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Abstract. We develop a general convergence theory for adaptive discontinuous Galerkin methods for elliptic PDEs covering the popular SIPG, NIPG and LDG schemes as well as all practically relevant marking strategies. Another key feature of the presented result is, that it holds for penalty parameters only necessary for the standard analysis of the respective scheme. The analysis is based on a quasi interpolation into a newly developed limit space of the adaptively created non-conforming discrete spaces, which enables to generalise the basic convergence result for conforming adaptive finite element methods by Morin, Siebert, and Veeser [A basic convergence result for conforming adaptive finite elements, Math. Models Methods Appl. Sci., 2008, 18(5), 707–737].

1. Introduction

Discontinuous Galerkin finite element methods (DGFEM) have enjoyed considerable attention during the last two decades, especially in the context of adaptive algorithms (ADGMs): the absence of any conformity requirements across element interfaces characterizing DGFEM approximations allows for extremely general adaptive meshes and/or an easy implementation of variable local polynomial degrees in the finite element spaces. There has been a substantial activity in recent years for the derivation of a posteriori bounds for discontinuous Galerkin methods for elliptic problems [KP03, BHL03, Ain07, HSW07, CGJ09, EV09, ESV10, ZGHS11, DPE12]. Such a posteriori estimates are an essential building block in the context of adaptive algorithms, which typically consist of a loop

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}. \tag{1.1}$$

The convergence theory, however, for the 'extreme' non-conformity case of ADGMs had been a particularly challenging problem due to the presence of a negative power of the mesh-size h stemming from the discontinuity-penalization term. As a consequence, the error is not necessarily monotone under refinement. Indeed, consulting the unprecedented developments of convergence and optimality theory of conforming adaptive finite element methods (AFEMs) during the last two decades, the strict reduction of some error quantity appears to be fundamental for most of the results. In fact, Dörfler's marking strategy typically ensures that the error is uniformly reduced in each iteration [Dör96, MNS00, MNS02] and leads to optimal convergence rates [Ste07, CKNS08, KS11, DK08, BDK12]; compare also with the monographs [NSV09, CFP14] and the references therein. Showing that the error reduction is proportional to the estimator on the refined elements, instance optimality of an adaptive finite element method was shown recently for an AFEM with modified marking strategy in [DKS16, KS16]. A different approach was, however, taken in

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[MSV08, Sie11], where convergence of the AFEM is proved, exploiting that the approximations converge to a solution in the closure of the adaptively created finite element spaces in the trial space together with standard properties of the a posteriori bounds. The result covers a large class of inf-sup stable PDEs and all practically relevant marking strategies without yielding convergence rates though.

Karakashian and Pascal [KP07] gave the first proof of convergence for an adaptive DGFEM based on a symmetric interior penalty scheme (SIPG) with Dörfler marking for Poisson's problem. Their proof addresses the challenge of negative power of h in the estimator, by showing that the discontinuity-penalization term can be controlled by the element and jump residuals only, provided that the DGFEM discontinuity-penalisation parameter, henceforth denoted by σ, is chosen to be *sufficiently large*; the element and jump residuals involve only positive powers of h and, therefore, can be controlled similarly as for conforming methods. The optimality of the adaptive SIPG was shown in [BN10]; see also [HKW09].

The standard error analysis of the SIPG requires that σ is sufficiently large for the respective bilinear from to be coercive with respect to an energy-like norm. It is not known in general, however, whether the choice of σ required for coercivity of the interior penalty DGFEM bilinear form is large enough to ensure that the discontinuity-penalization term can be controlled by the element and jump residuals only. Therefore, the convergence of SIPG is still open for values of σ large enough for coercivity but, perhaps, *not large enough* for the crucial result from [KP07] to hold. To the best of our knowledge, the only result in this direction is the proof of convergence of a weakly overpenalized ADGM for linear elements [GG14], utilizing the intimate relation between this method and the lowest order Crouzeix-Raviart elements.

This work is concerned with proving that the ADGM converges for *all* values of σ for which the method is coercive, thereby settling the above discrepancy between the magnitude of σ required for coercivity and the, typically much larger, values required for proof of convergence of ADGM. Apart from settling this open problem theoretically, this new result has some important consequences in practical computations: it is well known that as σ grows, the condition number of the respective stiffness matrix also grows. Therefore, the magnitude of the discontinuity-penalization parameter σ affects the performance of iterative linear solvers, whose complexity is also typically included in algorithmic optimality discussions of adaptive finite elements. In addition, the theory presented here includes a large class of practically relevant marking strategies and covers popular discontinuous Galerkin methods like the local discontinuous Galerkin method (LDG) and even the *nonsymmetric* interior penalty method (NIPG), which are coercive for any $\sigma > 0$. Moreover, we expect that it can be generalised to non-conforming discretisations for a number of other problems like the Stokes equations or fourth order elliptic problems. However, as for the conforming counterpart [MSV08], no convergence rates are guaranteed.

The proof of convergence of the ADGM, discussed below, is motivated by the basic convergence for the conforming adaptive finite element framework of Morin, Siebert and Veeser [MSV08]. More specifically, we extend considerably the ideas from [MSV08] and [Gud10] to be able to address the crucial challenge that the limits of DGFEM solutions, constructed by the adaptive algorithm, do not necessarily belong to the energy space of the boundary value problem as well as to conclude convergence from a perturbed best approximation result.

To highlight the key theoretical developments without the need to resort to complicated notation, we prefer to focus on the simple setting of the Poisson problem with essential homogeneous boundary conditions and conforming shape regular triangulations. We believe, however, that the results presented below are valid for general elliptic PDEs including convection and reaction phenomena as well as for some classes of non conforming meshes; compare with [BN10].

The remainder of this work is structured as follows. In Section 2 we shall introduce the ADGM framework for Poisson's equation and state the main result, which is then proved in Section 5 after some auxiliary results, needed to generalise [MSV08], are provided in Sections 3 and 4. In particular, in Section 3 a space is presented, which is generated from limits of discrete discontinuous functions in the sequence of discontinuous Galerkin spaces constructed by ADGM. Section 4 is then concerned with proving that the sequence of discontinuous Galerkin solutions produced by ADGM converges indeed to a generalised Galerkin solution in this limit space. This follows from an (almost) best-approximation property, generalising the ideas in [Gud10].

2. The ADGM and the main result

Let a measurable set ω and a $m \in \mathbb{N}$. We consider the Lebesgue space $L^2(\omega;\mathbb{R}^m)$ of square integrable functions over ω with values in $\mathbb{R}^m$, with inner product $\langle\cdot,\,\cdot\rangle_\omega$ and associated norm $\|\cdot\|_\omega$. We also set $L^2(\omega) := L^2(\omega;\mathbb{R})$. The Sobolev space $H^1(\omega)$ is the space of all functions in $L^2(\omega)$ whose weak gradient is in $L^2(\omega;\mathbb{R}^d)$, for $d \in \mathbb{N}$. Thanks to the Poincaré-Friedrichs' inequality, the closure $H^1_0(\omega)$ of $C^\infty_0(\omega)$ in $H^1(\omega)$ is a Hilbert space with inner product $\langle\nabla\cdot,\,\nabla\cdot\rangle_\omega$ and norm $\|\nabla\cdot\|_\omega$. Also, we denote the dual space $H^{-1}(\omega)$ of $H^1_0(\omega)$, with the norm $\|v\|_{H^{-1}(\omega)} := \sup_{w\in H^1_0(\omega)} \frac{\langle v,\,w\rangle}{\|\nabla w\|_\omega}$, $v \in H^{-1}(\omega)$, with dual brackets defined by $\langle v,\,w\rangle := v(w)$, for $w \in H^1_0(\omega)$.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz domain. We consider the Poisson problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.1}$$

with $f \in L^2(\Omega)$. The weak formulation of (2.1) reads: find $u \in H^1_0(\Omega)$, such that

$$\langle\nabla u,\,\nabla v\rangle_\Omega = \langle f,\,v\rangle_\Omega \qquad \text{for all } v \in H^1_0(\Omega). \tag{2.2}$$

From the Riesz representation theorem, it follows that the solution u exists and is unique.

2.1. Discontinuous Galerkin method.

Let $\mathcal{G}$ be a conforming (that is, not containing any hanging nodes) subdivision of Ω into disjoint closed simplicial elements E so that $\bar{\Omega} = \bigcup\{E : E \in \mathcal{G}\}$ and set $h_E := |E|^{1/d}$. Let $\mathcal{S} = \mathcal{S}(\mathcal{G})$ be the set of $(d-1)$-dimensional element faces S associated with the subdivision $\mathcal{G}$ including $\partial\Omega$, and let $\mathring{\mathcal{S}} = \mathring{\mathcal{S}}(\mathcal{G}) \subset \mathcal{S}$ by the subset of interior faces only. We also introduce the *mesh size* function $h_\mathcal{G} : \Omega \to \mathbb{R}$, defined by $h_\mathcal{G}(x) := h_E$, if $x \in E\backslash\partial E$ and $h_\mathcal{G}(x) = h_S := |S|^{1/(d-1)}$, if $x \in S \in \mathcal{S}$ and set $\Gamma = \Gamma(\mathcal{G}) = \bigcup\{S : S \in \mathcal{S}\}$ and $\mathring{\Gamma} = \mathring{\Gamma}(\mathcal{G}) = \bigcup\{S : S \in \mathring{\mathcal{S}}\}$. We assume that $\mathcal{G}$ is derived by iterative or recursive newest vertex bisection of an initial conforming mesh $\mathcal{G}_0$; see [Bän91, Kos94, Mau95, Tra97]. We denote by $\mathbb{G}$ the family of shape regular triangulations consisting of such subdivisions of $\mathcal{G}_0$.

Let $\mathcal{P}_r(E)$ denote the the space of all polynomials on E of degree at most $r \in \mathbb{N}$, we define the discontinuous finite element space

$$\mathbb{V}(\mathcal{G}) := \prod_{E\in\mathcal{G}} \mathbb{P}_r(E) \subset \prod_{E\in\mathcal{G}} W^{1,p}(E) =: W^{1,p}(\mathcal{G}), \quad 1 \leqslant p \leqslant \infty, \tag{2.3}$$

and $H^1(\mathcal{G}) := W^{1,2}(\mathcal{G})$. Let $\mathcal{N} = \mathcal{N}(\mathcal{G})$ be the set of Lagrange nodes of $\mathbb{V}(\mathcal{G})$ and define the neighbourhood of a node $z \in \mathcal{N}(\mathcal{G})$ by $N_\mathcal{G}(z) := \{E' \in \mathcal{G} : z \in E'\}$, and the union of its elements by $\omega_\mathcal{G}(z) = \bigcup\{E' \in \mathcal{G} : z \in E'\}$. We also define the corresponding neighbourhoods for all elements $E \in \mathcal{G}$ by $N_\mathcal{G}(E) := \{E' \in \mathcal{G} : E \cap E' \neq \varnothing\}$ and $\omega_\mathcal{G}(E) = \bigcup\{E' \in \mathcal{G} : E' \cap E \neq \varnothing\} = \bigcup\{\omega_\mathcal{G}(z) : z \in \mathcal{N}(E) \cap E\}$, respectively, and set $\omega_\mathcal{G}(S) := \bigcup\{E \in \mathcal{G} : S \subset E\}$; compare with Figure 1. The numbers of neighbours $\#N_\mathcal{G}(z)$ and $\#N_\mathcal{G}(E)$ are uniformly bounded for all $z \in \mathcal{N}$, respectively $E \in \mathcal{G}$, depending on the shape regularity of $\mathcal{G}$ and, thus, on $\mathcal{G}_0$.

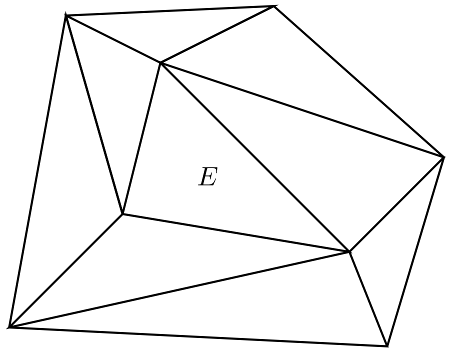

FIGURE 1. The neighbourhood $N_{\mathcal{G}}(E)$ of some $E \in \mathcal{G}$.

Let E^+, E^- be two generic elements sharing a face $S := E^+ \cap E^- \in \mathring{\mathcal{S}}$ and let $\boldsymbol{n}^+$ and $\boldsymbol{n}^-$ the outward normal vectors of E^+ respectively E^- on S. For $q : \Omega \to \mathbb{R}$ and $\boldsymbol{\phi} : \Omega \to \mathbb{R}^d$, let $q^\pm := q|_{S\cap\partial E^\pm}$ and $\boldsymbol{\phi}^\pm := \boldsymbol{\phi}|_{S\cap\partial E^\pm}$, and set

$$\{q\}|_S := \frac{1}{2}(q^+ + q^-), \qquad \{\boldsymbol{\phi}\}|_S := \frac{1}{2}(\boldsymbol{\phi}^+ + \boldsymbol{\phi}^-),$$
$$[\![q]\!]\,|_S := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \qquad [\![\boldsymbol{\phi}]\!]\,|_S := \boldsymbol{\phi}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\phi}^- \cdot \boldsymbol{n}^-;$$

if $S \subset \partial E \cap \partial\Omega$, we set $\{\boldsymbol{\phi}\}|_S := \boldsymbol{\phi}^+$ and $[\![q]\!]\,|_S := q^+\boldsymbol{n}^+$.

In order to define the discontinuous Galerkin schemes, we introduce the following local lifting operators. For $S \in \mathcal{S}$, we define $R^S_{\mathcal{G}} : L^2(S)^d \to \prod_{E\in\mathcal{G}} \mathbb{P}_\ell(E)^d$ and $L^S_{\mathcal{G}} : L^2(S) \to \prod_{E\in\mathcal{G}} \mathbb{P}_\ell(E)^d$ by

$$\int_\Omega R^S_{\mathcal{G}}(\boldsymbol{\phi}) \cdot \boldsymbol{\tau}\,\mathrm{d}x = \int_S \boldsymbol{\phi} \cdot \{\boldsymbol{\tau}\}\,\mathrm{d}s \qquad \forall \boldsymbol{\tau} \in \prod_{E\in\mathcal{G}} \mathbb{P}_\ell(E)^d \tag{2.4a}$$

and

$$\int_\Omega L^S_{\mathcal{G}}(q) \cdot \boldsymbol{\tau}\,\mathrm{d}x = \int_S q\,[\![\boldsymbol{\tau}]\!]\,\mathrm{d}s \qquad \forall \boldsymbol{\tau} \in \prod_{E\in\mathcal{G}} \mathbb{P}_\ell(E)^d, \tag{2.4b}$$

with $\ell \in \{r, r+1\}$. Note that $L^S_{\mathcal{G}}(q)$ and $R^S_{\mathcal{G}}(\boldsymbol{\phi})$ vanish outside $\omega_{\mathcal{G}}(S)$. Moreover, using the local definition and the boundedness of the lifting operators in a reference situation together with standard scaling arguments, we have for $\boldsymbol{\phi} \in \mathbb{P}_r(S)^d$ and $q \in \mathbb{P}_r(S)$ that

$$\|L^S_{\mathcal{G}}(\boldsymbol{\phi})\|_\Omega \lesssim \left\|h_{\mathcal{G}}^{-1/2}\boldsymbol{\phi}\right\|_S \quad \text{and} \quad \|R^S_{\mathcal{G}}(q)\|_\Omega \lesssim \left\|h_{\mathcal{G}}^{-1/2}q\right\|_S; \tag{2.4c}$$

compare with [ABCM02]. Also, here and below we write $a \lesssim b$ when $a \leqslant Cb$ for a constant C not depending on the local mesh size of $\mathcal{G}$ or other essential quantities for the arguments presented below. Observing that the sets $\omega_{\mathcal{G}}(S)$, $S \in \mathcal{S}$ do overlap at most $d+1$ times, we have for the global lifting operators $R_{\mathcal{G}} : L^2(\Gamma)^d \to \mathbb{V}(\mathcal{G})^d$ and $L_{\mathcal{G}} : L^2(\mathring{\Gamma}) \to \mathbb{V}(\mathcal{G})^d$ defined by

$$R_{\mathcal{G}}(\boldsymbol{\phi}) := \sum_{S\in\mathcal{S}} R^S_{\mathcal{G}}(\boldsymbol{\phi}) \qquad \text{and} \qquad L_{\mathcal{G}}(q) := \sum_{S\in\mathring{\mathcal{S}}} R^S_{\mathcal{G}}(q),$$

that

$$\|R_{\mathcal{G}}([\![v]\!])\|_\Omega \lesssim \left\|h_{\mathcal{G}}^{-1/2}v\right\|_\Gamma \quad \text{and} \quad \|L_{\mathcal{G}}(\boldsymbol{\beta}\cdot[\![v]\!])\|_\Omega \lesssim |\boldsymbol{\beta}| \left\|h_{\mathcal{G}}^{-1/2}v\right\|_{\mathring{\Gamma}}$$

for all $v \in \mathbb{V}(\mathcal{G})$ and $\boldsymbol{\beta} \in \mathbb{R}^d$.

