No. 596 February 2019

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ISSN: 2190-1767

JOURNAL ? (????), 1–36
DOI 10.1515/JOURNAL-????-??? © de Gruyter ????

Goal oriented a posteriori error estimators for problems with modified discrete formulations based on the dual weighted residual method

Dustin Kumor and Andreas Rademacher

Abstract. The article at hand focuses on finite element discretizations, where the continuous and the discrete formulations differ. We introduce a general approach based on the dual weighted residual method for estimating on the one hand the discretization error in a user specified quantity of interest and on the other hand the discrete model error induced by using different discrete techniques. Here, the usual error identities are obtained plus some additional terms. Furthermore, the numerical approximation of the error identities is discussed. As a simple example, we consider selective reduced integration for stabilizing the finite element discretization of linear elastic problems with nearly incompressible material behavior. This example fits well in the general setting. However, one has to be very careful in the numerical approximation of the error identities, where different reconstruction techniques have to be used for the additional terms due to the deviating discrete bi-linear form. Numerical examples substantiate the accuracy of the a posteriori error estimators and the efficiency of the adaptive methods based on them.

Keywords. structural mechanics, a posteriori error estimation, dual weighted residual method, adaptive finite element method, model adaptivity.

2010 Mathematics Subject Classification. 65N30; 65N15.

1 Introduction

Sheet-bulk metal forming is a comparatively new production process, which combines the classical techniques of sheet metal and bulk forming. We refer to [22] for a detailed description of this process. It is characterized in large parts of the workpiece by two dimensional stress states, which can accurately and efficiently be simulated by shell or solid-like shell elements. However, in some smaller parts three dimensional stress states occur, for instance if teeth are molded in the production of gear wheels. Thus, one wants to use different finite element types in

The authors gratefully acknowledge the financial support by the German Research Foundation (DFG) within the subproject A5 of the transregional collaborative research centre (Transregio) 73 “Sheet-Bulk Metal Forming”.

one mesh. In order to choose the element type and to obtain an efficient triangulation of the workpiece automatically, a posteriori error estimators and thereon based adaptive methods are a good choice.

In practical situations, one is interested in the estimation of the error in a user defined functional, which is often called quantity of interest. This case usually implies the use of dual or adjoint techniques, see, for instance, [15]. One approach is based on using the Cauchy-Schwarz-inequality and a posteriori error estimators, which estimate the error in the energy norm, for the primal and the dual problem. We refer, e.g., to [2, 28, 29]. In [6, 14, 24], the optimal convergence of adaptive methods based on this approach is proven. However, it is limited to linear problems and linear quantities of interest. In contrast, the dual weighted residual (DWR) method, see, e.g., [4, 7], can cope even with highly nonlinear problems. It is based on an analytic error identity, which has to be approximated numerically. Hence, the resulting error estimators are not reliable nor efficient. For linear problems, a safeguarded DWR method is proposed in [25], which ensures the reliability of the DWR method.

In this contribution, we consider also somehow model adaptive algorithms. Dimension adaptivity, which fits in the presented framework, is considered in [1, 8, 9, 33, 34]. Here, the combination of volume elements with shells or plates are discussed. Heterogeneous linear elastic models and their homogenization are in the focus of [26, 27], where the automatic selection of the local model is of major importance. The choice of the model is based on a posteriori error estimates with respect to the error measured in the energy norm as well as to linear quantities of interest. The adaptive coupling of different models used in electrocardialogy is presented in [23]. The basic result concerning the extension of the DWR method to control modeling errors and to adaptively choose the model is introduced in [10]. The estimation of the modeling error is basically given by entering the solution to the coarse model into the fine one weighted by the dual solution. The article [10] focuses on diffusion-reaction-equations with highly oscillating coefficients. The results are extended to time dependent problems in [11]. This approach has been applied on structural mechanic problem settings as well. The combination of linear and nonlinear elasticity is discussed in [17], while thermoplastic problems with damage are in the focus of [16]. Frictional contact problems with a model hierarchy consisting of several (nonlinear) friction laws are examined in [30].

As outlined in the beginning, a posteriori error estimators provide a good basis for the choice of the triangulation and of the element type in many situations. The focus of this article is on the development of a general approach to derive goal-oriented a posteriori error estimates in such problem settings based on the DWR method. To this end, we split the task into three different steps. At first, we consider the case, where only one element type is used. However, the continuous problem setting and the discrete one differ. We derive some error identities in this case involving, for instance, the primal residual as well as the primal and the dual one. They are marked by the fact that we get the usual residual terms of the DWR method plus some additional residuals connected to the difference between the continuous and the discrete semi-linear forms. In a second step, we examine the model error due to the use of different discrete semi-linear forms and corresponding function spaces. Here, one gets an analogous error identity to [10] in the discrete semi-linear forms plus some additional terms due to the varying discrete function spaces, where the additional terms simplify or vanish under certain additional assumptions on the discrete function spaces. As usual in the DWR method, the error identities cannot be evaluated numerically. Thus, in the third step, we discuss there numerical approximation in general, where we concentrate on the modeling error. Several considerations on the possibility of neglecting some terms are outlined.

The second part of the article is dedicated to the application of the general framework on a comparatively simple model problem. We consider selective reduced integration in linear elasticity in comparison to fully integrated elements. While the application of the general framework to estimate the discretization error is straight forward, the numerical results using the standard numerical approximation techniques are somehow disappointing. The reason behind can been seen by the interpretation of selective reduced integration as mixed method, which shows that the numerical reconstruction of the reduced integrated terms corresponding to the pressure have to be carried out using other techniques than for the displacement related terms. We apply an average technique rather than a patchwise bi-quadratic reconstruction. For the model error, we check that the error identity can be reduced to numerically evaluable terms. Furthermore, the resulting estimator leads to an accurate estimation of the model error and works fine as basis of a model adaptive algorithm.

The article consists of two main parts. In the first part, Section 2, the general approach for estimating the error in a user specified quantity of interest is discussed. In Section 2.1, an error identity with respect to the discretization error is derived. It is extended to the model error in Section 2.2. Finally, the numerical approximation of both error identities is discussed in Section 2.3. The second part, Section 3, focuses on the application of the general approach to a model problem setting, here a FE discretization of linear elasticity using selective reduced integration. After introducing the problem formulation, Section 3.1 is devoted to the derivation of the error estimator with respect to the discretization error. It is tested in some numerical examples in Section 3.2. In Section 3.3, an alternative approach based on the corresponding mixed formulation is presented and combined with the general framework. In Section 3.4 and 3.5, the estimation of the model error in comparison to the not reduced integrated bi-linear form is considered. Further, a model adaptive algorithm is proposed, whose applicability is substantiated by some numerical examples. The article closes with a conclusion and an outlook on further research topics.

2 General approach

For our investigations we consider a problem formulation of the following form: Find $u \in V$ so that

$$\mathcal{A}(u)(\varphi) = 0 \quad \forall \varphi \in W, \tag{1}$$

with $\mathcal{A} : V \times W \to \mathbb{R}$ being a semi-linear form representing the weak form of the underlying problem as well as with V and W suitable function spaces. Moreover, a quantity of interest is given by a possibly nonlinear functional $J : V \to \mathbb{R}$. The main goal is to determine the value $J(u)$ for u being a solution of equation (1). Assuming u to be unique the quantity of interest can be connected to the problem formulation by reformulating the whole framework into a trivial optimization problem, see [7, Section 2] for further details. Thus we have to find a stationary point of the Lagrangian functional

$$\mathsf{L}(y) = \mathcal{L}(\psi, \varphi) := J(\psi) - \mathcal{A}(\psi)(\varphi),$$

with $y = (\psi, \varphi) \in V \times W$.

Let us assume for now that the discrete problem formulation is not just given by replacing the continuous functions by its discrete counterparts but also by a modification of the semi-linear form $\mathcal{A}$, i.e. for finite dimensional subspaces $V_h \subset V$ and $W_h \subset W$ we get: Find $u_h \in V_h$ so that

$$\mathcal{A}_h(u_h)(\varphi_h) = 0 \quad \forall \varphi_h \in W_h. \tag{2}$$

With this modification the corresponding discrete Lagrangian functional is defined by

$$\mathsf{L}_h(y_h) = \mathcal{L}_h(\psi_h, \varphi_h) := J(\psi_h) - \mathcal{A}_h(\psi_h)(\varphi_h),$$

with $y_h = (\psi_h, \varphi_h) \in V_h \times W_h$. Similar investigations have been carried out in [7, Section 2.3] on different assumptions though. The main goal therein is the treatment of stabilization techniques demanding a "consistence" requirement of the form $\mathcal{A}_h(u)(\varphi) = 0$ for all $\varphi \in W$. In contrast to that the ongoing investigations ignore this consistence property and take into account possible numerical errors as well, i.e. equation (2) is not fulfilled exactly. Further we recall that for a stationary point $x = (u, z)$ of L the following identities hold:

$$\mathsf{L}(x) = J(u), \tag{3}$$

$$\mathsf{L}'(x)(y) = 0 \qquad \forall y \in W. \tag{4}$$

2.1 Error identity for discretization error

Within this section, we establish an error identity for the discretization error with respect to the quantity of interest J based on the framework of the dual weighted residual method. Since the discrete solution x_h is not a stationary point of the Lagrangian functional L but only of the discrete one L_h, the standard DWR approach, cf., for instance, [7, Section 2.1], is not applicable. Therefore, we introduce a functional $\delta\mathsf{L}$ describing the difference between the continuous and the discrete functional, i.e.

$$\delta\mathsf{L}(y) := \mathsf{L}(y) - \mathsf{L}_h(y) = \mathcal{A}_h(\psi)(\varphi) - \mathcal{A}(\psi)(\varphi),$$

and split L up into

$$\mathsf{L}(y) = \mathsf{L}_h(y) + \delta\mathsf{L}(y).$$

With this auxiliary functional we may affiliate the new situation to the standard framework and are able to establish an error identity of the following form:

Theorem 2.1. *Let J and $\mathcal{A}$ be three times as well as $\mathcal{A}_h$ at least once directional differentiable in the first argument. Furthermore, u and u_h are the solutions of equation* (1) *and* (2)*, respectively. Then we have the error representation*

$$\begin{aligned} J(u) - J(u_h) = {} & \frac{1}{2}\rho_h(u_h)(e_z) + \frac{1}{2}\rho_h^\star(u_h, z_h)(e_u) \\ & + \frac{1}{2}\Delta\rho(u_h)(e_z) + \frac{1}{2}\Delta\rho^\star(u_h, z_h)(e_u) \\ & + \rho_h(u_h)(z_h) + \Delta\rho(u_h)(z_h) + \mathcal{R}_h^{(3)} \end{aligned} \tag{5}$$

with

$$\rho_h(u_h)(\cdot) := -\mathcal{A}_h(u_h)(\cdot), \tag{6}$$

$$\rho_h^\star(u_h, z_h)(\cdot) := J'(u_h)(\cdot) - \mathcal{A}_h'(u_h)(\cdot, z_h), \tag{7}$$

$$\Delta\rho(u_h)(\cdot) := \mathcal{A}_h(u_h)(\cdot) - \mathcal{A}(u_h)(\cdot), \tag{8}$$

$$\Delta\rho^\star(u_h, z_h)(\cdot) := \mathcal{A}_h'(u_h)(\cdot, z_h) - \mathcal{A}'(u_h)(\cdot, z_h). \tag{9}$$

The remainder term $\mathcal{R}_h^{(3)}$ is cubic in the "primal" and "dual" error $e_u = u - u_h$

and $e_z = z - z_h$,

$$\mathcal{R}_h^{(3)} := \frac{1}{2} \int_0^1 \big[J''' (u_h + se_u) (e_u, e_u, e_u) \\ - \mathcal{A}''' (u_h + se_u) (e_u, e_u, e_u, z_h + se_z) \\ - 3\mathcal{A}'' (u_h + se_u) (e_u, e_u, e_z) \big] \, s (s - 1) \; ds. \tag{10}$$

Next to the components of the standard error identity, i.e. the primal residual $\rho_h (u_h) (e_z)$, the dual residual $\rho_h^\star (u_h, z_h) (e_u)$ and the remainder term $\mathcal{R}_h^{(3)}$, see [7, Sections 2.2 and 2.3], equation (5) consists of four additional terms. The first two, i.e. $\Delta\rho (u_h) (e_z)$ and $\Delta\rho^\star (u_h, z_h) (e_u)$, measure the difference between the respective continuous and the discrete residual weighted with a continuous function. The term $\rho_h (u_h) (z_h)$ represents the quality of the numerical solution process and the last one $\Delta\rho (u_h) (z_h)$ describes the difference between the two semi-linear forms weighted by the discrete dual solution.

