

# ON HOMOGENEOUS CONNECTIONS WITH EXOTIC HOLONOMY

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ABSTRACT.

In [Br], Bryant gave examples of torsion free connections on four-manifolds whose holonomy is *exotic*, i.e. is not contained on Berger's classical list of irreducible holonomy representations [Ber]. The holonomy in Bryant's examples is the irreducible four-dimensional representation of  $Sl(2, \mathbb{R})$  ( $Gl(2, \mathbb{R})$  resp.) and these connections are called  $H_3$ -connections ( $G_3$ -connections resp.).

In this paper we give a complete classification of homogeneous  $G_3$ -connections. The moduli space of these connections is four-dimensional, and the generic homogeneous  $G_3$ -connection is shown to be locally equivalent to a left-invariant connection on  $U(2)$ . Thus, we prove the existence of compact manifolds with  $G_3$ -connections. This contrasts a result in [Sch] which states that there are no compact manifolds with an  $H_3$ -connection.

## §0 Introduction.

Since its introduction by Èlie Cartan, the *holonomy* of a connection has played an important role in differential geometry. Most of the classical results are concerned with the holonomy of Levi Civita connections of Riemannian metrics. In 1955, Berger [Ber] classified the possible irreducible Riemannian holonomies and much work has been done since to study these holonomies and their applications. See [Bes] and [Sa] for a historical survey and also [J] for more recent results.

At the same time, Berger also partially classified the possible non-Riemannian holonomies of torsion free connections. However, his classification omits a finite number of possibilities, which are referred to as *exotic holonomies*. Until now, the complete list of exotic holonomies is still not known.

The incompleteness of Berger's list and therefore the existence of exotic holonomies was shown by Bryant [Br]. He investigated the irreducible representations of  $Sl(2, \mathbb{R})$ . For each  $d \geq 1$ , we can regard  $Sl(2, \mathbb{R})$  as a subgroup  $H_d \subseteq Gl(d+1, \mathbb{R})$  via the (unique)  $(d+1)$ -dimensional irreducible representation of  $Sl(2, \mathbb{R})$ . Moreover, if we let  $G_d \subseteq Gl(d+1, \mathbb{R})$  be the centralizer of  $H_d$ , then  $G_d$  may be regarded as a representation of  $Gl(2, \mathbb{R})$ . For  $d \geq 3$ , these representations do not occur on Berger's list of possible holonomies and are therefore candidates for exotic holonomies.

In his paper, Bryant showed that in the case  $d = 3$  torsion free connections with holonomies  $H_3$  and  $G_3$  do exist. We shall refer to them as  $H_3$ -connections ( $G_3$ -connections resp.).

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The “moduli space” of  $H_3$ -connections is the union of a one-dimensional space and six points. Moreover, there is exactly one homogeneous (non-flat)  $H_3$ -connection with a five-dimensional symmetry group. For other global properties of  $H_3$ -connections see [Sch]. On the other hand, the moduli space of  $G_3$ -connections is infinite dimensional; namely, the “generic”  $G_3$ -connection depends on four functions of three variables.

In this paper, we investigate certain “non-generic”  $G_3$ -connections. The generic condition in [Br] implies that the connection does not admit any non-zero infinitesimal symmetries. In a sense, we assume the exact opposite and consider the question if there exist any (*locally*) *homogeneous*  $G_3$ -connections besides the flat and the (unique) homogeneous  $H_3$ -connection. The answer to this question which had been raised in [Br, p.87] is affirmative. In fact, we shall arrive at a complete classification of homogeneous  $G_3$ -connections.

At this point, we shall state some consequences of this classification.

**Theorem 0.1.** *Any homogeneous  $G_3$ -connection whose holonomy is not contained in  $H_3$  is locally equivalent to a left-invariant connection on a four-dimensional Lie group.*

**Theorem 0.2.** *Up to isomorphism, there are twelve distinct possibilities for the Lie algebra of the symmetry group of a  $G_3$ -connection satisfying the hypothesis of Theorem 0.1. One of them is nilpotent, nine are solvable and the remaining two are  $\mathfrak{gl}(2, \mathbb{R})$  and  $\mathfrak{u}(2)$ .*

**Theorem 0.3.** *The moduli space of homogeneous  $G_3$ -connections is four-dimensional. More specifically, the moduli space has one four-dimensional component, seven one-dimensional components and fourteen points, including the flat connection and the homogeneous  $H_3$ -connection.*

**Theorem 0.4.** *The reduced holonomy of a homogeneous  $G_3$ -connection is either trivial, equal to  $H_3$  or equal to all of  $G_3$ .*

Here, the reduced holonomy stands for the identity component of the holonomy group. This result follows from a case-by-case investigation of all entries of the classification.

**Theorem 0.5.** *Generically, the symmetry group of a homogeneous  $G_3$ -connection has Lie algebra  $\mathfrak{u}(2)$ , i.e. the generic homogeneous  $G_3$ -connection is locally equivalent to a left-invariant connection on the (compact) Lie group  $U(2)$ .*

As a consequence, this yields the remarkable

**Corollary 0.6.** *There are  $G_3$ -connections on compact manifolds.*

Of course, there also exist compact  $G_3$ -manifolds with non-compact symmetry group  $G$ . Namely, if  $\Gamma \subseteq G$  is a cocompact lattice where the Lie algebra of  $G$  is as in Theorem 0.2, then there is an induced  $G_3$ -connection on the compact quotient  $M := \Gamma \backslash G$ . Note that, in general,  $M$  will not be globally homogeneous.

Corollary 0.6. contrasts a result which states that there are no  $H_3$ -connections on compact four-manifolds. [Sch, Th. 6.1]

In §1, we briefly recall the structure equations for a torsion free  $G_3$ -connection, following the notation of [Br].

In §2, the core of this paper, we first show that every connection other than the flat and the homogeneous  $H_3$ -connection has a symmetry group of dimension at most four. As a consequence, every homogeneous  $G_3$ -connection other than these two must be locally equivalent to a left-invariant connection on a four-dimensional Lie group. These connections are then shown to be in one-to-one correspondence with the orbit space of polynomials satisfying certain equations. Those polynomials can be completely classified.

Finally, in §3 we explicitly present the different polynomials that yield homogeneous  $G_3$ -connections. We also determine the Lie algebras of their symmetry groups which essentially determine, of course, the underlying manifolds.

The main part of the work presented here has been completed while the author was a visiting faculty member at Washington University in St. Louis, Mo, and he wishes to thank the Department of Mathematics there for its hospitality.