We define the bilinear form $\mathfrak{B}_{\mathcal{G}}[\cdot,\,\cdot]:\mathbb{V}(\mathcal{G})\times\mathbb{V}(\mathcal{G})\to\mathbb{R}$ by

$$
\begin{aligned}
\mathfrak{B}_{\mathcal{G}}[w,\,v] := &\int_{\mathcal{G}} \nabla w\cdot\nabla v\,\mathrm{d}x - \int_{\mathcal{S}} \big(\{\nabla w\}\cdot[\![v]\!] + \theta\{\nabla v\}\cdot[\![w]\!]\big)\,\mathrm{d}s \\
&+ \int_{\mathring{\mathcal{S}}} \big(\boldsymbol{\beta}\cdot[\![w]\!]\,[\![\nabla v]\!] + [\![\nabla w]\!]\,\boldsymbol{\beta}\cdot[\![v]\!]\big)\,\mathrm{d}s \\
&+ \int_{\Omega} \gamma\big(R_{\mathcal{G}}([\![w]\!]) + L_{\mathcal{G}}(\boldsymbol{\beta}\cdot[\![w]\!])\big)\cdot\big(R_{\mathcal{G}}([\![v]\!]) + L_{\mathcal{G}}(\boldsymbol{\beta}\cdot[\![v]\!])\big)\,\mathrm{d}x \\
&+ \int_{\mathcal{S}} \frac{\sigma}{h_{\mathcal{G}}}\,[\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s;
\end{aligned}
\tag{2.5}
$$

for $\theta\in\{\pm1\}$, $\gamma\in\{0,1\}$, $\boldsymbol{\beta}\in\mathbb{R}^d$ and $\sigma\geqslant 0$. Here we have used the short-hand notation

$$
\int_{\mathcal{G}}\cdot\,\mathrm{d}x := \sum_{E\in\mathcal{G}}\int_E \cdot\,\mathrm{d}x \qquad\text{and}\qquad \int_{\mathcal{S}}\cdot\,\mathrm{d}s := \sum_{S\in\mathcal{S}}\int_S \cdot\,\mathrm{d}s.
$$

We consider the choices $\theta=1$, $\boldsymbol{\beta}=\mathbf{0}$, and $\gamma=0$ yielding the symmetric interior penalty method (SIPG) [DD76], $\theta=-1$, $\boldsymbol{\beta}=\mathbf{0}$, and $\gamma=0$ which gives the nonsymmetric interior penalty methods (NIPG) [RWG99], and $\theta=1$, $\boldsymbol{\beta}\in\mathbb{R}^d$, and $\gamma=1$ which yields the local discontinuous Galerkin method (LDG) [CS98]; compare also with [ABCM02] and [JNS16].

In all three cases, the corresponding *discontinuous Galerkin finite element method* (DGFEM) then reads: find $u_{\mathcal{G}}\in\mathbb{V}(\mathcal{G})$ such that

$$
\mathfrak{B}_{\mathcal{G}}[u_{\mathcal{G}},\,v_{\mathcal{G}}] = \int_{\Omega} f v_{\mathcal{G}}\,\mathrm{d}x =: l(v_{\mathcal{G}}), \quad \text{for all}\ \ v_{\mathcal{G}}\in\mathbb{V}(\mathcal{G}).
\tag{2.6}
$$

Upon denoting by $\nabla_{\mathtt{pw}}v$ the piecewise gradient $\nabla_{\mathtt{pw}}v|_E=\nabla v|_E$ for all $E\in\mathcal{G}$, the corresponding energy norm $|\!|\!|\cdot|\!|\!|_{\mathcal{G}}$ is defined by

$$
|\!|\!|w|\!|\!|_{\mathcal{G}} := \Big(\left\|\nabla_{\mathtt{pw}}w\right\|_{\Omega}^2 + \bar\sigma\left\|h_{\mathcal{G}}^{-1/2}\,[\![w]\!]\right\|_{\Gamma}^2\Big)^{1/2},
$$

for $w|_E\in H^1(E)$, $E\in\mathcal{G}$. Here $\bar\sigma:=\max\{1,\sigma\}$. Also, for some subset $\mathcal{M}\subset\mathcal{G}$ with $\omega=\bigcup\{E\mid E\in\mathcal{M}\}$, we define

$$
|\!|\!|w|\!|\!|_{\mathcal{M}} := \Big(\left\|\nabla_{\mathtt{pw}}w\right\|_{\omega}^2 + \bar\sigma\left\|h_{\mathcal{G}}^{-1/2}\,[\![w]\!]\right\|_{\Gamma(\mathcal{M})}^2\Big)^{1/2}.
$$

If for SIPG we have $\sigma:=C_\sigma r^2$ for some constant $C_\sigma>0$ sufficiently large, $\sigma>0$ for NIPG and for LDG $\sigma>0$ when $\ell=r$ and $\sigma=0$ when $\ell=r+1$ ([JNS16]), then there exists $\alpha=\alpha(\sigma)>0$, such that

$$
\alpha\,|\!|\!|w|\!|\!|_{\mathcal{G}}^2 \leqslant \mathfrak{B}_{\mathcal{G}}[w,\,w] \qquad \forall w\in H^1(\mathcal{G}),
\tag{2.7}
$$

i.e. all three DGFEMs are coercive in $\mathbb{V}(\mathcal{G})$; see, e.g., [Arn82, ABCM02, JNS16] for details. Note that the choice $\bar\sigma=\max\{1,\sigma\}$ accounts for the fact that we can have $\sigma=0$ for the LDG in [JNS16].

From standard scaling arguments, we conclude the following local Poincaré-Friedrichs inequality from [Bre03, BO09].

Proposition 1 (Poincaré-$\mathbb{V}(\mathcal{G})$)**.** *Let $\mathcal{G}$ be a triangulation of Ω and $\mathcal{G}_\star$ some refinement of $\mathcal{G}$. Then, for $v\in\mathbb{V}(\mathcal{G}_\star)$, $E\in\mathcal{G}$ and $v_E:=|\omega_{\mathcal{G}}(E)|^{-1}\int_{\omega_{\mathcal{G}}(E)}v\,\mathrm{d}x$, we have*

$$
\|v-v_E\|_{\omega_{\mathcal{G}}(E)}^2 \lesssim \int_{\omega_{\mathcal{G}}(E)} h_{\mathcal{G}}^2|\nabla_{pw}v|^2\,\mathrm{d}x + \int_{S\in\mathcal{S}_\star,\,S\subset\omega_{\mathcal{G}}(E)} h_{\mathcal{G}}^2 h_{\mathcal{G}_\star}^{-1}\,[\![v]\!]^2\,\mathrm{d}s,
$$

where $\mathcal{S}_\star=\mathcal{S}(\mathcal{G}_\star)$ and the hidden constant depends on d and on the shape regularity of $N_{\mathcal{G}}(E)$.

The next important result from [KP03, Theorem 2.2] (compare also with [BN10, Lemma 6.9] and [BO09, Theorem 3.1]) quantifies the local distance of a discrete non-conforming function to the conforming subspace with the help of the of the scaled jump terms.

Proposition 2. *For $\mathcal{G} \in \mathbb{G}$, there exists an interpolation operator $\mathcal{I}_\mathcal{G} : H^1(\mathcal{G}) \to \mathbb{V}(\mathcal{G}) \cap H^1_0(\Omega)$, such that we have*

$$\left\| h_\mathcal{G}^{1/2}(v - \mathcal{I}_\mathcal{G} v) \right\|^2_{L^2(E)} + \|\nabla(v - \mathcal{I}_\mathcal{G} v)\|^2_{L^2(E)} \lesssim \int_{\partial E} h_\mathcal{G}^{-1} [\![v]\!]^2 \,\mathrm{d}s,$$

for all $E \in \mathcal{G}$ and $v \in \mathbb{V}(\mathcal{G})$.

From this, we can easily deduce the following broken Friedrichs type inequality; compare also with [BO09, (4.5)].

Corollary 3 (Friedrichs-$\mathbb{V}(\mathcal{G})$)**.** *Let $\mathcal{G} \in \mathbb{G}$, then*

$$\|v\|_{L^2(\Omega)} \lesssim |\!|\!| v |\!|\!|_\mathcal{G} \quad \text{for all } v \in \mathbb{V}(\mathcal{G}).$$

Let $BV(\Omega)$ denote the Banach space of functions with bounded variation equiped with the norm

$$\|v\|_{BV(\Omega)} = \|v\|_{L^1(\Omega)} + |Dv|(\Omega),$$

where Dv is the measure representing the distributional derivative of v with total variation

$$|Dv|(\Omega) = \sup_{\phi \in C^1_0(\Omega)^d, \|\phi\|_{L^\infty(\Omega)} \leqslant 1} \int_\Omega v \operatorname{div} \phi \,\mathrm{d}x.$$

Here the supremum is taken over the space $C^1_0(\Omega)^d$ of all vector valued continuously differentiable functions with compact support in Ω.

Another crucial result [BO09, Lemma 2] states then that the total variation of the distributional derivative of broken Sobolev functions is bounded by the discontiuous Galerkin norm.

Proposition 4. *For $\mathcal{G} \in \mathbb{G}$ we have that*

$$|Dv|(\Omega) \lesssim \|\nabla v\|_{L^1(\Omega)} + \int_\mathcal{S} |[\![v]\!]| \,\mathrm{d}s \lesssim |\!|\!| v |\!|\!|_\mathcal{G} \quad \text{for all } v \in H^1(\mathcal{G}).$$

2.2. **A posteriori error bound.** We recall the a posteriori results from [KP03, BN10, BGC05, BHL03]; compare also with [CGJ09].

For $v \in \mathbb{V}(\mathcal{G})$, we define the local error indicators for $E \in \mathcal{G}$ by

$$\mathcal{E}_\mathcal{G}(v, E) := \Big(\int_E h_\mathcal{G}^2 |f + \Delta v|^2 \,\mathrm{d}x + \int_{\partial E \cap \Omega} h_\mathcal{G} [\![\nabla v]\!]^2 \,\mathrm{d}s + \sigma \int_{\partial E} h_\mathcal{G}^{-1} [\![v]\!]^2 \,\mathrm{d}s \Big)^{1/2};$$

when $v = u_\mathcal{G}$, we shall write $\mathcal{E}_\mathcal{G}(E) := \mathcal{E}_\mathcal{G}(u_\mathcal{G}, E)$. Also, for $\mathcal{M} \subset \mathcal{G}$, we set

$$\mathcal{E}_\mathcal{G}(v, \mathcal{M}) := \Big(\sum_{E \in \mathcal{M}} \mathcal{E}(v, E)^2 \Big)^{1/2}.$$

Proposition 5. *Let $u \in H^1_0(\Omega)$ be the solution of* (2.2) *and $u_\mathcal{G} \in \mathbb{V}(\mathcal{G})$ its respective DGFEM approximation* (2.6) *on the grid $\mathcal{G} \in \mathbb{G}$. Then,*

$$\alpha\, |\!|\!| u - u_\mathcal{G} |\!|\!|^2_\mathcal{G} \leqslant \mathfrak{B}_\mathcal{G}[u - u_\mathcal{G},\, u - u_\mathcal{G}] \lesssim \sum_{E \in \mathcal{G}} \mathcal{E}_\mathcal{G}(E)^2,$$

The efficiency of the estimator follows with the standard bubble function technique of Verfürth [Ver96, Ver13]; compare also with [KP03, Theorem 3.2], [Gud10, Lemma 4.1] and Proposition 22 below.

Proposition 6. *Let $u \in H^1_0(\Omega)$ be the solution of* (2.2) *and let $\mathcal{G} \in \mathbb{G}$. Then, for all $v \in \mathbb{V}(\mathcal{G})$ and $E \in \mathcal{G}$, we have*

$$\int_E h_{\mathcal{G}}^2 |f + \Delta v|^2 \, dx + \int_{\partial E \cap \Omega} h_{\mathcal{G}} \, [\![\nabla v]\!]^2 \, ds \lesssim \|u - v\|^2_{\omega_{\mathcal{G}}(E)} + \|\nabla_{pw}(u - v)\|^2_{\omega_{\mathcal{G}}(E)} + \mathrm{osc}(N_{\mathcal{G}}(E), f)^2,$$

with data-oscillation defined by

$$\mathrm{osc}(\mathcal{M}, f) := \Big(\sum_{E' \in \mathcal{M}} \mathrm{osc}(E, f)^2 \Big)^{1/2}, \quad \textit{where} \quad \mathrm{osc}(E, f) := \inf_{f_E \in \mathbb{P}_{r-1}} \|h_{\mathcal{G}}(f - f_E)\|_E \,,$$

for all $\mathcal{M} \subset \mathcal{G}$. In particular, this implies

$$\mathcal{E}_{\mathcal{G}}(v, E) \lesssim |\!|\!| v - u |\!|\!|_{N_{\mathcal{G}}(E)} + \mathrm{osc}(N_{\mathcal{G}}(E), f).$$

Remark 7. *Note that the presented theory obviously applies to all locally equivalent estimators as well; compare e.g. with* [KP03, BN10, BGC05, BHL03, CGJ09]. *For the sake of a unified presentation, we restrict ourselves to the above representation.*

2.3. **Adaptive discontinuous Galerin finite element method (ADGM).** The adaptive algorithm, whose convergence will be shown below, reads as follows.

Algorithm 8 (ADGM)**.** Starting from an initial triangulation $\mathcal{G}_0$, the adaptive algorithm is an iteration of the following form

(1) $u_k = \mathsf{SOLVE}(\mathbb{V}(\mathcal{G}_k))$;
(2) $\{\mathcal{E}_k(E)\}_{E \in \mathcal{G}_k} = \mathsf{ESTIMATE}(u_k, \mathcal{G}_k)$;
(3) $\mathcal{M}_k = \mathsf{MARK}\big(\{\mathcal{E}_k(E)\}_{E \in \mathcal{G}_k}, \mathcal{G}_k\big)$;
(4) $\mathcal{G}_{k+1} = \mathsf{REFINE}(\mathcal{G}_k, \mathcal{M}_k)$; increment k.