Proof. With equation (3) and the definition of $\delta\mathsf{L}$, we get

$$\begin{aligned} J (u) - J (u_h) &= \mathsf{L} (x) - \mathsf{L} (x_h) + \delta\mathsf{L} (x_h) - \mathcal{A}_h (u_h) (z_h) \\ &= \int_0^1 \mathsf{L}' (x_h + se_x) (e_x) \; ds + \delta\mathsf{L} (x_h) - \mathcal{A}_h (u_h) (z_h) \\ &= \int_0^1 \mathsf{L}' (x_h + se_x) (e_x) \; ds + \Delta\rho (u_h) (z_h) + \rho_h (u_h) (z_h) \end{aligned}$$

by applying the fundamental theorem of calculus. Approximating the integral with the trapezoidal rule and its remainder term, we end up with

$$\begin{aligned} J (u) - J (u_h) = &\frac{1}{2}\mathsf{L}' (x_h) (e_x) + \frac{1}{2}\mathsf{L}' (x) (e_x) + \mathcal{R}_h^{(3)} \\ &+ \Delta\rho (u_h) (z_h) + \rho_h (u_h) (z_h) . \end{aligned}$$

Since x is a stationary point of L the term $\mathsf{L}' (x) (e_x)$ vanishes. Moreover, by splitting the continuous Lagrangian into the discrete and difference functional part and using the definition of the Lagrangian we have

$$\begin{aligned} \mathsf{L}' (x_h) (e_x) &= (\mathcal{L}_h)'_{u_h} (u_h, z_h) (e_u) + (\mathcal{L}_h)'_{z_h} (u_h, z_h) (e_z) + \delta\mathsf{L}' (x_h) (e_x) \\ &= J' (u_h) (e_u) - \mathcal{A}_h' (u_h) (e_u, z_h) - \mathcal{A}_h (u_h) (e_z) + \delta\mathsf{L}' (x_h) (e_x) \\ &= \rho_h^\star (u_h, z_h) (e_u) + \rho_h (u_h) (e_z) + \Delta\rho (u_h) (e_z) \\ &\quad + \Delta\rho^\star (u_h, z_h) (e_u) . \end{aligned}$$

At last we have to deal with the remainder term $\mathcal{R}_h^{(3)}$ that is given by

$$\mathcal{R}_h^{(3)} = \frac{1}{2}\int_0^1 \mathsf{L}'''(x_h + se_x)(e_x, e_x, e_x)\, s(s-1)\, ds$$

due to the applied trapezoidal rule. Calculating the third derivative completes the proof. □

Remark 2.2. Obviously, we can combine several terms in equation (5). Using definitions (6) – (9) the established error identity may be written as follows,

$$J(u) - J(u_h) = \frac{1}{2}\rho(u_h)(e_z) + \frac{1}{2}\rho^\star(u_h, z_h)(e_u) + \Delta(u_h)(z_h) + \mathcal{R}_h^{(3)},$$

with

$$\begin{aligned}
\rho(u_h)(e_z) &:= \rho_h(u_h)(e_z) + \Delta\rho(u_h)(e_z) \\
&= -\mathcal{A}(u_h)(e_z), \\
\rho^\star(u_h, z_h)(e_u) &:= \rho_h^\star(u_h, z_h)(e_u) + \Delta\rho^\star(u_h, z_h)(e_u) \\
&= J'(u_h)(e_u) - \mathcal{A}'(u_h)(e_u, z_h)
\end{aligned}$$

and

$$\Delta(u_h)(z_h) := \rho_h(u_h)(z_h) + \Delta\rho(u_h)(z_h) = -\mathcal{A}(u_h)(z_h).$$

However, the form presented in Theorem 2.1 points out the different sources of the error more precisely. Further, the detailed splitting becomes important when talking about the numerical approximation of the error identity for a specific problem as we will see in Section 3.

Remark 2.3. Assuming equation (2) is solved exactly and this holds for the corresponding dual problem as well then we also have

$$\mathsf{L}_h(x_h) = J(u_h), \tag{11}$$

$$\mathsf{L}'(x_h)(y_h) = 0 \qquad \forall y_h \in W_h. \tag{12}$$

Hence, it holds $\rho_h(u_h)(\varphi_h) = 0$ and $\rho_h^\star(u_h, z_h)(\psi_h) = 0$ for all $\varphi_h \in W_h, \psi_h \in V_h$ and equation (5) reduces to

$$\begin{aligned}
J(u) - J(u_h) = {}& \frac{1}{2}\rho_h(u_h)(z - \varphi_h) + \frac{1}{2}\rho_h^\star(u_h, z_h)(u - \psi_h) \\
&+ \frac{1}{2}\Delta\rho(u_h)(e_z) + \frac{1}{2}\Delta\rho^\star(u_h, z_h)(e_u) + \mathcal{R}_h^{(3)}
\end{aligned}$$

for arbitrary functions $\varphi_h \in W_h$ and $\psi_h \in V_h$.

Similar to the standard DWR approach there is a relationship between the primal and the dual residual. In contrast to the original identity, compare [7, Proposition 2.3] we have to consider the additional terms as well.

Theorem 2.4. *Assuming the same setting as described in Theorem* 2.1 *the dual residual can be expressed in terms of the primal residual, the two additional terms and the linearisation error* $\delta\rho$*,*

$$\begin{aligned} \rho_h^\star(u_h, z_h)(e_u) = {} & \rho_h(u_h)(e_z) + \Delta\rho(u_h)(e_z) \\ & - \Delta\rho^\star(u_h, z_h)(e_u) + \delta\rho \end{aligned} \tag{13}$$

with

$$\delta\rho := \int_0^1 \left[\mathcal{A}''(u_h + se_u)(e_u, e_u, z_h + se_z) - J''(u_h + se_u)(e_u, e_u)\right] ds.$$

Moreover we obtain a simplified error identity

$$J(u) - J(u_h) = \rho_h(u_h)(e_z) + \Delta\rho(u_h)(z) + \rho_h(u_h)(z_h) + \mathcal{R}_h^{(2)} \tag{14}$$

with a remainder term of second order in the primal error e_u

$$\mathcal{R}_h^{(2)} := \int_0^1 \left[\mathcal{A}''(u_h + se_u)(e_u, e_u, z) - J''(u_h + se_u)(e_u, e_u)\right] s\, ds.$$

Proof. We introduce a scalar auxiliary function g by

$$g(s) := \mathcal{L}_u'(u_h + se_u, z_h + se_z)(e_u)$$

with

$$g(1) = \mathcal{L}_u'(u, z)(e_u) = 0$$

due to equation (4) and

$$g'(s) = \mathcal{L}_{uu}''(u_h + se_u, z_h + se_z)(e_u, e_u) + \mathcal{L}_{uz}''(u_h + se_u, z_h + se_z)(e_u, e_z).$$

Using the definition of the Lagrangian functional and the possibility to split it into a discrete and a continuous part as well as the fundamental theorem of calculus leads to

$$\rho_h^\star(u_h, z_h)(e_u) = (\mathcal{L}_h)_{u_h}'(u_h, z_h)(e_u)$$

$$
\begin{aligned}
&= \mathcal{L}'_u(u_h, z_h)(e_u) - \delta\mathcal{L}'_u(u_h, z_h)(e_u) \\
&= g(0) - g(1) - \delta\mathcal{L}'_u(u_h, z_h)(e_u) \\
&= -\int_0^1 g'(s)\, ds - \delta\mathcal{L}'_u(u_h, z_h)(e_u) \\
&= -\int_0^1 \mathcal{L}''_{uu}(u_h + se_u, z_h + se_z)(e_u, e_u)\, ds \\
&\quad + \mathcal{L}'_z(u_h, z_h)(e_z) - \mathcal{L}'_z(u, z)(e_z) - \delta\mathcal{L}'_u(u_h, z_h)(e_u) \\
&= \delta\rho + (\mathcal{L}_h)'_{z_h}(u_h, z_h)(e_z) + \mathcal{A}_h(u_h)(e_z) \\
&\quad - \mathcal{A}(u_h)(e_z) - \mathcal{A}'_h(u_h)(e_u, z_h) + \mathcal{A}'(u_h)(e_u, z_h) \\
&= \delta\rho + \rho_h(u_h)(e_z) + \Delta\rho(u_h)(e_z) - \Delta\rho^\star(u_h, z_h)(e_u)
\end{aligned}
$$

and to equation (13). The simplified error representation can be deduced from the remainder term $\mathcal{R}_h^{(2)}$ by applying integration by parts and the identities (3) and (4)

$$
\begin{aligned}
\mathcal{R}_h^{(2)} &= -\int_0^1 \mathcal{L}''_{uu}(u_h + se_u, z)(e_u, e_u)\, s\, ds \\
&= \int_0^1 \mathcal{L}'_u(u_h + se_u, z)(e_u)\, ds - \left[\mathcal{L}'_u(u_h + se_u, z)(e_u)\, s\right]_0^1 \\
&= \mathcal{L}(u, z) - \mathcal{L}(u_h, z) - \mathcal{L}'_u(u, z)(e_u) \\
&= J(u) - \mathcal{L}_h(u_h, z) - \delta\mathcal{L}(u_h, z) \\
&= J(u) - J(u_h) - \rho_h(u_h)(e_z) - \rho_h(u_h)(z_h) - \Delta\rho(u_h)(z),
\end{aligned}
$$

which finishes the proof. □

Remark 2.5. The simplified error representation (14) can be reformulated by using definition (8) as

$$
J(u) - J(u_h) = \rho_h(u_h)(e_z) + \Delta\rho(u_h)(e_z) + \rho_h(u_h)(z_h) + \Delta\rho(u_h)(z_h) + \mathcal{R}_h^{(2)}.
$$

Remark 2.6. If we assume that the quantity of interest is a linear functional and that the semi-linear forms $\mathcal{A}$ and $\mathcal{A}_h$ are affine-linear in the first argument, the remainder terms $\mathcal{R}_h^{(3)}$ and $\mathcal{R}_h^{(2)}$ are zero just as the linearisation error $\delta\rho$.

2.2 Error identity concerning the model error

Hereafter, we are going to establish an error identity for the model error. Therefore, we need to specify what we mean by a model in this context and also by the error between two models. Since we talked about the different problem formulations for the continuous and discrete case in the previous section a natural choice would be to identify the model by the discrete semi-linear form used. Consequently, the aforementioned model error is defined by the difference of two discretization errors belonging to different discrete semi-linear forms each.

