11

Tight Models of de-Rham Algebras of Highly Connected Manifolds

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Abstract

The rational homotopy type of a closed oriented manifold M is determined by the weak equivalence class of its de Rham algebra $\Omega^*(M)$. In [3] Crowley and Nordström invented the Bianchi–Massey tensor of a DGCA which is invariant under quasi-isomorphisms. In fact, for $(r-1)$- connected $(r>1)$ manifolds of dimension $n \leq 5r-3$, this tensor, together with the cohomology ring, completely determines the rational homotopy type. In this chapter we show that each weak equivalence class of Poincaré DGCAs contains a tight graded differential algebra, by which we mean a finite dimensional algebra with a non-degenerate Poincaré pairing which does not contain any properly enclosed quasi-isomorphic subalgebra. This tight differential graded algebra can be described explicitly in terms of the Bianchi–Massey tensor.

11.1 Introduction

By the seminal work of Sullivan [9], it is known that two simply connected CW-complexes X_1, X_2 are rationally equivalent if and only if their rational homotopy algebras $\pi_*(X, \mathbb{Q})$ are weakly equivalent differential graded commutative algebras (DGCAs). In the case of closed simply connected manifolds M_1, M_2 this is equivalent to the weak equivalence of the de Rham algebras $\Omega^*(M_i)$.

There are numerous known invariants of DGCAs which are preserved under quasi-isomorphisms and hence may help to distinguish weak equivalence classes of DGCAs. One such invariant, called the *Bianchi–Massey tensor*, was introduced by Crowley and Nordström [3] for DGCAs of

Poincaré type, see Definition 11.3.1 below. This is a class which comprises the de-Rham algebras of closed oriented manifolds. In fact, they showed that, for $(r-1)$-connected $(r>1)$ Poincaré algebras of degree $n \leq 5r-3$, the Bianchi–Massey tensor is the only invariant (apart from the cohomology algebra), meaning that two such DGCAs are weakly equivalent if and only if their cohomology algebra and their Bianchi–Massey tensors coincide. The degree of the de Rham algebra $\Omega^*(M)$ equals the dimension of the closed oriented manifold M.

In [5], the Bianchi–Massey tensor was shown to be equivalent to a class in Harrison cohomology and hence to determine an A_3 algebra. Furthermore, it was shown there that each weak equivalence class of a simply connected Poincaré DGCA of Hodge type (see. Definition 11.3.4) contains a finite dimensional representative; moreover, any $(r-1)$-connected $(r>1)$ Poincaré DGCA of Hodge type is almost formal in the sense of [2] if its degree n satisfies $n \leq 4r-1$, so that e.g. any closed simply connected 7-manifold is almost formal; see Corollary 11.4.5 below.

In this chapter, we aim at identifying a "canonical" finite dimensional representative in each weak equivalence class, similar to the approach in [6]. For this, we introduce the notion of a *tight Poincaré DGCA* as a finite dimensional Poincaré DGCA with a non-degenerate Poincaré pairing that does not admit any proper quasi-isomorphically embedded subalgebra (Definition 11.5.1).

For a graded commutative algebra (GCA) H^*, we follow [3] in setting $\mathcal{K}^* \subset S^2(H^*)$ the kernel of the multiplication map $\cdot : S^2(H^*) \to H^*$.[1]

Theorem 11.1.1 *Let H^* be an $(r-1)$-connected $(r>1)$ Poincaré GCA of degree $n \leq 5r-3$. Then there is a bijective correspondence between symmetric bilinear forms*[2] *$\beta \in (S^2(\mathcal{K}^*))^\vee$ on $\mathcal{K}^*$ of degree $n+1$ and isomorphism classes of tight DGCAs Q^*_β with cohomology H^*.*

The construction of the finite dimensional models in [5] implies that each weak equivalence class of DGCAs with the restrictions in Theorem 11.1.1 contains a tight DGCA. However, different β may result in weakly equivalent DGCAs Q^*_β. A symmetric bilinear form $\hat{\beta}$ on $S^2(H^*)$ is said to be of *Riemannian type* if it satisfies

$$\hat{\beta}(h_1h_2, h_3h_4) = -(-1)^{|h_2||h_3|}\hat{\beta}(h_1h_3, h_2h_4).$$

[1] Here, $S^k(V^*)$ for a graded vector space V^* denotes the graded symmetric k-tensors on V^*

[2] Here $(S^2(\mathcal{K}^*))^\vee$ denotes the dual space of $S^2(\mathcal{K}^*)$.

