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Abstract. In this contribution, we shall be concerned with the question of wellposedness of thixo-viscoplastic flow problems in context of FEM approximations. We restrict our analysis to a quasi-Newtonian modeling approach with the aim to set foundations for an efficient monolithic Newton-multigrid solver. We present the wellposedness of viscoplastic subproblems and structure subproblems in parallel/independent fashion showing the possibility for a combined treatment. Then, we use the fixed point theorem for the coupled problem. For the numerical solutions, we choose 4:1 contraction configuration and use monolithic Newton-multigrid solver. We analyse the effect of taking into consideration thixotropic phenomena in viscoplastic material and opening up for more different coupling by inclusions of shear thickening and shear thinning behaviors for plastic viscosity and/or elastic behavior below the critical yield stress limit in more a general thixotropic models.

1 Introduction

Thixo-viscoplastic flows are introduced into yield stress flows by taking in consideration the internal material microstructure using a structure parameter λ. Firstly, the viscoplastic stress is modified to thixotropic stress dependent on the structure parameter

$$\begin{cases} \boldsymbol{\sigma}(\lambda) = 2\eta(\lambda)\mathbf{D}(\boldsymbol{u}) + \tau(\lambda)\dfrac{\mathbf{D}(\boldsymbol{u})}{\|\mathbf{D}(\boldsymbol{u})\|} & \text{if } \|\mathbf{D}(\boldsymbol{u})\| \neq 0, \\ \|\boldsymbol{\sigma}(\lambda)\| \leq \tau(\lambda) & \text{if } \|\mathbf{D}(\boldsymbol{u})\| = 0, \end{cases} \tag{1}$$

where $\mathbf{D}(\boldsymbol{u})$ denotes the strain rate tensor. The norm for a tensor Λ is given by $\|\Lambda\| = \sqrt{\mathrm{Tr}(\Lambda^2)}$. We use $\|\mathbf{D}(\boldsymbol{u})\|$ and $\|\mathbf{D}\|$ alternately. η denotes the plastic viscosity, and τ defines the yield stress that is a threshold parameter from which the material starts yielding. The shear stress has two contributions: a viscous part, and a strain rate independent part. Secondly, an evolution equation for the structure parameter is introduced to induce the time-dependent process of competition between the destruction (breakdown) and the construction (buildup) inhabited in the material

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right)\lambda = \mathcal{F} - \mathcal{G}, \tag{2}$$

where $\mathcal{F}$ and $\mathcal{G}$ are two nonlinear functions representing the buildup and breakdown of material microstructure. A collection of thixotropic models with various choices of η, τ, $\mathcal{F}$ and $\mathcal{G}$ are given in Table 1.

Table 1: Thixotropic models

	η	τ	$\mathcal{F}$	$\mathcal{G}$
Worrall et al. [11]	$\lambda\,\eta_0$	τ_0	$a(1-\lambda)\left\|\mathbf{D}\right\|$	$b\lambda\left\|\mathbf{D}\right\|$
Coussot et al.[3]	$\lambda^g\,\eta_0$		a	$b\lambda\left\|\mathbf{D}\right\|$
Houška [4]	$(\eta_0+\eta_1\lambda)\left\|\mathbf{D}\right\|^{n-1}$	$(\tau_0+\tau_1\lambda)$	$a(1-\lambda)$	$b\lambda^m\left\|\mathbf{D}\right\|$
Mujumbar et al. [5]	$(\eta_0+\eta_1\lambda)\left\|\mathbf{D}\right\|^{n-1}$	$\lambda^{g+1}G_0\Lambda_c$	$a(1-\lambda)$	$b\lambda\left\|\mathbf{D}\right\|$

Here η_0 and τ_0 are initial plastic viscosity and yield stress, respectively, in the absence of any thixotropic phenomena. η_1 and τ_1 are thixotropic plastic viscosity and yield stress. Λ_c is the critical elastic strain, and G_0 is the elastic modulus of unyielded material. a and b are buildup and breakage constants, and g, p, m, n are rate indices. We then concisely define the thixotropic model

$$\mathcal{M} := \mathcal{G} - \mathcal{F}. \tag{3}$$

In the quasi-Newtonian modeling approach for thixo-viscoplastic flows, an extended viscosity $\mu(\cdot,\cdot)$ is used for the generalized Navier-Stokes equations [8]. We use smooth and well definied approximations to $\left\|\mathbf{D}\right\|^{-1}$. Let $D_{\mathrm{II}} = \frac{1}{2}\left(\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{u})\right)$ be the second invariant of the strain rate tensor. For the smooth and well defined approximations to $\left\|\mathbf{D}\right\|^{-1}$, we use Papanastasiuo approximation [9]:

$$\frac{1}{\sqrt{D_{\mathrm{II},r}}} := \frac{1}{\sqrt{D_{\mathrm{II}}}}\left(1 - e^{-k\sqrt{D_{\mathrm{II}}}}\right), \tag{4}$$

where k is the regularization parameter. Then, the viscosity in generalized Navier-Stokes equations is given as follows

$$\mu(D_{\mathrm{II},r},\lambda) = \eta(D_{\mathrm{II}},\lambda) + \tau(\lambda)\frac{\sqrt{2}}{2}\frac{1}{\sqrt{D_{\mathrm{II},r}}} \tag{5}$$

Thus, the full set of equations for thixo-viscoplastic problems reads

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\boldsymbol{u} - \boldsymbol{\nabla}\cdot\left(2\mu(D_{\mathrm{II},r},\lambda)\mathbf{D}(\boldsymbol{u})\right) + \nabla p = \boldsymbol{f_u} \\ \nabla\cdot\boldsymbol{u} = 0 \\ \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\lambda + \mathcal{M}(D_{\mathrm{II}},\lambda) = f_\lambda \end{cases} \tag{6}$$

in Ω, where $\boldsymbol{u}$, p, and λ denote velocity, the pressure, and the structure parameter, respectively.

The main concern of this work is the question of existence and uniqueness of solutions of a quasi-Newtonian modeling approach of thixo-viscoplastic flow problems in context of FEM approximations. We use Browder-Minty theorem of monotone operator for viscoplastic subproblems and Lax-Milgram lemma for microstructure subproblems, then fixed point theorem for the coupled problem. The present paper has the following structure. In section 2, we derive the variational form for TVP flow problems and present it in a classical abstract setting of saddle point problems. Then, we tackle the main question of existence and uniqueness of solutions. In Section 3, we start by expressing the approximate TVP problems in a general setting of conforming approximations deducing similarly, to continuous TVP problems, the wellposedness of approximate TVP problems. In addition, we briefly introduce a higher order conforming FEM discretization based on stable two field Stokes elements supplemented with necessarily stabililizations. Next, in section 4, we present and analyse the numerical solutions of TVP flow problems in a 4:1 configuration obtained using monolithic Newton-multigrid solver. Section 5 highlights conclusions and reflections on some further coupling of the problem by introducing the shear thickening and shear thinning behaviors for the plastic viscosity and/or elastic behavior below the critical yield stress limit in more a general thixotropic models.