In order to derive the model error identity we append a second discrete setting to the problem formulations from above, i.e. in addition to equations (1) and (2) we consider the task to find $u_{hm} \in \overline{V}_h \subset V$ so that

$$\overline{\mathcal{A}}_h\left(u_{hm}\right)\left(\overline{\varphi}_h\right) = 0 \quad \forall \overline{\varphi}_h \in \overline{W}_h, \tag{15}$$

with a further discrete semi-linear form $\overline{\mathcal{A}}_h : V \times W \to \mathbb{R}$. Like before, the corresponding Lagrangian functional is given by

$$\overline{\mathsf{L}}_h\left(\overline{y}_h\right) = \overline{\mathcal{L}}_h\left(\overline{\psi}_h, \overline{\varphi}_h\right) := J\left(\overline{\psi}_h\right) - \overline{\mathcal{A}}_h\left(\overline{\psi}_h\right)\left(\overline{\varphi}_h\right)$$

for $\overline{y}_h = \left(\overline{\psi}_h, \overline{\varphi}_h\right) \in \overline{V}_h \times \overline{W}_h$. The stationary point $x_{hm} = \left(u_{hm}, z_{hm}\right) \in \overline{V}_h \times \overline{W}_h$ of $\overline{\mathsf{L}}_h$ fulfills

$$\overline{\mathsf{L}}_h\left(x_{hm}\right) = J\left(u_{hm}\right), \tag{16}$$

$$\overline{\mathsf{L}}_h'\left(x_{hm}\right)\left(\overline{y}_h\right) = 0 \quad \forall \overline{y}_h \in \overline{V}_h \times \overline{W}_h. \tag{17}$$

Moreover, we define

$$\delta \mathsf{L}_h\left(y\right) := \mathsf{L}_h\left(y\right) - \overline{\mathsf{L}}_h\left(y\right) = \overline{\mathcal{A}}_h\left(\psi\right)\left(\varphi\right) - \mathcal{A}_h\left(\psi\right)\left(\varphi\right)$$

for arbitrary $y = (\psi, \varphi) \in V \times W$.

Again, we allow the situation that the equations (15) – (17) are not fulfilled exactly. However, we assume that (2) holds exactly. Under these assumptions we formulate the model error identity describing the difference between the two discretization errors w.r.t. the quantity of interest J as follows.

Theorem 2.7. *Let J and $\mathcal{A}_h$ be three times and $\overline{\mathcal{A}}_h$ once directional differentiable in the first argument. Furthermore, $x_h = (u_h, z_h) \in V_h \times W_h$ and $x_{hm} = (u_{hm}, z_{hm}) \in \overline{V}_h \times \overline{W}_h$ are solutions of the primal and dual problem, respec-*

tively. Then we get the model error identity

$$
\begin{aligned}
J(u_h) - J(u_{hm}) &= [J(u) - J(u_{hm})] - [J(u) - J(u_h)] \\
&= \delta\mathsf{L}_h(x_{hm}) + \frac{1}{2}\delta\mathsf{L}_h'(x_{hm})(e_m) + \frac{1}{2}\mathsf{L}_h'(x_h)(e_m) \\
&\quad + \frac{1}{2}\overline{\mathsf{L}}_h'(x_{hm})(e_m) - \overline{\mathcal{A}}_h(u_{hm})(z_{hm}) + \mathcal{R}_m^{(3)}
\end{aligned}
\tag{18}
$$

with the remainder term of third order in the model error $e_m = x_h - x_{hm}$

$$
\mathcal{R}_m^{(3)} := \frac{1}{2}\int_0^1 \mathsf{L}_h'''(x_{hm} + se_m)(e_m, e_m, e_m)\, s(s-1)\; ds.
$$

Proof. We recall the idea used in the proof of Theorem 2.1 and replace L by L_h and L_h by $\overline{\mathsf{L}}_h$, respectively. However, we have to keep in mind that equations (15) – (17) are not fulfilled exactly and thus on one hand we have to add zeros to achieve the Lagrangian representation for the second discretizations and on the other hand the terms $\mathsf{L}_h'(x_h)(e_m)$ and $\overline{\mathsf{L}}_h'(x_{hm})(e_m)$ do not vanish. Hence, we end up with equation (18). □

It should be mentioned at this point that even if we have a numerical exact solution of the aforementioned equations the derivative terms do not necessarily vanish since in general neither $e_m \in V_h \times W_h$ nor $e_m \in \overline{V}_h \times \overline{W}_h$ holds. For this reason, we focus on these different situations in which the model error is an element of the discrete subspaces and adapt the identity (18) to each of them.

Corollary 2.8. *We assume the framework from Theorem* 2.7 *that the equations* (2)*,* (11)*,* (12)*, and* (15) – (17) *are fulfilled exactly and that additionally the correlation of the discrete subspaces is either*

$$
\overline{V}_h \subset V_h \quad \textit{and} \quad \overline{W}_h \subset W_h
$$

or

$$
V_h \subset \overline{V}_h \quad \textit{and} \quad W_h \subset \overline{W}_h.
$$

Then the model error identity (18) *can be reduced to*

$$
\begin{aligned}
J(u_h) - J(u_{hm}) &= \delta\mathsf{L}_h(x_{hm}) + \frac{1}{2}\delta\mathsf{L}_h'(x_{hm})(e_m) \\
&\quad + \frac{1}{2}\overline{\mathsf{L}}_h'(x_{hm})(e_m) + \mathcal{R}_m^{(3)}
\end{aligned}
\tag{19}
$$

or

$$J(u_h) - J(u_{hm}) = \delta \mathsf{L}_h(x_{hm}) + \frac{1}{2}\delta \mathsf{L}'_h(x_{hm})(e_m) + \frac{1}{2}\mathsf{L}'_h(x_h)(e_m) + \mathcal{R}^{(3)}_m. \tag{20}$$

Proof. We consider the first case. With $\overline{V}_h \subset V$ and $\overline{W}_h \subset W_h$ we have $e_m \in V_h \times W_h$ and thus equation (19) deduces from equation (18) using identity (12). The second case follows with analogous arguments. □

These subset relationships occur for example if we want to combine continuous and discontinuous Galerkin methods or use discretizations on different refinement levels.

If the two discrete problem formulations only differ in the semi-linear forms but the discrete subspaces are identical the model error identity can be further simplified:

Corollary 2.9. *We presume that the assumptions of Corollary* 2.8 *hold and that the discrete subspaces are additionally identical, i.e.*

$$\overline{V}_h = V_h \quad \textit{and} \quad \overline{W}_h = W_h.$$

Then the model error identity (18) *reduces to*

$$J(u_h) - J(u_{hm}) = \delta \mathsf{L}_h(x_{hm}) + \frac{1}{2}\delta \mathsf{L}'_h(x_{hm})(e_m) + \mathcal{R}^{(3)}_m. \tag{21}$$

Remark 2.10. The explicit dependency of the stationary point x_h and the assumption of identical discrete subspaces can be circumvented by applying the box quadrature rule instead of the trapezoidal rule. Requiring the relation $V_h \times W_h \subset \overline{V}_h \times \overline{W}_h$ the model error identity reads

$$J(u_h) - J(u_{hm}) = \delta \mathsf{L}_h(x_{hm}) + \delta \mathsf{L}'_h(x_{hm})(e_m) + \mathcal{R}^{(2)}_m. \tag{22}$$

Here, the remainder term is only of second order in the model error e_m and given by

$$\mathcal{R}^{(2)}_m := \int_0^1 \mathsf{L}''_h(x_{hm} + s e_m)(e_m, e_m)\, s\, ds.$$

Remark 2.11. The remainder terms $\mathcal{R}^{(3)}_m$ and $\mathcal{R}^{(2)}_m$ vanish, if the quantity of interest is a linear functional and and if the semi-linear forms $\mathcal{A}_h$ and $\overline{\mathcal{A}}_h$ are affine-linear in the first argument.

2.3 Numerical approximation of the error identity

After establishing error identities for the discretization and model error we consider the task to derive suitable error estimators and realize the numerical approximation and evaluation. An overview of different strategies is presented, e.g., in [4, Section 4.1] or [7, Section 5].

At first, we deal with the estimation of the discretization error. Since the remainder terms $\mathcal{R}_h^{(2)}$ and $\mathcal{R}_h^{(3)}$ are usually of higher order in the primal and dual error $e_x = (e_u, e_z)$ they are negligible. Nevertheless, there is still a dependency of the error identity on the analytical primal and dual solutions, which are not available in general. Therefore, we need adequate approximations of them. These can be provided with solving the problem by finite elements of higher polynomial degree or more efficiently by patch reconstruction techniques. For simplicity we do not distinguish between the different techniques but substitutionally denote the approximation of the unknown analytical solutions u and z by the operator Π. Assuming a problem formulation that fulfills the aforementioned conditions the discretization error can be estimated by

$$\begin{aligned}\eta_h := &\frac{1}{2}\rho_h(u_h)(\Pi(z) - z_h) + \frac{1}{2}\rho_h^\star(u_h, z_h)(\Pi(u) - u_h)\\ &+ \frac{1}{2}\Delta\rho(u_h)(\Pi(z) - z_h) + \frac{1}{2}\Delta\rho^\star(u_h, z_h)(\Pi(u) - u_h)\\ &+ \rho_h(u_h)(z_h) + \Delta\rho(u_h)(z_h)\end{aligned} \tag{23}$$

and

$$\eta_h^{\text{pr}} := \rho_h(u_h)(\Pi(z) - z_h) + \Delta\rho(u_h)(\Pi(z)) + \rho_h(u_h)(z_h) \tag{24}$$

based on equations (5) and (14), respectively.

In a second step, we consider the model error identity for numerically exact solved discrete problems. As before, we want to detect terms of higher order that are negligible within a model error estimation. Therefore, it is useful to obtain some a priori estimation of the various terms arising in equations (18) and (22). Assuming that the operator $\mathsf{L}_h' : V \times W \to (V \times W)'$ satisfies a stability property on $\mathsf{X}_h := (V_h \times W_h) \cup \left(\overline{V}_h \times \overline{W}_h\right)$, i.e. for $C_\text{s} > 0$ it holds

$$\|y_1 - y_2\|_{\mathsf{X}_h} \le C_\text{s}\|\mathsf{L}_h'(y_1) - \mathsf{L}_h'(y_2)\|_{\mathsf{X}_h'}, \tag{25}$$

we obtain an upper bound for the model error e_m of the form

$$\begin{aligned}\|e_m\|_{\mathsf{X}_h} = \|x_h - x_{hm}\|_{\mathsf{X}_h} &\le C_\text{s}\|\mathsf{L}_h'(x_h) - \mathsf{L}_h'(x_{hm})\|_{\mathsf{X}_h'}\\ &\le C_\text{s}\left(\|\mathsf{L}_h'(x_h)\|_{\mathsf{X}_h'} + \|\overline{\mathsf{L}}_h'(x_{hm})\|_{\mathsf{X}_h'} + \|\delta\mathsf{L}_h'(x_{hm})\|_{\mathsf{X}_h'}\right)\end{aligned}$$

Applying this to equation (18) leads to

$$\begin{aligned}|J(u_h) - J(u_{hm})| &\le |\delta \mathsf{L}_h(x_{hm})| + \frac{1}{2}\Big(\|\mathsf{L}'_h(x_h)\|_{\mathsf{X}'_h} + \|\overline{\mathsf{L}}'_h(x_{hm})\|_{\mathsf{X}'_h} \\ &\qquad + \|\delta \mathsf{L}'_h(x_{hm})\|_{\mathsf{X}'_h}\Big) \|e_m\|_{\mathsf{X}_h} + |\overline{\mathcal{A}}_h(u_{hm})(z_{hm})| + |\mathcal{R}_m^{(3)}| \\ &\le |\delta \mathsf{L}_h(x_{hm})| + \frac{1}{2} C_{\mathrm{s}} \Big(\|\mathsf{L}'_h(x_h)\|_{\mathsf{X}'_h} + \|\overline{\mathsf{L}}'_h(x_{hm})\|_{\mathsf{X}'_h} \\ &\qquad + \|\delta \mathsf{L}'_h(x_{hm})\|_{\mathsf{X}'_h}\Big)^2 + |\overline{\mathcal{A}}_h(u_{hm})(z_{hm})| + |\mathcal{R}_m^{(3)}|\end{aligned}$$

where $\mathcal{R}_m^{(3)}$ is of third order in the error e_m. Depending on the relationship of the discrete subspaces contributions containing L'_h or $\overline{\mathsf{L}}'_h$ may equal to zero. The term $\delta \mathsf{L}_h$ describes the difference between the two models and is given by the difference of the two semi-linear forms. Assuming that this perturbation and also its derivative are sufficiently small, then there is a constant $0 < C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right) \ll 1$ with

$$|\delta \mathsf{L}_h(x_{hm})| \le C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right) \|x_{hm}\|_{\mathsf{X}_h} \quad \text{and}$$
$$\|\delta \mathsf{L}'_h(x_{hm})\|_{\mathsf{X}'_h} \le C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right) \|x_{hm}\|_{\mathsf{X}_h}.$$

Referring to the situation described in Corollary 2.9 the inequality reduces to

$$|J(u_h) - J(u_{hm})| \le C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right) \|x_{hm}\|_{\mathsf{X}_h} + \frac{1}{2} C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right)^2 \|x_{hm}\|_{\mathsf{X}_h}^2 + |\mathcal{R}_m^{(3)}|.$$

Since the remainder term is of third order in the error e_m it is also of third order in the constant $C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right)$. As a result, the model error estimator η_m is establish by neglecting higher order terms w.r.t. $C\left(\mathcal{A}_h, \overline{\mathcal{A}}_h\right)$ and is given by

$$\eta_m := \delta \mathsf{L}_h(x_{hm}) - \overline{\mathcal{A}}_h(u_{hm})(z_{hm}) = -\mathcal{A}_h(u_{hm})(z_{hm}). \tag{26}$$

In the concrete setting detailed investigations on the different terms have to be performed taking into account the characteristics of the current problem formulation.