This terminology is chosen as such a tensor satisfies all (graded) symmetries of a Riemannian curvature tensor. Furthermore, a symmetric bilinear form β on $\mathcal{K}^*$ is said to be of Riemannian type if $\beta = \hat{\beta}_{|\mathcal{K}^*}$ for a symmetric bilinear form $\hat{\beta}$ of Riemannian type on $S^2(H^*)$. With this, we can show the following:

Theorem 11.1.2 *Let H^* be an $(r-1)$-connected $(r > 1)$ Poincaré GCA of degree $n \leq 5r - 3$.*

1. *Each weak equivalence class of DGCAs with cohomology H^* contains a tight DGCA Q^*_β, $\beta \in (S^2(\mathcal{K}^*))^\vee$. In fact, β may be chosen to be of Riemannian type.*
2. *Tight DGCAs $Q^*_{\beta_i}$, $i = 1, 2$, are weakly equivalent if and only if $(\beta_1 - \beta_2)_{|\mathcal{E}^*} = 0$, where $\mathcal{E}^* \subset S^2(\mathcal{K}^*)$ is the kernel of $S^2(\mathcal{K}^*) \hookrightarrow S^2(S^2(H^*)) \xrightarrow{mult} S^4(H^*)$.*

This chapter is organized as follows. In Section 11.2, we recall the relation between the rational homotopy equivalence of closed oriented manifolds or, more generally, of CW-complexes on the one hand and the weak equivalence of DGCAs on the other. In Section 11.3 we recall the definitions of Poincaré GCAs from [3] and Poincaré DGCAs of Hodge type from [5]. In Section 11.4 we recall from [5] that each simply connected Poincaré DGCA $\mathcal{A}^*$ of Hodge type is weakly equivalent to a finite dimensional DGCA $\mathcal{Q}_{\text{small}}$, called the *small quotient algebra of* $\mathcal{A}^*$, with a non-degenerate Poincaré pairing. In Section 11.5 we introduce the notion of *tight Poincaré DGCAs* and show Theorem 11.1.1. Once this is established, we recall in Section 11.6 the *Bianchi–Massey tensor* of [3] and compute it for the tight Poincaré DGCAs from Section 11.5 and show Theorem 11.1.2. We also given an explicit description of these models in the case of closed simply connected 7-manifolds.

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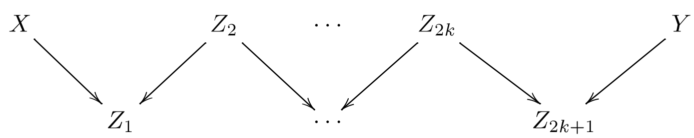
11.2 Rational and Weak Equivalence

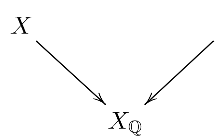
A continuous map $f : X \to Y$ between CW-complexes is called a *rational homotopy equivalence* if it induces an isomorphism on rational homotopy, i.e., if $f_* : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$ is an isomorphism. If X, Y are simply connected, then this is equivalently to requiring that f induces an isomorphisms on rational (co-)homology $f_* : H_*(X, \mathbb{Q}) \to H_*(Y, \mathbb{Q})$ or $f^* : H^*(Y, \mathbb{Q}) \to H^*(X, \mathbb{Q})$, respectively [4, Thm. 8.6]. Observe that in general, a rational homotopy equivalence does not admit an inverse, but it generates an equivalence relation by saying that X and Y are rationally equivalent if there are spaces and maps



(11.1)

where all arrows denote rational homotopy equivalences.[3]

As a special case of Sullivan's construction of the localization of topological spaces, it follows that, for each such space X, there is a CW-complex $X_{\mathbb{Q}}$, called the *rationalization of* X, together with a rational homotopy equivalence $X \to X_{\mathbb{Q}}$, and such that $X_{\mathbb{Q}}$ is a *rational space*, meaning that all homotopy groups $\pi_k(X_{\mathbb{Q}})$ are vector spaces over $\mathbb{Q}$. Moreover, $X_{\mathbb{Q}}$ is uniquely defined up to homotopy equivalence [4, Thm. 9.7], so that X and Y are rationally equivalent if and only if the diagram (11.1) may be simplified to



(11.2)

