Extended One-Step Schemes for Stiff and Non-Stiff Delay Differential Equations

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Abstract

The importance of delay differential equations (DDEs), in modelling mathematical biological, engineering and physical problems, has motivated searchers to provide efficient numerical methods for solving such important type of differential equations. Most of these types of differential models are stiff, and suitable numerical methods must be introduced to simulate the solutions. In this paper, we provide a reliable computational technique, based on a class of extended one-step methods for solving stiff and non-stiff DDEs. The efficiency and stability properties of this technique are studied. Numerical results and simulations are presented to demonstrate the effectiveness of the methodology.

Keywords: DDEs, Extended one-step schemes; Stability; Stiffness; Stability regions

1 Introduction

The theory of *delay differential equations* (DDEs) is of both theoretical and practical interest, as they provide a powerful model of many phenomena in applied sciences such as physics, biology, chemistry, economics, control theory and so on. The work reported in [1, 2, 3, 4, 5, 6, 7] indicates the scope for applications of DDEs. The delay(s), in such models, can be related to the duration of certain hidden processes like the stages of the life cycle, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the immune period and so on [8, 9]. Unfortunately, most of these models that represented by DDEs – especially in the study of chemical kinetics, or immune system interactions – are 'stiff', in the sense that they have properties that make them slow and expensive to solve using explicit numerical methods. The efficient use of reliable numerical methods (based in general on implicit formulae) for dealing with stiff models involves a degree of sophistication not necessarily available to nonspecialists.

The history of stiff differential equations goes back to more than 60 years to the very early days when the first identification of stiff equations as a special class of problems has been given by chemists Curtiss and Hirschfelder in 1952 [10]. Since then, stiff equations presented serious difficulties and were hard to solve, both in chemical problems (reaction kinetics) and increasingly in other areas (electrical engineering, mechanical engineering, etc) until around seventies century when a variety of methods began to appear in the literature [11, 12, 13]. The nature of the problems that leads to stiffness is the existence of physical phenomena with very different speeds (time constants) so that, while we may be interested in relative slow aspects of the model, there are features of the model that could change very rapidly. Prior to the availability of electronic computers, one could seldom solve problems that were large enough for this to be a problem, but once electronic computers became available and people began to apply them to all sorts of problems, we very quickly ran into stiff problems. In the literature of ODEs, various definitions are seen for the stiffness [14, 15, 16], one being somewhat more precise than another. The essence of stiffness is that the solution to be computed is slowly varying but that perturbations exist which are rapidly damped. The presence of such perturbations complicates the numerical computation of the slowly varying solution. The problem of stiffness also occurs in DDEs [17, 18, 19, 20]. However, the situation is more complicated than for ODEs because the existence of rapidly and slowly varying may not imply stiffness.

The *stiffness* with DDEs models is characterized by phenomena such as: strong contractivity of neighbouring solutions, multiple time scales (fast transient phases) and the fact that explicit numerical integrators are not able to reproduce a correct approximation of the solution in an efficient way. Another definition, stiff differential equation is an equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. In other words, the step size is restricted by stability and not accuracy considerations [21]. In [22], stiff equations are defined to be those equations where implicit methods perform tremendously better than explicit ones. Practically, the software code developers emphasis that stiff equation solvers are based on implicit methods and not explicit methods as codes based on explicit methods are much more computational expensive than those based on the implicit methods. One can also define a stiff solution of a DDE as one whose global accuracy of the numerical solution is determined by stability rather than local error and implicit methods are more appropriate for it.

Integrating of non-stiff problems with stiff method is very expensive, whereas non-stiff methods are much better suited for this purpose. Also, many problems may be stiff in some intervals and nons-tiff in others. Therefore, we need an efficient technique to be suitable for stiff and non-stiff problem. Explicit methods have lower computational costs, but with also lower accuracy, compared to implicit methods. If the problem can be solved with comparable accuracy with both explicit and implicit methods then explicit method is the choice. But what will happen if explicit methods have higher computational cost and lower accuracy results or even fail to get a solution as in stiff problems? In this case we have no choice but to use implicit methods.

Extended one step method (EOSM) is a combination of several *linear multi-step methods* (LMMs) [18, 22]. These methods are introduced by the authors of [23, 24, 25] to solve stiff and non-stiff ODEs. The authors showed that the EOSMs, depending on free parameters, are A/L-stable in the case of solving ODEs. In this paper we adapt a class of extended one step schemes for solving DDEs with constant and variable delays. We prove that EOSMs are suitable for stiff and non-stiff DDEs. Consider the general form of first order DDEs of the form

$$\begin{aligned} y'(t) &= f(t, y(t), y(t-\tau)), \quad t \in [t_0, T], \\ y(t) &= \psi(t), \quad t \in [-\tau, t_0], \end{aligned} \tag{1}$$

where the time-lag τ is assumed to be non-negative parameter. It could be constant, or variable as a function of t such that $0 \leq \tau(t) \leq \tau^*$, where τ^* is a constant. Because of the delay term it is no longer sufficient to supply an initial value, at time $t = t_0$, to completely define the problem, but an initial function $\psi(t)$ defined in interval $[-\tau, t_0]$. The function f is assumed to be sufficiently smooth with respect to its arguments, and $\psi(t)$ is also assumed to be continuous. We assume also that the function $f(t, u, v)$ satisfies the classical Lipschitz condition in the second and third variables u and v, i.e.

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \leq L(\|u - \bar{u}\| + \|v - \bar{v}\|), \quad L \in \mathbb{R}^+. \tag{2}$$

For stiff problems, however, L is typically very large, as the classical Lipschitz constant only measures variations of f but does not take into account if the direction field corresponding to the right-hand side f of (1) is diverging or converging. Therefore, for stiff problems, we should have the following Remark.

Remark 1 *Equation (1) has a unique smooth solution $y(t)$, satisfies the condition (2) and*

$$\Re\langle f(t,u,v)\ ,\ f(t,\bar{u},\bar{v})\rangle \leq M\|u-\bar{u}\|^2, \tag{3}$$

where $\langle .\ ,\ .\rangle$ denotes a given inner product. The constant M, representing the sensitivity of solution with respect to this initial perturbation, exists such that $M \ll L$ and possibly $M < 0$ applies.

The organization of this paper is as follows: In Section 2, we present extended one step schemes, up to order five, for the inial value probles of DDEs. In Section 3, we investigate the stability analysis the numerical schemes throughout P-stability and Q-stability. Stabilit regions are also deduced. The convergence of the scheme of order 3 is discussed in Section 4. Numerical simulations for differen types of DDEs are procided in Section 5 and conclusion in Section 6.

2 Extended One-Step Methods for DDEs

The main aim of this Section is to consider the application of the EOSM for the numerical solution of initial value problem for stiff and non-stiff delay differential equations DDEs. Let us first provide EOSM applied to the initial value problem of the ODE form

$$\begin{aligned} y\prime(t) &= f(t,y(t)), \quad 0 < t \leq b, \\ y(0) &= y_0, \qquad\qquad t = 0. \end{aligned} \tag{4}$$

It is well known that the order of a $k-$step method cannot exceed $k+2$, however an A-stable *linear multistep method* (LMM) can not exceed 2 [26]. To overcome this "order barrier" imposed by A-stability, so called extended one-step A-stable methods of order up to five had constructed by coupling several LMMs (see [25]. After discretization of the problem (4), one can get

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j f_{n+j}] + \kappa_n(h)\ , \tag{5}$$

with

$$y_{n+j} = \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} f_{n+i}] + E_{nj}(h). \tag{6}$$

The extended one-step scheme of such problem takes the form

$$\begin{aligned} y_{n+1} &= y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}] + T_n(h), \\ \hat{y}_{n+j} &= \beta_{j0} y_n + \beta_{j1} y_{n+1} + h[\ \gamma_{j0} f_n + \gamma_{j1} f_{n+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{n+i}] \end{aligned} \tag{7}$$

where $\alpha_j,\ \beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and $\gamma_{ji},\ j = 2, 3, \ldots m-1$ are real coefficients, $f_n = f(t_n, y_n)$ and y_n is an approximation to $y(t_n)$ at a sequence of equally spced points, $t_n = nh,\ n = 0, 1, \ldots, N$. One can refer to such a methods (omitting $T_n(h)$) by Table 1.