2 Continuous formulation of TVP problem

For the finite element discretization of thixo-viscoplastic flow problems (6), we shall constantly use Sobolev spaces which are introduced in context of shear rate independent viscosity i.e. $\eta = \eta(\lambda), (n = m = 1)$.

2.1 Notations and terminology

A zero order Sobolev space is the space of all square integrable real functions defined in Ω. This is the usual real Hilbert space denoted $L^2(\Omega) := H^0(\Omega)$. The first order Sobo1ev space, $H^1(\Omega)$ is that space of real functions whose generalized first spatial derivatives are in $H^0(\Omega)$ completed with $H^0(\Omega)$ functions. These Sobo1ev spaces can then be defined:

$$L^2(\Omega) = \left\{ v \mid \int_\Omega |v|^2 \; d\Omega < \infty \right\}, \tag{7}$$

$$H^1(\Omega) = \left\{ v \in \; L^2(\Omega) | \; \nabla \; v \; \in L^2(\Omega) \right\}. \tag{8}$$

We define in general for m integer

$$H^m(\Omega) = \left\{ v \in \; L^2(\Omega) | \; D^{|\alpha|} v := \frac{\partial^{|\alpha|} v}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \in L^2(\Omega), \forall \, |\alpha| \leq \; m \right\}, \tag{9}$$

and the space of all square integrable real functions with zero mean value in Ω

$$L_0^2(\Omega) = \left\{ v \in \; L^2(\Omega) \mid \int_\Omega v \; d\Omega = 0 \right\}. \tag{10}$$

For the trace of functions, we specifie different boundaries. Let $\partial\Omega$ denote the boundary of Ω, $\Gamma \subset \partial\Omega$ any section of it, Γ^- the inflow and Γ^+ the outflow:

$$\Gamma^- = \{ \boldsymbol{x} \in \Gamma \subset \; \Omega \mid \boldsymbol{u} \cdot \boldsymbol{n} < 0 \}, \; \Gamma^+ = \{ \boldsymbol{x} \in \Gamma \subset \; \Omega \mid \boldsymbol{u} \cdot \boldsymbol{n} > 0 \}. \tag{11}$$

The functions of $H^1(\Omega)$ with zero trace at boundaries are denoted by

$$H^1_\Gamma(\Omega) = \left\{ v \in \; H^1(\Omega) \mid v_{|\Gamma} = 0, \Gamma \subset \; \partial\Omega \right\}, \tag{12}$$

$$H^1_0(\Omega) = \left\{ v \in \; H^1(\Omega) \mid v_{|\partial\Omega} = 0 \right\}. \tag{13}$$

The following inner products and norms are defined (for steady state problems):

inner product	$(v, w) = \int_\Omega vw \; d\Omega$	(14)
zero norm/L^2-norm	$\|v\|_0 = (v, v)^{\frac{1}{2}}$	(15)
one norm/H^1-norm	$\|v\|_1 = [(v, v) + (\nabla \; v, \nabla \; v)]^{\frac{1}{2}}$	(16)
boundary inner product	$< v, w >_\pm = \int_{\Gamma^\pm} \|\boldsymbol{u} \cdot \boldsymbol{n}\| \; vwds$	(17)
boundary norm (semi-norm)	$< v >_\pm = < v, v >_\pm^{\frac{1}{2}}$	(18)
norm on $C^0(\bar{\Omega})$	$\|v\|_\infty = \sup_{\boldsymbol{x} \in \bar{\Omega}} \|\boldsymbol{v}(\boldsymbol{x})\|$	(19)

2.2 Continuous formulation of TVP problem

We set $\mathbb{T} := H^1_{\Gamma^-}(\Omega), \mathbb{V} := (H^1_0(\Omega))^2, \mathbb{W} := \mathbb{T} \times \mathbb{V}$, and $\mathbb{Q} := L^2_0(\Omega)$ associated, i.e. $\mathbb{W}$ and $\mathbb{Q}$, with the corresponding norms H^1-norm $\|\cdot\|_1$ and L^2-norm $\|\cdot\|_0$, respectively, and $\mathbb{T}'$, $\mathbb{V}'$ and $\mathbb{W}' := \mathbb{T}' \times \mathbb{V}'$ their corresponding dual spaces. We set $\tilde{\boldsymbol{u}} = (\lambda, \boldsymbol{u}), \tilde{\boldsymbol{v}} = (\xi, \boldsymbol{v})$, $\hat{\boldsymbol{u}} = (\tilde{\boldsymbol{u}}, \boldsymbol{u})$, and define on $\mathbb{W} \times \mathbb{W}$

$$a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = a_\lambda(\hat{\boldsymbol{u}})(\lambda, \xi) + a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) \quad \forall\, (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) \in \mathbb{W} \times \mathbb{W}. \tag{20}$$

The weak formulation for the thixo-viscoplastic flow problems (6) reads: *Find* $(\tilde{\boldsymbol{u}}, p) \in \mathbb{W} \times \mathbb{Q}$ *s. t.*

$$a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = l(\tilde{\boldsymbol{v}}), \quad \forall (\tilde{\boldsymbol{v}}, q) \in \mathbb{W} \times \mathbb{Q}, \tag{21}$$

equivalenty,

$$\begin{cases} a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) + b(\boldsymbol{v}, p) = \; l(\tilde{\boldsymbol{v}}), & \forall\, \tilde{\boldsymbol{v}} \in \mathbb{W} \\ b(\boldsymbol{u}, q) \qquad\qquad\qquad\quad = \; 0, & \forall\, q \in \mathbb{Q}, \end{cases} \tag{22}$$

and in operator form

$$\begin{cases} \mathcal{A}_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})\tilde{\boldsymbol{u}} + \mathcal{B}^{\mathrm{T}} p & = \boldsymbol{f}_{\tilde{\boldsymbol{u}}}, \quad \text{on } \mathbb{W}' \\ \mathcal{B}\boldsymbol{u} & = 0, \quad \text{on } \mathbb{Q}' \end{cases} \tag{23}$$

where $\mathcal{A}_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})$, $\mathcal{A}_\lambda(\hat{\boldsymbol{u}})$, $\mathcal{A}_{\boldsymbol{u}}(\hat{\boldsymbol{u}})$, $\mathcal{B}$, $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$, $a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $l(\cdot)$ are given as follows: For all $\tilde{\boldsymbol{u}} = (\lambda, \boldsymbol{u}) \in \mathbb{W}$, $\tilde{\boldsymbol{v}} = (\xi, \boldsymbol{v}) \in \mathbb{W}$ and $\hat{\boldsymbol{u}} = (\tilde{\boldsymbol{u}}, \boldsymbol{w}) \in \mathbb{W} \times \mathbb{V}$

$$\mathcal{A}_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})\tilde{\boldsymbol{u}} = \mathcal{A}_\lambda(\tilde{\boldsymbol{u}})\lambda + \mathcal{A}_{\boldsymbol{u}}(\hat{\boldsymbol{u}})\boldsymbol{u} \tag{24}$$