There is an algebraic analogue to this construction. Namely, recall that a differential commutative graded algebra (DGCA) is a graded vector space $\mathcal{A}^* = \bigoplus_{k=0}^{\infty} \mathcal{A}^k$ with an associative graded commutative product $\cdot$ and a differential $d : \mathcal{A}^* \to \mathcal{A}^*[-1]$, i.e., such that $\mathcal{A}^k \cdot \mathcal{A}^l \subset \mathcal{A}^{k+l}$ and $d\mathcal{A}^k \subset \mathcal{A}^{k+1}$, and with

[3] This is called the *localization of the category of simply connected spaces with respect to rational homotopy equivalences.*

$$\begin{aligned} \alpha \cdot \beta &= (-1)^{|\alpha||\beta|} \beta \cdot \alpha, \\ d(\alpha \cdot \beta) &= (d\alpha) \cdot \beta + (-1)^{|\alpha|} \alpha \cdot (d\beta), \\ d^2 &= 0, \end{aligned} \tag{11.3}$$

where $\alpha \in \mathcal{A}^{|\alpha|}$ and $\beta \in \mathcal{A}^{|\beta|}$ are homogeneous elements. Thus, the cohomology of $\mathcal{A}^*$ defined by

$$H^*(\mathcal{A}^*) := (\ker d)/(\mathrm{Im}\ d) \overset{\text{not.}}{\equiv} \mathcal{A}^*_d / d\mathcal{A}^*$$

has a well defined product $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$ which turns $H^*(\mathcal{A}^*)$ into a commutative graded algebra (GCA).

Given two DGCAs $\mathcal{A}^*$ and $\mathcal{B}^*$, a *DGCA-morphism* is a graded morphism of algebras $\varphi : \mathcal{A}^* \to \mathcal{B}^*$ which commutes with the differentials. Then φ induces a homomorphism on cohomologies

$$\varphi_* : H^*(\mathcal{A}^*) \longrightarrow H^*(\mathcal{B}^*),$$

and we call φ a *quasi-isomorphism* if φ_* is an isomorphism. Just as in the case of rational homotopy equivalences, a quasi-isomorphism does not posses an inverse in general, but it generates an equivalence relation by saying that DGCAs $\mathcal{A}^*$ and $\mathcal{B}^*$ are *weakly equivalent* if there are DGCAs $\mathcal{C}^*_i$ and quasi-isomorphisms

$$\begin{array}{ccccccccc} & \mathcal{C}^*_1 & & \cdots & & \mathcal{C}^*_{2k+1} & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \quad \swarrow & & \searrow \\ \mathcal{A}^* & & \mathcal{C}^*_2 & \cdots & \mathcal{C}^*_{2k} & & \mathcal{B}^* \end{array} \tag{11.4}$$

If $H^*(\mathcal{A}^*)$ is of *finite type*, i.e., all cohomologies $H^k(\mathcal{A}^*)$ are finite dimensional, then there is a DGCA $\mathcal{S}^*$, called the *Sullivan minimal model of* $\mathcal{A}^*$, and a quasi-isomorphism $\mathcal{S}^* \to \mathcal{A}^*$ such that a DGCA $\mathcal{B}^*$ is weakly equivalent to $\mathcal{A}^*$ if and only if $\mathcal{B}^*$ has the same minimal model $\mathcal{S}^*$, so that, in analogy to (11.2), any two weakly equivalent DGCAs $\mathcal{A}^*$ and B^* can be connected by quasi-isomorphisms

$$\begin{array}{ccc} & \mathcal{S}^* & \\ \swarrow & & \searrow \\ \mathcal{A}^* & & \mathcal{B}^* \end{array} \tag{11.5}$$

In fact, $\mathcal{S}^*$ is a free DGCA, generated by (possibly countably many) generators $\{x_n \mid n \in \mathbb{N}\}$ of positive degree, and where each dx_k is a decomposable polynomial in the generators [9, 8].

For instance, for a CW-complex X, the (singular) co-chains with coefficients in a field $\mathbb{F}$, together with the cup product, form a DGCA $C^*(X;\mathbb{F})$ with cohomology $H^*(X,\mathbb{F})$. In particular, a continuous map $f : X \to Y$ induces the DGCA-morphism $\varphi := f^* : C^*(Y,\mathbb{F}) \to C^*(X,\mathbb{F})$, whence in the case $\mathbb{F} = \mathbb{Q}$, φ is a quasi-isomorphim if and only if f is a rational homotopy equivalence. In particular, if X and Y are rationally equivalent, then $C^*(X,\mathbb{Q})$ and $C^*(Y,\mathbb{Q})$ are weakly equivalent, and $S^* := C^*(X_{\mathbb{Q}},\mathbb{Q})$ is the Sullivan minimal model of $C^*(X,\mathbb{Q})$.

