Uniqueness criteria for the adjoint equation in state-constrained elliptic optimal control problems

Christian Meyer, Lucia Panizzi, and Anton Schiela

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Abstract

The paper considers linear elliptic equations with regular Borel measures as inhomogeneity. Such equations frequently appear in state-constrained optimal control problems. By a counter-example of Serrin [18], it is known that, in the presence of non-smooth data, a standard weak formulation does not ensure uniqueness for such equations. Therefore several notions of solution have been developed that guarantee uniqueness. In this note, we compare different definitions of solutions, namely the ones of Stampacchia [19] and Boccardo-Galouët [4] and the two notions of solutions of [7] and [2], and show that they are equivalent. As side results, we reformulate the solution in the sense of [19], and prove the existence of solutions in the sense of [4], [7], and [2] in case of mixed boundary conditions.

1 Introduction

In this paper we investigate different notions of solution for linear elliptic partial differential equations (PDEs) with measure valued right hand sides. Our study is motivated by the analysis of state constrained optimal control problems, where such equations appear as adjoint equations in first order optimality conditions. To be more precise, we consider the following PDE with mixed boundary conditions

$$\begin{aligned} -\nabla \cdot a^T \nabla p &= \mu_\Omega && \text{in } \Omega \\ \nu \cdot a^T \nabla p &= \mu_\Gamma && \text{on } \Gamma \\ p &= 0 && \text{on } \partial\Omega \setminus \Gamma, \end{aligned} \tag{AE}$$

where Ω is a Lipschitz domain, Γ a relatively open part of its boundary with outward normal ν, and a a uniformly elliptic, but non-smooth coefficient. Moreover, the inhomogeneities μ_Ω and μ_Γ are regular Borel measures. The precise assumptions on the data will be made at the end of this introduction.

Naturally, (AE) in its strong form is to be understood only formally, and several different notions of weak solutions can be found in literature. We point out that the standard definition of the variational formulation in the energy space $H^1(\Omega)$ fails in case of (AE), since $H^1(\Omega) \not\subset C(\bar{\Omega})$ unless $\Omega \subset \mathbb{R}$. Therefore, an alternative definition of solutions with a modified notion of weak formulations is necessary in case of (AE). In the present paper, we will investigate four different definitions of weak solutions, namely the ones dating back to Stampacchia [19] and Boccardo-Galouët [4] and two notions of solutions based on distributional derivatives. We will show that all these definitions are equivalent in the sense that they yield the same unique solution to (AE). As a side result of this note, we prove the existence of a solution of (AE) in the presence of mixed boundary conditions. Concerning the definition of solutions in the spirit of Stampacchia, a corresponding result was already proven in [13], but, up to our best knowledge, this was unknown in case of the other notions of solutions.

Some words concerning the motivation for a detailed analysis of (AE) are in order. Linear PDEs involving measures as inhomogeneity frequently appear in state-constrained optimal control problems, see for instance [7] or [8] and the references therein. The reason is that the state constraints have to be considered in a space that allows the associated constraint set to have non-empty interior to guarantee the existence of Lagrange multipliers. Often this space is chosen to be $C(\bar{\Omega})$, and the resulting Lagrange multipliers therefore are only elements of $\mathcal{M}(\bar{\Omega}) = C(\bar{\Omega})^*$, where $\mathcal{M}(\bar{\Omega})$ denotes the space of regular Borel measures. These multipliers enter the adjoint equation in the right-hand side resulting in an equation of the form (AE), see Section 1.2 below. Moreover, mixed boundary conditions and non-smooth coefficients play an important role in various applications. We only cite the references [17], [14], and [20], where corresponding examples are given.

Let us put our work into perspective. First, we point out that, according to the counter-examples in [18] and [16], the homogeneous counterpart to (AE) may admit other solutions outside the energy space H^1, which satisfy the associated variational equality for a restricted set of test functions. Due to the weak regularity of the inhomogeneity in (AE), one cannot expect the solution to belong to H^1 so that this simple uniqueness criterion fails in this case. Therefore, several authors aimed to develop alternative criteria which ensure the existence of a unique solution to (AE). We only mention the works of [19], [4], [16], and [13] for linear elliptic equations with measures as inhomogeneity. Nonlinear equations involving measures are investigated in [3] and [6]. While a concept of solutions for (AE) based on the dual equation is developed in [19] and [13], the authors of [4] define the solution of (AE) as the limit of solutions of regularized elliptic equations. In [16] both concepts are compared showing that the two types of solutions are the same. In the optimal control literature different notions of solutions to (AE) are common, see Casas [7] and Alibert and Raymond [2]. While Casas developed a concept of very weak solutions for (AE) in [7], the solution of [2] is defined as the solution of a (standard) variational equation and uniqueness is guaranteed by imposing an additional formula of integration by parts. In this note, we will show that both concepts carry over to the problem with mixed boundary conditions and yield the same solution as the concept of Stampacchia [19].

The paper is organized as follows: after introducing the main assumptions and notation, we give a short introduction in state-constrained optimal control motivating our analysis of (AE). Then we turn to the notion of solution in the sense of Stampacchia [19] in Section 2. This section also involves a modified definition of the Stampacchia solution and shows its equivalence to the solution of Boccardo-Galouët. Afterwards Section 3 is then devoted to the definitions of solutions in the sense of Casas [7] and Alibert and Raymond [2]. Section 3 can be read independently of Subsections 2.2 and 2.3. Finally, we summarize our findings in a conclusion.

1.1 Preliminaries

Assumption 1.1. Throughout this paper, we impose the following quite mild assumptions on the data in (AE):

- The domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded Lipschitz domain in the sense of [12, Chapter 1.2]. Moreover, Γ is a relatively open part of the boundary $\partial\Omega$ of Ω, and the relatively closed complement $\Gamma_D := \partial\Omega \setminus \Gamma$ is assumed to have positive measure.

In addition, the set $\Omega \cup \Gamma$ is regular in the sense of Gröger, cf. [11]. That is, for every point $x \in \partial\Omega$, there exists an open neighborhood $\mathcal{U}_x \subset \mathbb{R}^d$ of x and a bi-Lipschitz map $\Psi_x : \mathcal{U}_x \to \mathbb{R}^d$ such that $\Psi_x(x) = 0 \in \mathbb{R}^d$ and $\Psi_x\big(\mathcal{U}_x \cap (\Omega \cup \Gamma)\big)$ equals one of the following sets:

$$\mathcal{E}_1 := \{y \in \mathbb{R}^d : |y| < 1,\ y_n < 0\},$$
$$\mathcal{E}_2 := \{y \in \mathbb{R}^d : |y| < 1,\ y_n \leq 0\},$$
$$\mathcal{E}_3 := \{y \in \mathcal{E}_2 : y_n < 0 \text{ or } y_1 > 0\}.$$

- Let a be a Lebesgue measurable, essentially bounded function on Ω, taking its values in the set of real $d \times d$ matrices, that additionally satisfies the usual (strong) ellipticity condition
$$y \cdot a(x)\, y \geq \alpha |y|^2, \quad y \in \mathbb{R}^d, \tag{1}$$
for almost all $x \in \Omega$ and some $\alpha > 0$.
- The inhomogeneity in (AE) is a regular Borel measure $\mu \in \mathcal{M}(\Omega \cup \Gamma)$, where $\mathcal{M}(\Omega \cup \Gamma)$ is the space of regular Borel measures with its usual norm. Moreover, μ_Ω and μ_Γ denote the restrictions of μ to Ω and Γ, respectively and $\cdot$ denotes the euclidean scalar product in $\mathbb{R}^d$.

Remark 1.2. In the case $n = 2$, there is a simple characterization of Gröger regular sets. It is shown in [13] that for $\Omega \cup \Gamma \subset \mathbb{R}^2$ to be regular in the sense of Gröger it is necessary and sufficient that Ω is a Lipschitz domain and $\partial\Omega \setminus \Gamma$ is a finite union of closed arc pieces of $\partial\Omega$, none of which degenerates to a single point. Unfortunately, there is no such simple characterization in case of $n = 3$, cf. [13]. If however $\Omega \subset \mathbb{R}^3$ is a Lipschitzian polyhedron and $\overline{\Gamma} \cap \partial\Omega \setminus \Gamma$ is a finite union of line segments, it can be shown that $\Omega \cup \Gamma$ is regular in the sense of Gröger, see [13].

Some words addressing our notation are in order. If X is a Banach space, we write X^* for its dual. The associated dual pairing will be denoted by $\langle \cdot\,,\,\cdot \rangle_{X^*}$ or $\langle \cdot\,,\,\cdot \rangle_X$ and, if there is no risk for misunderstanding, we sometimes neglect the index. If Y is another Banach space, the space of linear and continuous operators from X to Y is denoted by $L(X, Y)$.

