Metastability of the contact process on evolving networks: From fast to slow dynamics

Peter Mörters

joint work with

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Amitai Linker (Universidad Andres Bello)

The problem

Motivation: The contact process is a simple model of an infection spreading on a graph or network. When the network is scale-free the infection, once fully established, may survive for a long time due to a metastability effect. We study how time variability of the network can affect the metastable behaviour.

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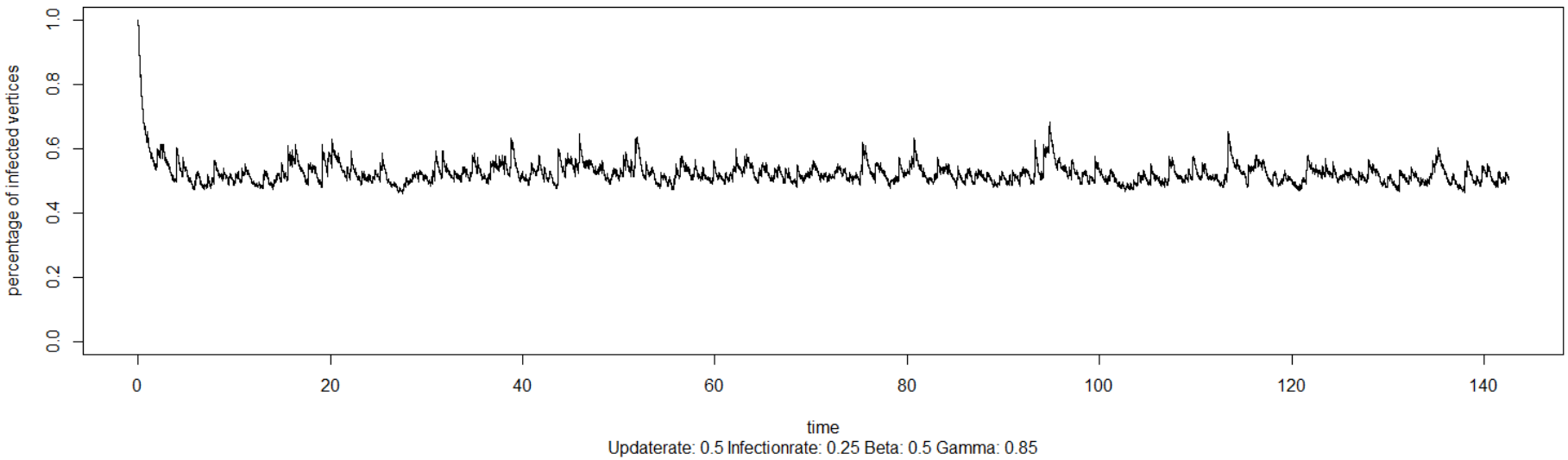

Figure: Simulation of the contact process on a scale-free graph, courtesy of Lucas Simon.

Network dynamics

Let

$$\kappa\colon (0,1] \to (0,\infty),$$
$$p\colon (0,1] \times (0,1] \to (0,\infty),$$

The vertex set of our graph is $\{1,\dots,N\}$ with small indices indicating large strength. The network is evolving by vertex updating.

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- new edges $i \leftrightarrow j$ are formed with probability

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independently for every $j \in \{1,\dots,N\} \setminus \{i\}$.

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We fix $0<\gamma<1$, $\beta>0$ and focus on the

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Then the graphs $\mathscr{G}_t$ are scale-free in the sense that

$$\lim_{N\to\infty} \text{proportion of vertices of degree } k =: \mu(k) = k^{-\tau + o(1)},$$

where the power law exponent is $\tau = 1 + \frac{1}{\gamma}$.

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With the parameter η we tune the speed of the network evolution. For large $\eta > 0$ powerful vertices update fast, for small $\eta < 0$ powerful vertices update slowly.

The contact process

The contact process on an (evolving) finite graph is defined as follows:

- Every vertex can either be infected or healthy.
- An infected vertex infects each of its neighbours at a fixed rate $\lambda > 0$.
- An infected vertex recovers with a fixed rate one.
- Once recovered, a vertex is again susceptible to infection by its neighbours.