The converse of this statement is also true provided that X and Y are simply connected; in this case X and Y are rationally equivalent if and only if $C^*(X,\mathbb{Q})$ and $C^*(Y,\mathbb{Q})$ are weakly equivalent.

Definition 11.2.1 A DGCA $\mathcal{A}^*$ is called *formal* if it is weakly equivalent to $(H^*(\mathcal{A}^*), d = 0)$. A topological space X is called formal if its rational singular co-chain algebra $C^*(X,\mathbb{Q})$ is formal.

That is, the rational homotopy type of a formal topological space X is determined by its cohomology ring $H^*(X,\mathbb{Q})$ only.

A DGCA over the field $\mathbb{F}$ is called *connected* if $H^0(\mathcal{A}^*) = \mathbb{F}$ and $(r-1)$*-connected* if it is connected and $H^k(\mathcal{A}^*) = 0$ for $k = 1, \ldots, r-1$. A 1-connected DGA is also called *simply connected*.

11.3 Poincaré DGCAs and DGCAs of Hodge Type

In this section we recall the terminology introduced in [5]. In general, for a graded vector space $V^* = \bigoplus_k V^k$ we say that a bilinear pairing $\langle -,- \rangle : V^* \times V^* \to \mathbb{F}$ is *of degree n*, if $\langle V^k, V^l \rangle = 0$ whenever $k + l \neq n$.

We begin with the following definition.

Definition 11.3.1 (see [3, Def. 2.7]) Let $H^* = \bigoplus_{k\geq 0} H^k$ be a GCA, and let $\int \in (H^n)^\vee$, where the latter denotes the dual of H^n. Then the bilinear pairing of degree n on H^* given by

$$\langle \alpha^k, \beta^l \rangle := \int \alpha^k \cdot \beta^l, \tag{11.6}$$

if $k + l = n$, is called the *Poincaré pairing of degree n induced by* $\int$.

We call H^* a *Poincaré algebra of degree* n if it is finite dimensional and admits a non-degenerate Poincaré pairing of degree n, i.e., such that $\langle \alpha, H^* \rangle = 0$ if and only if $\alpha = 0$.

Clearly, a Poincaré algebra of degree n is of the form $H^* = \bigoplus_{k=0}^{n} H^k$ whose Betti numbers $b^k = b^k(H^*) := \dim H^k$ satisfy $b^k = b^{n-k}$. If in addition $b^0 = b^n = 1$, then the pairing (11.6) is unique up to multiples.

Note that in [3, Def. 2.7], the degree of a Poincaré algebra is called the *dimension*, but as later we wish to consider the dimension of $\mathcal{A}^*$ as a graded vector space, the notion of degree seems more appropriate.

Definition 11.3.2 (see [3, Def. 2.7]) A *Poincaré DGCA of degree* n is a DGCA $\mathcal{A}^*$ whose cohomology algebra $H^* := H^*(\mathcal{A}^*)$ is a Poincaré algebra of degree n.

Proposition 11.3.3 *Let $\mathcal{A}^*$ be a Poincaré DGCA of degree n. Then there is a Poincaré pairing $\langle -, - \rangle$ of degree n on A^* such that*

$$\langle \alpha, \beta \rangle = \langle [\alpha], [\beta] \rangle_{H^*} \tag{11.7}$$

for all $\alpha, \beta \in \mathcal{A}^*_d$, where $[-] : \mathcal{A}^*_d \to H^*$ denotes the canonical projection and $\langle -, - \rangle_{H^*}$ is the Poincaré pairing on H^*. In particular,

$$\begin{aligned} \langle \alpha^k, \beta^l \rangle &= (-1)^{kl} \langle \beta^l, \alpha^k \rangle, \\ \langle \alpha^k \cdot \beta^l, \gamma^r \rangle &= \langle \alpha^k, \beta^l \cdot \gamma^r \rangle, \\ \langle d\alpha^k, \beta^l \rangle &= (-1)^{k+1} \langle \alpha^k, d\beta^l \rangle. \end{aligned} \tag{11.8}$$

Proof Pick $\int_{\mathcal{A}^*} \in (\mathcal{A}^n)^\vee$ such that $\int_{\mathcal{A}^*} \alpha_n = \int [\alpha_n]$ for all $\alpha_n \in \mathcal{A}^n$. Then it is straightforward to verify that the Poincaré pairing of degree n induced by $\int_{\mathcal{A}^*}$ has all the asserted properties. □

Strictly speaking, in order to extend the functional $\int_{\mathcal{A}^*}$ defined on $\mathcal{A}^n_d$ to all of $\mathcal{A}^n$ in the proof of Proposition 11.3.3 we need to make use of the axiom of choice. However, we shall later apply this to the case where either $\mathcal{A}^n$ is finite dimensional, or $\mathcal{A}^{n+1} = 0$, so that $\mathcal{A}^n_d = \mathcal{A}_n$; in these cases, the existence of this extension does not require the axiom of choice.

