Complete minors in randomly perturbed graphs

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Joint work with Elad Aigner-Horev and Michael Krivelevich.

The perturbed model

- Let $\mathcal{G}_n$ be some family of graphs with vertex-set $[n]$.
- Let $\mathcal{R}_n$ be some distribution over graphs with vertex-set $[n]$.
- The perturbed model is $G \cup R$, where $G \in \mathcal{G}_n$ and $R \sim \mathcal{R}_n$.
- Introduced in 2003 by Bohman, Frieze and Martin.
- Somewhat reminiscent of smoothed analysis (Spielman–Teng 2001).
- Typically (but not necessarily) $\mathcal{G}_n$ is a family of dense graphs and $\mathcal{R}_n$ is $\mathbb{G}(n,p)$.

Why study randomly perturbed structures?

- Extends random graph models (the case $G = \bar{K}_n$).
- Bridges (so to speak) probabilistic combinatorics and extremal combinatorics.
- May assist in the study of deterministic structures.

Complete minors in randomly perturbed graphs

Hadwiger's conjecture

Hadwiger's conjecture 1943.

$h(G) \geq \chi(G)$ for every graph G.

- $\chi(G)$ – chromatic number.
- $h(G)$ – Hadwiger number, i.e., largest integer r such that $G \succcurlyeq K_r$.

Hadwiger

Partial results

- Known only for $\chi(G) \in \{2, 3, 4, 5, 6\}$.
- The case $\chi(G) = 5$ is equivalent to the Four Colour Theorem.
- Kostochka, Thomason (1984):

$$h(G) = \Omega\left(\frac{\bar{d}(G)}{\sqrt{\log(\bar{d}(G))}}\right) = \Omega\left(\frac{\chi(G)}{\sqrt{\log(\chi(G))}}\right).$$

- Delcourt–Postle (2021): $h(G) = \Omega\left(\frac{\chi(G)}{\log\log(\chi(G))}\right)$.

Almost all graphs satisfy Hadwiger's conjecture

Essentially throughout the whole range of p:

$$h(G) = \Omega\left(\sqrt{\frac{e(G)}{\log(\overline{d}(G))}}\right) \quad \text{and} \quad \chi(G) = O\left(\frac{\overline{d}(G)}{\log(\overline{d}(G))}\right)$$

a.a.s. whenever $G \sim \mathbb{G}(n,p)$.

Hadwiger's conjecture holds rather strongly for most graphs.

Credits: Bollobás, Catlin, Erdős, Fountoulakis, Krivelevich, Kühn, Łuczak, Osthus, Pittel, Sudakov, Wierman.

Perturbing Hadwiger's conjecture

The following conjecture is a natural relaxation of Hadwiger's conjecture.

The perturbed Hadwiger's conjecture

$h(G \cup \mathbb{G}(n,p)) \geq \chi(G \cup \mathbb{G}(n,p))$ a.a.s. for any G, whenever $p = \omega(n^{-2})$.

An implication of Hadwiger's conjecture

Hadwiger's conjecture and the simple bound $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ imply

Implication of Hadwiger's conjecture

$h(G) \geq \frac{|V(G)|}{\alpha(G)}$ for any graph G.

If true, this is tight (G is the disjoint union of $\alpha(G)$ cliques).

An implication of Hadwiger's conjecture

Implication of Hadwiger's conjecture

$h(G) \geq \frac{|V(G)|}{\alpha(G)}$ for any graph G.

Current state of the art:

- Duchet and Meyniel (1982): $h(G) \geq \lceil \frac{|V(G)|}{2\alpha(G)-1} \rceil$.
- Progress by: Balogh, Fox, Kostochka, Kwawbarayashi, Lenz, Maffray, Pedersen, Plummer, Song, Steibitz, Toft, Wood, Woodall, Wu.
- Balogh and Kostochka (2011): $h(G) \geq \lceil \frac{|V(G)|}{(2-c)\alpha(G)} \rceil$, $c > 0.052$.

Complete minors in randomly perturbed graphs

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\forall \varepsilon > 0 \ \ \exists \alpha > 0$ s.t.

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right)$$

holds a.a.s., whenever $p := p(n) = \frac{1+\varepsilon}{n}$ and G is an n-vertex graph s.t. $\alpha(G) \leq \alpha n$.

Aigner-Horev

Krivelevich

Complete minors in randomly perturbed graphs

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\forall \varepsilon > 0 \ \ \exists \alpha > 0$ s.t.

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

whenever $p := p(n) = \frac{1+\varepsilon}{n}$ and $\alpha(G) \leq \alpha n$.

Main message:

$$\alpha(G) \geq \log^2 n \implies h(G + n/2 \text{ random edges}) = \tilde{\Omega}\left(n/\sqrt{\alpha(G)}\right).$$

Compare with: $h(G) \geq n/\alpha(G)$ (best possible).