$$\langle \mathcal{A}_\lambda(\tilde{\boldsymbol{u}})\lambda, \xi\rangle = a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi) = \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{w}))\lambda, \xi\Big) - \Big(\lambda, \boldsymbol{w}\cdot\nabla\xi\Big) + \langle\lambda, \xi\rangle_+ \tag{25}$$

$$= \Big(\mathcal{M}_a\,\lambda, \xi\Big) + \Big(\mathcal{M}_b\sqrt{D_{\mathrm{II}}(\boldsymbol{w})}\,\lambda, \xi\Big) - \Big(\lambda, \boldsymbol{w}\cdot\nabla\xi\Big) + \langle\lambda, \xi\rangle_+ \tag{26}$$

$$\langle \mathcal{A}_{\boldsymbol{u}}(\hat{\boldsymbol{u}})\boldsymbol{u}, \boldsymbol{v}\rangle = a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) = \Big(2\mu(D_{\mathrm{II},r}, \lambda)\mathbf{D}(\boldsymbol{u}), \mathbf{D}(\boldsymbol{v})\Big) + \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\Big) \tag{27}$$

$$\langle \mathcal{B}\boldsymbol{v}, q\rangle = b(\boldsymbol{v}, q) = -\Big(\nabla\cdot\boldsymbol{v}, q\Big) \tag{28}$$

$$l(\tilde{\boldsymbol{v}}) = l_\lambda(\xi) + l_{\boldsymbol{u}}(\boldsymbol{v}) \tag{29}$$

$$l_\lambda(\xi) = \Big(f_\lambda, \xi\Big) \tag{30}$$

$$l_{\boldsymbol{u}}(\boldsymbol{v}) = \Big(\boldsymbol{f}_{\boldsymbol{u}}, \boldsymbol{v}\Big) \tag{31}$$

2.3 Wellposedness results for TVP problem

The existence results are stated in the following theorem:

Theorem 1. *Let* $\boldsymbol{f}_{\boldsymbol{u}} \in (L^2(\Omega))^2$ *and* $f_\lambda \in L^2(\Omega)$*, the thixo-viscoplastic problem (21) has a unique solution* $(\tilde{\boldsymbol{u}}, p) = (\lambda, \boldsymbol{u}, p) \in \mathbb{W} \times \mathbb{Q}$ *with the following bound of the solution on the data*

$$\|\boldsymbol{u}\|_{\mathbb{V}} \le \frac{1}{\eta_0 \mathcal{C}_K}\,\|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \tag{32}$$

$$\|p\|_{\mathbb{Q}/\ker\mathcal{B}^T} \le \frac{1}{\beta}\Big(1 + \frac{2\,(\eta_\infty + k\tau_\infty) + \|\boldsymbol{u}\|_\infty}{\eta_0 \mathcal{C}_K}\Big)\,\|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \tag{33}$$

$$\|\lambda\|_0 \le \frac{1}{\mathcal{M}_a}\,\|f_\lambda\|_0 \tag{34}$$

$$\mathcal{M}_a\,\|\lambda\|_0^2 + \frac{1}{2}\langle\lambda\rangle^2 \le \frac{1}{\mathcal{M}_a}\,\|f_\lambda\|_0^2 \tag{35}$$

where $\mathcal{C}_K$ *denotes the Korn's inequality constant,* β *is the LBB constant.*

For the ease of the proof of the Theorem 1, we use the following intermediate propositions, segregating the steps for the proof. Keeping in mind the monolithic approach under investigation, the propositions are rank independent i.e. they can be merged in one. Let us introduce the following spaces

$$\mathbb{L}_0^s(\Omega) = \left\{ \boldsymbol{w} \in (L^s(\Omega))^d \,|\, \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega, \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega,\ s > 1 \right\} \tag{36}$$

$$\mathbb{H}_0^s(\Omega) = \left\{ \boldsymbol{w} \in (L^s(\Omega))^d \,|\, \nabla \boldsymbol{w} \in (L^s(\Omega))^d \,|\, \nabla \cdot \boldsymbol{w} = 0 \text{ in } \Omega, \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega,\ s > 1 \right\} \tag{37}$$

Proposition 1. *Let $\boldsymbol{f_u} \in (L^2(\Omega))^2$, for every $\zeta \in \mathbb{T}$, $\boldsymbol{w} \in \mathbb{L}_0^s$ and $\hat{\boldsymbol{u}} = (\zeta, \boldsymbol{u}, \boldsymbol{w})$ there exists a unique solution $\boldsymbol{u} = \boldsymbol{u}(\zeta, \boldsymbol{w})$ to the problem*

$$\begin{cases} a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = l_{\boldsymbol{u}}(\boldsymbol{v}), & \forall \boldsymbol{v} \in \mathbb{V} \\ b(\boldsymbol{u}, q) = 0, & \forall q \in \mathbb{Q}, \end{cases} \tag{38}$$

and it satisfies the estimate

$$\|\boldsymbol{u}\|_{\mathbb{V}} \le \frac{1}{\eta_0 \mathcal{C}_K} \|\boldsymbol{f_u}\|_0 \tag{39}$$

$$\|p\|_{\mathbb{Q}/\ker \mathcal{B}^T} \le \frac{1}{\beta}\Big(1 + \frac{2\,(\eta_\infty + k\tau_\infty) + \|\boldsymbol{w}\|_\infty}{\eta_0 \mathcal{C}_K}\Big) \|\boldsymbol{f_u}\|_0 \tag{40}$$

where $\mathcal{C}_K$ denotes the Korn's inequality constant and β is the LBB constant.

Proposition 2. *Let $f_\lambda \in L^2(\Omega)$, for every $\zeta \in \mathbb{T}$, $\boldsymbol{w} \in \mathbb{H}_0^s$ and $\hat{\boldsymbol{u}} = (\zeta, \boldsymbol{u}, \boldsymbol{w})$, there exists $\lambda = \lambda(\boldsymbol{w})$ a solution to the variational problem*

$$a_\lambda(\hat{\boldsymbol{u}})(\lambda, \xi) = l_\lambda(\xi), \quad \forall \xi \in \mathbb{T} \tag{41}$$

and it satisfies the estimate

$$\|\lambda\|_0 \le \frac{1}{\mathcal{M}_a} \|f_\lambda\|_0 \tag{42}$$

$$\mathcal{M}_a \|\lambda\|_0^2 + \frac{1}{2}\langle\lambda\rangle^2 \le \frac{1}{\mathcal{M}_a} \|f_\lambda\|_0^2 \tag{43}$$

Lemma 1. *(Monotonicity) For all $\boldsymbol{\Upsilon}, \boldsymbol{\zeta} \in \mathbb{R}^{d\times d}$, $\zeta \in \mathbb{R}^+$, and $\tau(\cdot) \in \mathcal{C}^0([0,1]; [\tau_0, \tau_\infty])$*

$$\Big(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}), \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\Big) \ge 0 \tag{44}$$

with $|\cdot|_{F,r}$ an approximation of the Frobenius norm $|\cdot|_F$ s.t.