Observe that this pairing induces another pairing of degree $n+1$ on $d\mathcal{A}^*$, given by

$$\langle\!\langle \alpha, \beta \rangle\!\rangle := \langle d^- \alpha, \beta \rangle, \qquad \alpha, \beta \in d\mathcal{A}^*, \tag{11.9}$$

where $d^-\alpha$ is an element such that $dd^-\alpha = \alpha$. Indeed, since $d^-\alpha$ is well defined up to adding an element of $\mathcal{A}^*_d$, and $\langle \mathcal{A}^*_d, d\mathcal{A}^* \rangle = 0$ by (11.8), it follows that the pairing (11.9) is well defined. From (11.6) and (11.8) we easily deduce that this pairing is also graded symmetric, i.e.,

$$\langle\langle \alpha^k, \beta^l \rangle\rangle = (-1)^{kl} \langle\langle \beta^l, \alpha^k \rangle\rangle. \tag{11.10}$$

Definition 11.3.4 ([5, Def. 2.2]) Let $\mathcal{A}^*$ be a connected Poincaré DGCA of degree n with a Poincaré pairing $\langle -, - \rangle$.

1 A *harmonic subspace of* $\mathcal{A}^*$ is a graded subspace $\mathcal{H}^* \subset \mathcal{A}^*_d$ complementary to $d\mathcal{A}^*$ and such that $1 \in \mathcal{H}^0$.
2 A *Hodge type decomposition of* $\mathcal{A}^*$ is a direct sum decomposition of the form

$$\mathcal{A}^* = d\mathcal{A}^* \oplus \mathcal{H}^* \oplus \mathcal{B}^* \tag{11.11}$$

where $\mathcal{H}^* \subset \mathcal{A}^*_d$ is harmonic, such that

$$\langle \mathcal{H}^* \oplus \mathcal{B}^*, \mathcal{B}^* \rangle = 0. \tag{11.12}$$

3 A Poincaré DGCA admitting a Hodge type decomposition is called a *Hodge type DGCA*.

By (11.7) the restriction of the projection $\mathcal{A}^*_d \to H^*(\mathcal{A}^*)$ yields an isometric isomorphism

$$(\mathcal{H}^*, \langle -, - \rangle) \longrightarrow (H^*(\mathcal{A}^*), \langle -, - \rangle_{H^*}). \tag{11.13}$$

In particular, $\mathcal{H}^*$ is finite dimensional and the restriction of the Poincaré pairing to $\mathcal{H}^*$ is non-degenerate.

If $\mathcal{H}^* \subset \mathcal{A}^*_d$ is a harmonic subspace, then we have the decomposition $\mathcal{A}^*_d = d\mathcal{A}^* \oplus \mathcal{H}^*$, and we define the *cocycle of* $\mathcal{H}^*$ to be the linear map

$$\xi_{\mathcal{H}^*} : S^2(H^*) \longrightarrow d\mathcal{A}^*, \qquad \xi_{\mathcal{H}^*}([h_1], [h_2]) := pr_{d\mathcal{A}^*}(h_1 \cdot h_2), h_i \in \mathcal{H}^*. \tag{11.14}$$

Proposition 11.3.5 *Let $\mathcal{A}^* = d\mathcal{A}^* \oplus \mathcal{H}^* \oplus \mathcal{B}^*$ be a Poincaré DGCA with a Hodge type decomposition. Then every harmonic subspace $\hat{\mathcal{H}}^* \subset \mathcal{A}^*_d$ is of the form*

$$\hat{\mathcal{H}}^* = \{v + \beta(v) \mid v \in \mathcal{H}^*\} \tag{11.15}$$

for some linear map $\beta : \mathcal{H}^* \to d\mathcal{A}^*$ with $\beta(1) = 0$. Defining

$$\hat{\mathcal{B}}^* := \{x - \beta^\dagger(x) - \frac{1}{2}\beta\beta^\dagger(x) \mid x \in \mathcal{B}^*\}, \tag{11.16}$$

