Wachmacher-Aufgabe 9

Seien $b, c \in \mathbb{R}$ beliebig. Für welchen Wert von $a \in \mathbb{R}$ gilt

$$b(a-1) + 2c(a-1) = A \quad \text{mit } A = a-1 \,?$$

Lösung:

$$b(a-1) + 2c(a-1) \overset{!}{=} (a-1)$$

$$\overset{-(a-1)}{\Leftrightarrow} b(a-1) + 2c(a-1) - (a-1) = 0$$

$$\Leftrightarrow (a-1)(b+2c-1) = 0 \underset{\text{beliebig}}{\overset{\text{Da } b,c\in\mathbb{R}}{\Longrightarrow}} a-1 = 0 \Rightarrow a = 1.$$

Bsp: Substitutionsregel

$$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx = \,?$$

Substituiere: $y = 1+x^2$ mit $\frac{dy}{dx} = 2x \Leftrightarrow dx = \frac{dy}{2x}$

NR: $$\text{"}\frac{x^3}{\sqrt{1+x^2}}\,dx = \frac{x^{\cancel{3}\,2}}{\sqrt{1+x^2}} \cdot \frac{dy}{\cancel{2x}} \underset{y=1+x^2}{=} \frac{x^2}{\sqrt{y}\cdot 2}\,dy \overset{x^2=y-1}{=} \frac{y-1}{2\sqrt{y}}\,dy\text{"}$$

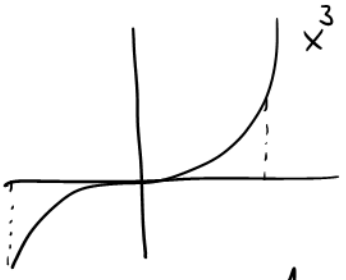
Also:

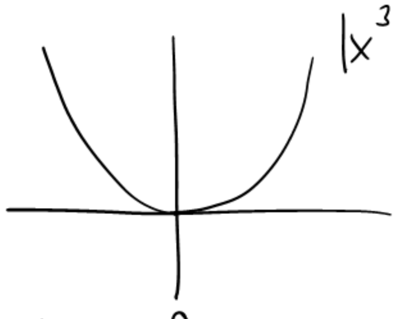
$$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx = \int_{1+0^2}^{1+1^2} \frac{y-1}{2\sqrt{y}}\,dy = \frac{1}{2}\int^2 \left(\frac{y}{\sqrt{y}} - \frac{1}{\sqrt{y}}\right)dy = \frac{1}{2}\int_1^2 \left(\underset{= y^{\frac{1}{2}}}{\sqrt{y}} - \frac{1}{\sqrt{y}}\right)dy$$

$$= \frac{1}{2}\left[\frac{2}{3}y^{\frac{3}{2}} - 2\sqrt{y}\right]_1^2 = \left[\frac{1}{3}y^{\frac{3}{2}} - \sqrt{y}\right]_1^2 = \frac{1}{3}2^{\frac{3}{2}} - \sqrt{2} - \Big(\underbrace{\frac{1}{3}1^{\frac{3}{2}} - \sqrt{1}}_{-\frac{2}{3}}\Big)$$

$$= \frac{1}{3}\underbrace{2^{\frac{3}{2}}}_{2\cdot\sqrt{2}} - \sqrt{2} + \frac{2}{3} = -\frac{1}{3}\sqrt{2} + \frac{2}{3}.$$

zu 9.12





$\frac{1}{4}x^4$ ist <u>nicht</u> die Stammfkt für $|x^3|$

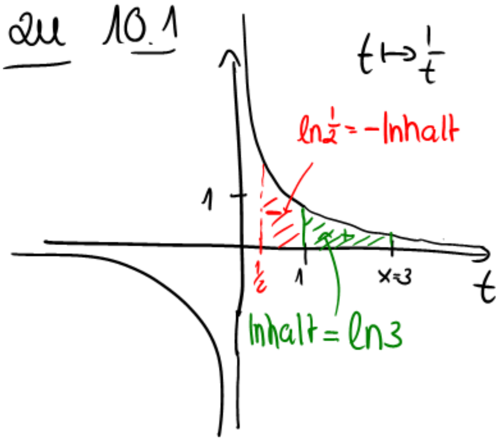
$$A_f(-1,1) = \int_{-1}^1 |x^3|\,dx = \int_{-1}^0 |x^3|\,dx + \int_0^1 |x^3|\,dx = \int_{-1}^0 (-x^3)\,dx + \int_0^1 x^3\,dx$$

