On Multigrid Convergence for Quadratic Finite Elements

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Abstract

Quadratic and higher order finite elements are interesting candidates for the numerical solution of (elliptic) partial differential equations (PDEs) due to their improved approximation properties in comparison to linear approaches. While the systems of equations that arise from the discretisation of the underlying PDEs are often solved by iterative schemes like preconditioned Krylow-space methods, multigrid solvers are still rarely used due the higher effort that is associated with the realization of appropriate smoothing and intergrid transfer operators. However, numerical tests indicate that quadratic FEM can provide even better convergence rates than linear finite elements: If m denotes the number of smoothing steps, the convergence rates behave asymptotically like $\mathcal{O}(\frac{1}{m^2})$ in contrast to $\mathcal{O}(\frac{1}{m})$ for linear FEM. We prove this new convergence result for quadratic conforming finite elements in a multigrid solver.

1   Introduction

In this paper, we analyse quadratic FEM if applied to elliptic 2nd order PDEs. We modify the classical multigrid proof of Hackbusch/Braess to obtain a sharper result for this situation. Moreover, the analysis indicates that these results may be valid for even higher order FEM, too, leading to our conjecture that multigrid convergence rates might further improve for higher order elements. The paper is organized as follows: In section 2 we introduce our notations. We formulate the smoothing property for the multigrid algorithm, as it was already formulated by other authors [1, 3, 7], and we repeat some of the the key ingredients of the classical multigrid W-cycle proof. Section 3 specialises this proof for the situation of quadratic finite elements and points out a possible generalisation for higher order finite elements. Finally, in section 4 we perform numerical tests which show that the results of the proof are sharp.

2   Notation and key ingredients

We consider a typical selfadjoint elliptic boundary value problem in a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$, for instance:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

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2.1 Assumptions and notations. *a)* *Throughout this paper, $c, c' > 0$ denote generic constants that can vary from equation to equation; $(\cdot, \cdot)$ represents the standard L_2 product.*

b) $\mathcal{T} = \{\mathcal{T}_h\}$, $\mathcal{T}_h \subseteq \Omega$ *with a mesh size parameter $h > 0$ denotes a family of uniform decompositions of Ω in the sense of [3], i.e. there is $\kappa > 0$ so that every $T \in \mathcal{T}_h$ contains a ball with radius $\rho_h \geq \frac{h}{\kappa}$ for every h.*

c) *Let $V := H_0^1(\Omega)$, and $\{V_h\}$ with $V_h \subset V$ denotes a nested family of affine conforming finite elements in the sense of [3]. This means, $V_{2h} \subset V_h \subset V$, V_h has dimension $n = n_h$ and two elements $T_1 \neq T_2 \in \mathcal{T}_h$ intersect at most in common corners or edges.*

d) *Let $a(\cdot, \cdot)$ be a positive definite, symmetric bilinear form, for instance*

$$a(v, w) \quad = \quad (\nabla v, \nabla w) \qquad \forall \, v, w \in V,$$

that is H_0^1-coercive and continuous, i.e. there exist $c, \alpha > 0$ such that

$$|a(v, w)| \leq c\|v\|_1 \|w\|_1, \qquad a(v, v) \geq \alpha\|v\|_1^2 \qquad \forall v, w \in H_0^1(\Omega).$$

By definition, the norm induced by this bilinear form $\|v\|_a := \sqrt{a(v, v)}$ is equivalent to the H^1-norm, i.e. with appropriate constants $c, c' > 0$ for all $v \in V$, the following estimate holds:

$$c\|v\|_1 \quad \leq \quad \|v\|_a \quad \leq \quad c'\|v\|_1 \quad (\text{or shorter: } \|v\|_1 \sim \|v\|_a)$$

e) *The problem is to find a weak solution $u \in V$ of the boundary value problem*

$$a(u, \varphi) \quad = \quad (f, \varphi) \qquad \forall \, \varphi \in V \tag{2.2}$$

for a given $f \in L_2(\Omega)$.

f) *This problem is replaced by a discrete formulation: Find $u_h \in V_h$ such that*

$$a(u_h, \varphi_h) \quad = \quad (f, \varphi_h) \qquad \forall \, \varphi_h \in V_h. \tag{2.3}$$

g) *We assume the problem to be $H^3(\Omega)$-regular and quadratic finite elements to be used.*

