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Existence and Regularity of the Plastic Multiplier in Static and Quasistatic Plasticity

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Existence of the plastic multiplier with L^1 spatial regularity for quasistatic and static plasticity is proved for arbitrary continuous and convex yield functions and linear hardening laws. L^2 regularity is shown in the particular cases of kinematic hardening, or combined kinematic and isotropic hardening.

1 Introduction

We consider rate-independent models of static and quasistatic plasticity with hardening. These models are usually stated in terms of variational inequalities of the first or second kind, for the primal and dual formulations, respectively. In the static case, the dual variational inequality can be interpreted as the necessary and sufficient optimality conditions for an underlying optimization problem which aims to minimize the energy associated with the generalized stresses $\mathbf{\Sigma}$. In this context, the plastic multiplier can be viewed as the Lagrange multiplier associated with the admissibility constraint for the generalized stresses, $\Phi(\mathbf{\Sigma}) \le 0$, where Φ denotes the yield function. This interpretation of the plastic multiplier is valid in case of smooth yield functions.

The existence of the plastic multiplier in a pointwise sense, even in the case of nonsmooth convex yield functions, was shown, for instance, in [1, Lemma 4.7]. Indeed, relations analogous to ours but in a pointwise setting can already be found in [1, Section 4.2]. The purpose of this paper is to extend these results into function space. Using standard assumptions, we show that the plastic multiplier λ belongs to $L^2(0,T;L^1(\Omega))$ in the quasistatic case. The proof uses arguments from convex analysis, which can be found, for instance, in [5, 3]. Indeed, for practical models and yield functions, $\lambda \in L^2(0,T;L^2(\Omega))$ holds. For the important special cases of linear kinematic hardening (including perfect plasticity), or combined linear kinematic and isotropic hardening (including pure isotropic hardening), a proof is included in Section 3. In the case of static, or incremental, plasticity, $\lambda \in L^1(\Omega)$ respectively $\lambda \in L^2(\Omega)$ holds. The latter result was also shown in [2] using optimization methods.

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The plastic multiplier is not only of theoretical interest but appears also in computational approaches, see, for instance, [4, 8] and [6, Section 3.4].

1.1 Problem Setting and Assumptions

We consider quasistatic plasticity with hardening in its dual variational form: Find generalized stresses $\boldsymbol{\Sigma}$ and displacements $\boldsymbol{u}$ satisfying $\boldsymbol{\Sigma}(t) \in \mathcal{K}$ for almost all t, such that

$$a(\dot{\boldsymbol{\Sigma}}(t), \boldsymbol{T} - \boldsymbol{\Sigma}(t)) + b(\boldsymbol{T} - \boldsymbol{\Sigma}(t), \dot{\boldsymbol{u}}(t)) \geq 0 \quad \text{for all } \boldsymbol{T} \in \mathcal{K} \tag{1.1a}$$

$$b(\boldsymbol{\Sigma}(t), \boldsymbol{v}) = \langle \ell(t)\,,\, \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in V \tag{1.1b}$$

hold for almost all t. The underlying domain Ω is a bounded, open connected subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$. We introduce the spaces for $\boldsymbol{\Sigma}$ and $\boldsymbol{u}$ as

$$\mathcal{T} = L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}}) \times L^2(\Omega; \mathbb{R}^m), \quad V = \text{closed subspace of } H^1(\Omega)^d.$$

A common example for V are functions with homogeneous boundary conditions on some part of the boundary $\partial\Omega$. The bilinear forms $a : \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ and $b : \mathcal{T} \times V \to \mathbb{R}$ are assumed bounded. And thus they induce bounded linear operators $A : \mathcal{T} \to \mathcal{T}$ and $B : \mathcal{T} \to V'$. Both A and B should not depend on the time variable $t \in (0, T)$.

We assume that the yield function $\Phi : \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m \to [0, \infty)$ is a convex, positively homogeneous and continuous function and that the set K of admissible generalized stresses can be written as $K := \{\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m : \Phi(\boldsymbol{\Sigma}) \leq \tilde{\sigma}_0\}$. As usual, $\tilde{\sigma}_0 > 0$ is the initial yield stress. The feasible set $\mathcal{K}$ for (1.1) should depend neither on x nor t, thus we define $\mathcal{K} := \{\boldsymbol{\Sigma} \in \mathcal{T} : \boldsymbol{\Sigma}(x) \in K \text{ for almost all } x \in \Omega\}$. Finally, the right hand side $\ell(t)$ in (1.1) is taken in $H^1(0, T; V')$.