α_0	α_1	α_2	$\dots$		α_{m-1}	
β_{20}	β_{21}	γ_{20}	γ_{21}			
β_{30}	β_{31}	γ_{30}	γ_{31}	γ_{32}		
$\vdots$			$\vdots$	$\vdots$	$\ddots$	
$\beta_{m-1,0}$	$\beta_{m-1,1}$	$\gamma_{m-1,0}$	$\gamma_{m-1,1}$	$\gamma_{m-1,2}$	$\dots$	$\gamma_{m-1,m-2}$

Table 1: Coefficeints of the extended one-step methods.

Usmani and Agarwall [27] deduced an extended one-step third order A-stable scheme by requiring that $E_{n2}(h) = O(h^3)$. Later, Jacques [23] modified the method of such schemes to obtain a one parameter family of third order L-stable method by requiring that $E_{n2}(h) = O(h^3)$. Chawla *et al.* [24] obtained a two-parameter family of fourth order and A-stable methods by requiring that $E_{n2}(h)$ and $E_{n3}(h) = O(h^3)$;there exists a one-parameter sub-family of these methods which are, in addition, L-stable. Chawla *et al.* [25] extended these ideas to obtain a two-parameter family of fifth order and gave sub-families of A-stable and L-stable methods.
The general idea, for the derivation of a methods of order m , we require that $\kappa_n(h)$ and $T_n(h) = O(h^{m+1})$ while $E_{nj}(h) = O(h^{m-1})$.

Tables 2 & 3 display the tubule of A-stable and L-stable of order four and five, respectively.

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
5	-4	2	4	
28	-27	12	18	0

$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
1	0	0	2	
2	-1	$-\frac{1}{2}$	4	$\frac{1}{2}$

Table 2: A-stable scheme (left) and L-stable scheme (right) of order four for the ODEs (4) [24].

$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$	
5	-4	2	4		
28	-27	12	18	0	
$\frac{1563}{19}$	$-\frac{1544}{19}$	$\frac{694}{19}$	$\frac{928}{19}$	$-\frac{2}{19}$	0

$\frac{251}{720}$	$\frac{323}{360}$	$-\frac{11}{30}$	$\frac{53}{360}$	$-\frac{19}{720}$	
5	-4	2	4		
28	-27	12	18	0	
$\frac{1611}{19}$	$-\frac{1592}{19}$	$\frac{712}{19}$	$\frac{966}{19}$	$-\frac{12}{19}$	$\frac{2}{19}$

Table 3: A-stable scheme (left) and L-stable scheme of order 5 for the ODEs (4) [25].

We extend the above schemes to the DDEs

$$\begin{aligned} y\prime(t) &= f(x, y(t), y(\alpha(t))), \quad && a \leq t \leq b, \\ y(t) &= g(t), && \nu \leq t \leq a. \end{aligned} \tag{8}$$

Here f , α and g denote given functions with $\alpha(t) \leq t$ for $t \geq a$, the function α is usually called the delay or lag function and y is unknown solution for $t > a$. If the delay is a constant, it is called the constant delay, if it is a function of only time, then it is called the time dependent delay, if it is a function of time and the solution $y(t)$, then it is called the state dependent delay. The existence, uniqueness, and continuation of solutions to the above problem have been studied by Driver [28].

The extended one-step scheme for DDE (8) is given by

$$y_{n+1} = y_n + h[\ \alpha_0 f_n + \alpha_1 f_{n+1} + \sum_{j=2}^{m-1} \alpha_j \hat{f}_{n+j}], n = 0, 1, \ldots, N-1, \tag{9}$$

where $\hat{f}_{n+j} = f(t_{n+j}, \hat{y}_{n+j}, y^h(\alpha(t_{n+j})))$ and $\alpha_j,\ j = 2, 3, \ldots m-1$ are real coefficients. The function y^h is computed from

$$\begin{cases} y^h(t) = g(t) \quad \text{for} \quad t \leq a \\ y^h(t) = \beta_{j0} y_k + \beta_{j1} y_{k+1} + h[\ \gamma_{j0} f_k \\ \qquad + \gamma_{j1} f_{k+1} + \sum_{i=2}^{j-1} \gamma_{ji} \hat{f}_{k+i}]\ , \\ \qquad\qquad t_k < t \leq t_{k+1} \quad k = 0, 1, \ldots \end{cases} \tag{10}$$

where $\beta_{j0},\ \beta_{j1},\ \gamma_{j0},\ \gamma_{j1}$ and γ_{ji} are real coefficients. The function $\hat{y}_{n+j}$ are computed from (10) when $t = t_{n+j}$. In this paper, we will use $\tilde{}$ for the coefficients of $\hat{y}_{n+j}$ as in the following form

$$\hat{y}_{n+j} = \tilde{\beta}_{j0} y_n + \tilde{\beta}_{j1} y_{n+1} + h[\ \tilde{\gamma}_{j0} f_n + \tilde{\gamma}_{j1} f_{n+1} + \sum_{i=2}^{j-1} \tilde{\gamma}_{ji} \hat{f}_{n+i}] \tag{11}$$

Scheme of third order ($m = 3$)

In order to determine the coefficients α_0, α_1 and α_2, we rewrite (9) for $m = 3$ in the exact form

$$\begin{aligned} y(t_{n+1}) = y(t_n) + h\,[&\alpha_0 f(t_n, y(t_n), y(\alpha(t_n))) \\ &+ \alpha_1 f(t_{n+1}, y(t_{n+1}), y(\alpha(t_{n+1}))) \\ &+ \alpha_2 f(t_{n+2}, y(t_{n+2}), y(\alpha(t_{n+2})))] + \kappa(t_{n+1}). \end{aligned} \tag{12}$$

We expand the left and right sides of (12) in the Taylor series at the point t_{n+1}, equate the coefficients up to the third order terms $O(h^3)$ and solving the resulting system of equations, we obtain

$$\alpha_0 = \frac{5}{12}, \quad \alpha_1 = \frac{2}{3}, \quad \alpha_2 = -\frac{1}{12} \tag{13}$$

and

$$\kappa(t_{n+1}) = \frac{h^4}{24} y^{(4)}(\xi) \tag{14}$$

where $t_n < \xi < t_{n+2}$. Substituting from (13) into (9) for $m = 3$, we obtain

$$y_{n+1} = y_n + \frac{h}{12}\left[5 f_n + 8 f_{n+1} - \hat{f}_{n+2}\right] \tag{15}$$

where

$$y^h(t) = g(t) \quad \text{for} \quad t \leq a \tag{16}$$

and $y^h(t)$ with $t > a$ is defined by

$$y^h(t) = \beta_{20} y_k + \beta_{21} y_{k+1} + h\left[\gamma_{20} f_k + \gamma_{21} f_{k+1}\right], \text{ for } t_k < t \leq t_{k+1};\ k = 0, 1, \ldots \tag{17}$$

In order to determine the coefficients β_{20}, β_{21}, γ_{20} and γ_{21}, we rewrite (17) in the exact form

$$\begin{aligned} y(t) =&\beta_{20}y(t_k) + \beta_{21}y(t_{k+1}) + h\left[\gamma_{20}f(t_k, y(t_k), y(\alpha(t_k)))\right. \\ &\left.+ \gamma_{21}f(t_{k+1}, y(t_{k+1}), y(\alpha(t_{k+1})))\right] + E(t_{k+1}). \end{aligned} \tag{18}$$

Similarly, we expand the left and right sides of (18) with Taylor series at point t_{k+1} and equate the coefficients up to the terms of second order $O(h^2)$. We obtain the resulting system of equations

$$\begin{cases} \beta_{20} + \beta_{21} = 1 \\ \beta_{20} - \gamma_{20} - \gamma_{21} = -\delta(t) \\ \beta_{20} - 2\gamma_{20} = \delta^2(t) \end{cases} \tag{19}$$

where

$$\delta(t) = \frac{1}{h}(t - t_{k+1}). \tag{20}$$

The solution of the above system (19) is

$$\begin{cases} \beta_{20} = 1 - \beta_{21} \\ \gamma_{20} = \dfrac{1}{2}(1 - \beta_{21} - \delta^2(t)) \\ \gamma_{21} = \dfrac{1}{2}(\delta^2(t) + 2\delta(t) - \beta_{21} + 1) \end{cases} \tag{21}$$

and

$$E(t_{k+1}) = \frac{h^3}{12}(2\delta^3(t) + 3\delta^2(t) + \beta_1 - 1)y^{(3)}(\eta) \tag{22}$$

where β_{21} is a free parameter and $t_k < \eta < t_{k+1}$. Substituting from (21) into (17), we obtain

$$\begin{aligned} y^h(t) =&(1 - \beta_{21})y_k + \beta_{21}y_{k+1} + \frac{h}{2}\left[(1 - \beta_{21} - \delta^2(t))f_k\right. \\ &\left.+(\delta^2(t) + 2\delta(t) - \beta_{21} + 1)f_{k+1}\right], \quad \text{for} \quad t_k < t \le t_{k+1};\ k = 0, 1, \dots, \end{aligned} \tag{23}$$

Finally, from (23), the approximation $\hat{y}_{n+2}$ is determined in the form

$$\hat{y}_{n+2} = (1 - \beta_{21})y_n + \beta_{21}y_{n+1} - \frac{h}{2}\big[\beta_{21}f_n + (\beta_{21} - 4)f_{n+1}\big]. \tag{24}$$

Equations (15), (23) and (24) are the basis of the third order methods (see [29]).

