Aufgabe: $((M_1 \cup M_2) \cap M_3) \setminus M_4 = ?$

Lsg: $(M_1 \cup M_2) \cap M_3 = \{4, 5, 7, 9, 10, 13\} \cap \{1, 3, 5, 7\} = \{5, 7\}$

$\Rightarrow ((M_1 \cup M_2) \cap M_3) \setminus M_4 = \{5, 7\} \setminus \{7, 13, 19\} = \{5\}$

zu 11.4

$$4 \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + 0 \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} + 2 \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}4\cdot 1 + 0\cdot 0 + 2\cdot 1\\ 4\cdot 1 + 0 \cdot 1 + 2 \cdot 0\\ 4\cdot 0 + 0\cdot 1 + 2\cdot 1\end{pmatrix} = \begin{pmatrix}6\\4\\2\end{pmatrix}$$

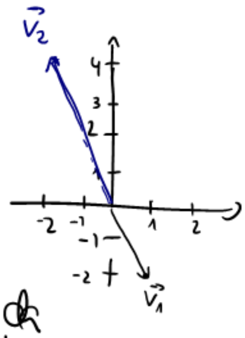
Kennt man die Koeffizienten nicht, so stellt man ein GLS auf:

$$\alpha_1 \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}6\\4\\2\end{pmatrix}$$

$$\Leftrightarrow \begin{aligned} 1\cdot\alpha_1 \qquad + 1\cdot\alpha_3 &= 6 \\ 1\cdot\alpha_1 + 1\cdot\alpha_2 \qquad &= 4 \\ 1\cdot\alpha_2 + 1\cdot\alpha_3 &= 2 \end{aligned}$$

Methode zur Lösung: Gauß Algorithmus.

Bsp zu 11.5



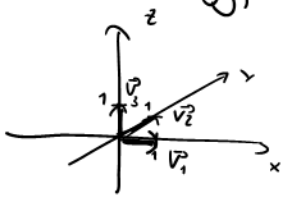
- Die Vektoren $\vec{v}_1 = \begin{pmatrix}1\\-2\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}-2\\4\end{pmatrix}$ sind linear abhängig, da

$$2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 = \vec{0}$$

- Die Vektoren $\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ sind linear unabhängig, da

$$\alpha_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

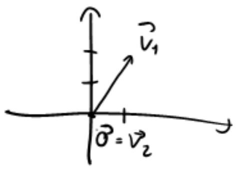
$$\Leftrightarrow \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Leftrightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$



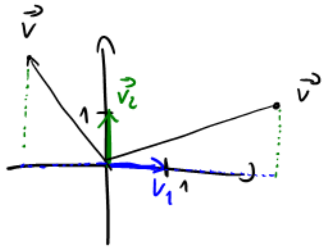
- Die Vektoren $\vec{v}_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\0\end{pmatrix}$ sind linear abhängig, denn

1 Mgl. $\underset{\alpha_1}{\underset{\shortparallel}{0}} \cdot \vec{v}_1 + \underset{\alpha_2}{\underset{\shortparallel}{1}} \cdot \vec{v}_2 = \vec{0}$

2 Mgl. $0 \cdot \vec{v}_1 + 5 \cdot \vec{v}_2 = \vec{0}$



Bsp zum Spann:



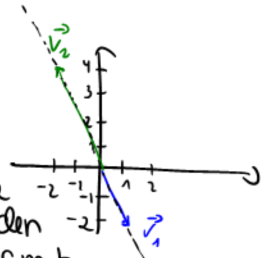
- $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$. Dann gilt $\underbrace{\operatorname{span}(\vec{v}_1, \vec{v}_2)}_{\text{Spann von } \vec{v}_1, \vec{v}_2} = \mathbb{R}^2$, denn

$$\operatorname{span}(\vec{v}_1, \vec{v}_2) = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}\} = \left\{ \begin{pmatrix}\alpha_1\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} \;\middle|\; \alpha_1, \alpha_2 \in \mathbb{R} \right\} = \mathbb{R}^2$$

- $\vec{v}_1 = \begin{pmatrix}1\\-2\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}-2\\4\end{pmatrix}$. Dann gilt $\operatorname{span}(\vec{v}_1, \vec{v}_2) = \left\{\alpha \cdot \begin{pmatrix}1\\-2\end{pmatrix} \middle| \alpha \in \mathbb{R}\right\}$, denn