Complete minors in randomly perturbed graphs

$\forall \varepsilon > 0 \;\; \exists \alpha > 0$ s.t.

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

whenever $p := p(n) \geq \frac{1+\varepsilon}{n}$ and $\alpha(G) \leq \alpha n$.

- Only meaningful if $\alpha(G) \leq c\frac{n}{\log n}$.
- Indeed, $h(\mathbb{G}(n, C/n)) = \Omega\left(\sqrt{n}\right)$, where $C > 1$ is a constant [Fountoulakis, Kühn, Osthus, 2008)].

Tightness

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

Up to logarithmic terms, the result is tight (proved on next slide).

Open problem

Get rid of the logarithmic terms.

All logarithmic terms involved are artefacts of our proof

- Union bounds;
- Application of the Kostochka–Thomason Theorem.

Tightness

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n}\cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

- $G := \bigcup K_{k+1}$ ($n/(k+1)$ disjoint K_{k+1}) .
  - $e(G) = \frac{nk}{2}$ (G is k-regular).
  - $\alpha(G) = \frac{n}{k+1}$.
- $R \sim \mathbb{G}(n,p)$, $p := p(n) \leq \frac{k}{2n}$.
  - $e(R) \leq \frac{nk}{2}$ a.a.s.
- $e(G \cup R) \leq nk$ a.a.s.
- $h(G) = O(\sqrt{e(G \cup R)}) = O(\sqrt{nk}) = O(n/\sqrt{\alpha(G)})$.

α vs. δ

- Most vertices v of an n-vertex graph G satisfy $\deg(v) = \Omega(n/\alpha(G))$.
- Hence, why not simply require a large minimum degree?
- $G = (A \cup B)$ complete bipartite graph, $|A| = \sqrt{n}$, $|B| = n - \sqrt{n}$;
- Then, $\delta(G) = \sqrt{n}$ and $h(G) = \sqrt{n} + 1$;
- Adding $\Theta(n)$ random edges results a.a.s. in $h(G \cup \mathbb{G}(n,p)) = \Theta(\sqrt{n})$;
- On the other hand, by our theorem, if $\alpha(G) = \sqrt{n}$, then $h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n^{3/4}}{\sqrt{\log n}}\right)$.

Known related results

- Fountoulakis, Kühn and Osthus 2009:
  - Proved: $h(\mathbb{G}_{n,3}) = \Theta(\sqrt{n})$.
  - Hamilton cycle + random matching.

- Kang, Kang, Kim, and Oum 2020+:

$\exists C > 0$ s.t. $\forall\ \omega(n^{-2}) = p := p(n) \leq 2/n$ if G is an n-vertex connected graph with $\Delta(G) \leq \Delta \leq pn^2/C$, then

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\min\left\{\sqrt{\frac{pn^2}{\log\Delta}}, \frac{pn^2}{\Delta\sqrt{\log\Delta}}\right\}\right)$$

holds a.a.s.

Below $1/n$ for the perturbation

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\exists c > 0$ s.t. given an integer $k := k(n) = o(n)$

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\} \cdot k}\right) \quad \text{a.a.s.}$$

whenever G is an n-vertex graph satisfying $\alpha(G) \leq \frac{cn}{k}$ and $p := p(n) = \frac{8}{nk}$.

- $p < 1/n \implies$ get rid of $\alpha(G) = O(n/\log n)$.
  - Perturbation no longer "brings" a $\Omega(\sqrt{n})$-minor.

Hadwiger's conjecture: perturbed vertex-transitive graphs

Definition: vertex-trasitive

$\forall (u, v) \in V(G) \times V(G)$ $\exists$ automorphism of G mapping u to v

Proposition (Aigner-Horev, H. and Krivelevich 2023+)

For every integer $8 < k := k(n)$ $\exists c_1, c_2 > 0$ such that if

- G is vertex-transitive and
- $c_1 k^2 \log^3 n \leq \alpha(G) \leq \frac{c_2 n}{k}$ and
- $p := p(n) = \frac{8}{nk}$,

then a.a.s. $G \cup \mathbb{G}(n, p)$ satisfies Hadwiger's conjecture.

Hadwiger's conjecture: perturbed vertex-transitive graphs

Proof sketch:

- G is vertex-transitive $\implies \chi(G) \leq (1+\log n)\frac{n}{\alpha(G)}$;
- $R \sim \mathbb{G}(n,p)$, $p < 1/n$;
- $\chi(R) \leq 3$ a.a.s. (unicyclic components);
- A.a.s. $\chi(G \cup R) \leq \chi(G)\chi(R) \leq 3\chi(G)$;
- By our theorem: a.a.s. $h(G \cup R) = \Omega\left(\frac{n}{k\sqrt{\alpha(G)\log n}}\right) \geq 3\chi(G)$;
  - Use $\alpha(G) = \Omega(k^2 \log^3 n)$.

