Getting started: CFD notation

PDE of p-th order $\quad f\left(u, \mathbf{x}, t, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^p u}{\partial t^p}\right) = 0$

scalar unknowns $\quad u = u(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \ t \in \mathbb{R}, \quad n = 1, 2, 3$

vector unknowns $\quad \mathbf{v} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{v} \in \mathbb{R}^m, \quad m = 1, 2, \dots$

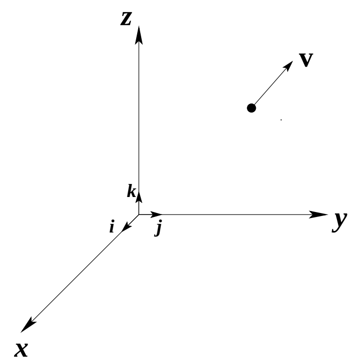
Nabla operator $\quad \boxed{\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}} \quad \mathbf{x} = (x, y, z), \quad \mathbf{v} = (v_x, v_y, v_z)$

$$\nabla u = \mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y} + \mathbf{k}\frac{\partial u}{\partial z} = \left[\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right]^T$$ gradient

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$ divergence

$$\nabla \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}$$ curl

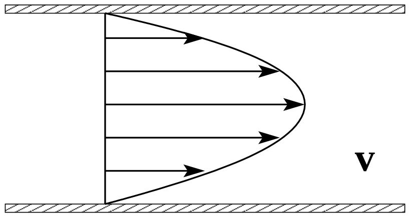
$$\Delta u = \nabla \cdot (\nabla u) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$ Laplacian



Tensorial quantities in fluid dynamics

Velocity gradient

$$\nabla \mathbf{v} = [\nabla v_x, \nabla v_y, \nabla v_z] = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$



Remark. The trace (sum of diagonal elements) of $\nabla \mathbf{v}$ equals $\nabla \cdot \mathbf{v}$.

Deformation rate tensor (symmetric part of $\nabla \mathbf{v}$)

$$\mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

Spin tensor $\quad \mathcal{S}(\mathbf{v}) = \nabla \mathbf{v} - \mathcal{D}(\mathbf{v}) \quad$ (skew-symmetric part of $\nabla \mathbf{v}$)

Vector multiplication rules

Scalar product of two vectors

$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \qquad \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} = [a_1 \;\; a_2 \;\; a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}$$

Example. $$\mathbf{v} \cdot \nabla u = v_x \frac{\partial u}{\partial x} + v_y \frac{\partial u}{\partial y} + v_z \frac{\partial u}{\partial z}$$ *convective derivative*

Dyadic product of two vectors

$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^3, \qquad \mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \;\; b_2 \;\; b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

Elementary tensor calculus

1. $\alpha \mathcal{T} = \{\alpha t_{ij}\}, \qquad \mathcal{T} = \{t_{ij}\} \in \mathbb{R}^{3\times 3},\ \alpha \in \mathbb{R}$

2. $\mathcal{T}^1 + \mathcal{T}^2 = \{t^1_{ij} + t^2_{ij}\}, \qquad \mathcal{T}^1, \mathcal{T}^2 \in \mathbb{R}^{3\times 3}, \quad \mathbf{a} \in \mathbb{R}^3$

3. $$\mathbf{a} \cdot \mathcal{T} = [a_1, a_2, a_3] \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} = \sum_{i=1}^{3} a_i \underbrace{[t_{i1}, t_{i2}, t_{i3}]}_{i\text{-th row}}$$

4. $$\mathcal{T} \cdot \mathbf{a} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \sum_{j=1}^{3} \begin{bmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{bmatrix} a_j \quad (j\text{-th column})$$

5. $$\mathcal{T}^1 \cdot \mathcal{T}^2 = \begin{bmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ t^1_{21} & t^1_{22} & t^1_{23} \\ t^1_{31} & t^1_{32} & t^1_{33} \end{bmatrix} \begin{bmatrix} t^2_{11} & t^2_{12} & t^2_{13} \\ t^2_{21} & t^2_{22} & t^2_{23} \\ t^2_{31} & t^2_{32} & t^2_{33} \end{bmatrix} = \left\{ \sum_{k=1}^{3} t^1_{ik} t^2_{kj} \right\}$$

6. $$\mathcal{T}^1 : \mathcal{T}^2 = tr(\mathcal{T}^1 \cdot (\mathcal{T}^2)^T) = \sum_{i=1}^{3} \sum_{k=1}^{3} t^1_{ik} t^2_{ik}$$

Divergence theorem of Gauß

Let $\Omega \in \mathbb{R}^3$ and $\mathbf{n}$ be the outward unit normal to the boundary $\Gamma = \bar{\Omega} \backslash \Omega$.

