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REGULARITY OF DISPLACEMENT AND STRESSES IN LINEAR AND NONLINEAR ELASTICITY WITH MIXED BOUNDARY CONDITIONS

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Abstract. Equations of linear and nonlinear elasticity with mixed boundary conditions are considered. The bounded domain is assumed to be regular in the sense of Gröger and to have a Lipschitz boundary. $\mathcal{W}^{1,p}$ regularity for the displacements and $\mathcal{L}^p$ regularity for the stresses is proved for some $p > 2$.

1 Introduction

In the present work, we establish regularity properties of solutions to nonlinear elasticity systems with nonsmooth data and mixed boundary conditions in nonsmooth domains. To be more precise, we prove $\mathcal{W}^{1,p}$ regularity of the displacement vector field $\boldsymbol{u}$ and $\mathcal{L}^p$ regularity of the stress tensor field $\boldsymbol{\sigma}$ with some $p > 2$, provided that the right hand side belongs to $\mathcal{W}^{-1,p}(\Omega)$. Moreover, the solution depends Lipschitz continuously on the data, and we trace the dependence of the Lipschitz constant on p. The specific maximum value of p depends only on the domain, the partition of its boundary into Dirichlet and Neumann parts, and the coercivity and boundedness constants of the nonlinear stress-strain relation.

Let us briefly review the existing literature concerning regularity in *linear* elasticity. We only mention Dahlberg et al. [1988], Dahlberg and Kenig [1990], Nicaise [1992] and the references therein. We point out that a result on $\mathcal{W}^{1,p}$ regularity of the displacement field in linear elasticity could also be obtained by a perturbation result in Šneĭberg [1974]. It is well known that one cannot expect $\mathcal{W}^{1,\infty}$ regularity of the displacement field even for smooth domains, cf. Nečas and Štípl [1976].

Recently, based on Šneĭberg's result, Brown and Mitrea [2009] showed $\mathcal{W}^{1,p}$ regularity for the solution of the Lamé system accounting for mixed boundary conditions in nonsmooth domains. They allow non-homogeneous Dirichlet boundary conditions but no volume forces. In contrast to their analysis, we consider also *nonlinear* equations and give bounds for p which depend only on the coercivity and boundedness constants of the operator. This result is of importance for instance for the discussion of elasticity systems with temperature dependent Lamé coefficients. Given that the temperature varies between certain bounds, our result implies the existence of a unique solution in $\mathcal{W}^{1,p}$ with p independent of the actual temperature field.

Shi and Wright [1994] proved results analogous to ours, but for domains with C^1 boundary. Their proof relies on reverse Hölder inequalities. Lipschitz dependence on the data is not shown. By contrast, we require only domains with Lipschitz boundary and apply the techniques of Gröger [1989] who proved $\mathcal{W}^{1,p}$ regularity for mixed boundary value problems with operators of divergence type.

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The paper is organized as follows. The main result and assumptions are summarized in the remainder of this section. Section 2 is devoted to the proof of Theorem 1.4 concerning a linear reference problem. The nonlinear problem and the proof of Theorem 1.1 is addressed in Section 3.

1.1. Main Result and Assumptions. We present our main results for bounded regular domains $\Omega \subset \mathbb{R}^n$ of arbitrary dimension $n \geq 1$. The boundary Γ is Lipschitz and it consists of a Dirichlet part Γ_D (of positive measure) and a Neumann part Γ_N. The assumptions and notation are summarized below.

Theorem 1.1 (Main Theorem)**.** *Suppose that Assumption 1.5 holds and define* $B_p\colon \mathcal{W}^{1,p}_D(\Omega) \to \mathcal{W}^{-1,p}(\Omega)$ *as*

$$\langle B_p(\boldsymbol{u}), \boldsymbol{v}\rangle = \int_\Omega \boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) : \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x \quad \forall \boldsymbol{u} \in \mathcal{W}^{1,p}_D(\Omega), \boldsymbol{v} \in \mathcal{W}^{1,p'}_D(\Omega). \tag{1.1}$$

Then there exists $p > 2$ such that B_p is continuously invertible. Moreover, the inverse is globally Lipschitz .

In fact, our proof is constructive in the sense that we give sufficient conditions for p to be admissible for Theorem 1.1. These values of p depend only on the domain, the partition of its boundary into Dirichlet and Neumann parts, and the coercivity and boundedness constants of the nonlinearity $\boldsymbol{b}$. This is made precise in the following proposition.

Proposition 1.2. *Let Assumption 1.5 (1)–(2) be fulfilled. Let $\{\boldsymbol{b}\}_\iota$ be a family of nonlinearities which satisfy Assumption 1.5 (3) and suppose that (1.5c)–(1.5d) holds uniformly for all ι. Then there exists $p > 2$ such that all induced operators $\{B_p\}_\iota$ are continuously invertible. Their inverses share a uniform Lipschitz constant.*

In particular, Theorem 1.1 states that the nonlinear system (note that $\boldsymbol{n}$ is the normal vector on Γ_N)

$$\left.\begin{aligned} -\operatorname{\mathbf{div}} \boldsymbol{b}(x, \boldsymbol{\varepsilon}(\boldsymbol{u})(x)) &= \boldsymbol{f}(x) \text{ in } \Omega \\ \boldsymbol{n}(x) \cdot \boldsymbol{b}(x, \boldsymbol{\varepsilon}(\boldsymbol{u})(x)) &= \boldsymbol{g}(x) \text{ on } \Gamma_N \\ \boldsymbol{u} &= 0 \quad\ \text{ on } \Gamma_D = \Gamma \setminus \Gamma_N \end{aligned}\right\} \tag{1.2}$$

has a unique solution $\boldsymbol{u} \in \mathcal{W}^{1,p}_D(\Omega)$ for all $\boldsymbol{f}$ and $\boldsymbol{g}$ which define elements of $\mathcal{W}^{-1,p}(\Omega)$. Moreover, the solution depends Lipschitz continuously on the data, i.e.,

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\mathcal{W}^{1,p}} \leq L\left(\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{\mathcal{W}^{-1,p}} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{\mathcal{W}^{-1,p}}\right).$$

In terms of the nonlinear stress-strain relation $\boldsymbol{\sigma} = \boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u}))$, (1.2) becomes

$$\begin{aligned} -\operatorname{\mathbf{div}} \boldsymbol{\sigma} &= \boldsymbol{f} \text{ in } \Omega \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} &= \boldsymbol{g} \text{ on } \Gamma_N \\ \boldsymbol{u} &= 0 \text{ on } \Gamma_D. \end{aligned}$$

Note that Theorem 1.1 implies that $\boldsymbol{\sigma} \in L^p(\Omega)^{n\times n}$ since $\boldsymbol{b}$ maps L^p to L^p, see (1.5). In case of the linear stress-strain relation $\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathbb{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u})$, (1.2) is the standard system of linear elasticity.