Complete topological minors in randomly perturbed graphs

Hajós' Conjecture

Hajós' Conjecture

$\mathrm{tcl}(G) \geq \chi(G)$.

$\mathrm{tcl}(G)$: largest integer r such that subdivision of $K_r \subseteq G$.

Hajós

Disproved by Catlin (1979).

Theorem (Erdős and Fajtlowicz - 1981)

Almost all graphs do not satisfy Hajós' Conjecture.

Almost all graphs do not satisfy Hajós' conjecture

Theorem (Bollobás – 1988)

$\chi(\mathbb{G}(n,1/2)) = (1+o(1))\frac{n}{2\log_2 n}$ a.a.s.

Theorem (Erdős and Fajtlowicz – 1981)

$\mathrm{tcl}(\mathbb{G}(n,1/2)) = O(\sqrt{n})$ a.a.s.

- **Main argument:** space limitation (too many non-trivial paths require too many vertices).

Complete topological minors in random graphs

Observation: $\mathrm{tcl}(G) \leq \Delta(G) + 1$ for any G.

Theorem

$\mathrm{tcl}(\mathbb{G}(n,p)) = \Theta\left(\min\{\Delta(\mathbb{G}(n,p)), \sqrt{n}\}\right)$ a.a.s. whenever $p := p(n) > 1/n$ is bounded away from 1.

Credits: Ajtai, Bollobás, Catlin, Erdős, Fajtlowicz, Komlós, Szemerédi.

Complete topological minors in randomly perturbed graphs

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\exists C > 0$ such that

$$\mathrm{tcl}(G \cup \mathbb{G}(n,p)) \geq \min\left\{\delta(G)/8, \sqrt{n/60\log_2 n}\right\}$$

holds a.a.s. whenever $p := p(n) \geq C/n$ and G is an n-vertex graph satisfying $\delta(G) = \omega(1)$.

Tightness

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\mathrm{tcl}(G \cup \mathbb{G}(n,p)) \geq \min\left\{\delta(G)/8, \sqrt{n/60\log_2 n}\right\}$ a.a.s. whenever $p := p(n) \geq C/n$ and $\delta(G) = \omega(1)$.

Tight up to logarithmic terms.

Tightness with respect to $\delta(G)$:

- $k := k(n)$;
- G - k-regular;
- $\frac{2\log n}{n} \leq p := p(n) = o(k/n)$ (lower bounds k);
- $\Delta(\mathbb{G}(n,p)) = o(k)$ a.a.s.;
- $\mathrm{tcl}(G \cup \mathbb{G}(n,p)) \leq \Delta(G \cup \mathbb{G}(n,p)) + 1 = (1+o(1))k = (1+o(1))\delta(G)$.

Tightness

Theorem (Aigner-Horev, H. and Krivelevich 2023+)

$\mathrm{tcl}((G \cup \mathbb{G}(n,p)) \geq \min\left\{\delta(G)/8, \sqrt{n/60\log_2 n}\right\}$ a.a.s. whenever $p := p(n) \geq C/n$ and $\delta(G) = \omega(1)$.

Tight up to logarithmic terms.

Tightness with respect to $\sqrt{n}$:

- $G \sim \mathbb{G}(n, 1/3)$;
- $R \sim \mathbb{G}(n, o(1))$;
- Then $G \cup R \sim \mathbb{G}(n,p)$ for some $1/3 \leq p \leq 1/2$;
- $\mathrm{tcl}(G \cup R) = O(\sqrt{n})$ a.a.s. (Erdős–Fajtlowicz).

Complete Minors

PROOF SKETCHES

Minors - sketch: uniform sets

G - a graph.

Definition (ℓ-set):

- $X \subseteq V(G)$;
- $|X| = \ell$;
- Elements chosen u.a.r. from $V(G)$ without replacement.

Minors - sketch: ensembles

G - a graph.

Definition ((k,ℓ)-ensemble):

- $(X_1, \ldots, X_k)$ sequence of ℓ-sets;
- Sampled in succession;
- X_i is sampled from $V(G) \setminus \bigcup_{j=1}^{i-1} X_j$.

Notation: $\mathcal{X} = \bigcup_{i=1}^{k} X_i$.

