A SET OF FINITE ELEMENT SPACES FOR THE MIXED FORMULATION OF TWOBODY CONTACT PROBLEMS

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Abstract. A set of stable finite element spaces for the mixed formulation of twobody contact problems on non matching meshes is presented. The stabilisation of the mixed problem is achieved by balancing the spaces with respect to the mesh size and the polynomial grade of the finite element functions. A numerical convergence study is done, which confirms the estimated convergence order.

1. Introduction

This paper presents a scheme to create stable discretizations for twobody contact problems on non matching meshes. The use of non matching meshes is of advantage in many applications where a matching contact boundary can not be aquired. The idea is to use a mixed formulation where the contact constraints are satisfied only in a weak sense.The use of a mixed formulation has another advantage. The Lagrange multiplier can directly be interpreted as an approximation of the normal contact force, which for example in engineering applications, is a quantity of interest. For the choice of the Lagrange multiplier space different approaches are available like the mortar discretization, see [2],[8], or stabilization technics for example in [7]. The mortar technic requires additional constraints on the boundary of the contact zone and stabilization technics naturally lead to additional terms. The choice of discrete spaces in this paper leads to a stable mixed problem with no additional constraints. The idea is based on an extension of the approach suggested in [6] for contact with a rigid obstacle. For the two dimensional case with piecewise constant functions for the Lagrange multiplier the twobody contact problem has also been studied in [10]. Here we will cover also the three dimensional case and, following the ideas for one sided contact in [11], we will show stability also for higher order spaces so that *hp*-methods can be applied.
The paper is organized as follows: After introducing the necessary notation we state the problem. Next we explain the stability argument. Finally several numerical examples are given to show the estimated order of convergence.

2. problem setting

We refer to the contact problem between two linear elastic bodies. The domains are given by $\Omega^l \in \mathbb{R}^d$ with $l = 1, 2$ and $d = 2, 3$. On the boundary $\Gamma^l = \partial\Omega^l$ subsets $\Gamma^l_D, \Gamma^l_N, \Gamma^l_C$ are given where Dirichlet or Neumann data are applied. The possible contact area is given by Γ^l_C. These three subsets are assumed to be disjoint and to have positive measures.

The linearized strain operator is given by $\varepsilon(u^l) := \frac{1}{2}\left(\nabla u^l + \nabla(u^l)^T\right)$, where ∇u^l is the gradient of the displacement u^l. The stress $\sigma(u^l)$ of linear elasticity depends on Young's modulus of elasticity E^l and on the Poisson ratio ν^l.
Let $n^l(x)$ be the outer normal vector of $x \in \partial\Omega^l$. The displacement on the boundary in normal direction is given by $\gamma_n(u^l)$, and $\sigma_n(u^l) = (n^l)^T\sigma(u^l)n^l$ is the stress in normal direction on the boundary. We define the tangential contact stress by $\sigma_t(u^l) = \sigma(u^l)\cdot n^l - \sigma_n(u^l)\cdot n^l$, see [9] for details.
The displacements are assumed to posess weak derivatives in $(L^2(\Omega^l))^d$ and thus are in the Sobolev spaces $\mathcal{H}^1(\Omega^l) := (H^1(\Omega^l))^d$.
The subspaces

$$\mathcal{H}^1_D(\Omega^l) = \left\{v \ \mid v \in \mathcal{H}^1(\Omega^l),\ v\,|_{\Gamma^l_D} = 0\right\}$$

are used to apply homogeneous dirichlet boundaries and we set $\mathcal{H}^1_D := \mathcal{H}^1_D(\Omega^1) \times \mathcal{H}^1_D(\Omega^2)$.
Scalar products are written in the form $(\cdot,\cdot)_{2,\Omega^l} := (\cdot,\cdot)_{(L^2(\Omega^l))^d}$ and $(\cdot,\cdot)_{1,\Omega^l} := (\cdot,\cdot)_{\mathcal{H}^1(\Omega^l)}$ with the induced norms

$$\|v\|^2_{2,\Omega^l} = (v,v)_{2,\Omega^l} \text{ and } \|v\|^2_{1,\Omega^l} = (v,v)_{1,\Omega^l}.$$