W.r.t. adaptive mesh refinement as well as a local adaptive choice of the model given by the different semi-linear forms, a localization of the estimated discretization and model error to each element is indispensable to get local indicators. Several different techniques to realize this localization are known: The classic approach is based on integration by parts on element level, see, for instance, [7, Section 3.1]. Whereas filtering techniques are introduced in [10]. A quite new kind of localization is developed in [32], which uses a partition of unity. In this article, we employ filtering techniques.

3 Application to selective reduced integration

In this section, we apply the general framework introduced in the last section to a representative model problem to present its functionality and point out possible difficulties. For sake of simplicity, we focus on the problem of linear elasticity for nearly incompressible materials taking into account the plain strain assumption to end up in a two-dimensional problem. The semi-linear form $\mathcal{A}$ is given by an additive composition of a bi-linear form a and a linear form l with

$$\mathcal{A}(\psi)(\varphi) = a(\psi, \varphi) - l(\varphi), \tag{27}$$

$$a(\psi, \varphi) = (\mathbb{C} : \mathrm{D}\,\psi,\, \mathrm{D}\,\varphi)_0, \tag{28}$$

$$l(\varphi) = (f,\, \varphi)_0 + (g,\, \gamma(\varphi))_{\Gamma_\mathrm{N}}, \tag{29}$$

$$V = W = \left\{ \varphi \in H^1\left(\Omega, \mathbb{R}^2\right) \;\middle|\; \gamma(\varphi) = 0 \text{ on } \Gamma_\mathrm{D} \right\}. \tag{30}$$

Here, $(\cdot,\, \cdot)_0$ is the standard L^2 scalar product on the polygonal domain $\Omega \subset \mathbb{R}^2$. Its boundary $\partial\Omega$ is subdivided into two parts Γ_D and Γ_N, where we assume that the Hausdorff measure of Γ_D is greater than zero. The L^2 scalar product on Γ_N is denoted by $(\cdot,\, \cdot)_{\Gamma_\mathrm{N}}$. The symmetric gradient is given by D, : stands for the tensor double contraction and γ denotes the trace operator. The body and the traction force are described by $f \in L^2\left(\Omega, \mathbb{R}^2\right)$ and $g \in L^2\left(\Gamma_\mathrm{N}, \mathbb{R}^2\right)$, respectively. The linear form is composed by

$$l(\varphi) = (f,\, \varphi)_0 + (g,\, \gamma(\varphi))_{\Gamma_\mathrm{N}}.$$

For linear isotropic material behavior, the elasticity tensor $\mathbb{C}$ is given by

$$\mathbb{C} := 2\mu \left(\mathbb{I} - \frac{1}{3} \mathsf{I} \otimes \mathsf{I} \right) + K \mathsf{I} \otimes \mathsf{I},$$

with the fourth and second order identity tensors $\mathbb{I}, \mathsf{I}$ and the shear and bulk moduli $\mu, K \in L^\infty_{>0}(\Omega) := \{ v \in L^\infty(\Omega) \mid v(x) > 0 \text{ a.e. in } \Omega \}$, respectively. In correlation with nearly incompressible material behavior it is more common to talk about the Poisson's ratio ν than about the bulk modulus K. Given the Poisson's ratio $\nu \in L^\infty_{>0}(\Omega)$ the bulk modulus is given by

$$K = \frac{2\mu(1+\nu)}{3(1-2\nu)}.$$

Although, the bi-linear form a is H^1-elliptic – due to Korn's inequality – as well as continuous, i.e.

$$\alpha \|\psi\|^2_{H^1} \leq a(\psi, \psi) \quad \text{and} \quad a(\psi, \varphi) \leq C \|\psi\|_{H^1} \|\varphi\|_{H^1}, \quad \forall \psi, \varphi \in V,$$

the constants α and C depend on the material parameter K and μ. More precisely, we have $\alpha \leq \mu$ and $C \geq K + \mu$. Recalling Céa's Lemma, we have to take into account the quotient C/α for the error estimation. In the case of nearly incompressible materials ν is very close to 0.5, resulting in a significantly greater bulk modulus K than shear modulus μ, i.e. $K \gg \mu$. Thus we end up in a bad estimation due to an increasing quotient that is proportional to K. This phenomena can also be observed within finite element computations and is known as volumetric or Poisson's locking. There are different possibilities to overcome the drawback of this formulation. One of them is the modification of the bi-linear form a another will be discussed later on.

The aforementioned modification is based on a special treatment of the volumetric part of the bi-linear form a in the discrete problem formulation. In order to achieve a consistent presentation, we reformulate the problem formulation stated in equations (27) - (29). Starting from a two field formulation based on the Hellinger-Reissner functional, see e.g., [3], we split up the stress field σ into the volumetric and deviatoric part. Recalling that $\sigma^{\text{vol}} = -p\mathsf{I}$ and requiring that $\sigma^{\text{dev}} = (\mathbb{C} : \mathrm{D}\, u)^{\text{dev}}$ we get a formulation in the displacement field u and the mechanical pressure p. Further we define an operator $\mathrm{P} : H^1\left(\Omega, \mathbb{R}^2\right) \to L^2\left(\Omega\right)$ such that P fulfills

$$\mathsf{I} : (\mathrm{D}\,\varphi)^{\text{vol}} + \mathsf{I} : \mathbb{A} : \mathrm{P}\left(\varphi\right)\mathsf{I} = 0 \quad \text{a.e in } \Omega, \tag{31}$$

with the compliance tensor

$$\mathbb{A} := \mathbb{C}^{-1} = \frac{1}{2\mu}\left(\mathbb{I} - \frac{1}{3}\mathsf{I} \otimes \mathsf{I}\right) + \frac{1}{9K}\mathsf{I} \otimes \mathsf{I}.$$

Replacing the mechanical pressure p by the operator P leads to a bi-linear form

$$\begin{aligned} \bar{a}\left(\psi, \varphi\right) &= \left(\mathbb{C} : (\mathrm{D}\,\psi)^{\text{dev}}, (\mathrm{D}\,\varphi)^{\text{dev}}\right)_0 + \left(\mathbb{A} : \mathrm{P}\left(\psi\right)\mathsf{I}, \mathrm{P}\left(\varphi\right)\mathsf{I}\right)_0 \\ &= \left(\mathbb{C} : (\mathrm{D}\,\psi)^{\text{dev}}, (\mathrm{D}\,\varphi)^{\text{dev}}\right)_0 + \left(\mathrm{P}\left(\psi\right), \mathrm{P}\left(\varphi\right)\right)_{K^{-1}} \end{aligned}$$

with the weighted scalar product $(\cdot, \cdot)_{K^{-1}} := \left(K^{-1}\cdot, \cdot\right)_0 = \left(\cdot, K^{-1}\cdot\right)_0$. Since $\mathrm{P}\left(\varphi\right) = -K \operatorname{div}\left(\varphi\right)$ is a feasible choice a can be seen as a specialization of $\bar{a}$.
To establish the discrete problem we choose a triangulation $\mathcal{T}_h$ of Ω using quadrilaterals with $\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} \overline{K}$ and a finite dimensional subspace

$$V_h := \left\{\psi_h \in V \;\middle|\; \psi_h|_K = \widehat{\psi_h} \circ F^{-1},\ \widehat{\psi_h} \in Q_1(\widehat{K}, \mathbb{R}^2),\ K \in \mathcal{T}_h\right\}, \tag{32}$$

where $Q_1(\widehat{K}, \mathbb{R}^2)$ is the space of vector valued functions on the reference element $\widehat{K}$ having bi-linear polynomials in each component. The function $F : \widehat{K} \to K$

maps the reference element to the respective physical element K and is bijective and orientation-preserving. Furthermore, we introduce the discrete subspace

$$M_h := \left\{ \xi_h \in M := L^2(\Omega) : \xi_h|_K = \widehat{\xi_h} \circ F^{-1},\ \widehat{\xi_h} \in P_0(\widehat{K}),\ K \in \mathcal{T}_h \right\},$$

where $P_0(\widehat{K})$ is the space of constant scalar functions. With this subspace we define a discrete operator $\mathrm{P_h} : H^1\left(\Omega, \mathbb{R}^2\right) \to M_h$ such that $\mathrm{P_h}$ fulfills equation (31) in a weak sense, i.e.

$$\left(\mathsf{I} : (\mathrm{D}\,\varphi)^{\mathrm{vol}} + \mathsf{I} : \mathbb{A} : \mathrm{P_h}(\varphi)\, \mathsf{I},\ \zeta_h \right)_0 = 0 \quad \forall \zeta_h \in M_h.$$

Straight forward algebraic calculations lead to the equivalent but more compact definition

$$\left(\mathrm{P_h}(\varphi), \zeta_h\right)_{K^{-1}} = \left(-K \operatorname{div}(\varphi), \zeta_h\right)_{K^{-1}} \quad \forall \zeta_h \in M_h. \tag{33}$$

Within a finite element computation the determination of $\mathrm{P_h}$ can be realized, e.g., by using a one point Gaussian quadrature rule only in the volumetric part. This is why the approach is also known as selective reduced integration, see, e.g., [18].