We assume that a solution $\boldsymbol{\Sigma} \in H^1(0, T; \mathcal{T})$ and $\boldsymbol{u} \in H^1(0, T; V)$ to (1.1) exists. However, we do not require the solution to be unique. An existence and uniqueness results can be found, for instance, in [1, Theorem 8.12] under additional assumptions.

1.2 Statement of the Main Result

The statement of our main results makes reference to the generalized plastic strain

$$\boldsymbol{P} = -A\boldsymbol{\Sigma} - B^*\boldsymbol{u} \in H^1(0, T; \mathcal{T}) \tag{1.2}$$

and to the subdifferential $\partial\Phi$ of the convex yield function Φ.

Theorem 1.1 *Under the assumptions stated above, there exists a plastic multiplier* $\lambda \in L^2(0, T; L^1(\Omega))$ *which satisfies* $\lambda \geq 0$ *and*

$$\dot{\boldsymbol{P}}(t, x) \in \lambda(t, x) \cdot \partial\Phi(\boldsymbol{\Sigma}(t, x)) \quad \textit{for a.a. } (t, x) \in (0, T) \times \Omega. \tag{1.3}$$

Moreover, $\lambda(t, x) = 0$ *where* $\Phi(\boldsymbol{\Sigma}(t, x)) < \tilde{\sigma}_0$ *holds. In addition, it can be represented as*

$$\lambda(t, x) = \frac{1}{\tilde{\sigma}_0} \langle \dot{\boldsymbol{P}}(t, x)\,,\, \boldsymbol{\Sigma}(t, x) \rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m} \quad \textit{for a.a. } (t, x) \in (0, T) \times \Omega. \tag{1.4}$$

Corollary 1.2 *The plastic multiplier satisfies* $\lambda \in L^2(0, T; L^2(\Omega))$ *in the following cases:*

(a) *linear kinematic hardening*

(b) *linear kinematic and isotropic hardening and the von Mises yield criterion.*

The precise assumptions for this corollary and a proof are given in Section 3.

Existence results for (1.1) often include the uniqueness of $\boldsymbol{\Sigma}$, see for instance, [1, Theorem 8.12]. The following remark states conditions under which the plastic multiplier and the displacement field $\boldsymbol{u}$ are unique, compare [1, pp. 200–202].

Remark 1.3 (Uniqueness) Suppose that $\partial\Phi$ is a singleton on the yield surface $\{\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m : \Phi(\boldsymbol{\Sigma}) = \tilde{\sigma}_0\}$. Suppose moreover that the solution $\boldsymbol{\Sigma}$ of (1.1) is unique. We split $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$ into stresses and conjugate forces and $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{\xi})$ into plastic strain and internal variables and assume that $B\boldsymbol{\Sigma}$ depends only on $\boldsymbol{\sigma}$, i.e., $B\boldsymbol{\Sigma} = B_1\boldsymbol{\sigma}$. This assumption is not restrictive since $\langle B\boldsymbol{\Sigma}\,,\, \boldsymbol{u}\rangle = -\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\sigma}\, \mathrm{d}x$ holds for systems in elastoplasticity.

Under these conditions, (1.2) implies the uniqueness of $\dot{\boldsymbol{\xi}}$. Then the second row of (1.3), i.e., $\dot{\boldsymbol{\xi}} = \lambda\, \partial_{\boldsymbol{\chi}} \Phi(\boldsymbol{\Sigma})$ shows that the plastic multiplier λ is unique. Now (1.3) implies the uniqueness of $\dot{\boldsymbol{P}}$, and the uniqueness of $\dot{\boldsymbol{u}}$ follows from (1.2).