The norm for $\mathcal{H}^1_D$ is given through

$$|||u|||^2 := \left\|u^1\right\|^2_{1,\Omega^1} + \left\|u^2\right\|^2_{1,\Omega^2},$$

where $u := (u^1, u^2)$.
Following the common notation we denote the trace space on $\Gamma^l_C \subset \partial\Omega^l$ of $H^1(\Omega^l)$ as $H^{\frac{1}{2}}(\Gamma^l_C)$ and dual by $H^{-\frac{1}{2}}(\Gamma^l_C)$, see [9].
We define the norm for functions $\lambda \in H^{1/2}(\Gamma_C)$ by:

$$||\lambda||_{1/2} = \inf\{\|u\|_{1,\Omega^l} \quad \mid u \in \mathcal{H}^1(\Omega^l) \text{ and } \gamma_n(u) = \lambda\}$$

and the norm of its dual space is given by

$$||\mu||_{-1/2} = \sup_{v \in \mathcal{H}^1_D(\Omega^l)} \frac{< \mu, \gamma_n(v) >_{-\frac{1}{2},\frac{1}{2}}}{||v||_{1,\Omega_i}},$$

where $< \cdot,\cdot >_{-\frac{1}{2},\frac{1}{2}}$ is the dual pairing between $(H^{-1/2})(\Gamma_C)$ and $H^{1/2})(\Gamma_C)$.
If the two domains are in contact in the reference configuration then there exists a common contact boundary Γ_C with $\Gamma_C = \Gamma^1_C \cap \Gamma^2_C$ and the contact conditions can be directly applied on this part of the boundary: At the contact interface the two bodies may come into contact but must not penetrate each other which leads to the non-penetration condition

$$[u\cdot n](x) = u^1(x)\cdot n^1(x) + u^2(x)\cdot n^2(x) \leq\ 0.$$

Note that for $x \in \Gamma_C$ there holds $n^1(x) = -n^2(x)$.
If the two bodies are not in contact we assume a bijective mapping $\Phi : \Gamma^1_C \to \Gamma^2_C$ between the two possible contact surfaces to be given. Further we define

$$n_\Phi = \begin{cases} \frac{\Phi(x)-x}{|\Phi(x)-x|}, & \text{if } x \neq \Phi(x) \\ n^1(x) = -n^2(x), & \text{if } x = \Phi(x) \end{cases}$$

as the normal vector in the contact area. The non-penetration condition for $x \in \Gamma_C^1$ then reads as:

$$[u \cdot n]_\Phi (x) = u^1(x) \cdot n_\Phi(x) - u^2(\Phi(x)) \cdot n_\Phi(x) \leq g.$$

Here g defines the gap function and is given via Φ by:

$$\Gamma_C^1 \ni x \to g(x) = |x - \Phi(x)| \in \mathbb{R}.$$

Under certain assumptions on Φ and on the geometry of the deformed configuration the above defined non-penetration condition is a close approximation of the geometrical non-penetration condition, see [4]. We will call $[u \cdot n]_\Phi$ the jump of the displacements.

Problem 2.1. The strong formulation of the problem is given by: Find u with

$$\begin{array}{rcll} -\text{div } \sigma(u^l) & = & f^l & \text{in } \Omega^l \\ u^l & = & 0 & \text{on } \Gamma_D^l, \\ \sigma(u^l)n^l & = & p^l & \text{on } \Gamma_N^l, \\ \sigma_t(u^l) & = & 0 & \\ \sigma_{n_\Phi}(u^1) = -\Phi^* \sigma_{n_\Phi}(u^2) & \leq & 0 & \\ [u \cdot n]_\Phi & \leq & g & \text{on } \Gamma_C^l \\ \sigma_{n_\Phi}(u) \cdot ([u \cdot n]_\Phi - g) & = & 0 & \end{array}$$

In this paper we assume that the gap $g = 0$. That means we do not need the mapping Φ and we have $\Gamma_C^1 = \Gamma_C^2$. However we will define the multiplier on one domain only, the choice is of course arbitrary.
The strong problem can be formulated as a mixed variational problem where the contact condition is satisfied in a weak sense, see e.g. [6]. In the mixed formulation the Lagrange multiplier can be interpreted as the normal force in the contact area. Therefore we define the bilinear form $a(\cdot, \cdot)$ as

$$a(v, w) = \sum_{k=1,2} \int_{\Omega^l} \sigma(v^l) : \varepsilon(w^l)\; dx, \qquad v, w \in \mathcal{H}_D^1 \tag{2.1}$$

The weak contact condition is now defined by:

$$b(\lambda, u) = < \lambda, [u \cdot n]_\Phi >_{-\frac{1}{2}, \frac{1}{2}} . \tag{2.2}$$

