Semi-smooth Newton methods for mixed FEM discretizations of higher-order for frictional, elasto-plastic two-body contact problems

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Abstract

In this acticle a semi-smooth Newton method for frictional two-body contact problems and a solution algorithm for the resulting sequence of linear systems is presented. It is based on a mixed variational formulation of the problem and a discretization by finite elements of higher-order. General friction laws depending on the normal stresses and elasto-plastic material behaviour with linear isotropic hardening are considered. Numerical results show the efficiency of the presented algorithm.

Keywords *higher-order FEM, frictional contact, semi-smooth Newton method, complementarity function, plasticity, hardening*

1 Introduction

Frictional, elasto-plastic multi-body contact problems play an important role in mechanical engineering [16, 23, 26]. The nonlinearities caused by geometric contact and frictional constraints combined with the nonlinearity in the material law result in challenging numerical problems in forms of variational inequalities and therefore efficient solving methods are needed. In the mortar context active set strategies have been established as powerful methods for solving contact problems of various kind [25]. As shown in [11] these strategies can be interpreted as semi-smooth Newton methods. Active set strategies for geometrical contact and frictional constraints as well as multi-body contact are analyzed in [12, 15]. Linear and quadratic finite elements have been regarded in [14] whereas plastic yield conditions are described by NCP functions in [9]. Besides Mortar methods there exist a series of strategies for the numerical treatment of contact problems. One approach form classical fixpoint methods [7]. General ideas with penalty methods and lagrange multiplier techniques are described in [16]. A combination of both are augmented lagrage multiplier techniques [24]. Domain decomposition methods as FETI techniques [8] or Dirichlet-Neumann algorithms in combination with Mortar discetizations [18] are efficient parallel computable solvers. By monotone multigrid constructed global convergent solvers are suggested in [19]. A cascadic multigrid algorithm for variational inequalities is presented in [3]. Under the use of higher-order DG-discretizations active set strategies have been applied to linear-elastic obstacle problems [1].

We transfer the approach of active-set strategies that have been developed for mortar methods [12, 15] to mixed finite elements introduced by Haslinger [10] and higher-order discretizations presented in [4, 22]. Lagrange multipliers capture the geometrical contact and frictional constraints. They are discretized on a coarser mesh and with ansatz functions of different polynomial degree than the primal variable. This ansatz is conforming if d-linear or d-quadratic finite elements are chosen to discretize the primal variable and becomes nonconforming for higher degrees. In [17, 22] a solution scheme for linear-elastic, frictional multi-body contact problems and higher-order discretizations is suggested. It is based on the dual formulation of the discrete mixed variational formulation and leads to an optimization problem in the lagrange multipliers. This problem is solved by a SQP method, which instantiates the contact constraints as sign- and the frictional constraints as nonlinear constraints. To include nonlinear material behaviour inside a Newton iteration a contact problem has to be solved in every step to full accuracy. A more convenient ansatz in the context of nonlinear material is the application of an inexact, monolitic Newton method based on active sets.

In the presented work the discrete weak formulations of the constraint inequalities are reformulated by NCP functions in terms of equations. As premis for this proceeding we choose a convenient combination of basis functions for the higher-order Lagrange ansatz spaces and quadrature rules. Within the regarded discretization the coupling matrices are nonquadratic and nondiagonal in contrast to the Mortar ansatz. On the other hand the construction of basis functions is easier. Concerning plasticity we use a primal-mixed formulation in the displacements and project the stresses onto the admissible set [21]. The nonlinearity in the material is processed by linearization and damping strategies. We present efficient inexact strategies to solve the arising full saddle point systems. A block triangular preconditioner is adapted from [2] and a cheap but effective preconditioning method for the Schur complement matrix is suggested. Representative for one time step of a quasi-static process a static problem is regarded. A generalization to quasi-static or dynamic multi-body problems can easily be performed [13, 17].

This paper is organized as follows. In Section 2 we present the regarded frictional, elasto-plastic two-body contact problem in its strong as well as mixed variational formulation. A higher-order discretization is given in Section 3 whereas in Section 4 semi-smooth Newton methods for contact and general frictional contraints are developed for the described mixed finite elements. Numerical results of problems in two and three space dimensions are shown in Section 5.

2 Problem formulation

In this section we give the strong and weak formulations of the regarded frictional two-body contact problem with an elasto-plastic material law and linear isotropic hardening. We consider two deformable bodies Ω^m with $m = 1, 2$ in $d = 2$ or 3 spatial dimensions, on which volume forces $f^m \in L^2(\Omega^m, \mathbb{R}^d)$ are acting. Their boundaries are denoted by Γ^m, $m = 1, 2$. With the outer normal vector n on Γ^m one can define the surface stresses $\sigma_n(u^m) := \sigma(u^m)n$. The normal part of these stresses is given by the scalar value $\sigma_{nn} := n^\top\sigma(u^m)n$, whereas the tangential stress vectors are calculated by $\sigma_{nt,i} := n^\top\sigma(u^m)t_i$. The matrix $t \in \mathbb{R}^{d\times d-1}$ contains the tangential vectors on Γ_m that build an orthonormal system with the outer normal n. Define the trace operators $\gamma^m_M : H^1(\Omega^m) \to L^2(M)$ for $M \subset \Gamma^m$ and

$$H_D(\Omega^m) := \left\{ v \in H^1(\Omega^m) \,\middle|\, \gamma_{\Gamma^m_D}(v) = 0 \right\} \ .$$

Their d-dimensional cartesian product space is denoted by $V := \left(H_D(\Omega^1)\right)^d \times \left(H_D(\Omega^2)\right)^d$. We are interested in the displacements $u = (u^1, u^2) \in V$, which fulfill the following conditions for $m = 1, 2$:

$$\varepsilon(u^m) = A^m\sigma(u^m) + \varepsilon^{m,P} \qquad \text{in } \Omega^m \tag{1}$$

$$-\operatorname{div}\sigma(u^m) \ = \ f^m \qquad \text{in } \Omega^m \tag{2}$$

$$\varepsilon^{m,P}(\tau - \sigma(u^m)) \geq 0 \ \ \forall\tau \text{ with } \mathcal{F}^{m,\mathrm{iso}}(\tau, |\varepsilon^{m,P}|_F) \leq 0 \qquad \text{in } \Omega^m \tag{3}$$

$$u^m = 0 \qquad \text{on } \Gamma^m_D \tag{4}$$

$$\sigma_n(u^m) = p^m \qquad \text{on } \Gamma^m_N \ . \tag{5}$$

Relation (1) describes the material law, the relation between the linearized strain $\varepsilon(u^m) = \frac{1}{2}\left(\nabla u^m + \nabla u^{m,\top}\right)$ and the stress $\sigma(u^m)$. The strain is split up into an elastic part $A^m\sigma(u^m)$ with the fourth order compliance tensor A^m corresponding to isotropic material and a plastic part

$\varepsilon^{m,P}$. Equation (2) ensures that outer and inner force are balanced. The deviatoric part of a tensor τ is denoted by $\tau^D := \tau - \frac{1}{3}\operatorname{trace}(\tau) I_{d\times d}$ and $|\cdot|_F$ indicates the Frobenius norm. Defining the flow function $\mathcal{F}^{m,\text{iso}}(\tau,\eta) = |\tau^D|_F - (\sigma_0^m + \gamma_{\text{iso}}^m \eta)$ with the yield stress σ_0^m and the isotropic hardening parameter γ_{iso}^m the complementarity condition (3) ensures that plastic strain only may occur if the flow function is zero. The bodies are fixed at some closed subset of their boundary $\Gamma_D^m \subset \Gamma^m$ with positive measure. These Dirichlet boundary conditions are described in equation (4). At parts $\Gamma_C^m \subset \Gamma^m$ the two bodies may come into contact. We assume that the open set Γ_C^m fulfills $\bar{\Gamma}_C^m \subsetneq \Gamma^m \backslash \Gamma_D^m$. On the remaining part of the boundaries $\Gamma_N^m := \Gamma^m \backslash \left(\Gamma_D^m \cup \bar{\Gamma}_C^m\right)$ surface stresses $p^m \in L^2(\Gamma_N^m, \mathbb{R}^d)$ act on each body Ω^m by relation (5). Let $\Phi : \Gamma_C^1 \to \Gamma_C^2$ be an appropriate, bijective, sufficiently smooth mapping between the contact boundaries of the slave body Ω^1 and the master body Ω^2. In order to model contact conditions we define a generalized normal vector for $x \in \Gamma^1$ by

$$n_\delta(x) := \begin{cases} \frac{\Phi(x)-x}{\|\Phi(x)-x\|} & , x \neq \Phi(x) \\ n^1(x) = -n^2(x) & , x = \Phi(x) \end{cases}$$

and corresponding tangential vectors $t \in L^2(\Gamma^1)^{d\times(d-1)}$ such that $(n_\delta(x), t_\delta(x))$ form an orthonormal system. We define the normal jump

$$[v]_{n_\delta}(x) := \gamma_{\Gamma_C^1}(v^1)(x)\cdot n_\delta(x) - \gamma_{\Gamma_C^2}(v^2)(\Phi(x))\cdot n_\delta(x)$$

and the tangential jump

$$[v]_{t_\delta}(x) := t_\delta(x)^\top \gamma_{\Gamma_C^1}(v^1)(x) - t_\delta(x)^\top \gamma_{\Gamma_C^2}(v^2)(\Phi(x))$$

on Γ_C^1. The distance of Ω^1 and Ω^2 is given by the gap function $g(x) := |\Phi(x) - x|$. With this notion the contact conditions, which are defined on the slave body, read as follows:

$$[u]_{n_\delta} \leq g \qquad \text{on } \Gamma_C^1 \tag{6}$$

$$\sigma_{n_\delta n_\delta}(u^1) \leq 0 \qquad \text{on } \Gamma_C^1 \tag{7}$$

$$\sigma_{n_\delta n_\delta}(u^1)([u]_{n_\delta} - g) = 0 \qquad \text{on } \Gamma_C^1 \tag{8}$$

$$\sigma_{n_\delta}(u^1) = -\Theta^* \sigma_{n_\delta}(u^2) \qquad \text{on } \Gamma_C^1\ . \tag{9}$$