Here we have used the notation $\mathcal{E}_k(E) := \mathcal{E}_{\mathcal{G}_k}(E)$, for brevity.

SOLVE. We assume that the output

$$u_{\mathcal{G}} = \mathsf{SOLVE}(\mathbb{V}(\mathcal{G}))$$

is the DGFEM approximation (2.6) of u with respect to $\mathbb{V}(\mathcal{G})$.

ESTIMATE. We suppose that

$$\{\mathcal{E}_{\mathcal{G}}(E)\}_{E \in \mathcal{G}} := \mathsf{ESTIMATE}(u_{\mathcal{G}}, \mathcal{G})$$

computes the error indicators from Section 2.2.

MARK. We assume that the output

$$\mathcal{M} := \mathsf{MARK}(\{\mathcal{E}_{\mathcal{G}}(E)\}_{E \in \mathcal{G}}, \mathcal{G})$$

of marked elements satisfies

$$\mathcal{E}_{\mathcal{G}}(E) \leqslant g(\mathcal{E}_{\mathcal{G}}(\mathcal{M})), \qquad \text{for all } E \in \mathcal{G} \backslash \mathcal{M}. \tag{2.8}$$

Here $g : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a fixed function, which is continuous in 0 with $g(0) = 0$, i.e. $\lim_{\epsilon \to 0} g(\epsilon) = 0$.

REFINE. We assume for $\mathcal{M} \subset \mathcal{G} \in \mathbb{G}$, that for the refined grid

$$\tilde{\mathcal{G}} := \mathsf{REFINE}(\mathcal{G}, \mathcal{M})$$

we have

$$E \in \mathcal{M} \quad \Rightarrow \quad E \in \mathcal{G} \backslash \tilde{\mathcal{G}}, \tag{2.9}$$

i.e., each marked element is refined at least once.

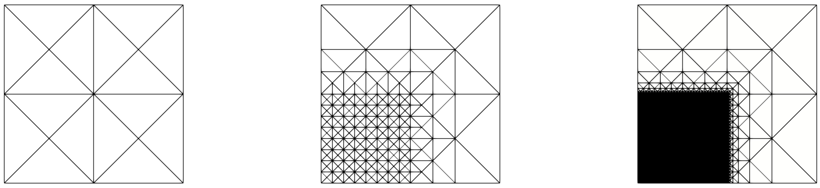

FIGURE 2. Selection of a sequence of triangulations of $\Omega = (0,1)^2$, where in each iteration the elements in $\Omega^- = [0,0.5] \times [0,0.5]$ are marked for refinement. The elements $\mathcal{G}^+$ in the remaining domain $\Omega \backslash \Omega^-$ are, after some iteration, not refined anymore. Moreover, after some iteration, their whole neighbourhood is not refined anymore.

2.4. **The main result.** The main result of this work states that the sequence of discontinuous Galerkin approxiations, produced by ADGM, converges to the exact solution of (2.1).

Theorem 9. *We have that*

$$\mathcal{E}_k(\mathcal{G}_k) \to 0 \quad \text{as } k \to \infty.$$

In particular, this implies that

$$\|u - u_k\|_k \to 0 \quad \text{as } k \to \infty.$$

3. A limit space and quasi-interpolation

In this section we shall first introduce a new limit space $\mathbb{V}_\infty$ of the sequence of adaptively constructed discontinuous finite element spaces $\{\mathbb{V}(\mathcal{G}_k)\}_{k \in \mathbb{N}}$. A new quasi-interpolation operator is then introduced in Section 3.3 in order to to prove that there exists a unique Galerkin solution u_∞ of a generalised discontinuous Galerkin problem in $\mathbb{V}_\infty$.

3.1. **Sequence of partitions.** The ADGM produces a sequence $\{\mathcal{G}_k\}_{k \in \mathbb{N}_0}$ of nested admissible partitions of Ω. Following [MSV08], we define

$$\mathcal{G}^+ := \bigcup_{k \geqslant 0} \bigcap_{j \geqslant k} \mathcal{G}_j, \qquad \text{and} \qquad \Omega^+ := \Omega(\mathcal{G}^+)$$

to be the set and domain of all elements, respectively, which eventually will not be refined any more; here $\Omega(X) := \text{interior}\left(\bigcup\{E : E \in X\}\right)$ for a collection of elements X. We also define the complementary domain $\Omega^- := \text{interior}(\Omega \backslash \Omega^+)$. For the ease of presentation, in what follows, we shall replace subscripts $\mathcal{G}_k$ by k to indicate the underlying triangulation, e.g. we write $N_k(E)$ instead of $N_{\mathcal{G}_k}(E)$.

The following result states that neighbours of elements in $\mathcal{G}^+$ are eventually also elements of $\mathcal{G}^+$; cf., [MSV08, Lemma 4.1].

Lemma 10. *For $E \in \mathcal{G}^+$ there exists a constant $K = K(E) \in \mathbb{N}_0$, such that*

$$N_k(E) = N_K(E) \qquad \text{for all } k \geqslant K,$$

i.e., we have $N_k(E) \subset \mathcal{G}^+$ for all $k \geqslant K$.

Next, for a fixed $k \in \mathbb{N}_0$, we set

$$\begin{aligned} \mathcal{G}_k^- &:= \{E \in \mathcal{G}_k : \omega_k(E) \subset \overline{\Omega^-}\}, & \Omega_k^- &:= \Omega(\mathcal{G}_k^-), \\ \mathcal{G}_k^+ &:= \mathcal{G}_k \cap \mathcal{G}^+, & \Omega_k^+ &:= \Omega(\mathcal{G}_k^+), \\ \mathcal{G}_k^{++} &:= \{E \in \mathcal{G}_k : N_k(E) \subset \mathcal{G}^+\}, & \Omega_k^{++} &:= \Omega(\mathcal{G}_k^{++}), \\ \mathcal{G}_k^\star &:= \mathcal{G}_k \backslash (\mathcal{G}_k^{++} \cup \mathcal{G}_k^-), & \Omega_k^\star &:= \Omega(\mathcal{G}_k^\star); \end{aligned}$$

compare also with Figure 2. This notation is also adopted for the corresponding faces, e.g., we denote $\mathcal{S}_k^+ := \mathcal{S}(\mathcal{G}_k^+)$ and $\mathring{\mathcal{S}}_k^+ := \mathring{\mathcal{S}}(\mathcal{G}_k^+)$ and correspondingly for all other above sub-triangulations of $\mathcal{G}_k$.

The next lemma is related to [MSV08, (4.15) and Corollary 4.1]. However, the definitions of $\Omega_k^\star$ and Ω_k^- differ from the corresponding ones in [MSV08], which requires some modifications in the proof.

Lemma 11. *We have that* $\lim_{k\to\infty} |\Omega_k^\star| = 0$ *and* $\lim_{k\to\infty} \|h_k \chi_{\Omega_k^-}\|_{L^\infty(\Omega)} = 0$*, with* $\chi_{\Omega_k^-}$ *denoting the characteristic function of* Ω_k^-.

Proof. In order to prove the first claim, we begin by observing that $|\Omega_k^\star| \leqslant |\Omega^- \backslash \Omega_k^-| + |\Omega^+ \backslash \Omega_k^{++}|$ and consider the two terms on the right-hand side separately.

Since $\#\mathcal{G}_k^+ < \infty$, we have thanks to Lemma 10 that for all $k \in \mathbb{N}$ there exists $K = K(k) \geqslant k$, such that $\mathcal{G}_K^{++} \supset \mathcal{G}_k^+$. Consequently, we have

$$|\Omega^+ \backslash \Omega_{K(k)}^{++}| \leqslant |\Omega^+ \backslash \Omega_k^+| = \sum_{E \in \mathcal{G}^+ \backslash \mathcal{G}_k^+} |E| \to 0,$$

as $k \to \infty$. This holds because the right-hand side is a tail of the series $\sum_{E \in \mathcal{G}^+} |E|$, which is convergent, as $|E| > 0$ and all partial sums are bounded by $|\Omega|$. Since $|\Omega^+ \backslash \Omega_k^{++}|$ is monotonically decreasing, we conclude that $|\Omega^+ \backslash \Omega_k^{++}| \to 0$ as $k \to \infty$.

We observe that the sequence $\{\Omega_k^-\}_{k\in\mathbb{N}}$ is nested, i.e. $\Omega_0^- \subset \Omega_1^- \subset \Omega_2^- \subset \ldots \subset \overline{\Omega^-}$. Therefore, we have that the sequence $\{|\Omega^- \backslash \Omega_k^-|\}_{k\in\mathbb{N}}$ is converging, because it is monotonically decreasing. Assume that $\lim_{k\to\infty} |\Omega^- \backslash \Omega_k^-| \neq 0$, then we have by the continuity of the Lebesgue measure that

$$0 \neq \lim_{k\to\infty} |\Omega^- \backslash \Omega_k^-| = \left|\Omega^- \backslash \bigcup_{k \geqslant 0} \Omega_k^-\right|.$$

Consequently, there exists a ball B_ρ with some radius $\rho > 0$ such that $B_\rho \subset \Omega^- \backslash \bigcup_{k\geqslant 0} \Omega_k^-$. For $k \in \mathbb{N}$ let $\mathcal{G}_k^{B_\rho} := \{E \in \mathcal{G}_k : E \cap B_\rho \neq \varnothing\}$, then there exists $E \in \mathcal{G}_k^{B_\rho}$ with $|E| \gtrsim \rho$ independent of k. This follows from the fact that, since $B_\rho \subset \Omega^- \backslash \Omega_k^-$, there exists no $E \in \mathcal{G}_k$ with $\Omega(N_k(E)) \subset B_\rho$, together with the local quasi uniformity of $\mathcal{G}_k$. Thanks to the fact that the size of an element is reduced under refinement by a factor $2^{-1/d}$ and that the grids $\mathcal{G}_k$ are nested, we have that there is some $K > 0$, such that there exists $E \in \mathcal{G}_k^{B^\rho}$ with $E \in \mathcal{G}_K$ for all $k \geqslant K$, i.e. $E \in \mathcal{G}^+$. This is the contradiction since $\varnothing \neq E \cap B^\rho \subset E \cap \Omega^-$.

The second claim follows from [MSV08, Corollary 4.1] noting that $\Omega^- \subset \Omega_k^0$ with Ω_k^0 as in [MSV08]. □

3.2. **The limit space.** In this section, we shall investigate the limit of the finite element spaces $\mathbb{V}_k := \mathbb{V}(\mathcal{G}_k)$, $k \in \mathbb{N}$. To this end, we define

$$\begin{aligned} \mathbb{V}_\infty := \big\{ v \in BV(\Omega) : v|_{\Omega^-} \in H^1_{\partial\Omega\cap\partial\Omega^-}(\Omega^-) \text{ and } v|_E \in \mathbb{P}_r \ \forall E \in \mathcal{G}^+ \\ \text{such that } \exists \{v_k\}_{k\in\mathbb{N}}, v_k \in \mathbb{V}_k \text{ with } \lim_{k\to\infty} \|v - v_k\|_k = 0 \\ \text{and } \limsup_{k\to\infty} \|v_k\|_k < \infty \big\}; \end{aligned}$$

here $H^1_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$ denotes the space of functions from $H^1_0(\Omega)$ restricted to Ω^-. Note that for $v \in BV(\Omega)$ there exists the L^1-trace of v on $\Gamma_k = \bigcup\{S : S \in \mathcal{S}_k\}$;

compare e.g. with the trace theorem [BO09, Theorem 4.2]. In other words, v is measurable with respect to the $(d-1)$-dimensional Hausdorff measure on $\mathcal{S}_k$ and, therefore, the term $|\!|\!|v|\!|\!|_k$, $v \in \mathbb{V}_\infty$, makes sense. Obviously, we have $\mathbb{V}_k \cap C(\Omega) \subset \mathbb{V}_\infty$ for all $k \in \mathbb{N}$ and, thus, $\mathbb{V}_\infty$ is not empty.

Setting $h_+ := h_{\mathcal{G}^+}$ and $\mathcal{S}^+ := \mathcal{S}(\mathcal{G}^+)$, we define

$$\langle v, w \rangle_\infty := \int_{\Omega^-} \nabla v \cdot \nabla w \,\mathrm{d}x + \int_{\mathcal{G}^+} \nabla v \cdot \nabla w \,\mathrm{d}x + \bar{\sigma} \int_{\mathcal{S}^+} h_+^{-1} [\![v]\!] [\![w]\!] \,\mathrm{d}s,$$

and $|\!|\!|v|\!|\!|_\infty := \langle v, v \rangle_\infty^{1/2}$, for all $v, w \in \mathbb{V}_\infty$. For brevity, we shall frequently use the notation

$$\int_\Omega \nabla_{\mathtt{pw}} v \cdot \nabla_{\mathtt{pw}} w \,\mathrm{d}x \equiv \int_{\Omega^-} \nabla v \cdot \nabla w \,\mathrm{d}x + \int_{\mathcal{G}^+} \nabla v \cdot \nabla w \,\mathrm{d}x.$$

We shall next list some basic properties of the space $\mathbb{V}_\infty$.

Proposition 12. *For $v \in \mathbb{V}_\infty$, we have*

$$|\!|\!|v|\!|\!|_k \nearrow |\!|\!|v|\!|\!|_\infty < \infty \qquad \text{as } k \to \infty.$$

In particular, for fixed $\ell \in \mathbb{N}$, let $E \in \mathcal{G}_\ell$; then, we have

$$\int_{\{S \in \mathcal{S}_k : S \subset E\}} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \nearrow \int_{\{S \in \mathcal{S}^+ : S \subset E\}} h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s, \quad \text{as } k \to \infty.$$

Proof. Since $v \in \mathbb{V}_\infty$, there exists $\{v_k\}_{k\in\mathbb{N}}, v_k \in \mathbb{V}_k$ with $\lim_{k\to\infty} |\!|\!|v - v_k|\!|\!|_k = 0$ and $\limsup_{k\to\infty} |\!|\!|v_k|\!|\!|_k < \infty$. We first observe that

$$|\!|\!|v|\!|\!|_k \leqslant |\!|\!|v - v_k|\!|\!|_k + |\!|\!|v_k|\!|\!|_k < \infty$$

uniformly in k. Thanks to the mesh-size reduction, i.e. $h_m \leqslant h_k$ for all $m \geqslant k$, we conclude that

$$\int_{\mathcal{S}_k} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \leqslant \int_{\mathcal{S}_k} h_m^{-1} [\![v]\!]^2 \,\mathrm{d}s \leqslant \int_{\mathcal{S}_m} h_m^{-1} [\![v]\!]^2 \,\mathrm{d}s,$$

thanks to the inclusion $\bigcup_{S \in \mathcal{S}_k} S \subset \bigcup_{S \in \mathcal{S}_m} S$. Therefore, we have $|\!|\!|v|\!|\!|_k \leqslant |\!|\!|v|\!|\!|_m$ for all $m \geqslant k$ and, thus, $\{|\!|\!|v|\!|\!|_k\}_{k\in\mathbb{N}}$ converges. Consequently, for $\epsilon > 0$ there exists $K = K(\epsilon)$, such that for all $k \geqslant K$ and $m > k$ large enough, we have

$$\begin{aligned}
\epsilon > \big|\, |\!|\!|v|\!|\!|_m^2 - |\!|\!|v|\!|\!|_k^2 \,\big| &= \bar{\sigma} \int_{\mathcal{S}_m \setminus (\mathcal{S}_m \cap \mathcal{S}_k)} h_m^{-1} [\![v]\!]^2 \,\mathrm{d}s - \bar{\sigma} \int_{\mathcal{S}_k \setminus (\mathcal{S}_m \cap \mathcal{S}_k)} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \\
&\geqslant (2^{1/(d-1)} - 1)\, \bar{\sigma} \int_{\mathcal{S}_k \setminus (\mathcal{S}_m \cap \mathcal{S}_k)} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \\
&\geqslant (2^{1/(d-1)} - 1)\, \bar{\sigma} \int_{\mathcal{S}_k \setminus \mathcal{S}_k^+} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s.
\end{aligned}$$

This follows from the fact that $h_m|_S \leqslant 2^{-1/(d-1)} h_k|_S$ for all $S \in \mathcal{S}_k \setminus (\mathcal{S}_m \cap \mathcal{S}_k)$ together with $\mathcal{S}_k^+ = \mathcal{S}_m \cap \mathcal{S}_k$ for sufficiently large $m > k$.