With this notation the mixed formulation of problem 2.1 is given by

Problem 2.2. Find $(u, \lambda) \in \mathcal{H}_D^1 \times H_+^{-\frac{1}{2}}(\Gamma_C)$ with

$$\begin{array}{rcll} a(u, v) + b(\lambda, v) & = & (f, v) & \forall v \in \mathcal{H}_D^1 \\ b(\mu - \lambda, u) & \leq & < \mu - \lambda, g >_{-\frac{1}{2}, \frac{1}{2}} & \forall \mu \in H_+^{-\frac{1}{2}}(\Gamma_C) \end{array}$$

The existence and uniqueness of this problem is given, see e.g. [6].

3. DISCRETIZATION

As usual in mixed formulations the choice of the discrete spaces is critical. They need to be balanced so that a discrete inf sup condition holds.
We assume a finite element mesh $\mathcal{T}_h^l$ consisting of rectangles in 2d and hexadrals in 3d with a meshwidth h_l on each domain Ω^l. On the contact boundary a finite element mesh $\mathcal{T}_{C,H}$ consisting of lines or rectangles with a meshwidth H is given.
Let $\Psi_T^l : [-1,1]^d \to T \in \mathcal{T}_h^l, \Psi_{C,T_C} : [-1,1]^{d-1} \to T_C \in \mathcal{T}_{C,H}$ be bijective and sufficiently smooth transformations and let $p_T^l, p_{C,T_C} \in \mathbb{N}$ be degree distributions on $\mathcal{T}_h^l$ and $\mathcal{T}_{C,H}$, respectively. Using the polynomial tensor product space S_k^q of order q on the reference element $[-1,1]^d$, we define

$$S_l^p(\mathcal{T}_h^l) := \{v \in \mathcal{H}_D^1(\Omega^l) \mid \forall T \in \mathcal{T}_h^l : v_{|T} \circ \Psi_T^l \in S_k^{p_T}\}.$$

With this definition the space for the displacements in Ω^l is given by:

$$V_{h_l}^{p_l} := \{u \in \mathcal{H}^1(\Omega^l) \mid u^i_{|_{T_h}} \in S_l^p(T_h^l)\} \tag{3.1}$$

And for the Lagrange multiplier we have :

$$M_H^{p_C} := \{\nu \in \mathrm{L}^2(\Gamma_C) \mid \forall T_C \in \mathcal{T}_{C,H} : \nu_{|T} \circ \Psi_{C,T_C} \in S_{k-1}^{p_{C,T_C}}\} \tag{3.2}$$

For $q = 0$ we use the space of piecewise constant discontinous function on a trinagulation T_H. The index $_H$ implies a possibly coarser triangulation on the boundary for the Lagrange multiplier. We denote V_h as the product space

$$V_h = V_{h_1}^{p_1} \times V_{h_2}^{p_2}.$$

And set the space of the Lagrange multiplier as $\Lambda_H := M_H^{p_C}$. We have the following

Lemma 3.1. Let h_l, H, p_l, q be chosen such that

$$c := \left(1 - \frac{c_1}{2}\left(\frac{\max\{1,q\}^r}{p_1}\frac{h_1}{H}\right)^\epsilon + \frac{c_2}{2}\left(\frac{\max\{1,q\}^r}{p_2}\frac{h_2}{H}\right)^\epsilon\right) > 0$$

independent of the meshwidth, then

$$c\,||\lambda_H||_{-1/2} \leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||},$$

holds.

The proof follows the argument in [6] and [11] for the simplified Signorini problem and extends it for the unilateral contact problem. Here we assume that the domains are in contact in the reference state, thus we do not need the mapping Φ. We use an approximation result for higher-order finite element methods(3.3) and a result for an inverse inequality for negative norms (3.4).

For every $\mu \in H^{-1/2}(\Gamma_C)$ we define a Neumann problem:

Problem 3.2. Find $\bar{v} \in \mathcal{H}_D^l$ with

$$(\bar{v}, w)_1 =< \mu, \gamma_n(w) >_{-\frac{1}{2},\frac{1}{2}} \quad \forall w \in \mathcal{H}_D^l,$$