$$|\cdot|_F \le |\cdot|_{F,r} \tag{45}$$

$$\Big| |\boldsymbol{\Upsilon}|_{F,r} - |\boldsymbol{\zeta}|_{F,r} \Big| \le |\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F \tag{46}$$

Remark 1. *The conditions on the approximation norm (45) and (46) are sufficient but not necessary. Indeed, one may use exclusively the approximated norm. This is what might be opted in simulations codes with user-given regularizations.*

Proof. (Lemma 1) Let $\boldsymbol{\Upsilon}, \boldsymbol{\zeta} \in \mathbb{R}^{d\times d}$ and $\xi \in \mathbb{R}^+$

$$\left(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{47}$$

$$= \left(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{48}$$

$$- \left(\tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{49}$$

$$= \left(\frac{\tau(\zeta)}{|\boldsymbol{\Upsilon}|_{F,r}}\boldsymbol{\Upsilon} - \boldsymbol{\zeta}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{50}$$

$$- \left(\tau(\zeta)\frac{|\boldsymbol{\Upsilon}|_{F,r}\,\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}\,|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{|\boldsymbol{\zeta}|_{F,r}\,\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}\,|\boldsymbol{\Upsilon}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{51}$$

$$= \left(\frac{\tau(\zeta)}{|\boldsymbol{\Upsilon}|_{F,r}}\boldsymbol{\Upsilon} - \boldsymbol{\zeta}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{52}$$

$$- \left(\frac{\tau(\zeta)}{|\boldsymbol{\Upsilon}|_{F,r}}\left(|\boldsymbol{\Upsilon}|_{F,r} - |\boldsymbol{\zeta}|_{F,r}\right)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{53}$$

$$\geq \left(\frac{\tau(\zeta)}{|\boldsymbol{\Upsilon}|_{F,r}}\boldsymbol{\Upsilon} - \boldsymbol{\zeta}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{54}$$

$$- \left(\frac{\tau(\zeta)}{|\boldsymbol{\Upsilon}|_{F,r}}\,|\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F\,\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}, \boldsymbol{\Upsilon} - \boldsymbol{\zeta}\right) \tag{55}$$

$$\geq 0 \tag{56}$$

The term $\tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}$ is subtracted in (48) and added in (49), (55) is deduced from (53) using (46) and (45) for the term $\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}$. □

Lemma 2. *(Continuity) For all $\boldsymbol{\Upsilon}, \boldsymbol{\zeta}, \boldsymbol{\xi} \in \mathbb{R}^{d\times d}$, $\zeta \in \mathbb{R}^+$, and $\tau(\cdot) \in \mathcal{C}^0([0,1]; [\tau_0, \tau_\infty])$*

$$\left(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}, \boldsymbol{\xi}\right) \leq 2\tau_\infty k\,|\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F\,|\boldsymbol{\xi}|_F \tag{57}$$

with $|\cdot|_{F,r}$ an approximation of the Frobenius norm $|\cdot|_F$ s.t.

$$\frac{1}{|\boldsymbol{\Upsilon}|_{F,r}} \leq k \tag{58}$$

where $k \geq 0$ is a regularization parameter.

Proof. (Lemma 2) let $\boldsymbol{\Upsilon}, \boldsymbol{\zeta}, \boldsymbol{\xi} \in \mathbb{R}^{d\times d}$ and $\zeta \in \mathbb{R}^{+}$

$$\Big(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}}, \boldsymbol{\xi}\Big) \tag{59}$$

$$= \Big(\tau(\zeta)\frac{\boldsymbol{\Upsilon}}{|\boldsymbol{\Upsilon}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}, \boldsymbol{\xi}\Big) \tag{60}$$

$$- \Big(\tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|_{F,r}} - \tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}, \boldsymbol{\xi}\Big) \tag{61}$$

$$\leq \tau_\infty \frac{1}{|\boldsymbol{\Upsilon}|_{F,r}} |\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F \, |\boldsymbol{\xi}|_F \tag{62}$$

$$+ \tau_\infty \frac{1}{|\boldsymbol{\Upsilon}|_{F,r}} \Big| |\boldsymbol{\Upsilon}|_{F,r} - |\boldsymbol{\zeta}|_{F,r} \Big| \frac{|\boldsymbol{\zeta}|_F}{|\boldsymbol{\zeta}|_{F,r}} |\boldsymbol{\xi}|_F \tag{63}$$

$$\leq \tau_\infty \frac{1}{|\boldsymbol{\Upsilon}|_{F,r}} |\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F \, |\boldsymbol{\xi}|_F \tag{64}$$

$$+ \tau_\infty \frac{1}{|\boldsymbol{\Upsilon}|_{F,r}} |\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F \frac{|\boldsymbol{\zeta}|_F}{|\boldsymbol{\zeta}|_{F,r}} |\boldsymbol{\xi}|_F \tag{65}$$

$$\leq 2\tau_\infty k \, |\boldsymbol{\Upsilon} - \boldsymbol{\zeta}|_F \, |\boldsymbol{\xi}|_F \tag{66}$$

The term $\tau(\zeta)\frac{\boldsymbol{\zeta}}{|\boldsymbol{\Upsilon}|_{F,r}}$ is subtracted in (60) and added in (61), taking the upper bound for $\tau(\cdot)$ to get (62, 63), and using conditions on the norm (45), (46), and (58) for terms $\frac{|\boldsymbol{\Upsilon}|_F}{|\boldsymbol{\Upsilon}|_{F,r}}$, $\Big||\boldsymbol{\Upsilon}|_{F,r} - |\boldsymbol{\zeta}|_{F,r}\Big|$, and $\frac{1}{|\boldsymbol{\Upsilon}|_{F,r}}$ respectively. □

Now, we proceed with the proof of propositions (1) and (2).

Proof. (Proposition 1) Firstly, we introduce the subspace $\mathbb{V}_0 \subset \mathbb{V}$

$$\mathbb{V}_0 = \{\boldsymbol{v} \in \mathbb{V} \mid b(v, q) = 0, \forall q \in \mathbb{Q}\}. \tag{67}$$