For the static plasticity problem, the variational inequality (1.1) becomes

$$a(\boldsymbol{\Sigma}, \boldsymbol{T} - \boldsymbol{\Sigma}) + b(\boldsymbol{T} - \boldsymbol{\Sigma}, \boldsymbol{u}) \geq 0 \quad \text{for all } \boldsymbol{T} \in \mathcal{K} \tag{1.5a}$$

$$b(\boldsymbol{\Sigma}, \boldsymbol{v}) = \langle \ell\,,\, \boldsymbol{v}\rangle \quad \text{for all } \boldsymbol{v} \in V \tag{1.5b}$$

with $\ell \in V'$. A solution $\boldsymbol{\Sigma} \in \mathcal{T}$ and $\boldsymbol{u} \in V$ is assumed to exist.

Theorem 1.4 *In the static case, there exists a plastic multiplier $\lambda \in L^1(\Omega)$ which satisfies $\lambda \geq 0$ and $\boldsymbol{P}(x) \in \lambda(x) \cdot \partial\Phi(\boldsymbol{\Sigma}(x))$. Moreover, $\lambda(x) = 0$ where $\Phi(\boldsymbol{\Sigma}(x)) < \tilde{\sigma}_0$ holds. In addition, the relation $\lambda(x) = \frac{1}{\tilde{\sigma}_0}\langle \boldsymbol{P}(x)\,,\, \boldsymbol{\Sigma}(x)\rangle_{\mathbb{R}^{d\times d}_{\text{sym}}\times\mathbb{R}^m}$ holds almost everywhere in Ω.*

The static version of Corollary 1.2 holds as well.

2 Existence of the Plastic Multiplier

This section is devoted to the existence proof for the plastic multiplier λ and its $L^2(0,T;L^1(\Omega))$ regularity, as stated in Theorem 1.1. With $\boldsymbol{P} := -A\boldsymbol{\Sigma} - B^\star\boldsymbol{u}$, (1.1a) reads

$$\langle \dot{\boldsymbol{P}}(t)\,,\, \boldsymbol{T} - \boldsymbol{\Sigma}(t)\rangle_{\mathcal{T}} \leq 0 \quad \text{for all } \boldsymbol{T} \in \mathcal{K},$$

which is the maximum plastic work inequality. Here, $\langle \cdot\,,\, \cdot\rangle_{\mathcal{T}}$ is the inner product of $\mathcal{T}$. This inequality is equivalent to the statement

$$\dot{\boldsymbol{P}}(t) \in N_{\mathcal{K}}(\boldsymbol{\Sigma}(t)), \tag{2.1}$$

where $N_{\mathcal{K}}(\boldsymbol{\Sigma}(t))$ denotes the normal cone of $\mathcal{K}$ at $\boldsymbol{\Sigma}(t)$. We can thus continue to argue pointwise in time and omit the time variable for the time being.

Lemma 2.1 ([1, Lemma 4.5] or [3, p. 186]) *Let Z be a locally convex, linear topological Hausdorff space and let $f : Z \to \mathbb{R}$ be a proper convex function. Suppose that there exist $z_0, z_1 \in Z$ such that f is continuous at z_0 and $f(z_1) < f(z_0)$. Then the normal cone to the level set $\mathcal{L} = \{z \in Z : f(z) \leq f(z_0)\}$ at z_0 is equal to* $\operatorname{cone}\partial f(z_0)$*, the cone spanned by $\partial f(z_0)$, which consists exactly of all nonnegative multiples of elements in $\partial f(z_0)$.*

With this statement at hand we can calculate $N_{\mathcal{K}}(\widetilde{\boldsymbol{\Sigma}}) \subset \mathcal{T}'$ for arbitrary $\widetilde{\boldsymbol{\Sigma}} \in \mathcal{T}$. Since $\mathcal{T}$ is a Hilbert space, we identify it with $\mathcal{T}'$.