We can estimate the parameters for schemes of order 4 and order 5 in the same manner.

Scheme of fourth order ($m = 4$)

We can then obtain a two-parameter family of extended fourth order one-step methods, which we will refer it by $PM_4(\gamma_{20}, \gamma_{32})$ and these method are consisting of

$$y_{n+1} = y_n + \frac{h}{24}\left[9f_n + 19f_{n+1} - 5\hat{f}_{n+2} + \hat{f}_{n+3}\right], \tag{25}$$

if $\alpha(t) \in (t_n, t_{n+2})$ then the function y^h is computed from

$$\begin{aligned} y^h(t) = (2\gamma_{20} + \delta_1^2(t))y_k + (1 - \delta_1^2(t) - 2\gamma_{20})y_{k+1} + h\left[\gamma_{20}f_k + (\gamma_{20} + (\delta_1^2(t) + \delta_1(t))f_{k+1}\right] \\ \text{for} \quad t_k < t \le t_{k+1};\ k = 0, 1, \dots, \end{aligned} \tag{26}$$

where $\delta_1(t) = \frac{1}{h}(t - t_{k+1})$. The function $\hat{y}_{n+2}$ is computed from (26) when $t = t_{n+2}$ and will take the following form

$$\hat{y}_{n+2} = (1 + 2\gamma_{20})y_n - 2\gamma_{20}y_{n+1} + h[\gamma_{20}f_n + (2 + \gamma_{20})f_{n+1}], \tag{27}$$

if $\alpha(t) \in (t_{n+2}, t_{n+3})$ then the function y^h is computed from

$$\begin{aligned} y^h(t) = {} & (2\gamma_{31} + 4\gamma_{32} - 2\delta_2(t) - \delta_2^2(t))y_k + (1 + \delta_2^2(t) + 2\delta_2(t) - 2\gamma_{31} \\ & - 4\gamma_{32})y_{k+1} + h[(-\delta_2(t) - \delta_2^2(t) + \gamma_{31} + 3\gamma_{32})f_k \\ & + \gamma_{31}\hat{f}_{k+1} + \gamma_{32}\hat{f}_{k+2}] \qquad \text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1, \ldots, \end{aligned} \tag{28}$$

where $\delta_2(t) = \frac{1}{h}(t - t_{k+1})$ and

$$\gamma_{31} = 5\gamma_{20} + \delta_2^3(\alpha(t_{n+3})) + \delta_2(\alpha(t_{n+3})) + 2\delta_2^2(\alpha(t_{n+3})) - 8\gamma_{32} - 5\delta_1^2(\alpha(t_{n+2})) - 5\delta_1^3(\alpha(t_{n+2}))$$

with a free parameter γ_{32}. The function $\hat{y}_{n+3}$ is computed from (28) when $t = t_{n+3}$ and will take the following form

$$\begin{aligned} \hat{y}_{n+3} = {} & 2(4 + 5\gamma_{20} - 6\gamma_{32})y_n + (-7 - 10\gamma_{20} + 12\gamma_{32})y_{n+1} + \\ & h[(2 + 5\gamma_{20} - 5\gamma_{32})f_n + (8 + 5\gamma_{20} - 8\gamma_{32})f_{n+1} + \gamma_{32}\hat{f}_{n+2}]. \end{aligned} \tag{29}$$

Scheme of fifth order ($m = 5$)

We will refer it by $PM_5(\gamma_{32}, \gamma_{43})$ and these methods are consisting of :

$$y_{n+1} = y_n + \frac{h}{720}\left[251f_n + 646f_{n+1} - 264\hat{f}_{n+2} + 106\hat{f}_{n+3} - 19\hat{f}_{n+4}\right], \tag{30}$$

if $\alpha(t) \in (t_n, t_{n+2})$ then the function y^h is computed from

$$\begin{aligned} y^h(t) = {} & (2\delta_1^3(t) - 3\delta_1^2(t) + 1)y_k + (3\delta_1^2(t) - 2\delta_1^3(t))y_{k+1} \\ & + h\left[(\delta_1^3(t) - 2\delta_1^2(t) + \delta_1(t))f_k + (\delta_1^3(t) - \delta_1^2(t))f_{k+1}\right] \\ & \text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1, \ldots, \end{aligned} \tag{31}$$

The function $\hat{y}_{n+2}$ computed from (31) when $t = t_{n+2}$ and will take the following form

$$\hat{y}_{n+2} = 5y_n - 4y_{n+1} + h\left[2f_n + 4f_{n+1}\right], \tag{32}$$

if $\alpha(t) \in (t_{n+2}, t_{n+3})$ then the function y^h is computed from

$$\begin{aligned} y^h(t) = {} & (2\delta_2^3(t) - 3\delta_2^2(t) - 12\gamma_{32} + 1)y_k + (3\delta_2^2(t) - 2\delta_2^3(t) + 12\gamma_{32})y_{k+1} \\ & + h[(\delta_2^3(t) - 2\delta_2^2(t) + \delta_2(t) - 5\gamma_{32})f_k + (\delta_2^3(t) - \delta_2^2(t) \\ & - 8\gamma_{32})f_{k+1} + \gamma_{32}\hat{f}_{k+2}], \quad \text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1, \ldots. \end{aligned}$$

The function $\hat{y}_{n+3}$ is computed from the above equation when $t = t_{n+3}$ and will take the following form

$$\begin{aligned} \hat{y}_{n+3} = {} & (28 - 12\gamma_{32})y_n - (27 - 12\gamma_{32})y_{n+1} + h[\ (12 - 5\gamma_{32})f_n \\ & +(18 - 8\gamma_{32})f_{n+1} + \gamma_{32}\hat{f}_{n+2}] \end{aligned} \tag{33}$$

if $\alpha(t) \in (t_{n+3}, t_{n+4})$ then the function y^h is computed from

$$y^h(t) = (2\delta_3^3(t) - 3\delta_3^2(t) - 12\gamma_{42} - 36\gamma_{43} + 1)y_k + (3\delta_3^2(t) - 2\delta_3^3(t) + 12\gamma_{42} + 36\gamma_{43})y_{k+1} + h[(\delta_3^3(t) - 2\delta_3^2(t) + \delta_3(t) - 5\gamma_{42} - 16\gamma_{43})f_k + (\delta_3^3(t) - \delta_3^2(t) - 8\gamma_{42} - 21\gamma_{43})f_{k+1} + \gamma_{42}\hat{f}_{k+2} + \gamma_{43}\hat{f}_{k+3}] \quad \text{for} \quad t_k < t \leq t_{k+1};\ k = 0, 1, \ldots,$$

and $\hat{y}_{n+4}$ is computed from the above equation when $t = t_{n+4}$

$$\begin{aligned}\hat{y}_{n+4} =& (81 - 12\tilde{\gamma}_{42} - 36\gamma_{43})y_n - (80 - 12\tilde{\gamma}_{42} + 36\gamma_{43})y_{n+1} + \\ & h[\ (36 - 5\tilde{\gamma}_{42} - 16\gamma_{43})f_n + (48 - 8\tilde{\gamma}_{42} - 21\gamma_{43})f_{n+1} + \tilde{\gamma}_{42}\hat{f}_{n+2} + \gamma_{43}\hat{f}_{n+3}]\end{aligned} \tag{34}$$

where $\delta_3(t) = \frac{1}{h}(t - t_{k+1})$, $\tilde{\gamma}_{42} = -\frac{1}{19}(2 - 106\gamma_{32} + 95\gamma_{43})$, and

$$\begin{aligned}\gamma_{42} = & \frac{264}{228}(\delta_1^4(\alpha(t_{n+2})) + \delta_1^2(\alpha(t_{n+2}) - 2\delta_1^3(\alpha(t_{n+2})))) - \frac{106}{228}(\delta_2^4(\alpha(t_{n+3})) - 2\delta_2^3(\alpha(t_{n+3})) \\ & + \delta_2^2(\alpha(t_{n+3})) - 12\gamma_{32}) + \frac{19}{228}(\delta_3^4(\alpha(t_{n+4})) + \delta_3^2(\alpha(t_{n+4})) - 2\delta_3^3(\alpha(t_{n+4})) - 60\gamma_{43}).\end{aligned}$$