Minors - sketch: ensemble lemma

The ensemble lemma

$\gamma \in (0, 1/12)$ fixed, G graph on n vertices,

$$\ell := \max\left\{\frac{2}{\gamma}\sqrt{\alpha(G)},\ \frac{32}{\gamma}\log n\right\} \quad \text{and} \quad k := \gamma\frac{n}{\ell}$$

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G (candidate branch sets)
$(U_1, \ldots, U_k)$ - (k, ℓ)-ensemble of $G \setminus \mathcal{X}$ (more candidate branch sets)

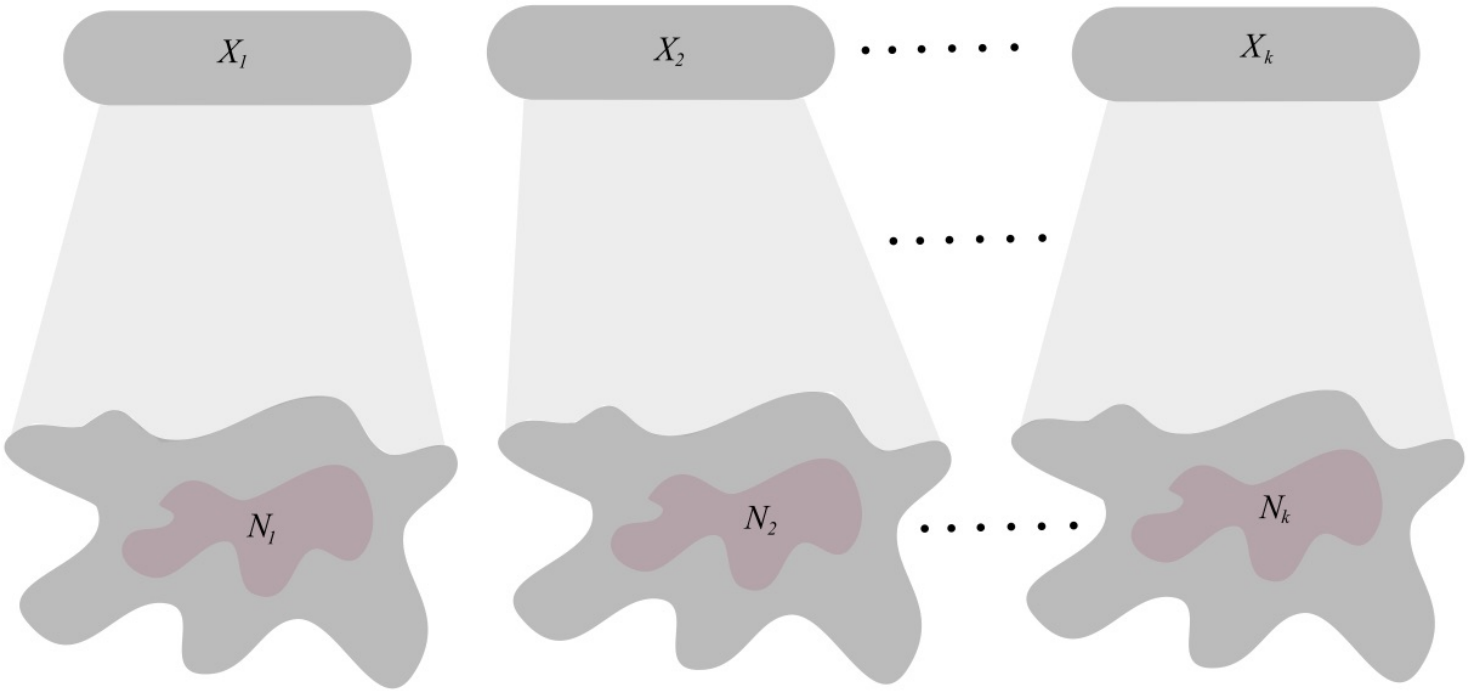
The following holds a.a.s. simultaneously.

- $\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.
- $|\{j \in [k] : N_i \cap U_j \neq \emptyset\}| = \Omega_\gamma(k)$ for every $i \in [k]$.

Ensemble lemma - the first ensemble

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G;