For every $\zeta \in \mathbb{T}$, $\boldsymbol{w} \in \mathbb{L}_0^s$ and $\hat{\boldsymbol{u}} = (\zeta, \boldsymbol{u}, \boldsymbol{w})$ apply Browder-Minty Theorem of monotone operator to $a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$ on $\mathbb{V}_0 \times \mathbb{V}_0$ to show existence and uniqueness of the solution $\boldsymbol{u} = \boldsymbol{u}(\zeta, \boldsymbol{w}) \in \mathbb{V}_0$. Secondly, the existance and uniqueness of the pressure in $L_0^2(\Omega)$ is due to the LBB condition. Indeed, $a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$ is coercive, continuous, and monotone on $\mathbb{V}_0 \times \mathbb{V}_0$:
Coerciveness: For all $\boldsymbol{v} \in \mathbb{V}_0$ we have

$$a_{\boldsymbol{u}}(\hat{\boldsymbol{v}})(\boldsymbol{v}, \boldsymbol{v}) = \Big(2\mu(D_{\mathrm{II},r}, \zeta)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v})\Big) + \Big(\boldsymbol{w} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}\Big) \tag{68}$$

$$= \Big(2\mu(D_{\mathrm{II},r}, \zeta)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v})\Big) - \Big(\boldsymbol{v}, \boldsymbol{w} \cdot \nabla \boldsymbol{v}\Big) \tag{69}$$

$$\geq 2\Big(\eta(\zeta)\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v})\Big) \tag{70}$$

$$\geq 2\eta_0\Big(\mathbf{D}(\boldsymbol{v}), \mathbf{D}(\boldsymbol{v})\Big) \tag{71}$$

$$\geq \eta_0 \mathcal{C}_K \|\boldsymbol{v}\|_1^2 \tag{72}$$

(69) is deduced from (68) with integration by part of the convective term and imposing the homogeneous boundary condition. (70) is due to the positivity of the plastic contribution (see Lemma 1). Taking η_0 the lower value of $\eta(\cdot)$, we get the inequality (71). (72) is deduced from (71) using the Korn's first inequality [6]. Thus, we get the coerciveness of $a_{\boldsymbol{u}}(\hat{\boldsymbol{v}})(\boldsymbol{v}, \boldsymbol{v})$.

Continuity: For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\eta} \in \mathbb{V}_0$,and for all $\zeta \in \mathbb{T}$ we have

$$
\begin{aligned}
& a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{\eta}) - a_{\boldsymbol{u}}(\hat{\boldsymbol{v}})(\boldsymbol{v},\boldsymbol{\eta}) && (73)\\
&\quad = \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{u}),\zeta)\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) + \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{\eta}\Big) - \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{v}),\zeta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{v},\boldsymbol{\eta}\Big) && (74)\\
&\quad = \Big(2\eta(\zeta)(\mathbf{D}(\boldsymbol{u})-\mathbf{D}(\boldsymbol{v})),\mathbf{D}(\boldsymbol{\eta})\Big) && (75)\\
&\qquad + \Big(\tau(\zeta)\frac{\sqrt{2}}{\sqrt{D_{\mathrm{II},r}(\boldsymbol{u})}}\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(\tau(\zeta)\frac{\sqrt{2}}{\sqrt{D_{\mathrm{II},r}(\boldsymbol{v})}}\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) && (76)\\
&\qquad + \Big(\boldsymbol{w}\cdot\nabla(\boldsymbol{u}-\boldsymbol{v}),\boldsymbol{\eta}\Big) && (77)\\
&\quad \le 2\eta_\infty \|\mathbf{D}(\boldsymbol{u})-\mathbf{D}(\boldsymbol{v})\|_0 \|\mathbf{D}(\boldsymbol{\eta})\|_0 && (78)\\
&\qquad + 2\tau_\infty k \|\mathbf{D}(\boldsymbol{u})-\mathbf{D}(\boldsymbol{v})\|_0 \|\mathbf{D}(\boldsymbol{\eta})\|_0 && (79)\\
&\qquad + \|\boldsymbol{w}\|_\infty \|\boldsymbol{u}-\boldsymbol{v}\|_1 \|\boldsymbol{\eta}\|_0 && (80)\\
&\quad \le (2\eta_\infty + 2k\tau_\infty + \|\boldsymbol{w}\|_\infty) \|\boldsymbol{u}-\boldsymbol{v}\|_1 \|\boldsymbol{\eta}\|_1 && (81)
\end{aligned}
$$

(78) is deduced from (75) by taking upper bound for $\eta(\cdot)$ and using the Hölder inequality. (79) is a results of Lemma 2 applied to (76). Using Hölder inequality for (77), we get (80). we set $\mathcal{C}_1 = 2(\eta_\infty + k\tau_\infty) + \|\boldsymbol{w}\|_\infty$.
Monotonocity: Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{V}_0$, for all $\zeta \in \mathbb{T}, \boldsymbol{w} \in \mathbb{L}_0^s$ and set $\hat{\boldsymbol{u}} = (\zeta, \boldsymbol{u}, \boldsymbol{w}), \boldsymbol{\eta} = \boldsymbol{u} - \boldsymbol{v}$

$$
\begin{aligned}
& a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{v}) - a_{\boldsymbol{u}}(\hat{\boldsymbol{v}})(\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}) && (82)\\
&\quad = \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{u}),\zeta)\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) + \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{\eta}\Big) - \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{v}),\zeta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{v},\boldsymbol{\eta}\Big) && (83)\\
&\quad = \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{u}),\zeta)\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{v}),\zeta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) + \Big(\boldsymbol{w}\cdot\nabla\boldsymbol{\eta},\boldsymbol{\eta}\Big) && (84)\\
&\quad = \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{u}),\zeta)\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(2\mu(D_{\mathrm{II},r}(\boldsymbol{v}),\zeta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) && (85)\\
&\quad = \Big(2\eta(\zeta)\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(2\eta(\zeta)\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) && (86)\\
&\qquad + \Big(\tau(\zeta)\frac{\sqrt{2}}{\sqrt{D_{\mathrm{II},r}(\boldsymbol{u})}}\mathbf{D}(\boldsymbol{u}),\mathbf{D}(\boldsymbol{\eta})\Big) - \Big(\tau(\zeta)\frac{\sqrt{2}}{\sqrt{D_{\mathrm{II},r}(\boldsymbol{v})}}\mathbf{D}(\boldsymbol{v}),\mathbf{D}(\boldsymbol{\eta})\Big) && (87)\\
&\quad \ge \eta_0 \mathcal{C}_K \|\boldsymbol{u}-\boldsymbol{v}\|_1^2 && (88)
\end{aligned}
$$

(85) is deduced from (84) with integration by part of the convective term and imposing the homogeneous boundary condition. (88) is a result of Korn's inequality applied to (86) and the positivity of the term (87) see Lemma 1.
By using Browder-Minty Theorem of monotone operator to $a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$ on $\mathbb{V}_0 \times \mathbb{V}_0$, there exists a unique solution $\boldsymbol{u} = \boldsymbol{u}(\zeta, \boldsymbol{w})$. Moreover, using the coercivity of $a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$ and the continuity of $l_{\boldsymbol{u}}(\cdot)$, we get

$$\eta_0 \mathcal{C}_K \|\boldsymbol{u}\|_1^2 \le a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{u}) = l_{\boldsymbol{u}}(\boldsymbol{u}) \le \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \|\boldsymbol{u}\|_0 \le \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \|\boldsymbol{u}\|_1 \tag{89}$$

Thus, the solution satisfies the estimate

$$\|\boldsymbol{u}\|_1 \le \frac{1}{\eta_0 \mathcal{C}_K} \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \tag{90}$$

The existence of the pressure is due to Lagrange multiplier argument. Indeed, let us consider the problem

$$l_{\boldsymbol{u}}(\boldsymbol{v}) = \langle \boldsymbol{f}_{\boldsymbol{u}}, \boldsymbol{v}\rangle - a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbb{V}. \tag{91}$$