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Metastability exponents

Starting with every vertex infected, denote

$$I_N(t) = \frac{1}{N}\,\mathbb{E}\big[\text{ number of infected vertices at time } t\big].$$

We say that $\rho(\lambda)$ is the metastable density if, whenever t_N is going to infinity slower than exponentially, we have

$$\lim_{N\to\infty} I_N(t_N) = \rho(\lambda) > 0.$$

The metastability exponents are defined as

$$\xi(\tau,\eta) := \lim_{\lambda\downarrow 0} \frac{\log \rho(\lambda)}{\log \lambda}.$$

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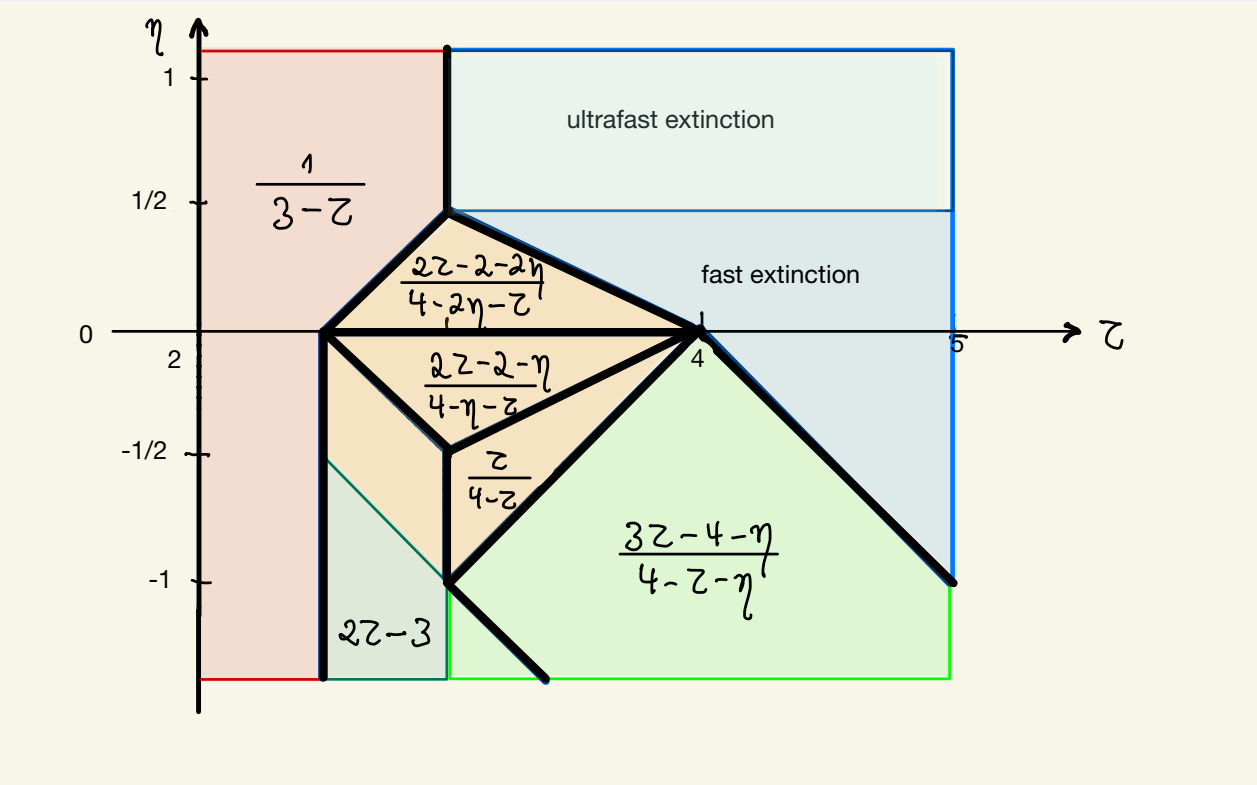