### §1 The structure equations.

We begin with a brief description of the irreducible  $Sl(2, \mathbb{R})$ -representations.

For  $d \in \mathbb{N}$ , let  $\mathcal{V}_d \subseteq \mathbb{R}[x, y]$  be the  $(d+1)$ -dimensional subspace of homogeneous polynomials of degree  $d$ . There is an  $Sl(2, \mathbb{R})$ -action ( $Gl(2, \mathbb{R})$ -action resp.) on  $\mathcal{V}_d$  induced by the transposed action of  $Sl(2, \mathbb{R})$  ( $Gl(2, \mathbb{R})$  resp.) on  $\mathbb{R}^2$ , i.e. if  $p \in \mathcal{V}_d$  and  $A \in Sl(2, \mathbb{R})$  ( $A \in Gl(2, \mathbb{R})$  resp.) then

$$(A \cdot p)(x, y) := p(u, v) \quad \text{with} \quad (u, v) = (x, y)A.$$

It is well known that this action is irreducible for every  $d$  and moreover that - up to equivalence - this is the only irreducible  $(d+1)$ -dimensional representation of  $Sl(2, \mathbb{R})$  ( $Gl(2, \mathbb{R})$  resp.) (cf. [H, p.33]). We let  $H_d \subseteq Gl(\mathcal{V}_d)$  ( $G_d \subseteq Gl(\mathcal{V}_d)$  resp.) be the image of this representation and let  $\mathfrak{h}_d \subseteq \mathfrak{gl}(\mathcal{V}_d)$  ( $\mathfrak{g}_d \subseteq \mathfrak{gl}(\mathcal{V}_d)$  resp.) be the Lie algebra of  $H_d$  ( $G_d$  resp.).

The *Clebsch-Gordan formula* [H, p.126] describes the irreducible decomposition of a tensor product of irreducible  $Sl(2, \mathbb{R})$ -modules:

$$\mathcal{V}_m \otimes \mathcal{V}_n = \mathcal{V}_{|m-n|} \oplus \mathcal{V}_{|m-n|+2} \oplus \cdots \oplus \mathcal{V}_{m+n-2} \oplus \mathcal{V}_{m+n}.$$

A convenient tool to compute the decomposition of polynomials into their irreducible components are the bilinear pairings

$$\langle \cdot, \cdot \rangle_p : V_n \otimes V_m \longrightarrow V_{n+m-2p}$$

$$\langle u, v \rangle_p = \frac{1}{p!} \sum_{k=0}^p (-1)^k \binom{p}{k} \frac{\partial^p u}{\partial^k x \partial^{p-k} y} \frac{\partial^p v}{\partial^{p-k} x \partial^k y} \quad \text{for} \quad u \in V_n, v \in V_m.$$

It can be shown that these pairings are  $Sl(2, \mathbb{R})$ -equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula (cf. [Br, p.37]).

Now we shall describe the structure equations for  $G_3$ -connections as in [Br, §2]. Let  $M$  be a four-manifold and let  $\pi : \mathfrak{F} \rightarrow M$  be the  $\mathcal{V}_3$ -coframe bundle, i.e. each  $u \in \mathfrak{F}$  is a linear isomorphism  $u : T_{\pi(u)}M \xrightarrow{\sim} \mathcal{V}_3$ . Then  $\mathfrak{F}$  is naturally a principal

right  $Gl(\mathcal{V}_3)$ -bundle over  $M$ , where the right action  $R_g : \mathfrak{F} \rightarrow \mathfrak{F}$  is defined by  $R_g(u) = g^{-1} \circ u$ . The *tautological* 1-form  $\omega$  on  $\mathfrak{F}$  with values in  $\mathcal{V}_3$  is defined by letting  $\omega(v) = u(\pi_*(v))$  for  $v \in T_u\mathfrak{F}$ . For  $\omega$ , we have the  $Gl(\mathcal{V}_3)$ -equivariance  $R_g^*(\omega) = g^{-1}\omega$ .

A  $G_3$ -structure on  $M$  is, by definition, a  $G_3$ -subbundle  $F \subseteq \mathfrak{F}$ . For any  $G_3$ -structure, we will denote the restrictions of  $\pi$  and  $\omega$  to  $F$  by the same letters.

We now turn to describe connections on  $F$ . Since  $G_3$  is canonically isomorphic to  $Gl(2, \mathbb{R})$ , we may regard the  $Gl(2, \mathbb{R})$ -representations  $\mathcal{V}_d$  equally well as  $G_3$ -representations. Moreover, it is easily seen that the map  $\rho_d : \mathcal{V}_2 \oplus \mathcal{V}_0 \rightarrow \text{End}(\mathcal{V}_d)$  defined by  $\rho_d(a^2 + a^0)(a^d) := \langle a^0, a^d \rangle_0 + \langle a^2, a^d \rangle_1$  for all  $a^i \in \mathcal{V}_i$  establishes an isomorphism  $\mathcal{V}_2 \oplus \mathcal{V}_0 \xrightarrow{\sim} \mathfrak{g}_d$ . We will use this to regard a *connection* on  $F$  as a  $G_3$ -equivariant,  $\mathcal{V}_2 \oplus \mathcal{V}_0$ -valued 1-form  $\varphi = \phi + \lambda$  on  $F$  where  $\phi$  and  $\lambda$  take values in  $\mathcal{V}_2$  and  $\mathcal{V}_0$  resp. The *torsion* of  $\varphi$  is then represented by the  $\mathcal{V}_3$ -valued 2-form  $T(\varphi) = d\omega + \langle \phi, \omega \rangle_1 + \langle \lambda, \omega \rangle_0$  and the *curvature* of  $\varphi$  by the  $(\mathcal{V}_2 \oplus \mathcal{V}_0)$ -valued 2-form  $R(\varphi) = d\varphi + \frac{1}{2} \langle \varphi, \varphi \rangle_1 = d\lambda + d\phi + \frac{1}{2} \langle \phi, \phi \rangle_1$ .