The bodies are not allowed to penetrate each other (6) and only negative or vanishing contact forces are allowed (7). By the complementatity condition (8) the property that either contact occurs or the normal contact forces vanish is modeled. Equation (9) contains the adjoint Θ^* of a transfer operator $\Theta : L^2(\Gamma_C^2) \to L^2(\Gamma_C^1)$ which is defined by $\Theta(v^2)(x^1) := v(\Phi(x^1))$ and ensures equality of the contact forces on Γ_C^1 and Γ_C^2. Besides the described normal contact we regard frictional constraints on Γ_C^1 :

$$\left\|\sigma_{n_\delta t_\delta}(u^1)\right\| \leq s(\sigma_{n_\delta n_\delta}(u^1)) \tag{10}$$

$$\left\|\sigma_{n_\delta t_\delta}(u^1)\right\| < s(\sigma_{n_\delta n_\delta}(u^1)) \Rightarrow [u]_{t_\delta} = 0 \tag{11}$$

$$\left\|\sigma_{n_\delta t_\delta}(u^1)\right\| = s(\sigma_{n_\delta n_\delta}(u^1)) \Rightarrow \exists \alpha \in \mathbb{R}_{\geq 0} : [u]_{t_\delta} = \alpha \sigma_{n_\delta t_\delta}(u) \tag{12}$$

with the euclidean norm $\|\cdot\|$. The tangential stresses are bounded by a functional s which represents a general friction law depending on the normal stresses of the slave body. If this threshold is not achieved the bodies stick (11). Otherwise the slip condition (12) holds and a tangential movement occurs, that is proportional to the tangential stresses. Equations (1)-(12) generate a strong formulation of the regarded static, frictional, elasto-plastic two-body contact problem.

Following [21] we introduce a primal-mixed formulation of the frictional elasto-plastic two-body contact problem (1)-(12) and project the stresses onto the admissible set by the projector

$$P_\Pi(\tau) := \begin{cases} \tau & , |\tau^D|_F \leq \sigma_0^m \\ \left(\frac{\gamma_{\text{iso}}^m}{2\mu^m + \gamma_{\text{iso}}^m} + \left(1 - \frac{\gamma_{\text{iso}}^m}{2\mu^m + \gamma_{\text{iso}}^m}\right)\frac{\sigma_0^m}{|\tau^D|_F}\right)\tau^D + \frac{1}{3}\operatorname{trace}(\tau) I_{d\times d} & , |\tau^D|_F > \sigma_0^m \end{cases}$$

with the shear modulus μ^m of the m-th body material. This yields a variational inequality in the displacements

$$a(u)(\varphi-u)-f_{\text{ext}}(\varphi-u)+j(\varphi)-j(u)\geq 0 \quad \forall\varphi\in K$$

on the konvex set

$$K:=\{\, v\in V \,|\, [v]_n\leq g\} \;\subset\; V$$

containing the contact conditions and with the functional

$$j(\varphi):=\int_{\Gamma_C^1} s\|[\varphi]_t\| d\sigma$$

describing the frictional constraints. The semi-linearform a is defined by

$$a: V\times V\to\mathbb{R}, \qquad\qquad a(v)(w):=\sum_{m=1}^{2}\int_{\Omega^m} P_\Pi\left((A^m)^{-1}\varepsilon(v^m)\right):\varepsilon(w^m)dx \;,$$

and the linear form

$$f_{\text{ext}}(v) \;:=\; \sum_{m=1}^{2}(f^m,v^m)+(p^m,v^m)_{\Gamma_N^m}$$

corresponds to the external energy. Introducing the Lagrange functional,

$$\mathscr{L}(v,\mu_n,\mu_t):=\frac{1}{2}a(v)(v)-f_{\text{ext}}(v)+\langle\mu_n,[v]_n-g\rangle+(\mu_t,s[v]_t)_{0,\Gamma_C^1}$$

the dual cone

$$\Lambda_n:=H_+^{-\frac{1}{2}}(\Gamma_C^1):=\left(\left\{\mu\in H^{\frac{1}{2}}(\Gamma_C^1)\;\middle|\;\mu\geq 0 \text{ a.e.}\right\}\right)'$$

and

$$\Lambda_t:=\left\{\mu\in L^2(\Gamma_C^1)^{d-1}\;\middle|\;\|\mu\|\leq s\right\}$$

we replace the contact and the frictional constraints and end up in a mixed formulation of the form: Find $(u,\lambda_n,\lambda_t)\in V\times\Lambda_n\times\Lambda_t$ such that

$$a(u)(v)+b_n(\lambda_n,v)+b_t(\lambda_t,v)=f_{\text{ext}}(v) \qquad \forall v\in V \tag{13}$$

$$b_n(\mu_n-\lambda_n,u)+b_t(\mu_t-\lambda_t,u)\leq\langle\mu_n-\lambda_n,g\rangle \qquad \forall\mu_n\in\Lambda_n,\;\forall\mu_t\in\Lambda_t \;. \tag{14}$$

Here, the bilinear forms that include the contact and frictional conditions are defined by

$$b_n:\Lambda_n\times V\to\mathbb{R}, \qquad\qquad b_n(\mu,w):=\langle\mu,[w]_n\rangle,$$
$$b_t:\Lambda_t\times V\to\mathbb{R}, \qquad\qquad b_t(\mu,w):=(\mu,s[w]_t)_{0,\Gamma_C} \;.$$

3 Discretization

In this section we introduce a higher-order discretization for the described problem (13)-(14) that was mentioned in [4]. Denote by $\mathscr{T}_{h,m}$ and $\mathscr{B}_H$ triangulations of $\Omega_m,\ m=1,2$ respectively Γ_C with corresponding affine transformations

$$F_{m,T}:\hat{T}:=[-1,1]^d\to T\in\mathscr{T}_{h,m}$$

and

$$F_E:\hat{E}:=[-1,1]^{d-1}\to E\in\mathscr{B}_H.$$

Let S_l^r be the polynomial tensor product space of order r on the reference element $[-1,1]^l$. We define ansatz functions of polynomial degree $p \in \mathbb{N}$ on body $m = 1,2$ by

$$V_{h,m}^p \;:=\; \{v \in (H_D(\Omega_m))^d \;\mid \forall T \in \mathscr{T}_{h,m} :\; v|_T \circ F_{m,T} \in S_d^p(T) \;\} \;=:\; \operatorname{span}\{\phi_i^m\}_{i=1}^{n_m}$$

and the tensor space

$$V_h \;=\; V_{h,1}^{p_1} \times V_{h,2}^{p_2} \;.$$

The cumulated dimensions of the discrete spaces $V_{h,m}^p$ are named $n := n_1 + n_2$. By introduction of the space

$$\mathcal{M}_H^q \;:=\; \left\{ v \in L^2(\Gamma_C^1) \;\middle|\; \forall E \in \mathscr{B}_H : v|_E \circ\; F_E \in S_{d-1}^q(E) \right\}$$

with a polynomial degree $q \in \mathbb{N}$ the admissible set for the lagrange multipliers concerning geometrical contact is defined as follows:

$$\Lambda_{n,H} \;:=\; \{ v \in \mathcal{M}_H^q | \;\; \forall E \in \mathscr{B}_H : \forall x \in \mathcal{C}_q : v(F_E(x)) \le 0 \;\; \} \;=:\; \operatorname{span}\{\psi_i^n\}_{i=1}^{m_1} \;.$$

The sign condition $v(F_E(x)) \le 0$ is only defined on the finite set $\mathcal{C}_q \subset [-1,1]^{d-1}$ which consists of the $(q+1)^{d-1}$ Gauss-quadrature points because for polynomials of higher degree a uniform condition is hard to satisfy. For constant ansatz functions we define $\mathcal{C}_0 := \{0_{d-1}\}$, whereas for $q = 1$ the set $\mathcal{C}_1$ consists of the corners of the reference element $[-1,1]^{d-1}$. The non-conforming ansatz was proposed in [6] and convergence is shown for an elastic two-body problem in [4]. In the case of lower polynomial degrees $q = 0,1$ this ansatz becomes conforming, since the choice of $\mathcal{C}_q$ leads to a uniform sign condition on the elements $E \in \mathscr{B}_H$. Accordingly the discrete space for the lagrange multiplier concerning friction is defined by the non-conforming ansatz

$$\Lambda_{t,H} \;:=\; \{ v \in (\mathcal{M}_H^q)^{d-1} \,|\, \forall E \in \mathscr{B}_H : \forall x \in \mathcal{C}_q :\; v(F_E(x)) \le 1 \;\} \;:=\; \operatorname{span}\{\psi_i^t\}_{i=1}^{m_2}$$

and becomes conforming again for $q = 0,1$. Here, the dimension is given by $m_2 := (d-1)m_1$. In two space dimensions we chose coinciding bases for $\Lambda_{n,H}$ and $\Lambda_{t,H}$ whereas in three space dimensions the choice is as follows

$$\psi_i^t = \begin{cases} \begin{pmatrix} \psi^n_{\frac{i}{2}} \\ 0 \end{pmatrix} & ,\ \text{i even} \\ \begin{pmatrix} 0 \\ \psi^n_{\frac{i+1}{2}} \end{pmatrix} & ,\ \text{i odd}\ . \end{cases}$$