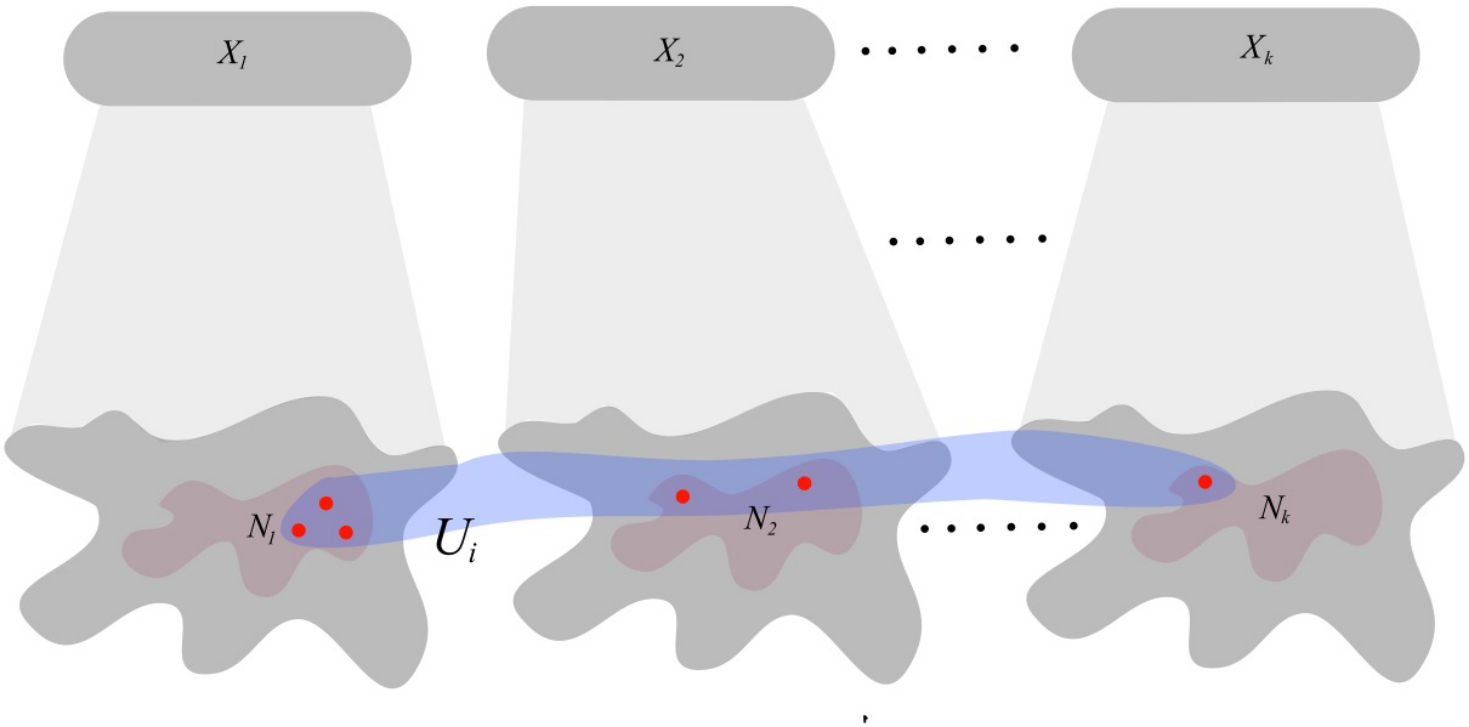
$\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.



Ensemble lemma - the second ensemble

$(U_1, \ldots, U_k)$ - (k, ℓ)-ensemble of $G \setminus \mathcal{X}$;

$|\{j \in [k] : N_i \cap U_j \neq \emptyset\}| = \Omega_\gamma(k)$ for every $i \in [k]$.



Randomness shift

Observation

Any subgraph K of $R \sim \mathbb{G}(n,p)$ is distributed uniformly over all copies of K in K_n.

Proof:

- Generate $R' \sim \mathbb{G}(n,p)$;
- Sample a permutation $\pi \in S_n$ u.a.r.;
- Set $R := ([n], \{\pi(u)\pi(v) : uv \in E(R')\})$;
- Observe that $R \sim \mathbb{G}(n,p)$.

Using the ensemble lemma I

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

Proof:

- A.a.s. $R \sim \mathbb{G}(n,p)$ contains an $\Omega(n)$-long path P;
- Partition P into subpaths $P_1, \ldots, P_{2k}$, on ℓ vertices each (ℓ and k as in ensemble lemma);
- By randomness shift:
  - $(V(P_1), \ldots, V(P_k))$ – (k,ℓ)-ensemble;
  - $(V(P_{k+1}), \ldots, V(P_{2k}))$ – (k,ℓ)-ensemble of $G \setminus \bigcup_{i=1}^{k} V(P_i)$;
- Apply ensemble lemma;

Using the ensemble lemma II

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n}\cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\}}\right) \text{ a.a.s.}$$

- $G \succcurlyeq H$: $v(H) = 2k,\ e(H) = \Omega(k^2) \implies \bar{d}(H) = \Omega(k)$.
- Kostochka–Thomason Theorem: $H \succcurlyeq K_r$ with

$$r = \Omega\left(\frac{\bar{d}(H)}{\sqrt{\log \bar{d}(H)}}\right) = \Omega\left(\frac{k}{\sqrt{\log k}}\right) = \Omega\left(\frac{n}{\sqrt{\log n}\cdot \ell}\right)$$
$$= \Omega\left(\frac{n}{\sqrt{\log n \cdot \alpha(G)}}\right).$$

The ensemble lemma - proof sketch

The ensemble lemma - first property

$\gamma \in (0, 1/12)$ fixed, G graph on n vertices,

$$\ell := \max\left\{\frac{2}{\gamma}\sqrt{\alpha(G)},\ \frac{32}{\gamma}\log n\right\} \quad \text{and} \quad k := \gamma\frac{n}{\ell}$$

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G;
$(U_1, \ldots, U_k)$ - (k, ℓ)-ensemble of $G \setminus \mathcal{X}$;

The following holds a.a.s. simultaneously.