Next we introduce the function spaces that will be used throughout the paper. We define

$$\mathcal{D}(\Omega) = \big\{ v|_\Omega \,:\, v \in C^\infty(\mathbb{R}^d),\ \operatorname{supp} v \cap \partial\Omega = \emptyset \big\} \tag{2}$$
$$\mathcal{D}_\Gamma(\Omega) = \big\{ v|_\Omega \,:\, v \in C^\infty(\mathbb{R}^d),\ \operatorname{supp} v \cap \Gamma_D = \emptyset \big\} \tag{3}$$
$$\tilde{C}_\Gamma(\Omega) = \big\{ v \in C(\bar{\Omega}),\, v = 0 \text{ on } \Gamma_D \big\} \tag{4}$$

By the Riesz representation theorem $\mathcal{M}(\bar{\Omega}) \cong C(\bar{\Omega})^*$, and we conclude $\mathcal{M}(\Omega \cup \Gamma) \cong \tilde{C}_\Gamma(\Omega)^*$. The next lemma addresses a density result for $\tilde{C}_\Gamma(\Omega)$.

Lemma 1.3. *The set $\mathcal{D}_\Gamma(\Omega)$ is dense in $\tilde{C}_\Gamma(\Omega)$.*

Proof. See Lemma A.1. □

Now, let $2 \leq q < \infty$ (this restriction will be imposed throughout the paper) be given, and define q' as the conjugate exponent via $q'^{-1} + q^{-1} = 1$. We define

$$W^{1,q}_\Gamma(\Omega) = \overline{\mathcal{D}_\Gamma(\Omega)}^{W^{1,q}}, \quad W^{-1,q}_\Gamma(\Omega) := (W^{1,q'}_\Gamma(\Omega))^*. \tag{5}$$

In the following, we write $W_\Gamma^{1,2}(\Omega) = H_\Gamma^1(\Omega)$ and, to simplify the notation, also $W_\Gamma^{1,q} = W_\Gamma^{1,q}(\Omega)$ et cetera in case no confusion can arise. In view of a well-known Sobolev embedding theorem, we have that for $q > d$, which will be always assumed in the following, it holds

$$E : W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$$

By E_Γ we denote the restriction of this embedding:

$$E_\Gamma : W_\Gamma^{1,q}(\Omega) \hookrightarrow \tilde{C}_\Gamma(\Omega). \tag{6}$$

By Lemma 1.3 these embeddings have dense range. Note that if $v \in W_\Gamma^{1,q}(\Omega)$ then $v|_{\Gamma_D} = 0$. We pass now to define the bilinear form

$$\begin{aligned} a : W_\Gamma^{1,q} \times W_\Gamma^{1,q'} &\to \mathbb{R} \\ (v, w) \mapsto a(v, w) &:= \int_\Omega a\nabla v \cdot \nabla w \, dx. \end{aligned}$$

and the continuous mapping

$$\begin{aligned} A_q : W_\Gamma^{1,q} &\to W_\Gamma^{-1,q} \\ v \mapsto A_q v &:= a(v, \cdot), \end{aligned} \tag{7}$$

where $W_\Gamma^{-1,q} := (W^{1,q'})^*$. Because of our assumption on the ellipticity of $a(\cdot,\cdot)$, we have that $a(v, v) \ge \alpha \|v\|_{H_\Gamma^1}^2$. Then A_q is injective, since $A_q v = 0$ implies $0 = a(v, v) \ge \alpha \|v\|_{H_\Gamma^1}^2$, and thus $v = 0$. We notice that $A_2 : H_\Gamma^1 \to H_\Gamma^1$ is by the Lax-Milgram theorem even an isomorphism. However, for $q > 2$ continuous invertibility of A_q depends on regularity of the coefficients and the boundary. If A_q is an isomorphism, then we say that A_q enjoys *maximal regularity* with respect to q.

1.2 The adjoint equation for state constrained optimal control

In order to motivate our study of partial differential equations with measures, we exemplarily consider the following linear quadratic optimal control problem with pointwise constraints on the state:

$$\left.\begin{aligned} &\min \quad j(y, u) := \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 \\ &\text{s.t.} \quad a(y, v) = \int_\Omega u\, v\, dx \quad \forall\, v \in H_\Gamma^1 \\ &\text{and} \quad y(x) \le 1 \quad \text{a.e. in } \Omega \end{aligned}\right\} \tag{P}$$

with given $\alpha > 0$ und $y_d \in L^2(\Omega)$. It is well known that the derivation of first-order necessary conditions for problems of this type requires to consider the pointwise state constraints in a space Y whose topology allows the set $\{y \in Y : y(x) \le 1 \text{ a.e. in } \Omega\}$ to contain an interior point (also known as Slater point), cf. e.g. [7]. This is clearly the case if $Y \hookrightarrow \tilde{C}_\Gamma(\Omega)$, and indeed, under the hypotheses made in Assumption 1.1, the unique solution of the state equation in (P) is continuous, cf. Theorem 2.9 below. Thus we can introduce a linear and continuous control-to-state-mapping $S : L^2(\Omega) \to \tilde{C}_\Gamma(\Omega)$, and the reduced optimization problem reads

$$\text{(P)} \qquad \Leftrightarrow \qquad \left\{\begin{aligned} &\min \quad j(E_2 S u, u) \\ &\text{s.t.} \quad (Su)(x) \le 1 \quad \forall\, x \in \Omega \cup \Gamma, \end{aligned}\right.$$

where $E_2 : \tilde{C}_\Gamma(\Omega) \hookrightarrow L^2(\Omega)$ is the associated embedding operator. By means of the generalized Karush-Kuhn-Tucker theory, one shows in a standard way the existence of an adjoint state $p \in L^2(\Omega)$ and multiplier $\mu \in \mathcal{M}(\Omega \cup \Gamma) \cong \tilde{C}_\Gamma(\Omega)^*$ such that the following first-order optimality system is satisfied (cf. [7, Theorem 5.2])

$$p + \alpha u = 0 \tag{8a}$$

$$p = S^*(E_2^*(Su - y_d) - \mu) \tag{8b}$$

$$\langle \mu\,,\, Su\rangle_{\mathcal{M}(\Omega\cup\Gamma)} = 0, \quad (Su)(x) \le 1 \;\forall x \in \Omega \cup \Gamma \tag{8c}$$

$$\langle \mu\,,\, v\rangle_{\mathcal{M}(\Omega\cup\Gamma)} \;\forall\, v \in \tilde{C}_\Gamma(\Omega), v(x) \ge 0 \;\forall x \in \Omega \cup \Gamma. \tag{8d}$$

The crucial question is now how to interpret the abstract operator equation in (8b). If A_q enjoys maximal regularity, then $S = E_\Gamma A_q^{-1}\tilde{E}$, where E_Γ is defined as in (6) and $\tilde{E} : L^2(\Omega) \to W_\Gamma^{-1,q}$ is the embedding operator.

Consequently, we obtain $S^* = \tilde{E}^*(A_q^{-1})^* E_\Gamma^*$ with $A_q^* : W_\Gamma^{1,q'} \to W_\Gamma^{-1,q'}$, and therefore p solves the weak formulation of (AE) associated to the operator A_q^*, cf. Theorem 2.1 below. If however A_q is not longer maximal regular, then the situation changes and another notion of weak solutions associated to S^* is required. In the following, we will see that different notions of solutions exist and show that they are all equivalent in the sense that they deliver the same unique solution.

2 Solutions in the sense of Stampacchia

Our first approach to unique solvability of (AE) starts with the observation of the dual nature of this problem. The associated analysis proceeds in three steps. First, we formulate (AE) as an equation involving an adjoint operator A_q^*, then we show bijectivity of the pre-adjoint A_q in an $\|\cdot\|_\infty$ setting, and finally, we conclude bijectivity of A_q^* via the closed range theorem of functional analysis. We demonstrate this procedure for the comfortable case when A_q is maximal regular:

Theorem 2.1. *For some $q > d$ assume that A_q as defined in (7) enjoys maximal regularity. Then the equation*

$$a(v,p) = \int_\Omega v\, d\mu_\Omega + \int_\Gamma v\, d\mu_\Gamma \quad \forall v \in \mathcal{D}_\Gamma \tag{D0}$$

has a unique solution $p \in W_\Gamma^{1,q'}$.