Then $$\int_\Omega \nabla \cdot \mathbf{f} \, d\mathbf{x} = \int_\Gamma \mathbf{f} \cdot \mathbf{n} \, ds$$ for any differentiable function $f(\mathbf{x})$

Example. A sphere: $\Omega = \{\mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x}|| < 1\}, \quad \Gamma = \{\mathbf{x} \in \mathbb{R}^3 : ||\mathbf{x}|| = 1\}$

where $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean norm of $\mathbf{x}$

Consider $\mathbf{f}(\mathbf{x}) = \mathbf{x}$ so that $\nabla \cdot \mathbf{f} \equiv 3$ in Ω and $\mathbf{n} = \frac{\mathbf{x}}{||\mathbf{x}||}$ on Γ

volume integral: $$\int_\Omega \nabla \cdot \mathbf{f} \, d\mathbf{x} = 3 \int_\Omega d\mathbf{x} = 3|\Omega| = 3 \left[\frac{4}{3} \pi 1^3 \right] = 4\pi$$

surface integral: $$\int_\Gamma \mathbf{f} \cdot \mathbf{n} \, ds = \int_\Gamma \frac{\mathbf{x} \cdot \mathbf{x}}{||\mathbf{x}||} \, ds = \int_\Gamma ||\mathbf{x}|| \, ds = \int_\Gamma ds = 4\pi$$

Governing equations of fluid dynamics

Physical principles

1. Mass is conserved
2. Newton's second law
3. Energy is conserved

Mathematical equations

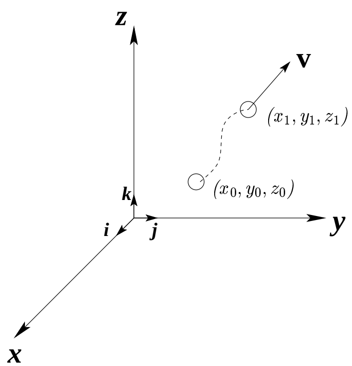
- continuity equation
- momentum equations
- energy equation

It is important to understand the meaning and significance of each equation in order to develop a good numerical method and properly interpret the results

Description of fluid motion

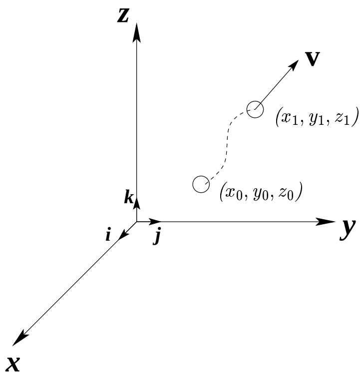
Eulerian monitor the flow characteristics in a fixed control volume

Lagrangian track individual fluid particles as they move through the flow field



Description of fluid motion

Trajectory of a fluid particle



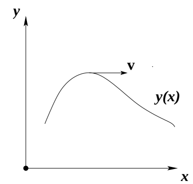
$$\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$$

$$x = x(x_0, y_0, z_0, t)$$
$$y = y(x_0, y_0, z_0, t)$$
$$z = z(x_0, y_0, z_0, t)$$

$$\frac{dx}{dt} = v_x(x, y, z, t), \quad x|_{t_0} = x_0$$
$$\frac{dy}{dt} = v_y(x, y, z, t), \quad y|_{t_0} = y_0$$
$$\frac{dz}{dt} = v_z(x, y, z, t), \quad z|_{t_0} = z_0$$

Definition. *A streamline is a curve which is tangent to the velocity vector* $\mathbf{v} = (v_x, v_y, v_z)$ *at every point. It is given by the relation*