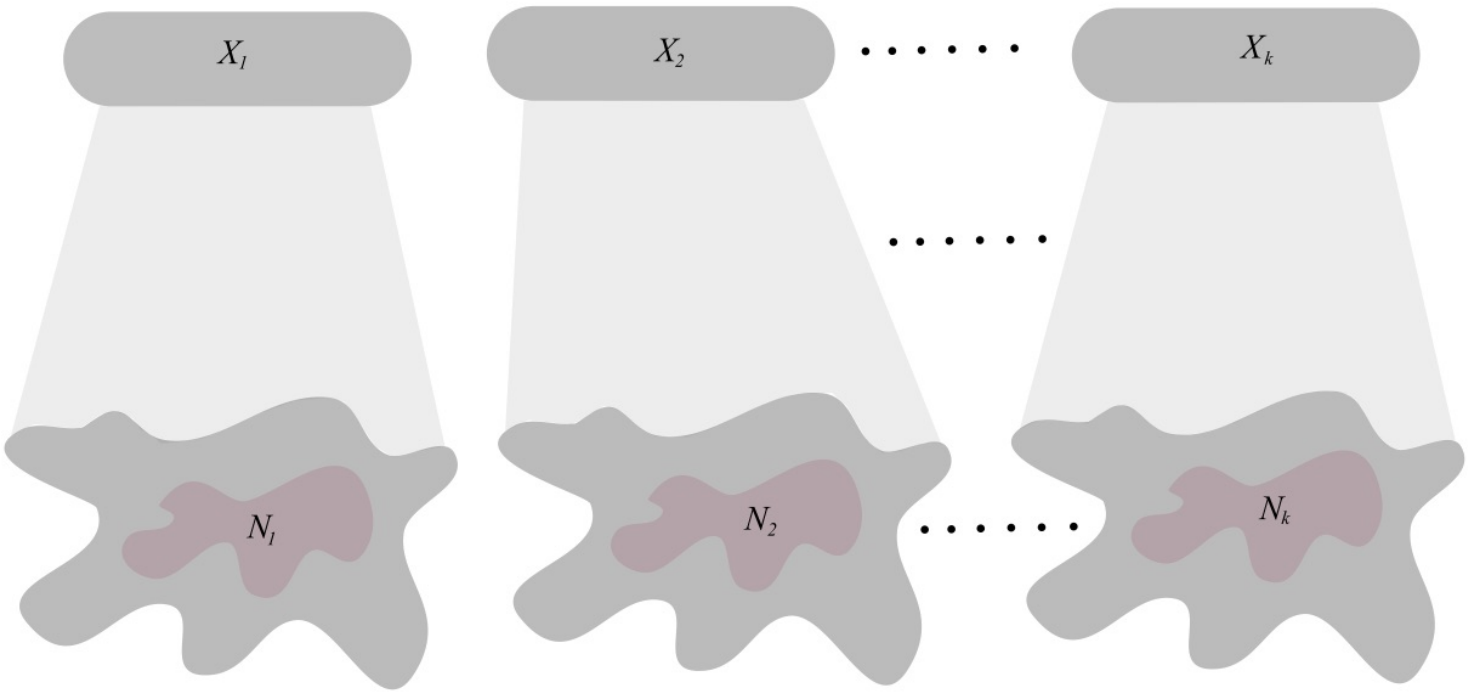
- $\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.
- $|\{j \in [k] : N_i \cap U_j \neq \emptyset\}| = \Omega_\gamma(k)$ for every $i \in [k]$.

Defining the sets N_i

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G;

$\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.

Reminder:



Defining the sets N_i

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G;

$\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.

- Consider the formation of X_i:
- $X_i^{(j)}$:= the set X_i after jth vertex is added;
- $N_i := A_i \setminus \mathcal{X}$;
- A_i is defined through the nested sequence

$$\emptyset = A_i^{(0)} \subseteq A_i^{(1)} \subseteq \ldots \subseteq A_i^{(\ell)} = A_i \subseteq N_G(X_i).$$

Defining A_i

$(X_1, \ldots, X_k)$ - (k, ℓ)-ensemble of G;

$\forall i \in [k]$, $\exists N_i \subseteq N_G(X_i)$ s.t. $|N_i| \geq k/3$ and $N_i \cap \mathcal{X} = \emptyset$.