Proof. Since $q > d$, $E_\Gamma : W_\Gamma^{1,q} \hookrightarrow \tilde{C}_\Gamma(\Omega)$ exists as a continuous embedding (cf. (6)), and thus by density of $\mathcal{D}_\Gamma$ in $W_\Gamma^{1,q}$, (D0) is equivalent to the operator equation

$$\langle A_q v, p\rangle_{W_\Gamma^{1,q'}} = \langle \mu, E_\Gamma v\rangle_{\tilde{C}_\Gamma(\Omega)} \quad \forall v \in W_\Gamma^{1,q}$$

or, involving adjoint operators: $A_q^* p = E_\Gamma^*\mu$, where E_Γ^* inherits continuity from E_Γ. By our assumption of maximal regularity, A_q is an isomorphism, and hence by the closed range theorem, A_q^* is an isomorphism as well. Thus, $p := (A_q^*)^{-1}E_\Gamma^*\mu$ is the unique solution of (D0). ☐

Remark 2.2. It is important to observe that our theorem only states uniqueness of p in the space $W_\Gamma^{1,q'}$ for the particular choice q for which A_q enjoys maximal regularity. If there is $\hat{q} > q$ such that $A_{\hat{q}}$ lacks maximal regularity, then there may be some $\hat{p} \in W_\Gamma^{1,\hat{q}'}$ that solves

(D0), as well. Thus, for uniqueness the *proper choice of space of candidates plays a decisive role*. This has already been pointed out in the work of Serrin [18]. Serrin constructed a Dirichlet problem with discontinuous coefficient a, and a pathologic solution $w \notin H_0^1$, such that $a(v,w) = 0$ forall $v \in \mathcal{D}$. By [16] it is possible for every $q' < 2$ to construct a problem with a pathologic solution $w \in W_0^{1,q'}$. A similar counter-example can be found in [10].

Remark 2.3. Due to the results of [11], A_q is always maximal regular for some $q > 2$. This means, provided that $d = 2$, i.e., in the two-dimensional setting, we always find some $q > d$ such that A_q enjoys maximal regularity. However, for each $q > 2$, there exists a problem such that A_q is not maximal regular, as the above mentioned counter-examples demonstrate. Therefore, in the important case $d = 3$, it is in general impossible to find a suitable choice of q.

For the rest of the paper, we will consider the much more delicate case, where A_q lacks maximal regularity for all $q > d$. In this case, the lack of surjectivity of A_q inhibits a direct application of the proof technique in Theorem 2.1. Therefore we will next define a "surjective variant" of A_q by considering A_q in a modified domain contained in the next definition.

Definition 2.4. We define the subspace D_q of $H_\Gamma^1(\Omega)$:

$$D_q := \{v \in H_\Gamma^1(\Omega) : \exists\, c_v \geq 0 \text{ with } |a(v,w)| \leq c_v \,\|w\|_{W_\Gamma^{1,q'}} \,\forall w \in H_\Gamma^1\}. \tag{9}$$

This space is often called *maximal domain* of A_q. Some observations about D_q are contained in the following remark.

Remark 2.5. Let D_q be defined as in (9). The following facts hold:

1. It is clear from the definition of a that $W_\Gamma^{1,q} \subset D_q \subset H_\Gamma^1$.
2. A_q enjoys maximal regularity (and thus Theorem 2.1 applies), if and only if $W_\Gamma^{1,q} = D_q$.
3. The density of $\mathcal{D}_\Gamma$ in H_Γ^1 implies

$$D_q = \{v \in H_\Gamma^1 : \exists\, c_v \geq 0 \text{ with } |a(v,\varphi)| \leq c_v \,\|\varphi\|_{W_\Gamma^{1,q'}} \,\forall \varphi \in \mathcal{D}_\Gamma\}.$$

Since H_Γ^1 is dense in $W_\Gamma^{1,q'}$, the definition of D_q implies that, for every $v \in D_q$, there is a unique continuous extension $f^{(v)} \in W_\Gamma^{-1,q}$ of $a(v,\cdot) \in H_\Gamma^{-1}$. Moreover, it is easily seen that $f^{(v)}$ is linear in v so that we can introduce an extended bilinear form

$$\begin{aligned} \bar{a} : D_q \times W_\Gamma^{1,q'} &\to \mathbb{R} \\ (v,w) &\mapsto \bar{a}(v,w) := \langle f^{(v)}\,,\, w\rangle_{W_\Gamma^{1,q'}} \end{aligned}$$

This bilinear form fulfills

$$\bar{a}(v,w) = a(v,w) \quad \forall\, v \in D_q, w \in H_\Gamma^1,$$

and can also be written in the explicit form

$$\bar{a}(v,w) = \lim_{w_k \to w} \int_\Omega a\nabla v \cdot \nabla w_k \, dx \quad \forall\, v \in D_q, \forall w \in W^{1,q'}(\Omega),$$

where w_k is an arbitrary sequence in H_Γ^1 that converges to w in $W^{1,q'}$. By definition of D_q the limit on the right hand side only depends on w, but not on the particular sequence $w_k \to w$.

Remark 2.6. One can in general not expect $\bar{a}$ to be expressed in form of a Lebesgue integral since $D_q \not\subset W^{1,q}_\Gamma$ unless A_q enjoys maximal regularity. This type of extension of integral expressions is also used in other branches of mathematics. Two important examples are the Fourier-Plancherel transform as continuous extension of the Fourier transform and the Itô integral for Brownian processes.

Based on the bilinear form $\bar{a}$, we introduce the following mapping:

Definition 2.7. The linear mapping from D_q to $W^{-1,q}_\Gamma$ induced by the bilinear form $\bar{a}$ is denoted by

$$\begin{aligned} \bar{A}_q : D_q &\to W^{-1,q}_\Gamma \\ v &\mapsto \bar{a}(v, \cdot). \end{aligned} \tag{10}$$

For $\bar{A}_q$, we find

Lemma 2.8. *The operator* $\bar{A}_q : D_q \to W^{-1,q}_\Gamma$ *is bijective.*

Proof. Injectivity follows just as for A_q. By construction of D_q as maximal domain, $\bar{A}_q$ inherits surjectivity from A_2. □

2.1 Solutions via a limit from

The question of continuity of $\bar{A}_q$ depends on the topology we use for D_q. Since $W^{1,q}_\Gamma \subset D_q$ and the inclusion is in general strict, see [13], we cannot use $\|\cdot\|_{W^{1,q}_\Gamma}$. However, by bijectivity of $\bar{A}_q$, we can equip D_q with an *initial topology* by assigning to each element of D_q the norm of its image:

$$\|v\|_{D_q} := \left\|\bar{A}_q v\right\|_{W^{-1,q}_\Gamma}. \tag{11}$$

This automatically makes $\bar{A}_q$ an isomorphism. Moreover, $(D_q, \|\cdot\|_{D_q})$ is complete, since $W^{-1,q}_\Gamma$, i.e. the image space of $\bar{A}_q$, is complete and $\bar{A}_q$ is an isomorphism. To make the definition of $\bar{A}_q$ and D_q useable for the discussion of (AE), we need the following regularity result proven in [13, Thm. 3.3]:

Theorem 2.9. *Let* $q > d$. *Then under Assumption 1.1 there is a continuous, compact, and dense embedding* $E_q : D_q \hookrightarrow \tilde{C}_\Gamma(\Omega)$.

Proof. In [13, Thm. 3.3] is shown that D_q is contained in a space of Hölder continuous functions, which is compactly embedded into $\tilde{C}_\Gamma(\Omega)$ by the Arzéla-Ascoli theorem. Density follows from $D_q \supset \mathcal{D}_\Gamma$, and Lemma 1.3. □

Observe that this theorem holds under weaker assumptions than maximal regularity for $q > d$, from which we might also conclude continuity of solutions via the Sobolev embedding (6). *From now on, we fix a* $q > d$ *such that the assertion of Theorem 2.9 holds.* With this result at hand, we can define our first notion of solutions:

Definition 2.10. Let $\mu \in \mathcal{M}(\Omega \cup \Gamma)$ be given. A function $p \in W^{1,q'}_\Gamma(\Omega)$ is a solution of (AE), if the equation

$$\bar{a}(v, p) = \int_\Omega v \, d\mu_\Omega + \int_\Gamma v \, d\mu_\Gamma \quad \forall \, v \in D_q \tag{D1}$$

is satisfied.