3.1 Estimation of discretization error

We consider the situation described above in the framework presented in Section 2.1. The continuous problem formulation with homogeneous Dirichlet boundary conditions reads: Find $u \in V$ so that

$$\mathcal{A}(u)(\varphi) = 0 \quad \forall \varphi \in V,$$

in which $\mathcal{A}$ and V are defined using equations (27) - (30).
Keeping in mind that a defined in equation (28) can be interpreted as a specialization of $\bar{a}$ the discrete problem can be derived from the continuous one by replacing the operator P with its discrete counterpart $\mathrm{P_h}$ defined in equation (33). Thus, the problem formulation is given by:
Find $u_h \in V_h$ so that

$$\mathcal{A}_h(u_h)(\varphi_h) = 0 \quad \forall \varphi_h \in V_h, \tag{34}$$

with V_h as in equation (32) and

$$\mathcal{A}_h(\psi_h)(\varphi_h) = a_h(\psi_h, \varphi_h) - l(\varphi_h), \tag{35}$$

$$a_h(\psi_h, \varphi_h) = \left(\mathbb{C} : (\mathrm{D}\,\psi)^{\mathrm{dev}}, (\mathrm{D}\,\varphi)^{\mathrm{dev}} \right)_0 + \left(\mathrm{P_h}(\psi_h), \mathrm{P_h}(\varphi_h)\right)_{K^{-1}}. \tag{36}$$

Recalling Theorem 2.1 we may identify the different parts as

$$\begin{aligned}
\rho_h(u_h)(\cdot) &= l(\cdot) - a_h(u_h, \cdot),\\
\rho_h^\star(u_h, z_h)(\cdot) &= J'(u_h)(\cdot) - a_h(\cdot, z_h),\\
\Delta\rho(u_h)(\cdot) &= (\mathrm{P_h}(u_h), \mathrm{P_h}(\cdot))_{K^{-1}} - (K \operatorname{div}(u_h), \operatorname{div}(\cdot))_0\\
&= (\mathrm{P_h}(u_h), \mathrm{P_h}(\cdot))_{K^{-1}} - (\mathrm{P}(u_h), \mathrm{P}(\cdot))_{K^{-1}}\\
&= (\mathrm{P_h}(u_h) - \mathrm{P}(u_h), \mathrm{P}(\cdot))_{K^{-1}}\\
\Delta\rho^\star(u_h, z_h)(\cdot) &= (\operatorname{div}(\cdot), \mathrm{P_h}(z_h) - K \operatorname{div}(z_h))_0,
\end{aligned}$$

using the definition of $\mathrm{P_h}$ in (33). Hence, the error identity reads

$$\begin{aligned}
&J(u) - J(u_h)\\
&= \frac{1}{2}\left[\rho_h(u_h)(e_z) + \rho_h^\star(u_h, z_h)(e_u) + \Delta\rho(u_h)(e_z) + \Delta\rho^\star(u_h, z_h)(e_u)\right]\\
&\quad + \rho_h(u_h)(z_h) + \Delta\rho(u_h)(z_h) + \mathcal{R}_h^{(3)}.
\end{aligned}$$

Further on, rearranging the term

$$\begin{aligned}
\Delta\rho(u_h)(z_h) &= \frac{1}{2}\left[(\mathrm{P_h}(u_h) - K \operatorname{div}(u_h), \operatorname{div}(z_h))_0\right]\\
&\quad + \frac{1}{2}\left[(\operatorname{div}(u_h), \mathrm{P_h}(z_h) - K \operatorname{div}(z_h))_0\right]\\
&= \frac{1}{2}\Delta\rho(u_h)(z_h) + \frac{1}{2}\Delta\rho^\star(u_h, z_h)(u_h)
\end{aligned}$$

we have

$$\begin{aligned}
&J(u) - J(u_h)\\
&= \frac{1}{2}\left[\rho_h(u_h)(e_z) + \rho_h^\star(u_h, z_h)(e_u) + \Delta\rho(u_h)(z) + \Delta\rho^\star(u_h, z_h)(u)\right]\\
&\quad + \rho_h(u_h)(z_h) + \mathcal{R}_h^{(3)}.
\end{aligned}$$

Because of $\mathcal{A}$ being affine-linear in the first argument the remainder term only depends on the quantity of interest J, i.e.

$$\mathcal{R}_h^{(3)} = \frac{1}{2}\int_0^1 J'''(u_h + s e_u)(e_u, e_u, e_u)\, s(s-1)\, ds.$$

Neglecting the remainder term of higher order the error estimator η_h reads

$$\begin{aligned}\eta_h = \frac{1}{2}\left[\rho_h\left(u_h\right)\left(\Pi(z)-z_h\right)+\rho_h^\star\left(u_h,z_h\right)\left(\Pi(u)-u_h\right)\right.\\ \left.+\Delta\rho\left(u_h\right)\left(\Pi(z)\right)+\Delta\rho^\star\left(u_h,z_h\right)\left(\Pi(u)\right)\right]+\rho_h\left(u_h\right)\left(z_h\right)\end{aligned} \tag{37}$$

with $\Pi(\cdot)$ being the approximation of the primal and dual solution, respectively.

3.2 Numerical example

At first, we consider a smooth problem and choose an analytic primal solution $u \in C^\infty\left(\Omega, \mathbb{R}^2\right)$ given by

$$u(x,y) := \frac{1}{14}\begin{bmatrix} -\frac{2\pi}{3}\left[\sin\left(\pi\left(x-0.5\right)\right)+1\right]\cos\left(2\pi\left(y-0.25\right)\right) \\ \frac{\pi}{2(1+\nu)}\cos\left(\pi\left(x-0.5\right)\right)\left[\sin\left(2\pi\left(y-0.25\right)\right)+1\right]\end{bmatrix}$$

for $\Omega := (0,2)\times(0,1) \subset \mathbb{R}^2$. The material parameters are chosen as constant functions, i.e. $\nu\left(x,y\right)=\nu=0.25$ and $\mu\left(x,y\right)=\mu=1.0$. With this definition it holds $u=0$ on $\partial\Omega$ and

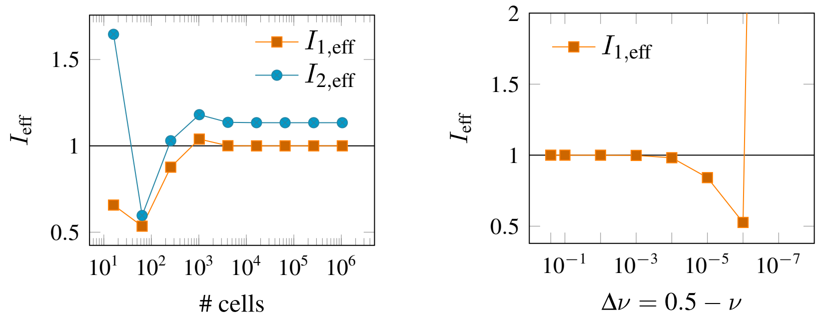
$$\operatorname{div}\left(u(x,y)\right) = \frac{\pi^2\left(1-2\nu\right)}{42\left(1+\nu\right)}\cos\left(\pi\left(x-0.5\right)\right)\cos\left(2\pi\left(y-0.25\right)\right).$$

Thus, $\operatorname{div}(u)$ tends to zero for $\nu \to 0.5$. The volume force f results from the strong form of the partial differential equation, i.e. $f := -\operatorname{div}\left(\sigma(u)\right)$, and for the linear quantity of interest J we choose the integral mean value over a certain subset $B := [1.4, 1.6]\times[0.15, 0.65]$. However, instead of using the non-smooth characteristic function of the subset B we replace it by a continuously differentiable approximation χ_{B_α} based on $B_\alpha := [1.4-\alpha, 1.6+\alpha]\times[0.15-\alpha, 0.65+\alpha]$ and cubic cut off polynomials. Thus, setting $\widetilde{B} := B_{0.05}$ we define the quantity of interest by

$$J\left(u\right) = \frac{1}{|\widetilde{B}|}\int_{\widetilde{B}} u_1+u_2\, d(x,y) = \frac{1}{|\widetilde{B}|}\int_{\Omega}\left(u_1+u_2\right)\chi_{\widetilde{B}}\, d\left(x,y\right).$$

In order to evaluate the error estimator presented in equation (37) we need to specify the reconstruction operator Π. Assuming that the triangulation has a patch structure, the higher order approximations of the analytical quantities can be achieved by a bi-quadratic patch interpolant operator $I_{2h}^{(2)}$, cf., for instance, [7, Section 5]. Thus, the evaluation of the discretization error estimation relies on the formula

$$\eta_h = \widetilde{\eta}_h + \eta_{\mathrm{A}} + \eta_{\mathrm{N}} \tag{38}$$



(a) Effectivity indices for error estimation with and without additional terms.

(b) Effectivity index taking into account additional terms for $\nu \to 0.5$.

Figure 1. Effectivity indices in dependence of the number of cells and of $\Delta\nu$.

with

$$\widetilde{\eta}_h = \frac{1}{2}\left[\rho_h\left(u_h\right)\left(I_{2h}^{(2)}(z_h) - z_h\right) + \rho_h^\star\left(u_h, z_h\right)\left(I_{2h}^{(2)}(u_h) - u_h\right)\right],$$
$$\eta_{\text{A}} = \frac{1}{2}\left[\Delta\rho\left(u_h\right)\left(I_{2h}^{(2)}(z_h)\right) + \Delta\rho^\star\left(u_h, z_h\right)\left(I_{2h}^{(2)}(u_h)\right)\right],$$
$$\eta_{\text{N}} = \rho_h\left(u_h\right)\left(z_h\right).$$

The quality of the derived error estimator η_h may be investigated by considering the effectivity index I_{eff}. It is given by the quotient of the estimated and the real error, i.e. $I_{\text{eff}} = \eta_h/(J(u)-J(u_h))$. In Figure 1a we compare error estimations with and without additional terms, i.e.

$$I_{1,\text{eff}} = \frac{\eta_h}{J\left(u\right) - J\left(u_h\right)} \quad \text{and} \quad I_{2,\text{eff}} = \frac{\widetilde{\eta}_h + \eta_{\text{N}}}{J\left(u\right) - J\left(u_h\right)}.$$

We observe that using the full error estimator including the additional terms gives an error representation, which converges to an effectivity index of 1 on the considered meshes, whereas the restriction to the conventional terms does not. Thus the additional terms are indispensable.
After that, we consider the nearly incompressible case and study the error estimation for $\nu \to 0.5$. We choose a fixed discretization on a uniform refined mesh with 262 144 elements. Figure 1b depicts the development of the effectivity index dependent on $\Delta\nu = 0.5 - \nu$, the difference of Poisson's ratio ν to 0.5. It can be observed that the error estimation becomes worse the closer ν is to 0.5.

3.3 Estimation of discretization error using a mixed formulation

In Section 3.2 we have seen that the additional terms have to be taken into account to get an accurate error estimation. However, the quality of the error estimation still depends on the Poisson's ratio ν and becomes worse when ν tends to 0.5. Therefore, we deal with a second possibility to overcome the drawback of volumetric locking and consider the equivalent mixed formulation. Further details can be found in [20]. Instead of replacing the mechanical pressure $p \in L^2(\Omega)$ with the operator P we keep it as an independent field and add equation (31) in a weak sense. Thus the mixed problem reads: Find $U \in \mathsf{V}$ so that

$$\mathcal{B}(U)(\Phi) = 0 \quad \forall \Phi \in \mathsf{V} \tag{39}$$

with $U = (u, p), \Phi = (\varphi, \zeta) \in \mathsf{V} = V \times M$ and

$$\mathcal{B}(U)(\Phi) = b(u, \varphi) + c(\varphi, p) - l(\varphi) + c(u, \zeta) - d(p, \zeta).$$

The bi-linear forms b, c and d are given by

$$\begin{aligned} b &: V \times V \to \mathbb{R}, & b(u, \varphi) &= \left(\mathbb{C} : (\mathrm{D}\, u)^{\mathrm{dev}}, (\mathrm{D}\, \varphi)^{\mathrm{dev}}\right)_0, \\ c &: V \times M \to \mathbb{R}, & c(u, \zeta) &= (\mathrm{div}(u), \zeta)_0, \\ d &: M \times M \to \mathbb{R}, & d(p, \zeta) &= (p, \zeta)_{K^{-1}}. \end{aligned}$$

The resulting formulation is a saddle point problem with penalty term for which a unique solution exist. For further details on this topic we refer to [12, Chapter III, §4].