If we assume that  $\varphi$  describes a *torsion free* connection, then the *first structure equation*  $T(\varphi) = 0$  reads

$$(1) \quad d\omega = -\lambda \wedge \omega - \langle \phi, \omega \rangle_1.$$

Differentiating (1) yields

$$d\lambda \wedge \omega + \left\langle d\phi + \frac{1}{2} \langle \phi, \phi \rangle_1, \omega \right\rangle_1 = 0.$$

This equation, which is the first Bianchi identity, can be solved to show that there is a  $(\mathcal{V}_2 \oplus \mathcal{V}_4)$ -valued function  $\mathbf{a} = a^2 + a^4$  on  $F$  with  $a^i : F \rightarrow \mathcal{V}_i$ , such that the *second structure equations* hold:

$$(2) \quad \begin{aligned} d\lambda &= \langle a^4, \langle \omega, \omega \rangle_1 \rangle_4, \\ d\phi &= -\frac{1}{2} \langle \phi, \phi \rangle_1 + a^2 \langle \omega, \omega \rangle_3 - \frac{1}{12} \langle a^2, \langle \omega, \omega \rangle_1 \rangle_2 + \frac{1}{12} \langle a^4, \langle \omega, \omega \rangle_1 \rangle_3. \end{aligned}$$

Note that, in particular, we obtain as a formula for the curvature

$$(3) \quad R(\varphi) = \langle a^4, \langle \omega, \omega \rangle_1 \rangle_4 + a^2 \langle \omega, \omega \rangle_3 - \frac{1}{12} \langle a^2, \langle \omega, \omega \rangle_1 \rangle_2 + \frac{1}{12} \langle a^4, \langle \omega, \omega \rangle_1 \rangle_3.$$

Differentiating these equations once again and solving for  $\mathbf{a}$  we find that there is a  $(\mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7)$ -valued function  $\mathbf{b} = b^1 + b^3 + b^5 + b^7$  on  $F$  with  $b^i : F \rightarrow \mathcal{V}_i$ , such that the *third structure equations* hold:

$$(4) \quad \begin{aligned} da^2 &= 2\lambda \wedge a^2 - \langle \phi, a^2 \rangle_1 + 10 \langle b^1, \omega \rangle_1 + \langle b^3, \omega \rangle_2 + 14 \langle b^5, \omega \rangle_3, \\ da^4 &= 2\lambda \wedge a^4 - \langle \phi, a^4 \rangle_1 + 9 \langle b^1, \omega \rangle_0 - 5 \langle b^5, \omega \rangle_2 + \langle b^7, \omega \rangle_3. \end{aligned}$$

The function  $\mathbf{b}$  represents the covariant derivative  $\nabla R$  of the curvature. In particular, we emphasize that  $(\nabla R)(x) = 0$  at some  $x \in M$  if and only if  $\mathbf{b}(u) = 0$  for all  $u \in \pi^{-1}(x)$ .

We can also obtain formulas for  $d\mathbf{b}$  by differentiating (4). Since these formulas are fairly involved we shall not write them in full. However, we can describe the  $G_3$ -equivariance of  $\mathbf{b}$  by the equations

$$(5) \quad db^i \equiv 3\lambda \wedge b^i - \langle \phi, b^i \rangle_1 \pmod{\omega}, \quad \text{for } i = 1, 3, 5, 7.$$

A  $G_3$ -connection on  $M$  is, by definition, a  $G_3$ -structure on  $M$  which carries a torsion free connection. A manifold  $M$  with a  $G_3$ -connection will be called a  $G_3$ -manifold.

## §2 Homogeneous $G_3$ -structures.

Throughout this section, we shall assume that  $M$  is a connected  $G_3$ -manifold. We begin with some definitions.

**Definition 2.1.** Let  $M$  be a connected  $G_3$ -manifold with connection  $\nabla$  and let  $\pi : F \rightarrow M$  be the associated  $G_3$ -structure.

- (1) A (local) symmetry on  $M$  is a (local) diffeomorphism  $\underline{\alpha} : M \rightarrow M$  such that  $\underline{\alpha}_*(\nabla_X Y) = \nabla_{\underline{\alpha}_*(X)} \underline{\alpha}_*(Y)$  for all  $X, Y \in \mathfrak{X}(M)$ .
- (2) A (local) symmetry on  $F$  is a (local) diffeomorphism  $\alpha : F \rightarrow F$  such that  $\alpha^*(\omega) = \omega$  and  $\alpha^*(\varphi) = \varphi$ .

There is a one-to-one correspondence between symmetries on  $M$  and on  $F$ . In fact, given a (local) symmetry  $\underline{\alpha}$  on  $M$ , there is a unique (local) symmetry  $\alpha$  on  $F$  making the diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha} & F \\ \pi \downarrow & & \pi \downarrow \\ M & \xrightarrow{\underline{\alpha}} & M \end{array}$$

commute, and vice versa.

**Definition 2.2.** Let  $M$  and  $\pi : F \rightarrow M$  as before.

- (1) An infinitesimal symmetry on  $M$  is a vector field  $\underline{S} \in \mathfrak{X}(M)$  such that  $\mathfrak{L}_{\underline{S}}\nabla = 0$ , i.e.  $[\underline{S}, \nabla_X Y] - \nabla_{[\underline{S}, X]} Y - \nabla_X [\underline{S}, Y] = 0$  for all  $X, Y \in \mathfrak{X}(M)$ .
- (2) An infinitesimal symmetry on  $F$  is a vector field  $S \in \mathfrak{X}(F)$  such that  $\mathfrak{L}_S \omega = \mathfrak{L}_S \varphi = 0$ .

Again, there is a one-to-one correspondence between infinitesimal symmetries on  $M$  and on  $F$ : in fact, given an infinitesimal symmetry  $\underline{S} \in \mathfrak{X}(M)$ , then there is a unique infinitesimal symmetry  $S$  on  $F$  s.th.  $\underline{S} = \pi_*(S)$ . Conversely, given an infinitesimal symmetry  $S \in \mathfrak{X}(F)$  then the vector field  $\pi_*(S)$  is well defined and is an infinitesimal symmetry on  $M$ .

Note that the flow along an infinitesimal symmetry on  $M$  (on  $F$  resp.) yields a one-parameter family of local symmetries on  $M$  (on  $F$  resp.). In fact, the infinitesimal symmetries form the Lie algebra of the group of (local) symmetries.

Due to the above mentioned one-to-one correspondences, we will frequently speak of symmetries (local, infinitesimal symmetries resp.) of the  $G_3$ -connection without specifying whether they are regarded as symmetries (local, infinitesimal symmetries resp.) on  $M$  or on  $F$ .

The group of (local) symmetries will be denoted by  $G$  and its Lie algebra of infinitesimal symmetries by  $\mathfrak{g}$ .

It is worth remarking that as a consequence of the structure equations (2) and (4) we have  $d\mathbf{a}(S) = d\mathbf{b}(S) = 0$  for any infinitesimal symmetry  $S$  on  $F$ .