Suppose: $A_i^{(0)} \subseteq A_i^{(1)} \subseteq \ldots \subseteq A_i^{(j)}$ are defined, $0 \leq j < \ell$.

Define $A_i^{(j+1)}$:

- If $|A_i^{(j)}| \geq k/2$, set $A_i^{(j+1)} := A_i^{(j)}$.
- Otherwise, do the following:

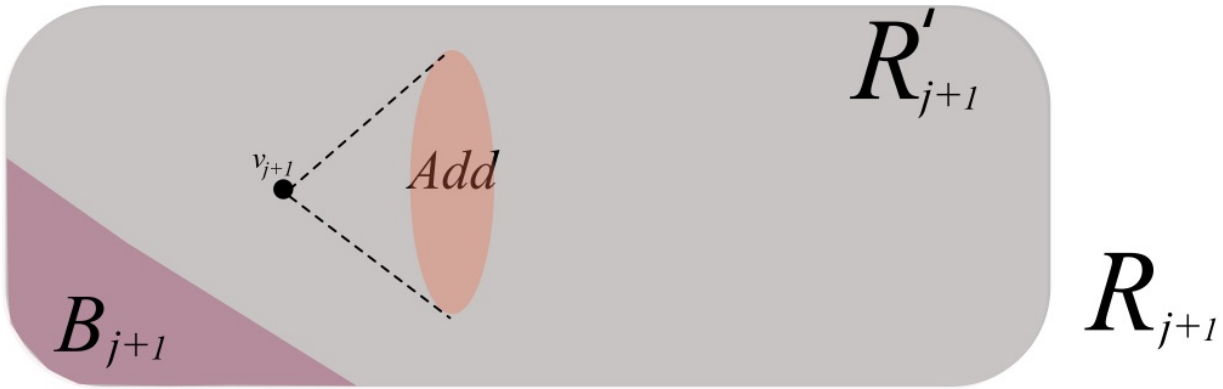
Defining $A_i^{(j+1)}$

Set

- $R_{j+1} := V(G) \setminus \left(\left(\bigcup_{r=1}^{i-1} X_r \right) \cup X_i^{(j)} \cup A_i^{(j)} \right)$
- $B_{j+1} := \left\{ v \in R_{j+1} : \deg_{G[R_{j+1}]}(v) < \gamma \frac{|R_{j+1}|}{\alpha(G[R_{j+1}])} \right\}$
- $R'_{j+1} := R_{j+1} \setminus B_{j+1}$

Defining $A_i^{(j+1)}$

- v_{j+1} - the $(j+1)$st vertex added to X_i.
- If $v_{j+1} \in R'_{j+1}$: $A_i^{(j+1)} := A_i^{(j)} \cup N_{G[R_{j+1}]}(v_{j+1})$ SUCCESS
- If $v_{j+1} \notin R'_{j+1}$: $A_i^{(j+1)} := A_i^{(j)}$ FAILURE

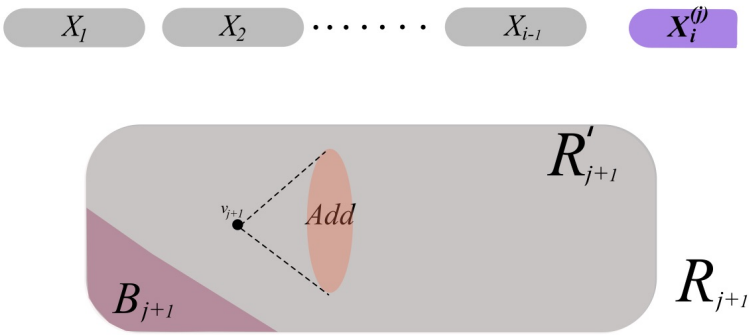


Defining $A_i^{(j+1)}$

Claim: *Throughout the process* $|R'_{j+1}| = \Omega_\gamma(n)$.

Proof:

- $\left|\left(\bigcup_{r=1}^{i-1} X_r\right) \cup X_i^{(j)} \cup A_i^{(j)}\right|$ is small by design.
- Why should B_{j+1} be small?
  - Recall: $B_{j+1} := \left\{v \in R_{j+1} : \deg_{G[R_{j+1}]}(v) < \gamma \frac{|R_{j+1}|}{\alpha(G[R_{j+1}])} =: r\right\}$;
  - $\alpha(G[R_{j+1}]) \geq \alpha(G[B_{j+1}]) \geq |B_{j+1}|/(\Delta(G[B_{j+1}]) + 1) \geq |B_{j+1}|/r$;
  - Then: $|B_{j+1}| \leq r\alpha(G[R_{j+1}]) = \gamma \frac{|R_{j+1}|}{\alpha(G[R_{j+1}])} \alpha(G[R_{j+1}]) = \gamma |R_{j+1}|$.