Next, we formulate a standard multigrid algorithm which follows the two-grid approach in the usual way by recursively replacing the solution process of the coarse grid problem:

2.2 Algorithm (One multigrid iteration step on V_h).
Purpose: For an iterate u_h^0 and a right hand side f_h, compute a new iterate u_h^* approximating the solution $u_h \in V_h$. The algorithm is configured by three parameters: The pair (μ, ν) denotes the number of pre- and postsmoothing steps, while $p \in \mathbb{N}$ configures the cycle ($p = 2$ results in the W-cycle). The operator $\mathcal{S}_h : V_h \to V_h$ denotes a special mapping, called *smoothing operator*, and $h_{\max}$ is the resolution of the coarsest grid in the family $\{V_h\}$. Set $f_h := f$ on the finest mesh level h.

FUNCTION MultigridCycle(h, u_h^0, f_h) **:** u_h^*

BEGIN

a) *Coarse grid solution:* If $h = h_{\max}$, then solve the coarse grid problem

$$a(u_{h_{\max}}, \varphi_{h_{\max}}) = (f_{h_{\max}}, \varphi_{h_{\max}}) \qquad \forall\, \varphi_{h_{\max}} \in V_{h_{\max}}.$$

Return $u_h^* = u_{h_{\max}}$. Otherwise

b) *Presmoothing:* Compute $\quad u_h^i := \mathcal{S}_h u_h^{i-1}, \quad i = 1, ..., \mu.$

c) *Coarse grid correction:* Set up the *coarse grid problem* to find $u_{2h} \in V_{2h}$,

$$a(u_{2h}, \varphi_{2h}) = (f_{2h}, \varphi_{2h}) \qquad \forall\, \varphi_{2h} \in V_{2h},$$

for the function $f_{2h} \in V$ defined by

$$(f_{2h}, \varphi_{2h}) := (f_h, \varphi_{2h}) - a(u_h^\mu, \varphi_{2h}) \qquad \forall\, \varphi_{2h} \in V_{2h}.$$

Set $u_{2h}^0 := 0$. Solve the coarse grid problem approximately and recursively with p multigrid steps on the lower level and correct the current approximative solution:

$$
\begin{aligned}
u_{2h}^i &:= \textbf{MultigridCycle}(2h, u_{2h}^{i-1}, f_{2h}) \qquad i = 1, ..., p, \\
u_h^{\mu+1} &:= u_h^\mu + u_{2h}^p
\end{aligned}
$$

d) *Postsmoothing:* Compute $\quad u_h^{\mu+1+i} := \mathcal{S}_h u_h^{\mu+i}, \quad i = 1, ..., \nu.$
Return $u_h^* := u_h^{\mu+\nu+1}.$

END MultigridCycle

In the following, we only focus on the case of $\mu > 0$ and $\nu = 0$, thus ignoring any postsmoothing. Furthermore, we restrict ourselves to the case of a two-grid algorithm, as the multigrid proof for the W-cycle follows by a perturbation argument (cf. [1, 3, 4, 5]). The assumptions above allow to formulate a couple of (algebraic) statements that hold for all conforming FEM spaces. We will briefly summarise the results in the following. For additional and more detailed information, see [1, 3, 5].

2.3 Definitions (Scale of norms). *We define a linear continuous operator* $\mathcal{A}_h : V_h \to V_h$ *by:*

$$(\mathcal{A}_h v_h, w_h) = a(v_h, w_h) \qquad \forall\, v_h, w_h \in V_h \tag{2.4}$$

For $s \in \mathbb{R}$ and $v_h = \sum c_i \psi_h^i \in V_h$ with $\{\psi_h^i\} \subset V_h$ a set of orthonormal eigenfunctions to eigenvalues $\{\lambda_i\}$ of $\mathcal{A}_h$, using a spectral decomposition [1], we define the following scale of norms:

$$|||v_h|||_s := \sqrt{(\mathcal{A}_h^s v_h, v_h)} = \sqrt{\sum_i \lambda_i^s |c_i|^2} \tag{2.5}$$