In this paper we will be concerned with *homogeneous  $G_3$ -manifolds*, i.e. those  $G_3$ -manifolds whose symmetry group acts transitively on  $M$ . First, we will prove a Lemma which will yield some relation between the isotropy and the curvature at a point of  $M$ .

**Lemma 2.3.** *Let  $M$  be a  $G_3$ -manifold, let  $\pi : F \rightarrow M$  as before and let  $x \in M$  be a point such that neither the curvature  $R$  nor its covariant derivative  $\nabla R$  vanish at  $x$ , and let  $\mathfrak{g}_x \subseteq \mathfrak{g}$  be the set of infinitesimal symmetries on  $M$  which vanish at  $x$ . If  $\mathfrak{g}_x \neq 0$  then there exists a point  $u_0 \in \pi^{-1}(x)$  such that*

*either*

$$a^2(u_0) = r_2x^2, \quad a^4(u_0) = r_4x^3y, \\ b^1(u_0) = 0, \quad b^3(u_0) = s_3x^3, \quad b^5(u_0) = s_5x^4y, \quad b^7(u_0) = s_7x^5y^2,$$

*or*

$$a^i(u_0) = r_ix^i, \quad \text{for } i = 2, 4 \quad b^i(u_0) = s_ix^i, \quad \text{for } i = 1, 3, 5, 7.$$

*for some constants  $r_i, s_i \in \mathbb{R}$ .*

*Proof.* The hypothesis that  $R$  and  $\nabla R$  do not vanish at  $x$  implies that  $\mathbf{a}(u) \neq 0$  and  $\mathbf{b}(u) \neq 0$  for all  $u \in \pi^{-1}(x)$ .

Now let  $0 \neq \underline{S} \in \mathfrak{g}_x$ , and let  $S \in \mathfrak{X}(F)$  be the corresponding infinitesimal symmetry on  $F$ . Clearly,  $\pi_*(S_u) = 0$  and hence  $\omega(S_u) = 0$  for all  $u \in \pi^{-1}(x)$ . Since  $\varphi + \omega$  is a coframe on  $F$  and  $S \neq 0$ , we have  $\varphi(S) \neq 0$ . Moreover, since  $S$  is an infinitesimal symmetry, we also have  $d\mathbf{a}(S) = d\mathbf{b}(S) = 0$ .

Applying (4) and (5) to  $S$ , we obtain that at any point  $u \in \pi^{-1}(x)$ , we have

$$(6-1) \quad 2\lambda(S)a^i - \langle \phi(S), a^i \rangle_1 = 0 \quad \text{for } i = 2, 4,$$

$$(6-2) \quad 3\lambda(S)b^i - \langle \phi(S), b^i \rangle_1 = 0 \quad \text{for } i = 1, 3, 5, 7.$$

Now consider the following two cases:

- (1) case 1:  $\phi(S) \in \mathcal{V}_2$  factors into two independent linear factors over  $\mathbb{C}$ . Then there is some  $g \in \text{Sl}(2, \mathbb{C})$  such that  $\rho_2^{\mathbb{C}}(g) \cdot \phi(S) = cxy$  for some  $c \in \mathbb{C}$ .

We deduce from (6-1) and  $\mathbf{a} \neq 0$  that  $2\lambda(S) = kc$  with  $k \in \{0, \pm 2, \pm 4\}$ . Likewise, from (6-2) and  $\mathbf{b} \neq 0$  we deduce that  $3\lambda(S) = kc$  with  $k \in \{\pm 1, \pm 3, \pm 5, \pm 7\}$ .

The only possibility for these to hold simultaneously is that  $\lambda(S) = \pm c$ . In particular,  $c \in \mathbb{R}$ . From here we can conclude that  $\phi(S)$  factors *over*  $\mathbb{R}$ , hence at some point  $u_0 \in \pi^{-1}(x)$  we have  $\phi(S_{u_0}) = \lambda(S_{u_0})xy$  and  $\lambda(S_{u_0}) \neq 0$ . From (6-1) and (6-2) we obtain that  $\mathbf{a}(u_0)$  and  $\mathbf{b}(u_0)$  are of the first form presented above.

- (2) case 2:  $\phi(S)$  is the square of a linear polynomial. Then there is some  $u_0 \in \pi^{-1}(x)$  such that  $\phi(S_{u_0}) = x^2$ .

We deduce from (6-1) and  $\mathbf{a} \neq 0$  that  $\lambda(S_{u_0}) = 0$ . Again, equations (6-1) and (6-2) imply that  $\mathbf{a}(u_0)$  and  $\mathbf{b}(u_0)$  are of the second form presented above.

- (3) case 3:  $\phi(S) = 0$ . Then we deduce from (6-1) and  $\mathbf{a} \neq 0$  that  $\lambda(S) = 0$ , i.e.  $\varphi(S) = 0$  which is impossible.  $\square$

Now we obtain the following remarkable

**Theorem 2.4.** *Let  $M$  be a (locally) homogeneous  $G_3$ -manifold. Then either*

- (1) *the  $G_3$ -connection is flat, or*
- (2)  *$M$  is locally equivalent to the unique homogeneous  $H_3$ -manifold, or*
- (3) *the isotropy group of a point in  $M$  is discrete, hence the (local) symmetry group has dimension four.*

*Proof.* Let  $G$  denote the group of symmetries and suppose that  $G$  acts transitively on  $M$ .

First of all, note that there cannot be a locally symmetric non-flat  $G_3$ -connection: the isotropy of a symmetric  $G_3$ -connection contains all of  $G_3$ . But the map  $\mathbf{a} : F \rightarrow \mathcal{V}_2 \oplus \mathcal{V}_4$  must be invariant under the action of the isotropy group, therefore we must have  $\mathbf{a} = 0$ , i.e. the connection is *flat*.

We will now assume that the  $G_3$ -connection is neither flat nor locally symmetric and that the isotropy group at each point is at least one-dimensional. We shall conclude from these assumptions that the holonomy is contained in  $H_3$ , and this will complete the proof.