where $\beta^\dagger : \mathcal{B}^* \to \mathcal{H}^*$ is the unique map satisfying $\langle \beta^\dagger(x), h \rangle = \langle x, \beta(h) \rangle$ for all $x \in \mathcal{B}^*, h \in \mathcal{H}^*$, the decomposition $\mathcal{A}^* = d\mathcal{A}^* \oplus \hat{\mathcal{H}}^* \oplus \hat{\mathcal{B}}^*$ is a Hodge *type decomposition. Moreover, the cocycles of $\mathcal{H}^*$ and $\hat{\mathcal{H}}^*$ are related by the formula*

$$\begin{aligned}\xi_{\hat{\mathcal{H}}^*}([h_1],[h_2]) = \xi_{\mathcal{H}^*}([h_1],[h_2])\\ + h_1 \cdot \beta([h_2]) - \beta([h_1]\cdot[h_2]) + \beta(]h_1]) \cdot h_2\\ + \beta([h_1]) \cdot \beta([h_2]).\end{aligned} \tag{11.17}$$

Proof Since $\mathcal{A}^*_d = d\mathcal{A}^* \oplus \mathcal{H}^* = d\mathcal{A}^* \oplus \hat{\mathcal{H}}^*$, it follows that $\hat{\mathcal{H}}^*$ is of the form (11.16), and it is straightforward to verify that $\mathcal{A}^* = d\mathcal{A}^* \oplus \hat{\mathcal{H}}^* \oplus \hat{\mathcal{B}}^*$ is a Hodge type decomposition and that the cocycle $\xi_{\hat{\mathcal{H}}^*}$ is of the asserted form. □

We define $\mathcal{K}^* \subset S^2(H^*)$ as the kernel of the multiplication map, i.e., via the short exact sequence

$$0 \longrightarrow \mathcal{K}^* \longrightarrow S^2(H^*) \stackrel{\cdot}{\longrightarrow} H^* \longrightarrow 0. \tag{11.18}$$

Given a Hodge type decomposition (11.11), the restriction $d : \mathcal{B}^* \to d\mathcal{A}^*[-1]$ is a linear isomorphism, whence there is an inverse $d^- : d\mathcal{A}^* \to \mathcal{B}^*[1]$. We may extend d^- to all of $\mathcal{A}^*$ by defining $d^-_{|\mathcal{H}^*\oplus\mathcal{B}^*} = 0$. Thus, $(d^-)^2 = 0$, and

$$dd^-d = d, \qquad d^-dd^- = d^-. \tag{11.19}$$

It follows that the projections in (11.11) are given by

$$pr_{\mathcal{H}^*} = 1 - [d, d^-], \qquad pr_{d\mathcal{A}^*} = dd^-, \qquad pr_{\mathcal{B}^*=d^-\mathcal{A}^*} = d^-d, \tag{11.20}$$

where $[d, d^-] = dd^- + d^-d$ is the super-commutator. Therefore, (11.11) may be written as

$$\mathcal{A}^* = d\mathcal{A}^* \oplus \mathcal{H}^* \oplus d^-\mathcal{A}^* = dd^-\mathcal{A}^* \oplus \mathcal{H}^* \oplus d^-d\mathcal{A}^*, \tag{11.21}$$

and, setting $\mathcal{A}^*_{d^-} := \ker d^- = \mathcal{H}^* \oplus d^-\mathcal{A}^*$, (11.12) implies

$$\langle \mathcal{A}^*_d, d\mathcal{A}^* \rangle = \langle \mathcal{A}^*_{d^-}, d^-\mathcal{A}^* \rangle = 0. \tag{11.22}$$

Example 11.3.6 The quintessential example of a Poincaré algebra of degree n of Hodge type (which motivates our terminology) is the de Rham algebra $(\Omega^*(M), d)$ of a closed smooth oriented manifold M, with $\int$ given by the integration of n-forms. The Hodge decomposition w.r.t. some Riemannian metric g on M is then a Hodge type decomposition in the sense of Definition 11.3.4 whose harmonic subspace is the space $\mathcal{H}^*(M)$ of $\triangle_g$-harmonic forms. Note that the maps d^* and d^- are related by the formula $d^* =_{\triangle_g} d^-$.