Following the procedure already described in Section 2, we rewrite the basic task from equation (39) into a trivial optimization problem. Consequently, we have to find a stationary point $X = (U, Z) \in \mathsf{V} \times \mathsf{V}$ of the Lagrangian functional

$$\mathsf{L}(X) = \mathcal{L}(U, Z) = J(u) - \mathcal{B}(U)(Z)$$

with $Z = (z, q) \in \mathsf{V}$. Within this formulation the discretization is straight forward. We choose a discrete subspace $\mathsf{V}_h \subset \mathsf{V}$ and try to find a stationary point $X_h = (U_h, Z_h) \in \mathsf{V}_h \times \mathsf{V}_h$ of L that fulfills

$$\mathsf{L}'(X_h)(Y_h) = 0 \quad \forall Y_h = (\Psi_h, \Phi_h) \in \mathsf{V}_h \times \mathsf{V}_h.$$

In contrast to Section 3.1, we do not need to modify the Lagrangian functional since the discrete problem formulation establishes only from restriction to the discrete spaces. Thus we can rely on the classic approach to derive the error identity for the discretization error,

$$J(u) - J(u_h) = \frac{1}{2}\rho(U_h)(e_Z) + \frac{1}{2}\rho^\star(U_h, Z_h)(e_U) + \rho(U_h)(Z_h) + \mathcal{R}_h^{(3)},$$

with

$$\begin{aligned}
\rho\left(U_h\right)(\cdot) &:= -\mathcal{B}\left(U_h\right)(\cdot) \\
&= l(\cdot) - b\left(u_h, \cdot\right) - c\left(\cdot, p_h\right) - c\left(u_h, \cdot\right) + d\left(p_h, \cdot\right), \\
\rho^{\star}\left(U_h, Z_h\right)(\cdot) &:= J'\left(u_h\right)(\cdot) - \mathcal{B}'\left(U_h\right)\left(\cdot, Z_h\right) \\
&= J'\left(u_h\right)(\cdot) - b\left(\cdot, z_h\right) - c\left(z_h, \cdot\right) - c\left(\cdot, q_h\right) + d\left(\cdot, q_h\right).
\end{aligned}$$

The remainder term is of third order in the error

$$e_X = \left(e_U, e_Z\right) = \left(e_u, e_p, e_z, e_q\right) = \left(u - u_h, p - p_h, z - z_h, q - q_h\right).$$

and is only affected by the quantity of interest since $\mathcal{B}$ is affine-linear in the first argument, i.e.

$$\mathcal{R}_h^{(3)} := \frac{1}{2} \int_0^1 J'''\left(u_h + s e_u\right)\left(e_u\right) s\,(s-1)\; ds.$$

For the error estimation we neglect the remainder term and use a suitable approximation of the primal and dual solution denoted with the operator Π. Then, the discretization error estimator η_h reads

$$\eta_h = \frac{1}{2}\rho\left(U_h\right)\left(\Pi(Z) - Z_h\right) + \frac{1}{2}\rho^{\star}\left(U_h, Z_h\right)\left(\Pi(U) - U_h\right) + \rho\left(U_h\right)\left(Z_h\right).$$

The next step is to specify the operator Π to perform a numerical evaluation of the established error estimator. The displacement part u and z of the analytical solution U and Z, respectively, are approximated using the bi-quadratic patch interpolant $I_{2h}^{(2)}$ already introduced in Section 3.2. For the pressure variables p and q we choose the operator I_h^{ZZ}. It is based on average techniques [5, 13] also known as ZZ-approach [35], which are usually applied for a posteriori error estimation. Thus, the operator Π is defined by

$$\Pi(\Psi) := \left(I_{2h}^{(2)}(\psi_h), I_h^{\mathrm{ZZ}}(\xi_h)\right).$$

The primal and dual residual are evaluated using the representations

$$\begin{aligned}
&\rho\left(U_h\right)\left(\Pi(Z) - Z_h\right) \\
&= l\left(I_{2h}^{(2)}(z_h) - z_h\right) - b\left(u_h, I_{2h}^{(2)}(z_h) - z_h\right) - c\left(I_{2h}^{(2)}(z_h) - z_h, p_h\right) \\
&\quad - c\left(u_h, I_h^{\mathrm{ZZ}}(q_h) - q_h\right) + d\left(p_h, I_h^{\mathrm{ZZ}}(q_h) - q_h\right)
\end{aligned} \tag{40}$$

and

$$
\begin{aligned}
&\rho^\star\left(U_h, Z_h\right)\left(\Pi(U) - U_h\right) \\
&= J'\left(u_h\right)\left(I_{2h}^{(2)}(u_h) - u_h\right) - b\left(I_{2h}^{(2)}(u_h) - u_h, z_h\right) - c\left(z_h, I_h^{\mathrm{ZZ}}(p_h) - p_h\right) \\
&\quad - c\left(I_{2h}^{(2)}(u_h) - u_h, q_h\right) + d\left(I_h^{\mathrm{ZZ}}(p_h) - p_h, q_h\right).
\end{aligned} \tag{41}
$$

At this point, we may compare the error estimators derived from the pure primal and the mixed problem. Therefore, we re-substitute $p_h = \mathrm{P_h}\left(u_h\right)$ and $q_h = \mathrm{P_h}\left(z_h\right)$ using the operator $\mathrm{P_h}$ defined via (33).
At first, we consider the primal residual presented in equation (40) with $U_h^{\mathrm{P_h}} = \left(u_h, \mathrm{P_h}\left(u_h\right)\right)$ and $Z_h^{\mathrm{P_h}} = \left(z_h, \mathrm{P_h}\left(z_h\right)\right)$. Applying the definitions of the bi-linear forms c and d and recalling the definition of $\mathrm{P_h}$ we get

$$
\begin{aligned}
c\left(I_{2h}^{(2)}(z_h) - z_h, \mathrm{P_h}\left(u_h\right)\right) &= \left(K \operatorname{div}\left(I_{2h}^{(2)}(z_h) - z_h\right), \mathrm{P_h}\left(u_h\right)\right)_{K^{-1}} \\
&= \left(\mathrm{P_h}\left(I_{2h}^{(2)}(z_h) - z_h\right), \mathrm{P_h}\left(u_h\right)\right)_{K^{-1}}
\end{aligned}
$$

and

$$
\begin{aligned}
&d\left(\mathrm{P_h}\left(u_h\right), I_h^{\mathrm{ZZ}}\left(\mathrm{P_h}\left(z_h\right)\right) - \mathrm{P_h}\left(z_h\right)\right) - c\left(u_h, I_h^{\mathrm{ZZ}}\left(\mathrm{P_h}\left(z_h\right)\right) - \mathrm{P_h}\left(z_h\right)\right) \\
&= \left(\mathrm{P_h}\left(u_h\right) - K \operatorname{div}\left(u_h\right), I_h^{\mathrm{ZZ}}\left(\mathrm{P_h}\left(z_h\right)\right)\right)_{K^{-1}}.
\end{aligned}
$$

Finally, we end up with

$$
\begin{aligned}
\rho\left(U_h^{\mathrm{P_h}}\right)\left(\Pi(Z) - Z_h^{\mathrm{P_h}}\right) &= \rho_h\left(u_h\right)\left(I_{2h}^{(2)}(z_h) - z_h\right) \\
&\quad + \left(\mathrm{P_h}\left(u_h\right) - K \operatorname{div}\left(u_h\right), I_h^{\mathrm{ZZ}}\left(\mathrm{P_h}\left(z_h\right)\right)\right)_{K^{-1}}.
\end{aligned}
$$

Analogously, we have the identity

$$
\begin{aligned}
\rho^\star\left(U_h^{\mathrm{P_h}}, Z_h^{\mathrm{P_h}}\right)\left(\Pi(U) - U_h^{\mathrm{P_h}}\right) &= \rho_h^\star\left(u_h, z_h\right)\left(I_{2h}^{(2)}(u_h) - u_h\right) \\
&\quad + \left(I_h^{\mathrm{ZZ}}\left(\mathrm{P_h}\left(p_h\right)\right), \mathrm{P_h}\left(z_h\right) - K \operatorname{div}\left(z_h\right)\right)_{K^{-1}}
\end{aligned}
$$

for the dual residual (41). At last it holds

$$
\rho\left(U_h^{\mathrm{P_h}}\right)\left(Z_h^{\mathrm{P_h}}\right) = \rho_h\left(u_h\right)\left(z_h\right)
$$

due to equation (33). In contrast to the error estimator presented in equation (37) based on the problem formulation in the displacement only, we figure out that within the additional terms a higher order approximation of $\mathrm{P}\left(u\right)$ and $\mathrm{P}\left(z\right)$ is

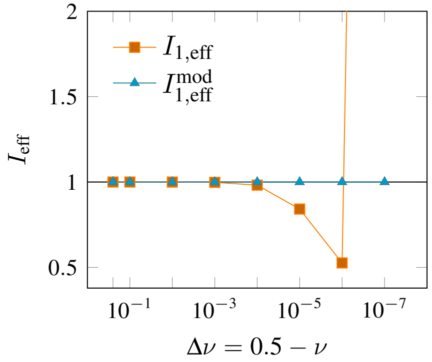


Figure 2. Effectivity index of modified and unmodified error estimator for $\nu \to 0.5$.

needed instead of u and z, respectively. Thus, the modified a posteriori error estimator in the purely displacement formulation reads

$$\begin{aligned} \eta_h^{\text{mod}} &= \widetilde{\eta}_h + \eta_{\text{A}}^{\text{mod}} + \eta_{\text{N}} \\ &= \frac{1}{2}\left[\rho_h\left(u_h\right)\left(I_{2h}^{(2)}(z_h) - z_h\right) + \rho_h^\star\left(u_h, z_h\right)\left(I_{2h}^{(2)}(u_h) - u_h\right)\right] \\ &\quad + \frac{1}{2}\left[\Delta\rho\left(u_h\right)\left(I_h^{\text{ZZ}}\left(\text{P}_\text{h}\left(z_h\right)\right)\right) + \Delta\rho^\star\left(u_h, z_h\right)\left(I_h^{\text{ZZ}}\left(\text{P}_\text{h}\left(u_h\right)\right)\right)\right] \\ &\quad + \rho_h\left(u_h\right)\left(z_h\right) \end{aligned} \tag{42}$$

with

$$\begin{aligned} \Delta\rho\left(u_h\right)(\cdot) &= \left(\text{P}_\text{h}\left(u_h\right) - K\,\text{div}\left(u_h\right),\, \cdot\right)_{K^{-1}}, \\ \Delta\rho^\star\left(u_h, z_h\right)(\cdot) &= \left(\cdot,\, \text{P}_\text{h}\left(z_h\right) - K\,\text{div}\left(z_h\right)\right)_{K^{-1}}. \end{aligned}$$

In order to investigate the effects of the modified reconstruction technique on the quality of the error estimation we recall the example presented in Section 3.2. The results of the non-modified error estimator η_h and of the modified error estimator η_h^{mod} are illustrated in Figure 2. With the improved version given in equation (42) a precise error estimation is achieved even for a Poisson's ratio ν very close to 0.5.

After dealing with a smooth problem having an optimal rate of convergence already for uniform refinement, we consider a framework with less regularity. A typical example is an L-shaped domain with a corner singularity. We adapt this to the framework of linear elasticity prescribing an analytical primal solution based on a singularity function presented in [21]. Using polar coordinates $(r,t) \in \mathbb{R}^2$ the solution is given by

$$u(r,t) := \frac{r^z}{13} \begin{bmatrix} -(z+1)\sin(t)\,\theta(t) - \cos(t)\,\partial_t\theta(t) \\ (z+1)\cos(t)\,\theta(t) - \sin(t)\,\partial_t\theta(t) \end{bmatrix}, \quad \frac{\pi}{2} \leq t \leq 2\pi.$$

and fulfills $\operatorname{div}(u) = 0$. The function $\theta : \mathbb{R} \to \mathbb{R}$ reads

$$\theta(t) = c_0 \sin((z-1)\,t) + c_1 \cos((z-1)\,t) + c_2 \sin((z+1)\,t) + c_3 \cos((z+1)\,t)$$

with constants

$$c_0 = 0.857971843963184, \qquad c_1 = 0.190068891083326,$$
$$c_2 = 0.103221773043934, \qquad c_3 = -0.465943555785929$$

and $z = 0.544483736782463$. The domain is given by

$$\Omega = (-0.5, 0.5)^2 \setminus ([0, 0.5] \times [0, 0.5]) \subset \mathbb{R}^2.$$