From Lemma 2.3. we conclude that there are  $Gl(2, \mathbb{R})$ -equivariant functions  $v_i : F \rightarrow \mathcal{V}_1$  for  $i = 1, 2$  such that  $\langle v_1, v_2 \rangle_1 \equiv 1$  and functions  $\underline{r}_i, \underline{s}_i : M \rightarrow \mathbb{R}$  such that

either

$$(*) \quad \begin{aligned} a^2 &= r_2 v_1^2, & a^4 &= r_4 v_1^3 v_2, \\ b^1 &= 0, & b^3 &= s_3 v_1^3, & b^5 &= s_5 v_1^4 v_2, & b^7 &= s_7 v_1^5 v_2^2, \end{aligned}$$

or

$$(**) \quad a^i = r_i v_1^i, \quad \text{for } i = 2, 4 \quad b^i = s_i v_1^i, \quad \text{for } i = 1, 3, 5, 7,$$

where  $r_i = \underline{r}_i \circ \pi$  and  $s_i = \underline{s}_i \circ \pi$ .

Since the connection is *homogeneous* we may assume that  $v_i$  is  $G$ -invariant for  $i = 1, 2$  and that  $\underline{r}_i, \underline{s}_i$  are *constant* for all  $i$ . Thus, so are  $r_i$  and  $s_i$ .

If (\*) holds, then the structure equations (4) imply that

$$\begin{aligned} 2r_2 dv_1 &= 2r_2 \lambda \wedge v_1 - 2r_2 \langle \phi, v_1 \rangle_1 \\ &\quad + (3s_3 + 56s_5) v_2 \langle v_1^3, \omega \rangle_3 + 3(28s_5 - s_3) v_1 \langle v_1^2 v_2, \omega \rangle_3 \\ 2r_2 r_4 v_1 dv_2 &= -2r_2 r_4 \lambda \wedge v_1 v_2 - 2r_2 r_4 v_1 \langle \phi, v_2 \rangle_1 \\ &\quad + (-9r_4 s_3 - 60r_2 s_5 - 168r_4 s_5 + 20r_2 s_7) v_2^2 \langle v_1^3, \omega \rangle_3 \\ &\quad + (9r_4 s_3 + 20r_2 s_5 - 252r_4 s_5 + 40r_2 s_7) v_1 v_2 \langle v_1^2 v_2, \omega \rangle_3 \\ &\quad + 10r_2 (4s_5 + s_7) v_1^2 \langle v_1 v_2^2, \omega \rangle_3 \end{aligned}$$

Taking the latter equation modulo  $v_1$ , we conclude that

$$-9r_4 s_3 - 60r_2 s_5 - 168r_4 s_5 + 20r_2 s_7 = 0.$$

If we furthermore assume that  $r_2 r_4 \neq 0$  then we get for the exterior derivatives

$$\begin{aligned} dv_1 &= \lambda \wedge v_1 - \langle \phi, v_1 \rangle_1 + \frac{3s_3 + 56s_5}{2r_2} v_2 \langle v_1^3, \omega \rangle_3 + 3 \frac{28s_5 - s_3}{2r_2} v_1 \langle v_1^2 v_2, \omega \rangle_3 \\ dv_2 &= -\lambda \wedge v_2 - \langle \phi, v_2 \rangle_1 + \frac{9r_4 s_3 + 20r_2 s_5 - 252r_4 s_5 + 40r_2 s_7}{2r_2 r_4} v_2 \langle v_1^2 v_2, \omega \rangle_3 \\ &\quad + 5 \frac{4s_5 + s_7}{r_4} v_1 \langle v_1 v_2^2, \omega \rangle_3 \end{aligned}$$

However, taking exterior derivatives of these equations we conclude that  $r_2 r_4 = 0$  which is impossible.

The remaining cases can be dealt with in a similar fashion: if  $r_2 = 0$  then the above equations imply that  $s_3 = s_5 = 0$ . From (4) and the equation

$$0 = \langle dv_1, v_2 \rangle_1 + \langle v_1, dv_2 \rangle_1,$$

we can get explicit expressions for  $dv_1$  and  $dv_2$ . If we take exterior derivatives then we conclude that  $r_4 = 0$ , i.e. the connection is *flat*.

If we assume that  $r_4 = 0$  then we conclude that  $s_5 = s_7 = 0$ , and from there it follows that the holonomy group is contained in the subgroup  $H_3 \subseteq G_3$ .

Finally, if (\*\*) holds then, by a similar analysis, we can conclude that the holonomy of the  $G_3$ -connection is contained in  $H_3$ .  $\square$

We turn now to the problem of classifying the homogeneous  $G_3$ -manifolds. By the preceding Theorem, it will suffice to consider left-invariant  $G_3$ -connections on four-dimensional Lie groups. In fact, if  $M$  is a (locally) homogeneous  $G_3$ -manifold with a four-dimensional (local) symmetry group  $G$  then, for some fixed point  $p \in M$ , the map  $g \mapsto g \cdot p$  yields a local diffeomorphism from (an open subset of)  $G$  into  $M$  which can be used to define a left-invariant  $G_3$ -connection on  $G$ . By construction, this connection is locally equivalent to the connection on  $M$ .

Now let us describe left-invariant  $G_3$ -structures on a Lie group  $G$ .

**Proposition 2.5.** *Let  $G$  be a four-dimensional Lie group with Lie algebra  $\mathfrak{g}$ . Then there is a one-to-one correspondence between  $G_3$ -structures on  $G$  which are invariant under the natural left-action of  $G$ , and the set of equivalence classes of linear isomorphisms  $\{\iota : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3\} / \sim$ , where  $\iota \sim g \circ \iota$  for all  $g \in G_3$ .*

*Proof.* Fix a linear isomorphism  $\iota : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$ . For any  $p \in G$  and  $g \in G_3$ , we let  $\alpha_{(p,g)} : T_p G \xrightarrow{\sim} \mathcal{V}_3$  be the linear isomorphism  $\alpha_{(p,g)} := g^{-1} \circ \iota \circ (\omega_G)_p$ , where  $\omega_G$  denotes the Maurer-Cartan form of  $G$ . Then  $F := \{\alpha_{(p,g)} : T_p G \xrightarrow{\sim} \mathcal{V}_3 \mid g \in G_3, p \in G\} \subseteq \mathfrak{F}$  defines a left-invariant  $G_3$ -structure on  $G$ . Note that if we replace the isomorphism  $\iota : \mathfrak{g} \mapsto \mathcal{V}_3$  by  $g \circ \iota$  for any  $g \in G_3$  then the  $G_3$ -structure remains unchanged.

Conversely, given a left-invariant  $G_3$ -structure  $\pi : F \rightarrow G$ , pick any  $\iota \in \pi^{-1}(e)$  and regard it as an isomorphism  $\iota : T_e G = \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$ . It is left to the reader to verify that this establishes the desired one-to-one correspondence.  $\square$

Now suppose that we are given a left-invariant  $G_3$ -connection on  $G$ . We want to find explicit expressions for the tautological and the connection 1-forms.