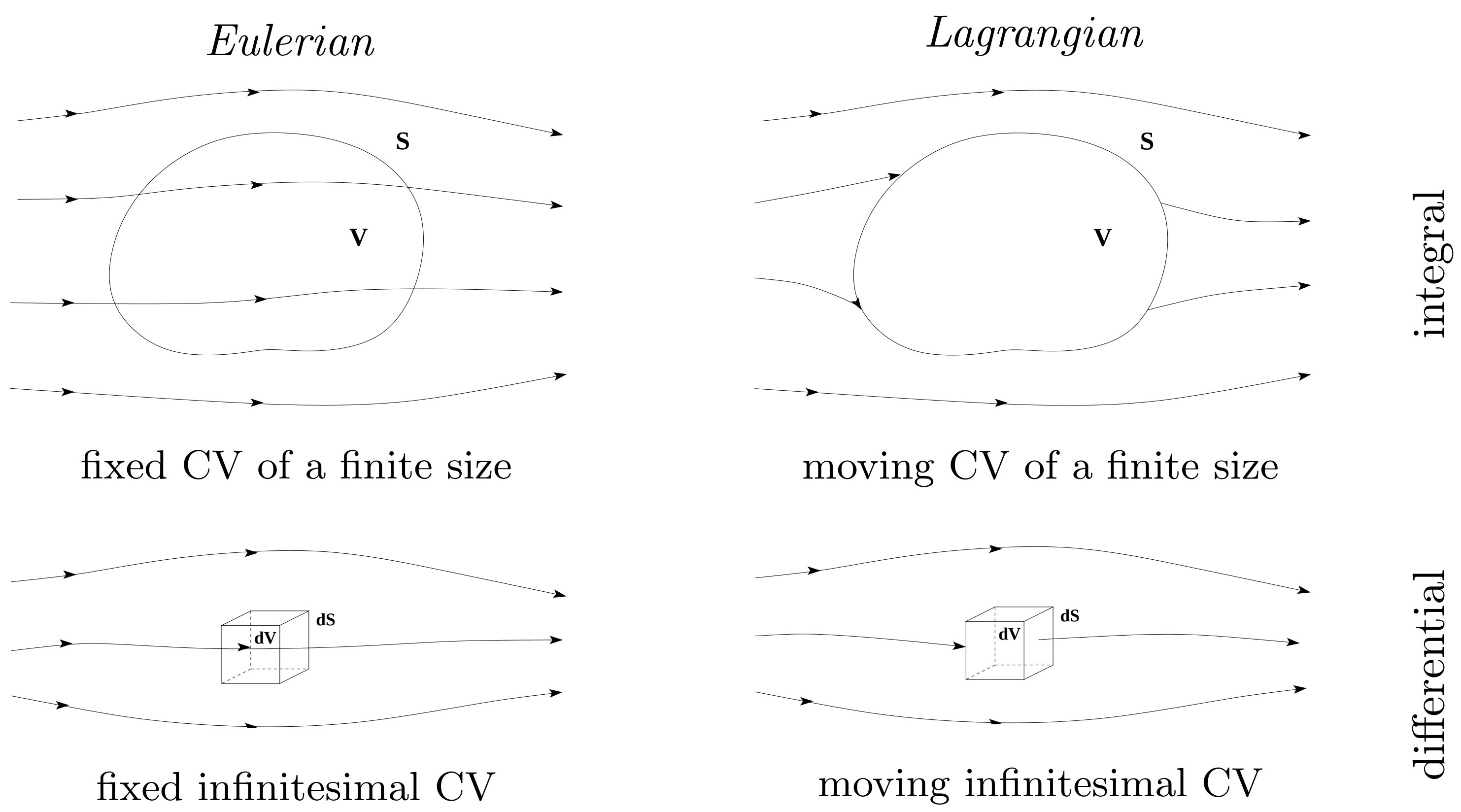
$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}$$



$$\frac{dy}{dx} = \frac{v_y}{v_x}$$

Streamlines can be visualized by injecting tracer particles into the flow field.

Flow models and reference frames



Good news: all flow models lead to the same equations

Eulerian vs. Lagrangian viewpoint

Definition. *Substantial time derivative* $\frac{d}{dt}$ *is the rate of change for a moving fluid particle. Local time derivative* $\frac{\partial}{\partial t}$ *is the rate of change at a fixed point.*

Let $u = u(\mathbf{x}, t)$, where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$. The chain rule yields

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} = \frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u$$

substantial derivative = local derivative + convective derivative

Reynolds transport theorem

$$\frac{d}{dt}\int_{V_t} u(\mathbf{x}, t)\, dV = \int_{V \equiv V_t} \frac{\partial u(\mathbf{x}, t)}{\partial t}\, dV + \int_{S \equiv S_t} u(\mathbf{x}, t)\mathbf{v} \cdot \mathbf{n}\, dS$$

rate of change in a moving volume = *rate of change in a fixed volume* + *convective transfer through the surface*