Remark 2.11. Let us shortly comment this definition. Suppose we would choose the space of test functions to be the smaller space $W_\Gamma^{1,q}$, then equation (D1) would read simpler:

$$a(v,p) = \int_\Omega v\,d\mu_\Omega + \int_\Gamma v\,d\mu_\Gamma \quad \forall\, v \in W_\Gamma^{1,q} \tag{12}$$

because on $W_\Gamma^{1,q}$, we have $a = \bar{a}$ by definition so that we would avoid the limit expression. However, (12) is not suitable for a definition of solutions in $W_\Gamma^{1,q'}$ since there is no uniqueness of the solution in this setting, as the counter-examples of Serrin [18] and Prignet [16] demonstrate. The reason is basically that $W_\Gamma^{1,q}$ is not dense in $(D_q, \|\cdot\|_{D_q})$. Therefore the definition with test functions in D_q is necessary. The drawback of the Definition 2.10 is that it is in general not possible to express equation (D1) through a limit relation that preserves the structure of the bilinear form as a Lebesgue integral. More precisely, by definition of $\bar{a}$, (D1) is equivalent to

$$\lim_{\substack{p_k \in H_\Gamma^1 \\ \|p_k - p\|_{W_\Gamma^{1,q'}} \to 0}} \int_\Omega a\nabla v \cdot \nabla p_k dx = \int_\Omega v\,d\mu_\Omega + \int_\Gamma v\,d\mu_\Gamma \quad \forall\, v \in D_q, \tag{13}$$

but, by the passage to the limit for $k \to \infty$ in the above equation, the bilinear form might not longer be expressed by a Lebesgue integral since $D_q \not\subset W_\Gamma^{1,q}$. Another different definition of solution which tries to overcome this drawback is discussed separately in the Subsection 2.3.

Theorem 2.12. *For each $\mu \in \mathcal{M}(\Omega \cup \Gamma)$, there is a unique solution of* (AE) *in the sense of Definition 2.10. It is also a (possibly non-unique) solution of* (12).

Proof. Equation (D1) can be expressed as

$$\langle \bar{A}_q v, p\rangle_{D_q} = \langle \mu, E_q v\rangle_{\tilde{C}_\Gamma(\Omega)} \quad \forall v \in D_q,$$

or, involving adjoints $\bar{A}_q^* p = E_q^* \mu$. Since $\bar{A}_q$ is an isomorphism, it follows by the closed range theorem that $\bar{A}_q^*$ is an isomorphism, too. Hence, $p = (\bar{A}_q^*)^{-1} E_q^* \mu$ is the unique solution of (AE) in the sense of Definition 2.10. Clearly, if p satisfies(13), then also (12). ☐

Remark 2.13. The solution of (D1) induced by the above theorem, is in the spirit of [19] (cf. also [13, Section 6.2]). Thus we will frequently term it as solution of Stampacchia in the following.

Remark 2.14. We may reduce the space of test functions D_q in (D1) to a smaller subspace. Since $\mathcal{D}$ is dense in $W_\Gamma^{-1,q}$, and $\bar{A}_q$ is an isomorphism,

$$\mathcal{D}_q := \bar{A}_q^{-1}(\mathcal{D}) = \{v \in D_q : \bar{A}_q v \in \mathcal{D}\}$$

is a dense subspace of D_q. So in (D1) and also in the definitions that follow, D_q may be replaced by the dense subspace $\mathcal{D}_q$. However, even for this smaller space $\mathcal{D}_q \not\subset W_\Gamma^{1,q}$.

2.2 Well-posedness and solutions of Boccardo-Galouët

The next lemma shows that p is well posed with respect to perturbations of μ.

Lemma 2.15. *Assume that $\mu_k \in \mathcal{M}(\Omega\cup\Gamma)$ converges to $\mu \in \mathcal{M}(\Omega\cup\Gamma)$ in the weak* sense. Let p_k be the solutions of* (D1) *with respect to μ_k. Then $p_k \to p$ strongly in $W_\Gamma^{1,q'}$ and p solves* (D1) *with right hand side μ.*

Proof. Since the embedding $E_q : D_q \hookrightarrow \tilde{C}_\Gamma(\Omega)$ is compact, so is its adjoint $E_q^* : \tilde{C}_\Gamma(\Omega)^* \to D_q^*$. Hence weak* convergence $\mu_k \rightharpoonup \mu$ implies strong convergence $E_q^*\mu_k \to E_q^*\mu$. Hence by continuous invertibility of $\bar{A}_q^*$, we obtain $p_k \to p$ in $W_\Gamma^{1,q'}$, and by continuity $\bar{A}_q^* p = E_q^*\mu$. □

Based on this lemma, we can now turn to an alternative notion of solutions to (AE) in the sense of Boccardo-Galouët [4], who defined p as limit of H_Γ^1-solutions p_k of (12) for a sequence μ_k of smooth right-hand sides that converges to μ in $\mathcal{M}(\Omega\cup\Gamma)$ in the weak* sense (cf. also [16]):

Corollary 2.16. *Let $\mu \in \mathcal{M}(\Omega\cup\Gamma)$ be given. Then there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset L^2(\Omega)$ converging weakly* to μ, i.e.*

$$\lim_{k\to\infty} \int_\Omega f_k\, v\, dx \to \langle \mu\,,\, v\rangle_{\mathcal{M}(\Omega\cup\Gamma)} \quad \forall\, v \in \tilde{C}_\Gamma(\Omega).$$

Moreover, there is a unique element $p \in W_\Gamma^{1,q'}$ such that $A_2^{-1} f_k \to p$ in $W_\Gamma^{1,q'}$. In addition, p solves (12) *with right hand side μ and is a solution in the sense of Definition 2.10.*

Proof. The density of $L^2(\Omega)$ in $\mathcal{M}(\Omega\cup\Gamma)$ w.r.t. the weak* topology follows from the assertion of Proposition A.2. Let $p_k := A_2^{-1} f_k \in H_\Gamma^1$. Then, in view of $\bar{a}(v, p_k) = a(v, p_k)$ for all $v \in D_q$ and all $k \in \mathbb{N}$, p_k also solves (D1) with right-hand side f_k. Thus, the result is an immediate consequence of Lemma 2.15. □

Remark 2.17. The equivalence of the Stampacchia solution to the solution of Boccardo-Galouët was already shown in [16] for the case of homogeneous Dirichlet boundary conditions.

Remark 2.18. Lemma 2.15 and Corollary 2.16 are essential for numerical path-following methods applied to state-constrained problems. These methods usually construct a sequence of smooth approximations of μ. Thus the results ensure that their adjoint states will converge to the correct weak solution.

2.3 Uniqueness in a tailored subspace

We consider here again the notion of solution given in Definition 2.10 and look at possibilities of getting rid of the limit expression $\bar{a}$. The idea is to consider the operator $\bar{A}_q$ in a different topological setting. To this end we embed D_q densely into $\tilde{C}_\Gamma(\Omega)$ and thus equip D_q with the norm $\|\cdot\|_\infty$. To avoid confusion, let us denote this space by $\tilde{D}_q$. Note that, by Theorem 2.9, we know that the embedding $E_q : \tilde{D}_q \hookrightarrow \tilde{C}_\Gamma(\Omega)$ is well defined for $q > d$ and dense. In this setting, we reconsider $\bar{A}_q$ as the operator

$$\begin{aligned} \tilde{A}_q : \tilde{C}_\Gamma(\Omega) \supset \tilde{D}_q &\to W_\Gamma^{-1,q} \\ v \mapsto \tilde{A}_q v &:= \bar{a}(v, \cdot), \end{aligned} \tag{14}$$

This definition is identical to (10) from an algebraic point of view. The only difference is that $\tilde{D}_q$ is now embedded and retopologized by $\tilde{C}_\Gamma(\Omega)$.

In order to define an adjoint operator for $\tilde{A}_q$ in this setting we define the set

$$D(\tilde{A}_q^*) := \{w \in W_\Gamma^{1,q'} : \exists\, c_w \text{ so that } \bar{a}(v, w) \le c_w \|v\|_\infty\, \forall\, v \in \tilde{D}_q\}.$$

For each $w \in D(\tilde{A}_q^*)$ the linear functional $f^{(w)} : v \mapsto \bar{a}(v,w)$ is well defined and continuous on $\tilde{D}_q$, and thus has a unique continuous extension to a functional $\bar{f}^{(w)} \in \tilde{C}_\Gamma(\Omega)^*$. Then we can define the adjoint operator (see e.g. [5, Section II.6]) via

$$\begin{aligned} \tilde{A}_q^* : W_\Gamma^{1,q'} \supset D(\tilde{A}_q^*) &\to \tilde{C}_\Gamma(\Omega)^* \\ w &\mapsto \tilde{A}_q^* w := \bar{f}^{(w)}, \end{aligned}$$

which satisfies:

$$\langle v, \tilde{A}_q^* w\rangle_{\tilde{C}_\Gamma(\Omega)} = \langle \tilde{A}_q v, w\rangle_{W_\Gamma^{1,q'}} = \bar{a}(v,w) \quad \forall\, v \in \tilde{D}_q, w \in D(\tilde{A}_q^*). \tag{15}$$

The definition of $D(\tilde{A}_q^*)$ yields a second notion of solutions:

Definition 2.19. Let $\mu \in \mathcal{M}(\Omega \cup \Gamma)$ be given. A function $p \in D(\tilde{A}_q^*) \subset W_\Gamma^{1,q'}$ is a solution of (AE), if it satisfies

$$a(v,p) = \int_\Omega v\, d\mu_\Omega + \int_\Gamma v\, d\mu_\Gamma \quad \forall\, v \in \mathcal{D}_\Gamma. \tag{D2}$$

In comparison to Definition 2.10, the space of solutions has been restricted here, which, as we will see, allows us to dispense with the limit formulation $\bar{a}(\cdot,\cdot)$.