Remark 1.3. *We briefly comment on the case of nonhomogeneous Dirichlet boundary conditions $\boldsymbol{u} = \boldsymbol{u}_D$ on Γ_D. Suppose that an extension $\bar{\boldsymbol{u}}_D \in \mathcal{W}^{1,p}(\Omega)$ of the Dirichlet data $\boldsymbol{u}_D$ exists. Clearly, the domain of B_p in (1.1) can be extended to elements $\boldsymbol{u} \in \mathcal{W}^{1,p}(\Omega)$. We thus consider the problem*

$$\left.\begin{aligned} B_p(\boldsymbol{u}) &= \boldsymbol{F} \in \mathcal{W}^{-1,p}(\Omega) \\ \boldsymbol{u}(x) &= \boldsymbol{u}_D(x) \quad \text{on } \Gamma_D. \end{aligned}\right\} \tag{1.3}$$

We then define the shifted operator $\widetilde{B}_p \colon \mathcal{W}^{1,p}_D(\Omega) \to \mathcal{W}^{-1,p}(\Omega)$ by $\widetilde{B}_p(\cdot) := B_p(\cdot + \bar{\boldsymbol{u}}_D)$. Now, (1.3) is equivalent to $\widetilde{B}_p(\boldsymbol{u} - \bar{\boldsymbol{u}}_D) = \boldsymbol{F}$, and thus $\boldsymbol{u}$ is unique. It is easy to check that $\boldsymbol{b}(\cdot\,, \cdot + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_D))$ satisfies Assumptions (1.5b)–(1.5d) with the same constants as $\boldsymbol{b}$. Moreover, $\boldsymbol{b}(\cdot, 0 + \boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}_D)) \in \mathcal{L}^p_{n^2}(\Omega)$ holds which is sufficient according to Remark 1.6 (3). We conclude that we can prove the same regularity for solutions to (1.3) as for solutions to (1.2) and the solution operators $\widetilde{B}_p^{-1}$ and B_p^{-1} share the same Lipschitz constant.

We prove Theorem 1.1 by applying the technique of Gröger [1989]. To this end, we need a linear reference problem induced by the Riesz isomorphism for $\mathcal{W}^{1,2}_D(\Omega)$, endowed with the norm generated by $\boldsymbol{\varepsilon}(\cdot)$.

Theorem 1.4 (Reference problem)**.** *Suppose that Assumption 1.5 (1)–(2) holds and define $J_p \colon \mathcal{W}^{1,p}_D(\Omega) \to \mathcal{W}^{-1,p}(\Omega)$ by $J_p(\boldsymbol{u}) = -\,\mathbf{div}\,\boldsymbol{\varepsilon}(\boldsymbol{u})$. Then there exists $q > 2$ such that J_q is continuously invertible.*

Similarly as above, Theorem 1.4 states that system of linear elasticity

$$\left.\begin{aligned} -\,\mathbf{div}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) &= \boldsymbol{f}(x) \text{ in } \Omega \\ \boldsymbol{n} \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) &= \boldsymbol{g}(x) \text{ on } \Gamma_N \\ \boldsymbol{u} &= 0 \qquad \text{on } \Gamma_D \end{aligned}\right\} \tag{1.4}$$

has a unique solution $\boldsymbol{u} \in \mathcal{W}^{1,p}_D(\Omega)$ in particular for all $\boldsymbol{f} \in \mathcal{L}^2_n(\Omega)$ and $\boldsymbol{g} \in \mathcal{L}^2_n(\Gamma_N)$, which depends continuously on the data. Note that this is the standard elasticity system with Lamé coefficients $\mu = 1/2$ and $\lambda = 0$.

Assumption 1.5.

(1) *$\Omega \subset \mathbb{R}^n$ is a bounded domain of dimension $n \geq 1$, which is assumed to have a Lipschitz boundary, see [Grisvard, 1985, Def. 1.2.1.1]. The boundary Γ consists of disjoint subsets Γ_D and Γ_N. Γ_D has positive surface measure.*

(2) *The union $G := \Omega \cup \Gamma_N$ is assumed to be regular in the sense of [Gröger, 1989, Def. 2.2]. That is, G is bounded and for every $y \in \partial G$, there exist subsets U and $\widetilde{U}$ in $\mathbb{R}^n$ and a Lipschitz transformation (a Lipschitz bijective map with Lipschitz inverse) $\Phi \colon U \to \widetilde{U}$ such that U is an open neighborhood of y in $\mathbb{R}^n$ and that $\Phi(U \cap G)$ is one of the following sets:*

$$\begin{aligned} E_1 &:= \{x \in \mathbb{R}^n : |x| < 1,\ x_n < 0\} \\ E_2 &:= \{x \in \mathbb{R}^n : |x| < 1,\ x_n \leq 0\} \\ E_3 &:= \{x \in E_2 : x_n < 0 \text{ or } x_1 > 0\}. \end{aligned}$$

(3) *The nonlinear function $\boldsymbol{b} \colon \Omega \times \mathbb{R}^{n\times n}_{\text{sym}} \to \mathbb{R}^{n\times n}_{\text{sym}}$ satisfies*

$$\boldsymbol{b}(\cdot, 0) \in \mathcal{L}^\infty_{n^2}(\Omega) \tag{1.5a}$$

$$\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable} \tag{1.5b}$$

$$(\boldsymbol{b}(x, \boldsymbol{\varepsilon}) - \boldsymbol{b}(x, \hat{\boldsymbol{\varepsilon}})) : (\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}) \geq m\, |\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}|^2 \tag{1.5c}$$

$$|\boldsymbol{b}(x, \boldsymbol{\varepsilon}) - \boldsymbol{b}(x, \hat{\boldsymbol{\varepsilon}})| \leq M\, |\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}| \tag{1.5d}$$

for almost all $x \in \Omega$ and all $\boldsymbol{\varepsilon}, \hat{\boldsymbol{\varepsilon}} \in \mathbb{R}^{n\times n}_{\text{sym}}$ with constants $0 < m \leq M$. When applied to matrices, $|\cdot|$ denotes their Frobenius norm.

Remark 1.6.

(1) *Assumption (1) is equivalent to the uniform cone condition, see [Grisvard, 1985, Theorem 1.2.2.2]. In [Ting, 1972, Theorem 3], this assumption*

is used in the following way (see also [Kikuchi and Oden, 1988, Theorem 5.13]): There exists a finite open covering $\{U_\alpha\}$ of $\overline{\Omega}$ with the following properties:

(i) $U_0 \subset \Omega$ has a positive distance to Γ.

(ii) For every α, there exists a cone $K_\alpha = K(0, R_\alpha, \omega_\alpha)$—the segment of a ball in $\mathbb{R}^n$ with vertex at zero, radius R_α and aperture angle ω_α—such that $x + K_\alpha \subset \overline{\Omega}$ for all $x \in U_\alpha \cap \overline{\Omega}$.

Moreover, by refinement of the covering, it can be assumed that the cap of the cone $x + K_\alpha$ lies outside of U_α. That is, $x + (K_\alpha \cap \{y \in \mathbb{R}^n : |y| = R_\alpha\})$ does not intersect U_α, for all $x \in U_\alpha \cap \overline{\Omega}$. We refer to Figure 1.1 for an illustration.

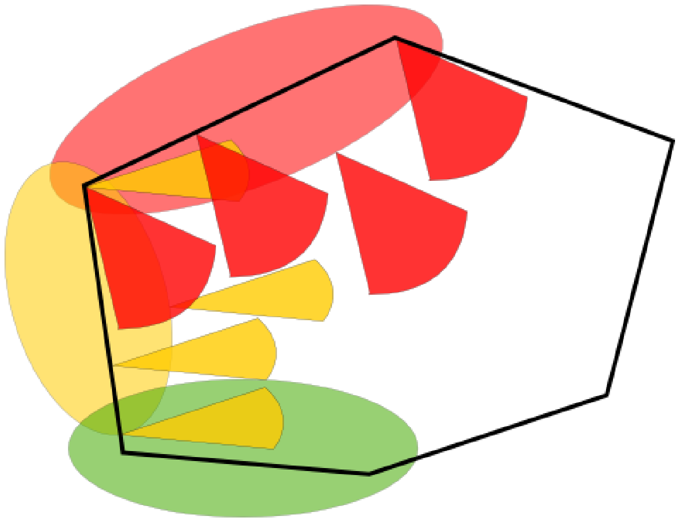

FIGURE 1.1. Domain Ω with some of its covering sets U_α and corresponding cones $x + K_\alpha$ for some $x \in U_\alpha$.