Stability of the described discretization is proven for the elastic case and balanced h, H, p, and q in [4]. It is shown that if

$$\sum_{m=1}^{2} \left(hH^{-1} \max\{1,q\}^2 p^{-1} \right)^{\theta_m} \le \epsilon$$

holds for an $\epsilon > 0$ sufficiently small and $1 + \theta_m$-regularity of the solution u there exists an $\alpha \in \mathbb{R}_{\ge 0}$ independent of h, H, p and q such that the Brezzi-Babuska condition

$$\alpha \, \|(\mu_{n,H}, \mu_{t,H})\|_{-1/2} \le \sup_{v_h \in V_h, \|v_h\|=1} \left(b_n(\mu_{n,H}, v_h) + b_t(\mu_{t,H}, v_h) \right)$$

is fulfilled for all $(\mu_{n,H}, \mu_{t,H}) \in \Lambda_{n,H} \times \Lambda_{t,H}$. Practical investigations show that the choice $q = \max\{p^m\} - 1$ and $H = 2\max\{h^m\}$ leads to a stable discretization, c.f. [17]. Eventually the discrete problem is to find $(u_h, \lambda_{n,H}, \lambda_{t,H}) \in V_h \times \Lambda_{n,H} \times \Lambda_{t,H}$ with

$$\begin{aligned} a(u_h)(v_h) + b_n(\lambda_{n,H}, v) + b_t(\lambda_{t,H}, v_h) &= f_{\text{ext}}(v_h) \\ b_n(\mu_{n,H} - \lambda_{n,H}, u_h) + b_t(\mu_{t,H} - \lambda_{t,H}, u_h) &\le \langle \mu_{n,H} - \lambda_{n,H}, g \rangle \end{aligned} \tag{15}$$

for all $v_h \in V_h,\ \mu_{n,H} \in \Lambda_{n,H}$ and $\mu_{t,H} \in \Lambda_{t,H}$.

4 Semi-smooth Newton methods

Following the work [12] we transfer the presented active-set strategies for mortar methods to mixed finite elements, which were introduced by Haslinger [10] for low order. We generalize the solving methods to higher-order discretizations that are proposed in [4] and general friction laws depending on the contact forces.

4.1 Active-set strategy for contact

We begin with presenting a semi-smooth Newton method for geometrical contact constaints. For simplicity we skip the subscript δ of the generalized normals n_δ in the two-body contact formulations. The point of departure is formed by the weak version of the contact conditions (6)-(8)

$$\begin{aligned} \int_{\Gamma_C^1} ([u_h]_n - g)\psi_H \; do \leq 0 \\ \int_{\Gamma_C^1} \lambda_{n,H}\psi_H \; do \leq 0 \\ \int_{\Gamma_C^1} \lambda_{n,H}([u_h]_n - g)\psi_H \; do = 0 \; . \end{aligned} \tag{16}$$

Define the coupling matrix $N = [N_1 \; N_2]$ of size $N_m \in \mathbb{R}^{m_1 \times n_m}$ concerning the bases of V_h and $\Lambda_{H,n}$ by

$$\begin{aligned} (N_1)_{ij} &:= \int_{\Gamma_C^1} \psi_i^n(x)\gamma_{\Gamma_C^1}(\phi_j^1)(x) \cdot n \; do, \\ (N_2)_{ij} &:= \int_{\Gamma_C^1} \psi_i^n(x)\gamma_{\Gamma_C^2}(\phi_j^2)(\Phi(x)) \cdot n \; do \end{aligned}$$

as well as the mass matrix

$$M \in \mathbb{R}^{m_1 \times m_1}, \quad M_{ij} := \int_{\Gamma_C^1} \psi_j^n \psi_i^n \; do$$

and the gap vector

$$\bar{g}_i = \int_{\Gamma_C^1} g\psi_i^n \; do \; .$$

In the following a bar generally indicates the vector-valued representation of the corresponding discrete function in the belonging basis. The vector-valued formulation of (16) reads

$$N\bar{u}_h - \bar{g} \leq 0, \; M\bar{\lambda}_{n,H} \leq 0, \; \left(M\bar{\lambda}_{n,H}\right)_i (N\bar{u}_h - \bar{g})_i = 0 \; \forall i = 1, \ldots, m_1 \; . \tag{17}$$

To reformulate these in terms of an equation we define the NCP function

$$\begin{aligned} C_N &: \mathbb{R}^n \times \mathbb{R}^{m_1} \to \mathbb{R}^{m_1} \\ C_N(\bar{u}_h, \bar{\lambda}_{n,H})_i &:= (M\bar{\lambda}_{n,H})_i - \max\left\{0, \; (M\bar{\lambda}_{n,H})_i + c_n(N\bar{u}_h - \bar{g})_i\right\} \end{aligned}$$

with a positive constant c_n . Following the arguments in [12, Chapter 4] the discrete, weak contact conditions (17) can equivalently be expressed by

$$C_N(\bar{u}_h, \bar{\lambda}_{n,H}) = 0 \; . \tag{18}$$

With variations $\bar{\delta u}_h^k, \bar{\delta\lambda}_{n,H}^k$ and characteristical functions χ_i defined by

$$\chi_i := \begin{cases} 1 & , \; \left(M\bar{\lambda}_{n,H} + c_n(N\bar{u}_h - \bar{g})\right)_i > 0 \\ 0 & , \; \left(M\bar{\lambda}_{n,H} + c_n(N\bar{u}_h - \bar{g})\right)_i \leq 0 \end{cases}$$

the generalized derivative of the NCP function reads

$$C'_N(\bar{u}_h, \bar{\lambda}_{n,H})(\bar{\delta u}_h, \bar{\delta\lambda}_{n,H})_i = -\chi_i \left(M\bar{\delta\lambda}_{n,H} + c_n N\bar{\delta u}_h\right)_i + \left(M\bar{\delta\lambda}_{n,H}\right)_i \ .$$

We apply a semi-smooth Newton's method

$$C'_N(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1})(\bar{\delta u}_h^k, \bar{\delta\lambda}_{n,H}^k) = -C_N(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1})$$

to solve (18). The new iterates are calculated by $\bar{u}_h^k := \bar{u}_h^{k-1} + \bar{\delta u}_h^k$ and $\bar{\lambda}_{n,H}^k := \bar{\lambda}_{n,H}^{k-1} + \bar{\delta\lambda}_{n,H}^k$. Let active and inactive indices in Newton step k be determined in the following way

$$\mathcal{A}_n^k := \left\{ i \in \{1, \ldots, m_1\} \ \middle|\ \left(M\bar{\lambda}_{n,H} + c_n(N\bar{u}_h - \bar{g})\right)_i > 0 \right\} \tag{19}$$

$$\mathcal{I}_n^k := \left\{ i \in \{1, \ldots, m_1\} \ \middle|\ \left(M\bar{\lambda}_{n,H} + c_n(N\bar{u}_h - \bar{g})\right)_i \leq 0 \right\} \ . \tag{20}$$

Then the iterate $(\bar{u}_h^k, \bar{\lambda}_{n,H}^k)$ of Newton's method can equivalently be expressed by

$$\begin{aligned} (N\bar{u}_h^k)_i &= \ \bar{g}_i, \ i \in \mathcal{A}_n^k \\ (M\bar{\lambda}_{n,H}^k)_i &= \ 0, \ \ i \in \mathcal{I}_n^k \end{aligned}$$

for all $i = 1, \ldots, m_1$.

4.2 Active-set strategy for friction

In this subsection we present an active-set strategy for frictional constraints in the context of higher-order finite elements. The discrete analogon of the frictional constraints (10)-(12) read

$$\begin{aligned} &\|\sigma_{nt}(u_h)\| \leq s(\sigma_{nn}(u_h)) \\ &\|\sigma_{nt}(u_h)\| < s(\sigma_{nn}(u_h)) \Rightarrow [u_h]_t = 0 \\ &\|\sigma_{nt}(u_h)\| = s(\sigma_{nn}(u_h)) \Rightarrow \exists \alpha \in \mathbb{R}_{\geq 0} : [u_h]_t = \alpha\sigma_{nt}(u_h) \ . \end{aligned} \tag{21}$$

Instead of these pointwise conditions (21) we use a weak formulation with test functions $\psi_n \in \Lambda_n$ resp. $\psi_t \in \Lambda_t$:

$$\begin{aligned} &\int_{\Gamma_C^1} \|\lambda_{t,H}\| \, \psi_n \ do \leq \int_{\Gamma_C^1} s(\lambda_{n,H})\psi_n \ do \\ &\int_{\Gamma_C^1} \|\lambda_{t,H}\| \, \psi_n \ do < \int_{\Gamma_C^1} s(\lambda_{n,H})\psi_n \ do \Rightarrow \int_{\Gamma_C^1} [u_h]_t \psi_t \ do = 0 \\ &\int_{\Gamma_C^1} \|\lambda_{t,H}\| \, \psi_n \ do = \int_{\Gamma_C^1} s(\lambda_{n,H})\psi_n \ do \Rightarrow \exists \alpha \in \mathbb{R}_{\geq 0} : \int_{\Gamma_C^1} [u_h]_t \psi_t \ do = \alpha \int_{\Gamma_C^1} \lambda_{t,H}\psi_t \ do \ . \end{aligned}$$