2.4 Remarks. a) For $v_h \in V_h$, there is the norm equivalence

$$|||v_h|||_1 = \sqrt{(\mathcal{A}_h v_h, v_h)} = \sqrt{a(v_h, v_h)} = \|v_h\|_a \sim \|v_h\|_1, \tag{2.6}$$

and by definition, there holds

$$|||v_h|||_0 = \|v_h\|_0. \tag{2.7}$$

b) For $v_h \in V_h$ and $s \in \mathbb{R}$, the symmetry of the matrix leads to

$$|||\mathcal{A}_h^{\frac{s}{2}} v_h|||_0 = \sqrt{(\mathcal{A}_h^{\frac{s}{2}} v_h, \mathcal{A}_h^{\frac{s}{2}} v_h)} = \sqrt{(\mathcal{A}_h^s v_h, v_h)} = |||v_h|||_s. \tag{2.8}$$

c) For $r, t \in \mathbb{R}$, $s = \frac{r+t}{2}$ and $v_h, w_h \in V_h$ we have the *logarithmic convexity*:

$$|(\mathcal{A}_h^s v_h, w_h)| = |(\mathcal{A}_h^{\frac{r}{2}} v_h, \mathcal{A}_h^{\frac{t}{2}} w_h)| \leq |||v_h|||_r \, |||w_h|||_t \tag{2.9}$$

d) The eigenvalues $\{\lambda\}$ of $\mathcal{A}_h$ and eigenfunctions $v_h \in V_h$ which are determined by the generalised eigenvalue problem

$$a(v_h, \varphi_h) = \lambda(v_h, \varphi_h) \qquad \forall \varphi_h \in V_h$$

can be bounded as follows (cf. [1]):

$$c \leq \lambda(\mathcal{A}_h) \leq c' h^{-2}$$

One of the key ingredients of the multigrid proof is a characterisation of the *smoothing operator*:

2.5 Lemma (Smoothing property of damped Richardson iteration). *Let* $\lambda_{max} = \lambda_{max}(\mathcal{A}_h)$ *denote the maximal eigenvalue of the operator* $\mathcal{A}_h$ *and* $u_h \in V_h$ *the exact solution of problem (2.3). Furthermore, with arbitrary* $u_h^0 \in V_h$, *for* $i \in \mathbb{N}_0$ *we define the* damped Richardson smoother $\mathcal{S}_h : V_h \to V_h$ *with* $u_h^{i+1} := \mathcal{S}_h u_h^i$ *being the solution of*

$$(u_h^{i+1}, \varphi_h) = (u_h^i, \varphi_h) - \lambda_{max}^{-1}((f, \varphi_h) - a(u_h^i, \varphi_h)) \qquad \forall \varphi_h \in V_h.$$

Then, with $e_i = e_h^i := u_h^i - u_h$, *for arbitrary* $s, t \in \mathbb{R}$, $s \geq t$, *the Richardson smoother satisfies the* smoothing property, *i.e. after* n *Richardson smoothing steps:*

$$|||e_n|||_s \leq \frac{c}{n^{\frac{s-t}{2}}} h^{-(s-t)} |||e_0|||_t \tag{2.10}$$

Proof: Following [1, 3], let $\{z_i\} \subset V_h$ denote an orthonormal basis of eigenvectors of V_h, $\{\lambda_i\}$ the corresponding eigenvalues and $e_0 = \sum_i c_i z_i$ the representation of e_0 w.r.t. this basis. By induction,

$$e_n = \sum_i (1 - \lambda_i/\lambda_{\max})^n c_i z_i.$$

Then, from $0 < \lambda_i/\lambda_{\max} \le 1$, it follows:

$$
\begin{aligned}
|||e_n|||_s^2 \overset{(2.5)}{=} \; (\mathcal{A}_h^s e_n, e_n) &= \sum_i \lambda_i^s [(1 - \lambda_i/\lambda_{\max})^n c_i]^2 \\
&= \lambda_{\max}^{s-t} \sum_i (\lambda_i/\lambda_{\max})^{s-t} [(1 - \lambda_i/\lambda_{\max})^{2n}] \, \lambda_i^t c_i^2 \\
&\le \lambda_{\max}^{s-t} \max_{0 \le \xi \le 1} [\xi^{s-t}(1 - \xi)^{2n}] \sum_i \lambda_i^t c_i^2
\end{aligned}
$$