For the solution $\bar{v}$ of problem 3.2 then holds:

$$||\mu||_{-1/2} = ||\bar{v}||_{1,\Omega^l}$$

We call problem 3.2 regular, if $\bar{v} \in \mathcal{H}_D^1(\Omega^l) \cap H_D^{1+\epsilon}(\Omega^l)$ and

$$||\bar{v}||_{1+\epsilon,\Omega_i} \leq C||\mu||_{-\frac{1}{2}+\epsilon}.$$

for all $\mu \in H^{-1/2+\epsilon}(\Gamma_C)$

Lemma 3.3. Let $\mu \in \mathrm{L}^2(\Gamma_C)$ and $u^\mu \in \mathcal{H}_D^1(\Omega^l) \cap H_D^{1+\epsilon}(\Omega^l)$ be the solution of 3.2, then there exists a function $u_I^\mu \in S_p^l(\mathcal{T}_h^l$ and a constant $C > 0$, independent of u^μ, h and p, such that

$$||u^\mu - u_I^\mu||_{1,\Omega^l} \leq C\frac{h^\epsilon}{p^\epsilon}||u^\mu||_{1+\epsilon,\Omega^l}.$$

Proof: See[[1], Thm. 4.6].

Lemma 3.4. There exists a constant $C > 0$ which is independent of H and p_C, such that

$$||\mu_H||_{-\frac{1}{2}+\epsilon,\Gamma_C} \leq C\frac{\max\{1, p_C\}^{2\epsilon}}{H^\epsilon}||\lambda_H||_{-1/2} \tag{3.3}$$

for all $\mu \in M_H^{p_C}$.

Proof. See [[5],Thm. 3.5, Thm. 3.9] and [12].

Now we regard again the Neumann solution $\bar{v}^i$ of problem 3.2: For a given λ_H on the domain Ω_i and a finite element solution $\bar{v}_{h_i}^i \in V_{h_i}^{p_i}$ of problem 3.2 holds:

$$||\lambda_H||_{-\frac{1}{2}} = ||\bar{v}^i||_{1,\Omega^i} \leq ||\bar{v}_{h_i}^i||_{1,\Omega^i} + ||\bar{v}^i - \bar{v}_{h_i}^i||_{1,\Omega^i}$$

In the next step we expand the finite element solution $\bar{v}_{h_i}^i$ of problem 3.2 to the other domain in the following way

$$\hat{v}_h^i := \begin{cases} \bar{v}_{h_i}^i(x) \ , \ x \in \Omega_i \\ 0 \ , \ x \in \Omega_j \setminus \Gamma_C \ , \ i \neq j \end{cases}$$

Obviously for $\hat{v}_h^i$

$$|||\hat{v}_h^i||| = ||\bar{v}_{h_i}^i||_{1,\Omega^i}$$

holds.
Further we have:

Lemma 3.5. For $\bar{v}_{h_l}^i \in V_{h_l^{p_l}}$ holds:

$$||\bar{v}_{h_i}^i||_{1,\Omega^i} \leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||}.$$

Proof:

$$\begin{aligned}
||\bar{v}^i_{h_i}||_{1,\Omega^i} &= \frac{<\lambda_H, \gamma^i_n(\bar{v}^i_{h_i})>_{-\frac{1}{2},\frac{1}{2}}}{||\bar{v}^i_{h_i}||_{1,\Omega^i}} \\
&= \frac{<\lambda_H, \gamma^i_n(\bar{v}^i_{h_i})>_{-\frac{1}{2},\frac{1}{2}} + <\lambda_H, \gamma^j_n(0)>_{-\frac{1}{2},\frac{1}{2}}}{|||\hat{v}^i_h|||} \quad, i \neq j \\
&= \frac{b(\lambda_H, \hat{v}^i_h)}{|||\hat{v}^i_h|||} \\
&\leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||}
\end{aligned}$$

Now we are able to state the proof of Lemma 3.1: For a given $\lambda_H \in \Lambda_H$ and solutions $\bar{v}^1 \in \mathcal{H}^1_D$ and $\bar{v}^2 \in \mathcal{H}^2_D$ of the problem 3.2 we have:

$$\begin{aligned}
||\lambda_H||_{-\frac{1}{2}} &= \frac{1}{2}||\bar{v}^1||_{1,\Omega^1} + \frac{1}{2}||\bar{v}^2||_{1,\Omega^2} \\
&\leq \frac{1}{2}(||\bar{v}^1_{h_1}||_{1,\Omega^1} + ||\bar{v}^1 - \bar{v}^1_{h_1}||_{1,\Omega^1}) \\
&\quad + \frac{1}{2}(||\bar{v}^2_{h_2}||_{1,\Omega^2} + ||\bar{v}^2 - \bar{v}^2_{h_2}||_{1,\Omega^2}) \\
&= \frac{1}{2}||\bar{v}^1_{h_1}||_{1,\Omega^1} + \frac{1}{2}||\bar{v}^2_{h_2}||_{1,\Omega^2} + \frac{1}{2}\Sigma^2_{i=1}||\bar{v}^i - \bar{v}^i_{h_i}||_{1,\Omega^i} \\
&\leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||} + \frac{1}{2}\sum_{i=1}^{2} ||\bar{v}^i - \bar{v}^i_{h_i}||_{1,\Omega^i} \text{ using 3.5} \\
&\leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||} + \frac{1}{2}\sum_{i=1}^{2} c_i \frac{h^\epsilon_i}{p^\epsilon_i}||\bar{v}^i||_{1+\epsilon,\Omega^i} \quad \text{using 3.3} \\
&\leq \sup_{v_h \in V_h} \frac{b(\lambda_H, v_h)}{|||v_h|||} \\
&\quad + \frac{1}{2}\sum_{i=1}^{2} c_i \left(\frac{\max\{1,p_C\}^r}{p_i}\frac{h_i}{H}\right)^\epsilon ||\lambda_H||_{-1/2} \text{ using 3.4}
\end{aligned}$$

finally we combine all terms containing λ_H and gain

$$\begin{aligned}
&\left(1 - \tfrac{c_1}{2}\left(\tfrac{\max\{1,p_C\}^r}{p_1}\tfrac{h_1}{H}\right)^\epsilon - \tfrac{c_2}{2}\left(\tfrac{\max\{1,p_C\}^r}{p_2}\tfrac{h_2}{H}\right)^\epsilon\right)||\lambda_H||_{-1/2} \\
\leq\ & \sup_{v_h \in V_h} \tfrac{b(\lambda_H, v_h)}{|||v_h|||}
\end{aligned}$$

which proofs Lemma 3.1.

Remark: One drawback of this stability condition is, that it does not provide an indicator for which choice of spaces the discretization is stable. In numerical tests though one can detect stable behaviour for the Lagrange multiplier, as for unbalanced spaces the a checkerboard behaviour occurs. For example a choice $p^l = 1, q = 0$ and $H \approxeq 2h^l$ implies stable behaviour. In contrast to the mortar discretization technic we cut the projection step. This leads to a dependency on both discretizations, but equilibrating the mesh size for both domains in the contact area is not complicated.

Of course the choice of a coarser mesh for the Lagrange multiplier is done

by $c_1^l \cdot h^l \leq H \leq c_2^l \cdot h^l$ with c_1^l, c_2^l independent of h^l.
We conclude by citing a priori error estimates for the discretization:

Lemma 3.6. For the choice $p^l = 1, q = 0$,$c_1^l \cdot h^l \leq H \leq c_2^l \cdot h^l$ and $u^l \in \left(H^2(\Omega^l)\right)^d$, an a priori estimate of for (u_h, λ_H) of order

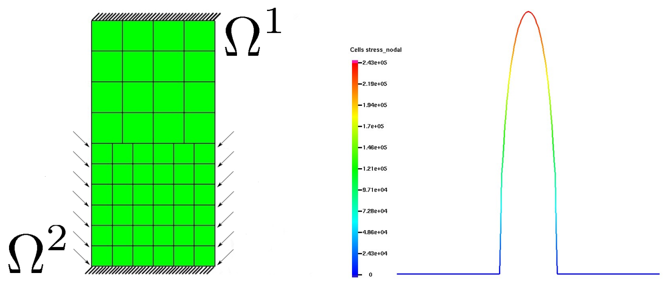
$$|||u - u_h|||_V + ||\lambda - \lambda_H||_{-\frac{1}{2},\Gamma_C} \leq c(u)h^{\frac{3}{4}}$$

can be established.

The proof can be found in [10].
Remark: By assuming a stronger regularity for the jump of the displacements $[u]$ on the boundary this estimate can be improved to $O(h)$ follwing the arguments in [3].