Therefore, we have $\int_{\mathcal{S}_k \setminus \mathcal{S}_k^+} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \to 0$ as $k \to \infty$ and, thus,

$$|\!|\!|v|\!|\!|_k^2 = \int_\Omega |\nabla_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \bar{\sigma} \int_{\mathcal{S}_k^+} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s + \bar{\sigma} \int_{\mathcal{S}_k \setminus \mathcal{S}_k^+} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s \to |\!|\!|v|\!|\!|_\infty^2 + 0.$$

This proves the first claim. The second claim is a localised version and follows completely analogously. □

Lemma 13 (Poincaré-$\mathbb{V}_\infty$)**.** *Fix $k \in \mathbb{N}$ and let $E \in \mathcal{G}_k$. Then for $v \in \mathbb{V}_\infty$ and $v_E := \frac{1}{|\omega_k(E)|} \int_{\omega_k(E)} v \,\mathrm{d}x$, we have*

$$\|v - v_E\|_{\omega_k(E)}^2 \lesssim \|h_k \nabla_{\mathtt{pw}} v\|_{\omega_k(E)}^2 + \int_{\{S \in \mathcal{S}^+ : S \subset \omega_k(E)\}} h_k^2 h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s.$$

Proof. By the definition of $\mathbb{V}_\infty$, there exists $v_\ell \in \mathbb{V}_\ell$, $\ell \in \mathbb{N}_0$, with $\lim_{\ell\to\infty} \|v - v_\ell\|_\ell = 0$ and $\limsup_{\ell\to\infty} \|v_\ell\|_\ell < \infty$. Therefore, we have

$$\left\|\nabla_{\mathtt{pw}} v_\ell\right\|^2_{\omega_k(E)} + \int_{\{S\in\mathcal{S}_\ell : S\subset\omega_k(E)\}} h_\ell^{-1} [\![v_\ell]\!]^2 \,\mathrm{d}s \to \left\|\nabla_{\mathtt{pw}} v\right\|^2_{\omega_k(E)} + \int_{\{S\in\mathcal{S}^+ : S\subset\omega_k(E)\}} h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s \quad \text{as } \ell\to\infty;$$

see Proposition 12. Moreover, we have

$$\|v_E - v_{\ell,E}\|_{\omega_k(E)} \leqslant \|v - v_\ell\|_{\omega_k(E)} \leqslant \|v - v_\ell\|_\ell \to 0 \quad \text{as } \ell\to\infty,$$

where $v_{\ell,E} := \frac{1}{|\omega_k(E)|}\int_{\omega_k(E)} v_\ell \,\mathrm{d}x$. We conclude with Proposition 1 that

$$\begin{aligned}\|v - v_E\|^2_{\omega_k(E)} &\leftarrow \|v_\ell - v_{\ell,E}\|^2_{\omega_k(E)} \\ &\lesssim \left\|h_k \nabla_{\mathtt{pw}} v_\ell\right\|^2_{\omega_k(E)} + \int_{\{S\in\mathcal{S}_\ell : S\subset\omega_k(E)\}} h_k^2 h_\ell^{-1} [\![v_\ell]\!]^2 \,\mathrm{d}s \\ &\to \left\|h_k \nabla_{\mathtt{pw}} v\right\|^2_{\omega_k(E)} + \int_{\{S\in\mathcal{S}^+ : S\subset\omega_k(E)\}} h_k^2 h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s,\end{aligned}$$

as $\ell\to\infty$. □

In order to extend the dG bilinear form (2.5) to $\mathbb{V}_\infty$, we need to define appropriate lifting operators. For each $S \in \mathcal{S}^+$, there exists $\ell = \ell(S) \in \mathbb{N}$, such that $S \in \mathcal{S}_\ell^{++}$. We define the local lifting operators $R_\infty^S : L^2(S)^d \to L^2(\Omega)^d$ and $L_\infty^S : L^2(S) \to L^2(\Omega)^d$ by

$$R_\infty^S = R_\ell^S := R_{\mathcal{G}_\ell}^S \qquad \text{and} \qquad L_\infty^S = L_\ell^S := L_{\mathcal{G}_\ell}^S. \tag{3.1}$$

From (2.4) it is easy to see, that R_ℓ^S and L_ℓ^S depend only on S and the at most two adjacent elements $E, E' \in \mathcal{G}_\ell^+$ with $S \subset E\cap E'$. Therefore, and thanks to the fact that the $\mathcal{G}_k^+$ are nested, we have that $R_\ell^S = R_k^S$ for all $k \geqslant \ell$ and, thus, the definition is unique. We formally define the global lifting operators by

$$R_\infty := \sum_{S\in\mathcal{S}^+} R_\infty^S \qquad \text{and} \qquad L_\infty := \sum_{S\in\mathring{\mathcal{S}}^+} L_\infty^S;$$

here $\mathring{\mathcal{S}}^+ := \{S \in \mathcal{S}^+ : S \notin \partial\Omega\}$.

Moreover, from the local estimates (2.4c), it is easy to see that for $v \in \mathbb{V}_\infty$ and $\boldsymbol{\beta} \in \mathbb{R}^d$, we have that $\sum_{S\in\mathcal{S}_k^+} R_\infty^S([\![v]\!])$ and $\sum_{S\in\mathring{\mathcal{S}}_k^+} L_\infty^S(\boldsymbol{\beta}\cdot[\![v]\!])$ are Cauchy sequences in $L^2(\Omega)^d$. Consequently, $R_\infty([\![v]\!]), L_\infty(\boldsymbol{\beta}\cdot[\![v]\!]) \in L^2(\Omega)$ are well posed and we have

$$\|R_\infty([\![v]\!])\|_\Omega \lesssim \left\|h_+^{-1/2} v\right\|_{\Gamma^+} \quad \text{and} \quad \|L_\infty(\boldsymbol{\beta}\cdot[\![v]\!])\|_\Omega \lesssim |\boldsymbol{\beta}| \left\|h_+^{-1/2} v\right\|_{\mathring{\Gamma}^+}, \tag{3.2}$$

where $\Gamma^+ = \bigcup\{S : S \in \mathcal{S}^+\}$ and $\mathring{\Gamma}^+ = \bigcup\{S : S \in \mathring{\mathcal{S}}^+\}$. This enables us to generalise the discontinuous Galerkin bilinear form to $\mathbb{V}_\infty$ setting

$$\begin{aligned}\mathfrak{B}_\infty[w, v] := &\int_\Omega \nabla_{\mathtt{pw}} w\cdot\nabla_{\mathtt{pw}} v \,\mathrm{d}x - \int_{\mathcal{S}^+} \big(\{\nabla w\}\cdot[\![v]\!] + \theta\{\nabla v\}\cdot[\![w]\!]\big)\,\mathrm{d}s \\ &+ \int_{\mathring{\mathcal{S}}^+} \big(\boldsymbol{\beta}\cdot[\![w]\!]\,[\![\nabla v]\!] + [\![\nabla w]\!]\,\boldsymbol{\beta}\cdot[\![v]\!]\big)\,\mathrm{d}s \\ &+ \int_\Omega \gamma\big(R_\infty([\![w]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![w]\!])\big)\cdot\big(R_\infty([\![v]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![v]\!])\big)\,\mathrm{d}x \\ &+ \int_{\mathcal{S}^+} \frac{\sigma}{h_+}[\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s,\end{aligned}$$

for $v, w \in \mathbb{V}_\infty$.

Lemma 14. *The space* $\big(\mathbb{V}_\infty, \langle\cdot,\cdot\rangle_\infty\big)$ *is a Hilbert space.*

Corollary 15. *There exists a unique* $u_\infty \in \mathbb{V}_\infty$*, such that*

$$\mathfrak{B}_\infty[u_\infty,\, v] = \int_\Omega f v \,\mathrm{d}x \qquad \text{for all } v \in \mathbb{V}_\infty. \tag{3.3}$$

In order to prove the last two statements, we introduce a new quasi-interpolation, which is designed in due consideration of the future refinements. The proofs of Lemma 14 and Corollary 15 are postponed to the end of Section 3.3.

3.3. **Quasi-interpolation.** We shall now define a quasi-interpolation operator Π_k, which maps into $\mathbb{V}_\infty \cap \mathbb{V}_k$; this will be a key technical tool in the analysis. On the one hand, membership in $\mathbb{V}_\infty \cap \mathbb{V}_k$ suggests to use some Clément type interpolation since the mapped functions need to be continuous in Ω^-. On the other hand, the fact that the ADGM may leave some elements (namely $\mathcal{G}_k^+ \supset \mathcal{G}_k^{++}$) unrefined, suggests to define Π_k to be the identity on these elements. Note that the quasi-interpolation operator from [CGS13] is motivated by a similar idea in order to map from one Crouzeix-Raviart space into its intersection with a finer one.

For fixed $k \in \mathbb{N}$, let $\{\Phi_z^E : E \in \mathcal{G}_k,\ z \in \mathcal{N}_k(E)\}$ be the Lagrange basis of $\mathbb{V}_k := \mathbb{V}(\mathcal{G}_k)$, i.e., Φ_z^E is a piecewise polynomial of degree r with $\operatorname{supp}(\Phi_z^E) = E$ and

$$\Phi_z^E(y) = \delta_{zy} \qquad \text{for all } z, y \in \mathcal{N}_k.$$

Its dual basis is then the set $\{\Psi_z^E : E \in \mathcal{G}_k,\ z \in \mathcal{N}_k(E)\}$ of piecewise polynomials of degree r, such that $\operatorname{supp}(\Psi_z^E) = E$ and

$$\left\langle \Psi_y^E,\, \Phi_z^E \right\rangle_\Omega = \delta_{zy} \qquad \text{for all } z, y \in \mathcal{N}_k(E).$$

For all $\ell \geqslant k$, we define $\Pi_k : L^1(\Omega) \to L^1(\Omega)$ by

$$\Pi_k v := \sum_{E \in \mathcal{G}_k} \sum_{z \in \mathcal{N}_k(E)} (\Pi_k v)|_E(z)\, \Phi_z^E, \tag{3.4}$$

where for $z \in \mathcal{N}_k(E)$ we have that

$$(\Pi_k v)|_E(z) := \begin{cases} \int_E v \Psi_z^E \,\mathrm{d}x, & \text{if } N_k(z) \cap \mathcal{G}_k^{++} \neq \varnothing \\ 0, & \text{else if } z \in \partial\Omega \\ \sum_{E' \in N_k(z)} \frac{|E'|}{|\omega_k(z)|} \int_{E'} v \Psi_z^{E'} \,\mathrm{d}x, & \text{else.} \end{cases} \tag{3.5}$$

Lemma 16 (Properties of Π_k)**.** *The operator* $\Pi_k : L^1(\Omega) \to L^1(\Omega)$ *defined in* (3.4) *has the following properties:*

(1) $\Pi_k : L^p(\Omega) \to L^p(\Omega)$ *is a linear and bounded projection for all* $1 \leqslant p \leqslant \infty$. *In particular, we have that*

$$\|\Pi_k v\|_{L^p(E)} \lesssim \|v\|_{L^p(\omega_k(E))},$$

where the constant solely depends on p, r, d, *and the shape regularity of* $\mathcal{G}_0$.

(2) $\Pi_k v \in \mathbb{V}_k$ *for all* $v \in L^1(\Omega)$;

(3) $\Pi_k v|_E = v|_E$, *if* $E \in \mathcal{G}_k$ *and* $v|_{\omega_k(E)} \in \mathbb{P}_r(\omega_k(E))$;

(4) $\Pi_k v|_E = v|_E$, *if* $E \in \mathcal{G}_k^{++}$ *and* $v|_E \in \mathbb{P}_r(E)$; *if moreover* $v \in \mathbb{V}_k$, *then also* $[\![v - \Pi_k v]\!]\,|_S \equiv 0$ *for all* $S \in \mathcal{S}_k^{++}$.

(5) $\Pi_k v|_{\Omega \setminus \Omega_k^+} \in C(\overline{\Omega \setminus \Omega_k^+})$ *and* $[\![\Pi_k v]\!] = 0$ *on* $\partial(\Omega \setminus \Omega_k^+)$;

(6) $\Pi_k v = v$, *for all* $v \in \mathbb{V}_k$ *with* $v|_{\Omega \setminus \Omega_k^{++}} \in C(\Omega \setminus \Omega_k^{++})$;

(7) $\Pi_k v \in \mathbb{V}_\infty$, *and we have* $|\!|\!|\Pi_k v|\!|\!|_k = |\!|\!|\Pi_k v|\!|\!|_\infty$.

Proof. Claims (1)–(3) follow by standard estimates for the Scott-Zhang operator [SZ90, DG12].

Assertion (4) is a consequence of the definition (3.5) of Π_k since $E \in \mathcal{G}_k^{++}$ implies that $N_k(E) \cap \mathcal{G}_k^{++} = N_k(E)$. Note that $v \in \mathbb{V}(\mathcal{G})$ implies $v|_E \in \mathbb{P}_r(E)$ for all $E \in \mathcal{G}_k$ and thus $(\Pi_k v)|_E(z) = v|_E(z)$ for all $E \in N_k(z)$ if $N_k(z) \cap \mathcal{G}_k^{++} \neq \varnothing$. This is in particular the case when $z \in S \cap \mathcal{N}_k$ with $S \in \mathcal{S}_k^{++}$.

For $E \in \mathcal{G}_k \backslash \mathcal{G}_k^+$, we have that $N_k(z) \cap \mathcal{G}_k^{++} = \varnothing$ since otherwise there exists $E' \in N_k(E) \cap \mathcal{G}_k^{++}$ and thus $E \in N_k(E')$, which implies $E \in \mathcal{G}_k^+$, thanks to the definition of $\mathcal{G}_k^{++}$. Therefore, (3.5) implies that $\Pi_k v$ is continuous on $\Omega \backslash \Omega_k^+$. Moreover, for $z \in \mathcal{N}_k(E) \cap \Omega \backslash \Omega_k^+$, definition (3.5) is independent of E and thus $\Pi_k v$ does not jump across the boundary $\Omega \backslash \Omega_k^+$. This completes the proof of (5).

On the one hand, if $v \in \mathbb{V}_k$ with $v|_{\Omega \backslash \Omega_k^+} \in C(\overline{\Omega \backslash \Omega_k^{++}})$ then we have clearly $\Pi_k v|_{\Omega \backslash \Omega_k^+} = v|_{\Omega \backslash \Omega_k^+}$. On the other hand, we can conclude $\Pi_k v|_{\Omega_k^{++}} = v|_{\Omega_k^{++}}$ from (4). This yields (6).