Now using $\lambda_{\max}^{s-t} \le ch^{-2(s-t)}$, $\max_{0\le\xi\le1}[\xi^{s-t}(1-\xi)^{2n}] \le 1/n^{s-t}$ and $\sum_i \lambda_i^t c_i^2 = |||e_0|||_t^2$, taking the square root we obtain the desired result. $\qquad\square$

2.6 Corollary. *After $\mu \in \mathbb{N}$ smoothing steps, we obtain:*

$$
|||e_\mu|||_3 \;\le\; \frac{c}{\mu^2} h^{-4} |||e_0|||_{-1} \tag{2.11}
$$

The other key ingredient of the multigrid proof is the analysis of the *coarse grid correction*:

2.7 Definition (Coarse grid operator). *The* coarse grid operator *is defined as the mapping* $\mathcal{P}_h^{2h} : V_h \to V_{2h}$ *with*

$$
a(\mathcal{P}_h^{2h} v_h, v_{2h}) = a(v_h, \mathcal{I}_{2h}^h v_{2h}) \qquad \forall\, v_h \in V_h, \; v_{2h} \in V_{2h} \tag{2.12}
$$

where $\mathcal{I}_{2h}^h : V_{2h} \to V_h$, $\mathcal{I}_{2h}^h v_{2h} = v_{2h} \in V_h \;\forall\, v_{2h} \in V_{2h}$ *denotes the natural injection.*

2.8 Remarks. The coarse grid correction in step c) of the given two-grid algorithm is defined as $u_h^{\mu+1} = u_h^\mu + u_{2h}$ with $u_{2h} = -\mathcal{P}_h^{2h}(e_h^\mu)$. Subtracting u_h leads to

$$
e_h^{\mu+1} = e_h^\mu - \mathcal{P}_h^{2h} e_h^\mu \tag{2.13}
$$

and with the definition of $\mathcal{I}_{2h}^h$, this leads to the following essential *orthogonality relation*:

$$
a(e_h^{\mu+1}, \varphi_{2h}) = a(e_h^\mu - \mathcal{P}_h^{2h} e_h^\mu, \varphi_{2h}) = 0 \qquad \forall\, \varphi_{2h} \in V_{2h} \tag{2.14}
$$

3 Approximation property for quadratic FEM

The aim of this section is to proof the approximation property

$$
|||e_h^{\mu+1}|||_{-1} \le ch^4 |||e_h^\mu|||_3,
$$

which can be combined with the smoothing property (2.11) to obtain the two grid convergence characterisation for quadratic finite elements:

$$
|||e_h^{\mu+1}|||_{-1} \le \frac{c}{\mu^2} |||e_h^0|||_{-1}
$$

We start our investigation with the following Lemma:

3.1 Lemma.

$$|||e_h^{\mu+1}|||_1^2 \;\leq\; |||e_h^{\mu+1}|||_{-1}|||e_h^{\mu}|||_3 \tag{3.1}$$

Proof: Using the logarithmic convexity and the orthogonality of the Ritz-projection $a(e_h^{\mu+1}, \varphi_{2h}) = 0 \; \forall \varphi_{2h} \in V_{2h}$ leads to:

$$|||e_h^{\mu+1}|||_1^2 \quad = \quad a(e_h^{\mu+1}, e_h^{\mu} - \mathcal{P}_h^{2h} e_h^{\mu}) \quad = \quad a(e_h^{\mu+1}, e_h^{\mu}) \quad = \quad (\mathcal{A}_h e_h^{\mu+1}, e_h^{\mu})$$

$$\overset{(2.9)}{\leq} \quad |||e_h^{\mu+1}|||_{-1}|||e_h^{\mu}|||_3$$

$\square$

Next, we formulate an essential inequality for the $||| \cdot |||_{-1}$-norm:

3.2 Proposition (Norm estimate). *There is a constant $c > 0$ such that for all $v_h \in V_h$ the following inequality holds:*

$$|||v_h|||_{-1} \;\leq\; c\,\|v_h\|_{-1} \tag{3.2}$$

Proof: The $\|\cdot\|_{-1}$ norm is defined by $\|v\|_{-1} = \sup_{\varphi \in H^1(\Omega)} \frac{(v,\varphi)}{\|\varphi\|_1}$. Without loss of generality, we assume $v_h \neq 0$. Then, we can write:

$$|||v_h|||_{-1}^2 \quad \overset{(2.5)}{=} \quad (v_h, \mathcal{A}_h^{-1} v_h) \;=\; \frac{(v_h, \mathcal{A}_h^{-1} v_h)}{\|\mathcal{A}_h^{-1} v_h\|_1}\, \|\mathcal{A}_h^{-1} v_h\|_1$$

$$\overset{\mathcal{A}_h^{-1} v_h \in V_h \subset H^1(\Omega)}{\leq} \quad \sup_{w \in H^1(\Omega)} \frac{(v_h, w)}{\|w\|_1}\, \|\mathcal{A}_h^{-1} v_h\|_1 \quad \overset{\text{Def.}}{=} \quad \|v_h\|_{-1}\, \|\mathcal{A}_h^{-1} v_h\|_1$$

$$\overset{(2.6)}{\leq} \quad c\,\|v_h\|_{-1}\, |||\mathcal{A}_h^{-1} v_h|||_1 \quad \overset{(2.8)}{=} \quad c\,\|v_h\|_{-1}\, |||v_h|||_{-1}$$

$\square$

Furthermore, we need a duality argument for quadratic finite elements:

3.3 Proposition (Duality argument for quadratic finite elements). *Let $u_h \in V_h$ be the piecewise quadratic approximation to $u \in V$ in the sense of (2.3). Then, the following estimate holds for a constant $c > 0$:*

$$\|u - u_h\|_{-1} \leq ch^2 \|u - u_h\|_1 \tag{3.3}$$

Proof: For arbitrary $g \in H^1(\Omega)$, let $z \in V$ be the solution of the auxiliary problem

$$a(z, \varphi) = (g, \varphi) \qquad \forall\, \varphi \in V.$$

As the primal problem is assumed to be $H^3(\Omega)$-regular, this dual problem is also $H^3(\Omega)$-regular, so we have $z \in H^3(\Omega)$. Let $z_h \in V_h$ the piecewise quadratic solution of the corresponding discrete problem

$$a(z_h, \varphi_h) = (g, \varphi_h) \qquad \forall\, \varphi_h \in V_h.$$

By using $\varphi := u - u_h \in V$, $a(u - u_h, \varphi_h) = 0 \; \forall \varphi_h \in V_h$ and the Bramble-Hilbert Lemma for quadratic finite elements, we derive

$$(g, u - u_h) \quad = \quad a(z - z_h, u - u_h) \;\leq\; ch^2 \|z\|_3\, \|u - u_h\|_1 \;\leq\; ch^2 \|g\|_1\, \|u - u_h\|_1$$

for all $g \in H^1(\Omega)$. Therefore, we obtain

$$\|u - u_h\|_{-1} = \sup_{g \in H^1(\Omega)} \frac{(u - u_h, g)}{\|g\|_1} \leq ch^2 \|u - u_h\|_1.$$

$\square$

3.4 Corollary. *By formally setting setting $V := V_h$, $u := e_h^\mu \in V_h$, $u_h := \mathcal{P}_h^{2h} e_h^\mu \in V_{2h}$ we obtain with (2.13)*

$$\|e_h^{\mu+1}\|_{-1} \leq ch^2 \ \|e_h^{\mu+1}\|_1. \tag{3.4}$$

After these preparations, we can formulate the desired approximation property:

3.5 Proposition (Approximation property). *Assuming $H^3(\Omega)$-regularity for problem (2.2) and $\mu \in \mathbb{N}$ large enough, the error $e_h^{\mu+1}$ after one two-grid cycle behaves like*

$$|||e_h^{\mu+1}|||_{-1} \leq ch^4 |||e_h^\mu|||_3 \tag{3.5}$$

for a constant $c > 0$.

