Holonomy

Lorenz J. Schwachhöfer*

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1 Introduction

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined as the subgroup $Hol_p(M) \subset \mathrm{Aut}(T_pM)$ consisting of all automorphisms of the tangent space T_pM at $p \in M$ induced by parallel translations along p-based loops.

The notion of *holonomy* first arose in classical mechanics at the end of the 19th century. It was Heinrich Hertz who used the terms *'holonomic' and 'non-holonomic' constraints* in his magnum opus *Die Prinzipien der Mechanik, in neuen Zusammenhängen dargestellt* ("The principles of mechanics presented in a new form") which appeared one year after his death in 1895.

The notion of holonomy in the mathematical context seems to have appeared for the first time in the work of E.Cartan [Car1, Car2, Car4]. He considered the Levi-Civita connection of a Riemannian manifold M, so that the holonomy group is contained in the orthogonal group. He showed that in this case, the connected component of a Riemannian holonomy group is always a Lie subgroup, and that this group is always connected if M is simply connected. Moreover, he observed that $Hol_p(M)$ and $Hol_q(M)$ are conjugate via parallel translation along any path from p to q, so the holonomy group $Hol(M) \subset GL(n, \mathbb{R})$ is well defined up to conjugation. For further comments on the history of holonomy, see [Br5].

2 Basic results on holonomy

2.1 Holonomy and coverings

Let (M, ∇) be a manifold with an affine connection which we shall always assume to be torsion free. Then the universal cover of M carries a unique connection such that the covering map $\pi : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ is locally affine. We denote by $Hol_p^0(M)$ the *reduced holonomy group* which is defined as the subgroup of holonomy transformations given by parallel translation along *contractible* loops.

*Fachbereich Mathematik, Universität Dortmund, Vogelpothsweg 87, 44221 Dortmund, Germany; e-mail: lschwach@math.uni-dortmund.de

Fix $\tilde{p} \in \tilde{M}$ with $\pi(\tilde{p}) = p$. Let $\tilde{c}$ be a $\tilde{p}$-based loop in $\tilde{M}$ and denote its holonomy transformation by $P_{\tilde{c}} \in Hol_{\tilde{p}}(\tilde{M})$. Then $c = \pi \circ \tilde{c}$ is a *contractible* p-based loop in M, and since π is locally affine, it follows that parallel vector fields along $\tilde{c}$ and c are π-related. Therefore, denoting the holonomy transformation of c by $P_c \in Hol_p^0(M)$, we have the relation

$$d\pi_{\tilde{p}} \circ P_{\tilde{c}} = P_c \circ d\pi_{\tilde{p}}.$$

But the contractible p-based loops in M are those which admit a $\tilde{p}$-based lift $\tilde{c}$ with $c = \pi \circ \tilde{c}$. Thus, the above relation shows:

Proposition 2.1 *Let $\pi : (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ be the universal cover of (M, ∇), and let $p \in M$, $\tilde{p} \in \tilde{M}$ be such that $p = \pi(\tilde{p})$. Then*

$$Hol_p^0(M) = (d\pi_{\tilde{p}})\ Hol_{\tilde{p}}(\tilde{M})\ (d\pi_{\tilde{p}})^{-1}.$$

By the very definition, we have an epimorphism $\pi_1(M, p) \to Hol_p(M)/Hol_p^0(M)$ which shows that $Hol_p(M)/Hol_p^0(M)$ is countable. Therefore, we conclude

Corollary 2.2 *Let (M, ∇) be a manifold with an affine connection and let $p \in M$. Then $Hol_p^0(M)$ is the identity component of $Hol_p(M)$.*

It may well happen that $Hol_p^0(M) \subset \mathrm{Aut}(T_pM)$ is not closed. This happens e.g. for Levi-Civita connections of certain pseudo-Riemannian metrics ([Wu2], [BI1]). But even in the Riemannian case, where $Hol_p^0(M)$ is always compact, the number of components of $Hol_p(M)$ may be infinite. This occurs, for example, if M is the quotient of $\mathbb{R}^3$ with the standard connection by the group generated by an affine map whose linear part is rotation around an axis with irrational rotation angle. However, this quotient is non-compact.