(2) *For details concerning the regularity condition of Gröger [1989], we refer to [Haller-Dintelmann et al., 2009, Section 5]. In particular, it implies that Ω is a Lipschitz domain (in the sense of [Grisvard, 1985, Chapter 1.2]). For $n = 2$, the regularity condition is equivalent to Ω being a bounded Lipschitz domain with Γ_N a relatively open subset of Γ such that $\overline{\Gamma}_N \cap \Gamma_D$ is finite and no connected component of Γ_D consists of a single point.*

(3) *Assumptions (1.5b) and (1.5d) ensure that $\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\cdot))$ is measurable for every measurable function $\boldsymbol{\varepsilon}$ with values in $\mathbb{R}^{n\times n}_{\text{sym}}$. Together with Assumption (1.5a) they imply that $\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\cdot)) \in \mathcal{L}^p_{n^2}(\Omega)$ whenever $\boldsymbol{\varepsilon} \in \mathcal{L}^p(\Omega)$, see [Precup, 2002, Section 5.1]. In fact, it is sufficient to require $\boldsymbol{b}(\cdot, 0) \in \mathcal{L}^q_{n^2}(\Omega)$ for some $q > 2$, which imposes an upper bound for p in Theorem 1.1.*

(4) *In the case of linear elasticity $\boldsymbol{b}(\cdot, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathbb{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u})$, (1.5c) indeed can hold only for symmetric matrices $\boldsymbol{\varepsilon}$ since $\mathbb{C}\,\boldsymbol{\varepsilon} = 0$ for skew-symmetric $\boldsymbol{\varepsilon}$. For the standard Lamé system with $\mathbb{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\,\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{tr}\,\boldsymbol{\varepsilon}(\boldsymbol{u})\,\boldsymbol{I}$ it is sufficient that $\mu > 0$ and $2\,\mu + n\,\lambda > 0$ for (1.5c) to hold.*

1.2. Notation. In this section, we define the spaces and norms. For $1 \le p \le \infty$, as usual p' denotes the dual or conjugate exponent such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 1.7. *We denote by $L^p(\Omega)$ the standard Lebesgue space on Ω. The space $\mathcal{L}^p_m(\Omega)$ is the space of vectors of $L^p(\Omega)$ functions of length m, or equivalently*

$L^p(\Omega \times \{1, \dots, m\})$. Its norm is given by

$$\|\boldsymbol{u}\|_{\mathcal{L}^p_m} := \left(\int_\Omega \sum_{i=1}^m |u_i(x)|^p \,\mathrm{d}x \right)^{1/p}. \tag{1.6}$$

For brevity we also denote this norm by $\|\cdot\|_p$ when the integer $m \in \{n, n^2, n^2+n\}$ is clear from the context.

Definition 1.8. *We denote by $W^{1,p}(\Omega)$ the usual Sobolev space on Ω equipped with the norm*

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |u(x)|^p + \sum_{i=1}^n |D_i u(x)|^p \,\mathrm{d}x \right)^{1/p}. \tag{1.7}$$

We define $W^{1,p}_D(\Omega)$ as the following subspace of $W^{1,p}(\Omega)$:

$$W^{1,p}_D(\Omega) = \mathrm{cl}\,\{u_{|\Omega} : u \in C_0^\infty(\mathbb{R}^n),\ \mathrm{supp}\, u \cap \Gamma_D = \emptyset\}$$

with the closure in $W^{1,p}(\Omega)$. In Gröger [1989], the same space is denoted by $W^{1,p}_0(\Omega)$.

Definition 1.9. *We define $\mathcal{W}^{1,p}(\Omega)$ as the space of vectors of $W^{1,p}(\Omega)$ functions of length n equipped with the norm*

$$\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}} := \left\| \begin{pmatrix} \boldsymbol{u} \\ \nabla \boldsymbol{u} \end{pmatrix} \right\|_{\mathcal{L}^p_{n+n^2}} = \left(\sum_{i=1}^n \|u_i\|^p_{W^{1,p}} \right)^{1/p}. \tag{1.8}$$

By $\mathcal{W}^{1,p}_D(\Omega)$ we denote the space of vectors of $W^{1,p}_D(\Omega)$ functions, which is a closed subspace of $\mathcal{W}^{1,p}(\Omega)$. Moreover, we denote by $\mathcal{W}^{-1,p'}(\Omega)$ the dual space of $\mathcal{W}^{1,p}_D(\Omega)$.

Remark 1.10. *We emphasize that Gröger [1989] uses different norms for the spaces $\mathcal{L}^p_n(\Omega)$ (denoted by Y_p there) and $W^{1,p}$, namely*

$$\|\boldsymbol{u}\|_{Y_p} := \left(\int_\Omega |\boldsymbol{u}(x)|_2^p \right)^{1/p}, \qquad \|u\|_{W^{1,p}} := \left\| \begin{pmatrix} u \\ \nabla u \end{pmatrix} \right\|_{Y_p},$$

where $|\boldsymbol{u}(x)|_2$ denotes the 2-norm of the vector $\boldsymbol{u}(x)$. However, all proofs of Gröger [1989], except for Theorem 1, will also work with our norms. Since we are using only the results of Theorem 3 and Lemma 1, the use of other norms is justified. We comment further on this issue when the results are used.

2 Proof of Theorem 1.4 (Reference Problem)

2.1. Preliminary Results. In this section we state three lemmas needed for the proof of Theorem 1.4.

Lemma 2.1. *For all $t, d \in (0,1)$, the function*

$$g\colon [2,\infty) \to \mathbb{R}, \quad g\colon p \mapsto \left(1 - t + t\, d^{1/p}\right)^p$$

is monotone decreasing.

Proof. We calculate the derivative

$$g'(p) = (1 - t + t\, d^{1/p})^{p-1} \left((1 - t + t\, d^{1/p}) \ln(1 - t + t\, d^{1/p}) - t\, d^{1/p} \ln(d^{1/p}) \right)$$

and observe that the first factor is positive for $t \in (0,1)$. With $h(x) = x \ln x$ we rewrite the second factor and use the convexity of h, which leads to

$$h((1-t) + t\, d^{1/p}) - t\, h(d^{1/p}) - (1-t)\, h(1) \le 0$$

since $h(1) = 0$. Thus, g is monotone decreasing. □

Now we are ready to prove

Lemma 2.2 (Coffee Break Inequality)**.** *For all* $t, k \in (0,1)$ *there exists a constant* $c = c(t,k) < 1$ *such that*

$$(1-t)^p a^p + \Big[(1-t)\, b + t\ (k\, b^p + a^p)^{1/p}\Big]^p \le c\, (a^p + b^p) \tag{2.1}$$

holds for all $p \ge 2$ *and* $a, b \in [0, \infty)$*.*

Proof. It is sufficient to prove the inequality for the case $a^p + b^p = 1$. For convenience, we set

$$T := (1-t)^p (1-b^p) + \Big[(1-t)\, b + t\ (k\, b^p + 1 - b^p)^{1/p}\Big]^p .$$

Now, choose $D \in (0,1)$ such that

$$U := \Big((1-t) + t\,\big(1-(1-k)\,D\big)^{1/2}\Big)^2 \le c \quad \text{and}$$
$$V := (1-t)^2 + \Big[(1-t)\, D^{1/2} + t\Big]^2 \le c,$$

with some $c < 1$. This is possible since $U = 1$ for $D = 0$ and U is strictly decreasing w.r.t. D and V is smaller than 1 for $D = 0$ and continuous w.r.t. D. One may equilibrate the two terms in order to get the best constant c. Now we show $T \le c$ by distinguishing two cases.