Proof: By the preceding lemmas and the duality argument for quadratic finite elements it follows

$$|||e_h^{\mu+1}|||_{-1} \overset{(3.2)}{\leq} c\|e_h^{\mu+1}\|_{-1} \overset{(3.3)}{\leq} ch^2\|e_h^{\mu+1}\|_1 \overset{(2.6)}{\leq} ch^2|||e_h^{\mu+1}|||_1$$

and therefore

$$|||e_h^{\mu+1}|||_{-1}^2 \leq ch^4|||e_h^{\mu+1}|||_1^2 \overset{(3.1)}{\leq} ch^4|||e_h^{\mu+1}|||_{-1} \ |||e_h^\mu|||_3.$$

Cancelling redundant terms gives the desired formula. $\square$

Finally, we combine the previous results and achieve the proof for the optimal two-grid convergence for quadratic finite elements:

3.6 Theorem (Two-grid convergence with quadratic finite elements). *Assuming $H^3(\Omega)$-regularity for problem (2.2) and $\mu \in \mathbb{N}$ large enough, the error $e_h^{\mu+1}$ after one two-grid cycle behaves like*

$$|||e_h^{\mu+1}|||_{-1} \leq \frac{c}{\mu^2}|||e_h^0|||_{-1} \tag{3.6}$$

for a constant $c > 0$.

Proof:

$$|||e_h^{\mu+1}|||_{-1} \overset{(3.5)}{\leq} ch^4|||e_h^\mu|||_3 \overset{(2.11)}{\leq} ch^4\frac{c}{\mu^2}h^{-4}|||e_h^0|||_{-1}$$

$\square$

The convergence of the W-cycle multigrid algorithm with the same inequality for the error follows again by a perturbation argument similar to the linear case [1, 3, 4, 5].

3.7 Remarks. We finish this section with some final remarks about the case of higher order approximations. For this purpose, we assume the problem (2.2) to be even $H^{s+1}(\Omega)$-regular

with $s > 2$ and the discretisation to be done with piecewise polynomials of order s. Then, formula (3.6) suggests the inequality

$$|||e_h^{\mu+1}|||_{1-s} \leq \frac{c}{\mu^s}|||e_h^0|||_{1-s}$$

in this situation. To derive this, we take $t := s - 1$ and assume the inequality

$$\|v_h\|_t \leq c |||v_h|||_t \qquad \forall\, v_h \in V_h \tag{3.7}$$

which holds for $t = 0, 1$ even in the stronger sense (2.6) and (2.7). The same technique as in Proposition 3.2 allows to derive the inequality

$$|||v_h|||_{-t} \leq c \|v_h\|_{-t} \qquad \forall\, v_h \in V_h.$$

With the help of a similar duality argument, this leads to

$$|||e_h^{\mu+1}|||_{1-s}^2 \;\leq\; c\|e_h^{\mu+1}\|_{1-s}^2 \;\leq\; ch^{2s}\|e_h^{\mu+1}\|_1^2 \;\leq\; ch^{2s}|||e_h^{\mu+1}|||_{1-s}|||e_h^{\mu}|||_{1+s}.$$

Then, using the general formulation of the smoothing property (2.10) would indeed result in the above optimal two-grid convergence for finite element approximations of order s.

4   Numerical examples

In this section we perform some numerical tests in order to illustrate the above results. For this purpose, we define the *smoothing efficiency index*:

4.1 Definition (Smoothing efficiency index). *For $m \in \mathbb{N}$ we denote by Φ_m the multigrid algorithm with m smoothing steps. Let $\rho(\Phi_i)$ denote the asymptotic convergence rate which is measured from the last three multigrid iterations in the solution process. After a sufficiently large number of iterations, this is approximately a constant number. Then, we define*

$$G(i,j) := \left(\frac{\rho(\Phi_i)}{\rho(\Phi_j)}\right)^{\frac{1}{t}} \tag{4.1}$$

where $t := \log_2(j/i)$, $i, j \in \mathbb{N}$, $i < j$.

Thus, the smoothing efficiency index describes approximatively the mean improvement of the convergence rate when doubling the number of smoothing steps. In particular, if $j = 2^k i$ for $k \in \mathbb{N}$, $t = k$ gives the number of doubling. We expect $G(i,j) \approx 2$ for an approximation with linear finite elements and $G(i,j) \approx 4$ if quadratic finite elements are used. For the numerical experiments below, we use $k = 3$.