It remained an open question for a long time whether the holonomy of a *compact* Riemannian manifold is necessarily compact (where the connection considered here is of course the Levi-Civita connection).

As it turns out, the answer is negative. In fact, B.Wilking ([Wi]) constructed examples of compact Riemannian manifolds with noncompact holonomy. He also showed that any such manifold must be finitely covered by a torus bundle over a compact manifold, where the dimension of the torus fiber is at least four.

2.2 The de Rham Splitting theorem

Let us consider the representation of $Hol_p(M) \subset Aut(T_pM)$ on the tangent space T_pM. We call the holonomy group *decomposable* if there is a $Hol_p(M)$-invariant decomposition

$$T_pM = V_1 \oplus \ldots \oplus V_r \tag{1}$$

with $r \geq 2$ and $V_j \neq 0$ for all j; if there is no such splitting, we call the representation *indecomposable.*

Evidently, (in-)decomposability of the holonomy group is independent of the choice of $p \in M$ since all holonomy groups are conjugate. Parallel translation of the subspaces V_j yields an integrable totally geodesic distribution. The splitting of the holonomy group does not imply the splitting of the connection in general (see [KN] p.290 for an example); however, for metric connections, this is the case. Namely, we have the following

Theorem 2.3 (de Rham Splitting Theorem [dR], [Wu1]) *Let (M, g) be a (pseudo-)Riemannian manifold, and suppose that the holonomy group of its Levi-Civita connection is decomposable. Then locally, (M, g) is isometric to a product metric $(\mathbb{R}^{k_1}, g_1) \times \ldots \times (\mathbb{R}^{k_r}, g_r)$ with $k_j = \dim V_j$, and $Hol_p^0(M) = H_1 \times \ldots \times H_r$ with $H_j \subset O(V_j, g_j)$.*

Moreover, if M is simply connected and ∇ is geodesically complete, then there is a splitting $(M, g) = (M_1, g_1) \times \ldots \times (M_r, g_r)$, where the holonomy of (M_j, g_j) is H_j.

By virtue of this theorem, it is reasonable to regard (local) product connections as *"trivial"* compositions and hence to assume that the holonomy is indecomposable.

2.3 Holonomy and Symmetric Spaces

A *locally affine symmetric space* is a manifold with an affine connection (M, ∇) which admits for each $p \in M$ an affine involution $\imath_p$ defined in a neighborhood of p which has p as an isolated fixed point. If these involutions can be extended to all of M, then (M, ∇) is called an *affine symmetric space.* For these, one can show the following

Theorem 2.4 *[Car4] Let (M, ∇) be an affine symmetric space without Euclidean factor. Then for each $p \in M$, the isotropy group of p, i.e., the group of affine maps of M which fix p, and the holonomy group $Hol_p(M)$ have the same identity component.*

We should remark here that Cartan proved this result merely in the case of *Riemannian* symmetric spaces. However, his proof goes through immediately for affine symmetric spaces as well.

In fact, Cartan even succeeded in providing a classification of *simply connected irreducible Riemannian symmetric spaces* ([Car3]), i.e., those symmetric spaces whose holonomy group is contained in the orthogonal group. In this case, irreducibility and indecomposability are equivalent.

Later, M.Berger ([Ber2]) gave a classification of simply connected affine symmetric spaces whose holonomy group is semi-simple.

In general, the classification of affine symmetric spaces is far from complete. In the case of *pseudo-Riemannian symmetric spaces*, this classification was established for metrics of signature $(1, n)$ by M.Cahen and N.Wallach ([CW]), and recently by I.Kath and M.Olbrich for general signatures ([KO]).