3 Stability Analysis

There are many concepts of stability of numerical methods for DDEs based on different test equations and the delay terms. Barwell [30] considered the below scalar equation for $\lambda = 0$ and $\mu \in \mathbb{C}$ and also considered the case, where λ and μ are complex using the linear DDEs

$$\begin{aligned} \acute{y}(t) &= \lambda y(t) + \mu y(t - \tau), \quad t \geq 0 \\ y(t) &= g(t), \quad -1 \leq t \leq 0. \end{aligned} \tag{35}$$

In order to find the asymptotical stability region of (35) (which depends on the lag term τ), we suppose, without any loosing of generality, that $\tau = 1$ in (35). We search for (λ, μ) values for which the first solution s crosses the imaginary axis ($Re(s) = 0$), i.e., $s = i\theta$ for θ real. If we insert this into $h(s) = s - \lambda - \mu e^{-s\tau}$, we obtain

$$\lambda = -\mu \quad \text{for } \theta = 0 \quad \text{(s real)},$$

$$\lambda = i\theta - \mu e^{-i\theta} \quad \text{for } \theta \neq 0.$$

By separating real and imaginary parts, we get $\lambda = \dfrac{\theta\cos\theta}{\sin\theta}$, $\mu = -\dfrac{\theta}{\sin\theta}$ is valid for all real values λ and μ. Thus the stability region of $y'(t) = \lambda y(t) + \mu y(t-1)$ is bounded by $\mu = -\lambda$ and the parametrized curve $\lambda = \theta cot(\theta)$, $\mu = -\theta/sin(\theta)$; see Figure 1.

Definition 1 *(P-stability)*
A numerical method, applied to (35) is said to be P-stable if under the condition $Re(\lambda) < -|\mu|$, *the numerical solution satisfies* $y(t_n) \longrightarrow 0$ *as* $t \longrightarrow \infty$ *for all stepsizes* $h = \dfrac{\tau}{m}$, *where* m *is positive integer.*

Definition 2 *(P-stability Region)*
If λ *and* μ *are real in (35), the region* S_P *in the* $(\lambda\ ,\mu\ )$ *plane is called the* $P-$ *stability region if for any* $(\lambda\ ,\mu) \in S_P$ *the numerical solution of (35) satisfies* $y(t_n) \longrightarrow 0$ *as* $t \longrightarrow \infty$ *for the stepsize* h

Definition 3 *(Q-stability Region)*
If $\lambda = 0$ and μ is complex in (35), the region $Q(\mu)$ in the $\mu-$plane is called the $Q-$ stability region if for any $\mu \in Q(\mu)$ the numerical solution $y(t_n) \longrightarrow 0$ as $t_n \longrightarrow \infty$. It is clear that if the method is P-stable then it is A-stable.

Applying the third order EOSM to the test problem (35), yields

$$[24 - 2\lambda h(8 - \beta_{21}) + (\lambda h)^2(4 - \beta_{21})]y_{n+1} = [24 + 2\lambda h(4 + \beta_{21}) + (\lambda h)^2\beta_{21}]y_n + \mu h[(10 + \lambda h\beta_{21})y(t_n - \tau) + (16 - \lambda h(4 - \beta_{21}))y(t_{n+1} - \tau) - 2\mu hy(t_{n+2} - \tau)]. \tag{36}$$

The characteristic polynomial associated with (36) takes the form

$$\begin{aligned} W_s(z) &= [24 - 2X(8 - \beta_{21}) + X^2(4 - \beta_{21})]z^{s+1} \\ &- [24 + 2X(4 + \beta_{21}) + X^2\beta_{21}]z^s \\ &- Y[10 + X\beta_{21} + (16 - X(4 - \beta_{21}))z - 2z^2] \\ &= 0, \quad s = 1, 2, \ldots \end{aligned} \tag{37}$$

where $X = \lambda h$ and $Y = \mu h$. The left banner of Figure 2 shows a bounded Q-stability region ($\lambda = 0$, and μ is complex) to the test problem (35), when $\tau = 1$ with the third order and $\beta = 0$. However, the right banner shows the bounded P-stability region of the same order, with $\beta = 10$.

Remark 2 *To calculate the stability region, we take different values of (λ, μ) along the co-ordinate axes and find the roots of the stability polynomial. If all the roots have magnitude less than one we accept the value of (λ, μ) as part of the stability region.*

Similarly, we can estimate the stability regions of Q-stability and P-stability of EOSMs of orders 4 and 5 which are displayed in Figures 3 and 4, respectively. We may notice that the P-stability for order 4 and 5 are unbounded and similar to the analytical stability region given in Figure 1, but Q-stability are bounded for such schemes.

4 Convergence of the Method

We investigate, in this section, the convergence factor of the third order EOSMs, that can be expressed in the forms

$$y_{n+1}^{(j+1)} = y_n + \frac{h}{12}[5f_n + 8f(x_{n+1}, y_{n+1}^{(j)}, y^{h(j)}(\alpha(x_{n+1}))) - f(x_{n+2}, \hat{y}_{n+2}^{(j)}, y^{h(j)}(\alpha(x_{n+2})))] \quad j = 1, \ldots \tag{38}$$

and

$$\begin{aligned} y^{h(j)}(x) = (1 - \beta_{21})y_k + \beta_{21}y_{k+1} + \frac{h}{2}[(1 - \beta_{21} - \delta^2(x)f_k + (\delta^2(x) + 2\delta(x) \\ -\beta_{21} + 1)f_{k+1}], \quad \text{for} \quad x_k < x \leq x_{k+1};\ k = 0, 1, \ldots \end{aligned} \tag{39}$$

where $y_{n+1}^{(0)}$ is an initial approximation to the solution y at x_{n+1} and $y_{n+1}^{(j)}$, $j \geq 1$ are Picard iterations.

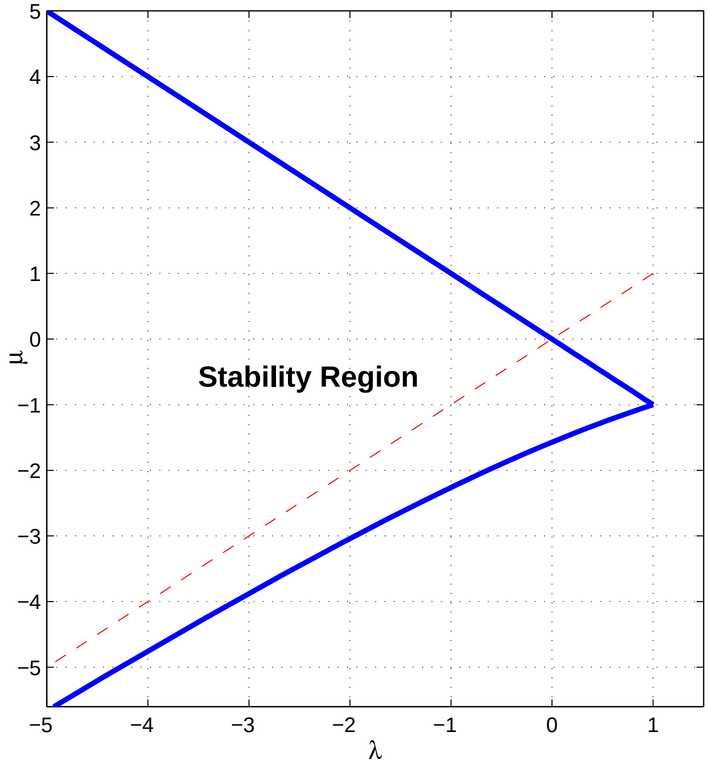

Figure 1: Analytical stability region for the test problem $y'(t) = \lambda y(t) + \mu y(t-1)$.