4.2 Example. We make the following experiment: On the unit square $\Omega = [0,1]^2 \subset \mathbb{R}^2$ we numerically solve

$$a(u,v) = (\nabla u, \nabla v) = (f,v) \quad \forall\, v \in V.$$

With $u_{|\partial\Omega} := 0$, following [6, p. 203] this problem is $H^{4-\alpha}$-regular, $0 < \alpha < 1$. As right hand side, we prescribe f corresponding to the analytical solution $u(x,y) := \sin(xy)\,\sin((1-x)(1-y))$. We use a two-grid algorithm on level 6/7, i.e. we regularly refine the unit square 5

and 6 times, respectively. For implementational reasons, we use quadrilateral finite elements, as we make use of the discretisation and multigrid techniques in the FEAT library [2]. The iteration is stopped if the relative residual (measured in the l_2-norm) drops below 10^{-28}, or if a maximum of 30 two-grid iterations is reached. Instead of the Richardson smoother as used for the proof, we apply the damped Jacobi-iteration as smoother (with damping parameter $\omega = 0.7$).

Table 4.1 shows the resulting number of iteration steps ITE, convergence rates ρ and smoothing efficiency indices $G(\cdot, \cdot)$, corresponding to the number of presmoothing steps m (no postsmoothing is performed). For this table, we use the (quadrilateral) bilinear space Q_1 and the biquadratic space Q_2 for the spatial discretisation. As intergrid transfer operator we use basically the natural injection. In the last two columns we make a test with Q_2 where we use the *piecewise bilinear interpolation on a once more refined mesh*.

When the natural injection is used, the predicted results agree very well with the results from theory: For Q_1 we obtain a smoothing efficiency index of ≈ 2, while for Q_2 we obtain the predicted value ≈ 4. Nevertheless, the last two columns where we used only the *piecewise bilinear interpolation on a once more refined mesh* as grid transfer operator show a quite different behavior. It is clearly seen that the property of "doubling the number of smoothing steps quarters the convergence rate" is lost in this case! Only a factor of 2 can be seen here. So, as a rule of thumb we can state what many practitioners have already observed in numerical simulations:

Using biquadratic finite elements with piecewise bilinear grid transfer in multigrid, the convergence rates asymptotically behave like $\mathcal{O}(1/m)$, m denoting the number of smoothing steps, thus loosing the optimal convergence rates for quadratic finite elements.

m	Q_1, bilinear interp.		Q_2, biquadratic interp.		Q_2, bilinear interp.	
	ρ	**G(m/8,m)**	ρ	**G(m/8,m)**	ρ	**G(m/8,m)**
4	7,712E-02		1,426E-01		2,931E-01	
6	5,282E-02		8,112E-02		2,258E-01	
8	4,022E-02		6,391E-02		1,808E-01	
12	2,719E-02		4,032E-02		1,287E-01	
16	2,044E-02		2,554E-02		1,032E-01	
24	1,340E-02		1,011E-02		7,368E-02	
32	1,030E-02	1,957	4,971E-03	3,062	5,702E-02	1,726
48	6,536E-03	2,007	1,990E-03	3,442	3,916E-02	1,793
64	5,026E-03	2,000	1,167E-03	3,797	2,983E-02	1,823
96	2,757E-03	2,144	5,515E-04	4,181	2,021E-02	1,854
128	2,200E-03	2,102	3,370E-04	4,232	1,503E-02	1,901
192	1,463E-03	2,092	1,576E-04	4,003	1,032E-02	1,926

Table 4.1: Numerical test, Q_1 with bilinear interpolation (i.e. natural injection), Q_2 with biquadratic interpolation (i.e. natural injection) and Q_2 with bilinear interpolation

5 Conclusions

We have shown that using quadratic finite elements for the discretisation of PDEs is not only advantageous for the accuracy, but also for the solution process of the discretised linear systems using standard (geometrical) multigrid algorithms. In fact, the convergence rates behave like $\mathcal{O}(1/m^2)$, for m being the number of smoothing steps, in contrast to the factor $\mathcal{O}(1/m)$ which is well-known for linear FEM. This property can be clearly seen in practical examples, but only if *'fully biquadratic grid transfer'*, i.e. natural injection, is used; a lower order interpolation destroys this property which unfortunately happens in many existing codes. The theoretical considerations indicate that this result could be extended to even higher order finite elements. If this is the case, geometrical multigrid solvers for the FEM discretisation of elliptic PDEs could asymptotically lead to much faster convergence rates of order $\mathcal{O}(1/m^s)$, with s denoting the polynomial order.

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