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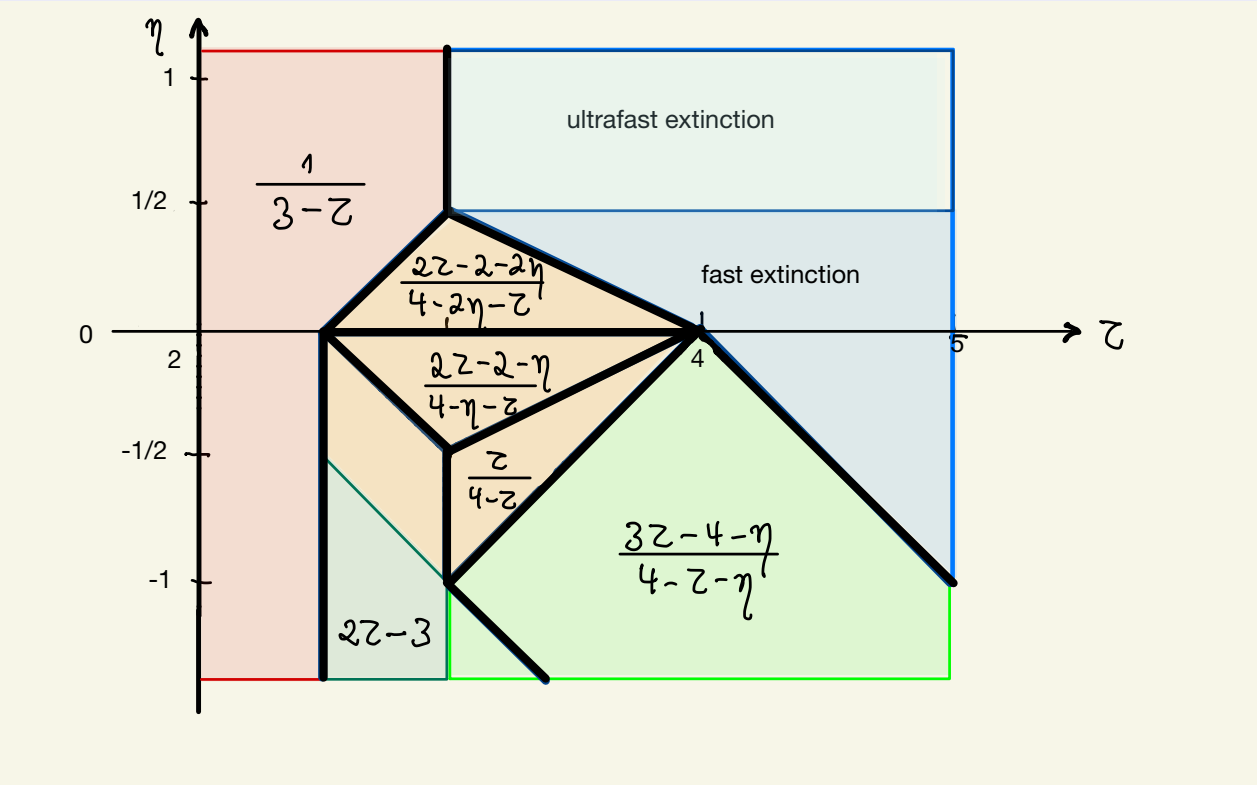

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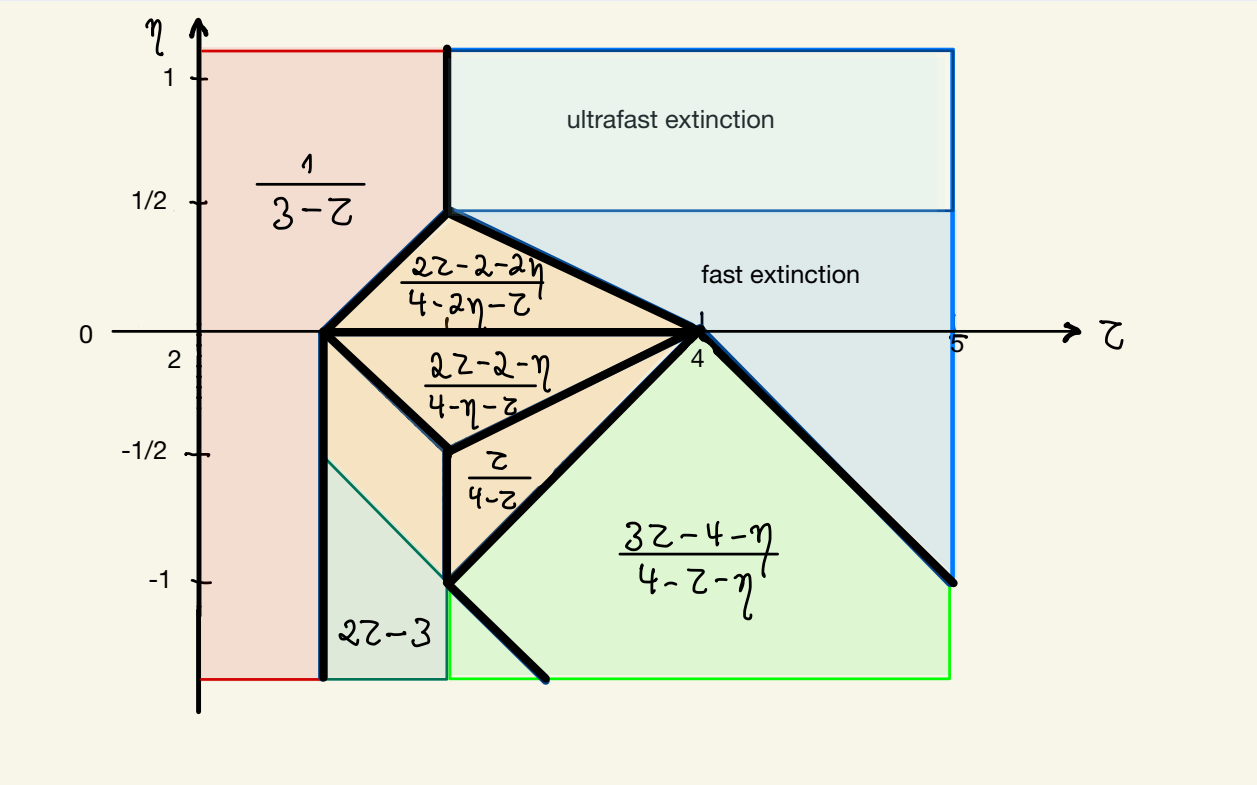