2.4 The Ambrose-Singer Holonomy Theorem

Let $F : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to M$ be differentiable with $p := F(0, 0)$, and let $v := \partial F/\partial t|_{(0,0)}$ and $w := \partial F/\partial s|_{(0,0)}$. Let $P(s, t) : T_pM \to T_pM$ be the parallel translation along the path $F \circ c_{(s,t)}$ where $c_{(s,t)}$ is the rectangle in the (s, t)-plane with edges parallel to the coordinate axes and with a diagonal from the origin to (s, t). Then the curvature endomorphism at p can be described as

$$R_p(v, w) = -\frac{\partial^2 P}{\partial s \partial t}|_{(0,0)}.$$

In particular, it follows that the curvature endomorphisms $R(v, w)$ are contained in the Lie algebra $\mathfrak{hol}_p$ of Hol_p. Likewise, when conjugating the paths $F \circ c_{(s,t)}$ from above with a fixed path α joining a point $q \in M$ with p, we also conclude that

$$(P_\alpha \cdot R_q)(v, w) := P_\alpha \circ R_q(P_\alpha^{-1} v, P_\alpha^{-1} w) \circ P_\alpha^{-1} \in \mathfrak{hol}_p \quad \text{for all } v, w \in T_pM, \tag{2}$$

where $P_\alpha : T_qM \to T_pM$ denotes parallel translation along α.

Theorem 2.5 (Ambrose-Singer holonomy theorem [AS]) *The Lie algebra* $\mathfrak{hol}_p$ *of the holonomy group* Hol_p *is generated by the curvature endomorphisms* $(P_\alpha \cdot R_q)(v, w)$ *from (2) with* $v, w \in T_pM$ *and* α *any path ending in* p*.*

This description of the holonomy algebra proves to be an important tool for the classification of holonomy groups. Namely, the curvature of a torsion free connection always satisfies the *first and second Bianchi identity*

$$R(x,y)z + R(y,z)x + R(z,x)y = 0 \quad \text{and} \quad (\nabla_x R)(y,z) + (\nabla_y R)(z,x) + (\nabla_z R)(x,y) = 0. \quad (3)$$

Let V be a vector space and $\mathfrak{h} \subset \mathrm{End}(V)$ a Lie subalgebra. We define the *space of formal curvature maps*

$$K(\mathfrak{h}) := \left\{ R \in \Lambda^2 V^* \otimes \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0 \text{ for all } x, y, z \in V \right\},$$

and the *space of formal curvature derivatives*

$$K^1(\mathfrak{h}) := \{ \phi \in V^* \otimes K(\mathfrak{h}) \mid \phi(x)(y,z) + \phi(y)(z,x) + \phi(z)(x,y) = 0 \text{ for all } x, y, z \in V \}.$$

We also let $\underline{\mathfrak{h}} := \{ R(x,y) \mid R \in K(\mathfrak{h}),\ x, y \in V \} \subset \mathfrak{h}$. Evidently, $\underline{\mathfrak{h}} \lhd \mathfrak{h}$.

From (3) it follows that $P_\alpha R_p \in K(\mathfrak{hol}_p)$ for all path α with end point p; hence the Ambrose-Singer Holonomy Theorem implies that $\underline{\mathfrak{hol}}_p = \mathfrak{hol}_p$. Likewise, (3) implies that the map $x \mapsto (\nabla_x R)_p$ lies in $K^1(\mathfrak{hol}_p)$. Thus, if $K^1(\mathfrak{hol}_p) = 0$ then $\nabla R \equiv 0$, i.e., the connection is locally symmetric. These facts motivate the following definition.

Definition 2.6 *A Lie subalgebra* $\mathfrak{h} \subset End(V)$ *is called a* Berger algebra *if* $\underline{\mathfrak{h}} = \mathfrak{h}$*. A Berger algebra* $\mathfrak{h} \subset End(V)$ *is called* symmetric *if* $K^1(\mathfrak{h}) = 0$ *and* non-symmetric *otherwise.*

A Lie subgroup $H \subset Aut(V)$ *is called a* (symmetric respectively non-symmetric) Berger group *if its Lie algebra* $\mathfrak{h} \subset End(V)$ *is a (symmetric respectively non-symmetric) Berger algebra.*

In the literature, the two criteria for a non-symmetric Berger algebra are usually referred to as *Berger's first and second criterion.* Our discussion from above now yields the following.