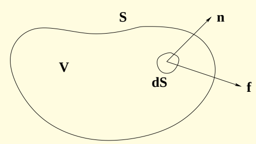
Derivation of the governing equations

Modeling philosophy

1. Choose a physical principle
   - conservation of mass
   - conservation of momentum
   - conservation of energy
2. Apply it to a suitable flow model
   - Eulerian/Lagrangian approach
   - for a finite/infinitesimal CV
3. Extract integral relations or PDEs which embody the physical principle

Generic conservation law

$$\frac{\partial}{\partial t}\int_V u\,dV + \int_S \mathbf{f}\cdot\mathbf{n}\,dS = \int_V q\,dV$$



$$\mathbf{f} = \mathbf{v}u - d\nabla u$$

flux function

Divergence theorem yields

$$\int_V \frac{\partial u}{\partial t}\,dV + \int_V \nabla\cdot\mathbf{f}\,dV = \int_V q\,dV$$

Partial differential equation

$$\frac{\partial u}{\partial t} + \nabla\cdot\mathbf{f} = q \qquad \text{in } V$$

Derivation of the continuity equation

Physical principle: conservation of mass

$$\frac{dm}{dt} = \frac{d}{dt}\int_{V_t} \rho \, dV = \int_{V\equiv V_t} \frac{\partial \rho}{\partial t} \, dV + \int_{S\equiv S_t} \rho \mathbf{v} \cdot \mathbf{n} \, dS = 0$$

accumulation of mass inside CV = net influx through the surface

Divergence theorem yields

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0 \quad \Rightarrow$$

Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

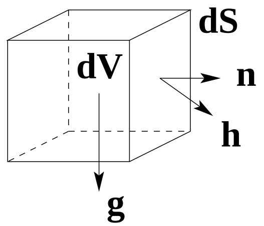
Lagrangian representation

$$\nabla \cdot (\rho \mathbf{v}) = \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} \quad \Rightarrow \quad \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0$$

Incompressible flows: $\frac{d\rho}{dt} = \nabla \cdot \mathbf{v} = 0$ (constant density)

Conservation of momentum

Physical principle: $\mathbf{f} = m\mathbf{a}$ (Newton's second law)



total force	$\mathbf{f} = \rho\mathbf{g}\,dV + \mathbf{h}\,dS,$	where $\mathbf{h} = \sigma \cdot \mathbf{n}$
body forces	$\mathbf{g}$	gravitational, electromagnetic,…
surface forces	$\mathbf{h}$	pressure + viscous stress

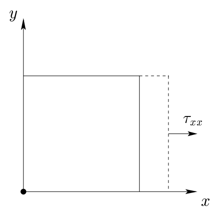
Stress tensor $\boxed{\sigma = -p\mathcal{I} + \tau}$ momentum flux

For a *newtonian fluid* viscous stress is proportional to velocity gradients:

$$\tau = (\lambda \nabla \cdot \mathbf{v})\mathcal{I} + 2\mu \mathcal{D}(\mathbf{v}), \quad \text{where} \quad \mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T), \quad \lambda \approx -\frac{2}{3}\mu$$

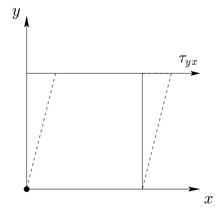
Normal stress: *stretching*

$$\tau_{xx} = \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial v_x}{\partial x}$$
$$\tau_{yy} = \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial v_y}{\partial y}$$
$$\tau_{zz} = \lambda \nabla \cdot \mathbf{v} + 2\mu \frac{\partial v_z}{\partial z}$$



Shear stress: *deformation*

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right)$$
$$\tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)$$
$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right)$$



Derivation of the momentum equations

Newton's law for a moving volume

$$\begin{aligned}\frac{d}{dt}\int_{V_t}\rho\mathbf{v}\,dV &= \int_{V\equiv V_t}\frac{\partial(\rho\mathbf{v})}{\partial t}\,dV+\int_{S\equiv S_t}(\rho\mathbf{v}\otimes\mathbf{v})\cdot\mathbf{n}\,dS\\ &= \int_{V\equiv V_t}\rho\mathbf{g}\,dV+\int_{S\equiv S_t}\sigma\cdot\mathbf{n}\,dS\end{aligned}$$

Transformation of surface integrals

$$\int_V\left[\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})\right]dV=\int_V\left[\nabla\cdot\sigma+\rho\mathbf{g}\right]\,dV,\qquad \sigma=-p\mathcal{I}+\tau$$