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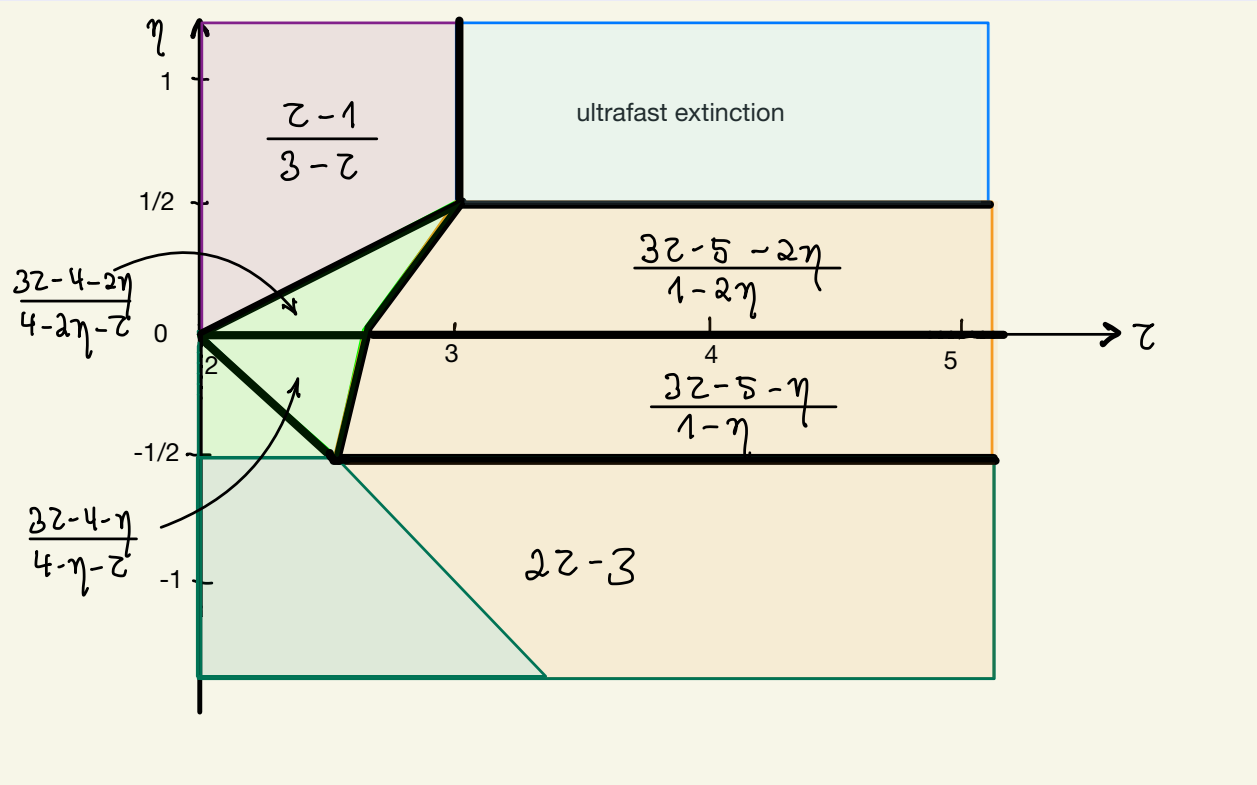

