frac{(-2)^{\frac{d-1}{2}} a^{\frac{d+2}{2}}}{\sqrt{\det(\Sigma)}\kappa_{d-1}(d-1)!!p} \langle \Sigma^{-1}\mu, x\rangle h^{((d-1)/2)}(ax^T\Sigma^{-1}x)$$

and for even $d = 2, 4, \ldots$ and $x \in \mathbb{R}^d$,

$$\widetilde{f}_Z(x) = \frac{(-2)^{\frac{d}{2}} a^{\frac{d+3}{2}}}{\kappa_d d!!p} \langle \Sigma^{-1}\mu, x\rangle \int_{-\infty}^{\infty} h^{(d/2)}(a(x^T\Sigma^{-1}x + z^2))\, dz.$$

Here $h(s) = e^{-s}/\sqrt{s}$ for $s > 0$, $h^{(k)}$ is its kth derivative, and κ_d the volume of the d-dimensional unit ball.

Convergence of second moments

Proposition: Foss/S. 2020

i) If $\mathbb{E}\tau^{(m)} \to \infty$ as $m \to \infty$, then

$$\lim_{m\to\infty} \mathbb{E}\langle u, Y^{(m)}\rangle^2 = \frac{\mathbb{E}\langle u, \xi\rangle^2}{2a}$$

for all $u \in \mathbb{R}^d$.

ii) If $\mathbb{E}\tau^{(m)} \to \widehat{\tau} \in [1, \infty)$ as $m \to \infty$ and if there exists a random variable τ on the same probability space such that $\tau^{(m)} \xrightarrow{\mathbb{P}} \tau$ as $m \to \infty$, then

$$\lim_{m\to\infty} \mathbb{E}\langle u, Y^{(m)}\rangle^2 = \left(1 - \frac{\mathbb{E}\tau}{\widehat{\tau}}\right)\frac{\mathbb{E}\langle u, \xi\rangle^2}{2a}$$

for all $u \in \mathbb{R}^d$.

Proof of the limit theorem

i) From

$$Y^{(m)} \stackrel{d}{=} \alpha_m Y^{(m)} \mathbb{1}\{Y^{(m)} \notin \gamma_m A\} + \beta_m \xi$$

we obtain for $u \in \mathbb{R}^d$,

$$\begin{aligned}\varphi_{Y^{(m)}}(u) &= \mathbb{E}\mathbb{1}\{Y^{(m)} \notin \gamma_m A\} e^{\mathrm{i}\langle u, \alpha_m Y^{(m)} + \beta_m \xi\rangle} + \mathbb{E}\mathbb{1}\{Y^{(m)} \in \gamma_m A\} e^{\mathrm{i}\beta_m \langle u, \xi\rangle} \\ &= \varphi_\xi(\beta_m u)\big(\mathbb{E}\varphi_{Y^{(m)}}(\alpha_m u) + \mathbb{E}\mathbb{1}\{Y^{(m)} \in \gamma_m A\}(1 - e^{\mathrm{i}\alpha_m \langle u, Y^{(m)}\rangle})\big),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\frac{1 - \varphi_\xi(\beta_m u)}{\varphi_\xi(\beta_m u)} \varphi_{Y^{(m)}}(u) &= \mathbb{E}\varphi_{Y^{(m)}}(\alpha_m u) - \varphi_{Y^{(m)}}(u) \\ &\quad + \mathbb{E}\mathbb{1}\{Y^{(m)} \in \gamma_m A\}(1 - e^{\mathrm{i}\alpha_m \langle u, Y^{(m)}\rangle}).\end{aligned}$$

Assume that $Y^{(m)} \xrightarrow{d} Y$ as $m \to \infty$. Then, one can show

$$\frac{\mathbb{E}\langle u, \xi\rangle^2}{2} \varphi_Y(u) = -a\langle u, \varphi_Y'(u)\rangle + \mathrm{i}a\langle u, \mu\rangle + \frac{1-p}{2}\mathbb{E}\langle u, \xi\rangle^2.$$

Proof of the limit theorem

ii) The distribution of the one-dimensional projections $\langle v, Z \rangle$ follows from the formula for the characteristic function of Z.

iii) There is an inversion formula to compute the density from the characteristic function, which leads to some involved integrals. We guessed the density and could verify that it has the correct one-dimensional projections.

Hitting time

Recall that $\tau^{(m)} = \min\{t \geq 1 : X_t^{(m)} \in \gamma_m A\}$.