If $\alpha(x_{n+1}) \in (x_k, x_{k+1}]$, $k = 0, 1, \ldots, n-1$, Eq. (39) will take the following form:

$$\begin{aligned} y^{h(j)}(\alpha(x_{n+1})) &= (1-\beta_{21})y_k + \beta_{21}y_{k+1} + \frac{h}{2}[(1-\beta_{21}-\delta^2(\alpha(x_{n+1}))f_k + (\delta^2(\alpha(x_{n+1})) \\ &+2\delta(\alpha(x_{n+1})) - \beta_{21}+1)f_{k+1}], \quad \text{for} \quad x_k < x \leq x_{k+1};\ k = 0, 1, \ldots \end{aligned} \tag{40}$$

If $\alpha(x_{n+1}) \in (x_n, x_{n+1}]$, we put

$$\begin{aligned} y^{h}(\alpha(x_{n+1})) &= (1-\beta_{21})y_n + \beta_{21}y_{n+1} + \frac{h}{2}[(1-\beta_{21}-\delta^2(\alpha(x_{n+1}))f_n \\ &+(\delta^2(\alpha(x_{n+1})) + 2\delta(\alpha(x_{n+1})) - \beta_{21}+1)f_{n+1}] \end{aligned} \tag{41}$$

and

$$\begin{aligned} y^{h(j)}(\alpha(x_{n+1})) &= (1-\beta_{21})y_n + \beta_{21}y^{(j)}_{n+1} + \frac{h}{2}[(1-\beta_{21}-\delta^2(\alpha(x_{n+1}))f_n + (\delta^2(\alpha(x_{n+1})) \\ &+2\delta(\alpha(x_{n+1})) - \beta_{21}+1)f(x_{n+1}, y^{(j)}_{n+1}, y^{h(j)}(\alpha(x_{n+1})))]. \end{aligned} \tag{42}$$

Since $\alpha(x_{n+1}) - x_n \leq h$, we let $\alpha(x_{n+1}) - x_n = r_1 h$ with $r_1 \in (0, 1]$. Then, from (41) and (42) and by using the Lipschitz condition, it follows that

$$\left| y^{h(j)}(\alpha(x_{n+1})) - y^{h}(\alpha(x_{n+1})) \right| \leq \frac{2|\beta_{21}| + hL|r_1^2 - \beta_{21}|}{2 - hL|r_1^2 - \beta_{21}|} \left| y^{(j)}_{n+1} - y_{n+1} \right|. \tag{43}$$

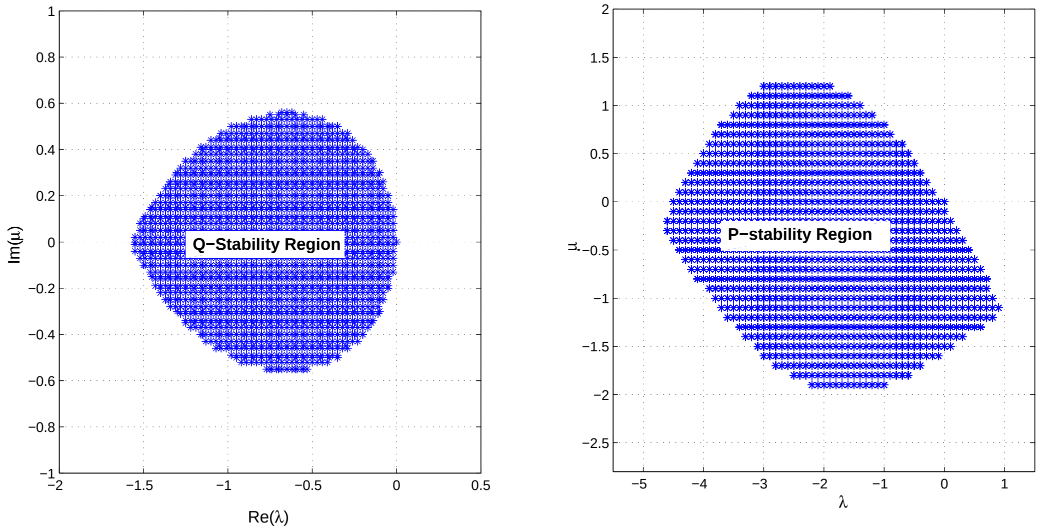

Figure 2: Left banner shows bounded Q-stability region for the test problem (35) with $\tau = 1$, using the third order scheme ($s = 3$) and $\beta_{21} = 0$. The right banner shows the bounded P-stability region, with $\beta_{21} = 10$.

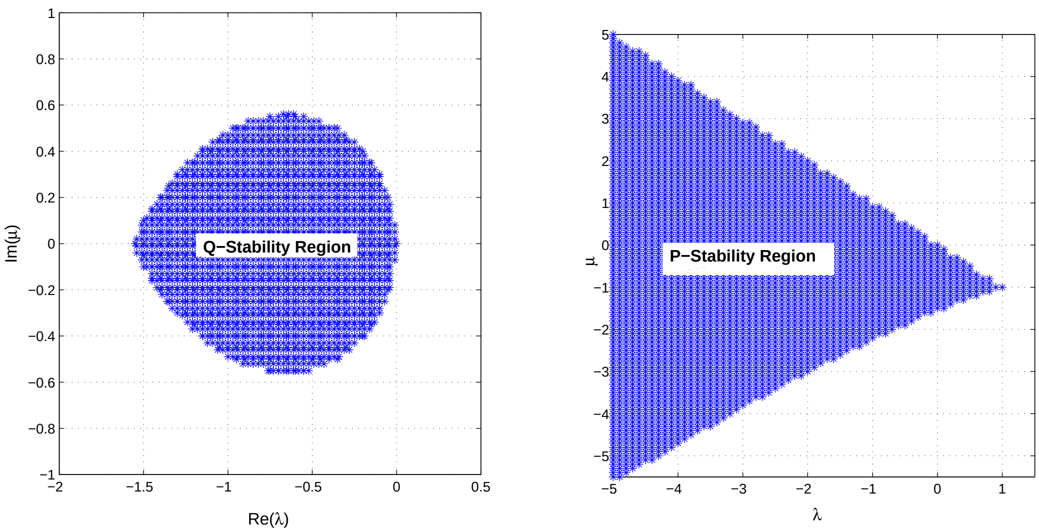

Figure 3: Left banner shows bounded Q-stability region for the test problem (35) with $\tau = 1$, using the fourth order scheme ($s = 4$) and $\gamma_{32} = 0.5$, $\gamma_{20} = 0$. However, the right banner shows unbounded P-stability region.

where $L = \max\{L_1, L_2\}$.By the same way, if $\alpha(x_{n+2}) - x_{n+1} = r_2 h$ with $r_2 \in (0, 1]$, we get

$$\left| y^{h(j)}(\alpha(x_{n+2})) - y^h(\alpha(x_{n+2})) \right| \leq \frac{R_1}{2 - hL|r_1^2 - \beta_{21}|} \left| y_{n+1}^{(j)} - y_{n+1} \right|. \tag{44}$$

where

$$R_1 = 2|\beta_{21}| - hL|\beta_{21}||r_1^2 - \beta_{21}| + hL|r_2^2 + 2r_2 - \beta_{21} + 1| + hL|\beta_{21}||r_2^2 + 2r_2 - \beta_{21} + 1|$$