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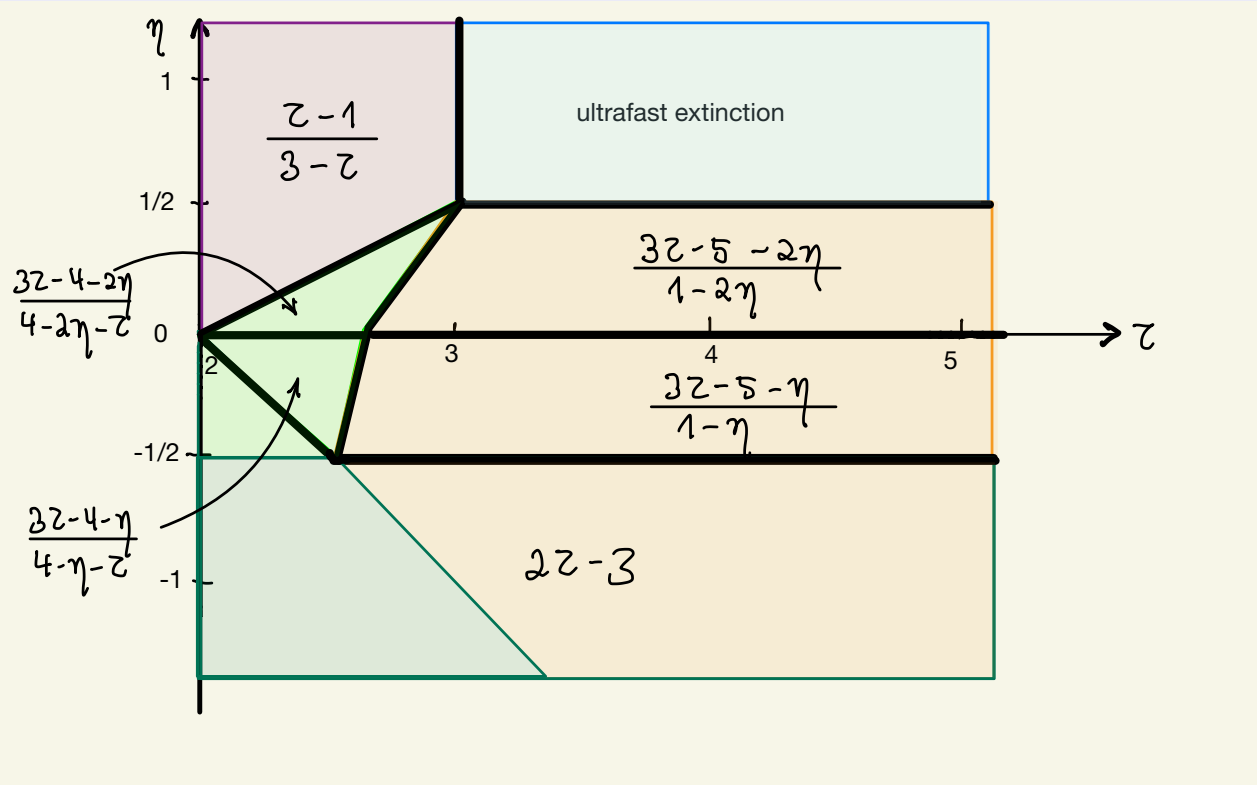

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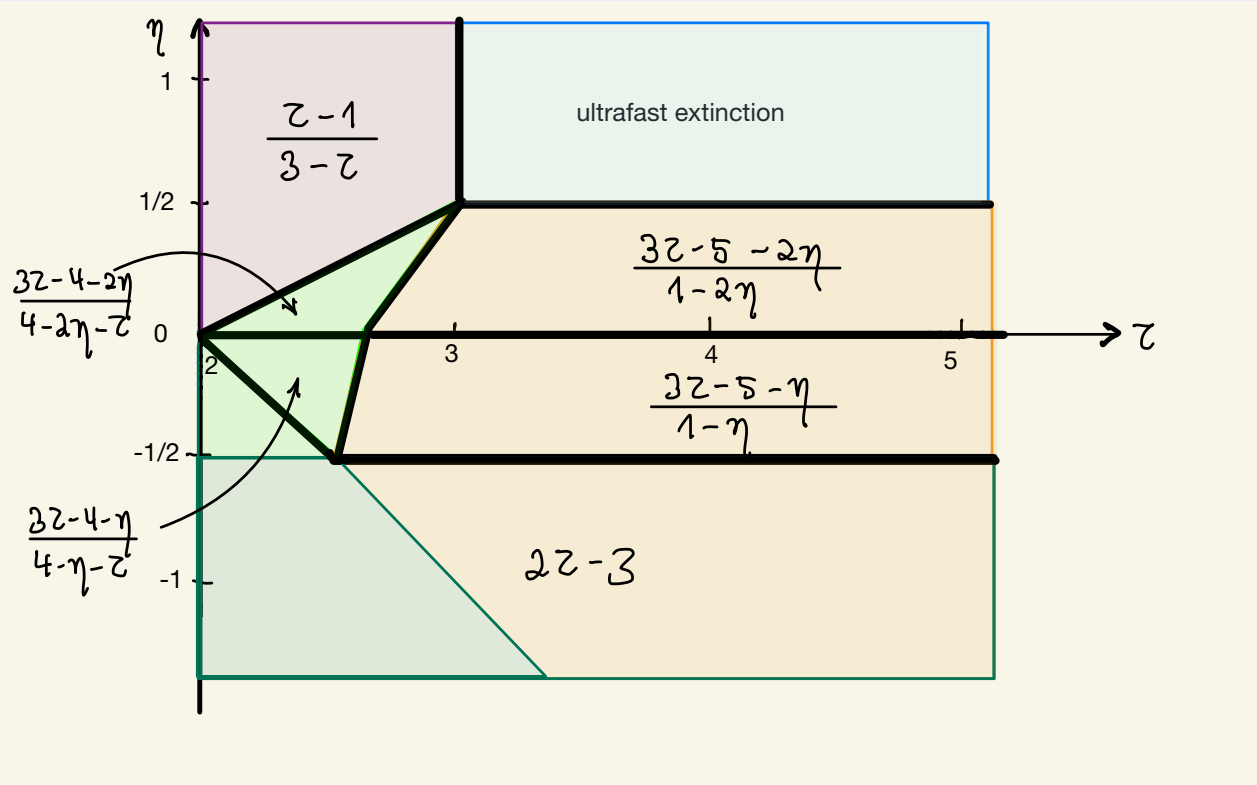