Proposition 2.7 *[Ber1] Let* $H \subset Aut(V)$ *be a Lie subgroup which occurs as the holonomy group of a torsion free affine connection on some manifold* M*. Then* H *must be a Berger group. If the connection is not locally symmetric, then* H *must be a non-symmetric Berger group.*

3 Classification results

3.1 Holonomy of Riemannian manifolds

Let (M, g) be a Riemannian manifold. In this case, the holonomy group is contained in the orthogonal group $O(n)$ or, equivalently, is compact. Moreover, indecomposability is equivalent to irreducibility of the group. Thus, the de Rham Theorem 2.3 and Proposition 2.7 imply that we need to classify all irreducible non-symmetric Berger groups which are contained in $O(n)$. This has been achieved by Berger ([Ber1]) where the below classification table was established.

Another important question is the determination of *parallel spinors*, i.e., parallel sections of the spinor bundle of a spin manifold M. If we assume that the holonomy of M is connected (e.g. if

M is simply connected, cf. Proposition 2.1), then the space of parallel spinors corresponds to the subspace of the spinor representation on which the holonomy algebra $\mathfrak{hol}(M) \subset \mathfrak{so}(n) \cong \mathfrak{spin}(n)$ acts trivially. These spaces have been described by M.Wang ([Wa]) for all entries in Berger's list, and we add the dimension of the space of parallel spinors for each of the holonomies in question.

Table 1:
CLASSIFICATION OF CONNECTED IRREDUCIBLE NON-SYMMETRIC HOLONOMIES CONTAINED IN SO(n)

n	H	associated geometry	dim. of space of parallel spinors
$n \geq 2$	SO(n)	generic Riemannian manifold	0
$2m \geq 4$	U(m)	generic Kähler manifold	0
$2m \geq 4$	SU(m)	special Kähler manifold	2
$4m \geq 8$	Sp(m) · Sp(1)	quaternionic Kähler manifold	0
$4m \geq 8$	Sp(m)	hyper-Kähler manifold	$m+1$
7	G_2	exceptional holonomy	1
8	Spin(7)	exceptional holonomy	1

It was noted immediately that this list is contained in the list of subgroups of the orthogonal group which act transitively on the unit sphere. This fact was later proven directly by J.Simons ([Si]) in an algebraic way. Recently, C.Olmos gave a beautiful simple argument showing this transitivity using only submanifold theory ([O]).

As it turns out, *all* of the groups in Table 1 do occur as holonomy of Riemannian connections.

1. $SO(n)$ is the reduced holonomy of a "generic" Riemannian manifold.

2. If $Hol \subset U(m)$, then the metric g is called *Kähler*. Kähler metrics form a natural class of complex manifolds, and the "generic" Kähler manifold has holonomy equal to $U(m)$.

3. If $Hol \subset SU(m)$ then the metric is called a *Calabi-Yau metric*. Since $SU(m) \subset U(m)$, each Calabi-Yau metric is necessarily Kähler. In fact, a Kähler metric with connected holonomy group is Calabi-Yau iff its Ricci curvature vanishes.

   The first examples of *complete* Calabi-Yau metrics were given by E.Calabi ([Cal]). Later, S.T.Yau's solution to the Calabi conjecture ([Y]) showed that a compact Kähler manifold with trivial canonical line bundle or, equivalently, with vanishing first Chern class admits a unique Calabi-Yau metric whose Kähler form represents the same cohomology class as the Kähler form of the original Kähler metric. For explicit examples, we refer to the books by A.Besse ([Bes]), S.Salamon ([Sa1]) and D.Joyce ([J3]).