Defining N_i: final argument

Recall:

- We have defined A_i (probabilistically);
- We have set $N_i := A_i \setminus \mathcal{X}$.

It remains to prove that

$$\Pr\left[\ \mathcal{E}_i^{(1)} \text{ fails for some } i \in [k]\ \right] + \sum_{i=1}^{k} \Pr\left[\mathcal{E}_i^{(2)} \text{ fails} \,\middle|\, \mathcal{E}_i^{(1)} \text{ holds}\right] = o(1),$$

where

$$\mathcal{E}_i^{(1)} : |A_i| \geq k/2 \quad \text{and} \quad \mathcal{E}_i^{(2)} : |A_i \cap \mathcal{X}| \leq 2\gamma |A_i|$$

Final argument

- First term is $o(1)$:
  - Fix $i \in [k]$;
  - Use concentration of the binomial distribution to prove that there are many successes when constructing A_i;
  - Union bound over $i \in [k]$.
- Second term is $o(1)$:
  - Fix $i \in [k]$;
  - Upper bound the probability that when adding a vertex to $\mathcal{X}$ it is in A_i (failure);
  - Use concentration of the binomial distribution;
  - Union bound over $i \in [k]$.

The ensemble lemma - second property

The ensemble lemma

$\gamma \in (0, 1/12)$ fixed, G graph on n vertices,

$$\ell := \max\left\{\frac{2}{\gamma}\sqrt{\alpha(G)},\ \frac{32}{\gamma}\log n\right\} \quad \text{and} \quad k := \gamma\frac{n}{\ell}$$

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- $|\{j \in [k] : N_i \cap U_j \neq \emptyset\}| = \Omega_\gamma(k)$ for every $i \in [k]$.

Second property

- Fix $i \in [k]$;
- For an ensemble $(U_1, \ldots, U_k)$ set $u_i = |\{j \in [k] : U_j \cap N_i \neq \emptyset\}|$;
- Goal: prove that $\mathbb{P}(u_i < ck \mid |N_i| \geq k/3) = o(1/k)$ for some constant $c > 0$;
- Sample $U_1, \ldots, U_k$ one by one;
- Given $U_1, \ldots, U_{j-1}$, sample U_j.

Second property

- Use concentration of the hypergeometric distribution to show that $\mathbb{P}(|N_i \setminus (U_1 \cup \ldots \cup U_{j-1})| \geq k/4 \mid |N_i| \geq k/3)$ is large;
- Show that $\mathbb{P}(N_i \cap U_j \neq \emptyset \mid |N_i \setminus (U_1 \cup \ldots \cup U_{j-1})| \geq k/4) \geq \varepsilon > 0$;
- Then $\mathbb{P}(N_i \cap U_j \neq \emptyset \mid |N_i| \geq k/3) \geq \varepsilon' > 0$;
- Use concentration of the binomial distribution;
- Union bound over $i \in [k]$.

Proof of second theorem

$k := k(n) = o(n)$, $\alpha(G) \leq cn/k$ and $p := p(n) = 8/nk \implies$

$$h(G \cup \mathbb{G}(n,p)) = \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\left\{\sqrt{\alpha(G)},\ \log n\right\} \cdot k}\right) \quad \text{a.a.s.}$$

Proof sketch:

- Set $r = \frac{n}{2k}$;
- Find in G vertex-disjoint paths $P_1, \ldots, P_r$, each on k vertices;
  - Recall: small $\alpha \implies$ large average degree in large induced subgraphs.

Proof of second theorem

- Define auxiliary graphs Γ_G and Γ_R:
  - $V(\Gamma_G) = V(\Gamma_R) = [r]$;
  - $ij \in E(\Gamma_G) \iff E_G(P_i, P_j) \neq \emptyset$;
  - $ij \in E(\Gamma_R) \iff E_R(P_i, P_j) \neq \emptyset$.
- Note that
  - $\alpha(\Gamma_G) \leq \alpha(G) \leq cn/k \leq \alpha r$;
  - $\Gamma_R \sim \mathbb{G}(r, q)$, $q \geq (1+\varepsilon)/r$.
- Therefore

$$h(G \cup R) \geq h(\Gamma_G \cup \Gamma_R) \geq \Omega\left(\frac{r}{\sqrt{\log r} \cdot \max\{\sqrt{\alpha(\Gamma_G)}, \log r\}}\right)$$
$$= \Omega\left(\frac{n}{\sqrt{\log n} \cdot \max\{\sqrt{\alpha(G)}, \log n\} \cdot k}\right).$$

Thank you for your attention!