Given a Poincaré pairing on $\mathcal{A}^*$, as a consequence of (11.8), the null-space

$$\mathcal{A}^*_\perp := \{\alpha \in \mathcal{A}^* \mid \langle \alpha, \mathcal{A}^* \rangle = 0\}$$

is a differential ideal of $\mathcal{A}^*$, whence the quotient $\mathcal{Q}^* := \mathcal{A}^*/\mathcal{A}^*_\perp$ is again a DGCA, fitting into the short exact sequence

$$0 \longrightarrow \mathcal{A}^*_\perp \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{Q}^* \longrightarrow 0. \tag{11.23}$$

Then there is an induced non-degenerate pairing on $\mathcal{Q}^*$, satisfying

$$\langle [\alpha], [\beta] \rangle_{\mathcal{Q}^*} = \langle \alpha, \beta \rangle_{\mathcal{A}^*}, \tag{11.24}$$

where $[\cdot] : \mathcal{A}^* \to \mathcal{Q}^*$ is the canonical projection.

Lemma 11.3.7 *If the Poincaré DGCA $\mathcal{A}^*$ admits a Hodge type decomposition, then the differential ideal $\mathcal{A}^*_\perp$ is invariant under d^-, and has the decomposition*

$$\mathcal{A}^*_\perp = dd^- \mathcal{A}^*_\perp \oplus d^- d\mathcal{A}^*_\perp. \tag{11.25}$$

In particular, $(\mathcal{A}^*_\perp, d)$ is acyclic, i.e., has trivial cohomology.

Proof The orthogonality relations (11.21) and (11.22) imply that $\mathcal{A}^*_\perp \subset \mathcal{H}^*_\perp = dd^- \mathcal{A}^* \oplus d^- d\mathcal{A}^*$.

If $\alpha^k \in \mathcal{A}^k_\perp$ and $\beta^{n-k+1} \in \mathcal{A}^{n-k+1}$, then by (11.8)

$$\begin{aligned}
\langle d^- d\alpha, d\beta^{n-k+1} \rangle &= (-1)^{k+1} \langle dd^- d\alpha, \beta^{n-k+1} \rangle \\
&= (-1)^{k+1} \langle d\alpha, \beta^{n-k+1} \rangle \\
&= -\langle \alpha, d\beta^{n-k+1} \rangle = 0,
\end{aligned}$$

where we used $dd^- d = d$. That is, $\langle d^- d\mathcal{A}^*_\perp, d\mathcal{A}^* \rangle = 0$, and then the orthogonality relations (11.22) imply that $\langle d^- d\mathcal{A}^*_\perp, \mathcal{A}^* \rangle = 0$, i.e., $\mathcal{A}^*_\perp$ is invariant under $d^- d$. Thus, because $\mathcal{A}^*_\perp \subset dd^- \mathcal{A}^* \oplus d^- d\mathcal{A}^*$, (11.25) follows. In particular, $\mathcal{A}^*_\perp$ is also invariant under dd^-.

To see that $\mathcal{A}^*_\perp$ is also invariant under d^-, let $\alpha^k \in \mathcal{A}^k_\perp$ and $\beta^{n-k} \in \mathcal{A}^{n-k}$. Then

$$\langle d^- \alpha^k, d\beta^{n-k} \rangle = (-1)^k \langle dd^- \alpha^k, \beta^{n-k} \rangle = 0,$$

as $dd^- \alpha^k \in \mathcal{A}^k_\perp$ by the dd^--invariance of $\mathcal{A}^k_\perp$. Therefore, $\langle d^- \mathcal{A}^*_\perp, d\mathcal{A}^* \rangle = 0$, and then (11.22) implies $\langle d^- \mathcal{A}^*, \mathcal{A}^* \rangle = 0$, i.e., $\mathcal{A}^*_\perp$ is invariant under d^-. This together with (11.25) now implies that $\mathcal{A}^*_\perp$ is acyclic. □

Corollary 11.3.8 *If $\mathcal{A}^*$ is of Hodge type, then the projection $\pi : \mathcal{A}^* \to \mathcal{Q}^*$ is a quasi-isomorphism, $\mathcal{Q}^*$ is of Hodge type and $\langle -, - \rangle_{\mathcal{Q}^*}$ from (11.24) is a non-degenerate Poincaré pairing.*

Proof The first statement follows from the long exact sequence associated to (11.23) and $H^*(\mathcal{A}^*_\perp) = 0$ by Lemma 11.3.7. Furthermore, (11.21) and (11.25) imply that $d, d^- : \mathcal{Q}^* \to \mathcal{Q}^*$ are well defined, and that there is a decomposition

$$\mathcal{Q}^* \cong dd^-\mathcal{Q}^* \oplus \mathcal{H}^* \oplus d^-d\mathcal{Q}^*,$$

and (11.24) easily implies that this is a Hodge type decomposition and that $\langle -, - \rangle_{\mathcal{Q}^*}$ is the Poincaré pairing. □

Remark 11.3.9 As the induced pairing on $\mathcal{Q}^*$ is non-degenerate by construction, Corollary 11.3.8 implies that every Hodge type Poincaré DGCA is equivalent to a non-degenerate one. This has been shown in [6, Thm. 1.1] by slightly different means.