da $x^3 \leq 0$ in $[-1,0]$ da $x^3 \geq 0$ in $[0,1]$

$$= \left[-\frac{1}{4}x^4\right]_{-1}^0 + \left[\frac{1}{4}x^4\right]_0^1$$

$$= 0 + \frac{1}{4}(-1)^4 + \frac{1}{4}1^4 - 0 = \frac{1}{2}$$

zu 10.1



$$\boxed{\ln x := \int_1^x \frac{1}{t}\,dt}$$

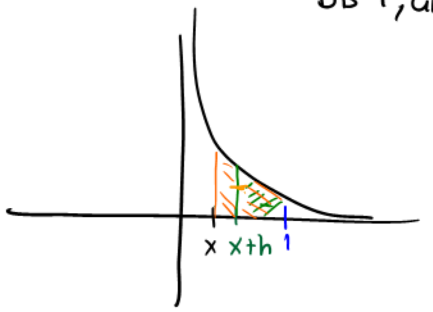
Für $x = \frac{1}{2}$ gilt: $\ln(\frac{1}{2}) = \int_1^{\frac{1}{2}} \frac{1}{t}\,dt = -\underbrace{\left(\int_{\frac{1}{2}}^{1} \frac{1}{t}\,dt\right)}_{\text{pos. Flächeninhalt von } \frac{1}{2} \text{ bis } 1, \text{ also } >0} < 0$

zu 10.2

Differenzenquotient in $x \in (0,\infty)$:

$$\frac{\ln(x+h) - \ln x}{h} = \frac{1}{h}\left(\int_1^{x+h} \frac{1}{t}\,dt - \int_1^{x} \frac{1}{t}\,dt\right)$$

$$= \frac{1}{h}\left(\cancel{\int_1^{x} \frac{1}{t}\,dt} + \int_x^{x+h} \frac{1}{t}\,dt - \cancel{\int_1^{x} \frac{1}{t}\,dt}\right) = \underbrace{\frac{1}{h}\int_x^{x+h} \frac{1}{t}\,dt}_{\text{Durchschnittswert von } \frac{1}{t} \text{ für } t \in [x, x+h]}$$



$$\Rightarrow \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} \frac{1}{t}\,dt = \frac{1}{x} = (\ln(x))'$$

zu 10.3

1. $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$

3. $\ln(x \cdot y) = \int_1^{x \cdot y} \frac{1}{t}\,dt = \underbrace{\int_1^{x} \frac{1}{t}\,dt}_{= \ln x} + \int_x^{xy} \frac{1}{t}\,dt = \ln x + \int_x^{xy} \frac{1}{t}\,dt \overset{NR}{=} \ln x + \ln y$

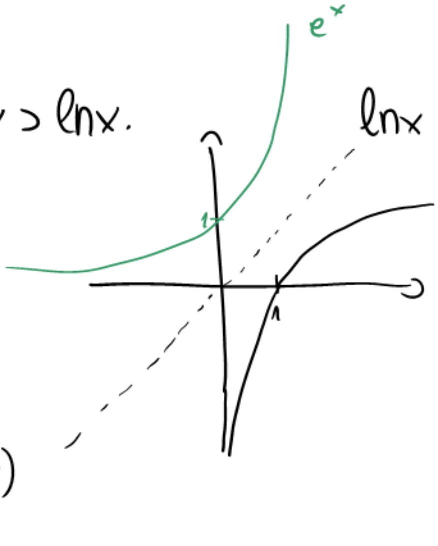
NR: $\int_x^{xy} \frac{1}{t}\,dt \underset{\frac{ds}{dt} = \frac{1}{x} \Leftrightarrow dt = x\,ds}{\overset{s = \frac{t}{x}}{=}} \int_{\frac{x}{x}=1}^{\frac{xy}{x}=y} \frac{1}{sx} \cdot x\,ds = \int_1^{y} \frac{1}{s}\,ds = \ln y$

2. Wähle in 3. $y = \frac{1}{x}$. Dann gilt $x \cdot y = x \cdot \frac{1}{x} = 1$

Also $0 \overset{1.}{=} \ln 1 = \ln(x \cdot \frac{1}{x}) \overset{3.}{=} \ln x + \ln\frac{1}{x} \Rightarrow 0 = \ln x + \ln\frac{1}{x}$
$\Rightarrow \ln\frac{1}{x} = -\ln x$

zu 5. Seien $0 < x < y$. Dann gilt

$\ln y - \ln x = \int_1^{y} \frac{1}{t}\,dt - \int_1^{x} \frac{1}{t}\,dt = \int_x^{y} \underbrace{\frac{1}{t}}_{>0}\,dt > 0. \Rightarrow \ln y > \ln x.$



zu 10.5

5. $x := \ln a$, $y = \ln b$, also $\exp(x) = a$, $\exp(y) = b$

$\Rightarrow x + y = \ln a + \ln b \overset{\ln-}{\underset{\text{Rechengesetz}}{=}} \ln(a \cdot b)$

$\Rightarrow a \cdot b = \exp(x+y)$, also $\exp(x+y) = \exp(x)\exp(y)$

4. $\underbrace{\exp}_{f^{-1}}'(x) = \dfrac{1}{\underbrace{\ln}_{f}'(\underbrace{\exp(x)}_{f^{-1}(x)})} = \dfrac{1}{\frac{1}{\exp(x)}} = \exp(x)$

$$\boxed{(\varphi^{-1})'(y) = \frac{1}{\varphi'(\varphi^{-1}(y))}}$$