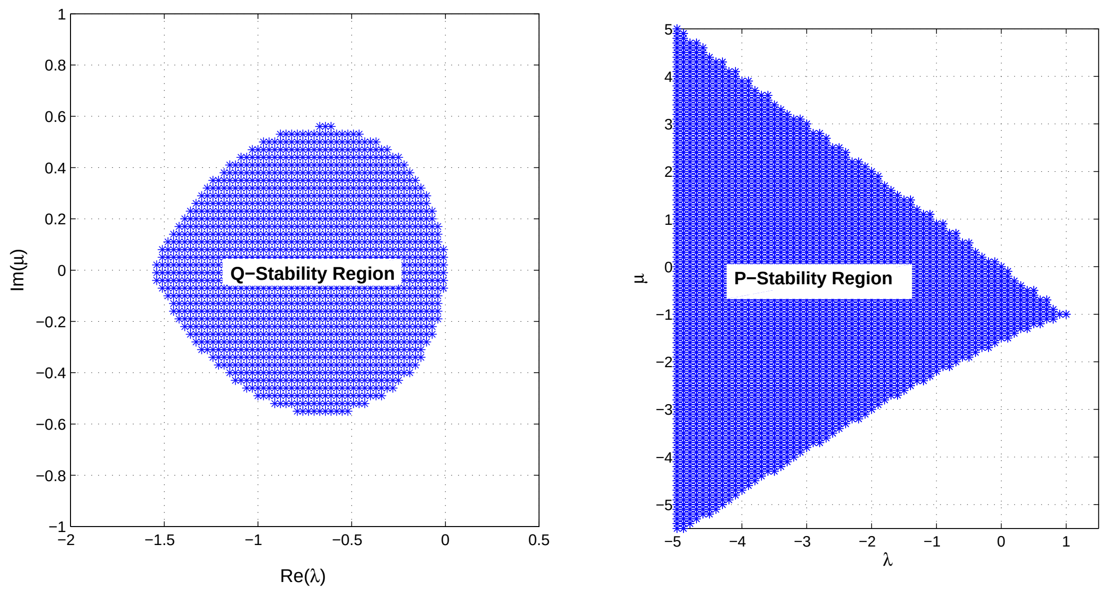

Figure 4: Left banner shows bounded Q-stability region for the test problem(35) with $\tau = 1$, using the fifth order scheme ($s = 5$) and $\gamma_{43} = 2/19$, $\gamma_{32} = 0$. However, the right banner shows unbounded P-stability region.

From (44), it follows

$$\left|\hat{y}_{n+2}^{(j)} - y_{n+2}\right| \leq \frac{2|\beta_{21}| - hL|\beta_{21}||r_1^2 - \beta_{21}| + hL|\beta_{21} - 4| + hL|\beta_{21}||\beta_{21} - 4|}{2 - hL|r_1^2 - \beta_{21}|} \left|y_{n+1}^{(j)} - y_{n+1}\right|. \tag{45}$$

Using (43), (44) and (45), we obtain

$$\left|y_{n+1}^{(j+1)} - y_{n+1}\right| \leq C \left|y_{n+1}^{(j)} - y_{n+1}\right|, \ j = 0, 1, \dots \tag{46}$$

where

$$C = \frac{hL\ R_2}{24 - 12hL|r_1^2 - \beta_{21}|}$$

with

$$\begin{aligned} R_2 = 16 - 7hL|r_1^2 - \beta_{21}| + 6|\beta_{21}| - 2hL|\beta_{21}||r_1^2 - \beta_{21}| + hL|\beta_{21} - 4| + hL|\beta_{21}||\beta_{21} - 4| \\ +hL|r_2^2 + 2r_2 - \beta_{21} + 1| + hL|\beta_{21}||r_2^2 + 2r_2 - \beta_{21} + 1|. \end{aligned} \tag{47}$$

The constant C is referred as the convergence factor. Thus, the iterative process (39) is convergent if $C < 1$. Now, we state the following theorem for β_{21}.

Theorem 3 *If the sequence* $\{y_{n+1}^{(j)}\}$ *for* $\beta_{21} = 0$ *given by (38)and(40) is bounded by a constant* C *and the condition*

$$\begin{aligned} hL &< \frac{-2R_3 + 2\sqrt{R_3^2 + 6R_4}}{R_4} \\ R_3 &= 4 + 3r_1^2 \\ R_4 &= r_2^2 + 2r_2 - 7r_1^2 + 5 \end{aligned} \tag{48}$$

is satisfied, where $r_1, r_2 \in (0,1]$ *and* $L = \max\{L_1, L_2\}$*. Then, the extended third order method is convergent.*

5 Numerical Simulations

In this section, we present various examples of constant, variable delay and steady state DDEs; stiff and non-stiff problems to show the efficiency of EOSMs for such types of DDEs. We compare our results with the numerical results obtained by DDE23 [31] which are based on the explicit RK schemes.

Example 1 *Consider the logistic DDE [32]*

$$\begin{aligned} y\prime(t) &= -\lambda y(t-1)(1+y(t)), \quad && t \geq 0, \\ \phi(t) &= 1, && t \leq 0. \end{aligned} \tag{49}$$

This problem has been suggested as a mathematical description of a fluctuating population of organism and control systems (see e.g. [33] and [34]). The exact solution of this problem is unknown. Figures 5-7 show the numerical simulations obtained by EOSMs and those of DDE23 with different values of $\lambda = 1.5$, 2.5 *and* 3, *respectively and for tolerance* $= 10^{-3}$*. For* $\lambda = 1.5$*, the results are almost identical. For* $\lambda = 2.5$ *and* 3*, the solution obtained by DDE23 starts to diverge away from the correct solution while the solution obtained by EOSMs still meets the known numerical solution [34].*

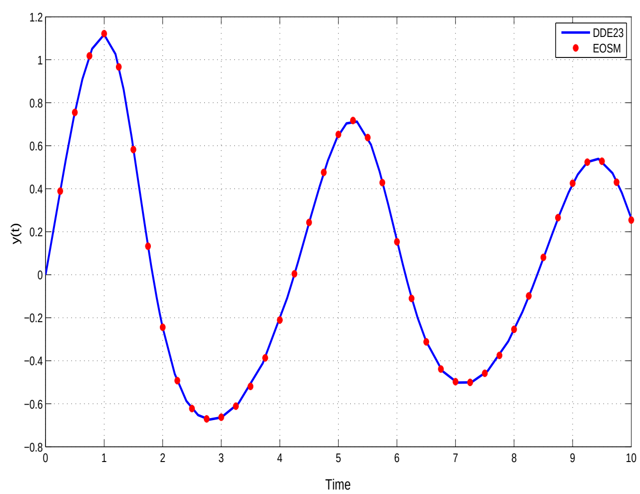

Figure 5: Solution for non-linear eqn (49)($\lambda = 1.5$) by EOSMs versus DDE23 [31].

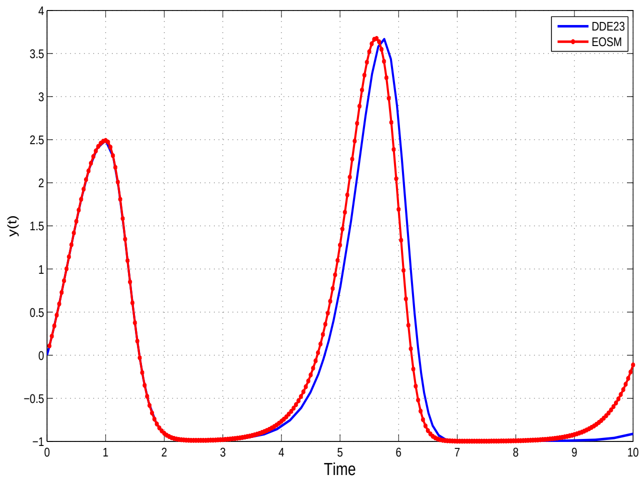

Figure 6: Solution for (49) ($\lambda = 2.5$) by EOSMs versus DDE23.

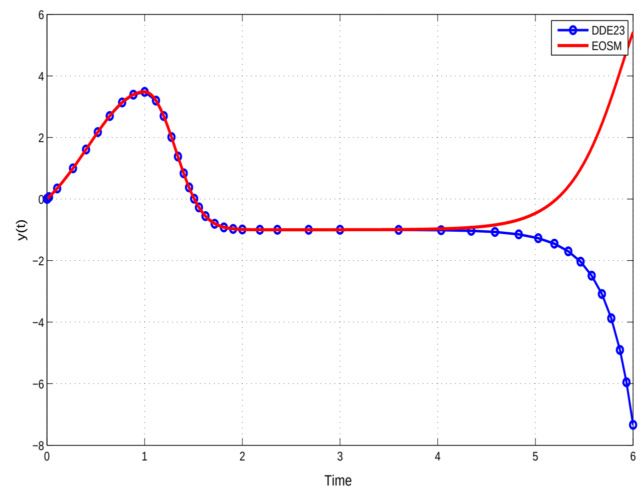

Figure 7: Solution for (49) ($\lambda = 3$) by EOSMs versus DDE23.