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Heuristics

In static scale-free networks if an infected vertex of large degree $k \gg \lambda^{-2}$ recovers, it typically has of order λk infected neighbours. The probability that none of these neighbours reinfects the vertex before it recovers is roughly

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$$(1 - \frac{\lambda}{\lambda + 1})^{\lambda k} \sim e^{-k\lambda^2}.$$

Hence the vertex can hold the infection for a time which is exponential in the vertex degree before a sustained recovery. In a static scale-free network there are sufficiently many vertices of high degree for this local survival mechanism to keep an established infection alive for a time exponential in the graph size, see for example Berger at al (2005).

Heuristics

In evolving scale-free networks the capacity of a vertex of large degree k to use the neighbourhood for local survival is significantly reduced compared to the static case. Even for *slow evolutions*, if $\eta < 0$, this degradation effect is significant. On recovery of a vertex of degree k the number of infected neighbours is still of order λk and therefore the probability that none of them reinfects the vertex before it updates is roughly

$$\frac{k^\eta}{\lambda^2 k + k^\eta} \sim \lambda^{-2} k^{\eta-1} \quad \text{if } \lambda^2 k \gg k^\eta.$$

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Hence the time that a vertex can survive locally is of order $\lambda^2 k^{1-\eta}$. Although this is just polynomial in the vertex degree and hence much smaller than in the static case, this local survival mechanism together with a spreading strategy can keep an established infection alive for a time exponential in the graph size.

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Updating of vertices also has the beneficial regeneration effect that during the time up to a sustained recovery updates of powerful vertices will regenerate their neighbourhood and the overall connectivity of the graph is improved, allowing it to spread the infection more efficiently.

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In a period of length $k^{-\eta}$ between two updates the probability that one of the λk infected neighbours of this vertex passes the infection to a second vertex of degree k is of order $\lambda^2 k^{1-\eta} \frac{k}{N}$ where $\frac{k}{N}$ is the order of the probability that there is an edge between a given neighbour of the first and the second vertex.

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As there are $\lambda^2 k$ updates before a sustained recovery the overall probability that an infected vertex with degree $k \gg \lambda^{-2}$ passes the infection to a vertex of the same degree via an intermediary vertex is $\lambda^4 k^{3-\eta} \frac{1}{N}$ and if the proportion $a(k)$ of vertices with degree at least k satisfies

$$a(k) \gg \lambda^{-4} k^{\eta-3}$$

the strategy of delayed indirect spreading, passing the infection from one powerful vertex to another via an intermediary vertex, yields survival of the infection.

Heuristics

A similar argument suggests that, as an infected vertex of degree k has of order $\lambda^2 k$ updates before a sustained recovery, if $\frac{p(k)}{N}$ denotes the probability that two vertices of degree k are connected by an edge, then the probability that by the time of its sustained recovery an infected vertex of degree k directly infects another vertex of degree k is

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Hence if the proportion $a(k)$ of vertices with degree at least k satisfies

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However, this is only correct if the update rate k^η of a vertex of degree k is at least of the order of the infection rate λ. If $k^\eta \ll \lambda$, then the dynamics of the subnetwork of powerful vertices is much slower than the spread of the infection. The infections passed from a powerful vertex will then typically reach vertices that are already infected, a depletion effect.

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In these cases the infection will spread at rate k^η instead of λ, so that overall delayed direct spreading can only give slow extinction if

$$a(k) \gg \lambda^{-2} k^{-1} p(k)^{-1}.$$

Heuristics

When the graph is very densely connected both spreading mechanisms are equally effective. We speak of delayed concurrent spreading. In this case the infection retains its metastable density even if either all edges between high degree vertices were removed or all low degree neighbouring vertices of high degree vertices were leaves.