4. Metrics with $Hol = Sp(m) \cdot Sp(1)$ are called *quaternionic Kähler*, although this terminology is somewhat misleading: quaternionic Kähler manifolds are *not* Kähler, as $Sp(1) \cdot Sp(m)$ is not contained in $U(m)$. Quaternionic Kähler manifolds are always Einstein, but not Ricci flat.

   Homogeneous quaternionic-Kähler manifolds were classified by D.Aleksevskii and V.Cortés ([Al], [Co]). For more details on the theory of these manifolds, see [Sa1], [Sa2], [Gali], [GL]. It is worth pointing out that there are so far no known examples of closed quaternionic Kähler manifolds with positive scalar curvature other than quaternionic projective space.

5. Metrics with $Hol \subset Sp(m)$ are called *hyper-Kähler*. These metrics are Kähler as $Sp(m) \subset SU(2m)$. In fact, hyper-Kähler metrics admit a whole two-sphere worth of Kähler structures

which induce the quaternionic structure. First explicit examples were found by Calabi ([Cal]). Compact examples were constructed using Yau's proof of the Calabi conjecture, see [Bea] for details.

6. The holonomy groups G_2 and $Spin(7)$ are called *exceptional holonomies* as they only occur in dimension 7 and 8, respectively. The existence of metrics with exceptional holonomy was shown locally by R.Bryant ([Br1]). Complete examples were given by Bryant and Salamon ([BS]), and compact examples were given by Joyce ([J1], [J2]) . See also [J3] for a more detailed exposition.

3.2 Holonomy groups of Pseudo-Riemannian manifolds

3.2.1 Irreducible holonomy groups

In [Ber1], Berger also classified all connected irreducible Berger groups which are subgroups of $SO(p,q)$ which are therefore candidates for the holonomy group of a pseudo-Riemanian manifold with a metric of signature (p,q). There were some minor omissions and errata on his list which were corrected by Bryant ([Br3]). As in the case of Riemannian holonomies, one can obtain the dimension of the space of parallel spinors in each of these cases which has been worked out by H.Baum and I.Kath ([BK]). Summarizing all these results, we obtain the following table:

Table 2: Classification of connected irreducible non-symmetric Holonomies contained in $\mathrm{SO}_0(r,s)$

$n = r + s$	H	associated geometry	Dim. of space of parallel spinors
$p+q \geq 2$	$\mathrm{SO}_0(p,q)$	generic	0
$2p \geq 4$	$\mathrm{SO}(p,\mathbb{C})$	generic complex	0
$2(p+q) \geq 4$	$\mathrm{U}(p,q)$	pseudo-Kähler	0
$2(p+q) \geq 4$	$\mathrm{SU}(p,q)$	special pseudo-Kähler	2
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)$	pseudo-hyper-Kähler	$p+q+1$
$4(p+q) \geq 8$	$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$	pseudo-quaternionic Kähler	0
$4p \geq 8$	$\mathrm{Sp}(p,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$		0
$8p \geq 16$	$\mathrm{Sp}(p,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$		0
7	G_2		1
7	G_2'		1
14	$\mathrm{G}_2^{\mathbb{C}}$		2
8	$\mathrm{Spin}(7)$		1
8	$\mathrm{Spin}(4,3)$		1
16	$\mathrm{Spin}(7,\mathbb{C})$		1

We should also point out that all of these Berger groups do occur as the holonomy of pseudo-Riemannian manifolds, as has been shown by Bryant ([Br3]).

3.2.2 Indecomposable Lorentzian holonomy groups

A *Lorentzian manifold* is a pseudo-Riemannian manifold of signature $(n,1)$. According to Berger's classification in Table 2, there is no proper irreducible subgroup of $SO_0(n,1)$ which can occur as the holonomy group of a Lorentzian manifold. In fact, there is no proper irreducible subgroup of

$SO_0(n,1)$ at all - this fact has been shown in a purely geometric manner by Di Scala and Olmos ([DO]).