Thus, $l_{\boldsymbol{u}}(\boldsymbol{v}) = 0$ on $\mathbb{V}_0 = \ker \mathcal{B}$, $l_{\boldsymbol{u}} \in \operatorname{Im} \mathcal{B}$, and there exists a $p \in \mathbb{Q}$ s.t.

$$l_{\boldsymbol{u}}(\boldsymbol{v}) = b(\boldsymbol{v}, p), \quad \forall \boldsymbol{v} \in \mathbb{V}. \tag{92}$$

Then, using the LBB condition, the estimate for velocity (90) and the upper bound, denoted $\|a_{\boldsymbol{u}}\|$, for of $a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\cdot,\cdot)$.

$$\begin{aligned}
\|p\|_{\mathbb{Q}/\ker \mathcal{B}^{\mathrm{T}}} &\leq \frac{1}{\beta} \sup_{\boldsymbol{v}\in\mathbb{V}} \frac{b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_{\mathbb{V}}} && (93)\\
&\leq \frac{1}{\beta} \|l_{\boldsymbol{u}}\|_0 && (94)\\
&\leq \frac{1}{\beta}\Big(\|a_{\boldsymbol{u}}\| \, \|\boldsymbol{u}\|_{\mathbb{V}} + \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \Big) && (95)\\
&\leq \frac{1}{\beta}\Big(\frac{1}{\eta_0 \mathcal{C}_K} \|a_{\boldsymbol{u}}\| \, \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 + \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \Big) && (96)\\
&\leq \frac{1}{\beta}\Big(1 + \frac{\|a_{\boldsymbol{u}}\|}{\eta_0 \mathcal{C}_K} \Big) \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 && (97)\\
&\leq \frac{1}{\beta}\Big(1 + \frac{2\,(\eta_\infty + k\tau_\infty) + \|\boldsymbol{w}\|_\infty}{\eta_0 \mathcal{C}_K} \Big) \|\boldsymbol{f}_{\boldsymbol{u}}\|_0 && (98)
\end{aligned}$$

□

This concludes the proposition 1. Now, we investigate the proposition 2.

Proof. (Proposition 2) For every $\zeta \in \mathbb{T}$, $\boldsymbol{w} \in \mathbb{H}_0^s$, and $\hat{\boldsymbol{u}} = (\zeta, \boldsymbol{u}, \boldsymbol{w})$. In order to show the existence of unique solution $\lambda = \lambda(\boldsymbol{w})$ for the variational problem

$$a_\lambda(\hat{\boldsymbol{u}})(\lambda, \xi) = l_\lambda(\xi), \quad \forall \xi \in \mathbb{T} \tag{99}$$

We use Lax-Milgram lemma; we demonstrate the coerciveness and continuity of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$, and the continuity of $l_\lambda(\cdot)$.

Coerciveness of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$*:* Let $\xi \in \mathbb{T}$, we have

$$\begin{aligned}
a_\lambda(\hat{\boldsymbol{u}})(\xi,\xi) &= \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{w}))\xi, \xi\Big) - \Big(\xi, \boldsymbol{w}\cdot\nabla\xi\Big) + \langle \xi, \xi\rangle_+ && (100)\\
&= \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{w}))\xi, \xi\Big) + \Big(\xi, \boldsymbol{w}\cdot\nabla\xi\Big) - \langle \xi, \xi\rangle_+ + \langle \xi, \xi\rangle_- + \langle \xi, \xi\rangle_+ && (101)\\
&= \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{w}))\xi, \xi\Big) + \Big(\xi, \boldsymbol{w}\cdot\nabla\xi\Big) + \langle \xi, \xi\rangle_- && (102)\\
&= \Big(\mathcal{M}(D_{\mathrm{II}}(\boldsymbol{w}))\xi, \xi\Big) + \frac{1}{2}\left\{< \xi >_-^2 + < \xi >_+^2\right\} && (103)\\
&= \Big(\mathcal{M}_a \xi, \xi\Big) + \Big(\mathcal{M}_b \sqrt{D_{\mathrm{II}}(\boldsymbol{w})}\xi, \xi\Big) + \frac{1}{2}\left\{< \xi >_-^2 + < \xi >_+^2\right\} && (104)\\
&\geq \mathcal{M}_a \|\xi\|_0^2 + \frac{1}{2}\left\{< \xi >_-^2 + < \xi >_+^2\right\} && (105)
\end{aligned}$$

(101) is deduced from (100) using integration by part for the convective part and summing them up, we get (103). (105) is deduced from (104) due to the positivity of the second term.

Continuity of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$*:* Let $\lambda, \xi \in \mathbb{T}$, we have

$$\begin{aligned}
|a_\lambda(\hat{\boldsymbol{u}})(\lambda,\xi)| &= \Big|\Big(\mathcal{M}_a \lambda, \xi\Big) + \Big(\mathcal{M}_b \sqrt{D_{\mathrm{II}}(\boldsymbol{w})}\lambda, \xi\Big) - \Big(\lambda, \boldsymbol{w}\cdot\nabla\xi\Big) + \langle \lambda, \xi\rangle_+\Big| && (106)\\
&\leq \mathcal{M}_a \|\lambda\|_0 \|\xi\|_0 + \mathcal{M}_b \|\mathbf{D}(\boldsymbol{w})\|_\infty \|\lambda\|_0 \|\xi\|_0 + \|\boldsymbol{w}\|_\infty \|\lambda\|_0 \|\xi\|_1 + < \lambda, \xi >_+ && (107)
\end{aligned}$$

For the second and third terms in (106), we used (L^∞, L^2, L^2) Hölder's inequality.

Continuity of $l_\lambda(\cdot)$:

$$l_\lambda(\xi) = \Big(f_\lambda, \xi\Big) \leq \|f_\lambda\|_0 \|\xi\|_0 \leq \|f_\lambda\|_0 \|\xi\|_1 \tag{108}$$

Then, there exists $\lambda = \lambda(\boldsymbol{w})$ a solution to the variational problem (41). To express the continuous behavior of the solution on the data, we use the coercivity of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$ and the continuity of $l_\lambda(\cdot)$

$$\mathcal{M}_a \|\lambda\|_0^2 \leq \ a_\lambda(\hat{\boldsymbol{u}})(\lambda, \lambda) = \ l_\lambda(\lambda) \leq \ \|f_\lambda\|_0 \|\lambda\|_0 \tag{109}$$

then,

$$\|\lambda\|_0 \leq \frac{1}{\mathcal{M}_a} \|f_\lambda\|_0 \tag{110}$$

Since, we also have

$$\mathcal{M}_a \|\lambda\|_0^2 + \frac{1}{2}\langle\lambda\rangle^2 \leq \ \|f_\lambda\|_0 \|\lambda\|_0 \tag{111}$$

then,

$$\mathcal{M}_a \|\lambda\|_0^2 + \frac{1}{2}\langle\lambda\rangle^2 \leq \frac{1}{\mathcal{M}_a} \|f_\lambda\|_0^2 \tag{112}$$

which is the desired inequality (43). □

Remark 2. *The boundness of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$ from below by the L_2-norm and the boundary norms is a considerably weaker condition w.r.t. $\|\cdot\|_1$-norm which is the norm considered for the microstructure space $\mathbb{T}$.*