Remark 2.20. Let us compare the definition of $D(\tilde{A}_q^*)$ to the following, seemingly similar definition:

$$\hat{D} := \{w \in W_\Gamma^{1,q'} : \exists\, c_w \text{ so that } a(v,w) \le c_w \left\|v\right\|_\infty \forall\, v \in W_\Gamma^{1,q}\}.$$

The only difference is a more restricted choice of test-variables v, which allows to use a instead of $\bar{a}$. Clearly, $D(\tilde{A}_q^*) \subset \hat{D}$, and the counter-examples of [19] imply that this inclusion may be *strict* in general.

In order to show existence and uniqueness of solutions in the sense of Definition 2.19 we need some auxiliary results.

Lemma 2.21. *If $p \in D(\tilde{A}_q^*)$ and $\mu \in \tilde{C}_\Gamma(\Omega)^*$, then*

$$\tilde{A}_q^* p = \mu \quad \Leftrightarrow \quad a(v,p) = \langle \mu, v\rangle_{\tilde{C}_\Gamma(\Omega)} \forall v \in \mathcal{D}_\Gamma.$$

Proof. By (15) $\tilde{A}_q^* p = \mu$ is equivalent to

$$\langle \tilde{A}_q v\,,\, p\rangle_{W_\Gamma^{1,q'}} = \langle \mu, v\rangle_{\tilde{C}_\Gamma(\Omega)} \quad \forall\, v \in \tilde{D}_q.$$

By Lemma 1.3, $\mathcal{D}_\Gamma$ is dense in $\tilde{C}_\Gamma(\Omega)$ and thus in $\tilde{D}_q$ giving in turn that the above equation is equivalent to

$$\langle \tilde{A}_q v, p\rangle_{W_\Gamma^{1,q'}} = \langle \mu, v\rangle_{\tilde{C}_\Gamma(\Omega)} \quad \forall\, v \in \mathcal{D}_\Gamma.$$

Now our assertion follows from $a(v,p) = \langle \tilde{A}_q v, p\rangle_{W_\Gamma^{1,q'}}$, if $v \in \mathcal{D}_\Gamma$. □

Thus, we have reduced our problem to the study of invertibility of $\tilde{A}_q^*$. To this end, we first show closedness and invertibility of $\tilde{A}_q$, and then conclude the same for $\tilde{A}_q^*$.

Lemma 2.22. *The operator $\tilde{A}_q : \tilde{C}_\Gamma(\Omega) \supset \tilde{D}_q \to W_\Gamma^{-1,q}$ continuously invertible and closed.*

Proof. Since $\tilde{D}_q$ is the maximal domain of $\tilde{A}_q$, continuous invertibility of $\tilde{A}_q$ is a direct consequence of Theorem 2.9. To conclude closedness, let $u_n \in \tilde{D}_q$, with $u_n \overset{\tilde{C}_\Gamma(\Omega)}{\to} u$ for some $u \in \tilde{C}_\Gamma(\Omega)$ and $\tilde{A}_q u_n := f_n \overset{W_\Gamma^{-1,q}}{\to} f$ for some $f \in W_\Gamma^{-1,q}$. We must show that $u \in \tilde{D}_q$ and $\tilde{A}_q u = f$.

Since $f_n \to f$, and $\tilde{A}_q$ has a continuous inverse, it follows $u_n = \tilde{A}_q^{-1} f_n \to \tilde{A}_q^{-1} f$. By assumption $u = \lim u_n$ it follows that $u = \tilde{A}_q^{-1} f$, and thus $\tilde{A}_q u = f$ and in particular $u \in \tilde{D}_q$. □

Lemma 2.23. *The adjoint operator* $\tilde{A}_q^* : W_\Gamma^{1,q'} \supset D(A_q^*) \to \tilde{C}_\Gamma(\Omega)^*$ *is continuously invertible.*

Proof. First, we show that $\tilde{A}_q^*$ is bijective. Since $\tilde{A}_q$ is densely defined, closed, and surjective, $\tilde{A}_q^*$ is injective (cf. [5, Cor. II.17]). To show the surjectivity of $\tilde{A}_q^*$, we use the closed range theorem for closed operators [5, Thm. II.20]: if $\tilde{A}_q$ is closed, $\mathrm{Ker}(\tilde{A}_q) = \{0\}$, and $\mathrm{Rg}(\tilde{A}_q)$ is closed, then $\tilde{A}_q^*$ is surjective. While $\tilde{A}_q$ is closed by Lemma 2.22, the last two conditions are satisfied, since $\tilde{A}_q$ is bijective.

It remains to show that the inverse of $\tilde{A}_q^*$ is continuous. To this end, observe that $\tilde{A}_q^*$ as an adjoint operator is closed (cf. [5, Prop. II.16]). Then the open mapping theorem for closed operators, see [21, Thm. IV.4.4], implies that $(\tilde{A}_q^*)^{-1} : \tilde{C}_\Gamma(\Omega)^* \to W_\Gamma^{1,q'}$ is indeed continuous. □

Now we are in the position to prove the main result of this subsection:

Theorem 2.24. *For every* $\mu \in \mathcal{M}(\Omega \cup \Gamma)$*, there exists a unique solution in the sense of Definition 2.19, which coincides with the solution in the sense of Definition 2.10.*

Proof. By Lemma 2.21, (D2) is equivalent to $\tilde{A}_q^* w = \mu$ in $\tilde{C}_\Gamma(\Omega)^*$. Hence, the continuous invertibility of $\tilde{A}_q^* : W_\Gamma^{1,q'} \supset D(\tilde{A}_q^*) \to \tilde{C}_\Gamma(\Omega)^*$ by Lemma 2.23 yields the result.

Due to (15), the unique solution of (AE) in the sense of Definition 2.19 also satisfies (D1) and therefore coincides with the solution from Definition 2.10. □

Remark 2.25. At first glance Definition 2.19 appears to be more comfortable than Definition 2.10, since the equation can be written in a form involving a Lebesgue integral and smooth test functions. However, the solution in the sense of Definition 2.19 is only unique in $D(\tilde{A}_q^*)$, and there may be other solutions in $W_\Gamma^{1,q'}$. So, in other words, a non-standard *test* space in (D1) has been exchanged by a non-standard *solution* space in (D2).

3 Solutions based on Distributional Derivatives

As anticipated in the introduction, we now turn to different notions of solutions to (AE) that are commonly used in the optimal control literature. The first definition of solutions dates back to Casas [7], while the second notion of solutions was introduced by Alibert and Raymond in [2]. As we will see in the following, both notions of solutions are in fact equivalent to the Stampacchia solution in the sense that they yield the same (unique) solution. Both concepts rely on the distributional divergence and distributional normal trace defined in the following:

Definition 3.1. (Distributional derivatives)

1. The distributional divergence of a function $\omega \in L^1_{\text{loc}}(\Omega;\mathbb{R}^d)$ is defined in a standard way by

$$\langle -\operatorname{div}\omega\,,\,\varphi\rangle_{\mathcal{D}'} := \int_\Omega \omega\cdot\nabla\varphi\,dx \quad \forall\,\varphi\in\mathcal{D},$$

   where $\mathcal{D}$ is given as in (2), equipped with the standard notion of convergence. Furthermore we define the set

$$W_{\text{div}}(\Omega;\mathbb{R}^d) := \{\omega\in L^1(\Omega;\mathbb{R}^d)\ :\ \exists\,\mu\in\mathcal{M}(\Omega):\ \langle\operatorname{div}\omega\,,\,\varphi\rangle_{\mathcal{D}'} = \langle\mu\,,\,\varphi\rangle_{\mathcal{M}(\Omega)}\ \forall\,\varphi\in\mathcal{D}\},$$

   For convenience, we identify $\operatorname{div}\omega=\mu$ if $\omega\in W_{\text{div}}(\Omega;\mathbb{R}^d)$ in the following.