4. Numerical Examples

In our numerical examples we will make use of the stable set of spaces: $p_l = 1, q = 0$, with $2h = H$ and $p_l = 2, q = 1$ with $h = H$, where $h = h_l$ and Ω^l being the domain on which the Lagrange multiplier is defined. As mentioned before the stability relies on the discretization on both domains, so we need also $h_1 = c \cdot h_2$ with a constant c so that the problem remains stable.
We should mention that the assumption on regularity made in [3] will usually not be satisfied in applications, so that optimal convergence order can not be expected. However, we start the numerical investigation with a smooth example to show that asymptotical optimal order can be estabplished. This example is taken from [8] where it is used for the mortar discretization technic.
As a reference solution we use solution on a finer mesh with $h_{ref} \leq \frac{1}{4}h$.
example 1: In this example we consider the problem shown in figure (number). The two domains are given in their reference configuration by $\Omega^1 = [0, 10] \times [0, 10]$ and $\Omega^2 = [0, 10] \times [10, 20]$. A Neumann force of $(10^4, -10^5)$ on the left and $(-10^4, -10^5)$ on the right side of Ω^1 is applied and the lower boundary is clamped. On the upper boundary of Ω^2 inhomogeneous Dirichlet data is given by $(0, -5 \cdot 10^{-4})$. The material parameters are $E = 15 \cdot 10^8$ and $\nu = 0.2$ for Ω^1 and $E = 20 \cdot 10^8$ and $\nu = 0.4$ for Ω^2. In

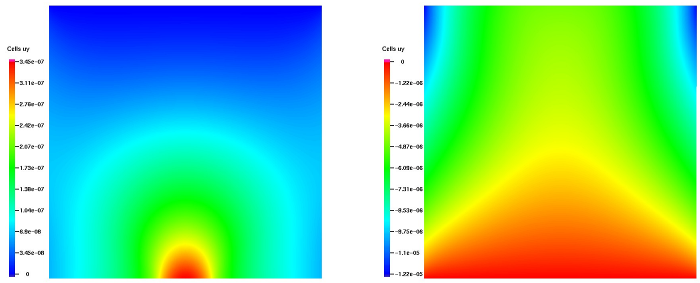


(a) domains on level 1

(b) normal stress by λ_H on reference solution

FIGURE 4.1. problem setting / resulting contact stress

picture 4.1 (a) the reference state is illustrated, on the right side the stress field provided by the Lagrangemultiplier is shown. The displacement in y-direction is given in picture 4.2 for the two domains. As the displacements on domain 2 are bigger, the domains are shown separately.



(a) displacements in uy-direction for domain 1 (b) displacements in uy-direction for domain 2

FIGURE 4.2. resulting diplacements in u_y-direction

Error in H1-seminorm and L2-norm for the p1,q0 discretization:

Degrees of Freedom	energy	conv	L2	conv
124	2.357103E-1	-	4.941582E-2	-
456	1.315057E-1	0.841889	1.680251E-2	1.556297
1744	7.049462E-2	0.899540	2.832984E-3	2.568282
6816	3.750949E-2	0.910257	7.355484E-4	1.945430
26944	1.973351E-2	0.926608	2.879982E-4	1.352761
107136	1.007051E-2	0.970511	6.448745E-5	2.158969

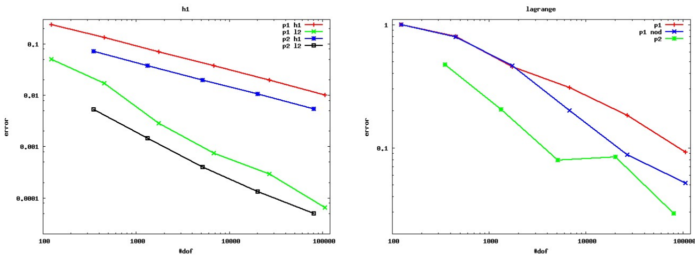
Error in H1-seminorm and L2-norm for the p2,q1 discretization:

Degrees of Freedom	energy	conv	L2	conv
352	7.212148E-2	-	5.322303E-3	-
1328	3.707915E-2	0.959821	1.457529E-3	1.868526
5152	1.960744E-2	0.919207	3.964319E-4	1.878379
20288	1.043609E-2	0.909819	1.319053E-4	1.587570
80512	5.383400E-3	0.954992	4.932157E-5	1.419212

The error of the Lagrangemultiplier measured in the L_2-norm on Γ_C can be seen in picture 4.3.

We see that for this example our discretization yields an asymptotically optimal order for the displacements. As the error for the Lagrange multiplier is estimated in the L_2-norm here, we cannot expect an optimal behavior as the suited norm would be the $H^{-\frac{1}{2}}$-norm. The reference solution has 525312 degrees of freedom on domain one and 1181184 on domain 2.