Proof. (Theorem 1) The proof for Theorem 1 is due to the fixed point theorem applied to the mapping of the solutions in the range of estimates of velocity, pressure, and microstructure parameter (39), (40), (42) respectively. □

3 Approximation of TVP problem

We start by setting the approximation TVP problem in its general abstract form

3.1 Abstract settings

For the approximation of TVP problem, let $\mathbb{T}_h \subset \mathbb{T}, \mathbb{V}_h \subset \mathbb{V}, \mathbb{W}_h \subset \mathbb{W}$, and $\mathbb{Q}_h \subset \mathbb{Q}$ be finite dimensional subspaces with the superscript h being a parameter dependent on the mesh spacing. We set $\tilde{\boldsymbol{u}}_h = (\lambda_h, \boldsymbol{u}_h), \tilde{\boldsymbol{v}}_h = (\xi_h, \boldsymbol{v}_h)$, $\hat{\boldsymbol{u}}_h = (\tilde{\boldsymbol{u}}_h, \boldsymbol{u}_h)$. The approximated TVP problem is to seek an approximated solution $(\tilde{\boldsymbol{u}}_h, p_h) \in \mathbb{W}_h \times \mathbb{Q}_h$ s. t.

$$a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}}_h)(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) = l(\tilde{\boldsymbol{v}}_h), \quad \forall (\tilde{\boldsymbol{v}}_h, q_h) \in \mathbb{W}_h \times \mathbb{Q}_h, \tag{113}$$

equivalenty,

$$\begin{cases} a_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}}_h)(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\boldsymbol{v}_h, p_h) = \ l(\tilde{\boldsymbol{v}}_h), & \forall\, \tilde{\boldsymbol{v}}_h \in \mathbb{W}_h \\ b(\boldsymbol{u}_h, q_h) = \ 0, & \forall\, q_h \in \mathbb{Q}_h, \end{cases} \tag{114}$$

and in operator form

$$\begin{cases} \mathcal{A}_{\tilde{\boldsymbol{u}}}(\hat{\boldsymbol{u}}_h)\tilde{\boldsymbol{u}}_h + \mathcal{B}_h^{\mathrm{T}} p_h = \boldsymbol{f}_{\tilde{\boldsymbol{u}}}, & \text{on } \mathbb{W}_h' \\ \mathcal{B}_h \boldsymbol{u}_h = 0, & \text{on } \mathbb{Q}_h' \end{cases} \tag{115}$$

The problems that we have to solve here are the existence and uniqueness of the solution $(\tilde{\boldsymbol{u}}_h, p_h)$ and the estimation $\|\lambda - \lambda_h\|_0$, $\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$, and $\|p - p_h\|_0$. We assume that the inf − sup condition for the pair $(\mathbb{V}_h, \mathbb{Q}_h)$ is satisfied i.e.

$$\exists \beta > 0 \,\text{s.t.} \sup_{\boldsymbol{v}_h \in \mathbb{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|} \geq \beta \, \|q_h\|_{\mathbb{Q}/\ker \mathcal{B}_h^{\mathrm{T}}} \quad \forall q_h \in \mathbb{Q}_h, \tag{116}$$

and β is independent of h.

Some results of section § 2.2 concerning existence and uniqueness are directly applied here due to the conforming approximations. We move on to use the higher order finite element discrcetization.

3.2 Finite element discretization

The finite element approximations of the problem (21) have to take care of its saddle point character, due to the bilinear form (28). Furthermore, since thixo-viscoplastic flows are usually slow, the only remaining issue is the coercivity of $a_\lambda(\hat{\boldsymbol{u}})(\cdot,\cdot)$ w.r.t. "L_2-norm and the boundary norms" which is considerably weaker for the microstructure space $\mathbb{T}$ equipped with $\|\cdot\|_1$-norm. We opt for higher order stable pair biquadratic for velocity and piecewise linear discontinuous for the pressure, Q_2/P_1^{disc}, and higher order quadratic for structure parameter Q_2 with the appropriate stabilization terms [8, 10]. Indeed, let the domain Ω be partitioned by a grid $K \in \mathcal{T}_h$ which are assumed to be open quadrilaterals such that $\Omega = \mathrm{int}\big(\bigcup_{k\in\mathcal{T}_h} \overline{K}\big)$. For an element $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K)$ the set of all 1-dimensional edges of K. Let $\mathcal{E}_i := \bigcup_{k\in\mathcal{T}_h} \mathcal{E}(K)$ be the set of all interior element edges of the grid $\mathcal{T}_h$.

We define the conforming finite element spaces $\mathbb{T}_h \subset \mathbb{T}$, $\mathbb{V}_h \subset \mathbb{V}$, and $\mathbb{Q} \subset \mathbb{Q}_h$ such that:

$$\mathbb{T}_h = \Big\{\xi_h \in \mathbb{T}, \xi_{h|K} \in \ Q_2(K) \,\forall\ K \in \mathcal{T}_h\Big\}, \tag{117}$$

$$\mathbb{V}_h = \big\{\boldsymbol{v}_h \in \mathbb{V}, \boldsymbol{v}_{h|K} \in \ (Q_2(K))^2 \,\forall\ K \in \mathcal{T}_h,\, \boldsymbol{v}_h = 0 \text{ on } \partial\Omega_h\big\}, \tag{118}$$

$$\mathbb{Q}_h = \Big\{q_h \in \mathbb{Q}, q_{h|K} \in \ P_1^{\mathrm{disc}}(K) \,\forall\ K \in \mathcal{T}_h\Big\}. \tag{119}$$

The approximate problem reads: *Find* $\tilde{\boldsymbol{u}}_h \in \mathbb{T}_h \times \mathbb{V}_h \times \mathbb{Q}_h$ *s. t.*

$$a_\lambda(\hat{\boldsymbol{u}})(\lambda_h, \xi_h) + j_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + a_{\boldsymbol{u}}(\hat{\boldsymbol{u}})(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) = 0, \quad \forall \tilde{\boldsymbol{v}}_h \in \mathbb{T}_h \times \mathbb{V}_h \times \mathbb{Q}_h. \tag{120}$$

The stabilization term $j_{\tilde{\boldsymbol{u}}}(\cdot,\cdot)$ is given as follows

$$j(\cdot,\cdot) := j_{\boldsymbol{u}}(\cdot,\cdot) + j_\lambda(\cdot,\cdot), \tag{121}$$

$$j_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) = \sum_{E\in\mathcal{E}_i} \gamma_{\boldsymbol{u}} |E|^2 \int_E [\nabla \boldsymbol{u}_h]\,[\nabla \boldsymbol{v}_h]\, d\sigma, \tag{122}$$

$$j_\lambda(\lambda_h, \xi_h) = \sum_{E\in\mathcal{E}_i} \gamma_\lambda |E|^2 \int_E [\nabla \lambda_h]\,[\nabla \xi_h]\, d\sigma. \tag{123}$$