2. The distributional normal trace on Γ of an element $\omega\in W_{\text{div}}(\Omega;\mathbb{R}^d)$ is given by

$$\langle \nu\cdot\omega\,,\,\varphi\rangle_{\mathcal{D}'_\Gamma} := \int_\Omega \omega\cdot\nabla\varphi\,dx + \langle\operatorname{div}\omega\,,\,\varphi\rangle_{\mathcal{M}(\Omega)} \quad \forall\,\varphi\in\mathcal{D}_\Gamma,$$

   where $\mathcal{D}_\Gamma$ is again the set given in (3) endowed with the same notion of convergence as $\mathcal{D}$.

Note that this formula represents a generalized formula of partial integration. Clearly, if $\omega\in C^1(\Omega;\mathbb{R}^d)$, then the distributional divergence and normal trace coincide with the divergence and normal trace of ω in a classical sense.

3.1 Very weak solutions

In the following subsection, we introduce the concept of very weak solutions for (AE) that is also used in [7]. Here, we extend the concept to the case of mixed boundary conditions by using a technique which is similar derivation of Stampacchia solutions in Section 2.1. As in Section 2.1, we again restrict the operator A_2 to a domain V which continuously embeds into $\tilde{C}_\Gamma(\Omega)$. By endowing V with the initial topology, the arising operator is again continuously invertible and, by the closed range theorem, this gives continuous invertibility of its adjoint. The space V thus plays a similar role as D_q in Section 2, and we define it with the help of distributional divergence and normal trace as follows:

Definition 3.2. For $q>d$ let $1<r,s<\infty$ and their conjugate exponents r',s' be given by

$$\frac{1}{s}=\frac{1}{q}+\frac{1}{d} \;\Leftrightarrow\; \frac{1}{s'}=\frac{1}{q'}-\frac{1}{d} \qquad\text{and}\qquad \frac{1}{r}=\frac{1}{q}\frac{d}{d-1} \;\Leftrightarrow\; \frac{1}{r'}=\frac{1}{q'}-\frac{1}{q(d-1)}$$

We introduce the linear space V by

$$\begin{aligned} V := \{v\in H^1_\Gamma\ :\ &\exists\,g_1\in L^s(\Omega):\ \langle\operatorname{div}a\nabla v\,,\,\varphi\rangle_{\mathcal{D}'} = \int_\Omega g_1\,\varphi\,dx\ \ \forall\,\varphi\in\mathcal{D},\\ &\exists\,g_2\in L^r(\Gamma):\ \langle\nu\cdot a\nabla v\,,\,\varphi\rangle_{\mathcal{D}'_\Gamma} = \int_\Gamma g_2\,\varphi\,ds\ \ \forall\,\varphi\in\mathcal{D}_\Gamma\}\end{aligned}$$

For every $v\in V$, we identify $\operatorname{div}a\nabla v = g_1\in L^s(\Omega)$ and $\nu\cdot a\nabla v = g_2\in L^r(\Gamma)$ in the following. Since $\operatorname{div}a\nabla v\in L^s(\Omega)$, we have $a\nabla v\in W_{\text{div}}(\Omega;\mathbb{R}^d)$. Therefore the normal trace in the sense of Definition 3.1, 2 is well defined.

In the following, we use the notation

$$L_{s,r} := L_s(\Omega) \times L_r(\Gamma)$$

and $L_{s',r'}$ is defined analogously. The choice of s and r is motivated by the following observation: There exists a continuous and dense Sobolev embedding and trace operator

$$\begin{aligned} E_L : W^{1,q'}_\Gamma &\hookrightarrow L_{s',r'} \\ w &\mapsto (w, \tau_\Gamma w). \end{aligned} \tag{16}$$

For the definition of the trace operator τ_Γ on Lipschitz domains and the associated trace theorem, we refer to [15, Chap. 2, Thm. 4.2]. The next lemma shows that we can define V similarly to D_q in (9):

Lemma 3.3. *The space V can equivalently be defined by*

$$V = \Big\{ v \in H^1_\Gamma \,:\, \exists\, c_v \geq 0 \text{ with } |a(v,w)| \leq c_v \Big(\|w\|_{L^{s'}(\Omega)} + \|\tau\, w\|_{L^{r'}(\Gamma)} \Big)\; \forall\, w \in H^1_\Gamma \Big\}.$$

Proof. We denote by M the set in the assertion of the lemma. First, let us show $V \subset M$. By definition of the distributional derivative and normal trace we find for an arbitrary $v \in V$

$$\begin{aligned} |a(v,\varphi)| &= |\langle \nu \cdot a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'_\Gamma} - \langle \operatorname{div} a\, \nabla v\,,\, \varphi \rangle_{\mathcal{M}(\Omega)}| \\ &\leq \|\operatorname{div} a\, \nabla v\|_{L^s(\Omega)}\, \|\varphi\|_{L^{s'}(\Omega)} + \|\nu \cdot a\, \nabla v\|_{L^r(\Omega)}\, \|\varphi\|_{L^{r'}(\Omega)} \quad \forall\, \varphi \in \mathcal{D}_\Gamma. \end{aligned}$$

Hence, by density of $\mathcal{D}_\Gamma$ in H^1_Γ, it follows that $V \subset M$. On the other hand, if $v \in M$, then

$$|\langle \operatorname{div} a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'}| = |a(v,\varphi)| \leq c_v \|\varphi\|_{L^{s'}(\Omega)} \quad \forall\, \varphi \in \mathcal{D},$$

and the Hahn-Banach theorem yields the existence of $g_1 \in L^s(\Omega)$ such that $\langle \operatorname{div} a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'} = \int_\Omega g_1\, \varphi\, dx$ for all $\varphi \in \mathcal{D}$. Thus $a\nabla v \in W_{\operatorname{div}}(\Omega; \mathbb{R}^d)$, and as above we identify $\operatorname{div} a\, \nabla v$ with g_1. Consequently its distributional normal trace is well defined and one finds

$$\Big| \langle \nu \cdot a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'_\Gamma} - \int_\Omega (\operatorname{div} a\, \nabla v) \varphi\, dx \Big| = |a(v,\varphi)| \leq c_v \Big(\|\varphi\|_{L^{s'}(\Omega)} + \|\tau\, \varphi\|_{L^{r'}(\Gamma)} \Big)\; \forall\, \varphi \in \mathcal{D}_\Gamma.$$

Hence the Hahn-Banach theorem again gives the existence of $(\tilde{g}_1, g_2) \in L_{s,r}$ so that

$$\langle \nu \cdot a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'_\Gamma} - \int_\Omega (\operatorname{div} a\, \nabla v) \varphi\, dx = \int_\Omega \tilde{g}_1\, \varphi\, dx + \int_\Gamma g_2\, \varphi\, ds\; \forall\, \varphi \in \mathcal{D}_\Gamma.$$

By testing the above equation with $\varphi \in \mathcal{D}$ we immediately find that $\operatorname{div} a\, \nabla v = \tilde{g}_1$ and consequently $\langle \nu \cdot a\, \nabla v\,,\, \varphi \rangle_{\mathcal{D}'_\Gamma} = \int_\Gamma g_2\, \varphi\, ds$ which implies $M \subset V$. ☐

On V we can define the linear operator

$$\begin{aligned} A_V : V &\to L_{s,r} \\ v &\mapsto (-\operatorname{div} a\nabla v, \nu \cdot a\nabla v). \end{aligned}$$

As in the proof of Lemma 3.3, the definition of the distributional derivative and the normal trace implies

$$\begin{aligned} \langle A_V v, \varphi \rangle_{L_{s,r}} &= \int_\Omega (-\operatorname{div} a\nabla v) \varphi\, dx + \int_\Gamma (\nu \cdot a\nabla v) \varphi\, ds \\ &= \int_\Omega a\nabla v \cdot \nabla\varphi\, dx = a(v,\varphi) \quad \forall\, \varphi \in \mathcal{D}_\Gamma. \end{aligned} \tag{17}$$

We thus conclude:

Lemma 3.4. *$A_V : V \to L_{s,r}$ is bijective.*

Proof. By (17) we see that A_V is a restriction of the bijective operator $A_2 : H^1_\Gamma \to H^{-1}_\Gamma$ and, by Lemma 3.3, V is its maximal domain with respect to the range space $L_{s,r}$. □

Just as we did with D_q in Section 2.1, we endow V with the norm

$$\|v\|_V = \|A_V v\|_{L_{r,s}} = \|\operatorname{div} a \nabla v\|_{L^s(\Omega)} + \|\nu \cdot a \nabla v\|_{L^r(\Gamma)},$$

which makes A_V an isomorphism and V a Banach space.