We replace the the normal and tangential stresses $\sigma_{nn}(u_h)$ and $\sigma_{nt}(u_h)$ by the corresponding lagrange multipliers λ_n and λ_t. The integrals $\int_{\Gamma_C^1} \|\lambda_{t,H}\| \, \psi_i^n \ do$ and $\int_{\Gamma_C^1} \lambda_t \psi_i^t \ do$ are approximated respectively integrated exactely by a $(q+1)^{d-1}$-point gaussian quadrature with weights α_i. The nodes of the lagrangian basis of $\Lambda_{n,H}$ and $\Lambda_{t,H}$ are placed in the these gaussian quadrature points $\mathcal{C}_q := \{\hat{x}_1, \ldots, \hat{x}_{(q+1)^{d-1}}\}$ on the reference element $[-1,1]^{d-1}$. Define the matrix $\bar{T} \in \mathbb{R}^{m_2 \times n}$ with partition $\bar{T} = [\bar{T}_1 \ \bar{T}_2]$ by

$$\left(\bar{T}_1\right)_{ij} := \frac{1}{\omega_i} \int_{\Gamma_C^1} \psi_i^t(x) \ t_\delta(x)^\top \gamma_{\Gamma_C^1}(\phi_j^1)(x) \ do$$

$$\left(\bar{T}_2\right)_{ij} := \frac{1}{\omega_i} \int_{\Gamma_C^1} \psi_i^t(x)) \ t_\delta(x)^\top \gamma_{\Gamma_C^2}(\phi_j^2)(\Phi(x)) \ do$$

The weight ω_i is given by $\omega_i := \alpha_i |\det \nabla F_E(\hat{x}_i)^\top \nabla F_E(\hat{x}_i)| \geq 0$ on $E = \operatorname{supp}(\psi_i^t)$. Let

$$s_i := \frac{1}{\omega_i} \int_{\Gamma_C^1} s(\lambda_{n,H}^-) \psi_i^n \, do \tag{22}$$

with

$$\lambda_{n,H}^- := \min\{0, \lambda_{n,H}\}$$

be the algebraic representation of the positive friction bound s. The resulting discrete friction constraints read

$$\begin{aligned}
&\left\|\bar{\lambda}_{t,H,\mathcal{I}(i)}\right\| \leq s_i \\
&\left\|\bar{\lambda}_{t,H,\mathcal{I}(i)}\right\| < s_i \Rightarrow \left(\bar{T}\bar{u}_h\right)_{\mathcal{I}(i)} = 0 \\
&\left\|\bar{\lambda}_{t,H,\mathcal{I}(i)}\right\| = s_i \Rightarrow \exists \alpha \in \mathbb{R}_{\geq 0} : \left(\bar{T}\bar{u}_h\right)_{\mathcal{I}(i)} = \alpha \bar{\lambda}_{t,H,\mathcal{I}(i)} \ .
\end{aligned} \tag{23}$$

for the dimension-dependent index map

$$\mathcal{I}(i) := \begin{cases} \{i\} & , \ d = 2 \\ \{2i-1, 2i\} & , \ d = 3 \end{cases} .$$

In this sense for a vector $u \in \mathbb{R}^{m_2}$ the selection $u_{\mathcal{I}(i)}$ is a scalar value in the case of two space dimensions and a vector in $\mathbb{R}^2$ if $d = 3$. Following [12, Section 5] we express the conditions (23) by the NCP function

$$C_T : \ \mathbb{R}^n \times \mathbb{R}^{m_1}_{\leq 0} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2},$$

$$\begin{aligned}
C_T(\bar{u}_h, \bar{\lambda}_{n,H}, \bar{\lambda}_{t,H})_{\mathcal{I}(i)} := &\max\left\{s_i, \left\|\left(\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}(\bar{u}_h)_{\mathcal{I}(i)}\right)\right\|\right\} \bar{\lambda}_{t,H,\mathcal{I}(i)} \\
&- \ s_i \left(\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}(\bar{u}_h)_{\mathcal{I}(i)}\right)
\end{aligned}$$

with $c_t > 0$ and $i \in \{1, \ldots, m_1\}$. Analog arguments to the proof in [12, Theorem 5.1] lead to the equivalence of the equation $C_T(\bar{u}_h, \bar{\lambda}_{n,H}, \bar{\lambda}_{t,H}) = 0$ and the frictional constraints (23). Defining the characteristical functions

$$\chi_{\mathcal{A}_{t,i}} := \begin{cases} 1 & , \ \left\|\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}\bar{u}_{h,\mathcal{I}(i)}\right\| - s_i \ > 0 \\ 0 & , \ \left\|\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}\bar{u}_{h,\mathcal{I}(i)}\right\| - s_i \ \leq 0 \end{cases}$$

and

$$\chi_{\mathcal{I}_{t,i}} := \begin{cases} 1 & , \ \left\|\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}\bar{u}_{h,\mathcal{I}(i)}\right\| - s_i \ \leq 0 \\ 0 & , \ \left\|\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t \bar{T}\bar{u}_{h,\mathcal{I}(i)}\right\| - s_i \ > 0 \end{cases}$$

the generalized derivative for $\bar{\lambda}_{n,H,i} < 0$ is given by

$$\begin{aligned}
&C_T'(\bar{u}_h, \bar{\lambda}_{n,H}, \bar{\lambda}_{t,H})(\bar{\delta u}_h, \bar{\delta\lambda}_{n,H}, \bar{\delta\lambda}_{t,H})_{\mathcal{I}(i)} \\
&= \chi_{\mathcal{A}_{t,i}} \frac{\bar{\lambda}_{t,H,\mathcal{I}(i)} \ \left(\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t(\bar{T}\bar{u}_h)_{\mathcal{I}(i)}\right)^\top}{\|\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t(\bar{T}\bar{u}_h)_{\mathcal{I}(i)}\|} (\bar{\delta\lambda}_{t,H,\mathcal{I}(i)} + c_t(\bar{T}\bar{\delta u}_h)_{\mathcal{I}(i)}) \\
&\quad + \bar{\delta\lambda}_{t,\mathcal{I}(i)} \max\left\{s_i, \left\|(\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t(\bar{T}\bar{u}_h)_{\mathcal{I}(i)})\right\|\right\} \\
&\quad - s_i(\bar{\delta\lambda}_{t,\mathcal{I}(i)} + c_t(\bar{T}\bar{\delta u}_h)_{\mathcal{I}(i)}) \\
&\quad - s'_{i,\lambda_{n,H}}(\bar{\delta\lambda}_{n,H})(\bar{\lambda}_{t,H,\mathcal{I}(i)} + c_t(\bar{T}\bar{u}_h)_{\mathcal{I}(i)}) \\
&\quad + \chi_{\mathcal{I}_{t,i}} s'_{i,\lambda_{n,H}}(\bar{\delta\lambda}_{n,H}) \ \bar{\lambda}_{t,\mathcal{I}(i)} \ .
\end{aligned}$$

We formulate the semi-smooth Newton method

$$C_T'(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})(\bar{\delta u}_h, \bar{\delta\lambda}_{n,H}, \bar{\delta\lambda}_{t,H}) = -C_T(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1}) \tag{24}$$

with increments $\delta\bar{u}_h^k = \bar{u}_h^k - \bar{u}_h^{k-1}$, $\delta\bar{\lambda}_{n,H}^k = \bar{\lambda}_{n,H}^k - \bar{\lambda}_{n,H}^{k-1}$ and $\delta\bar{\lambda}_{t,H}^k = \bar{\lambda}_{t,H}^k - \bar{\lambda}_{t,H}^{k-1}$. The Newton equation (24) is equivalent to

$$\begin{aligned}\int_{\Gamma_C^1} \mathscr{U}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i \; \bar{\lambda}_{t,H,\mathcal{I}(i)}^k \psi \; do + \int_{\Gamma_C^1} \mathscr{V}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i \; \bar{\lambda}_{n,H,i}^k \psi \; do \\ + \int_{\Gamma_C^1} \mathscr{W}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i \; (\bar{T}\bar{u}_h^k)_{\mathcal{I}(i)} \psi \; do = \int_{\Gamma_C^1} \mathscr{R}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i \psi \; do\end{aligned} \tag{25}$$

for all $i \in \{1, \ldots, m_1\}$ and $\psi \in \Lambda_{n,H}$. We define the dimension-dependent index map $\mathcal{J}(j) = \left\lceil \frac{j}{d-1} \right\rceil$ and seperate the set of indices $\{1, \ldots, m_2\}$ into sticky indices with contact

$$\mathcal{A}_t^k := \left\{ j \in \{1, \ldots, m_2\} \middle| \; i := \mathcal{J}(j) : \left\| \bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + \bar{T}(\bar{u}_h^{k-1})_{\mathcal{I}(i)} \right\| - s_i^{k-1} \; > 0, \; \bar{\lambda}_{n,H,i}^{k-1} < 0 \right\} , \tag{26}$$

slip indices in contact

$$\mathcal{I}_t^k := \left\{ j \in \{1, \ldots, m_2\} \middle| \; i := \mathcal{J}(j) : \; \left\| \bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + \bar{T}(\bar{u}_h^{k-1})_{\mathcal{I}(i)} \right\| - s_i^{k-1} \; \leq 0, \; \bar{\lambda}_{n,H,i}^{k-1} < 0 \right\} \tag{27}$$

and indices without contact

$$\mathcal{I}_{tn}^k := \left\{ j \in \{1, \ldots, m_2\} \mid \bar{\lambda}_{n,H,\mathcal{J}(j)}^{k-1} = 0 \right\} . \tag{28}$$