The claim (7) is an immediate consequence of (5). $\square$

Lemma 17 (Stability). *Let $v \in \mathbb{V}_\ell$ for some $k \leqslant \ell \in \mathbb{N}_0 \cup \{\infty\}$. Then for all $E \in \mathcal{G}_k$, we have*

$$\int_E |\nabla \Pi_k v|^2 \,\mathrm{d}x + \int_{\partial E} h_k^{-1} [\![\Pi_k v]\!]^2 \,\mathrm{d}s \lesssim \int_{\omega_k(E)} \left|\nabla_{pw} v\right|^2 \,\mathrm{d}x + \sum_{E' \in \mathcal{G}_\ell, E' \subset \omega_k(E)} \int_{\partial E'} h_\ell^{-1} [\![v]\!]^2 \,\mathrm{d}s,$$

setting $\mathcal{G}_\ell := \mathcal{G}^+$ and $h_\ell := h_+$, when $\ell = \infty$. In particular, we have $\|\Pi_k v\|_k \lesssim \|v\|_\ell$.

Proof. We begin by noting that, summing over all elements in $\mathcal{G}_k$ and accounting for the finite overlap of the domains $\omega_k(E)$, $E \in \mathcal{G}_k$, the global stability estimate is an immediate consequence of the corresponding local one.

We first assume $\ell < \infty$. Let $E \in \mathcal{G}_k^{++} \subset \mathcal{G}_\ell^{++}$. Then, thanks to Lemma 16(4), we have $\Pi_k v|_E = v|_E$. Moreover, let $E' \in \mathcal{G}_k$ such that $E \cap E' \in \mathcal{S}_k$; then $N_k(z) \ni E \in \mathcal{G}_k^{++}$ and thus $(\Pi_k v)|_{E'}(z) = v|_{E'}(z)$, for all $z \in \mathcal{N}_k(E) \cap \mathcal{N}_k(E')$. Consequently, we have $[\![\Pi_k v]\!] = [\![v]\!]$ on ∂E, in other words

$$\int_E |\nabla \Pi_k v|^2 \,\mathrm{d}x + \int_{\partial E} h_k^{-1} [\![\Pi_k v]\!]^2 \,\mathrm{d}s = \int_E |\nabla v|^2 \,\mathrm{d}x + \int_{\partial E} h_k^{-1} [\![v]\!]^2 \,\mathrm{d}s. \tag{3.6}$$

Let now $E \in \mathcal{G}_k$ be arbitrary. Then, an inverse estimate and the local stability (Lemma 16 (1) and (3)) for $v_E := \frac{1}{|\omega_k(E)|} \int_{\omega_k(E)} v \,\mathrm{d}x \in \mathbb{R}$, imply

$$\begin{aligned} \int_E |\nabla \Pi_k v|^2 \,\mathrm{d}x &\lesssim \int_E h_k^{-2} |\Pi_k (v - v_E)|^2 \,\mathrm{d}x \lesssim \int_{\omega_k(E)} h_k^{-2} |v - v_E|^2 \,\mathrm{d}x \\ &\lesssim \sum_{E' \subset \omega_k(E), E' \in \mathcal{G}_\ell} \int_{E'} |\nabla v|^2 \,\mathrm{d}x + \int_{\partial E'} h_\ell^{-1} [\![v]\!]^2 \,\mathrm{d}s; \end{aligned} \tag{3.7}$$

here the last estimate follows from the broken Poincaré inequality, Proposition 1.

If now for all $E' \in \mathcal{G}_k$, with $E' \subset \omega_k(E)$, we have $E' \notin \mathcal{G}_k^{++}$, which implies $E \in \mathcal{G}_k \backslash \mathcal{G}_k^{++}$. Then, thanks to Lemma 16(5), we have that $\Pi_k v$ is continuous across ∂E, i.e., $[\![\Pi_k v]\!]\,|_{\partial E} = 0$. On the contrary, assuming that there exists $E' \in \mathcal{G}_k^{++}$, with $E' \in N_k(E)$, we conclude that $E \in N_k(E')$ and thus $E \in \mathcal{G}^+$. From the local quasi uniformity, we thus have for all $E'' \in \mathcal{G}_\ell$ with $E'' \cap E \neq \varnothing$ that $|E''| \eqsim |E|$. Let $z \in \mathcal{N}_k(E)$; then, according to (3.5), we have that

$$[\![\Pi_k v]\!]\,|_{\partial E}(z) = \begin{cases} [\![v]\!]\,|_{\partial E}(z), & \text{if } \exists E' \in N_k(z) \cap \mathcal{G}_k^{++}; \\ 0, & \text{else.} \end{cases}$$

Using standard scaling arguments, this implies

$$\begin{aligned} \int_{\partial E} [\![\Pi_k v]\!]^2 \,\mathrm{d}s &\eqsim |\partial E| \sum_{z \in \mathcal{N}_k \cap \partial E} \left([\![\Pi_k v]\!]\,|_{\partial E}(z)\right)^2 = |\partial E| \sum_{z \in \mathcal{N}_k \cap \partial E} \left([\![v]\!]\,|_{\partial E}(z)\right)^2 \\ &\leqslant |\partial E| \sum_{z \in \mathcal{N}_\ell \cap \partial E} \left([\![v]\!]\,|_{\partial E}(z)\right)^2 \eqsim \int_{\partial E} [\![v]\!]^2 \,\mathrm{d}s. \end{aligned}$$

Combining this with (3.7) proves the local bound in the case $\ell < \infty$.

For $\ell = \infty$, we observe that a bound similar to (3.7) can be obtained with Lemma 13 instead of Proposition 1. The local bound follows then by arguing as in the case $\ell < \infty$. □

Corollary 18 (Interpolation estimate)**.** *For $v \in \mathbb{V}_\ell$, $k \leqslant \ell \in \mathbb{N} \cup \{\infty\}$, we have that*

$$\int_E |\nabla_{pw} v - \nabla_{pw} \Pi_k v|^2 \,\mathrm{d}x + \int_E h_k^{-2} |v - \Pi_k v|^2 + \int_{\partial E} h_k^{-1} [\![v - \Pi_k v]\!]^2 \lesssim \int_{\omega_k(E)} |\nabla_{pw} v|^2 \,\mathrm{d}x + \sum_{S \in \mathcal{S}_\ell, S \subset \omega_k(E)} \int_S h_k^{-1} [\![v]\!]^2 ,$$

where we set $\mathcal{G}_\ell := \mathcal{G}^+$ and $h_\ell := h_+$, when $\ell = \infty$. The constant depends only on d, r and the shape regularity of $\mathcal{G}_0$.

Proof. The claim follows from Lemma 16(3), together with the stability Lemma 17 and the local Poincaré inequality from Proposition 1, respectively, Lemma 13. □

The next result concerns the convergence of the quasi-interpolation.

Lemma 19. *Let $v \in \mathbb{V}_\infty$; then,*

$$|\!|\!| v - \Pi_k v |\!|\!|_k \to 0 \qquad \textit{and} \qquad |\!|\!| v - \Pi_k v |\!|\!|_\infty \to 0$$

as $k \to \infty$.

Proof. For brevity, set $v_k := \Pi_k v \in \mathbb{V}_k$. Thanks to Lemma 13 and Lemma 16(4) and (5), we have that

$$\begin{aligned} |\!|\!| v - v_k |\!|\!|_k^2 &\lesssim \int_{\mathcal{G}_k \setminus \mathcal{G}_k^{++}} |\nabla_{\mathtt{pw}} v - \nabla_{\mathtt{pw}} v_k|^2 \,\mathrm{d}x + \int_{\mathcal{S}_k \setminus \mathcal{S}_k^{++}} h_k^{-1} \left| [\![v - v_k]\!] \right|^2 \,\mathrm{d}s \\ &\leqslant \int_{\mathcal{G}_k^-} |\nabla_{\mathtt{pw}} v - \nabla_{\mathtt{pw}} v_k|^2 \,\mathrm{d}x + \int_{\mathcal{G}_k^\star} |\nabla_{\mathtt{pw}} v - \nabla_{\mathtt{pw}} v_k|^2 \,\mathrm{d}x \\ &\quad + \int_{\mathcal{S}_k^-} h_k^{-1} \left| [\![v - v_k]\!] \right|^2 \,\mathrm{d}s + \int_{\mathcal{S}_k^\star} h_k^{-1} \left| [\![v - v_k]\!] \right|^2 \,\mathrm{d}s \\ &= I_k^- + I_k^\star + II_k^- + II_k^\star. \end{aligned}$$

We first observe that $II_k^- = 0$ since $v, v_k \in H^1(\Omega_k^-)$ (note that $[\![v]\!] = [\![v_k]\!] = 0$ even on the boundary $\partial\Omega_k^-$ since $\overline{\Omega_k^-} \subset \Omega^-$). We conclude from Lemma 17 that

$$\begin{aligned} I_k^\star + II_k^\star &= \int_{\mathcal{G}_k^\star} |\nabla_{\mathtt{pw}} v - \nabla_{\mathtt{pw}} v_k|^2 \,\mathrm{d}x + \int_{\mathcal{S}_k^\star} h_k^{-1} \left| [\![v - v_k]\!] \right|^2 \,\mathrm{d}s \\ &\lesssim \sum_{E \in \mathcal{G}_k^\star} \Big(\int_{\omega_k(E)} |\nabla_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \sum_{E' \in \mathcal{G}^+, E' \subset \omega_k(E)} \int_{\partial E'} h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s \Big) \\ &\lesssim \sum_{E \in \mathcal{G}_k^\star} \int_{\omega_k(E)} |\nabla_{\mathtt{pw}} v|^2 \,\mathrm{d}x + \int_{\mathcal{S}^+ \setminus \mathcal{S}_k^{++}} h_+^{-1} [\![v]\!]^2 \,\mathrm{d}s. \end{aligned}$$

The first term on the right-hand side vanishes in the limit $k \to \infty$, from Lemma 11. The second term is the tail of a convergent series, since it is bounded thanks to $|\!|\!| v |\!|\!|_\infty < \infty$ and all of its summands are positive. Therefore, $I_k^\star + II_k^\star \to 0$ as $k \to \infty$.

Thus, it remains to prove that $I_k^- \to 0$ as $k \to \infty$. To this end, recall that $H^1_{\partial\Omega \cap \partial\Omega^-}(\Omega^-)$ is the space of restrictions of $H^1_0(\Omega)$-functions to Ω^-. Since $H^2_0(\Omega)$ is dense in $H^1_0(\Omega)$, for $\epsilon > 0$, there exists $v_\epsilon \in H^2_0(\Omega)$ such that $\|v - v_\epsilon\|_{H^1(\Omega^-)} \leqslant \|v - v_\epsilon\|_{H^1(\Omega)} < \epsilon$. Combining Lemma 16(3) and (1) with the Bramble-Hilbert

Lemma (see, e.g., [BS02]), we obtain with standard arguments that

$$\begin{aligned}\int_{\mathcal{G}_k^-} |\nabla v - \nabla v_k|^2 \,\mathrm{d}x &\lesssim \epsilon^2 + \int_{\mathcal{G}_k^-} |\nabla v_\epsilon - \nabla \Pi_k v_\epsilon|^2 \,\mathrm{d}x \\ &\lesssim \epsilon^2 + \int_{N_k(\mathcal{G}_k^-)} h_k^2 \sum_{|\alpha|=2} |D^\alpha v_\epsilon|^2 \,\mathrm{d}x \\ &\lesssim \epsilon^2 + \|h_k \chi_{\Omega_k^-}\|_{L^\infty(\Omega)}^2 \int_\Omega \sum_{|\alpha|=2} |D^\alpha v_\epsilon|^2 \,\mathrm{d}x,\end{aligned}$$

where we have used that $\|h_k\|_{L^\infty(\Omega(N_k(\mathcal{G}_k^-)))} \lesssim \|h_k \chi_{\Omega_k^-}\|_{L^\infty(\Omega)}$, thanks to the local quasi-uniformity of $\mathcal{G}_k$. Thus, we have $\|h_k \chi_{\Omega_k^-}\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$ from Lemma 11 and, therefore, we can conclude that $\lim_{k\to\infty} I_k^- \lesssim \epsilon$. This completes the proof of the first claim, since $\epsilon > 0$ is arbitrary.

The second claim follows similarly by replacing $\mathcal{S}_k$ by $\mathcal{S}^+$ and noting that $\|\!|\Pi_k v|\!\|_k = \|\!|\Pi_k v|\!\|_\infty$, since $\Pi_k v$ is continuous in $\Omega\backslash\Omega^+$. □

Proof of Lemma 14. The positivity of $\|\!|\cdot|\!\|_\infty$ on $\mathbb{V}_\infty$ follows from Lemma 19 together with $\|w\|_{BV(\Omega)} \leqslant \|\!|w|\!\|_\ell$ for all $w \in \mathbb{V}_\ell$; see Corollary 3 and Proposition 4.

In order to prove that $\mathbb{V}_\infty$ is complete with respect to $\|\!|\cdot|\!\|_\infty$, let $0 \neq v \in \overline{\mathbb{V}_\infty}^{\|\!|\cdot|\!\|_\infty}$, i.e. there exists a sequence $\{v^\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{V}_\infty$, such that $\|\!|v - v^\ell|\!\|_\infty \to 0$ as $\ell \to \infty$. Note that $v^\ell|_E \in \mathbb{P}_r$ for all $E \in \mathcal{G}^+$ and thus it follows from the definition of $\|\!|\cdot|\!\|_\infty$ that also $v|_E \in \mathbb{P}_r$ for all $E \in \mathcal{G}^+$.

For each $\ell, m \in \mathbb{N}$, we define $v_m^\ell := \Pi_m v^\ell \in \mathbb{V}_m$ and since $v_m^\ell \in C(\Omega\backslash\Omega^+)$ (see Lemma 16(5)), we have that $\|\!|v_m^\ell|\!\|_\ell = \|\!|v_m^\ell|\!\|_\infty$ for all $\ell \geqslant m \in \mathbb{N}$. Thanks to Lemma 19, for each $\ell \in \mathbb{N}$, there exists a monotone sequence $\{m_\ell\}_\ell \in \mathbb{N}$, such that $\|\!|v^\ell - v_{m_\ell}^\ell|\!\|_\infty \leqslant \frac{1}{\ell}$ and thus

$$\|\!|v - v_{m_\ell}^\ell|\!\|_{m_\ell} \leqslant \|\!|v - v_{m_\ell}^\ell|\!\|_\infty \leqslant \|\!|v - v^\ell|\!\|_\infty + \|\!|v^\ell - v_{m_\ell}^\ell|\!\|_\infty \to 0 \qquad \text{as } \ell \to \infty.$$

Consequently, we have that

$$\|\!|v_{m_\ell}^\ell|\!\|_{m_\ell} = \|\!|v_{m_\ell}^\ell|\!\|_\infty \to \|\!|v|\!\|_\infty < \infty \qquad \text{as } \ell \to \infty.$$

Thanks to Corollary 3 and Proposition 4, we can extract another subsequence of $\{v_{m_\ell}^\ell\}_{\ell\in\mathbb{N}}$ which is weakly-$*$ converging in $BV(\Omega)$. Therefore, $v \in BV(\Omega)$, and we have in the distributional sense, that

$$Dv(\phi) = \int_\Omega \nabla_{\mathtt{pw}} v \cdot \phi \,\mathrm{d}x + \int_{\mathcal{S}^+} [\![v]\!] \cdot \phi \,\mathrm{d}s \qquad \forall \phi \in C_0^\infty(\Omega)^d.$$

Note that $\mathbb{V}_k \subset \mathbb{V}_j$ for $j \geqslant k$ and thus $w_k := v_{m_\ell}^\ell \in \mathbb{V}_k$, $k \in \{m_\ell, \ldots, m_{\ell+1} - 1\}$. Consequently, we have $\|\!|v - w_k|\!\|_k \leqslant \|\!|v - w_k|\!\|_\infty = \|\!|v - v_{m_\ell}^\ell|\!\|_\infty \to 0$ as $k \to \infty$.