Case 1: $1 \ge b^p \ge D$

We estimate the innermost term of T and obtain

$$k\, b^p + 1 - b^p = 1 - (1-k)\, b^p \le 1 - (1-k)\, D =: \tilde{k} < 1.$$

Now, we insert this estimate into T and have

$$T \le (1-t)^p (1-b^p) + \Big[(1-t)\, b + t\, \tilde{k}^{1/p}\Big]^p .$$

It is easy to see that the right hand side is monotone increasing w.r.t. b by taking its derivative. And thus we have

$$T \le (1-t)^p (1-1^p) + \Big[(1-t)\, 1 + t\, \tilde{k}^{1/p}\Big]^p \le \Big[(1-t) + t\, \tilde{k}^{1/p}\Big]^p .$$

By using Lemma 2.1 we finally obtain

$$T \le \Big[(1-t) + t\, \tilde{k}^{1/2}\Big]^2 = U \le c.$$

Case 2: $b^p \le D$

We have

$$\begin{aligned} T &= (1-t)^p (1-b^p) + \Big[(1-t)\, b + t\ (k\, b^p + 1 - b^p)^{1/p}\Big]^p \\ &\le (1-t)^p + [(1-t)\, b + t]^p \\ &\le (1-t)^p + \Big[(1-t)\, D^{1/p} + t\Big]^p . \end{aligned}$$

By Lemma 2.1 we know that this function is monotone decreasing w.r.t. p and thus

$$T \le (1-t)^2 + \Big[(1-t)\, D^{1/2} + t\Big]^2 = V \le c.$$

To summarize, we have shown that $T \le c < 1$, which concludes the proof. □

2.2. Korn's Inequality. Korn's Inequality (of the second kind) states that for every $p \in (1, \infty)$, there exists a constant $K(n, p)$ such that

$$\|\nabla \boldsymbol{u}\|_p^p \leq K(n,p)\left(\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p + \|\boldsymbol{u}\|_p^p\right) \tag{2.2}$$

$$\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}}^p \leq \left(K(n,p)+1\right)\left(\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p + \|\boldsymbol{u}\|_p^p\right) \tag{2.3}$$

holds for all $\boldsymbol{u} \in \mathcal{W}^{1,p}(\Omega)$. Equation (2.2) is proved in Ting [1972] for bounded domains with Lipschitz boundary, and (2.3) is a trivial consequence. We need to verify that the constant is actually bounded w.r.t. p on compact intervals. This is the purpose of the following lemma.

Lemma 2.3. *For any $1 < q_1 \leq q_2 < \infty$ and with $n \in \mathbb{N}$ fixed, there exists $K > 0$ such that the constant $K(n,p) \leq K$ for every $p \in [q_1, q_2]$.*

Proof. Inequality (2.2) is proved in [Ting, 1972, Theorem 3] for $p \in (1, \infty)$. We trace the dependence of $K(n,p)$ on p throughout the paper to verify the assertion. For the remainder of this proof, references to equations and theorems in Ting [1972] appear in [brackets].

In [eq. (2.2)], we find the estimate of Calderon-Zygmund type

$$\|R_j f\|_p \leq C(n,p)\,\|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$, where R_j denotes the Riesz transform. The Riesz-Thorin interpolation theorem (see for instance Vogt [1998], Riesz [1927]) implies that

$$\|R_j\|_p \leq \|R_j\|_{q_1}^{1-\theta}\,\|R_j\|_{q_2}^{\theta} \leq C(n,q_1)^{1-\theta}\,C(n,q_2)^{\theta} \tag{$*$}$$

holds for all $p \in [q_1, q_2]$ and all $f \in L^p(\mathbb{R}^n)$, where $1/p = (1-\theta)/q_1 + \theta/q_2$. And hence $C(n,p)$ is bounded by $\max\{C(n,q_1), C(n,q_2)\}$ for all $p \in [q_1, q_2]$.

Next we verify that the constant in [Theorem 1, eq. (2.6)] is uniform w.r.t. p. We start from [eq. (2.8)], i.e.,

$$u_{i,j} = e_{ij} + \sum_{k=1}^{n}\left(R_k R_j e_{ki} - R_k R_i e_{jk}\right).$$

Here, $e_{ij} = (u_{i,j} + u_{j,i})/2$ is one component of $\boldsymbol{\varepsilon}(\boldsymbol{u})$. Taking norms, we get from ($*$)

$$\|u_{i,j}\|_p \leq C'(n,p)\big(\|e_{ij}\|_p + \sum_{k=1}^{n}(\|e_{ki}\|_p + \|e_{jk}\|_p)\big),$$

where $C'(n,p)$ is bounded for $p \in [q_1, q_2]$. By summation, we obtain the desired result [eq. (2.6)], i.e.,

$$\sum_{i,j=1}^{n}\int_\Omega |u_{i,j}|^p\,dx \leq C''(n,p)\sum_{i,j=1}^{n}\int_\Omega |e_{ij}(u)|^p\,dx,$$

where $C''(n,p)$ is bounded for $p \in [q_1, q_2]$.

We now turn to [Theorem 3]. The constant in [eq. (4.3)] depends only on the L^∞-norms of the derivatives of the partition of unity functions η_α and on n but not on p. Passing from [eq. (4.3)] to [eq. (4.5)], the constant changes and it will depend on p (see [eq. (4.4)]), but it remains bounded. In the proof of [eq. (4.6)], we need to apply again the Calderon-Zygmund type inequality after [eq. (4.9)], whose constant was already shown to be bounded w.r.t. $p \in [q_1, q_2]$. □

2.3. The Operator L_p and its Adjoint. We follow the notation in Gröger [1989] and define the mapping $L_p \colon \mathcal{W}_D^{1,p}(\Omega) \to \mathcal{L}^p_{n+n^2}(\Omega)$

$$L_p \boldsymbol{u} = \begin{pmatrix} \boldsymbol{u} \\ \nabla \boldsymbol{u} \end{pmatrix}.$$