We obtain for $j \in \mathcal{A}_t^k$ and $i := \mathcal{J}(j)$ the functions

$$\begin{aligned}\mathscr{U}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = \; & \frac{\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} (\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)})^\top}{\|(\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)})\|} \\ & + I_{(d-1)\times(d-1)} \left(\|(\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)})\| - s_i^{k-1} \right) ,\end{aligned}$$

$$\mathscr{V}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = -(s'_{i,\lambda_{n,H}})^{k-1} (\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c\bar{T}(\bar{u}_h^{k-1})_{\mathcal{I}(i)}) ,$$

$$\mathscr{W}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = c_t \frac{\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} (\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)})^\top}{\|\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)}\|} - c_t I_{(d-1)\times(d-1)} s_i^{k-1}$$

and

$$\begin{aligned}\mathscr{R}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = c_t & \frac{\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} \left(\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)} \right)^\top}{\|\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)}\|} (\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)}) \\ & - (s'_{i,\lambda_{n,H}})^{k-1} (\bar{\lambda}_{n,i}^{k-1}) (\bar{\lambda}_{t,H,\mathcal{I}(i)}^{k-1} + c_t (\bar{T}\bar{u}_h^{k-1})_{\mathcal{I}(i)}) .\end{aligned}$$

In the slip case $j \in \mathcal{I}_t^k$ the functions $w_i = 0$,

$$\mathscr{U}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = \; \frac{1}{\omega_i} I_{(d-1)\times(d-1)} ,$$

$$\mathscr{V}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = \; (\bar{T}\bar{u}^{k-1})_{\mathcal{I}(i)} \; \frac{(s'_{i,\lambda_{n,H}})^{k-1}}{s_i^{k-1}}$$

and

$$\mathscr{R}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_i = \; (\bar{T}\bar{u}^{k-1})_{\mathcal{I}(i)} \frac{(s'_{i,\lambda_{n,H}})^{k-1} \bar{\lambda}_{n,H,i}^{k-1}}{s_i^{k-1}}$$

determine the Newton equation (24). In the limit case $j \in \mathcal{I}_{tn}^k$ of no contact we choose $\bar{\lambda}_{t,H,\mathcal{I}(i)} = 0$ and therefore vanishing $\mathscr{U}_i, \mathscr{V}_i$ and $\mathscr{R}_i$ as well as $\mathscr{W}_i = I_{(d-1)\times(d-1)}$. We define the matrices $U = [U_1\ U_2] \in \mathbb{R}^{m_2 \times n}, V \in \mathbb{R}^{m_2 \times m_1},\ W \in \mathbb{R}^{m_2 \times m_2}$ and the vector $\bar{r} \in \mathbb{R}^{m_2}$ by

$$(U_1)_{ji} := \int_{\Gamma_C^1} \mathscr{U}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_{\mathcal{J}(j)}\ t_\delta(x)^\top \gamma_{\Gamma_C^1}(\phi_i^1)(x)\psi_j^t(x)d\sigma$$

$$(U_2)_{ji} := \int_{\Gamma_C^1} \mathscr{U}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_{\mathcal{J}(j)}\ t_\delta(x)^\top \gamma_{\Gamma_C^1}(\phi_i^2)(x)\psi_j^t(\Phi(x))d\sigma$$

$$V_{ji} := \int_{\Gamma_C^1} \mathscr{V}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_{\mathcal{J}(j)}\ \psi_i^n \psi_j^t d\sigma$$

$$W_{ji} := \int_{\Gamma_C^1} \mathscr{W}(\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_{\mathcal{J}(j)}\ \psi_i^t \psi_j^t d\sigma$$

$$\bar{r}_j := \int_{\Gamma_C^1} \mathscr{R}(\bar{T}\bar{u}_h^{k-1}, \bar{\lambda}_{n,H}^{k-1}, \bar{\lambda}_{t,H}^{k-1})_{\mathcal{J}(j)}\ \psi_j^t d\sigma\ .$$

Then the Newton equation (25) is equivalent to the vector-valued equation

$$U\bar{u}_h^k + V\bar{\lambda}_{n,H}^k + W\bar{\lambda}_{t,H}^k = \bar{r}\ .$$

4.3 Algebraic representation of the saddle-point system

In this subsection we give the linear systems that have to be solved during the Newton iteration. Furthermore we present the resulting algorithm and the used solving techniques. According to $\bar{T}$ let the unscaled matrix $T = [T_1\ T_2] \in \mathbb{R}^{m_2 \times n}$ be defined by

$$(T_1)_{ij} := \int_{\Gamma_C^1} \psi_i^t(x)\ t_\delta(x)^\top \gamma_{\Gamma_C^1}(\phi_j^1)(x)\ do\ ,$$
$$(T_2)_{ij} := \int_{\Gamma_C^1} \psi_i^t(x)\ t_\delta(x)^\top \gamma_{\Gamma_C^2}(\phi_j^2)(\Phi(x))\ do\ ,$$

as well as the block-diagonal matrix

$$K^k\ =\ \mathrm{diag}(K_1^k, K_2^k) =\ \left(a'(u_h^{k-1})(\phi_j^m, \phi_i^m)\right)_{i,j=1,\dots,n}$$

by the Frechét derivative of the semi-linearform in direction u_h^{k-1}. Define the vector

$$L^k(u_h^{k-1})\ =\ [L_1^k\ L_2^k]^\top\ =\ \left(a(u_h^{k-1})(\phi_i^m)\right)_{i=1,\dots,n}$$

and the Newton right hand side of the plastic problem

$$\tilde{f}_m^k\ :=\ K_m^k \bar{u}_{h,m}^{k-1} - L_m^k(u_h^{k-1}) + \bar{f}_m\ .$$

With this linearization of the plasticity and the described introduction of active-sets for contact and friction we approximate the solution of the discrete mixed formulation (15) by a semi-smooth Newton method in which step k corresponds to solving the linear system

$$\left[\begin{array}{cc|cc|ccc} K_1^k & 0 & N_{1,\mathcal{A}_n^k}^\top & N_{1,\mathcal{I}_n^k}^\top & T_{1,\mathcal{A}_t^k}^\top & T_{1,\mathcal{I}_t^k}^\top & T_{1,\mathcal{I}_{tn}^k}^\top \\ 0 & K_2^k & N_{2,\mathcal{A}_n^k}^\top & N_{2,\mathcal{I}_n^k}^\top & T_{2,\mathcal{A}_t^k}^\top & T_{2,\mathcal{I}_t^k}^\top & T_{2,\mathcal{I}_{tn}^k}^\top \\ \hline N_{1,\mathcal{A}_n^k} & N_{2,\mathcal{A}_n^k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{\mathcal{A}_n^k\mathcal{I}_n^k} & M_{\mathcal{I}_n^k\mathcal{I}_n^k} & 0 & 0 & 0 \\ \hline U_{1,\mathcal{A}_t^k} & U_{2,\mathcal{A}_t^k} & V_{\mathcal{A}_t^k\mathcal{A}_n^k} & V_{\mathcal{A}_t^k\mathcal{I}_n^k} & W_{\mathcal{A}_t^k\mathcal{A}_t^k} & W_{\mathcal{A}_t^k\mathcal{I}_t^k} & W_{\mathcal{A}_t^k\mathcal{I}_{tn}^k} \\ \bar{T}_{1,\mathcal{I}_t^k} & \bar{T}_{2,\mathcal{I}_t^k} & V_{\mathcal{I}_t^k\mathcal{A}_n^k} & V_{\mathcal{I}_t^k\mathcal{I}_n^k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{c} \bar{u}_{h,1}^k \\ \bar{u}_{h,2}^k \\ \hline \bar{\lambda}_{n,H,\mathcal{A}_n^k}^k \\ \bar{\lambda}_{n,H,\mathcal{I}_n^k}^k \\ \hline \bar{\lambda}_{t,H,\mathcal{A}_t^k}^k \\ \bar{\lambda}_{t,H,\mathcal{I}_t^k}^k \\ \bar{\lambda}_{t,H,\mathcal{I}_{tn}^k}^k \end{array}\right] = \left[\begin{array}{c} \tilde{f}_1 \\ \tilde{f}_2 \\ \hline \bar{g}_{\mathcal{A}_n^k} \\ 0 \\ \hline \bar{r}_{\mathcal{A}_t^k} \\ \bar{r}_{\mathcal{I}_t^k} \\ 0 \end{array}\right] . \tag{29}$$

Thereby for a matrix $A \in \mathbb{R}^{k\times l}$ the submatrix

$$A_{\mathcal{I}\mathcal{J}} := [A_{i,j}]_{i\in\mathcal{I},j\in\mathcal{J}} \in \mathbb{R}^{|\mathcal{I}|\times|\mathcal{J}|}$$

consists of the rows and columns of A, whose indices belong to the sets $\mathcal{I}$ resp. $\mathcal{J}$ of indices. If only rows of $\mathcal{I}$ are selected the notation is

$$A_{\mathcal{I}} := [A_{i,\cdot}]_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|\times l} .$$