It remains to verify that $v|_{\Omega^-} \in H^1_{\partial\Omega\cap\partial\Omega^-}(\Omega^-)$, i.e., that v is a restriction of a function from $H_0^1(\Omega)$ to Ω^-. To this end, we consider the conforming interpolation $\mathcal{I}_k w_k \in \mathbb{V}_k \cap H_0^1(\Omega)$ from Proposition 2, which also implies that $\|\nabla \mathcal{I}_k w_k\|_{L^2(\Omega)} \lesssim \|\!|w_k|\!\|_\infty < \infty$ uniformly in k, i.e., there exists a weak limit $\tilde{v} \in H_0^1(\Omega)$ of a subsequence of $\{\mathcal{I}_k w_k\}_{k\in\mathbb{N}}$. On the other hand, it follows from Lemma 16(5) that $[\![w_k]\!]|_{\partial E} = 0$ for all $E \in \mathcal{G}_k^+$ (recall that $\Omega_{m_\ell}^+ \subset \Omega_k^+$ for $k \geqslant m_\ell$). Consequently, the local estimate in Proposition 2 implies $\mathcal{I}_k w_k = w_k$ in $\Omega^- \subset (\Omega\backslash\Omega_k^+)$. Therefore, we have

$$\|\nabla v - \nabla \mathcal{I}_k w_k\|_{L^2(\Omega^-)} = \|\nabla v - \nabla_{\mathtt{pw}} w_k\|_{L^2(\Omega^-)} \leqslant \|\!|v - w_k|\!\|_\infty \to 0$$

as $k \to \infty$ and thus $v|_{\Omega^-} = \tilde{v}|_{\Omega^-}$.

Concluding, we have proved $v \in \mathbb{V}_\infty$, which implies Lemma 14. □

Proof of Corollary 15. The assertion follows from Lemma 14 and the observation that

$$|||v|||_\infty^2 \lesssim \mathfrak{B}_\infty[v,\, v] \qquad \text{and} \qquad \mathfrak{B}_\infty[v,\, w] \lesssim |||v|||_\infty\, |||w|||_\infty$$

for all $v, w \in \mathbb{V}_\infty$. Indeed, the continuity follows with standard techniques using (3.2) and the coercivity is a consequence of

$$|||\Pi_k v|||_\infty^2 = |||\Pi_k v|||_k^2 \lesssim \mathfrak{B}_k[\Pi_k v,\, \Pi_k v] = \mathfrak{B}_\infty[\Pi_k v,\, \Pi_k v]$$

and Lemma 19. □

4. (Almost) best approximation property

In this section we shall prove that the solution $u_\infty \in \mathbb{V}_\infty$ of (3.3) is indeed the limit of the discontinuous Galerkin solutions produced by ADGM. This is a consequence of the density of spaces $\{\mathbb{V}_k\}_{k\in\mathbb{N}_0}$ in $\mathbb{V}_\infty$ and the (almost) best approximation property of discontinuous Galerkin solutions; the latter generalises [Gud10].

Lemma 20. *Let $u_\infty \in \mathbb{V}_\infty$ be the solution of* (3.3) *and $u_k \in \mathbb{V}_k$ be the DGFEM approximation from* (2.6) *on $\mathcal{G}_k$ for some $k \in \mathbb{N}$ and u_∞ the unique solution of the limit problem from Corollary 15. Then, we have*

$$|||u_\infty - u_k|||_k \lesssim |||u_\infty - \Pi_k u_\infty|||_\infty + \frac{\langle f,\, u_k - \Pi_k u_k\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, u_k - \Pi_k u_k]}{|||u_k - \Pi_k u_\infty|||_k}.$$

Proof. Assume that $u_k \neq \Pi_k u_\infty \in \mathbb{V}_k \cap \mathbb{V}_\infty$ and set $\psi = u_k - \Pi_k u_\infty$. Then, we have from (2.7) that

$$\begin{aligned}
\alpha\, |||u_k - \Pi_k u_\infty|||_k^2 &\leqslant \mathfrak{B}_k[u_k - \Pi_k u_\infty,\, \psi] = \langle f,\, \psi\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, \psi] \\
&= \langle f,\, \Pi_k\psi\rangle_\Omega + \langle f,\, \psi - \Pi_k\psi\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, \psi] \\
&= \big(\mathfrak{B}_\infty[u_\infty,\, \Pi_k\psi] - \mathfrak{B}_k[\Pi_k u_\infty,\, \Pi_k\psi]\big) \\
&\quad + \big(\langle f,\, \psi - \Pi_k\psi\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, \psi - \Pi_k\psi]\big) \equiv (I) + (II),
\end{aligned}$$

using that $\Pi_k\psi \in \mathbb{V}_k \cap \mathbb{V}_\infty$ from Lemma 16(7). For (I), we have, respectively,

$$\begin{aligned}
(I) &= \int_\Omega \nabla_{\mathtt{pw}} u_\infty \cdot \nabla_{\mathtt{pw}}\Pi_k\psi \,\mathrm{d}x - \int_{\mathcal{S}^+} \big(\{\nabla u_\infty\}\cdot [\![\Pi_k\psi]\!] + \theta\{\nabla\Pi_k\psi\}\cdot[\![u_\infty]\!]\big)\,\mathrm{d}s \\
&\quad + \int_{\mathring{\mathcal{S}}^+} \big(\boldsymbol{\beta}\cdot[\![u_\infty]\!]\,[\![\nabla\Pi_k\psi]\!] + [\![\nabla u_\infty]\!]\,\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!]\big)\,\mathrm{d}s \\
&\quad + \int_\Omega \gamma\big(R_\infty([\![u_\infty]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![u_\infty]\!])\big)\cdot\big(R_\infty([\![\Pi_k\psi]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!])\big)\,\mathrm{d}x \\
&\quad + \int_{\mathcal{S}^+} \frac{\sigma}{h_+}\,[\![u_\infty]\!]\cdot[\![\Pi_k\psi]\!]\,\mathrm{d}s \\
&\quad - \int_\Omega \nabla_{\mathtt{pw}}\Pi_k u_\infty\cdot\nabla_{\mathtt{pw}}\Pi_k\psi\,\mathrm{d}x + \int_{\mathcal{S}_k}\big(\{\nabla\Pi_k u_\infty\}\cdot[\![\Pi_k\psi]\!] + \theta\{\nabla\Pi_k\psi\}\cdot[\![\Pi_k u_\infty]\!]\big)\,\mathrm{d}s \\
&\quad - \int_{\mathring{\mathcal{S}}^+}\big(\boldsymbol{\beta}\cdot[\![\Pi_k u_\infty]\!]\,[\![\nabla\Pi_k\psi]\!] + [\![\nabla\Pi_k u_\infty]\!]\,\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!]\big)\,\mathrm{d}s \\
&\quad - \int_\Omega \gamma\big(R_k([\![\Pi_k u_\infty]\!]) + L_k(\boldsymbol{\beta}\cdot[\![\Pi_k u_\infty]\!])\big)\cdot\big(R_k([\![\Pi_k\psi]\!]) + L_k(\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!])\big)\,\mathrm{d}x \\
&\quad - \int_{\mathcal{S}^+}\frac{\sigma}{h_k}\,[\![\Pi_k u_\infty]\!]\cdot[\![\Pi_k\psi]\!]\,\mathrm{d}s \\
&= \int_\Omega \nabla_{\mathtt{pw}}(u_\infty - \Pi_k u_\infty)\cdot\nabla_{\mathtt{pw}}\Pi_k\psi\,\mathrm{d}x \\
&\quad - \int_{\mathcal{S}_k^+}\{\nabla(u_\infty - \Pi_k u_\infty)\}\cdot[\![\Pi_k\psi]\!]\,\mathrm{d}s - \theta\int_{\mathcal{S}^+}\{\nabla\Pi_k\psi\}\cdot[\![u_\infty - \Pi_k u_\infty]\!]\,\mathrm{d}s \\
&\quad + \int_{\mathring{\mathcal{S}}^+}\big(\boldsymbol{\beta}\cdot[\![u_\infty - \Pi_k u_\infty]\!]\,[\![\nabla\Pi_k\psi]\!] + [\![\nabla u_\infty - \nabla\Pi_k u_\infty]\!]\,\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!]\big)\,\mathrm{d}s \\
&\quad + \int_\Omega\gamma\big(R_\infty([\![u_\infty - \Pi_k u_\infty]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![u_\infty - \Pi_k u_\infty]\!])\big) \\
&\qquad \cdot\big(R_\infty([\![\Pi_k\psi]\!]) + L_\infty(\boldsymbol{\beta}\cdot[\![\Pi_k\psi]\!])\big)\,\mathrm{d}x \\
&\quad + \int_{\mathcal{S}_k^+}\frac{\sigma}{h_k}\,[\![u_\infty - \Pi_k u_\infty]\!]\cdot[\![\Pi_k\psi]\!]\,\mathrm{d}s \\
&\lesssim |\!|\!| u_\infty - \Pi_k u_\infty |\!|\!|_\infty\, |\!|\!|\Pi_k\psi|\!|\!|_\infty = |\!|\!| u_\infty - \Pi_k u_\infty |\!|\!|_\infty\, |\!|\!|\Pi_k\psi|\!|\!|_k \\
&\lesssim |\!|\!| u_\infty - \Pi_k u_\infty |\!|\!|_\infty\, |\!|\!| u_k - \Pi_k u_\infty|\!|\!|_k\,;
\end{aligned}$$

here we used that $\Pi_k u_\infty, \Pi_k\psi \in \mathbb{V}_k \cap \mathbb{V}_\infty$, $h_\infty = h_k$ on $\mathcal{S}_k^+$ and that $\Pi_k u_\infty$ and $\Pi_k\psi$ are continuous on $\Omega\backslash\Omega_k^+$, i.e., $[\![\Pi_k u_\infty]\!] = [\![\Pi_k\psi]\!] = 0$ on $\mathcal{S}^+\backslash\mathcal{S}_k^+$, which follows from Lemma 16. Note that this and $[\![\Pi_k u_\infty]\!] = [\![\Pi_k\psi]\!] = 0$ on $\partial(\Omega\backslash\Omega_k^+)$ from Lemma 16 also implies that $L_k(\Pi_k\psi) = L_\infty(\Pi_k\psi)$ and $L_k(\Pi_k u_\infty) = L_\infty(\Pi_k u_\infty)$ as well as the corresponding relations between R_k and R_∞; compare with (3.1). Thus, the above estimate follows from the Cauchy-Schwarz inequality, application of inverse inequalities in conjunction with the stability of the lifting operators (3.2), and Lemma 17.

Consequently, triangle inequality and the above imply

$$\begin{aligned}
|\!|\!| u_\infty - u_k|\!|\!|_k &\leqslant |\!|\!| u_\infty - \Pi_k u_\infty|\!|\!|_k + |\!|\!| u_k - \Pi_k u_\infty|\!|\!|_k \\
&\lesssim |\!|\!| u_\infty - \Pi_k u_\infty|\!|\!|_k + |\!|\!| u_\infty - \Pi_k u_\infty|\!|\!|_\infty \\
&\qquad + \frac{\langle f,\, \psi - \Pi_k\psi\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, \psi - \Pi_k\psi]}{|\!|\!| u_k - \Pi_k u_\infty|\!|\!|_k}.
\end{aligned}$$

Thanks to $|\!|\!| u_\infty - \Pi_k u_\infty|\!|\!|_k \leqslant |\!|\!| u_\infty - \Pi_k u_\infty|\!|\!|_\infty$, this proves the assertion. □

The properties of the quasi-interpolation (3.4) allow for the consistency term in Lemma 20 to be bounded by the a posteriori indicators of essentially the elements, which will experience further refinements.

Lemma 21. *Let $u_\infty \in \mathbb{V}_\infty$ be the solution of (3.3) and $u_k \in \mathbb{V}_k$ be the DGFEM approximation from (2.6) on $\mathcal{G}_k$ for some $k \in \mathbb{N}$. Then, we have*

$$\frac{\langle f,\, u_k - \Pi_k u_k\rangle_\Omega - \mathfrak{B}_k[\Pi_k u_\infty,\, u_k - \Pi_k u_k]}{\|u_k - \Pi_k u_\infty\|_k} \lesssim \Big(\sum_{E\in\mathcal{G}_k\setminus\mathcal{G}_k^{3+}} \mathcal{E}_k(\Pi_k u_\infty, E)^2\Big)^{1/2},$$

where $\mathcal{G}_k^{3+} := \{E \in \mathcal{G}_k : N_k(E) \subset \mathcal{G}_k^{++}\}$.

Proof. Let $v_k := \Pi_k u_\infty$ and $\phi := u_k - \Pi_k u_k = u_k - \Pi_k u_\infty - \Pi_k(u_k - \Pi_k u_\infty)$. Then, using integration by parts, we have

$$\begin{aligned}
&\langle f,\, \phi\rangle_\Omega - \mathfrak{B}_k[v_k,\, \phi]\\
&\quad= \int_{\mathcal{G}_k} (f + \Delta v_k)\phi \,\mathrm{d}x - \int_{\mathcal{S}_k} [\![\nabla v_k]\!]\, \{\phi\}\,\mathrm{d}s + \int_{\mathcal{S}_k} \theta\{\nabla\phi\}\, [\![v_k]\!]\,\mathrm{d}s\\
&\qquad - \int_{\mathring{\mathcal{S}}_k} \big(\boldsymbol{\beta}\cdot[\![v_k]\!]\,[\![\nabla\phi]\!] + [\![\nabla v_k]\!]\,\boldsymbol{\beta}\cdot[\![\phi]\!]\,\big)\,\mathrm{d}s\\
&\qquad - \int_\Omega \gamma\big(R_k([\![v_k]\!]) + L_k(\boldsymbol{\beta}\cdot[\![v_k]\!])\big)\cdot\big(R_k([\![\phi]\!]) + L_k(\boldsymbol{\beta}\cdot[\![\phi]\!])\big)\,\mathrm{d}x\\
&\qquad - \sigma\int_{\mathcal{S}_k} h_k^{-1}\,[\![v_k]\!]\,[\![\phi]\!]\,\mathrm{d}s.
\end{aligned}$$

Thanks to properties of Π_k (see Lemma 16), we have that $[\![v_k]\!]\,|_S \equiv 0$ for $S \in \mathcal{S}_k\setminus\mathcal{S}_k^+$, $[\![v_k]\!]\,|_{\Omega\setminus\Omega_k^+} \equiv 0$, $\phi|_E \equiv 0$ for $E \in \mathcal{G}_k^{++}$, and $[\![\phi]\!]\,|_S \equiv 0$ for $S \in \mathcal{S}_k^{++}$. Therefore, we have

$$\begin{aligned}
&\langle f,\, \phi\rangle_\Omega - \mathfrak{B}_k[v_k,\, \phi]\\
&\quad= \int_{\mathcal{G}_k\setminus\mathcal{G}_k^{++}} (f + \Delta v_k)\phi \,\mathrm{d}x - \int_{\mathcal{S}_k\setminus\mathcal{S}_k^{++}} [\![\nabla v_k]\!]\, \{\phi\}\,\mathrm{d}s\\
&\qquad + \theta\int_{\mathcal{S}_k^+} \{\nabla\phi\}\, [\![v_k]\!]\,\mathrm{d}s\\
&\qquad - \int_{\mathring{\mathcal{S}}_k^+} \boldsymbol{\beta}\cdot[\![v_k]\!]\,[\![\nabla\phi]\!]\,\mathrm{d}s - \int_{\mathring{\mathcal{S}}_k\setminus\mathcal{S}_k^{++}} [\![\nabla v_k]\!]\,\boldsymbol{\beta}\cdot[\![\phi]\!]\,\mathrm{d}s\\
&\qquad - \int_\Omega \gamma\big(R_k([\![v_k]\!]) + L_k(\boldsymbol{\beta}\cdot[\![v_k]\!])\big)\cdot\big(R_k([\![\phi]\!]) + L_k(\boldsymbol{\beta}\cdot[\![\phi]\!])\big)\,\mathrm{d}x\\
&\qquad - \sigma\int_{\mathcal{S}_k^+\setminus\mathcal{S}_k^{++}} h_k^{-1}\,[\![v_k]\!]\,[\![\phi]\!]\,\mathrm{d}s
\end{aligned} \qquad (4.1)$$