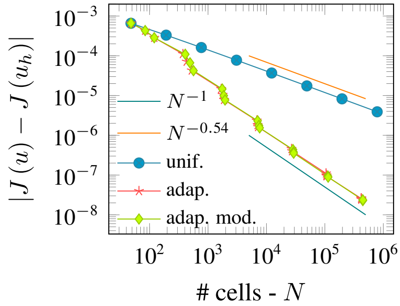
We demand homogeneous Dirichlet boundary conditions on

$$\Gamma_{\mathrm{D}} = \{0\} \times [0, 0.5] \cup [0, 0.5] \times \{0\},$$

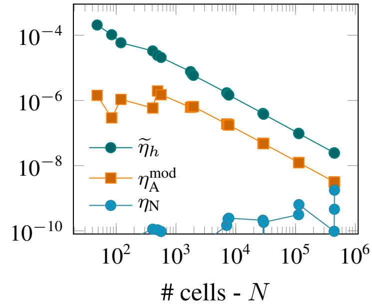
and the Neumann boundary condition on $\Gamma_{\mathrm{N}} = \partial\Omega \setminus \Gamma_{\mathrm{D}}$ reads $f_{\mathrm{N}} = \sigma(u)\,n$. The volume force is given by the strong form of the partial differential equation again, i.e. $f = -\operatorname{div}(\sigma(u))$. Recalling that u fulfills $\operatorname{div}(u) = 0$ the traction and volume force reduce to $f_{\mathrm{N}} = \sigma(u)^{\mathrm{dev}}\,n$ and $f = -\operatorname{div}\left(\sigma(u)^{\mathrm{dev}}\right)$, respectively. For the quantity of interest we restrict our investigations to a linear functional, i.e. the integral mean value of the first component of the solution u over the whole domain Ω,

$$J(u) = \frac{1}{|\Omega|} \int_{\Omega} u_1 \, d\Omega.$$

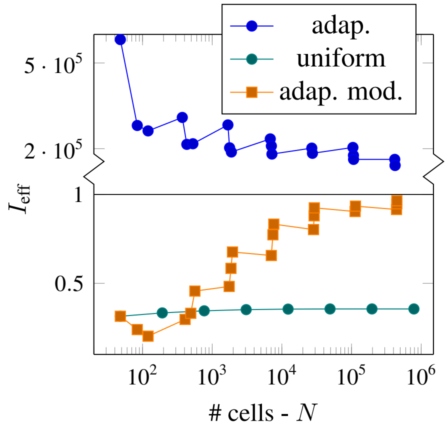
As marking strategy in the mesh adaptive algorithm, we employ the optimal mesh strategy introduced in [31, Section 4.3]. During the adaptive refinement, we allow only one hanging node per edge and ensure the patch property of the mesh. Figure 3a depicts the error in the quantity of interest J for uniformly and adaptively refined meshes. Within the adaptive case we distinguish between the refinement based on the localization of the original error estimator presented in equation (38) and the modified one given by equation (42). We observe a reduced convergence rate of order $N^{-0.54}$ for uniform mesh refinement. For both adaptive approaches,



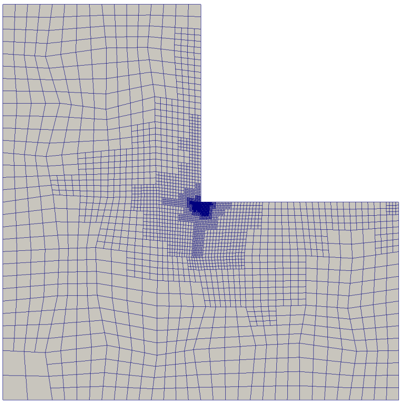
(a) Error w.r.t. the quantity of interest J for uniform, adaptive and adaptive refinement with modified higher order reconstruction.



(b) Development of the different parts of the error estimator η_h^{mod}.



(c) Development of the effectivity indices.



(d) Adaptively refined mesh in the 7th iteration for the modified reconstruction.

Figure 3. Convergence analysis results for a Poisson's ratio $\nu = 0.5 - 10^{-7}$.

we recover the optimal convergence rate of order N^{-1}, with almost identical results. Also, the adaptive meshes are similar. Hence, we show only an adaptively refined mesh for the modified error estimator, cf. Figure 3d. We observe strong refinements in the re-entrant corner as expected. Further, Figure 3b illustrates the development of the different components of the error estimator η_h^{mod}. We observe an almost parallel reduction of $\widetilde{\eta}_h$ and η_A^{mod} differing by a factor of approximately 10. From a number of 500 000 elements, the influence of the numerical error becomes significant, although we use a direct solver. The reason lies in the ill conditioning of the resulting linear system of equations. Since the estimated numerical error has the same order of magnitude as $\eta_{\mathrm{A}}^{\mathrm{mod}}$, this part has to be taken into account as well. In Figure 3c, the effectivity indices for the different approaches are compared. We find that the original error estimator presented in equation (38) leads to very bad effectivity indices in the order of 10^5. The modified estimator given by equation (42) leads to approximately constant effectivity indices in the order of 0.36 using uniform mesh refinement as expected. Whereas we find effectivity indices converging to 1 for the adaptive approach based on the modified estimator.

Recollecting the results of the previous examples we conclude that in frameworks in which continuous and discrete semi-linear form differ the additional terms in the error identity presented in Theorem 2.1 are not only of theoretical interest. Much more they are indispensable to achieve an accurate error estimation. However, special care must be taken to the higher order approximations of the analytical primal and dual solution. We have figured out that an adjustment is necessary dependent on the currently considered problem. Here, a higher order reconstruction of the divergence is required in the additional terms to get a correct error representation independent of the choice of ν. Further, the adaptive algorithms based on both estimators show an optimal convergence order.

3.4 Estimation of the model error

After dealing with the a posteriori estimation of the discretization error we focus on the model error. Referring to Section 2.2, the two models are given by the modified bi-linear form (36) using the operator $\mathrm{P_h}$ on the one hand and the original bi-linear form (28) on the other, i.e.

$$\mathcal{A}_h\left(\psi_h\right)\left(\varphi_h\right) = a_h\left(\psi_h, \varphi_h\right) - l\left(\varphi_h\right)$$

and

$$\overline{\mathcal{A}}_h\left(\overline{\psi}_h\right)\left(\overline{\varphi}_h\right) = a\left(\overline{\psi}_h, \overline{\varphi}_h\right) - l\left(\overline{\varphi}_h\right).$$

Since the discretization is the same in both models, i.e. $V_h = \overline{V}_h$, we can rely on the simplified model error identity (21). For the sake of simplicity, we assume the quantity of interest to be a linear functional. Then the derivative of the Lagrangian functional $\mathcal{L}_h'$ defines a linear operator $L : \mathsf{X}_h = V_h \times V_h \to \mathsf{X}_h'$ with

$$\left(\psi_h, \varphi_h\right) \mapsto L\left(\psi_h, \varphi_h\right) := \left(a_h\left(\cdot, \varphi_h\right), a_h\left(\psi_h, \cdot\right)\right).$$

and the problem can be reformulated using operator notation: Find $\left(u_h, z_h\right) \in V_h \times V_h$ so that

$$L\left(u_h, z_h\right) = \left(J, l\right)$$

holds. The bi-linear form

$$A\left(u_h, z_h; \psi_h, \varphi_h\right) := \left\langle L\left(u_h, z_h\right), \left(\psi_h, \varphi_h\right)\right\rangle_{\mathsf{X}_h', \mathsf{X}_h} = a_h\left(\psi_h, z_h\right) + a_h\left(u_h, \varphi_h\right)$$

induced by L is elliptic. Hence, the operator L fulfills the stability property (25).

For the perturbation $\delta\mathsf{L}_h$ we get

$$\begin{aligned}\delta\mathsf{L}_h\left(x_{hm}\right) &= \left(\mathrm{P}\left(u_{hm}\right), \mathrm{P}\left(z_{hm}\right)\right)_{K^{-1}} - \left(\mathrm{P_h}\left(u_{hm}\right), \mathrm{P_h}\left(z_{hm}\right)\right)_{K^{-1}} \\ &= \left(\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right), \mathrm{P}\left(z_{hm}\right)\right)_{K^{-1}}\end{aligned}$$

and for the derivative

$$\begin{aligned}\delta\mathsf{L}_h'\left(x_{hm}\right)\left(y_h\right) = &\left(\mathrm{P}\left(\psi_h\right), \mathrm{P}\left(z_{hm}\right) - \mathrm{P_h}\left(z_{hm}\right)\right)_{K^{-1}} \\ &+ \left(\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right), \mathrm{P}\left(\varphi_h\right)\right)_{K^{-1}}.\end{aligned}$$

Further, we have

$$\begin{aligned}\left|\delta\mathsf{L}_h\left(x_{hm}\right)\right| &\leq \left\|\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \left\|\mathrm{P}\left(z_{hm}\right)\right\|_{L^2_{K^{-1}}} \\ &\leq \left\|\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \|K\|_{L^\infty}^{1/2} \left\|z_{hm}\right\|_{H^1}\end{aligned}$$

and

$$\begin{aligned}\left|\delta\mathsf{L}_h'\left(x_{hm}\right)\left(y_h\right)\right| \leq &\left\|\mathrm{P}\left(\psi_h\right)\right\|_{L^2_{K^{-1}}} \left\|\mathrm{P}\left(z_{hm}\right) - \mathrm{P_h}\left(z_{hm}\right)\right\|_{L^2_{K^{-1}}} \\ &+ \left\|\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \left\|\mathrm{P}\left(\varphi_h\right)\right\|_{L^2_{K^{-1}}} \\ \leq &\Big(\left\|\mathrm{P}\left(z_{hm}\right) - \mathrm{P_h}\left(z_{hm}\right)\right\|_{L^2_{K^{-1}}} \\ &+ \left\|\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}}\Big) \|K\|_{L^\infty}^{1/2} \left\|y_h\right\|_{\mathsf{X}}\end{aligned}$$

with the weighted L^2-norm $\|\cdot\|_{L^2_{K^{-1}}} := (\cdot, \cdot)_{K^{-1}}^{1/2}$. At this point, we need to estimate the difference between the two operators P and $\mathrm{P_h}$. It holds

$$\begin{aligned}\left\|\mathrm{P}\left(u_{hm}\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \leq &\left\|\mathrm{P}\left(u\right) - \mathrm{P}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \\ &+ \left\|\mathrm{P}\left(u\right) - \mathrm{P_h}\left(u\right)\right\|_{L^2_{K^{-1}}} + \left\|\mathrm{P_h}\left(u\right) - \mathrm{P_h}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}}.\end{aligned}$$

Assuming the primal and dual solution to be in $V \cap H^{1+\theta}\left(\Omega, \mathbb{R}^2\right)$ for $\theta > 0$, respectively and applying a prior error estimates leads to

$$\left\|\mathrm{P}\left(u\right) - \mathrm{P}\left(u_{hm}\right)\right\|_{L^2_{K^{-1}}} \leq \|K\|_{L^\infty}^{1/2} \left\|u - u_{hm}\right\|_{H^1} \leq \|K\|_{L^\infty}^{1/2} C_A h^\theta \|u\|_{H^{1+\theta}}$$

for the first term. For the third term we refer to the result of the first term and use equation (33)

$$
\begin{aligned}
&\|\mathrm{P_h}\left(u\right)-\mathrm{P_h}\left(u_{hm}\right)\|^2_{L^2_{K^{-1}}}\\
=&\left(\mathrm{P_h}\left(u\right)-\mathrm{P_h}\left(u_{hm}\right),\, \mathrm{P}\left(u\right)-\mathrm{P}\left(u_{hm}\right)\right)_{K^{-1}}\\
\leq&\|\mathrm{P_h}\left(u\right)-\mathrm{P_h}\left(u_{hm}\right)\|_{L^2_{K^{-1}}}\|\mathrm{P}\left(u\right)-\mathrm{P}\left(u_{hm}\right)\|_{L^2_{K^{-1}}}.
\end{aligned}
$$

Thus

$$
\|\mathrm{P_h}\left(u\right)-\mathrm{P_h}\left(u_{hm}\right)\|_{L^2_{K^{-1}}} \leq \|\mathrm{P}\left(u\right)-\mathrm{P}\left(u_{hm}\right)\|_{L^2_{K^{-1}}} \leq \|K\|^{1/2}_{L^\infty} C_A h^\theta \|u\|_{H^{1+\theta}}.
$$