The linear system corresponds to a saddle point problem of the structure

$$\begin{bmatrix} K^k & B_1^\top \\ B_2^k & -C^k \end{bmatrix} \begin{bmatrix} \bar{u}_h^k \\ \bar{\lambda}^k \end{bmatrix} = b \tag{30}$$

with partitions

$$\bar{\lambda}^k := \begin{bmatrix} \bar{\lambda}_{n,H}^k \\ \bar{\lambda}_{t,H}^k \end{bmatrix}, \; B_1^\top := \begin{bmatrix} N_1^\top & T_1^\top \\ N_2^\top & T_2^\top \end{bmatrix}, \; B_2^k := \left[\begin{array}{cc} N_{1,\mathcal{A}_n^k} & N_{2,\mathcal{A}_n^k} \\ 0 & 0 \\ \hline U_{1,\mathcal{A}_t^k} & U_{2,\mathcal{A}_t^k} \\ \bar{T}_{1,\mathcal{I}_t^k} & \bar{T}_{2,\mathcal{I}_t^k} \\ 0 & 0 \end{array}\right]$$

and

$$-C^k := \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ M_{\mathcal{A}_n^k\mathcal{I}_n^k} & M_{\mathcal{I}_n^k\mathcal{I}_n^k} & 0 & 0 & 0 \\ \hline V_{\mathcal{A}_t^k\mathcal{A}_n^k} & V_{\mathcal{A}_t^k\mathcal{I}_n^k} & W_{\mathcal{A}_t^k\mathcal{A}_t^k} & W_{\mathcal{A}_t^k\mathcal{I}_t^k} & W_{\mathcal{A}_t^k\mathcal{I}_{tn}^k} \\ V_{\mathcal{I}_t^k\mathcal{A}_n^k} & V_{\mathcal{I}_t^k\mathcal{I}_n^k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right], \; b := \left[\begin{array}{c} \tilde{f}_1 \\ \tilde{f}_2 \\ \hline \bar{g}_{\mathcal{A}_n^k} \\ 0 \\ \hline \bar{r}_{\mathcal{A}_t^k} \\ \bar{r}_{\mathcal{I}_t^k} \\ 0 \end{array}\right] .$$

In general the system matrix of the problem (30) is unsymmetric. The matrices B_1 and B_2^k are nonquadratic whereas the quadratic matrix C^k might be singular. To solve the discrete contact problem (15) we have to solve a sequence of saddle point problems (30). This semi-smooth Newton method can be summarized in the following Algorithm 1.

Algorithm 1: Semi-smooth Newton method

for $k = 0$ to maxIter **do**
- assemble linearization K^k and right hand side $\tilde{f}^k$
- calculate active and inactive sets $\mathcal{A}_n^k$, $\mathcal{I}_n^k$ for contact (19, 20)
- calculate friction bound (22)
- calculate active and inactive sets $\mathcal{A}_t^k$, $\mathcal{I}_t^k$, $\mathcal{I}_{tn}^k$ for friction (26, 27, 28)
- assemble Newton matrices U^k, V^k, W^k and right hand side $\bar{r}^k$
- solve linear system (30) to tolerance $\mathrm{tol_k}$
- calculate residuals

$$\begin{aligned} \mathrm{res_{plast}} &:= \sum_{m=1}^{2} \| K_m^k \bar{u}_{h,m}^k - L_m^k(\bar{u}_{h,m}^k) + \bar{f}^m \| \\ C_N^k &:= C_N(\bar{u}_h^k, \bar{\lambda}_{n,H}^k) \\ C_T^k &:= C_T(\bar{u}_h^k, \bar{\lambda}_{n,H}^k, \bar{\lambda}_{t,H}^k) \end{aligned}$$

- **if** $\|C_N^k\| + \|C_T^k\| + \mathrm{res_{plast}} \leq \mathrm{tol}$ **then**
  - stop
- perform **damping** if $k > 2$:
  $\tilde{u}_m^0 := \bar{u}_{h,m}^k$, $\mathrm{res_0} := \mathrm{res_{plast}}$, $\omega_0 := 0.5$
  for $j = 0$ to $\mathrm{n_{damp}}$ **do**
  - $\tilde{u}_m^j := \omega_j \bar{u}_{h,m}^k + (1 - \omega_j)\bar{u}_{h,m}^{k-1}$
  - $\mathrm{res_j} := \sum_{m=1}^{2} \| K_m^k \tilde{u}_m^j - L_m^k(\tilde{u}_m^j) + \bar{f}^m \|$
  - **if** $\mathrm{res_j} \leq \mathrm{res_{j-1}}$ **then**
    - $\omega_j := 0.5\ \omega_{j-1}$
  - **else**
    - $\bar{u}_h^k := \tilde{u}_m^{j-1}$
    - break

In every Newton step the linear system (30) is solved to a tolerance $\mathrm{tol_k}$. We apply an inexact strategy and start with a relatively high value of $\mathrm{tol_k}$ und succesively reduce it. Because the system matrix is unsymmetric we use a GMRES solver. Furthermore we specify a block triangular preconditioner of the form

$$\mathcal{P}^k = \begin{bmatrix} K^k & B_1^\top \\ 0 & S^k \end{bmatrix} . \tag{31}$$

The matrix S^k is defined as the generalized Schur complement matrix

$$S^k = -(B_2^k (K^k)^{-1} B_1^\top + C^k) .$$

This approach was proposed in the work [2, Section 10.1.2]. We choose an approximation $\hat{S}^k$ to S^k implicitly by the action of $(\hat{S}^k)^{-1}$ on a given vector v. The Schur complement matrix is unsymmetric and not given explicitly, so the approximation $\hat{S}^k$ to S^k is given by the solution with GMRES to a given tolerance. In an analogous manner we define an approximation $\hat{K}^k$ to K^k. Because the approximation $(\hat{K}^k)^{-1}$ has to be calculated once per GMRES step at the solving process with $\hat{S}^k$ we calculate a LU-decomposition of K^k and define $(\hat{K}^k)^{-1}v$ as a direct solution. This direct solution can be computed in parallel on the bodies because the system is not coupled. Due to the bad condition of the Schur complement matrix we precondition it with

$$\mathcal{P}_S^k = \left[\operatorname{diag}\left(-(B_2^k\left(\operatorname{diag}(K^k)\right)^{-1} B_1^\top + C^k)\right)\right]^{-1} \quad . \tag{32}$$

The explicit form of $\left(\mathcal{P}^k\right)^{-1}$ reads

$$\left(\mathcal{P}^k\right)^{-1} = \begin{bmatrix} (\hat{K}^k)^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & B_1^\top \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -(\hat{S}^k)^{-1} \end{bmatrix} \,.$$

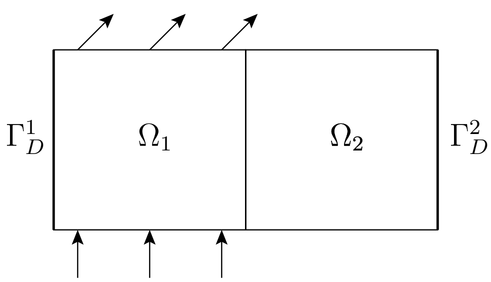
Basically the effort of applying $\mathcal{P}_S^k$ equates to one solution with the Schur complement matrix and one direct solution with K^k. The solution process of (30) is terminated if the residuals concerning contact, friction and plasticity C_N^k, C_T^k and $\mathrm{res}_{\mathrm{plast}}$ as defined in Algorithm 1 underrun a given tolerance tol. Due to the plastic material we damp the displacements by a line search strategy [20] beginning in the step $k = 3$ of the algorithm.

5 Numerical Examples

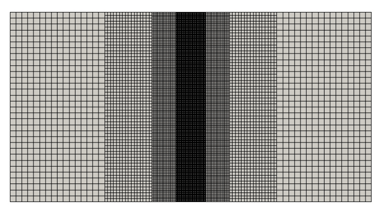
In this section we apply the presented algorithms to concrete selected numerical examples in two and three space dimensions. We can validate the successful operation of the method and the effectivity of the proposed preconditioning methods.

5.1 Twodimensional example

We consider the contact of an elasto-plastic and a linear-elastic body, which are represented by $\Omega^1 = [0,1] \times [0,1]$ and $\Omega^2 = [0.91, 1.91] \times [0,1]$, cf. Figure 1 (a). The body Ω^1 is subjected to Neumann



(a)



(b)

Figure 1: 2D Twobody contact Problem (a) and FE meshes (b)

boundary conditions given by $p^1 = (0, 30)$ on its lower side and by $p^1 = (30, 30)$ on its upper side. Homogeneous Dirichlet boundary conditions hold on $\Gamma_D^1 = \{0\} \times [0,1]$ and $\Gamma_D^2 = \{1.91\} \times [0,1]$. The two bodies may come into contact on the contact boundaries $\Gamma_C^1 = \{1\} \times [0,1]$ respectively $\Gamma_C^2 = \{0.91\} \times [0,1]$, which is caused by the overlapping of Ω_1 and Ω_2. Resulting displacements calculated on a globally refined mesh of size $n = 197.632$ and $m_1 = m_2 = 128$ with polynomial degrees $p = 2, q = 1$ are depicted in Figure 2. For stability reasons we choose a meshsize of Γ_C^1 that fulfills $H = 2h$. The Young's moduli $E^1 = 10^4$, $E^2 = 10^3$ and Poisson's ratios $\nu^1 = 0.25$, $\nu^2 = 0.22$ define the compliance tensors A^1 and A^2. We choose hardening parameters $\gamma_{\mathrm{iso}}^1 = 0.01, \gamma_{\mathrm{iso}}^2 = 1.0$ and yield stresses $\sigma_0^1 = 10^2$, $\sigma_0^2 = 10^{10}$. The norm of the deviatoric part of the stresses $|\sigma(u_h^1)^D|_F$ and the portion of plastified quadrature points are shown in Figure 3. A friction law is given by the functional

$$s(\lambda_n) := \mathrm{Y}_0 \left(\tanh\left(\left[\frac{\mu|\lambda_n|}{\mathrm{Y}_0}\right]^p\right)\right)^{\frac{1}{p}}$$