The last term on the right-hand side of (4.1) can be estimated using Cauchy-Schwarz' inequality; for the first two terms we use the interpolation estimates from Corollary 18 for $\phi = \psi - \Pi_k\psi$ with $\psi = u_k - \Pi_k u_\infty \in \mathbb{V}_k$ as to obtain

$$\begin{aligned}
&\int_{\mathcal{G}_k\setminus\mathcal{G}_k^{++}} (f + \Delta v_k)\phi \,\mathrm{d}x - \int_{\mathcal{S}_k\setminus\mathcal{S}_k^{++}} [\![\nabla v_k]\!]\, \{\phi\}\,\mathrm{d}s\\
&\lesssim \left[\Big(\int_{\mathcal{G}_k\setminus\mathcal{G}_k^{++}} h_k^2|f+\Delta v_k|^2\,\mathrm{d}x\Big)^{1/2} + \Big(\int_{\mathcal{S}_k\setminus\mathcal{S}_k^{++}} h_k\,[\![\nabla v_k]\!]^2\,\mathrm{d}s\Big)^{1/2}\right]\|u_k - \Pi_k u_\infty\|_k\,.
\end{aligned}$$

Moreover, from $\phi|_E \equiv 0$, $E \in \mathcal{G}_k^{++}$, we have that $\phi|_{\omega_k(S)} \equiv 0$ and thus $\{\nabla\phi\}|_S \equiv 0$ for all $S \in \mathcal{S}_k^{3+} = \mathcal{S}(\mathcal{G}_k^{3+})$. Therefore, by standard trace inequalities, inverse estimates and Corollary 18, we have that

$$\int_{\mathcal{S}_k^+} \{\nabla\phi\}\,[\![v_k]\!]\,\mathrm{d}s = \int_{\mathcal{S}_k^+\setminus\mathcal{S}_k^{3+}} \{\nabla\phi\}\,[\![v_k]\!]\,\mathrm{d}s \lesssim \Big(\int_{\mathcal{S}_k^+\setminus\mathcal{S}_k^{3+}} h_k^{-1}\,[\![v_k]\!]^2\,\mathrm{d}s\Big)^{1/2}\|\phi\|_k\,.$$

A similar argument yields

$$\int_{\mathring{\mathcal{S}}_k^+} \boldsymbol{\beta} \cdot [\![v_k]\!] \, [\![\nabla \phi]\!] \, \mathrm{d}s = \int_{\mathring{\mathcal{S}}_k^+ \setminus \mathcal{S}_k^{3+}} \boldsymbol{\beta} \cdot [\![v_k]\!] \, [\![\nabla \phi]\!] \, \mathrm{d}s$$
$$\lesssim |\boldsymbol{\beta}| \Big(\int_{\mathring{\mathcal{S}}_k^+ \setminus \mathcal{S}_k^{3+}} h_k^{-1} \, [\![v_k]\!]^2 \, \mathrm{d}s \Big)^{1/2} \, |\!|\!| \phi |\!|\!|_k .$$

Finally we have with (2.4c) and the local support of the local liftings, that

$$\int_\Omega R_k([\![v_k]\!]) \cdot R_k([\![\phi]\!]) \, \mathrm{d}x = \int_\Omega \big(\sum_{S \in \mathcal{S}_k^+} R_k^S([\![v_k]\!]) \big) \cdot \big(\sum_{S \in \mathcal{S}_k \setminus \mathcal{S}_k^{++}} R_k^S([\![\phi]\!]) \big) \, \mathrm{d}x$$
$$= \int_{\mathcal{G}_k^+ \setminus \mathcal{G}_k^{++}} R_k([\![v_k]\!]) \cdot R_k([\![\phi]\!]) \, \mathrm{d}x$$
$$\lesssim \Big(\int_{\mathcal{S}_k^+ \setminus \mathcal{S}_k^{3+}} h_k^{-1} \, [\![v_k]\!]^2 \, \mathrm{d}s \Big)^{1/2} \, |\!|\!| \phi |\!|\!|_k .$$

Similar bounds hold for the remaining terms in (4.1). Combining the above observations proves the desired assertion. □

In order to conclude convergence of the sequence of discrete discontinuous Galerkin approximations from Lemma 21, we need to control the error estimator. To this end, we shall use Verfürth's bubble function technique.

Proposition 22. *Let u_∞ be the solution of* (3.3)*. Then, for every $E \in \mathcal{G}_k^-$ and $v \in \mathbb{V}_k$, $k \in \mathbb{N}$, we have*

$$\int_E h_k^2 |f + \Delta v|^2 \, \mathrm{d}x + \int_{\partial E \cap \Omega} h_k \, [\![\nabla_{pw} v]\!]^2 \, \mathrm{d}s$$
$$\lesssim \| \nabla_{pw}(u_\infty - v) \|_{\omega_k(E)}^2 + \int_{\{S \in \mathcal{S}^+ : S \subset \omega_k(E)\}} h_+^{-1} \, [\![u_\infty - v]\!]^2 \, \mathrm{d}s$$
$$+ \operatorname{osc}(N_k(E), f)^2;$$

in particular, we also have

$$\sum_{E \in \mathcal{G}_k^-} \int_E h_k^2 |f + \Delta v|^2 \, \mathrm{d}x + \int_{\partial E \cap \Omega} h_k \, [\![\nabla_{pw} v]\!]^2 \, \mathrm{d}s$$
$$\lesssim |\!|\!| u_\infty - v |\!|\!|_\infty^2 + \sum_{E \in \mathcal{G}_k^-} \sum_{E' \in \omega_k(E)} \operatorname{osc}(E', f)^2.$$

Note that since $v \in \mathbb{V}_k \not\subset \mathbb{V}_\infty$ in general, the above terms may be equal to infinity.

Proof. The proof follows from standard techniques; compare e.g. [KP03, BN10]. However, in order to keep the presentation self-contained, we provide a sketch of the proof. For $E \in \mathcal{G}_k^-$, let $\phi_E \in H_0^1(E)$ be Verfürth's element bubble function with

$$h_E^d \, \| \nabla q \phi \|_{L^\infty(E)}^2 \lesssim \| \nabla q \phi \|_E^2 \lesssim h_E^{-2} \, \| q \|_E^2 \quad \text{for all } q \in \mathbb{P}_{r-1}(E). \tag{4.2}$$

Note that extending ϕ_E by zero to the whole domain Ω, we have that $\phi_E \in \mathbb{V}_\infty$, since $E \subset \Omega^-$. Let $f_E \in \mathbb{P}_{r-1}(E)$ an arbitrary polynomial. Observing that $(f_E + \Delta v)\phi_E \in C(\Omega)$ and thus does not jump across faces, we have by equivalence of norms on finite dimensional spaces and a scaled trace inequality, that

$$\begin{aligned}
&\int_E |f_E + \Delta v|^2 \,\mathrm{d}x \\
&\quad\lesssim \int_E (f_E + \Delta v)(f_E + \Delta v)\phi_E \,\mathrm{d}x \\
&\quad= \mathfrak{B}_\infty[u_\infty - v,\, (f_E + \Delta v)\phi_E] - \int_E (f - f_E)(f_E + \Delta v)\phi_E \,\mathrm{d}x \\
&\quad\lesssim \|\nabla_{\mathtt{pw}}(u_\infty - v)\|_E \,\|\nabla(f_E + \Delta v)\phi_E\|_E - \int_{\mathcal{S}^+} [\![u_\infty - v]\!] \{\nabla(f_E + \Delta v)\phi_E\} \,\mathrm{d}s \\
&\qquad + \|f - f_E\|_E \,\|(f_E + \Delta v)\phi_E\|_E .
\end{aligned}$$

From (4.2) and standard inverse estimates, we conclude that

$$\begin{aligned}
&\left|\int_{\mathcal{S}^+} [\![u_\infty - v]\!] \{\nabla(f_E + \Delta v)\phi_E\} \,\mathrm{d}s\right| \\
&\qquad\leqslant \sum_{S\in\mathcal{S}^+, S\subset E} \int_S [\![u_\infty - v]\!]^2 \,\mathrm{d}s \,\|\nabla(f_E + \Delta v)\phi_E\|_{L^\infty(E)} \\
&\qquad\lesssim \Big(\int_{\mathcal{S}^+} h_+^{d-1} [\![u_\infty - v]\!]^2 \,\mathrm{d}s\Big)^{1/2} h_E^{-1-\frac{d}{2}} \|f_E + \Delta v\|_E \\
&\qquad\lesssim \Big(\int_{\mathcal{S}^+} h_+^{-1} [\![u_\infty - v]\!]^2 \,\mathrm{d}s\Big)^{1/2} h_E^{-1} \|f_E + \Delta v\|_E ,
\end{aligned}$$

since $h_+ \leqslant h_E$ on E. Therefore, we arrive at

$$\begin{aligned}
\int_E h_k^2 |f_E + \Delta v|^2 \,\mathrm{d}x \lesssim \|\nabla_{\mathtt{pw}}(u_\infty - v)\|_E^2 &+ \sum_{S\in\mathcal{S}^+, S\subset E} \int_S h_+^{-1} [\![u_\infty - v]\!]^2 \,\mathrm{d}s \\
&+ h_E^2 \|f - f_E\|_E^2 .
\end{aligned} \tag{4.3}$$

Thanks to the definition of $\mathcal{G}_k^-$, the same bound applies for all $E' \in N_k(E)$.

We now turn to investigate the jump terms. To this end, we fix one $S \in \mathring{\mathcal{S}}_k$, $S \subset E$ and let $E' \in N_k(E)$ with $S = E \cap E'$. Let $\phi_S \in H_0^1(\omega_k(S))$ be Verfürth's face bubble function. Note that extending ϕ_S by zero to Ω, we have $\phi_S \in \mathbb{V}_\infty$ since $\omega_k(S) \subset \Omega^-$. For each $q \in \mathbb{P}_{r-1}(S)$, there exists some extension $\tilde{q} \in \mathbb{P}_{r-1}(\omega_k(S))$ such that

$$h_E^d \,\|\nabla\tilde{q}\phi_S\|_{L^\infty(\omega_k(S))} \lesssim \|\tilde{q}\phi_S\|_{\omega_k(S)}^2 \lesssim h_E \int_S |q|^2 \,\mathrm{d}s. \tag{4.4}$$

Noting that $[\![\nabla v]\!] \in \mathbb{P}_{r-1}(S)$, we have, by the equivalence of norms on finite dimensional spaces, that

$$\begin{aligned}
\int_S [\![\nabla v]\!]^2 \,\mathrm{d}s &\lesssim \int_S [\![\nabla v]\!]^2 \phi_S \,\mathrm{d}s \\
&= \mathfrak{B}_\infty[u_\infty - v,\, \widetilde{[\![\nabla v]\!]}\phi_S] - \int_{\omega_k(S)} (f + \Delta v)\widetilde{[\![\nabla v]\!]}\phi_S \,\mathrm{d}x \\
&\lesssim \|\nabla_{\mathtt{pw}}(u_\infty - v)\|_{\omega_k(S)} \left\|\nabla\widetilde{[\![\nabla v]\!]}\phi_S\right\|_{\omega_k(S)} \\
&\quad + \int_{\mathcal{S}^+} [\![u_\infty - v]\!] \{\nabla\widetilde{[\![\nabla v]\!]}\phi_S\} \,\mathrm{d}s \\
&\quad + \big(\|f + \Delta v\|_E^2 + \|f + \Delta v\|_{E'}^2\big)^{\frac{1}{2}} \left\|\widetilde{[\![\nabla v]\!]}\phi_S\right\|_{\omega_k(S)} .
\end{aligned}$$

Similarly, as for the element residual, we have that

$$\int_{\mathcal{S}^+} [\![u_\infty - v]\!] \{\nabla \widetilde{[\![\nabla v]\!]}\phi_S\} \,\mathrm{d}s$$
$$\lesssim \Big(\sum_{S' \in \mathcal{S}^+, S' \subset \omega_k(S)} h_+^{-1} [\![u_\infty - v]\!]^2 \Big)^{\frac{1}{2}} \Big(\int_S h_E [\![\nabla v]\!]^2 \,\mathrm{d}s \Big)^{\frac{1}{2}},$$

using (4.4). Again with (4.4), we obtain

$$\int_S h_E [\![\nabla v]\!]^2 \,\mathrm{d}s \lesssim \|\nabla_{\mathtt{pw}}(u_\infty - v)\|^2_{\omega_k(S)} + \sum_{S' \in \mathcal{S}^+, S' \subset \omega_k(S)} \int_S h_+^{-1} [\![u_\infty - v]\!]^2 \,\mathrm{d}s$$
$$+ h_E^2 \|f + \Delta v\|_E^2 + h_{E'}^2 \|f + \Delta v\|_{E'}^2 .$$

Finally applying the bound (4.3) to $E, E' \in N_k(E)$, we have proved the first assertion.