Let  $\pi : F \rightarrow G$  be the underlying  $G_3$ -structure and let  $\iota : \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_3$  be a corresponding isomorphism. The map  $\alpha : G \times G_3 \rightarrow F$  given by  $(p, g) \mapsto \alpha_{(p,g)}$  is clearly a diffeomorphism, and we will use  $\alpha$  as a coordinate system of  $F$ . Pulling back the Maurer-Cartan form on  $G \times G_3$  to  $F$  via  $\alpha^{-1}$ , we obtain a natural  $\mathfrak{g} \oplus \mathfrak{g}_3$ -valued coframe on  $F$ . Using the isomorphism  $\iota + \rho_3^{-1} : \mathfrak{g} \oplus \mathfrak{g}_3 \xrightarrow{\sim} \mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$ , with  $\rho_3$  from the previous section, we get a  $\mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$ -valued coframe on  $F$  which we denote by  $\underline{\omega} + \underline{\varphi}$  where  $\underline{\omega}$  and  $\underline{\varphi}$  take values in  $\mathcal{V}_3$  and  $\mathcal{V}_2 \oplus \mathcal{V}_0$  resp. We also let  $\underline{\varphi} = \underline{\phi} + \underline{\lambda}$  be the decomposition of  $\underline{\varphi}$  into its components. In this notation the tautological 1-form  $\omega$  on  $F$  is given by  $\omega = g^{-1}\underline{\omega}$ .

The connection 1-form  $\varphi := \phi + \lambda$  on  $F$  takes values in  $\mathcal{V}_2 \oplus \mathcal{V}_0$ , and  $\omega + \phi + \lambda$  yields another  $\mathcal{V}_3 \oplus \mathcal{V}_2 \oplus \mathcal{V}_0$ -valued coframe on  $F$ . In fact, the left-invariance of the connection implies that there is a linear map  $\varphi_0 : \mathcal{V}_3 \rightarrow \mathcal{V}_2 \oplus \mathcal{V}_0$  such that at a point  $\alpha_{(p,g)} \in F$ , we have the relations

$$(7) \quad \begin{aligned} \omega &= g^{-1}\underline{\omega} \\ \varphi &= \underline{\varphi} + g^{-1}(\varphi_0 \circ \underline{\omega}). \end{aligned}$$

We can decompose  $\varphi_0 = \phi_0 + \lambda_0$  with  $\phi_0 \in \mathcal{V}_3^* \otimes \mathcal{V}_2 \cong \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5$  and  $\lambda_0 \in \mathcal{V}_3^* \otimes \mathcal{V}_0 \cong \mathcal{V}_3$ . It follows that there is a polynomial  $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$  such that if we let  $\mathbf{r} = r^1 + r^3 + r^5 + s^3$  with  $r^i \in \mathcal{V}_i$  and  $s^3 \in \mathcal{V}_3$  then  $\phi_0(v) = \langle r^1, v \rangle_1 + \langle r^3, v \rangle_2 + \langle r^5, v \rangle_3$  and  $\lambda_0(v) = \langle s^3, v \rangle_3$  for all  $v \in \mathcal{V}_3$ .

Note that if we replace the isomorphism  $\iota : \mathfrak{g} \rightarrow \mathcal{V}_3$  by  $g \circ \iota$  for  $g \in G_3$  then the connection will be given by the polynomial  $g \cdot \mathbf{r}$ .

Let us now compute the torsion of the connection. We have

$$\begin{aligned} T(\varphi) &= d\omega + \phi \wedge \omega \\ &= g^{-1}(d\underline{\omega} + \langle \phi_0 \circ \underline{\omega}, \underline{\omega} \rangle_2 + \langle \lambda_0 \circ \underline{\omega}, \underline{\omega} \rangle_0) \\ &= g^{-1}(d\underline{\omega} + \langle \langle r^1, \underline{\omega} \rangle_1 + \langle r^3, \underline{\omega} \rangle_2 + \langle r^5, \underline{\omega} \rangle_3, \underline{\omega} \rangle_2 + \langle \langle s^3, \underline{\omega} \rangle_3, \underline{\omega} \rangle_0). \end{aligned}$$

Thus, the connection is *torsion free* if and only if

$$(8) \quad d\underline{\omega} + \langle \langle r^1, \underline{\omega} \rangle_1 + \langle r^3, \underline{\omega} \rangle_2 + \langle r^5, \underline{\omega} \rangle_3, \underline{\omega} \rangle_2 + \langle \langle s^3, \underline{\omega} \rangle_3, \underline{\omega} \rangle_0 = 0.$$

Taking the exterior derivative of (8), a calculation yields

$$(9) \quad \frac{1}{12} \langle t^0, \beta \rangle_0 + \frac{1}{900} \langle t^2, \beta \rangle_2 + \frac{1}{540} \langle t^4, \beta \rangle_4 + \frac{1}{900} \langle t^6, \beta \rangle_6 = 0,$$

where  $\beta = \langle \underline{\omega}, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_2$  and

$$(10) \quad \begin{aligned} t^0 &= \langle r^3, s^3 \rangle_3, \\ t^2 &= -90(r^1)^2 - 15 \langle r^1, 9r^3 - s^3 \rangle_1 - 3 \langle r^3, 3r^3 + s^3 \rangle_2 - 7 \langle r^5, 9r^3 - s^3 \rangle_3 \\ &\quad + 26 \langle r^5, r^5 \rangle_4, \\ t^4 &= -9r^1(5r^3 - s^3) + 3 \langle r^5, 5r^3 - s^3 \rangle_2 + \langle r^3, s^3 \rangle_1, \\ t^6 &= 60r^1r^5 - 3r^3(3r^3 + s^3) + 2 \langle r^5, 9r^3 - s^3 \rangle_1 - 9 \langle r^5, r^5 \rangle_2. \end{aligned}$$

If we define the map

$$\begin{aligned} \tau : \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5 &\longrightarrow \mathcal{V}_0 \oplus \mathcal{V}_2 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \\ \mathbf{r} &\longmapsto t^0 + t^2 + t^4 + t^6 \end{aligned}$$

with  $t^i$  as in (10) then it is easy to see that (9) is satisfied if and only if  $\tau(\mathbf{r}) = 0$ .