Lemma 3.5. *There is a continuous embedding*

$$E_V : (V, \|\cdot\|_V) \hookrightarrow (D_q, \|\cdot\|_{D_q}),$$

which satisfies the equation

$$E_L^* A_V = \bar{A}_q E_V. \tag{18}$$

Proof. By (16) the adjoint $E_L^* : L_{s,r} \hookrightarrow W_\Gamma^{-1,q}$ exists, and thus $E_L^* A_V v \in W_\Gamma^{-1,q}$ for all $v \in V$. This implies that there exists a continuous mapping $E_V := \bar{A}_q^{-1} E_L^* A_V : V \to D_q$, which satisfies (18). It remains to show that $E_V v = v$ for all $v \in V$. By (18) we have

$$a(v, \varphi) = \langle A_V v, E_L \varphi \rangle_{L_{s,r}} = \langle \bar{A}_q E_V v, \varphi \rangle_{W_\Gamma^{1,q'}} = a(E_V v, \varphi) \quad \forall \varphi \in D_\Gamma.$$

This implies $E_V v = v$ in view of the injectivity of A_2. □

In particular, V is continuously embedded in $\tilde{C}_\Gamma(\Omega)$ by Theorem 2.9. Now we can conclude existence and uniqueness of very weak solutions of (AE):

Theorem 3.6. *The equation*

$$\int_\Omega (-\operatorname{div} a \nabla v) p \, dx + \int_\Gamma (\nu \cdot \nabla v) t \, ds = \int_\Omega v \, d\mu_\Omega + \int_\Gamma v \, d\mu_\Gamma \quad \forall v \in V \tag{D3}$$

has a unique solution $(p, t) \in L_{s',r'}$. Moreover, we have $p \in W_\Gamma^{1,q}$ and $t = \tau p$, and p coincides with the solution in the sense of Definition 2.10.

Proof. Equation (D3) can be written as $\langle A_V v, (p,t) \rangle_{L_{s',r'}} = \langle \mu, E_q E_V v \rangle_{\tilde{C}_\Gamma(\Omega)}$ for all $v \in V$, or

$$A_V^*(p, t) = E_V^* E_q^* \mu, \tag{19}$$

where $E_q : D_q \to \tilde{C}_\Gamma(\Omega)$ was defined in Theorem 2.9. By the closed range theorem, $A_V^* : L_{s',r'} \to V^*$ is an isomorphism, because A_V is one. This yields existence of a unique couple $(p, t) \in L_{s',r'}$ such that (D3) is fulfilled.

Now let $\tilde{p}$ be the solution of (D1), i.e.

$$\bar{A}_q^* \tilde{p} = E_q^* \mu. \tag{20}$$

Then inserting (20) into (19) yields $A_V^*(p,t) = E_V^* \bar{A}_q^* \tilde{p}$. From Lemma 3.5 we conclude $A_V^* E_L = E_V^* \bar{A}_q^*$, and hence $A_V^*(p,t) = A_V^* E_L \tilde{p}$. Thus $(p,t) = E_L \tilde{p}$, which implies $p = \tilde{p}$ a.e. in Ω and $t = \tau \tilde{p}$ a.e. on Γ by definition of E_L. □

Remark 3.7. Notice the analogy to Section 2: we replaced $W_\Gamma^{-1,q}$ by $L_{s,r}$ and D_q by V.

3.2 Solutions in the sense of Alibert and Raymond

This subsection is devoted to a notion of solutions to (AE) introduced by Alibert and Raymond in [2]. The basis for this definition is the variational formulation (D0) which is not sufficient to obtain uniqueness. To guarantee the uniqueness of solutions, Alibert and Raymond additionally required a certain formula of integration by parts (see (23) below). However, their analysis does not account for mixed boundary conditions which are incorporated here. In addition, we will show that this notion of solutions coincides with ones defined before.

Let us first recall the space for solutions of (AE) that is used in [2].

Definition 3.8. The set W is defined by

$$W := \{p \in W^{1,1}_\Gamma(\Omega)\ :\ a^\top \nabla p \in W_{\mathrm{div}}(\Omega;\mathbb{R}^d) \text{ and }\\ \exists\, \mu \in \mathcal{M}(\Gamma) : \langle \nu \cdot a^\top p\,,\, \varphi\rangle_{\mathcal{D}'_\Gamma} = \langle \mu\,,\, \varphi\rangle_{\mathcal{M}(\Gamma)}\ \forall\, \varphi \in \mathcal{D}_\Gamma\}.$$

Similarly to above, we identify the co-normal derivative on Γ by $\nu \cdot a^\top \nabla p = \mu \in \mathcal{M}(\Gamma)$ if $p \in W$.

With this definition we may define the linear operator

$$A_W : W \to \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) = \mathcal{M}(\Omega \cup \Gamma)\\ w \mapsto (-\operatorname{div} a^T \nabla w, \nu \cdot a^\top \nabla w).$$

By definition of the distributional derivative, for each $p \in W$, $A_W p$ is the unique continuous extension of the linear functional $\varphi \to a(\varphi, p)$ to a measure. This extension exists by definition of W. To be more precise, we have:

$$\langle A_W p, \varphi\rangle_{\tilde{C}_\Gamma(\Omega)} = a(\varphi, p) \quad \forall \varphi \in \mathcal{D}_\Gamma. \tag{21}$$

Now we have everything at hand to introduce the notion of solutions to (AE) according to Alibert and Raymond [2]:

Definition 3.9. A function $p \in W$ is called solution of (AE), if it satisfies

1. the following weak formulation of (AE)

$$a(\varphi, p) = \int_\Omega \varphi\, d\mu_\Omega + \int_\Gamma \varphi\, d\mu_\Gamma \quad \forall\, \varphi \in \mathcal{D}_\Gamma, \tag{22}$$

2. $p \in L^{s'}(\Omega)$ and $\tau p \in L^{r'}(\Gamma)$, where s and r are as defined in Definition 3.2,
3. and the following formula of integration by parts

$$-\int_\Omega (\operatorname{div} a\, \nabla v) p\, dx + \int_\Gamma (\nu \cdot a\, \nabla v) p\, ds\\ = -\langle \operatorname{div} a^\top p\,,\, v\rangle_{\mathcal{M}(\Omega)} + \langle \nu \cdot a^\top p\,,\, v\rangle_{\mathcal{M}(\Gamma)} \quad \forall\, v \in V. \tag{23}$$

Remark 3.10. If in (23) the test space V was replaced by $\mathcal{D}_\Gamma$, it would be again impossible to show uniqueness of a solution. Thus, this notion of solution also involves a non-standard test space, just as the previously defined ones.

Theorem 3.11. *There exists a solution of* (AE) *in the sense of Definition 3.9. Moreover, it is unique and coincides with the solution of* (AE) *in the sense of Definition 2.10.*

Proof. For ease of notation, define the continuous embedding $E := E_q E_V : V \hookrightarrow \tilde{C}_\Gamma(\Omega)$.

Let us translate Definition 3.9 into operator equations. By (21) and density of $\mathcal{D}_\Gamma$ in $\tilde{C}_\Gamma(\Omega)$ the weak formulation (22) is expressed by $A_W p = \mu$. The formula of partial integration (23) reads

$$\langle A_V v, (p, \tau p)\rangle_{L_{s',r'}} = \langle A_W p, Ev\rangle_{\tilde{C}_\Gamma(\Omega)} \, \forall v \in V$$

or with adjoints: $A_V^*(p, \tau p) = E^* A_W p$.

If p is a solution in the sense of (D1), it clearly solves (22). The definition of the distributional divergence together with (22) then implies

$$\langle - \operatorname{div} a^\top \nabla p \, , \, \varphi\rangle_{\mathcal{D}'} = a(\varphi, p) = \langle \mu_\Omega \, , \, \varphi\rangle_{\mathcal{M}(\Omega)} \quad \forall \, \varphi \in \mathcal{D},$$

which gives $a^\top \nabla \bar{p} \in W_{\mathrm{div}}(\Omega; \mathbb{R}^d)$. Applying again (22) yields

$$\langle \mu_\Gamma \, , \, \varphi\rangle_{\mathcal{M}(\Gamma)} = a(\varphi, p) + \langle \operatorname{div} a^\top \nabla p \, , \, \varphi\rangle_{\mathcal{M}(\Omega)} = \langle \nu \cdot a^\top \nabla p \, , \, \varphi\rangle_{\mathcal{D}'_\Gamma} \quad \forall \, \varphi \in \mathcal{D}_\Gamma,$$

by the definition of the distributional normal trace. Thus we have $p \in W$ and $A_W p = \mu$. Moreover, by Theorem 3.6, the solution p also satisfies $A_V^*(p, \tau p) = E^* \mu$. Thus,

$$A_V^*(p, \tau p) = E^* \mu = E^* A_W p,$$

which is the formula of partial integration.