Error of the Lagrange multiplier in L2-norm for the p1,q0 discretization:



(a) error of displacements (b) error of λ in L_2-norm

FIGURE 4.3. error in u and λ

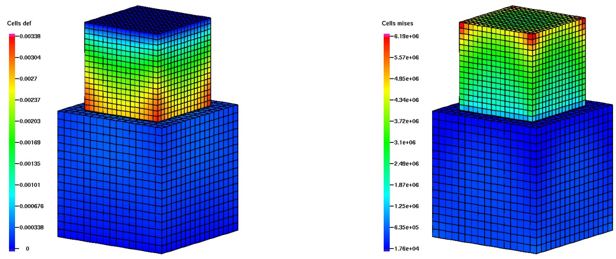
Degrees of Freedom	cellwise	conv	nodal	conv
2	1.0	-	9.984870E-1	-
4	7.983950E-1	0.324825	7.880209E-1	0.341510
8	4.507542E-1	0.824762	4.642795E-1	0.763240
16	3.083806E-1	0.547629	1.991357E-1	1.221242
32	1.823256E-1	0.758195	8.711818E-2	1.192706
64	9.204464E-2	0.986111	5.153962E-2	0.757292

Error of the Lagrange multiplier in L2-norm for the p2,q1 discretization:

Degrees of Freedom	cellwise	conv
5	4.715783E-1	-
9	2.040376E-1	1.208662
17	7.882273E-2	1.372152
33	8.404037E-2	-0.092471
65	2.917438E-2	1.526381

For a **second example** we take a 3d-setting similiar to the first example. The upper block (being domain 1) is clamped on it's upper boundary and subjected to a Neumann force $q := (0, 0, 10^6) - 10^6 \cdot n$ at it's sides, where n is the outer normal vector. The lower block (which is domain 2) is clamped on it's lower boundary. The blocks are overlapping in reference configuration and defined by $\Omega^1 := [0.3, 1.7] \times [0.3, 1.7] \times [2, 3.4]$ and $\Omega^2 := [0, 2] \times [0, 2] \times [0.003, 2.003]$. The possible contact boundary is given by the lower side of the upper block and the upper side of the lower block. As we see, the boundaries of the possible contact zones do not have to match to provide a stable discretization. For this example we use the $p = 1, q = 0,$ with $2h = H$ discretization which results in (tri-)linear elements for the displacements on domain one and two and piecewise constant elements for the Lagrange multiplier. The multiplier is defined on the possible contact boundary of the upper block Again the material parameters are $E = 15 \cdot 10^8$ and $\nu = 0.2$ for Ω^1 and $E = 20 \cdot 10^8$ and $\nu = 0.4$ for Ω^2. The deformed domains are shown in figure 4.4

For an analysis we use a reference solution on level 5 with 811200 degrees of freedom on each domain and 1024 cells for the Lagrange multiplier, again we see asymptotically optimal order (Though we only have three levels to



(a) Displacement on level 3 (b) von Mises Stress on level 3

FIGURE 4.4. example 2 in deformed state

measure - the fourth level is only one refinement away from the reference solution, so this has to be handled with caution !).

Error of the displacements

Degrees of Freedom	H^1 norm	conv	L_2 norm	conv
600	2.844037E-1		5.463880E-2	
3888	1.517031E-1	0.9066897	1.334490E-2	2.0336369
27744	7.858908E-2	0.9488498	3.960204E-3	1.7526420
209088	3.632390E-2	1.1134098	8.240026E-4	2.2648542

Again we measure the error of the Lagrangemultiplier in the L_2-norm of the boundary. In the third column we use a nodal interpolation of the cellwise constant multiplier, the resulting stress fields are shown in picture 4.6.

Error of the Lagrange multiplier in the L_2 norm

Degrees of Freedom	constant	conv	nodal	conv
4	5.532055E-1		5.416035E-1	
16	4.365887E-1	0.8877480	2.927135E-1	0.9066897
64	2.763904E-1	1.0074179	1.456061E-1	0.9488498
256	8.891592E-2	1.4293355	5.406383E-2	1.1134098

On level 1 the contact stresses are constant over the whole boundary. A good resolution can be found for for finer meshes. For completeness we mention that the second example can also be discretized with quadratic elements. The Lagrange multiplier is chosen as piecewise bilinear on the contact boundary. The resulting stress field of such a computation can be seen in picture 4.5.