To calculate its adjoint, we have to take into account the choice of boundary conditions. Since we have homogeneous Dirichlet conditions only on Γ_D, we have $\nabla^\star \neq -\mathbf{div}$ and also $\nabla^\star \nabla \neq -\Delta$. To find the adjoint of the gradient, we start with two test functions $\boldsymbol{u} \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $\boldsymbol{v} \in C_0^\infty(\mathbb{R}^n;\mathbb{R}^{n\times n})$ which are zero in a neighbourhood of Γ_D. We have

$$\begin{aligned}\langle \nabla \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathcal{L}^p,\mathcal{L}^{p'}} &= \int_\Omega \nabla \boldsymbol{u} : \boldsymbol{v} \, \mathrm{d}x \\ &= \int_\Omega -\boldsymbol{u}\cdot \mathbf{div}\, \boldsymbol{v}\, \mathrm{d}x + \int_{\Gamma_N} \boldsymbol{u}^\top \cdot \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}x = \langle \boldsymbol{u}, \nabla^\star \boldsymbol{v}\rangle_{\mathcal{W}_D^{1,p},\mathcal{W}^{-1,p'}},\end{aligned}$$

and thus formally, $\nabla^\star \boldsymbol{v} = -\mathbf{div}\, \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{n})|_{\Gamma_N}$ holds as elements in $\mathcal{W}^{-1,p'}(\Omega)$. Therefore the adjoint of ∇ contains not only the negative divergence, but also some information about the normal trace along the Neumann boundary. Let us write $-\nabla^\star = \overline{\mathbf{div}}$. In analogy to $\Delta = \mathbf{div}\,\nabla$, we write $\overline{\Delta} = \overline{\mathbf{div}}\,\nabla$. In its strong formulation, $-\overline{\Delta}\, \boldsymbol{u} = -\overline{\mathbf{div}}\,\nabla \boldsymbol{u} = \boldsymbol{f} + \boldsymbol{g}|_{\Gamma_N}$ corresponds to

$$-\Delta \boldsymbol{u} = -\mathbf{div}\,\nabla \boldsymbol{u} = \boldsymbol{f} \quad \text{in } \Omega, \qquad \frac{\partial}{\partial \boldsymbol{n}} \boldsymbol{u} = \boldsymbol{g} \quad \text{on } \Gamma_N.$$

Now it is easy to see that the adjoint of L_p is given by $L_p^\star \colon \mathcal{L}^{p'}_{n+n^2}(\Omega) \to \mathcal{W}^{-1,p'}(\Omega)$,

$$L_p^\star \begin{pmatrix} \boldsymbol{a} \\ A \end{pmatrix} = \boldsymbol{a} - \overline{\mathbf{div}}\, A.$$

Finally, we define

$$K_p := L_{p'}^\star L_p = I - \overline{\Delta},$$

which is a mapping from $\mathcal{W}_D^{1,p}(\Omega)$ to $\mathcal{W}^{-1,p}(\Omega)$. With a slight abuse of notation, we also denote the scalar version of $\overline{\Delta}$ by the same symbol, which maps $W_D^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$.

2.4. Solvability of an Uncoupled System. By [Gröger, 1989, Lemma 1], it is known that there exists some $q > 2$ such that $I - \overline{\Delta}$ is invertible as a mapping $W_D^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ for all $p \in [2, q]$. The norm of its inverse is denoted by M_p, which is bounded due to

$$M_p \le M_q^\theta, \tag{2.4}$$

where θ is defined by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$. Since the vector valued operator $K_p = I - \overline{\Delta}$ is not coupled, we have

$$(K_p)^{-1} = \begin{pmatrix} (I-\overline{\Delta})^{-1} & & 0 \\ & \ddots & \\ 0 & & (I-\overline{\Delta})^{-1} \end{pmatrix}$$

and

$$\|(K_p)^{-1}\|_{\mathcal{W}^{-1,p}\to\mathcal{W}_D^{1,p}} = M_p$$

holds also for the vector valued case. Note that this a consequence of the choice of the norm in $\mathcal{W}_D^{1,p}(\Omega)$. The next lemma follows from (2.4).

Lemma 2.4. *The operator norm of* $(I - \overline{\Delta})^{-1}$ *satisfies* $M_2 = 1$*, and for every* $\varepsilon > 0$ *there exists* $q > 2$ *such that* $M_p \le 1 + \varepsilon$ *for all* $p \in [2, q]$.

2.5. Proof of Theorem 1.4 (Reference Problem). The proof is done in two steps. In the first step, we use the idea of [Gröger, 1989, Theorem 1] to show that the operator $I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon}\colon \mathcal{W}^{1,q}_D(\Omega) \to \mathcal{W}^{-1,q}(\Omega)$ is boundedly invertible for some $q > 2$. In the second step, we show that $-\overline{\mathbf{div}}\,\boldsymbol{\varepsilon}$ is also boundedly invertible in the same spaces. Throughout this section, $t \in (0,1)$ is arbitrary.

Step 1: We define

$$B\begin{pmatrix}\boldsymbol{a}\\ A\end{pmatrix} = \begin{pmatrix}\boldsymbol{a} - t\,\boldsymbol{a}\\ A - \frac{t}{2}(A + A^\top)\end{pmatrix}$$

for $\boldsymbol{a} \in \mathcal{L}^p_n$ and $A \in \mathcal{L}^p_{n^2}$. Therefore we have

$$(L^\star_{p'} B L_p - K_p)\,\boldsymbol{u} = -t\,(I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon})\,\boldsymbol{u},$$

which is the operator under consideration (multiplied by $-t$). In order to use the proof of [Gröger, 1989, Theorem 1], we estimate $\|L^\star_{p'} B L_p\|$. Note that $L^\star_{p'} B L_p$ is an operator $\mathcal{W}^{1,p}_D(\Omega) \to \mathcal{W}^{-1,p}(\Omega)$. Thus we have

$$\begin{aligned}
\|L^\star_{p'} B L_p\| &= \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1} \|L^\star_{p'} B L_p \boldsymbol{u}\|_{\mathcal{W}^{-1,p}}\\
&= \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1}\ \sup_{\|\boldsymbol{v}\|_{\mathcal{W}^{1,p'}_D}\le 1} \langle L^\star_{p'} B L_p \boldsymbol{u}\,,\,\boldsymbol{v}\rangle_{\mathcal{W}^{-1,p},\mathcal{W}^{1,p'}_D}\\
&= \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1}\ \sup_{\|\boldsymbol{v}\|_{\mathcal{W}^{1,p'}_D}\le 1} \langle B L_p \boldsymbol{u}\,,\,L_{p'}\boldsymbol{v}\rangle_{\mathcal{L}^p,\mathcal{L}^{p'}}\\
&= \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1}\ \sup_{\|\boldsymbol{v}\|_{\mathcal{W}^{1,p'}_D}\le 1} \|B L_p \boldsymbol{u}\|_p \|L_{p'}\boldsymbol{v}\|_{p'}\\
&\le \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1} \left\|\begin{pmatrix}(1-t)\,\boldsymbol{u}\\ \nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\end{pmatrix}\right\|_p\\
&= \sup_{\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D}\le 1} \Big(\|(1-t)\,\boldsymbol{u}\|_p^p + \|\nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p\Big)^{1/p}.
\end{aligned} \tag{2.5}$$

In order to estimate the second term, we write $\nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = (1-t)\nabla\boldsymbol{u} + t\,\boldsymbol{r}(\boldsymbol{u})$ with the skew-symmetric part $\boldsymbol{r}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} - \nabla\boldsymbol{u}^\top)$ of the gradient. Therefore we have

$$\|\nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p \le (1-t)\|\nabla\boldsymbol{u}\|_p + t\,\|\boldsymbol{r}(\boldsymbol{u})\|_p. \tag{2.6}$$

Using Clarkson's Inequality

$$|\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}|^p + |\boldsymbol{u}_{i,j} - \boldsymbol{u}_{j,i}|^p \le 2^{p-1}\big(|\boldsymbol{u}_{i,j}|^p + |\boldsymbol{u}_{j,i}|^p\big),$$

taking sums for $i, j = 1, \ldots, n$ and integrating leads to

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p + \|\boldsymbol{r}(\boldsymbol{u})\|_p^p \le \|\nabla\boldsymbol{u}\|_p^p.$$

Korn's inequality (Lemma 2.3) implies

$$\|\boldsymbol{r}(\boldsymbol{u})\|_p^p \le \big(1 - K^{-1}\big)\,\|\nabla\boldsymbol{u}\|_p^p + \|\boldsymbol{u}\|_p^p.$$