The stabilization (121) is consistent, control the convective terms and makes the coercivity and continuity match in $\mathbb{T}_h$ equipped with H^1-norm equivalent to $|||\cdot|||$, where

$$|||\xi_h|||^2 = \|\xi_h\|_0^2 + j_\lambda(\xi_h, \xi_h). \tag{124}$$

Remark 3. *The essential part of finite element approximation of comparing the discrete solution* $(\lambda_h, \boldsymbol{u}_h, p_h)$ *of the approximated TVP problem (113) to the exact solution* $(\lambda, \boldsymbol{u}, p)$ *of the continuous TVP problem (21) will be reported exclusively in an upcoming work.*

4 Numerical solutions

We investigate numerical solutions of Houška's [4] thixo-viscoplastic material in a 4:1 curved contraction configuration. The fully-developed flow conditions according to Houška thixotropic model are imposed at entry, Γ^- , together with no-slip on top and bottom walls of reservoir, Γ.

The numerical solutions are obtained using a monolithic Newton-multigrid FEM solver. On one hand, we are using adaptive discrete Newton method to linearize the discrete nonlinear TVP problem, where the adaptive discrete Newton method is based on step-length in divided difference. The adaptive strategy is exclusively due to the residual convergences. On the other hand, the linearized systems inside the outer Newton loops are solved using a coupled geometrical multigrid solver based on local pressure Schur complement (LPSC) schemes which are generlization of Vanka-type smoothers for saddle point problems. They are simple iterative relaxation methods solving directly on element level and performing an outer block Gauss-Seidel iteration. The local character of this procedure together with a global defect-correction mechanism on one hand, and the choice of discontinuous FE approximations for pressure (P_1^{disc}) on the other hand, results in an efficient solver for TVP problems. For details, we refer to [1, 2, 7].

Our emphasis is to examine the transitions in shape and extent of unyielded zones w.r.t breakdown parameter as well as on isobands of material microstructuring level λ. Figure 1 illustrates the impact of thixotropy breakdown parameter $\mathcal{M}_b$. In fact, increasing the breakdown parameter induces more breakdown layers close to walls of downstream channel preventing the material from rest along pipelines, typically used in transportation of waxy crude oils. As consequence, the design for pipelines should take in consideration the thixotropic phenomena inherited in the material.

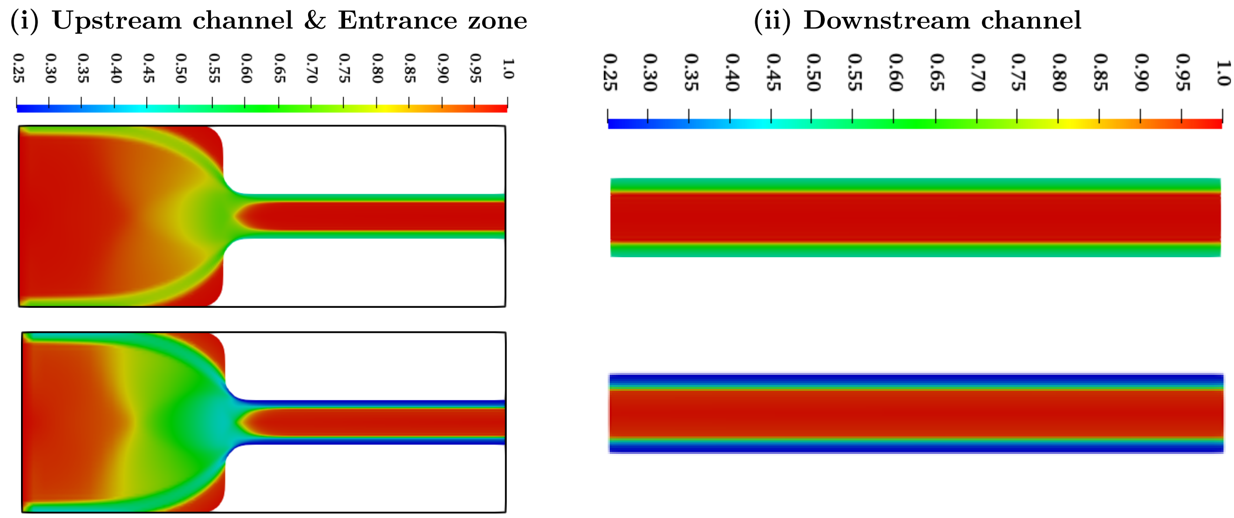

Figure 1: **Thixotropic flows in contractions:** Impact of breakdown parameter $\mathcal{M}_b$ on λ isobands for thixotropic flows. The other parameters are set to constants $\mathcal{M}_a = 1.0$, $\eta_0 = \eta_\infty = 1.0$, $\tau_0 = 0.0$, $\tau_\infty = 2.0$, and $k = 10^4$.

Furthermore, it is interesting to observe that unyielded zones in upstream and downstream parts of contraction domains do not merge. Physically, this happens because the unyielded material in the vicinity of center of reservoir becomes more rigid (or less flexible due to its inelastic nature). That means, the material can not deform elastically. Thus, when the material elements cross the contraction zone, they are not able to undergo even a small scale elastic crosswise extension, and thus remain disconnected and travel like a rigid-body in downstream channel. Further investigations to be considered, allowing for shear thickening and shear thinning behavior for plastic viscosity and/or elastic behaviour below the critical yield stress limit in more a general thixotropic models.

5 Summary

We presented the wellposedness of thixo-viscoplastic flow problems in context of finite element approximations. So, we started by introducing the variational principle and functional spaces in the context of shear rate independent plastic viscosity i.e. $\eta = \eta(\lambda)$. Then, we expressed the continuous as well as the approximated TVP problems in a classical abstract setting of saddle point problems based on the incompressibility constraint. The existence and uniqueness of coupling solutions are treated in the spirit of the ultimate goal to develop a corresponding monolithic solver. The microstructure parameter and velocity variables are treated simultaneously without loosing the clarity in exposing the wellposedness results. We used Browder-Minty theorem of monotone operator theorem for the viscoplastic subproblems and Lax-Milgram lemma for microstructure subproblems, then fixed point for the coupled problem.

We investigated numerically solutions of Houška's [4] thixo-viscoplastic material in a 4:1 curved contraction configuration using monolithic Newton-multigrid FEM Solver. We investigated the impact of thixotropy breakdown parameter $\mathcal{M}_b$ on transitions in shape and extent of unyielded zones and on isobands of material microstructuring level λ. In fact, increasing the breakdown parameter induces more breakdown layers close to walls of downstream channel preventing the material from rest along pipelines. Moreover, unyielded zones in upstream and downstream parts of contraction domains do not merge due the lack of elastic response of the material. Further undergoing investigations are the shear thickening and shear thinning behavior for the plastic viscosity and/or elastic behavior below the critical yield stress limit in more a general thixotropic models.

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