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The degradation effect is stronger for *fast evolutions*, i.e. when $\eta \geq 0$. On recovery of a vertex of degree k the number of infected neighbours is now

$$k \frac{\lambda}{\lambda + 1 + k^\eta} \sim \lambda k^{1-\eta}$$

and so the probability that none of them reinfects the vertex before it updates is

$$\frac{k^\eta}{\lambda^2 k^{1-\eta} + k^\eta} \sim \lambda^{-2} k^{2\eta - 1} \quad \text{if } \lambda^2 k^{1-\eta} \gg k^\eta.$$

Only if $\eta \leq \frac{1}{2}$ the local survival mechanism persists and vertices with degree k satisfying $k^{1-2\eta} \gg \lambda^{-2}$ can hold the infection for a time of order $\lambda^2 k^{1-2\eta}$.

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All these strategies are based on choosing a threshold $a(\lambda) \searrow 0$ and keeping a positive proportion of vertices in the set

$$\mathit{Stars} := \{1, \dots, \lceil a(\lambda) N \rceil\}.$$

infected. The stars have a large degree $\asymp a(\lambda)^{-\gamma}$ and the infected neighbours of the stars cover a proportion

$$\asymp \lambda\, a(\lambda)\, a(\lambda)^{-\gamma}$$

of vertices. The maximal proportion gives the *metastable density* up to logarithmic corrections.

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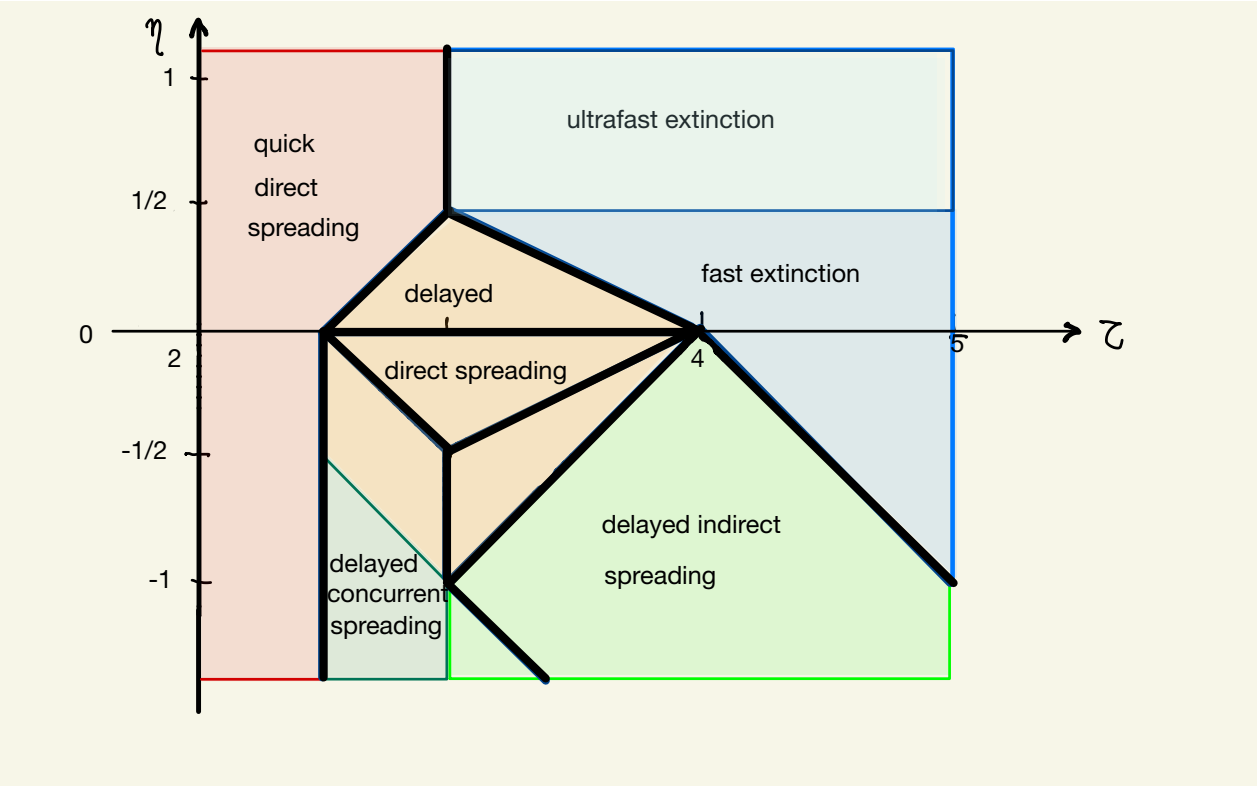