Now (2.6) yields

$$\|\nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p \le (1-t)\,\|\nabla\boldsymbol{u}\|_p + t\,\left[(1-K^{-1})\,\|\nabla\boldsymbol{u}\|_p^p + \|\boldsymbol{u}\|_p^p\right]^{1/p}.$$

We return to (2.5) and estimate the term in parantheses:

$$\begin{aligned}
&\|(1-t)\,\boldsymbol{u}\|_p^p + \|\nabla\boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p\\
&\qquad\le (1-t)^p\|\boldsymbol{u}\|_p^p + \Big((1-t)\,\|\nabla\boldsymbol{u}\|_p + t\,\left[(1-K^{-1})\,\|\nabla\boldsymbol{u}\|_p^p + \|\boldsymbol{u}\|_p^p\right]^{1/p}\Big)^p\\
&\qquad\le c\,\|\boldsymbol{u}\|^p_{\mathcal{W}^{1,p}},
\end{aligned}$$

where $c < 1$ by Lemma 2.2 with $a = \|\boldsymbol{u}\|_p$, $b = \|\nabla \boldsymbol{u}\|_p$ and $k = 1 - K^{-1}$. This implies

$$\begin{aligned}\|L_{p'}^\star B L_p\| &\le \sup_{\|\boldsymbol{u}\|_{\mathcal{W}_D^{1,p}} \le 1} \Big(\|(1-t)\,\boldsymbol{u}\|_p^p + \|\nabla \boldsymbol{u} - t\,\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p^p \Big)^{1/p} \\ &\le c^{1/p} < 1.\end{aligned}$$

Now we can estimate $\|K_p^{-1} L_{p'}^\star B L_p\| \le M_p\, c^{1/p}$. Due to Lemma 2.4 we conclude $M_q\, c^{1/q} < 1$ for some $q > 2$. This implies that $Q_{\boldsymbol{f}} \colon \mathcal{W}_D^{1,q}(\Omega) \to \mathcal{W}_D^{1,q}(\Omega)$, defined by

$$Q_{\boldsymbol{f}}(\boldsymbol{u}) := K_q^{-1}(L_{q'}^\star B L_q \boldsymbol{u} + t\,\boldsymbol{f}) = \boldsymbol{u} - t\,K_q^{-1}\big((I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon})\,\boldsymbol{u} - \boldsymbol{f}\big)$$

with $\boldsymbol{f} \in \mathcal{W}^{-1,q}(\Omega)$, is a contractive operator. Therefore, $Q_{\boldsymbol{f}}$ has a unique fixed point, which solves $(I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon})\,\boldsymbol{u} = \boldsymbol{f}$. And thus $I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon} \colon \mathcal{W}_D^{1,q}(\Omega) \to \mathcal{W}^{-1,q}(\Omega)$ is invertible. Boundedness of the inverse follows from

$$\|\boldsymbol{u}\|_{\mathcal{W}_D^{1,q}} \le M_q\, c^{1/q}\, \|\boldsymbol{u}\|_{\mathcal{W}_D^{1,q}} + t\, M_q\, \|\boldsymbol{f}\|_{\mathcal{W}^{-1,q}},$$

since $M_q\, c^{1/q} < 1$.

Step 2: We now address the reference problem (1.4) which is governed by the operator $-\overline{\mathbf{div}}\,\boldsymbol{\varepsilon}$. Let $q > 2$ be such that $I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon} \colon \mathcal{W}_D^{1,q}(\Omega) \to \mathcal{W}^{-1,q}(\Omega)$ is boundedly invertible according to Step 1. Let $\boldsymbol{f} \in \mathcal{W}^{-1,q}(\Omega)$ be given. Since $\boldsymbol{f} \in \mathcal{W}^{-1,2}(\Omega)$, there is a unique solution $\boldsymbol{u} \in \mathcal{W}_D^{1,2}(\Omega)$ to

$$-\overline{\mathbf{div}}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) = \boldsymbol{f},$$

which is a standard result for the linear elasticity system with Lamé coefficients $\mu = 1/2$ and $\lambda = 0$. Clearly, $\boldsymbol{u}$ satisfies

$$(I - \overline{\mathbf{div}}\,\boldsymbol{\varepsilon})\,\boldsymbol{u} = \boldsymbol{f} + \boldsymbol{u}.$$

Note that $\boldsymbol{u} \in \mathcal{W}^{-1,q}(\Omega)$ for $q \in [2, \infty)$ for $n \le 4$ and $q \in [2, 2n/(n-4)]$ for $n > 4$ by the Sobolev embedding theorem. And hence we infer from Step 1 that $\boldsymbol{u}$ belongs to $\mathcal{W}_D^{1,q}(\Omega)$ and that $\mathcal{W}^{-1,q}(\Omega) \ni \boldsymbol{f} \mapsto \boldsymbol{u} \in \mathcal{W}_D^{1,q}(\Omega)$ is continuous. ☐

Remark 2.5. *In the proof above, $t \in (0,1)$ was arbitrary. Note that the constant c in Step 1 depends on t and thus the interval of admissible q satisfying $M_q\, c^{1/q} < 1$ also depends on t. A careful analysis would reveal the optimal t which leads to the largest such interval. Admissible values of q for Theorem 1.4 thus depend only on the domain.*

Remark 2.6. *We point out that Theorem 1.4 could also be proved by means of an abstract perturbation result of Šneĭberg [1974] (cf. also [Auscher, 2007, Lemma 4.16]). Let us briefly sketch how to apply Šneĭberg's result. According to Griepentrog et al. [2002], $\mathcal{W}_D^{\pm 1,p}(\Omega)$ is a scale of complex interpolation spaces. That is, there holds $\mathcal{W}_D^{\pm 1,p}(\Omega) = [\mathcal{W}_D^{\pm 1,p_0}(\Omega), \mathcal{W}_D^{\pm 1,p_1}(\Omega)]_\theta$ with $p_0, p_1 \in (1,\infty)$, $\theta \in (0,1)$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. The continuous invertibility of $J_2 \colon \mathcal{W}_D^{1,2}(\Omega) \to \mathcal{W}_D^{-1,2}(\Omega)$ is a simple consequence of the standard elliptic theory. Now Šneĭberg's result implies the existence of $q > 2$ such that $J_q \colon \mathcal{W}_D^{1,q}(\Omega) \to \mathcal{W}^{-1,q}(\Omega)$ is likewise continuously invertible, which is the assertion of Theorem 1.4.*

Nevertheless the analysis presented in this section offers more insight since in principle it allows us to specify admissible values of q for a given domain Ω. Notice further that the assertion of Theorem 1.1 cannot *be directly proved by means of Šneĭberg's result since it does not apply to nonlinear operators. Even for linear operators $\boldsymbol{b}$, one cannot expect a result of the form of Proposition 1.2 based on Šneĭberg's result because it does not provide uniform bounds for p in terms of the coercivity and boundedness constants of $\boldsymbol{b}$.*

3 Proof of Theorem 1.1 (Nonlinear Problem)

In order to apply the technique of proof in Gröger [1989], we work with the Riesz isomorphism $\widehat{J}_2$ of $\mathcal{W}^{1,2}_D(\Omega)$ with respect to the scalar product

$$\langle \boldsymbol{u}\,,\,\boldsymbol{v}\rangle_\varepsilon = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x.$$

This implies that $\|\widehat{J}_2^{-1}\| = 1$, see Lemma 3.1 below. In order to use again the Riesz-Thorin interpolation results, we endow $\mathcal{W}^{1,p}_D(\Omega)$ with a new norm

$$\|\boldsymbol{u}\|_{\widehat{\mathcal{W}}^{1,p}_D} := \left(\int_\Omega |\boldsymbol{\varepsilon}(\boldsymbol{u})|_2^p\,\mathrm{d}x\right)^{1/p}.$$

To avoid confusion, we denote this space by $\widehat{\mathcal{W}}^{1,p}_D(\Omega)$. Due to a variant of Korn's Inequality, the norms $\|\cdot\|_{\widehat{\mathcal{W}}^{1,p}_D}$ and $\|\cdot\|_{\mathcal{W}^{1,p}_D}$ are equivalent on $\mathcal{W}^{1,p}_D(\Omega)$, with equivalence constants depending on p in an unspecific way. The proof is given in the appendix (Proposition A.2).