Therefore, we may assume that the holonomy representation is indecomposable but not irreducible. This implies that there must be a one-dimensional Hol-invariant subspace $\mathbb{R}\xi \subset \mathbb{R}^{n,1}$ with $\xi \neq 0$ such that $< \xi, \xi >= 0$. Let $\Xi := (\mathbb{R}\xi)^\perp/(\mathbb{R}\xi)$ which is well defined as $\mathbb{R}\xi \subset (\mathbb{R}\xi)^\perp$. Since Hol leaves $\mathbb{R}\xi$ and hence its orthogonal complement invariant, it follows that there is an induced action of Hol on Ξ which preserves the induced positive definite inner product. (Note that this action may fail to be irreducible).

Based on work of L.Bérard-Bergery and A.Ikemakhen ([BI1]), the following classification result was established recently by T.Leistner.

Theorem 3.1 *[L] Let $H \subset SO_0(n,1)$ be a connected indecomposable, but not irreducible subgroup, and let $\hat{H} \subset SO(\Xi)$ be the image of the induced representation described above. Then the following are equivalent.*

1. *H is a Berger group,*

2. *$\hat{H}$ is a Berger group, i.e., is either the isotropy group of an irreducible Riemannian symmetric space, or is one of the entries of Table 1, or the direct product of such groups.*

Moreover, if a Lorentzian Spin-manifold admits a parallel spinor, then $H = \hat{H} \ltimes \mathbb{R}^n$, and the dimension of the space of parallel spinors coincides with this dimension for a Riemannian Spin-manifold with holonomy $\hat{H}$.

Furthermore, each indecomposable Berger group which is contained in $SO_0(n,1)$ does occur as the holonomy group of a Lorentzian manifold ([L], [Gala2]).

3.2.3 Indecomposable holonomy groups of Pseudo-Riemannian manifolds of signature (p,q) with $p,q \geq 2$

In the non-Lorentzian case, there are a number of results which we cannot describe here in more detail. We already mentioned the classification of symmetric spaces ([KO]); another striking result is the classification of *Kählerian* holonomies of complex signature $(1,n)$ (hence of real signature $(2,2n)$) by Galaev ([Gala3]). Further results on signature $(2,n)$ may be found e.g. in [I], [Gala1], and for split signature (n,n) e.g. in [BI2].

3.3 Special Symplectic Holonomy groups

A *symplectic connection* is a torsion free connection on a symplectic manifold (M,ω) such that ω is parallel or, equivalently, $Hol \subset Sp(n,\mathbb{R})$. We say that this connection has *special symplectic holonomy* if Hol acts absolutely irreducibly on the tangent space or, equivalently, if Hol acts irreducibly and does not preserve any complex structure.

First special symplectic holonomies were given by Bryant ([Br2]) and by Q.-S.Chi, S.Merkulov and the author ([CMS1], [CMS2]). Finally, these holonomies were classified by Merkulov and the author ([MS], see also [Sc]), and the possible holonomies are listed in Table 3.

Table 3: Special Symplectic Holonomy Groups

Group H	Representation space	Group H	Representation space
$\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4 \simeq \odot^3\mathbb{R}^2$	E_7^5	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{C})$	$\mathbb{C}^4 \simeq \odot^3\mathbb{C}^2$	E_7^7	$\mathbb{R}^{56}$
$\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$	$\mathbb{R}^{2(p+q)} \simeq \mathbb{R}^2\otimes\mathbb{R}^{p+q}$, $p+q\geq 3$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
$\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^{2n} \simeq \mathbb{C}^2\otimes\mathbb{C}^n$, $n\geq 3$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$
$\mathrm{Sp}(1)\cdot\mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$, $n\geq 2$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20} \simeq \Lambda^3\mathbb{R}^6$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32}$
$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{C}^{32}$
$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$
$\mathrm{SL}(6,\mathbb{C})$	$\mathbb{C}^{20} \simeq \Lambda^3\mathbb{C}^6$	$\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$

As it turns out, all of these holonomies share striking rigidity properties. These were explained conceptually via a symplectic reduction process of parabolic contact manifolds by M.Cahen and the author ([CS]).