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = \alpha_1 \begin{pmatrix}1\\-2\end{pmatrix} + \alpha_2 \begin{pmatrix}-2\\4\end{pmatrix} = \begin{pmatrix}\alpha_1 - 2\alpha_2\\ -2\alpha_1 + 4\alpha_2\end{pmatrix} = (\alpha_1 - 2\alpha_2) \begin{pmatrix}1\\-2\end{pmatrix}$$

Gerade durch den Ursprung mit Richtungsvektor $\begin{pmatrix}1\\-2\end{pmatrix}$



Bsp zu Thema Basis:

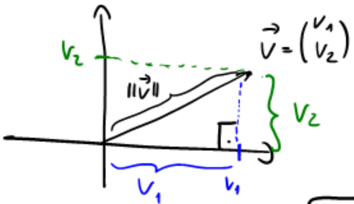
- $\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ bilden keine Basis des $\mathbb{R}^3$, da sich $\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ nicht als Linearkombination von $\vec{v}_1, \vec{v}_2$ schreiben lässt.
- $\vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ bilden keine Basis des $\mathbb{R}^3$, da weder $\vec{v} = \begin{pmatrix}0\\0\\1\end{pmatrix}$ nicht im Spann von $\vec{v}_1, \vec{v}_2, \vec{v}_3$ ist
- $\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}1\\2\end{pmatrix}$ bilden keine Basis des $\mathbb{R}^2$, da zwar $\operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \mathbb{R}^2$, aber die Darstellung von z.B. $\vec{v} = \begin{pmatrix}2\\4\end{pmatrix}$ ist nicht eindeutig, denn

$$\begin{pmatrix}2\\4\end{pmatrix} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + 2\cdot\vec{v}_3$$
$$\begin{pmatrix}2\\4\end{pmatrix} = 2\cdot\vec{v}_1 + 4\vec{v}_2 + 0\cdot\vec{v}_3$$

zu Def 12.1

- $\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\end{pmatrix} = 2\cdot 3 + 1\cdot(-1) = 5$, $\quad \begin{pmatrix}0\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix} = 0\cdot 2 + 1\cdot 1 + 1\cdot 0 = 1$

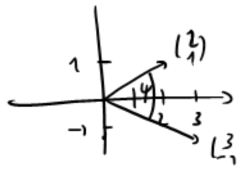
  ($v_1 = 2$, $v_2 = 1$, $w_1 = 3$, $w_2 = -1$)

- 

  Pythagoras: $\|\vec{v}\|^2 = v_1^2 + v_2^2 \Rightarrow \|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$

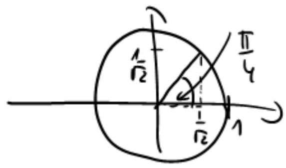
Bsp $\left\|\begin{pmatrix}2\\1\end{pmatrix}\right\| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $\quad \left\|\begin{pmatrix}3\\-1\end{pmatrix}\right\| = \sqrt{3^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

- Winkel zwischen $\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}3\\-1\end{pmatrix}$ ist gegeben durch

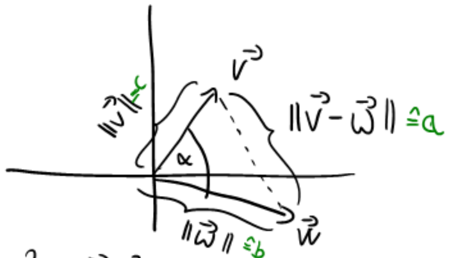


$$\cos\varphi = \frac{\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\-1\end{pmatrix}}{\left\|\begin{pmatrix}2\\1\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}3\\-1\end{pmatrix}\right\|} = \frac{2\cdot 3 + 1\cdot(-1)}{\sqrt{5}\cdot\sqrt{10}} = \frac{\cancel{5}}{\cancel{5}\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \varphi = \frac{\pi}{4} \mathrel{\hat{=}} 45°$$



zu Kosinussatz:



$$\underbrace{\|\vec{v} - \vec{w}\|^2}_{\cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\cdot\vec{v}\cdot\vec{w}} = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{v}\|\,\|w\|\cos\alpha$$

$$\cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\cdot\vec{v}\cdot\vec{w} = \cancel{\|\vec{w}\|^2} + \cancel{\|\vec{v}\|^2} - 2\,\|v\|\,\|w\|\cos\alpha$$

$$\Leftrightarrow \quad \cos\alpha = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$$