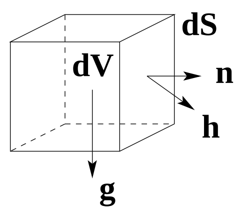
Momentum equations

$$\boxed{\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})=-\nabla p+\nabla\cdot\tau+\rho\mathbf{g}}$$

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})=\rho\underbrace{\left[\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right]}_{\text{substantial derivative}}+\mathbf{v}\underbrace{\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})\right]}_{\text{continuity equation}}=\rho\frac{d\mathbf{v}}{dt}$$

Conservation of energy

Physical principle: $\delta e = s + w$ (first law of thermodynamics)



δe	accumulation of internal energy
s	heat transmitted to the fluid particle
w	rate of work done by external forces

Heating: $s = \rho q \, dV - f_q \, dS$

q	internal heat sources
f_q	diffusive heat transfer
T	absolute temperature
κ	thermal conductivity

Fourier's law of heat conduction

$$\boxed{f_q = -\kappa \nabla T}$$

the heat flux is proportional to the local temperature gradient

Work done per unit time = total force × velocity

$$w = \mathbf{f} \cdot \mathbf{v} = \rho \mathbf{g} \cdot \mathbf{v} \, dV + \mathbf{v} \cdot (\sigma \cdot \mathbf{n}) \, dS, \qquad \sigma = -p\mathcal{I} + \tau$$

Derivation of the energy equation

Total energy per unit mass: $E = e + \frac{|\mathbf{v}|^2}{2}$

e	specific internal energy due to random molecular motion
$\frac{\vert\mathbf{v}\vert^2}{2}$	specific kinetic energy due to translational motion

Integral conservation law for a moving volume

$$
\begin{aligned}
\frac{d}{dt}\int_{V_t} \rho E \, dV &= \int_{V\equiv V_t} \frac{\partial(\rho E)}{\partial t} \, dV + \int_{S\equiv S_t} \rho E \, \mathbf{v}\cdot\mathbf{n} \, dS && \text{accumulation} \\
&= \int_{V\equiv V_t} \rho q \, dV + \int_{S\equiv S_t} \kappa \nabla T \cdot \mathbf{n} \, dS && \text{heating} \\
&+ \int_{V\equiv V_t} \rho \mathbf{g}\cdot\mathbf{v} \, dV + \int_{S\equiv S_t} \mathbf{v}\cdot(\sigma\cdot\mathbf{n}) \, dS && \text{work done}
\end{aligned}
$$

Transformation of surface integrals

$$\int_V \left[\frac{\partial(\rho E)}{\partial t} + \nabla\cdot(\rho E \mathbf{v})\right] dV = \int_V \left[\nabla\cdot(\kappa\nabla T) + \rho q + \nabla\cdot(\sigma\cdot\mathbf{v}) + \rho \mathbf{g}\cdot\mathbf{v}\right] \, dV,$$

where $\nabla\cdot(\sigma\cdot\mathbf{v}) = -\nabla\cdot(p\mathbf{v}) + \nabla\cdot(\tau\cdot\mathbf{v}) = -\nabla\cdot(p\mathbf{v}) + \mathbf{v}\cdot(\nabla\cdot\tau) + \nabla\mathbf{v} : \tau$

Different forms of the energy equation

Total energy equation

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - \nabla \cdot (p\mathbf{v}) + \mathbf{v} \cdot (\nabla \cdot \tau) + \nabla \mathbf{v} : \tau + \rho \mathbf{g} \cdot \mathbf{v}$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \rho \underbrace{\left[\frac{\partial E}{\partial t} + \mathbf{v} \cdot \nabla E\right]}_{\text{substantial derivative}} + E \underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right]}_{\text{continuity equation}} = \rho \frac{dE}{dt}$$

Momentum equations $\quad \rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g} \quad$ (Lagrangian form)

$$\rho \frac{dE}{dt} = \rho \frac{de}{dt} + \mathbf{v} \cdot \rho \frac{d\mathbf{v}}{dt} = \frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) + \mathbf{v} \cdot [-\nabla p + \nabla \cdot \tau + \rho \mathbf{g}]$$

Internal energy equation

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - p \nabla \cdot \mathbf{v} + \nabla \mathbf{v} : \tau$$

Summary of the governing equations

1. Continuity equation / conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

2. Momentum equations / Newton's second law

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = -\nabla p + \nabla \cdot \tau + \rho \mathbf{g}$$