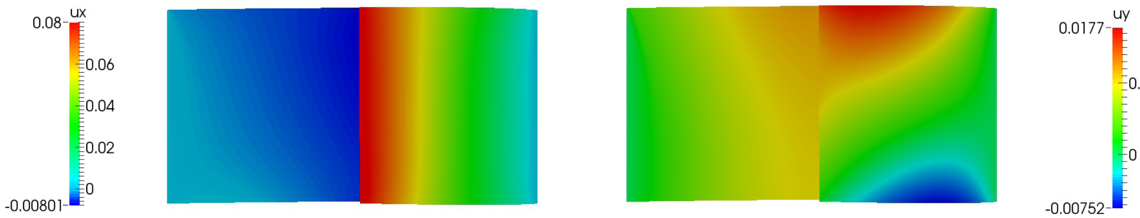
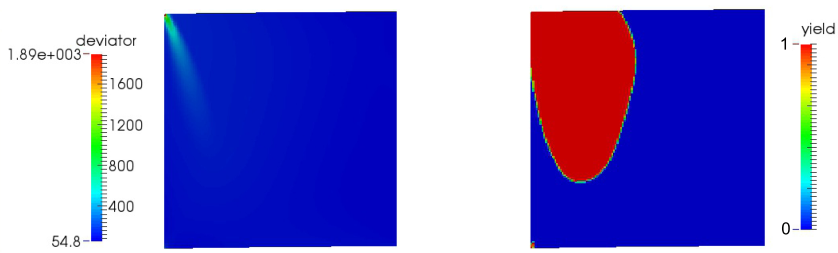


Figure 2: Deformations in x- and y-direction



(a) (b)

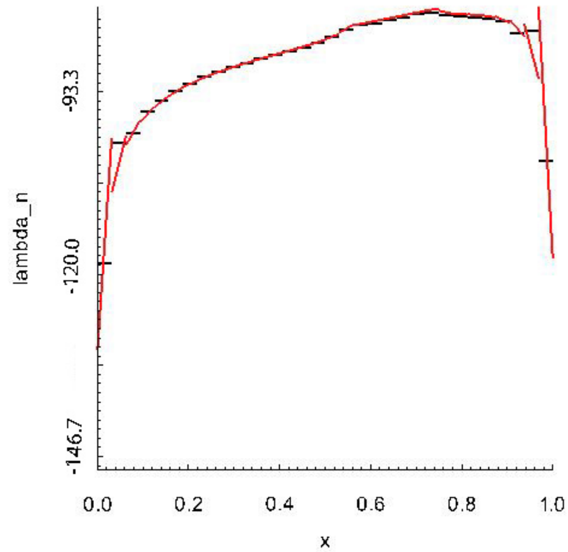
Figure 3: Deviator $|\sigma(u^1)^D|_F$ (a) and yielding part of Ω_1 (b)

with parameters $p = 2,\ \mu = 0.1$ and $\mathrm{Y}_0 = 10$. Furthermore the constants in the NCP functions are chosen by $c_n = 1$ and $c_t = E^1$, whereas the tolerance of Algorithm 1 is set to $\mathrm{tol} = 10^{-10}$. Calculated Lagrange multipliers λ_n and λ_t for selected polynomial degrees on a locally refined mesh (cf. Figure 1 (b)) are depicted in Figure 4. The nonoscillating course of these functions indicate the stability of the underlying mixed schemes. The development of the plastic, frictional and geometrical contact residuals for different polynomial degrees as defined in Algorithm 1 are presented in Figure 5. For both choices of discretizations $p = 2,\ q = 1$ and $p = 3, q = 2$ the frictional residual dominates the others permanently. Due to the simple geometric contact situation of exclusively active indices the quantity C_N^k tends to zero very fast. After some iterations in which the displacements have to be damped ($\alpha = 0.5$) the residuals C_T^k and res_{plast} also decrease rapidly. Overall the general behaviour appears to be similar for both discretizations.

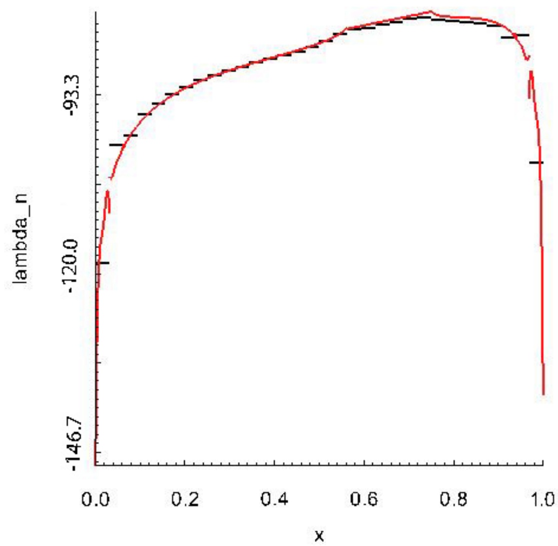
	step	1	2	3	4	5	6	7	8	9
no prec.	# iter	589	5.401	10.569	16.838	15.463	5.694	1.595	316	267
$\mathcal{P}^k$	# iter	5	4	4	4	4	3	3	2	/
	# iter$_\mathrm{S}$	19	382	1.036	54	36	27	105	1.016	/
$\mathcal{P}^k$, $\mathcal{P}_S^k$	# iter	3	3	3	3	3	3	3	2	/
	# iter$_\mathrm{S}$	13	27	27	27	27	27	27	18	/

Table 1: Comparison of preconditioners with $p = 1,\ q = 0$ on a uniform refined mesh, $n = 288,\ m_1 = m_2 = 4, \mathrm{tol} = 10^{-9}$

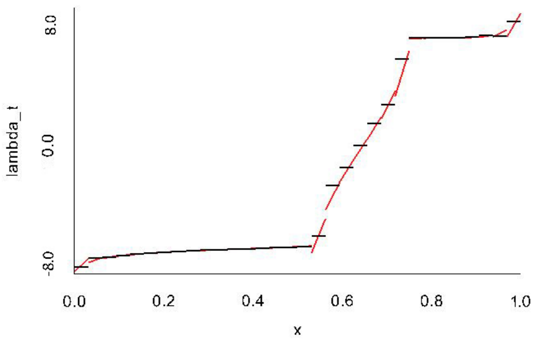
Table 1 shows a comparison of different choices for preconditioning methods applied at the solution process of a problem of very small size. The number of iterations of a Restarted GMRES solver



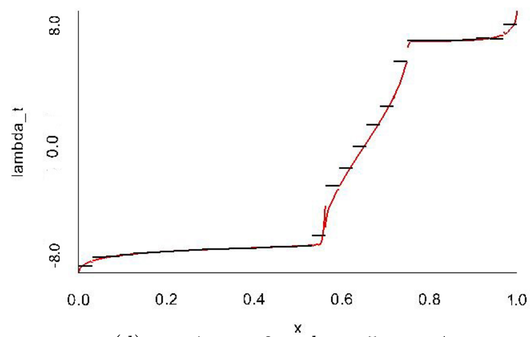
(a) $p = 1,\ q = 0$ and $p = 2,\ q = 1$



(b) $p = 1,\ q = 0$ and $p = 5,\ q = 4$

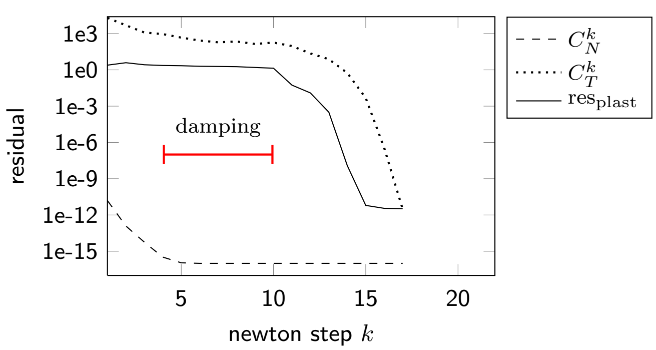


(c) $p = 1,\ q = 0$ and $p = 2,\ q = 1$

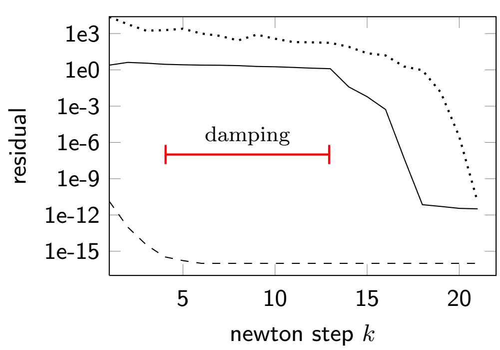


(d) $p = 1,\ q = 0$ and $p = 5,\ q = 4$

Figure 4: Lagrange multipliers λ_n and λ_t for different polynomial degrees



(a) $p = 2,\ q = 1,\ n = 141.952,\ m_1 = m_2 = 256$



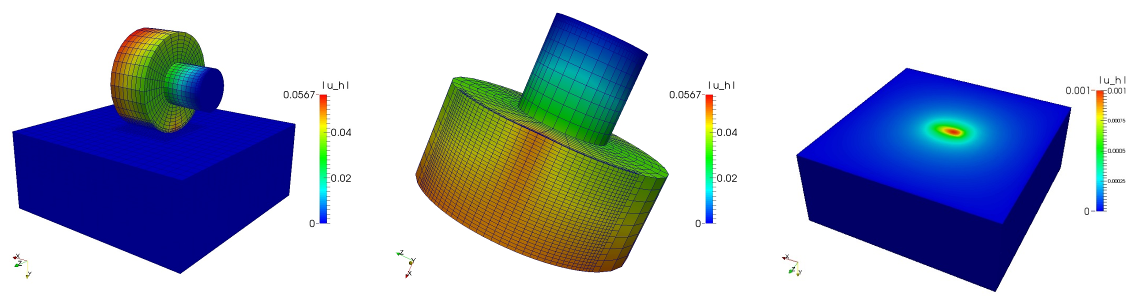
(b) $p = 3,\ q = 2,\ n = 236.480,\ m_1 = m_2 = 384$