3.4 Irreducible Holonomy groups

Holonomy groups which are irreducible, but of none of the above types, i.e., which preserve neither a (pseudo-)Riemannian nor a symplectic structure, have been investigated already by Berger ([Ber1]) and later by Bryant ([Br3], [Br4]). A complete classification was obtained by Merkulov and the author ([MS], see also [Sc]). We shall not deal much with the geometric content of these holonomies here, but rather conclude this survey with the classification table, referring the interested reader to the cited references.

Table 4: LIST OF NON-RIEMANNIAN, NON-SYMPLECTIC HOLONOMY GROUPS

NOTATIONS: $T_{\mathbb{F}}$ denotes any connected subgroup of $\mathbb{F}^*$.
$H_\lambda = \left\{ e^{t(\lambda+i)} \mid t \in \mathbb{R} \right\} \subset \mathbb{C}^*$ for $\lambda > 0$.
$\odot^p V$ denotes the symmetric tensors of V of degree p.

Group H	Representation space V	restrictions remarks
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\{A \in M_n(\mathbb{C}) \mid A = A^*\}$	$n \geq 3$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R}) \cdot \mathrm{SL}(m,\mathbb{R})$	$\mathbb{R}^n \otimes \mathbb{R}^m$	$n \geq m \geq 2, nm \neq 4$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H}) \cdot \mathrm{SL}(m,\mathbb{H})$	$\mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}^m$	$n \geq m \geq 1, nm \neq 1$
$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C}) \cdot \mathrm{SL}(m,\mathbb{C})$	$\mathbb{C}^n \otimes \mathbb{C}^m$	$n \geq m \geq 2, nm \neq 4$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geq 2$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 1$
$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\mathbb{C}^n$	$n \geq 2$
$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 2$
$\mathrm{U}(p,q)$	$\mathbb{C}^{p+q}$	$p+q \geq 2$
$\mathrm{SU}(p,q)$	$\mathbb{C}^{p+q}$	$p+q \geq 2,\ pq \neq 1$
$T_{\mathbb{C}} \cdot \mathrm{SU}(p,q)$	$\mathbb{C}^2$	$p+q = 2$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\Lambda^2\mathbb{R}^n$	$n \geq 5$
$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\Lambda^2\mathbb{C}^n$	$n \geq 5$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = A^*\}$	$n \geq 3$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\odot^2\mathbb{R}^n$	$n \geq 3$
$T_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n$	$n \geq 3$
$T_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\{A \in M_n(\mathbb{H}) \mid A = -A^*\}$	$n \geq 2$
$T_{\mathbb{R}} \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geq 3$
$T_{\mathbb{C}} \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^n$	$n \geq 3$
$T_{\mathbb{R}} \cdot \mathrm{Spin}(5,5)$	$\Delta^+_{(5,5)}$	
$T_{\mathbb{R}} \cdot \mathrm{Spin}(1,9)$	$\Delta^+_{(1,9)}$	
$T_{\mathbb{C}} \cdot \mathrm{Spin}(10,\mathbb{C})$	$(\Delta^+_{10})^{\mathbb{C}}$	
$T_{\mathbb{R}} \cdot \mathrm{E}_6^1$	$\mathbb{R}^{27}$	
$T_{\mathbb{R}} \cdot \mathrm{E}_6^4$	$\mathbb{R}^{27}$	
$T_{\mathbb{C}} \cdot \mathrm{E}_6^{\mathbb{C}}$	$\mathbb{C}^{27}$	
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$	$p+q \geq 3$
$\mathrm{Sp}(1) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n$	$n \geq 2$
$\mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geq 2$
$\mathbb{R}^* \cdot \mathrm{Sp}(2,\mathbb{R})$	$\mathbb{R}^4$	
$\mathrm{Sp}(p,q)$	$\mathbb{H}^{p+q}$	$p+q \geq 2$
$T_{\mathbb{R}} \cdot \mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2$	
$T_{\mathbb{C}} \cdot \mathrm{SL}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2$	

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