3. Energy equation / first law of thermodynamics

$$\frac{\partial (\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - \nabla \cdot (p \mathbf{v}) + \mathbf{v} \cdot (\nabla \cdot \tau) + \nabla \mathbf{v} : \tau + \rho \mathbf{g} \cdot \mathbf{v}$$

$$E = e + \frac{|\mathbf{v}|^2}{2}, \qquad \frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{v}) = \nabla \cdot (\kappa \nabla T) + \rho q - p \nabla \cdot \mathbf{v} + \nabla \mathbf{v} : \tau$$

This PDE system is referred to as the *compressible Navier-Stokes equations*

Conservation form of the governing equations

Generic conservation law for a scalar quantity

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f} = q, \qquad \text{where} \quad \mathbf{f} = \mathbf{f}(u, \mathbf{x}, t) \quad \text{is the flux function}$$

Conservative variables, fluxes and sources

$$U = \begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v} \otimes \mathbf{v} + p\mathcal{I} - \tau \\ (\rho E + p)\mathbf{v} - \kappa\nabla T - \tau \cdot \mathbf{v} \end{bmatrix}, \quad Q = \begin{bmatrix} 0 \\ \rho\mathbf{g} \\ \rho(q + \mathbf{g} \cdot \mathbf{v}) \end{bmatrix}$$

Navier-Stokes equations in divergence form

$$\boxed{\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F} = Q} \qquad U \in \mathbb{R}^5, \quad \mathbf{F} \in \mathbb{R}^{3\times 5}, \quad Q \in \mathbb{R}^5$$

- representing all equations in the same generic form simplifies the programming
- it suffices to develop discretization techniques for the generic conservation law

Constitutive relations

Variables: ρ, $\mathbf{v}$, e, p, τ, T *Equations:* continuity, momentum, energy

The number of unknowns exceeds the number of equations.

1. Newtonian stress tensor

$$\tau = (\lambda \nabla \cdot \mathbf{v})\mathcal{I} + 2\mu\mathcal{D}(\mathbf{v}), \qquad \mathcal{D}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^T), \quad \lambda \approx -\frac{2}{3}\mu$$

2. Thermodynamic relations, e.g.

$p = \rho RT$	ideal gas law	R	specific gas constant
$e = c_v T$	caloric equation of state	c_v	specific heat at constant volume

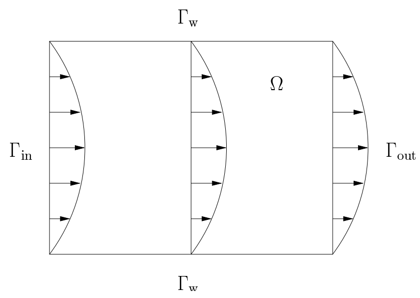
Now the system is closed: it contains five PDEs for five independent variables ρ, $\mathbf{v}$, e and algebraic formulae for the computation of p, τ and T. It remains to specify appropriate initial and boundary conditions.

Initial and boundary conditions

Initial conditions $\rho|_{t=0} = \rho_0(\mathbf{x}), \quad \mathbf{v}|_{t=0} = \mathbf{v}_0(\mathbf{x}), \quad e|_{t=0} = e_0(\mathbf{x}) \qquad \text{in } \Omega$

Boundary conditions

Let $\Gamma = \Gamma_{\rm in} \cup \Gamma_{\rm w} \cup \Gamma_{\rm out}$



Inlet $\quad \Gamma_{\rm in} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} < 0\}$

$$\rho = \rho_{in}, \quad \mathbf{v} = \mathbf{v}_{in}, \quad e = e_{in}$$

prescribed density, energy and velocity

Solid wall $\quad \Gamma_{\rm w} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} = 0\}$

$\mathbf{v} = 0$ no-slip condition

$T = T_w$ given temperature or

$\left(\frac{\partial T}{\partial n}\right) = -\frac{f_q}{\kappa}$ prescribed heat flux

Outlet $\quad \Gamma_{\rm out} = \{\mathbf{x} \in \Gamma : \mathbf{v} \cdot \mathbf{n} > 0\}$

$\mathbf{v} \cdot \mathbf{n} = v_n$ or $-p + \mathbf{n} \cdot \tau \cdot \mathbf{n} = 0$

$\mathbf{v} \cdot \mathbf{s} = v_s$ or $\mathbf{s} \cdot \tau \cdot \mathbf{n} = 0$

prescribed velocity vanishing stress

The problem is well-posed if the solution exists, is unique and depends continuously on IC and BC. Insufficient or incorrect IC/BC may lead to wrong results (if any).