If, in turn p is a solution in the sense of Definition 3.9, then $A_W p = \mu$ and $A_V^*(p, \tau p) = E^* A_W p$. Thus, $A_V^*(p, \tau p) = E^* \mu$, and so $(p, \tau p)$ is the solution in the sense of (D3) and thus also in the sense of (D1) by Theorem 3.6.

Therefore, both definitions of solutions are equivalent which establishes the assertion. □

4 Conclusion

To summarize the above analysis, we collect our results in a single theorem:

Theorem 4.1. *The following equation in the weak form*

$$a(v, p) = \int_\Omega v \, d\mu_\Omega + \int_\Gamma v \, d\mu_\Gamma \quad \forall v \in \mathcal{D}_\Gamma \tag{24}$$

admits solutions in $W_\Gamma^{1,1}$. Precisely one of these solutions is outstanding and characterized by one and hence all of the following equivalent additional conditions:

(i) $p \in W_\Gamma^{1,q'}$ *for some $q > d$ and satisfies the extended weak formulation*

$$\bar{a}(v, p) := \lim_{\substack{p_k \in H_\Gamma^1 \\ \|p_k - p\|_{W_\Gamma^{1,q'}} \to 0}} a(v, p_k) = \int_\Omega v \, d\mu_\Omega + \int_\Gamma v \, d\mu_\Gamma \quad \forall \, v \in D_q, \tag{25}$$

where D_q is the maximal domain of definition of A_q.

(ii) $p \in D(\bar{A}_q^*) := \{w \in W_\Gamma^{1,q'} : \exists \, c_w$ *so that* $\bar{a}(v, w) \le c_w \, \|v\|_\infty \, \forall \, v \in D_q\}$ *for some $q > d$.*

(iii) *p is the limit of solutions of* (24) *for a sequence of smooth right hand sides that converges to μ in $\mathcal{M}(\Omega \cup \Gamma)$ in the weak* sense.*

(iv) *$(p, \tau p) \in L_{s',r'}$ and p satisfies the very weak formulation*

$$\int\limits_{\Omega} (-\operatorname{div} a\nabla v) p \, dx + \int\limits_{\Gamma} (\nu \cdot \nabla v) p \, ds = \int_{\Omega} v \, d\mu_\Omega + \int_{\Gamma} v \, d\mu_\Gamma \quad \forall v \in V,$$

where V is the maximal domain of definition of A_V.

(v) *$(p, \tau p) \in L_{s',r'}$ and p satisfies the formula of partial integration:*

$$- \int\limits_{\Omega} (\operatorname{div} a \, \nabla v) p \, dx + \int\limits_{\Gamma} (\nu \cdot a \, \nabla v) p \, ds$$
$$= -\langle \operatorname{div} a^\top \, p \,, \, v \rangle_{\mathcal{M}(\Omega)} + \langle \nu \cdot a^\top \, p \,, \, v \rangle_{\mathcal{M}(\Gamma)} \quad \forall \, v \in V.$$

If the problem enjoys maximal regularity w.r.t. q, then (24) *is sufficient for p being unique in $W^{1,q'}_{\Gamma}$.*

All criteria ultimately rely on a regularity result, such as Theorem 2.9 and a duality technique, involving the closed range theorem. Except for *(iii)* a function space is involved, (D_q or V, resp.) which depends on the properties of a and cannot be described as a standard Sobolev space. In general it seems to be impossible to replace these spaces by more simple ones without loosing uniqueness. If the problem at hand enjoys maximal regularity for some $q > d$, then the non-standard space D_q becomes a Sobolev space, namely $W^{1,q'}_{\Gamma}$. In contrast, V is hard to characterize in general. One exception are H^2-regular Dirichlet problems, where $V = H^2 \cap H^1_0$ and $s = 2$.

A Density results for mixed boundary conditions

The following two density results were essential for the analysis underlying the proofs of Corollary 2.16, Lemma 2.21, and Theorem 3.11. Since the case of mixed boundary conditions is not covered in the standard literature, we provide the associated proofs for convenience of the reader.

Lemma A.1. *Let Ω be an open, bounded set and Γ a relatively open part of $\partial\Omega$. The set $\mathcal{D}_\Gamma(\Omega)$, defined in* (3)*, is dense in $\tilde{C}_\Gamma(\Omega)$, given in* (4)*.*

Proof. We will apply the Theorem of Stone-Weierstrass. First, we note that due to the Theorem of Tietze-Urysohn, each $v \in \tilde{C}_\Gamma(\Omega)$ can be extended to a function in $C(\mathbb{R}^d)$ with compact support in $\mathbb{R}^d$. For simplicity, this extension is also denoted by v. Now define $X := \mathbb{R}^d \setminus \Gamma_D$, which is a locally compact set, since Γ_D is closed. Then, because of $v \equiv 0$ on Γ_D, the extension of v is even contained in

$$C_0(X) := \{v \in C(\mathbb{R}^d) \, : \, \forall \epsilon > 0 \, \exists \text{ compact set } K \subset X \text{ so that } v(x) < \epsilon \, \forall \, x \in X \setminus K\}.$$

Moreover, we define $\mathcal{D}(X)$ by $\mathcal{D}(X) := \{v \in C^\infty(X), \; \operatorname{supp} v \cap \Gamma_D = \emptyset\}$. By taking the standard mollifier, we find for each $x \in X$ and each neighborhood $\mathcal{U}(x)$ of x a function $\phi \in \mathcal{D}(X)$ such that $\phi(x) \neq 0$ and $\phi \equiv 0$ in $X \setminus \mathcal{U}(x)$. Hence, $\mathcal{D}(X)$ is an algebra that separates points and vanishes nowhere.

Thus we are allowed to apply the locally compact version of the Stone-Weierstrass theorem, cf. [9, Corollary 8.3], which asserts density of $\mathcal{D}(X)$ in $C_0(X)$. Restriction to $\bar{\Omega}$, yields the desired result. □

Proposition A.2. *Let Ω be an open and bounded subset of $\mathbb{R}^d$. Then, for every $\mu \in \mathcal{M}(\overline{\Omega})$ there is a sequence $(f_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)$ that converges weakly* to μ, i.e.*

$$\int_\Omega f_n\, v\, dx \to \langle \mu\,,\, v\rangle_{\mathcal{M}(\overline{\Omega})} \quad \forall\, v \in C(\overline{\Omega}).$$

Proof. We have to show that $L^\infty(\Omega)$ is weak* sequentially dense in $\mathcal{M}(\overline{\Omega})$. By a general result of functional analysis (cf. e.g. [9, Theorem V.12.11]), a linear subspace U of the dual X^* of a separable Banach space X is weak* sequentially dense in X^*, if and only if there is a constant c such that, for every $x \in X$, there holds

$$\|v\|_X \leq \sup\{|\langle v, v^*\rangle| : v^* \in U,\ \|v^*\|_{X^*} \leq c\}. \tag{26}$$

Let us take $X = C(\overline{\Omega})$ (which is a separable Banach space), $X^* = \mathcal{M}(\overline{\Omega})$, and $U := L^\infty(\Omega)$. Then to verify (26), for each $v \in C(\overline{\Omega})$, we construct a sequence ϕ_k in $L^\infty(\Omega)$ with $\|\phi_k\|_{L^1(\Omega)} = 1$ and

$$\|v\|_{L^\infty(\Omega)} \leq \lim_{k\to\infty} |\int_\Omega v\, \phi_k\, dx|.$$

Indeed, for given $v \in C(\overline{\Omega})$, let $y \in \overline{\Omega}$ be a point, for which $|v(y)| = \|v\|_{L^\infty(\Omega)}$. (Note that such a y exists since $\overline{\Omega}$ is compact and v is continuous.) Furthermore, define for $\varepsilon > 0$

$$M_\varepsilon := \{x \in \Omega : |x - y| < \varepsilon\}.$$

Clearly, M_ε is non-empty and open as an intersection of two open sets, and therefore $|M_\varepsilon| > 0$. Now let $k \in \mathbb{N}$ and define $\phi_k := |M_{1/k}|^{-1}\chi_{M_{1/k}}$, where χ_{M_ε} is the characteristic function of the set M_ε. Then for each $k \in \mathbb{N}$, we have $\phi_k \in L^\infty(\Omega)$ and $\|\phi_k\|_{L^1(\Omega)} = 1$. Moreover, we find

$$\begin{aligned}\left|\int_\Omega \phi_k\, v\, dx\right| &= \left|v(y) + |M_{1/k}|^{-1}\int_{M_{1/k}} \big(v(x) - v(y)\big)\, dx\right| \\ &\geq |v(y)| - \sup_{x\in M_{1/k}} |v(x) - v(y)| \to |v(y)| = \|v\|_{L^\infty(\Omega)} \quad \text{for } k \to \infty,\end{aligned}$$

which proves the result. □

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