The second term describes the quality of the piecewise constant interpolation of $\mathrm{P}\left(u\right)$. Using the Bramble–Hilbert–Lemma and a scaling argument, we get

$$
\begin{aligned}
\|\mathrm{P}\left(u\right)-\mathrm{P_h}\left(u\right)\|_{L^2_{K^{-1}}} &\leq \|K^{-1/2}\|_{L^\infty}\|\mathrm{P}\left(u\right)-I^{(0)}_h\left(\mathrm{P}\left(u\right)\right)\|_{L^2}\\
&\leq \|K^{-1/2}\|_{L^\infty} C_I h^\theta \|\mathrm{P}\left(u\right)\|_{H^\theta}\\
&\leq \|K^{-1/2}\|_{L^\infty}\|K\|_{L^\infty} C_I h^\theta \|u\|_{H^{1+\theta}}
\end{aligned}
$$

and finally we end up with

$$
\begin{aligned}
&\|\mathrm{P}\left(u_{hm}\right)-\mathrm{P_h}\left(u_{hm}\right)\|_{L^2_{K^{-1}}}\\
\leq\;&\left(2C_A + C_I\|K^{-1/2}\|_{L^\infty}\|K\|^{1/2}_{L^\infty}\right)\|K\|^{1/2}_{L^\infty} h^\theta \|u\|_{H^{1+\theta}}.
\end{aligned}
$$

Carrying out the analogous argumentation it holds

$$
\begin{aligned}
&\|\mathrm{P}\left(z_{hm}\right)-\mathrm{P_h}\left(z_{hm}\right)\|_{L^2_{K^{-1}}}\\
\leq\;&\left(2C_A + C_I\|K^{-1/2}\|_{L^\infty}\|K\|^{1/2}_{L^\infty}\right)\|K\|^{1/2}_{L^\infty} h^\theta \|z\|_{H^{1+\theta}}.
\end{aligned}
$$

All in all, we have an upper estimation of $\delta\mathsf{L}_h$ and $\delta\mathsf{L}'_h$ of the following form,

$$
\begin{aligned}
|\delta\mathsf{L}_h\left(x_{hm}\right)| &\leq C'\|x_{hm}\|_{\mathsf{X}},\\
|\delta\mathsf{L}'_h\left(x_{hm}\right)\left(y_h\right)| &\leq C''\|y_h\|_{\mathsf{X}},
\end{aligned}
$$

with

$$
\begin{aligned}
C' &= \left(2C_A + C_I\|K^{-1/2}\|_{L^\infty}\|K\|^{1/2}_{L^\infty}\right)\|K\|_{L^\infty}\|u\|_{H^{1+\theta}} h^\theta,\\
C'' &= \left(\left(2C_A + C_I\|K^{-1/2}\|_{L^\infty}\|K\|^{1/2}_{L^\infty}\right)\left(\|u\|_{H^{1+\theta}} + \|z\|_{H^{1+\theta}}\right)\right)\|K\|_{L^\infty} h^\theta.
\end{aligned}
$$

Set $C = \max\{C', C''\}$ and recall that C' and C'' are of first order in h^θ we can achieve $C \ll 1$ for a sufficiently small mesh size h.
Recalling the simplified error identity from equation (21) we obtain that the first term is of first order and the derivative term of second order in the constant C. Moreover, the remainder term vanishes due to the linearity of the entire problem. Thus, neglecting the higher order terms, i.e. the derivative part, the model error estimator reads

$$\begin{aligned}\eta_m &= -\mathcal{A}_h(u_{hm})(z_{hm})\\ &= (\mathrm{P}(u_{hm}) - \mathrm{P_h}(u_{hm}), \mathrm{P}(z_{hm}))_{K^{-1}}\\ &= (K\,\mathrm{div}(u_{hm}) - \mathrm{P_h}(u_{hm}), \mathrm{div}(z_{hm}))_0 .\end{aligned}$$

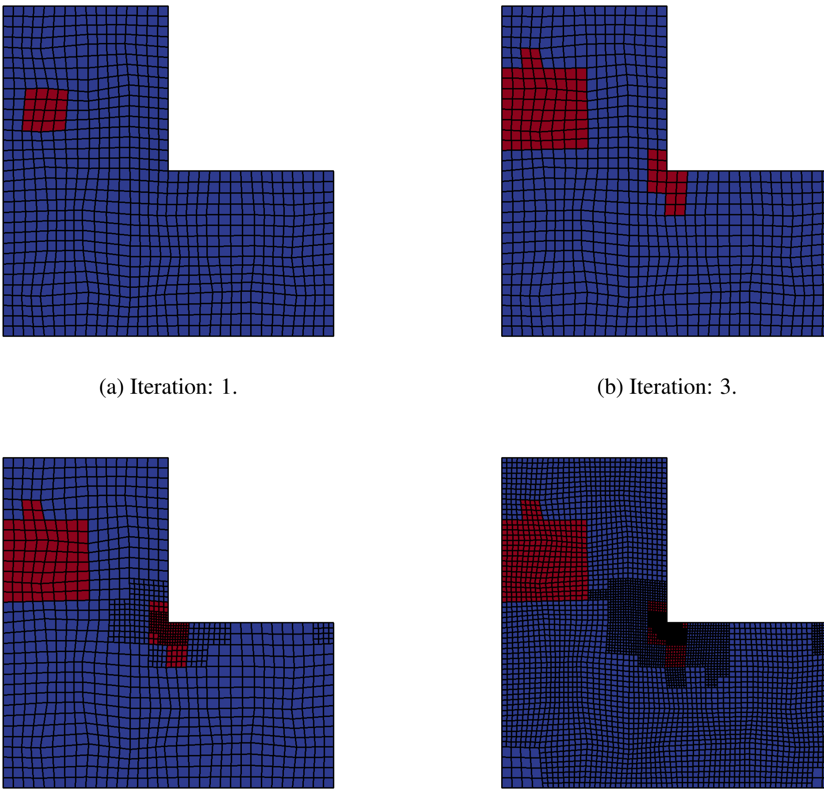
3.5 Numerical example

In this example we combine mesh and model adaptivity considering the second framework introduced in Section 3.2. In addition, we allow a non-constant Poisson's ratio ν to localize the nearly incompressible material in a certain region of Ω. More precisely, we define a circular subset with radius $R = 0.1$, centered in $M = (-0.37, 0.19)$ and assume the material to be nearly incompressible therein. On the complement, we choose a moderate value, i.e.

$$\nu(x,y) := \begin{cases} 0.25, & \sqrt{(x-M_x)^2 + (y-M_y)^2} \geq R \\ c_\nu, & \sqrt{(x-M_x)^2 + (y-M_y)^2} < R \end{cases}$$

with arbitrary $c_\nu \in [0, 0.5)$, here $c_\nu = 0.4999999$. As in Section 3.2 we replace $\nu(x,y)$ by a smooth approximation $\nu_\alpha(x,y)$ using a cubic polynomial in the transition zone $[R-\alpha, R+\alpha]$ with $\alpha = 0.02$.
The mesh and model adaptive algorithm is based on the following procedure. We start the algorithm computing a solution of the initial discrete problem using the semi-linear form $\overline{\mathcal{A}}_h$ everywhere in Ω. After estimating the discretization and model error we compare the global estimations. If $|\eta_m| > 0.4|\eta_h|$ only an adjustment of the model distribution is performed, otherwise there will be an adaptive mesh refinement only. The marking of cells refers to the optimal mesh strategy for mesh refinement, see [31, Section 4.3], and a fixed fraction approach, see [4, Section 4.2], with $\vartheta = 0.45$ for model adjustment, respectively.

Figure 4 shows sample iterations of the mesh and model adaptive algorithm. In the first four iterations only an adjustment of the model distribution is performed. Whereas during the ongoing iterations we have adaptive mesh refinement only. The change from the initial model, i.e. from the semi-linear form $\overline{\mathcal{A}}_h$ to $\mathcal{A}_h$ obviously occurs in the expected region. The nearly incompressible area is identified



(a) Iteration: 1.

(b) Iteration: 3.

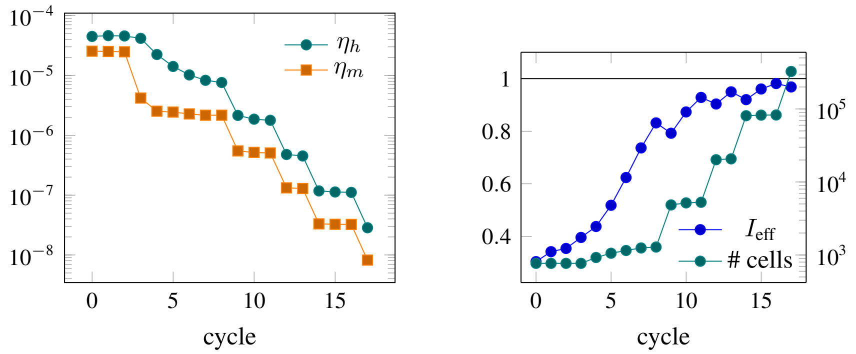
(c) Iteration: 5.

(d) Iteration: 9.

Figure 4. Different stages of the mesh and model adaptive algorithm with model distribution (red: $\mathcal{A}_h$, blue: $\overline{\mathcal{A}}_h$).

by the model error estimator, compare the red “square” on the left of the L-shaped domain in Figure 4b. Since the estimation basically depends on gradient information we find an unexpected model change around the re-entrant corner as well. This results from large gradients due to the corner singularity. The resulting mesh has a typical element distribution with concentrations around the re-entrant corner resolving the singularity and in the transition regions from Dirichlet to Neumann boundary conditions. However, it is not affected by the nearly incompressible region at all.

The progress of the model and discretization error estimator is shown in Fig-



(a) Progress of discretization and model error estimator within the adaptive algorithm.

(b) Effectivity of discretization error estimator and number of cells used.

Figure 5. Convergence analysis results for the mesh and model adaptive algorithm.

ure 5a. Up to the third cycle there is no real change within both estimators. Considering the fourth cycle, we have a significant reduction of the model error estimator leading to a mesh refinement this time. From this point on there is only mesh refinement due to the nearly parallel reduction behavior of both estimators. The numerical error is not significant within this example because it is only in the order of magnitude of 10^{-10}. In Figure 5b, we consider on the one hand the number of mesh cells showing again that we have model changes only in the first iterations. On the other hand, the effectivity indices of the model and discretization error estimator are depicted and we observe values converging to 1.

With this example, we substantiate the efficiency of the mesh and model adaptive algorithm as well as the accuracy of the estimator. It shows that the algorithm works as expected and detects the region of nearly incompressible material. Erroneously, a small additional region around the re-entrant corner is identified as such regions as well. Further, we observe an untypical mesh dependency of the model error estimator. However, this behavior is expected since the model error measures the difference between two discretisation errors which are mesh dependent themselves.

4 Conclusion and Outlook

In this article, we have extended the usual DWR framework to include also discretizations, where the continuous and the discrete semi-linear form differ and no further consistency assumptions hold. In this case, several additional terms have to be taken into account. Here, we study a comparatively simple model situation given by the selective reduced integration approach for nearly incompressible linear elasticity to test the introduced framework. Using a suitable approximation of the analytic variables in the error identity, we see an accurate error estimator and an optimal convergent adaptive algorithm in the numerical experiments. The second main contribution of the article at hand lies in the estimation of the modeling error arising from the use of two different discrete semi-linear forms. Extending the usual techniques on the continuous level to the discrete case, we also observe an accurate estimator as well as an efficient model and mesh adaptive algorithm in the numerical examples.

Ongoing research focuses on the more complex finite element formulation of solid-like shell elements. Here, similar problems occur due to several stabilization techniques which include without limitation an in-plane reduced integration scheme to prevent, e.g., membran locking. Within this context the model adaptive aspect becomes more important since using solid-like shell elements instead of volume elements leads to a remarkable reduction of the number of degrees of freedom and thus results in significant lower computational costs. First results discussed in [19] show great promise.

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Received **???**.

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