Figure 5: Residuals for different polynomial degrees

without any preconditioning is opposed to those of a supplemental application of a block triangular preconditioner $\mathcal{P}^k$ defined in (31). In the third version the generalized Schur complement matrix $\hat{S}^k$ within the evaluation of $\mathcal{P}^k$ is additionally preconditioned by $\mathcal{P}_S^k$ given in (32). The number of outer GMRES steps is named as $iter$, whereas $iter_S$ labels the accumulated number of GMRES iterations that are performed to solve with the Schur complement. Every evaluation of $\hat{S}^k$ includes a direct solution with the block-diagonal matrix $\hat{K}^k$. Due to the chosen friction law and the start value $\lambda_{n,H}^0 = 0$ the friction bound and therefore $\lambda_{t,H}^1$ vanishes. This results in a well-conditioned system matrix and a moderate amount of solving iterations in the first semi-smooth Newton step. In the following steps the number of outer iterations rises rapidly if no preconditioning is performed. This indicates the bad condition aroused by the Newton system concerning frictional contraints. In the later Newton steps due to the revising starting value the amount of solving iterations decreases again. The preconditioned versions reduce the number of Newton steps by one. Furthermore preconditioning the Schur complement matrix decreases $iter_S$ and therefore the costly amount of direct solutions with K^k considerably. The condition of the system matrix appears basically due to the frictional contraints to be very bad in this application, which necessitates preconditioning. An acceptable effort is achieved by the suggested combination of preconditioners.

5.2 Threedimensional example

As an application from mechanical engineering we consider a two-body contact problem that models a grinding process. The first body Ω_1 represents a cylindric mounted point with diameter 2.2. Its spindle is clamped at the end. A hexahedral body $\Omega_2 = [-2, 3] \times [1.1, 3.1] \times [-2.5, 2.5]$ represents a workpiece, which is machined by the mounted point. An overview of the contact situation and mesh as well as resulting deformations of both bodies are depicted in Figure 6. Around the contact zones and the clamped part of the shaft the meshes are locally refined. Dirichlet boundary conditions hold on its side surfaces and the bottom. We consider a Coulomb-Orowan friction law [26, Section 4.2.5]

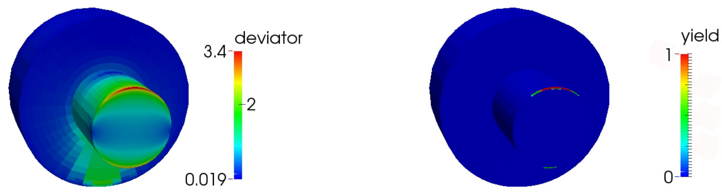


(a) (b) (c)

Figure 6: Overview of contact situation (a) and magnitude of deformations of mounted point Ω_1 (b) and workpiece Ω_2 (c)

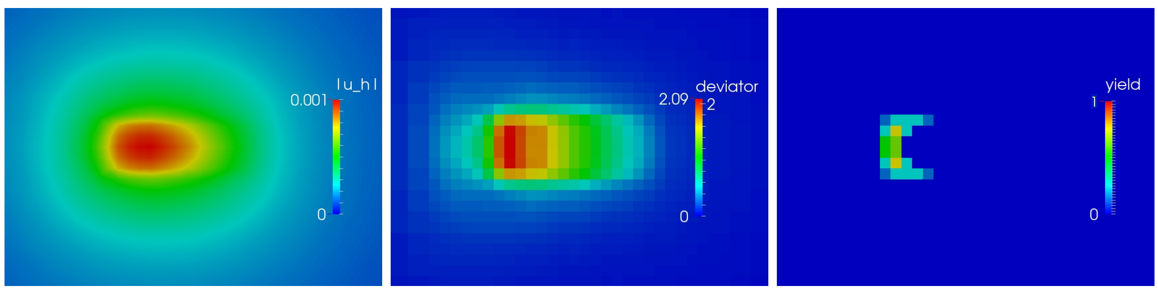
$$s(\lambda_n) := \min\{\mu|\lambda_n|, \mathrm{Y}_0\}$$

with constants $\mu = 0.5$ and $\mathrm{Y}_0 = 1.1$. In contrast to the twodimensional example the present meshes are nonmatching. The elasto-plastic material parameters are chosen by $E^1 = 10^2, E^2 = 10^3$, $\nu^1 = \nu^2 = 0.3$, $\sigma_0^1 = 3, \sigma_0^1 = 2$ and $\gamma_{\text{iso}}^1 = \gamma_{\text{iso}}^2 = 0.1$. Resulting deviators and portions of plastified quadrature points are pictured in Figures 7 and 8. As might be expected the workpiece plastifies at the contect zone. In contrast at the mounted point, the deviatoric part of the stresses gets maximal around the clamped part of the spindle. Resulting langrange multipliers for frictional and geometrical contact constraints are shown in Figure 9. The discretization appears to be stable again because no checkerboard patterns is observed.



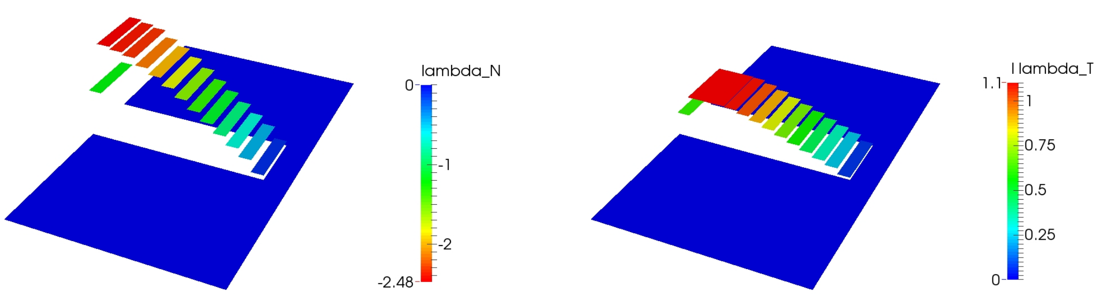
(a) (b)

Figure 7: Deviator $|\sigma(u^1)^D|_F$ (a) and portion of plastified quadrature points (b) of mounted point Ω_1



(a) (b) (c)

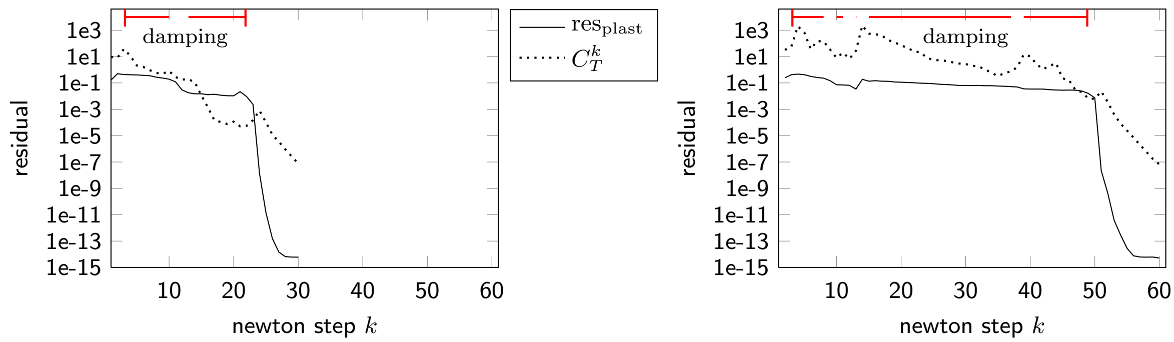
Figure 8: Magnitude of deformations u_h^2 (a), deviator $|\sigma(u^2)^D|_F$ (b) and portion of plastified quadrature points (c) at the contact zone of workpiece Ω_2



(a) λ_n (b) λ_t

Figure 9: Lagrange multipliers λ_n and λ_t for $p = 1,\ q = 0,\ n_2 = 65.895,\ n_2 = 20.358,\ m_1 = 256,\ m_2 = 512$

A comparison of different choices for the constant c_t inside the NCP function is shown in Figure (10). The development of the tangential and plastic residual is pictured. As values of c_t the Elasticity modula $E^1 = 10^2$ and $E^2 = 10^3$ are chosen. It is conspicuous that the number of Newton iterations is significantly larger for $c_t = 10^3$. Furthermore for this constant the development of the residual C_T^k appears more unstable. This behaviour indicates the sensitivity of the method concerning the choice of the constant c_t.



(a) $c_t = 10^2$ (b) $c_t = 10^3$

Figure 10: Development of C_T^k and C_N^k for $c_t = 10^2$ (a) and $c_t = 10^3$ (b) ($p = 1,\ q = 0,\ n_2 = 65.895,\ n_2 = 20.358,\ m_1 = 256,\ m_2 = 512$)

6 Conclusions and outlook

In this paper an active-set strategy for frictional two-body contact problems and mixed higher-order discretizations based on [10] and [4, 22] is presented. Compared to Mortar discretization for this ansatz a construction of higher-order dual basis functions is not necessary. We suggest a block triangular preconditioner for the the full saddle point system and an appropriate preconditioner for the generalized schur complement matrix. Numerical results show the successful operation of the semi-smooth Newton method and a considerable reduction of the condition by the preconditioners. In further work following [9] we will extend this solving method by reformulating plastic material behaviour in terms of an additional NCP function. This will end up in a larger and more ill-conditioned saddle point system and neccesitate the development of suitable preconditioners. Furthermore an extension of the contact model by including effects like adhesion [5] and seperation of two bodies [26] as well as the development of appropriate adaptive strategies are planed.

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