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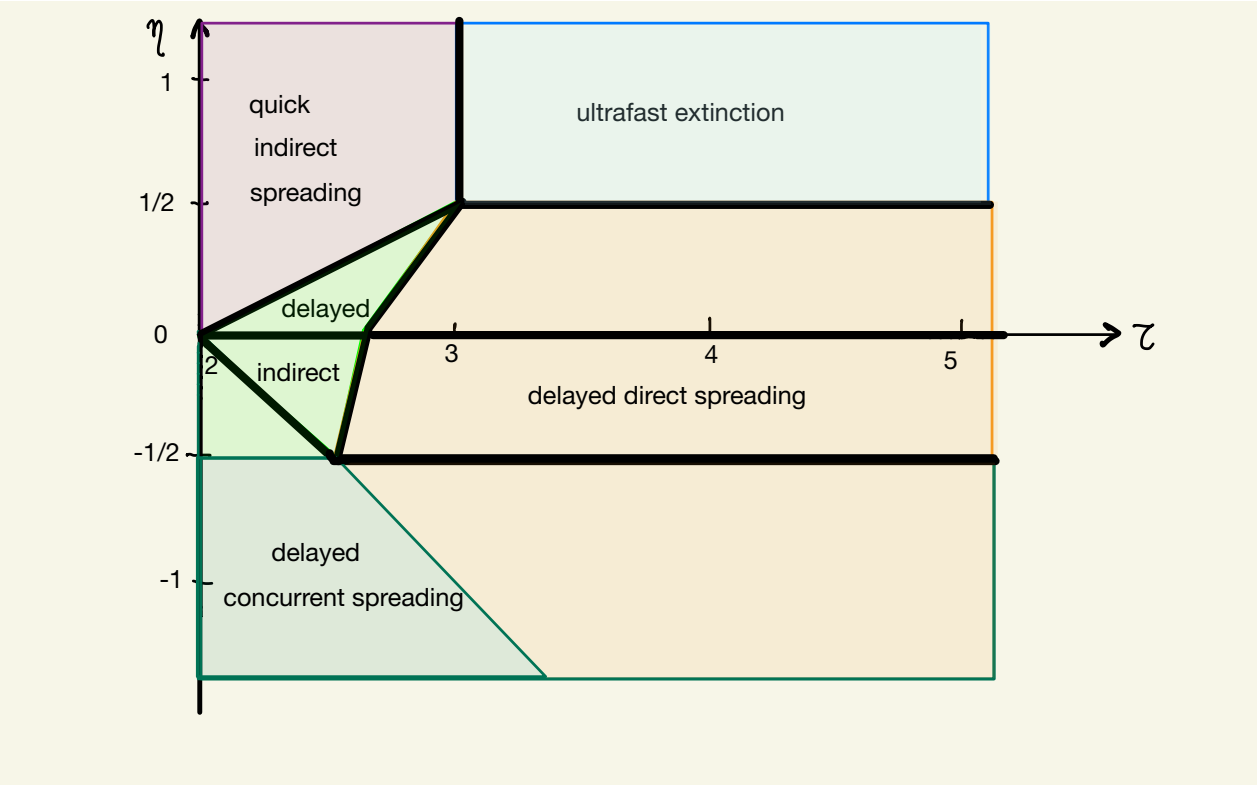

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  - Fast evolution: *First* the full model is coupled to a simpler model, the wait-and-see model, which has more infected vertices as certain unsustainable recoveries are omitted. For this stochastic upper bound we are able, in the *second* step, to associate a score to each configuration which eventually leads to the construction of a supermartingale, which gives the required upper bounds by application of the optional stopping theorem.

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  - Very slow evolution: Using self-duality, we bound the metastable density by the probability that the process, starting from one infected vertex x, survives up to some large given time t. Until time t, the contact process stays in a local dynamical neighbourhood of x in the graph, which is with high probability tree-like. The contact process on a static tree with degrees bounded by $\frac{1}{8\lambda^2}$ is locally subcritical: it may survive globally but not locally. We obtain a similar result for the local dynamical neighbourhood of x.

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  - Moderately slow evolution: A new technique is developed *combining local and global effects*. We split the vertices that are not stars into weak, strong quick and strong slow vertices. Weak vertices have degrees bounded by $\frac{1}{8\lambda^2}$. Strong quick ones have an update rate faster than λ. There is a score attached to weak infected vertices related to the size of the tree (with bounded degree) of weak vertices seen from this vertex. Strong vertices can be healthy, infected or saturated where saturated vertices cannot recover. Their score relates to their update rate and their weak and strong neighbours.

Thank you very much for your attention!