The second assertion follows, then, by summing over all $E \in \mathcal{G}_k^-$ together with an observation from [MSV08], which we sketch here in order to keep this work self-contained. Let $M := \max\{\#N_k(E) : E \in \mathcal{G}_k^-\}$ be the maximal number of neighbours, then $\mathcal{G}_k^-$ can be split into $M^2 + 1$ subsets $\mathcal{G}_{k,0}^-, \ldots, \mathcal{G}_{k,M^2}^-$ such that for each j, we have that $E', E \in \mathcal{G}_{k,j}^-$ with $E \neq E'$ implies that $N_k(E) \cap N_k(E') = \emptyset$. Consequently, we have

$$\sum_{E \in \mathcal{G}_k^-} \|\nabla_{\mathtt{pw}}(u_\infty - v)\|^2_{\omega_k(E)} \leqslant \sum_{j=0}^{M^2} \sum_{E \in \mathcal{G}_{k,j}^-} \|\nabla_{\mathtt{pw}}(u_\infty - v)\|^2_{\omega_k(E)}$$
$$\leqslant (M^2 + 1) \|\nabla_{\mathtt{pw}}(u_\infty - v)\|^2_{\Omega_k^-} .$$

Together with similar estimates for the jump terms and the oscillations the second assertion follows from the first one. □

Theorem 23. *Let u_∞ the solution of* (3.3) *and $u_k \in \mathbb{V}_k$ be the DGFEM approximation from* (2.6) *on $\mathcal{G}_k$ for some $k \in \mathbb{N}$. Then,*

$$|\!|\!| u_\infty - u_k |\!|\!|_k \to 0 \quad \text{as } k \to \infty.$$

Proof. Thanks to Lemma 20, Lemma 19 and Lemma 21, we have that

$$\lim_{k\to\infty} |\!|\!| u_\infty - u_k |\!|\!|_k^2 \lesssim \lim_{k\to\infty} |\!|\!| u_\infty - v_k |\!|\!|_\infty^2 + \sum_{E \in \mathcal{G}_k \setminus \mathcal{G}_k^{3+}} \mathcal{E}_k(v_k, E)^2$$
$$= \lim_{k\to\infty} \sum_{E \in \mathcal{G}_k \setminus \mathcal{G}_k^{3+}} \mathcal{E}_k(v_k, E)^2,$$

where $v_k := \Pi_k u_\infty$. Using Lemma 11, we have

$$|\Omega \setminus (\Omega_k^- \cup \Omega_k^{3+})| \leqslant |\Omega \setminus (\Omega_k^- \cup \Omega_k^{++})| + |\Omega_k^{++} \setminus \Omega_k^{3+}|$$
$$\leqslant |\Omega_k^\star| + |\Omega^+ \setminus \Omega_k^{3+}| \to 0,$$

as $k \to \infty$. Indeed, for $k \in \mathbb{N}$, it follows from Lemma 10 and $\#\mathcal{G}_k^+ < \infty$, that there exists $K = K(k)$, such that $\mathcal{G}_k^+ \subset \mathcal{G}_K^{3+}$, i.e. $|\Omega^+ \setminus \Omega_K^{3+}| \leqslant |\Omega^+ \setminus \Omega_k^+| \to 0$ as $k \to \infty$. Thanks to monotonicity we conclude that $|\Omega^+ \setminus \Omega_k^{3+}| \to 0$ as $k \to \infty$. We next show that this implies

$$\sum_{E \in \mathcal{G}_k \setminus (\mathcal{G}_k^- \cup \mathcal{G}_k^{3+})} \mathcal{E}_k(v_k, E)^2 \to 0.$$

Lemma 19 implies that $|\!|\!| u_\infty - v_k |\!|\!|_\infty \to 0$ and, thus, the interior residual and the gradient jumps part of the estimator vanish due to uniform integrability. Moreover, it follows from Proposition 12 that

$$\int_{\mathcal{S}(\mathcal{G}_k\setminus(\mathcal{G}_k^-\cup\mathcal{G}_k^{3+}))} h_k^{-1}\,[\![v_k]\!]^2\,\mathrm{d}s \lesssim \int_{\mathcal{S}(\mathcal{G}_k\setminus\mathcal{G}_k^{3+})} h_k^{-1}\,[\![u_\infty]\!]^2\,\mathrm{d}s + |\!|\!|u_\infty - v_k|\!|\!|_k^2$$
$$\leqslant \int_{\mathcal{S}(\mathcal{G}^+\setminus\mathcal{G}_k^{3+})} h_+^{-1}\,[\![u_\infty]\!]^2\,\mathrm{d}s + |\!|\!|u_\infty - v_k|\!|\!|_k^2.$$

The last term on the right-hand side of the above estimate vanishes thanks to Lemma 19. Again, letting $K = K(k)$, such that $\mathcal{G}_k^+ \subset \mathcal{G}_K^{3+}$, we have

$$\int_{\mathcal{S}(\mathcal{G}^+\setminus\mathcal{G}_{K(k)}^{3+})} h_+^{-1}\,[\![u_\infty]\!]^2\,\mathrm{d}s \leqslant \int_{\mathcal{S}(\mathcal{G}^+\setminus\mathcal{G}_k^{+})} h_+^{-1}\,[\![u_\infty]\!]^2\,\mathrm{d}s \to 0, \quad \text{as } k\to\infty.$$

Thanks to monotonicity, we thus conclude $\int_{\mathcal{S}(\mathcal{G}^+\setminus\mathcal{G}_k^{3+})} h_+^{-1}\,[\![u_\infty]\!]^2\,\mathrm{d}s \to 0$, as $k\to\infty$.

On the remaining elements $\mathcal{G}_k^-$, it follows from Proposition 22 that

$$\sum_{E\in\mathcal{G}_k^-} \mathcal{E}_k(v_k,E)^2 \lesssim |\!|\!|u_\infty - v_k|\!|\!|_\infty^2 + \sum_{E\in\mathcal{G}_k^-} \operatorname{osc}(N_k(E),f)^2.$$

The first term on the right-hand side vanishes due to Lemma 19. For the second term we observe that $|\bigcup\{\omega_k(E) : E\in\mathcal{G}_k^-\}| \lesssim |\Omega_k^-|$, depending on the shape regularity of $\mathcal{G}_0$ and, therefore, it vanishes since

$$\left\|h_k\chi_{\Omega_k^-}\right\|_{L^\infty(\Omega)} \to 0 \quad \text{as } k\to\infty, \tag{4.5}$$

thanks to Lemma 11. □

5. Proof of the main result

We are now in the position to prove that the error estimator vanishes, following the ideas of [MSV08]. This in turn implies that the sequence of discontinuous Galerkin approximations produced by ADGM indeed converges to the exact solution of (2.1).

Lemma 24. *We have that*

$$\mathcal{E}_k(\mathcal{G}_k^-) \to 0, \quad \textit{as } k\to\infty.$$

Proof. Thanks to Proposition 22, we have

$$\sum_{E\in\mathcal{G}_k^-} \int_E h_k^2|f+\Delta u_k|^2\,\mathrm{d}x + \int_{\partial E\cap\Omega} h_k\,[\![\nabla u_k]\!]^2\,\mathrm{d}s$$
$$\lesssim |\!|\!|u_\infty - u_k|\!|\!|_\infty^2 + \sum_{E\in\mathcal{G}_k^-} \operatorname{osc}(N_k(E),f)^2.$$

The right-hand side vanishes thanks to Theorem 23 and (4.5).

It remains to prove that

$$\int_{\mathcal{S}(\mathcal{G}_k^-)} h_k^{-1}\,[\![u_k]\!]^2\,\mathrm{d}s \to 0, \quad \text{as } k\to\infty.$$

By definition, $\Omega_k^- \subset \Omega\setminus\Omega_k^+$ and, thanks to Lemma 16(5), we have that $\Pi_k u_\infty \in C(\Omega\setminus\Omega_k^+)$. Therefore, we conclude

$$\int_{\mathcal{S}(\mathcal{G}_k^-)} h_k^{-1}\,[\![u_k]\!]^2\,\mathrm{d}s = \int_{\mathcal{S}(\mathcal{G}_k^-)} h_k^{-1}\,[\![u_k - \Pi_k u_\infty]\!]^2\,\mathrm{d}s \leqslant |\!|\!|u_k - \Pi_k u_\infty|\!|\!|_k \to 0$$

as $k\to\infty$; see Lemma 19 and Theorem 23. □

Lemma 25. *We have that*

$$\lim_{k\to\infty} \mathcal{E}_k(\mathcal{G}_k^\star) = 0.$$

Proof. We conclude from the lower bound (Proposition 6) that

$$\sum_{E\in\mathcal{G}_k^\star}\int_E h_k^2|f+\Delta u_k|^2\,\mathrm{d}x+\int_{\partial E}h_k\,[\![\nabla u_k]\!]^2\,\mathrm{d}s$$
$$\lesssim \sum_{E\in\mathcal{G}_k^\star}\|u-u_k\|^2_{\omega_k(E)}+\left\|\nabla u-\nabla_{\mathtt{pw}}u_k\right\|^2_{\omega_k(E)}+\operatorname{osc}(N_k(E),f)^2$$
$$\lesssim \sum_{E\in\mathcal{G}_k^\star}\Big\{\|u\|^2_{\omega_k(E)}+\|u_\infty-u_k\|^2_{\omega_k(E)}+\|u_\infty\|^2_{\omega_k(E)}$$
$$+\|\nabla u\|^2_{\omega_k(E)}+\left\|\nabla_{\mathtt{pw}}u_\infty-\nabla_{\mathtt{pw}}u_k\right\|^2_{\omega_k(E)}+\left\|\nabla_{\mathtt{pw}}u_\infty\right\|^2_{\omega_k(E)}$$
$$+\operatorname{osc}(N_k(E),f)^2\Big\}.$$

This vanishes as $k\to\infty$ thanks to Theorem 23 and Lemma 11, together with the uniform integrability of the terms involving u and u_∞. Note that $\left|\bigcup\{\omega_k(E):E\in\mathcal{G}_k^\star\}\right|\lesssim|\Omega_k^\star|$, with the constant depending on the shape regularity of $\mathcal{G}_0$.

It remains to prove

$$\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![u_k]\!]^2\,\mathrm{d}s\to 0,\quad \text{as } k\to\infty.$$

To this end, we observe that

$$\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![u_k]\!]^2\,\mathrm{d}s=\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![u_k-\Pi_k u_\infty]\!]^2\,\mathrm{d}s+\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![\Pi_k u_\infty]\!]^2\,\mathrm{d}s$$
$$\leqslant\frac{1}{\bar{\sigma}}\,|\!|\!|u_k-\Pi_k u_\infty|\!|\!|_k^2+\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![\Pi_k u_\infty]\!]^2\,\mathrm{d}s.$$

As in the proof of Lemma 24, we have that the first term vanishes as $k\to\infty$. Thanks to Lemma 10, there exists $\ell(k)\geqslant K(k)\geqslant k$ such that $\mathcal{G}_k^+\subset\mathcal{G}_{K(k)}^{++}$ and $\mathcal{G}_{K(k)}^+\subset\mathcal{G}_{\ell(k)}^{++}$. Consequently, we have that $[\![\Pi_\ell u_\infty]\!]\,|_S=0$ for all $S\in\mathcal{G}_k$; see Lemma 16(5). Therefore, we conclude from Lemma 19 that

$$\sigma\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![\Pi_k u_\infty]\!]^2\,\mathrm{d}s=\sigma\int_{\mathcal{S}(\mathcal{G}_k^\star)}h_k^{-1}\,[\![\Pi_k u_\infty-\Pi_\ell u_\infty]\!]^2\,\mathrm{d}s$$
$$\lesssim|\!|\!|\Pi_k u_\infty-u_\infty|\!|\!|_k^2+|\!|\!|u_\infty-\Pi_\ell u_\infty|\!|\!|_\ell^2\to 0,$$

as $k\to\infty$. □

Lemma 26. *We have*

$$\mathcal{E}_k(\mathcal{G}_k^{++})\to 0\quad \textit{as } k\to\infty.$$

Proof. **Step 1:** By definition, elements in $\mathcal{G}_k^{++}$ will not be subdivided, i.e. we have that $\mathcal{M}_k\subset\mathcal{G}_k\backslash\mathcal{G}_k^{++}$; compare with (2.9). As a consequence of Lemmas 24 and 25, we conclude from (2.8) for all $E\in\mathcal{G}_k^{++}$ that

$$\mathcal{E}_k(E)\leqslant\lim_{k\to\infty}g(\mathcal{E}_k(\mathcal{M}_k))=\lim_{k\to\infty}g(\mathcal{E}_k(\mathcal{G}_k^-\cup\mathcal{G}_k^\star))\to 0, \tag{5.1}$$

as $k\to\infty$. We shall reformulate the above element-wise convergence in an integral framework, in order to conclude $\mathcal{E}_k(\mathcal{G}_k^{++})\to 0$ as $k\to\infty$ via a generalised version of the dominated convergence theorem. To this end, we shall consider some properties of the error indicators.

Step 2: Thanks to the definition of $\mathcal{G}_k^{++}$, we have for all $E\in\mathcal{G}_k^{++}$, that $\omega_k(E)=\omega_\ell(E)=:\omega(E)$ and $N_k(E)=N_\ell(E)=N(E)$ for all $\ell\geqslant k$. Therefore, we obtain by the lower bound, Proposition 6, that

$$\begin{aligned}\mathcal{E}_k(E)^2 &\lesssim |||u_k - u|||^2_{N(E)} + \operatorname{osc}(N(E), f)^2 \\ &\lesssim |||u_k - u_\infty|||^2_{N(E)} + |||u_\infty|||^2_{N(E)} + \|u\|^2_{H^1(\omega(E))} + \|f\|^2_{\omega(E)} \\ &=: |||u_k - u_\infty|||^2_{N(E)} + C_E^2.\end{aligned} \tag{5.2}$$

Arguing as in the proof of Proposition 22, we can conclude from the local estimate that

$$\sum_{E \in \mathcal{G}_k^{++}} C_E^2 \lesssim |||u_\infty|||^2_\infty + \|u\|^2_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)} < \infty \tag{5.3}$$

independently of k.

Step 3: We shall now reformulate $\mathcal{E}_k(\mathcal{G}_k^{++})$ in integral form. Note that thanks to Lemma 10, we have that $\mathcal{G}^+ = \bigcup_{k\in\mathbb{N}_0} \mathcal{G}_k^+ = \bigcup_{k\in\mathbb{N}_0} \mathcal{G}_k^{++}$, and also that the sequence $\{\mathcal{G}_k^{++}\}_{k\in\mathbb{N}_0}$ is nested. For $x \in \Omega^+$, let

$$\ell = \ell(x) := \min\{k \in \mathbb{N}_0 : \text{ there exists } E \in \mathcal{G}_k^{++} \text{ such that } x \in E\}.$$

Then, we define

$$\epsilon_k(x) := M_k(x) := 0 \quad \text{for } k < \ell,$$

and

$$\epsilon_k(x) := \frac{1}{|E|}\mathcal{E}_k^2(E), \qquad M_k := \frac{1}{|E|}\Big(|||u_k - u_\infty|||^2_{N(E)} + C_E^2\Big) \quad \text{for } k \geqslant \ell.$$

Consequently, for any $k \in \mathbb{N}_0$, we have

$$\mathcal{E}_k(\mathcal{G}_k^{++})^2 = \int_{\Omega^+} \epsilon_k(x)\,\mathrm{d}x.$$

Moreover, thanks to the fact that the sequence $\{\mathcal{G}_k^{++}\}_{k\in\mathbb{N}_0}$ is nested, we conclude from (5.1) that

$$\lim_{k\to\infty} \epsilon_k(x) = \lim_{k\to\infty} \frac{1}{|E|}\mathcal{E}_k^2(E) = 0.$$

It follows from (5.2) and (5.3) that M_k is an integrable majorant for ϵ_k.

Step 4: We shall show that the majorants $\{M_k\}_{k\in\mathbb{N}_0}$ converge in $L^1(\Omega^+)$ to

$$M(x) := \frac{1}{|E|}C_E^2, \quad \text{for } x \in E \quad \text{and} \quad E \in \mathcal{G}^+.$$

Then the assertion follows from a generalised majorised convergence theorem; see [Zei90, Appendix (19a)]. In fact, by the definition of M_k, we have that

$$\|M_k - M\|_{L^1(\Omega^+)} = \sum_{E\in\mathcal{G}_k^{++}} \|M_k - M\|_{L^1(E)} + \sum_{E\in\mathcal{G}^+\setminus\mathcal{G}_k^{++}} \|M\|_{L^1(E)}\,.$$

The latter term vanishes since it is the tail of a converging series (compare with (5.3)) and for the former term, we have, thanks to Theorem 23, that

$$\sum_{E\in\mathcal{G}_k^{++}} \|M_k - M\|_{L^1(E)} = \sum_{E\in\mathcal{G}_k^{++}} |||u_k - u_\infty|||^2_{N(E)} \lesssim |||u_k - u_\infty|||_k \to 0$$

as $k \to \infty$. □

Proof of Theorem 9. We have

$$\mathcal{G}_k^{++} \cup \mathcal{G}_k^\star \cup \mathcal{G}_k^- = \mathcal{G}_k.$$

Therefore, the claim follows from Lemmas 24, 25, and 26 together with Proposition 5. □

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