Conversely, given  $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$  with  $\tau(\mathbf{r}) = 0$ , then (8) determines a Lie algebra structure on  $\mathcal{V}_3$  and  $\omega$  and  $\varphi$  defined as in (7) establish a left-invariant  $G_3$ -connection on the  $G_3$ -structure  $\pi : G \times G_3 =: F \rightarrow G$ .

Thus, we have the following

**Proposition 2.6.** *There is a one-to-one correspondence between left-invariant  $G_3$ -connections on connected, simply connected four-dimensional Lie groups and the set of  $Gl(2, \mathbb{R})$ -orbits of  $\tau^{-1}(0)$ .*

Of course, the condition that the Lie group be simply connected is only imposed to make this correspondence one-to-one.

Let us now compute the curvature of the connection determined by  $\mathbf{r}$ . The second structure equation and a calculation yields

$$\begin{aligned}
 R(\varphi) &= d\varphi + \varphi \wedge \varphi \\
 &= g^{-1}(d(\phi_0 \circ \underline{\omega}) + \frac{1}{2} \langle \phi_0 \circ \underline{\omega}, \phi_0 \circ \underline{\omega} \rangle_1) \\
 (11) \quad &= g^{-1}(\langle \underline{a}^4, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_4 + \underline{a}^2 \langle \underline{\omega}, \underline{\omega} \rangle_3 \\
 &\quad - \frac{1}{12} \langle \underline{a}^2, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_2 + \frac{1}{12} \langle \underline{a}^4, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_3 + \underline{T}),
 \end{aligned}$$

where

$$\begin{aligned}
 \underline{a}^2 &= \frac{1}{10}(40(r^1)^2 + 5 \langle r^1, 11r^3 - s^3 \rangle_1 - 4 \langle r^3, 7r^3 - s^3 \rangle_2 \\
 &\quad + \langle 21r^3 + s^3, r^5 \rangle_3 - 8 \langle r^5, r^5 \rangle_4), \\
 \underline{a}^4 &= \frac{1}{12}(-6r^1 s^3 + \langle r^3, s^3 \rangle_1 + 2 \langle r^5, s^3 \rangle_2), \quad \text{and} \\
 \underline{T} &= \frac{15}{4} t^0 \langle \underline{\omega}, \underline{\omega} \rangle_3 + \frac{1}{20} t^2 \langle \underline{\omega}, \underline{\omega} \rangle_3 + \frac{1}{144} \langle t^4, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_3 - \frac{1}{180} \langle t^6, \langle \underline{\omega}, \underline{\omega} \rangle_1 \rangle_4.
 \end{aligned}$$

Clearly, if  $\tau(\mathbf{r}) = 0$  then  $\underline{T} = 0$ . Also, comparing (11) with (3) yields

$$(12) \quad a^i = g^{-1} \underline{a}^i \quad \text{for } i = 2, 4.$$

As we mentioned earlier, the holonomy of the connection is contained in  $H_3 \subseteq G_3$  if and only if  $a^4 \equiv 0$ . Therefore, we have as a consequence of Proposition 2.6.

**Corollary 2.7.** *Let  $\mathbf{r} \in \mathcal{V}_1 \oplus 2\mathcal{V}_3 \oplus \mathcal{V}_5$  such that  $\tau(\mathbf{r}) = 0$  and  $-6r^1 s^3 + \langle r^3, s^3 \rangle_1 + 2 \langle r^5, s^3 \rangle_2 \neq 0$ . Then the holonomy of the  $G_3$ -connection defined by  $\mathbf{r}$  is not contained in  $H_3$ . In particular, the connection is not flat.*

Thus, in order to classify the homogeneous  $G_3$ -connections we have to classify the  $Gl(2, \mathbb{R})$ -orbits of  $\mathbf{r}$  which satisfy the two conditions of Corollary 2.7. This can be done by a careful case-by-case investigation.

The necessary calculations (all of which were performed by **MATHEMATICA**) are not presented here. However, the author shall provide the interested reader with copies of the **MATHEMATICA** files used to compile this classification. The results are presented in the following section.

### §3 Classification of Homogeneous $G_3$ -structures.

In this section we will state the result of the classification of homogeneous  $G_3$ -connections whose holonomy is not contained in  $H_3$ . As it turns out, this implies that the holonomy equals all of  $G_3$ .

Suppose that for a given  $\mathbf{r} = r^1 + r^3 + r^5 + s^3$  with  $r^i \in \mathcal{V}_i$ ,  $s^3 \in \mathcal{V}_3$  we have  $\tau(\mathbf{r}) = 0$  and  $a^4 \neq 0$ .

There are two cases to be distinguished.

**Case A:**  $\mathbf{r} = r^1 + r^3 + r^5 + s^3$  and  $s^3 \neq 5r^3$ .

In this case, the orbits of  $\mathbf{r}$  can be parametrized as follows:

	$r^3$	$r^5$	$s^3$
(A1)	$-\frac{6}{5}uv^2$	$\frac{1}{10}v^3(3u^2 + tv^2)$	$-v^2(6u + v)$
(A2)	$\frac{3}{5}u(7v^2 \mp u^2)$	$-\frac{3}{10}u^2v(v^2 \pm u^2)$	$-15u(v^2 \pm u^2)$
(A3)	$\frac{1}{5}u(v^2 \mp 17u^2)$	$-\frac{3}{10}u^2v(v^2 \mp u^2)$	$3u(7v^2 \pm u^2)$
(A4)	$\frac{1}{5}u(-u^2 \pm 2uv + v^2)$	$-\frac{1}{10}u^2v(u + v)(u + 3v)$	$u(u^2 + 6uv + 3v^2)$
(A5)	$\frac{1}{15}(5u^3 - 45u^2v + 90uv^2 - 54v^3)$	$\frac{1}{10}u(v - u)(2v - u)(3v - 2u)(3v - u)$	$5u^3 - 21u^2v + 30uv^2 - 18v^3$

Here we assume in each case that  $u, v \in \mathcal{V}_1$  is a basis with  $\langle u, v \rangle_1 = 1$ , and also that  $r^1 = v$ .

The Lie algebras of the symmetry groups can be represented as follows:

(A1)		$\mathfrak{g} = \begin{pmatrix} \pm a & 6a & 0 & b \\ -4ta & \pm a & 12a & c \\ 0 & -2ta & \pm a & d \\ 0 & 0 & 0 & 0 \end{pmatrix}$
(A2)		$\mathfrak{g} = \begin{pmatrix} 3a & b & c \\ & a & d \\ & & 0 \end{pmatrix}$
(A3)		$\mathfrak{g} = \begin{pmatrix} 5a & b & c \\ & 2a & d \\ & & 0 \end{pmatrix}$
(A4), (A5)		$\mathfrak{g} = \begin{pmatrix} * & 0 & * \\ & * & * \\ & & 0 \end{pmatrix}$

Thus, in all these cases the symmetry groups are *solvable*.