Lemma 2.2 *Let $\widetilde{\boldsymbol{\Sigma}} \in \mathcal{K}$ and $\boldsymbol{Q} \in N_{\mathcal{K}}(\widetilde{\boldsymbol{\Sigma}})$ be given. Then there exists $\widetilde{\lambda} \in L^1(\Omega)$, $\widetilde{\lambda} \geq 0$, such that*

$$\boldsymbol{Q}(x) \in \widetilde{\lambda}(x) \cdot \partial\Phi(\widetilde{\boldsymbol{\Sigma}}(x)) \quad \text{for almost all } x \in \Omega \tag{2.2a}$$

$$\widetilde{\lambda}(x) = 0 \quad \text{for almost all } x \in \Omega \text{ with } \Phi(\widetilde{\boldsymbol{\Sigma}}(x)) < \tilde{\sigma}_0. \tag{2.2b}$$

P r o o f. An element $\boldsymbol{Q} \in N_{\mathcal{K}}(\widetilde{\boldsymbol{\Sigma}})$ is characterized by the variational inequality

$$0 \geq \langle \boldsymbol{Q}\,, \boldsymbol{T} - \widetilde{\boldsymbol{\Sigma}} \rangle_{\mathcal{T}} \quad \text{for all } \boldsymbol{T} \in \mathcal{K}.$$

We can interpret this variational inequality pointwise, see for instance [7, p.54] for a proof. Thus we have

$$0 \geq \langle \boldsymbol{Q}(x)\,, \boldsymbol{T} - \widetilde{\boldsymbol{\Sigma}}(x) \rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m} \quad \text{for all } \boldsymbol{T} \in K \text{ and for almost all } x \in \Omega.$$

This implies $\boldsymbol{Q}(x) \in N_K(\widetilde{\boldsymbol{\Sigma}}(x))$ a.e. Let us denote by $\mathcal{A} = \{x \in \Omega : \Phi(\widetilde{\boldsymbol{\Sigma}}(x)) = \tilde{\sigma}_0\}$ the active set where $\widetilde{\boldsymbol{\Sigma}}$ lies on the yield surface. We employ Lemma 2.1 with $f = \Phi$ and $Z = \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m$, $z_0 = \widetilde{\boldsymbol{\Sigma}}(x)$, and we can take $z_1 = 0$. Note that at every $x \in \mathcal{A}$ the level set $\mathcal{L}$ coincides with K. We conclude that for every $x \in \mathcal{A}$ there exists a number $\widetilde{\lambda}(x) \geq 0$ with

$$\boldsymbol{Q}(x) \in \widetilde{\lambda}(x) \cdot \partial\Phi(\widetilde{\boldsymbol{\Sigma}}(x)).$$

This shows (2.2) on $\mathcal{A}$.

For $x \notin \mathcal{A}$, i.e., $\Phi(\widetilde{\boldsymbol{\Sigma}}(x)) < \tilde{\sigma}_0$, we have $\widetilde{\boldsymbol{\Sigma}}(x) \in \operatorname{int} K$ since Φ is continuous. Thus $N_K(\widetilde{\boldsymbol{\Sigma}}(x)) = \{0\}$ and we can choose $\widetilde{\lambda}(x) = 0$. Since $\partial\Phi(\widetilde{\boldsymbol{\Sigma}}(x)) \neq \emptyset$, both conditions of (2.2) hold also on $\Omega \setminus \mathcal{A}$.

It remains to show $\widetilde{\lambda} \in L^1(\Omega)$. From $\boldsymbol{Q}(x) \in \widetilde{\lambda}(x) \cdot \partial\Phi(\widetilde{\boldsymbol{\Sigma}}(x))$ a.e. on $\mathcal{A}$ we obtain the variational inequality

$$\widetilde{\lambda}(x)\,\Phi(\boldsymbol{T}) \geq \widetilde{\lambda}(x)\,\Phi(\widetilde{\boldsymbol{\Sigma}}(x)) + \langle \boldsymbol{Q}(x)\,, \boldsymbol{T} - \widetilde{\boldsymbol{\Sigma}}(x) \rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m} \quad \text{for all } \boldsymbol{T} \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m$$

for almost all $x \in \mathcal{A}$. We choose $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 2\,\widetilde{\boldsymbol{\Sigma}}(x)$ and use the positive homogeneity of Φ as well as $\Phi(\widetilde{\boldsymbol{\Sigma}}(x)) = \tilde{\sigma}_0$ to conclude

$$\widetilde{\lambda}(x)\,\tilde{\sigma}_0 = \langle \boldsymbol{Q}(x)\,, \widetilde{\boldsymbol{\Sigma}}(x) \rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m}$$

a.e. on $\mathcal{A}$. And thus $\widetilde{\lambda}$ coincides on $\mathcal{A}$ with the pointwise product of two L^2 functions while $\widetilde{\lambda} = 0$ holds on $\Omega \setminus \mathcal{A}$. Due to the measurability of $\mathcal{A}$ we have $\widetilde{\lambda} \in L^1(\Omega)$. □