In the same way, we equip $\mathcal{L}^p_{n^2}(\Omega)$ with a new norm

$$\|\boldsymbol{u}\|_{\widehat{\mathcal{L}}^p_{n^2}} := \left(\int_\Omega |\boldsymbol{u}|_2^p\,\mathrm{d}x\right)^{1/p}.$$

The space is then called $\widehat{\mathcal{L}}^p_{n^2}(\Omega)$.

In analogy to the operator L_p for the reference problem (see Section 2.3), we define $\widehat{L}_p \colon \widehat{\mathcal{W}}^{1,p}_D(\Omega) \to \widehat{\mathcal{L}}^p_{n^2}(\Omega)$ by $\widehat{L}_p := \boldsymbol{\varepsilon}$. Note that $\|\boldsymbol{u}\|_{\widehat{\mathcal{W}}^{1,p}_D} = \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{\widehat{\mathcal{L}}^p_{n^2}}$, and thus $\widehat{L}_p$ has norm one. Moreover, we define

$$\widehat{J}_p := -\,\overline{\mathbf{div}}\,\boldsymbol{\varepsilon} = \widehat{L}^\star_{p'}\widehat{L}_p \quad\text{and}\quad \widehat{M}_p = \|\widehat{J}_p^{-1}\|_{\widehat{\mathcal{W}}^{-1,p}\to\widehat{\mathcal{W}}^{1,p}_D}$$

for those $p \ge 2$ such that $\widehat{J}_p^{-1}$ exists and is bounded, see Lemma 3.2.

Lemma 3.1. *We have* $\widehat{M}_2 = 1$.

Proof. Let $\boldsymbol{u} \in \widehat{\mathcal{W}}^{1,2}_D(\Omega)$, $\boldsymbol{u} \neq 0$. We estimate

$$\|\widehat{J}_2\,\boldsymbol{u}\|_{\widehat{\mathcal{W}}^{-1,2}} = \sup\,\langle \widehat{J}_2\,\boldsymbol{u}\,,\,\boldsymbol{v}\rangle = \sup\,\langle \boldsymbol{u}\,,\,\boldsymbol{v}\rangle_\varepsilon = \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_2 = \|\boldsymbol{u}\|_{\widehat{\mathcal{W}}^{1,2}_D},$$

where the supremum extends over $\|\boldsymbol{v}\|_{\widehat{\mathcal{W}}^{1,2}_D} \le 1$. And thus $\widehat{J}_2$ is an isometric isomorphism. □

By Theorem 1.4 and the equivalence of the norms $\|\cdot\|_{\widehat{\mathcal{W}}^{1,p}_D}$ and $\|\cdot\|_{\mathcal{W}^{1,p}_D}$, there exists $p > 2$ such that $\widehat{J}_p$ is boundedly invertible. However, due to the unspecific p-dependent equivalence constants, the boundedness constant of $\widehat{J}_p^{-1}$ is yet unknown. This is addressed in the following lemma.

Lemma 3.2. *If $\widehat{J}_q$ is boundedly invertible for some $q \ge 2$, then $\widehat{J}_p$ is also boundedly invertible for all $p \in [2, q]$, and*

$$\widehat{M}_p \le \widehat{M}_q^\theta$$

holds, where $\theta \in [0,1]$ is given by $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{q}$.

Proof. The proof follows along the lines of [Gröger, 1989, Lemma 1]. The main difference is that we have $\widehat{L} = \boldsymbol{\varepsilon}$ while $L = (\mathrm{id}, \nabla)$ holds in Gröger [1989]. For convenience, we repeat the main arguments. The operators

$$\widehat{P}_2 := \widehat{L}_2\,\widehat{J}_2^{-1}\,\widehat{L}^\star_2 \quad\text{and}\quad \widehat{P}_q := \widehat{L}_q\,\widehat{J}_q^{-1}\,\widehat{L}^\star_{q'}$$

map $\widehat{\mathcal{L}}^2_{n^2}$ or $\widehat{\mathcal{L}}^q_{n^2}$, respectively, into themselves. It is easy to check that $\|\widehat{P}_2\| = 1$. Since $\|\widehat{L}_q\| = \|\widehat{L}_{q'}\| = 1$, we obtain $\|\widehat{P}_q\| \le \widehat{M}_q$. For $p \in [2, q]$, we define $\widehat{P}_p$ as the restriction of $\widehat{P}_2$ to $\widehat{\mathcal{L}}^p_{n^2}$. By the Riesz-Thorin interpolation theorem, $\widehat{P}_p$ maps $\widehat{\mathcal{L}}^p_{n^2}$ into itself. Moreover, we obtain

$$\|\widehat{P}_p\| \le \|\widehat{P}_2\|^{1-\theta} \|\widehat{P}_q\|^{\theta} \le \widehat{M}_q^{\theta}.$$

In a second step, we conclude the boundedness of $\widehat{J}_p^{-1}$ from the boundedness of $\widehat{P}_p$. To this end, let $\boldsymbol{f} \in \widehat{\mathcal{W}}^{-1,p}(\Omega)$. We define a functional $\boldsymbol{z}$ on the image of $\widehat{L}_{p'}$, which is a subset of $\widehat{\mathcal{L}}^{p'}_{n^2}$, as follows:

$$\langle \boldsymbol{z}, \widehat{L}_{p'} \boldsymbol{v} \rangle := \langle \boldsymbol{f}, \boldsymbol{v} \rangle$$

Note that $\boldsymbol{v} \in \widehat{\mathcal{W}}^{1,p'}_D$ is uniquely determined by $\widehat{L}_{p'} \boldsymbol{v} = \boldsymbol{\varepsilon}(\boldsymbol{v})$ by Korn's inequality, see Proposition A.2.

$$\|\boldsymbol{z}\|_p = \sup \frac{\langle \boldsymbol{z}, \widehat{L}_{p'} \boldsymbol{v} \rangle}{\|\widehat{L}_{p'} \boldsymbol{v}\|_{p'}} = \sup \frac{\langle \boldsymbol{f}, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|_{\widehat{\mathcal{W}}^{1,p'}_D}} = \|\boldsymbol{f}\|_{\widehat{\mathcal{W}}^{-1,p}},$$

where the supremum extends over all $\boldsymbol{v} \in \widehat{\mathcal{W}}^{1,p'}_D(\Omega)$, $\boldsymbol{v} \neq 0$. By the Hahn-Banach theorem, $\boldsymbol{z}$ can be extended to a continuous linear functional (again denoted by $\boldsymbol{z}$) on $\widehat{\mathcal{L}}^{p'}_{n^2}(\Omega)$ with the same norm. For $\boldsymbol{u} := \widehat{J}_2^{-1} \boldsymbol{f}$ we need to show $\boldsymbol{u} \in \widehat{\mathcal{W}}^{1,p}_D(\Omega)$ and establish an estimate for $\|\boldsymbol{u}\|_{\widehat{\mathcal{W}}^{1,p}_D}$.