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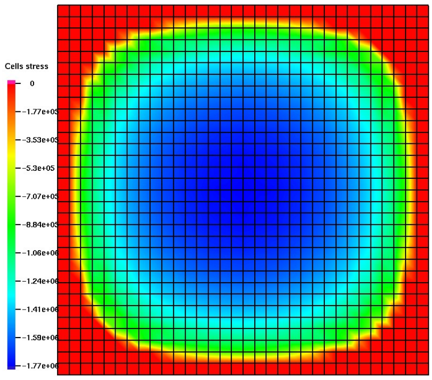

(a) stress field for linear Lagrangemultiplier

FIGURE 4.5. stress field for linear Lagrangemultiplier

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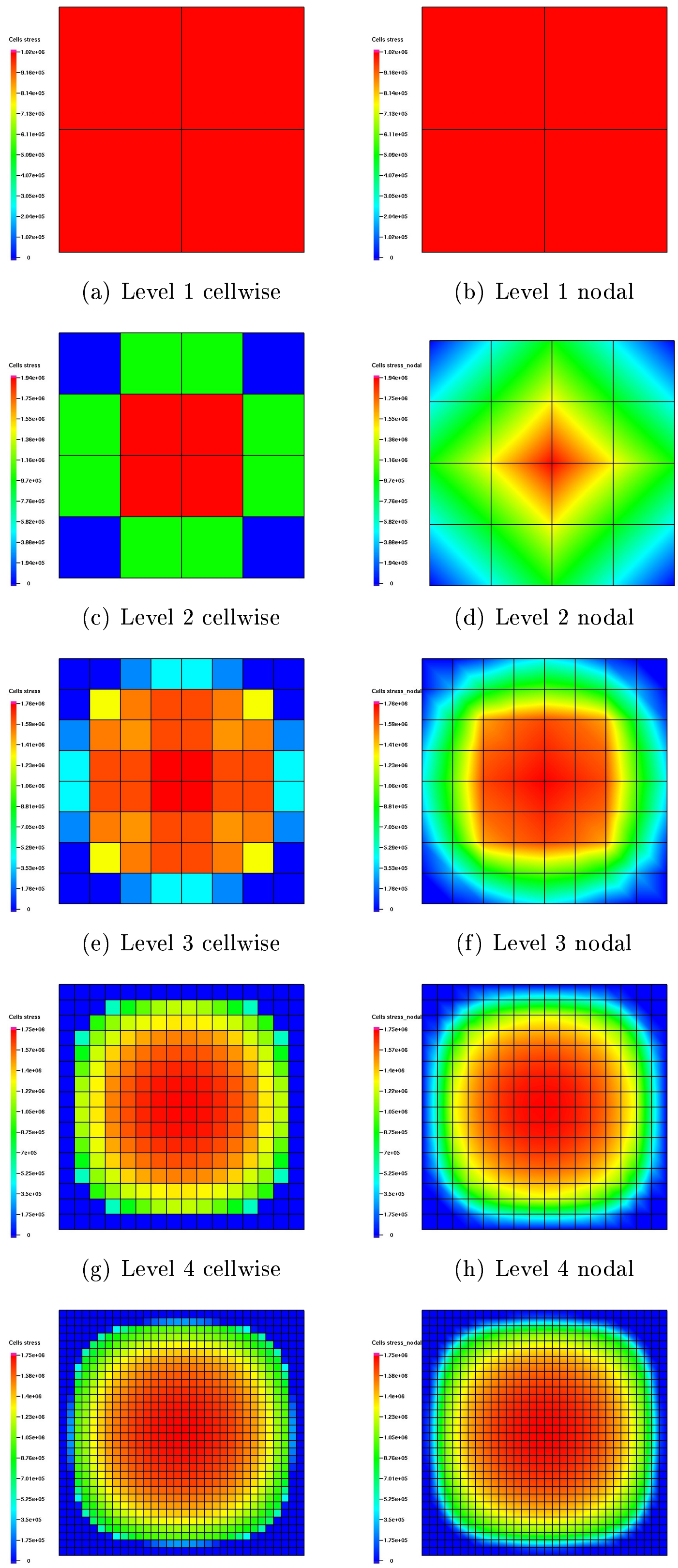

(a) Level 1 cellwise (b) Level 1 nodal

(c) Level 2 cellwise (d) Level 2 nodal

(e) Level 3 cellwise (f) Level 3 nodal

(g) Level 4 cellwise (h) Level 4 nodal

(i) Level 5 cellwise (j) Level 5 nodal

FIGURE 4.6. contact stress fields via λ_H