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. We recall from (2.1) that $\dot{\boldsymbol{P}}(t,\cdot) \in N_{\mathcal{K}}(\boldsymbol{\Sigma}(t,\cdot))$ holds. The previous lemma with $\boldsymbol{Q} = \dot{\boldsymbol{P}}(t)$ implies the existence of $\lambda(t,\cdot) \in L^1(\Omega)$ for almost all $t \in (0,T)$, such that $\dot{\boldsymbol{P}}(t,x) \in \lambda(t,x) \cdot \partial\Phi(\boldsymbol{\Sigma}(t,x))$ holds. Moreover, we obtain $\lambda(t,x) = 0$ where $\Phi(\boldsymbol{\Sigma}(t,x)) < \tilde{\sigma}_0$, and thus also $\dot{\boldsymbol{P}}(t,x) = 0$ there. The proof of Lemma 2.2 also shows that

$$\lambda(t,x) = \frac{1}{\tilde{\sigma}_0} \langle \dot{\boldsymbol{P}}(t,x)\,,\, \boldsymbol{\Sigma}(t,x)\rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^m} \tag{2.3}$$

is valid a.e. on $\{(t,x) : \Phi(\boldsymbol{\Sigma}(t,x)) = \tilde{\sigma}_0\}$. In fact, (2.3) holds a.e. in $(0,T) \times \Omega$. Since by assumption $\dot{\boldsymbol{P}} \in L^2(0,T;\mathcal{T})$ and $\boldsymbol{\Sigma} \in H^1(0,T;\mathcal{T}) \hookrightarrow L^\infty(0,T;\mathcal{T})$ hold, we conclude $\lambda \in L^2(0,T;L^1(\Omega))$. □

The proof for the static case (Theorem 1.4) is similar and therefore omitted.

3 Improved Regularity in Special Cases

We now turn to the proof of Corollary 1.2.

3.1 Case (a): Kinematic Hardening

In the case of kinematic hardening, the generalized stress takes the form $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi})$, where $\boldsymbol{\chi}$ is the back stress and $\mathcal{T} = \mathcal{S} \times \mathcal{S}$ holds with $\mathcal{S} = L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$. Moreover, the generalized plastic strain can be decomposed as $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{\xi})$. We assume that the set of generalized stresses does not depend on the spherical part of $\boldsymbol{\sigma} + \boldsymbol{\chi}$, i.e., we have

$$K := \left\{\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} : \varphi(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) \le \tilde{\sigma}_0\right\}.$$

As usual, $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - \frac{1}{d}\text{tr}(\boldsymbol{\sigma})$ denotes the deviatoric part of $\boldsymbol{\sigma}$. We also assume that φ grows linearly in all directions, i.e.,

$$\varphi(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D) \ge c\,|\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D|$$

with some $c > 0$, where $|\cdot|$ denotes the Frobenius norm of matrices. This assumption is satisfied, for instance, by the two most important yield functions, of von Mises and Tresca type. The feasibility of $\boldsymbol{\Sigma} \in \mathcal{K}$ then implies

$$\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}_{\text{sym}})). \tag{3.1}$$

The particular form of $\Phi(\boldsymbol{\Sigma}) = \varphi(\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D)$ implies the following structure of the subdifferential:

$$\partial\Phi(\boldsymbol{\Sigma}(x,t)) \subset \left\{\boldsymbol{T} = (\boldsymbol{\tau}, \boldsymbol{\mu}) \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} : \boldsymbol{\tau} = \boldsymbol{\mu},\ \text{tr}(\boldsymbol{\tau}) = \text{tr}(\boldsymbol{\mu}) = 0\right\}.$$

Then $\dot{\boldsymbol{P}}(t,x) = (\dot{\boldsymbol{p}}(t,x), \dot{\boldsymbol{\xi}}(t,x)) \in \lambda(t,x) \cdot \partial\Phi(\boldsymbol{\Sigma}(t,x))$ in turn implies

$$\dot{\boldsymbol{p}} = \dot{\boldsymbol{p}}^D = \dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\xi}}^D \quad \text{a.e. on } \Omega \times (0,T).$$