We have

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \widehat{L}_2 \boldsymbol{u} = \widehat{L}_2 \widehat{J}_2^{-1} \boldsymbol{f} = \widehat{L}_2 \widehat{J}_2^{-1} \widehat{L}^{\star}_{p'} \boldsymbol{z} = \widehat{P}_p \boldsymbol{z} \in \widehat{\mathcal{L}}^p_{n^2}.$$

A standard cut-off argument now shows $\boldsymbol{u} \in \mathcal{W}^{1,p}(\Omega)$. The verification of boundary conditions, i.e., $\boldsymbol{u} \in \widehat{\mathcal{W}}^{1,p}_D(\Omega)$, follows as in the proof of [Gröger, 1989, Lemma 1]. Thus we conclude that $\widehat{J}_p^{-1}$ maps $\widehat{\mathcal{W}}^{-1,p}(\Omega)$ continuously into $\widehat{\mathcal{W}}^{1,p}_D(\Omega)$. Finally, by

$$\widehat{P}_p = \widehat{L}_p \widehat{J}_p^{-1} \widehat{L}^{\star}_{p'}$$

and the isometry property of $\widehat{L}_p$ and $\widehat{L}^{\star}_{p'}$, we obtain

$$\widehat{M}_p = \|\widehat{J}_p^{-1}\| = \|\widehat{P}_p\| \le \widehat{M}_q^{\theta}.$$

□

We now have the necessary tools to conclude Theorem 1.1.

Proof of Theorem 1.1 and Proposition 1.2. Recall that m and M are defined in Assumption 1.5 and set $k := (1 - m^2/M^2)^{1/2}$. Moreover, let $q > 2$ be such that $\widehat{J}_q \colon \widehat{\mathcal{W}}^{1,q}_D(\Omega) \to \widehat{\mathcal{W}}^{-1,q}(\Omega)$ is continuously invertible, according to Theorem 1.4. Then following the proof of [Gröger, 1989, Theorem 1], we obtain the Lipschitz estimate

$$\|B_p^{-1}(\boldsymbol{f}) - B_p^{-1}(\boldsymbol{g})\|_{\widehat{\mathcal{W}}^{1,p}_D} \le m\, M^{-2}\, \widehat{M}_p\, (1 - \widehat{M}_p\, k)^{-1}\, \|\boldsymbol{f} - \boldsymbol{g}\|_{\widehat{\mathcal{W}}^{-1,p}(\Omega)}, \tag{3.1}$$

provided that $p \in [2, q]$ and $\widehat{M}_p\, k < 1$. In view of $0 \le k < 1$ and the estimate $\widehat{M}_p \le \widehat{M}_q^{\theta}$, the condition $\widehat{M}_p\, k < 1$ holds for some $p > 2$. This is precisely the statement of Theorem 1.1, except that equivalent norms are used, which does not change the assertion. Moreover, we observe that any value of $p > 2$ such that $\widehat{M}_p\, k < 1$ holds is valid uniformly for all operators $\boldsymbol{b}$ which satisfy (1.5c)–(1.5d) with the same constants m and M. Together with (3.1), this implies the assertion of Proposition 1.2. □

Notice that for $m = M$ and thus $k = 0$, every $p \in [2, q]$ is admissible. The same holds if $\widehat{M}_q \leq 1$. In case $k \in (0, 1)$ and $\widehat{M}_q > 1$, a sufficient condition for $\widehat{M}_p\, k < 1$ is

$$\frac{1}{p} > \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{q}\right) \frac{|\log(k)|}{\log(\widehat{M}_q)}. \tag{3.2}$$

A Equivalence of Norms

Lemma A.1. *Let $1 < p < \infty$. Let f be a continuous seminorm in $\mathcal{W}^{1,p}(\Omega)$ with the property*

$$f(\boldsymbol{v}) = 0 \text{ and } \boldsymbol{v} \in \ker(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \boldsymbol{v} = 0.$$

Then $\|\cdot\|^\star = f(\cdot) + \|\boldsymbol{\varepsilon}(\cdot)\|_p$ is a norm on $\mathcal{W}^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{\mathcal{W}^{1,p}}$.

Proof. The inequality $\|\cdot\|^\star \leq C_1 \|\cdot\|_{\mathcal{W}^{1,p}}$ is clear. Let us assume that there is no $C_2 > 0$ such that $\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}} \leq C_2 \|\boldsymbol{u}\|^\star$ holds for all $\boldsymbol{u} \in \mathcal{W}^{1,p}(\Omega)$. Then we find a sequence $\boldsymbol{u}_m$ with $\|\boldsymbol{u}_m\|^\star \to 0$ and $\|\boldsymbol{u}_m\|_{\mathcal{W}^{1,p}} = 1$. Since $\mathcal{W}^{1,p}(\Omega)$ is compactly embedded into $\mathcal{L}^p(\Omega)$, there is a subsequence (again denoted by $\boldsymbol{u}_m$) which converges strongly to some $\boldsymbol{u}$ in $\mathcal{L}^p(\Omega)$. Using that $\boldsymbol{u}_m$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}_m)$ are Cauchy sequences in $\mathcal{L}^p(\Omega)$, Korn's Inequality (2.3) implies that $\boldsymbol{u}_m$ is a Cauchy sequence in $\mathcal{W}^{1,p}(\Omega)$ and thus $\boldsymbol{u}_m \to \boldsymbol{u}$. By continuity we know $\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}} = 1$ and $\|\boldsymbol{u}\|^\star = 0$. This implies $\boldsymbol{\varepsilon}(\boldsymbol{u}) = 0$ and $f(\boldsymbol{u}) = 0$ and therefore $\boldsymbol{u} = 0$, which is a contradiction to $\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}} = 1$. □

Lemma A.1 implies another inequality of Korn type. In contrast to (2.3), $\|\boldsymbol{u}\|_p$ does not appear on the right hand side. The inequality holds for $\boldsymbol{u} \in \mathcal{W}^{1,p}_D(\Omega)$, and the constant depends on p in an unspecific way. Note that Proposition A.2 implies that the norms $\|\cdot\|_{\widehat{\mathcal{W}}^{1,p}_D}$ and $\|\cdot\|_{\mathcal{W}^{1,p}_D}$ are equivalent on $\mathcal{W}^{1,p}_D(\Omega)$ since the reverse inequality is obvious.

Proposition A.2. *Let $1 < p < \infty$. There exists a constant $C = C(p)$ such that*

$$\|\boldsymbol{u}\|_{\mathcal{W}^{1,p}_D} \leq C \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_p$$

holds for all $\boldsymbol{u} \in \mathcal{W}^{1,p}_D(\Omega)$.

Proof. Choose $f(\boldsymbol{u}) = \left(\int_{\Gamma_D} |\boldsymbol{u}|^p \,\mathrm{d}x\right)^{1/p}$, which is continuous w.r.t. $\mathcal{W}^{1,p}(\Omega)$ by properties of the trace operator. Note that the displacements $\boldsymbol{u} \in \mathcal{W}^{1,p}(\Omega) \cap \ker(\boldsymbol{\varepsilon})$ are the rigid body motions, see e.g., Friesecke et al. [2002]. Thus Lemma A.1 implies the assertion. □

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