Theorem: Foss/S. 2020

Assume that $\alpha_m \xrightarrow{\mathbb{P}} 1$ as $m \to \infty$ and that

$$\frac{\gamma_m}{\beta_m} \to \varrho \in [0, \infty) \quad \text{as} \quad m \to \infty.$$

If there exists an $\varepsilon > 0$ such that

$$\mathbb{E} \inf \left\{ t \geq 1 : \sum_{i=1}^{t} \xi_i \in \varrho A + B^d(0, \varepsilon) \right\} = \infty,$$

then

$$\liminf_{m \to \infty} \mathbb{P}(\tau^{(m)} > j) > 0, \quad j \in \mathbb{N}, \quad \text{and} \quad \widehat{\tau} \equiv \lim_{m \to \infty} \mathbb{E}\tau^{(m)} = \infty.$$

Idea of the proof: Compare

$$X_t^{(m)} = \beta_m \sum_{i=1}^{t} \prod_{j=i+1}^{t} \alpha_{j,m} \xi_i \quad \text{and} \quad S_t^{(m)} = \beta_m \sum_{i=1}^{t} \xi_i, \quad t \in \mathbb{N}_0.$$

Examples

Let $d = 1$, let $\alpha_m = 1 - \frac{a}{m}$ and $\gamma_m = \beta_m = \frac{1}{\sqrt{m}}$ for $m \in \mathbb{N}$ and let ξ be uniformly distributed on $[-1, 1]$.

i) $A = [-1/2, 1/2]$
By symmetry, $\mathbb{E} Y^{(m)} = 0$ and $\mu = \lim_{m \to \infty} \mathbb{E} Y^{(m)} = 0$. Then

$$Y^{(m)} \xrightarrow{d} N(0, 1/(2a)) \quad \text{as} \quad m \to \infty.$$

ii) A is not symmetric
Open problem: Show that $\mathbb{E} Y^{(m)} \to \mu \in \mathbb{R}$ as $m \to \infty$.
By tightness, each subsequence possesses a convergent subsequence such that the expectations converge. Our result can be applied along such subsequences.

iii) $A = [-1, 1/2]$
Here, the limiting distributions must be supported on $[0, \infty)$, whence there is a unique limiting distribution. Thus,

$$Y^{(m)} \xrightarrow{d} |N(0, 1/(2a))| \quad \text{as} \quad m \to \infty.$$

Examples

Proposition: Foss/S. 2020

Let $d = 1$. Let $\alpha_m = 1 - a/m$ and $\beta_m = 1/\sqrt{m}$ for $m \in \mathbb{N}$ and let ξ be such that $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$ and the distribution of ξ has a density. Assume that

$$A = [-2, 1] \text{ and } \xi \text{ has support } [-1, 1].$$

Then, for any $\mu \in [0, 1/\sqrt{\pi a}]$, there exists a sequence $\{\gamma_m\}_{m\in\mathbb{N}}$ such that

$$\mathbb{E}Y^{(m)} \to \mu \quad \text{and} \quad \mathbb{E}\tau^{(m)} \to \infty \quad \text{as} \quad m \to \infty$$

and

$$Y^{(m)} \xrightarrow{d} B|N| \quad \text{as} \quad m \to \infty$$

with independent $N \overset{d}{=} N(0, 1/(2a))$ and B such that

$$\mathbb{P}(B = 1) = \frac{1}{2} + \frac{\sqrt{\pi a}\mu}{2} \quad \text{and} \quad \mathbb{P}(B = -1) = \frac{1}{2} - \frac{\sqrt{\pi a}\mu}{2}.$$

Examples

Proposition:

Let $d = 1$. Let $\alpha_m = 1 - a/m$ and $\beta_m = 1/\sqrt{m}$ for $m \in \mathbb{N}$, let ξ be uniformly distributed on $[-1, 1]$ and let $A = [-1, 1]$. For each $\tau_0 \in [1, \infty)$ there exists a sequence $(\gamma_m)_{m\in\mathbb{N}}$ such that

$$\mathbb{E}\tau^{(m)} \to \tau_0 \quad \text{and} \quad \tau^{(m)} \xrightarrow{\mathbb{P}} 1 \quad \text{as} \quad m \to \infty.$$

Then,

$$Y^{(m)} \xrightarrow{d} B_1 N \quad \text{as} \quad m \to \infty$$

with independent $N \stackrel{d}{=} N(0, 1/(2a))$ and B_1 such that

$$\mathbb{P}(B_1 = 1) = 1 - \frac{1}{\tau_0} \quad \text{and} \quad \mathbb{P}(B_1 = 0) = \frac{1}{\tau_0}.$$

Thank you

Thank you!

S. Foss and M. Schulte: Non-standard limits for a family of autoregressive stochastic sequences, arXiv:2011.09948.