From (2.3), we infer

$$\tilde{\sigma}_0\,\lambda = \langle \dot{\boldsymbol{P}}\,,\, \boldsymbol{\Sigma}\rangle_{\mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}}} = \dot{\boldsymbol{p}} : \boldsymbol{\sigma} + \dot{\boldsymbol{\xi}} : \boldsymbol{\chi} = \dot{\boldsymbol{p}}^D : (\boldsymbol{\sigma} + \boldsymbol{\chi}) = \dot{\boldsymbol{p}}^D : (\boldsymbol{\sigma}^D + \boldsymbol{\chi}^D)$$

a.e. in $(0,T)\times\Omega$. Since $\dot{\boldsymbol{p}}^D \in L^2(0,T;\mathcal{S})$ and in view of (3.1), we conclude $\lambda \in L^2(0,T;L^2(\Omega))$.

Remark 3.1 Indeed, the same regularity for λ could be proved with the arguments above in the case of perfect plasticity, where $\boldsymbol{\Sigma} = \boldsymbol{\sigma}$ and $\Phi(\boldsymbol{\Sigma}) = \varphi(\boldsymbol{\sigma})$, provided that $\varphi(\boldsymbol{\sigma}^D) \ge c\,|\boldsymbol{\sigma}^D|$ holds. However, the standing regularity assumption $\boldsymbol{u} \in H^1(0,T;V)$ is usually not satisfied.

3.2 Case (b): Kinematic and Isotropic Hardening

In the case of kinematic and isotropic hardening, the generalized stress is $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \boldsymbol{\chi}, \alpha)$, where $\alpha \leq 0$ corresponds to an expansion of the yield surface. We use the space $\mathcal{T} = \mathcal{S} \times \mathcal{S} \times L^2(\Omega)$ here with $\mathcal{S} = L^2(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$. The generalized plastic strain is denoted by $\boldsymbol{P} = (\boldsymbol{p}, \boldsymbol{\xi}, \gamma)$. As before, a solution $\boldsymbol{\Sigma} \in H^1(0,T;\mathcal{T})$ is assumed to exist. Then the generalized plastic strain $\boldsymbol{P} = -A\boldsymbol{\Sigma} - B^*\boldsymbol{u}$ likewise belongs to $H^1(0,T;\mathcal{T})$. We assume a feasible set of the form

$$K := \left\{\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R} : \varphi(\boldsymbol{\sigma} + \boldsymbol{\chi}) + \alpha \leq \tilde{\sigma}_0\right\}$$

with $\varphi : \mathbb{R}^{d\times d}_{\text{sym}} \to \mathbb{R}$ convex. In order to verify the L^2 regularity of the plastic multiplier, we use the subdifferentiability of the yield function

$$\Phi(\boldsymbol{\Sigma}) = \varphi(\boldsymbol{\sigma} + \boldsymbol{\chi}) + \alpha \quad \text{and} \quad \partial\Phi(\boldsymbol{\Sigma}) = \big(\partial\varphi(\boldsymbol{\sigma} + \boldsymbol{\chi}), \partial\varphi(\boldsymbol{\sigma} + \boldsymbol{\chi}), 1\big)^\top.$$

Now $\dot{\boldsymbol{P}}(t,x) = (\dot{\boldsymbol{p}}(t,x), \dot{\boldsymbol{\xi}}(t,x), \dot{\gamma}(t,x)) \in \lambda(t,x) \cdot \partial\Phi(\boldsymbol{\Sigma}(t,x))$ implies $\lambda(t,x) = \dot{\gamma}(t,x) \in L^2(0,T;L^2(\Omega))$.

Remark 3.2 Indeed, the same regularity for λ can be proved with the arguments above in the case of pure isotropic hardening, where $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}, \alpha)$ and $\Phi(\boldsymbol{\Sigma}) = \varphi(\boldsymbol{\sigma}) + \alpha$, provided that the standing regularity assumption $\boldsymbol{u} \in H^1(0,T;V)$ is satisfied.

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