Example 2 *Consider the stiff DDE [32] of the form*

$$
\begin{aligned}
& y\prime(t) = -1000y(t) + q\ y(t-1) + c, && t \geq 0, \\
S1: \quad & q = 997e^{-3}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-3t)}, && t \geq 0, \textit{(exact solution)} \\
S1: \quad & q = 999e^{-1}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-t)}, && t \geq 0, \textit{(exact solution)} \\
S1: \quad & q = 999.99e^{-0.01}, \quad c = 1000 - q \\
& y(t) = 1 + e^{(-0.01t)}, && t \geq 0, \textit{(exact solution)}.
\end{aligned} \tag{50}
$$

This problem is derived from the linear stability DDE test equation. The choice of parameters *produces a stiff DDE. comparing the exact solution with the numerical results obtained by EOSMs for different values for the parameter q, we note that, both are identical (see Figs. 8-10)*

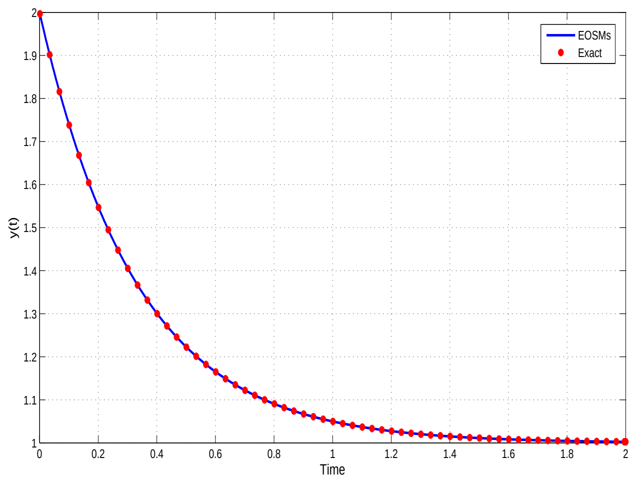

Figure 8: Solution for (50) ($q = 997e^{-3}$) using EOSMs versus exact solution.

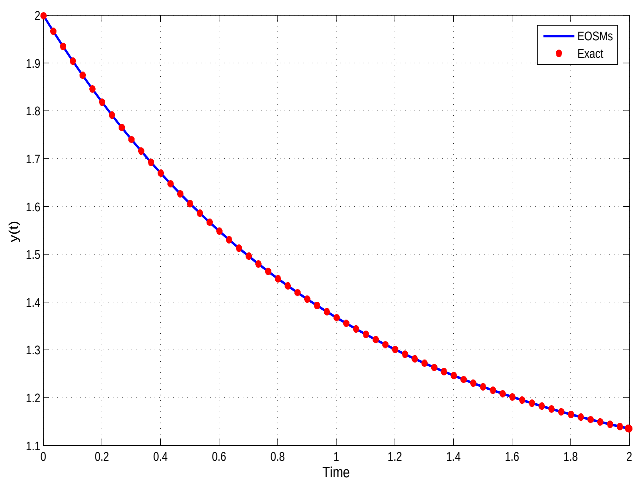

Figure 9: Solution for (50) ($q = 999e^{-1}$) using EOSMs versus exact solution.

Example 3 *Consider the varying-delay scalar DDE [32] of the form*

$$\begin{aligned} y\prime(t) &= 1 - y(1 - \boldsymbol{exp}(1 - \frac{1}{t})), \qquad & t \geq 1, \\ \phi(t) &= \ln(t), & 0 < t \leq 1, \end{aligned} \tag{51}$$

and analytical solution:

$$y(t) = \boldsymbol{ln}(t), t > 0$$

The numerical simulations are given in Figure 11.

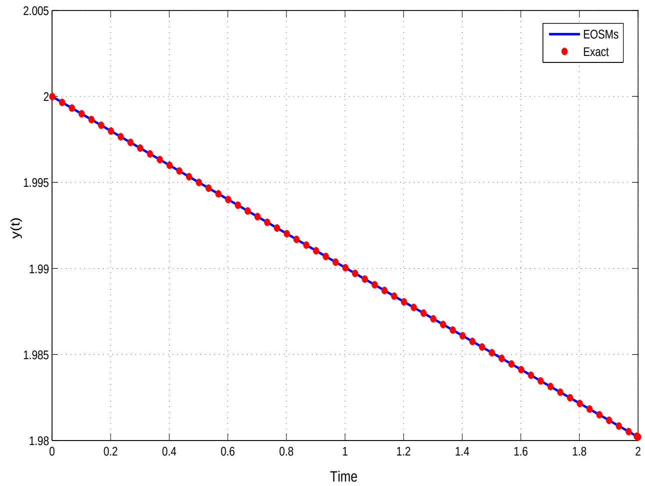

Figure 10: Solution for (50) ($q = 999.99e^{-0.01}$) using EOSMs versus exact solution.

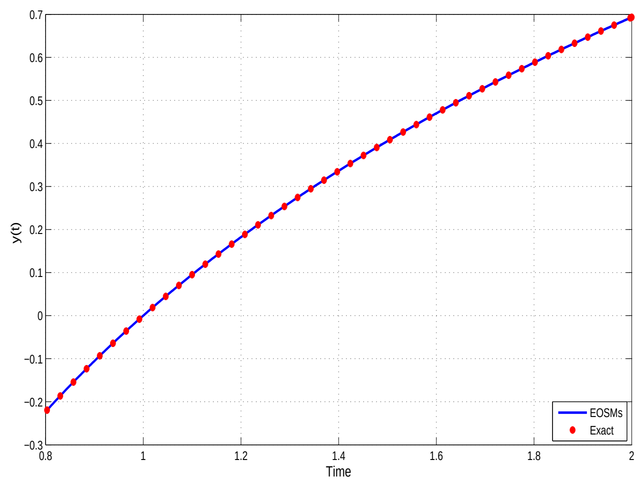

Figure 11: Solution for variable delay DDE (51) using EOSMs versus exact solution.

Example 4 *Finally (See Figure 12), consider the more general state-dependent DDE [32] of the form*

$$\begin{aligned} y\prime(t) &= \frac{y(y(t)-\sqrt{2}+1)}{2\sqrt{t}}, & t \geq 1, \\ \phi(t) &= 1, & 0 < t \leq 1. \end{aligned} \tag{52}$$

and analytical solution

$$y(t) = \sqrt{t}, \qquad 1 \leq t \leq 2.$$

Example 5 *We extend our analysis to solve a system of DDEs [32] of the form*

$$\begin{aligned} y\prime_1(t) &= y_2(t) \\ y\prime_2(t) &= 1 - y_2(t-1) - y_1(t) \end{aligned} \qquad t \geq 0, \tag{53}$$

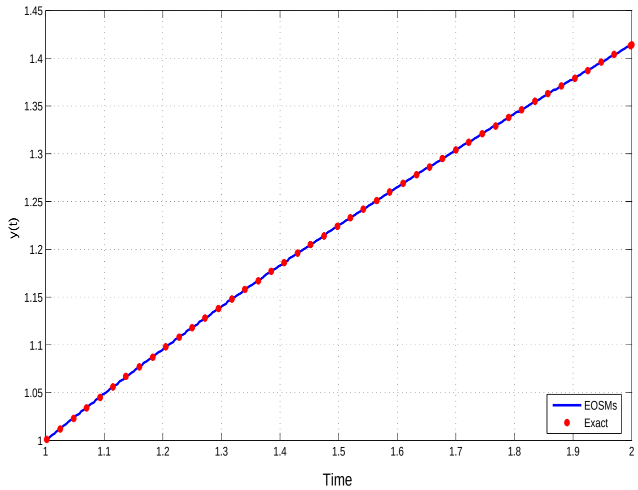

Figure 12: Solution for state dependent DDE (52) using EOSMs versus exact solution

with

$$y(0) = [y_1(0), y_2(0)]^T = [0, 0]^T, \qquad t \leq 0$$

and analytical solution

$$y_1(t) = \begin{cases} 1 - \cos(t), & 0 \leq t \leq 1; \\ 1 - \cos(t) + \frac{1}{2}(t-1) + \cos(t-1)\frac{1}{2}\sin(t-1), & 1 \leq t \leq 2. \end{cases}$$

$$y_2(t) = \begin{cases} \sin(t), & 0 \leq t \leq 1; \\ \sin(t) + \frac{1}{2}(1-t)\sin(t-1), & 1 \leq t \leq 2. \end{cases}$$