**Case B:**  $\mathbf{r} = r^1 + r^3 + r^5 + 5r^3$ .

In this case,  $t^0 = t^4 = 0$  follows immediately. The orbits of these  $\mathbf{r}$  with  $\tau(\mathbf{r}) = 0$  and  $a^4 \neq 0$  can be parametrized as follows:

	$r^3$	$r^5$
(B1)	$\pm \frac{1}{5}u^3$	$\frac{1}{30}v(v^2 \pm u^2)(2v^2 \pm 3u^2)$
(B2)	$\frac{3}{10}u(\pm u^2 - v^2)$	$-\frac{1}{10}u^2v(\pm u^2 + 3v^2)$
(B3)	$-\frac{3}{5}u(2v^2 \pm u^2)$	$\frac{3}{10}u^2v(v^2 \mp u^2)$
(B4)	$-\frac{6}{5}uv^2$	$\frac{3}{10}u^2v^3$
(B5)	$-\frac{1}{5}u(49u^2 + 36uv + 6v^2)$	$\frac{1}{30}u^2(8u + 3v)(47u^2 + 24uv + 3v^2)$
(B6)	$\frac{3}{5}uv^2$	$\frac{1}{10}u^2v^3$
(B7)	$\frac{3}{5}u(v^2 \pm u^2)$	$\frac{1}{10}u^2v(v^2 \pm 3u^2)$
(B8)	$\frac{3}{5}uv^2$	$\frac{1}{10}u^2(u^3 + v^3)$
(B9)	$\frac{3}{5}u(2cu^2 + v^2)$	$\frac{1}{10}u^2(6u^3 + 6cu^2v + v^3)$
(B10)	$\frac{3}{10}u((3c - 2)u^2 - 3uv - v^2)$	$-\frac{3}{10}u^2(u + v)(cu^2 + uv + v^2)$
(B11)	$\frac{3}{5}u(cu^2 - 6uv - 2v^2)$	$\frac{1}{10}u^2(4u + 3v)(8u^2 + cu^2 + 4uv + v^2)$
(B12)	$\frac{1}{10}v(-3u^2 + (3 - c)v^2)$	$\frac{1}{60c}(-u^2 - 2uv + (c - 1)v^2)$ $(u^3 + 3u^2v + 3(c + 1)uv^2 + (1 - 3c)v^3)$
(B13)	$\frac{3}{20}u((5 + c)u^2 + 6uv + v^2)$	$\frac{1}{40c}((c + 1)u^2 + 2uv + v^2)$ $((3 - c)u^3 + 3(3 + c)u^2v + 9uv^2 + 3v^3)$
(B14)	$\frac{12}{55c^2}u((28c^2 \mp 3)u^2 + 3c^2uv - c^2v^2)$	$\frac{\pm 1}{1100c^2}((80c^2 \mp 12)u^2 + 40c^2uv + 5c^2v^2)$ $(32(\pm 1 - 10c^2)u^3 + 30(\pm 1 - 8c^2)u^2v - 60c^2uv^2 - 5c^2v^3)$
(B15)	$\frac{1}{1350c_2c_4^2}u(c_1^2(600c_3^2 + 360c_2c_3^2 - 25c_2^3c_4 - 15c_2^3c_4 + 18c_2^4c_4)u^2 - 180c_1(10 + 3c_2)c_3c_4uv + 1350c_4^2v^2)$	$\frac{1}{24300c_1^2c_3^2c_4^4}(c_1^2(40c_3^2 + 5c_2^2c_4 + 6c_2^3c_4)u^2 - 120c_1c_3c_4uv + 90c_4^2v^2)$ $(10c_1^3c_3(-8c_3^2 + 3c_2^2c_4)u^3 + 9c_1^2c_4(40c_3^2 - 5c_2^2c_4 + 3c_2^3c_4)u^2v - 540c_1c_3c_4^2uv^2 + 270c_4^3v^3)$

Again, we assume in each case that  $u, v \in \mathcal{V}_1$  is a basis with  $\langle u, v \rangle_1 = 1$ , and that  $r^1 = v$ .

The Lie algebras of the symmetry groups in each case are as follows:

$$(B1) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } \pm = ' + ' \\ \mathfrak{gl}(2, \mathbb{R}) & \text{if } \pm = ' - ' \end{cases}$$

$$(B2), (B3) \quad \mathfrak{g} = \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix}$$

$$(B4) \quad \mathfrak{g} = \begin{pmatrix} 0 & a & b & c \\ & 0 & d & b \\ & & 0 & -a \\ & & & 0 \end{pmatrix}$$

$$(B5) \quad \mathfrak{g} = \mathfrak{u}(2)$$

$$(B6) - (B11) \quad \mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$$

$$(B12) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } c \neq 18 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } c = 18 \end{cases}$$

$$(B13) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } c \neq -\frac{8}{9} \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } c = -\frac{8}{9} \end{cases}$$

$$(B14) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } 1210c^2 \neq \pm 189 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } 1210c^2 = \pm 189 \end{cases}$$

$$(B15) \quad \mathfrak{g} = \begin{cases} \mathfrak{u}(2) & \text{if } 648c_3^2 + (3c_2 - 2)(6c_2 + 5)^2c_4 \neq 0 \\ \begin{pmatrix} 0 & * & * \\ & * & * \\ & & 0 \end{pmatrix} & \text{if } 648c_3^2 + (3c_2 - 2)(6c_2 + 5)^2c_4 = 0 \\ & \text{and } 6c_2 + 5 \neq 0 \\ \begin{pmatrix} a+b & 0 & c \\ 0 & a-b & d \\ 0 & 0 & a \end{pmatrix} & \text{if } c_3 = 6c_2 + 5 = 0 \text{ and } c_4 > 0 \\ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & a \end{pmatrix} & \text{if } c_3 = 6c_2 + 5 = 0 \text{ and } c_4 < 0 \end{cases}$$

From this we can conclude that the “moduli space” of homogeneous  $G_3$ -connections has one four-dimensional component (B15), seven one-dimensional components (A1) and (B9) – (B14), and fourteen points, including the flat connection and the homogeneous  $H_3$ -connection.

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