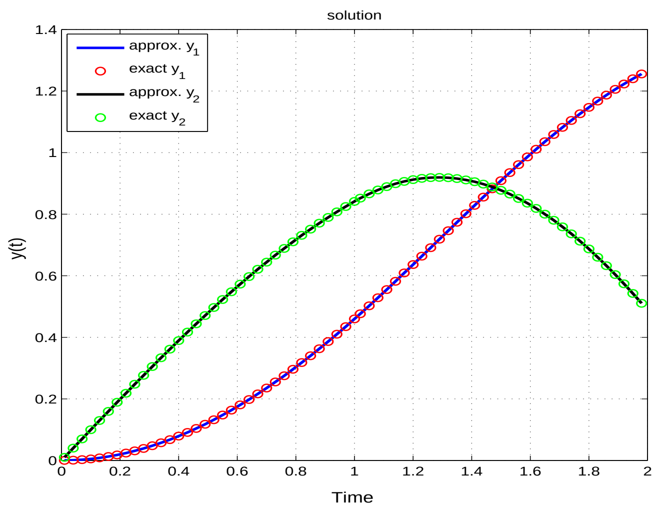

Figure 13: Solution for the system of DDEs (53) using EOSMs versus exact solution.

6 Conclusion

In this paper, a general form of extended one-step method for solving various types of DDEs has been provided. Stability properties of such schemes have been investigated. The suggested method is suitable and efficient for both non-stiff and stiff delay differential equations. It has been shown that the results obtained by EOSM schemes of order 3 are better, compared with those obtained by DDE23 Matlab code, which is based on explicit Runge-Kutta schemes, when solving stiff problems. This comparison emphasizes on the fact that, even if the explicit method gives a solution for the stiff problem, the computational cost for explicit methods is higher than that of EOSM method in case of stiff problem. The accuracy of explicit scheme is very low compared to those of EOSM. The suggested schemes may be able to solve many challenging stiff initial value problems in biology, chemistry and optimal control which cannot be solved by explicit schemes. The software of the given schemes of EOSM will be available very soon in a technical report.

This work is extendable to solve stiff and non-stiff integro-delay differential equations and also with variable step-size.

References

[1] C. T. H. Baker, G. Bocharov, E. Parmuzin, F. A. Rihan, Some aspects of causal & neutral equations used in modelling, J. Comput. Appl. Math. 229 (2009) 335–349.

[2] C. T. H. Baker, G. Bocharov, F. A. Rihan, Neutral delay differential equations in the modelling of cell growth, J. Egypt. Math. Soc. 16 (2) (2009) 133–160.

[3] C. T. H. Baker, G. Bocharov, C. Paul, F. Rihan, Models with delay for cell population dynamics: Identification, selection and analysis, Appl. Num. Math. 53 (2005) 107–129.

[4] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, New York, 2003.

[5] F. A. Rihan, Sensitivity analysis of dynamic systems with time lags, J. Comput. Appl. Math. 151 (2003) 445–462.

[6] F. A. Rihan, Numerical Treatment of Delay Differential Eguations in Bioscience, Ph.D. Thesis. University of Manchester, U.K., 2000.

[7] H. ZivaiPiran, Efficient Simulation, Accurate Sensitivity Analysis and Reliable Parameter Estimation for Delay Differential Equations, PhD. Thesis, University of Toronto (Canada), 2009.

[8] F. A. Rihan, B. F. Rihan, Numerical modelling of biological systems with memory using delay differential equations, Appl. Math. & Inf. Sci. 9 (3) (2015) 1615–1658.

[9] G. Bocharov, F. A. Rihan, Numerical modelling in biosciences using delay differential equations, J. Comput. Appl. Math. 125 (2000) 183–199.

[10] C. F. Curtiss, J. O. Hirschfelder, Integration of stiff equations, Proc Natl Acad Sci U S A. 38 (3) (1952) 235–2343.

[11] C. W. Gear, The automatic integration of stiff ordinary differential equations, in: A.J.H. Morrell, Ed. Information Processmg 68 (North-Holland, Amsterdam (1968) 187– 193.

[12] B. Ehle, J. Lawson, Generalized Runge-Kutta processes for stiff initial-value problems, J. Inst. Math, Appl. 16 (1975) 11–21.

[13] W. Enright, P. Muir, A Runge Kutta Boundary Value ODE Solver With Defect Control, Technical Report 267/93, June 1993, Department of Computer Science, University of Toronto.

[14] E. Hairer, G. Wanner, Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems, Springer-Verlag, Berlin, 1996.

[15] K. Dekker, J. Verwer, Stability of Runge- Kutta Methods for Stiff Nonlinear Differential Equations, CWI Monographs 2, Amsterdam, 1984.

[16] R. Aiken, ed., Stiff Computation, Oxford University Press, Oxford, U.K., 1985.

[17] K. Strehmel, R. Weiner, H. Claus, Stability analysis of linearly implicit one-step interpolation methods for stiff retarded differential equations, SIAM J. Numer. Anal. 26 (5) (1989) 1158–1174.

[18] G. A. Bocharov, G. I. Marchuk, A. A. Romanyukha, Numerical solution by LMMs of a stiff delay-differential system modelling an immune response, Numer. Math. 73 (1996) 131–148.

[19] N. Guglielmi, E. Hairer, Implementing Radau IIa methods for stiff delay differential equations, Computing 67 (2001) 1–12.

[20] N. Guglielmi, E. Hairer, Computing breaking points in implicit delay differential equations, Adv. Comput. Math. 29 (3) (2008) 229–247.

[21] C. T. H. Baker, C. A. H. Paul, Issues in the numerical solution of evolutionary delay differential equations, J. Adv. Comput. Math. 3 (1995) 171–196.

[22] G. Kirlinger, Linear multistep methods applied to stiff initial value problems, J. Math. Comput. Mod. 40 (2004) 1181–1192.

[23] I. B. Jacques, Extende one-step methods for the numerical solution of ordinary differential equations, Intern. J. Computer Math. 29 (1989) 247–255.

[24] M. M. Chawla, M. A. Al-Zannaidi, Stabilized fourth order extended methods for the numerical solution of ODEs, Computers Maths. Intern. J. Computer. Math. 52 (1994) 99–107.

[25] M. M. Chawla, M. A. Al-Zannaidi, M. S. Al-Sahhar, Class of stabilized extended one-step methods for the numerical solution of ODEs, Computers Math. Applic. 29 (1995) 79–84.

[26] J. D. Lambert, Computational methods in ordinary differentia equations, John Wiley, London, 1973.

[27] R. A. Usmani, R. P. Agarwal, An a-stable extended trapezoidal rule for numerical integration of ordinary equations, Computers Maths. Applic. 11 (1985) 1183–1191.

[28] R. D. Driver, Ordinary and Delay Differential equations, Springer-Verlag, New York, 1977.

[29] F. Ibrahim, A. Salama, A. Ouazzi, S. Turek, Extended one-step methods for solving delay-differential equations, Applied Mathematics and Information Sciences 8 (2014) 941–948.

[30] V. K. Barwell, Special stability problems for functional differential equations, BIT 15 (1975) 130–135.

[31] L. F. Shampine, S. Thompson, Solving DDEs in Matlab, Appl. Numer. Math. 37 (2001) 441–458.

[32] C. A. H. Paul, A test set of functional differential equations, Manchester Univ. Rep.No.243.

[33] H. J. Oberle, H. J. Pesch, Numerical treatment of delay differential equations by hermite interpolation, Numer. Math. 37 (1981) 235–255.

[34] G. Jones, On the nonlinear differential-difference equation $f\prime(t) = -\alpha f(t-1)(1+f(t)$, J. Math. Anal. Appl. 4 (1962) 440–469.