11.4 Small Algebras of Hodge Type DGCAs

Given a Poincaré DGCA $\mathcal{A}^*$ with a Hodge type decomposition (11.21), we consider the following class of subalgebras:

Definition 11.4.1 ([5, Def. 3.1]) Let $\mathcal{A}^*$ be a Poincaré DGCA with a Hodge type decomposition (11.21). A $\mathcal{H}^*$*-subalgebra* is a DG-subalgebra of $\mathcal{A}^*$ which is d^--invariant and contains $\mathcal{H}^*$.

If $\mathcal{C}^* \subset \mathcal{A}^*$ is such an $\mathcal{H}^*$-subalgebra, then – as it is closed under both d and d^- – it follows that it admits a Hodge type decomposition analogous to (11.21)

$$\mathcal{C}^* = d\mathcal{C}^* \oplus \mathcal{H}^* \oplus d^-\mathcal{C}^* = dd^-\mathcal{C}^* \oplus \mathcal{H}^* \oplus d^-d\mathcal{C}^*.$$

It is evident from here that $H^*(\mathcal{C}^*) \cong \mathcal{H}^* \cong H^*(\mathcal{A}^*)$, whence the inclusion $\mathcal{C}^* \hookrightarrow \mathcal{A}^*$ is a quasi-isomorphism. As the class of $\mathcal{H}^*$-subalgebras contains $\mathcal{A}^*$ itself and is invariant under arbitrary intersections, it follows that there is a minimal such algebra, namely the intersection of all $\mathcal{H}^*$-subalgebras of $\mathcal{A}^*$.

Definition 11.4.2 ([5, Def. 3.2]) For a Poincaré DGCA $\mathcal{A}^*$ with a Hodge type decomposition (11.21), we define the *small algebra* $\mathcal{A}^*_{\mathrm{small}}$ of $\mathcal{A}^*$ to be the (unique) smallest $\mathcal{H}^*$-subalgebra of $\mathcal{A}^*$. Furthermore, the small quotient of $\mathcal{A}^*$ is defined to be $\mathcal{Q}^*_{\mathrm{small}} := \mathcal{A}^*_{\mathrm{small}}/(\mathcal{A}^*_{\mathrm{small}})_\perp$.

In general, $\mathcal{A}^*_{\mathrm{small}}$ may be of infinite type. However, in the *simply connected* case, it is not, as the following shows.

Proposition 11.4.3 ([5, Prop. 3.3]) *Let $\mathcal{A}^*$ be a simply connected Poincaré DGCA with a Hodge type decomposition (11.21). Then the small algebra $\mathcal{A}^*_{\mathrm{small}}$ is given recursively as*

$$\begin{cases} \mathcal{A}^0_{\mathrm{small}} &= \mathbb{F}\cdot 1_{\mathcal{A}^*}, \\ \mathcal{A}^1_{\mathrm{small}} &= 0, \\ \mathcal{A}^k_{\mathrm{small}} &= dd^-\widehat{\mathcal{A}}^k \oplus \mathcal{H}^k \oplus d^-\widehat{\mathcal{A}}^k, \qquad k \geq 2, \end{cases} \tag{11.26}$$

where, for given $l \geq 2$,

$$\widehat{\mathcal{A}}^l := \mathrm{span}\{\mathcal{A}^{l_1}_{\mathrm{small}} \cdot \mathcal{A}^{l_2}_{\mathrm{small}} \mid l_1, l_2 \geq 2, l_1 + l_2 = l\}$$

depends on $\mathcal{A}^2_{\mathrm{small}}, \ldots, \mathcal{A}^{l-2}_{\mathrm{small}}$ only. In particular, $\mathcal{A}^*_{\mathrm{small}}$ is of finite type, that is, $\dim \mathcal{A}^k_{\mathrm{small}} < \infty$ for all k, and the small quotient $\mathcal{Q}^*_{\mathrm{small}}$ of $\mathcal{A}^*$ is finite dimensional.

Proof It is straightforward to verify that the space given in (11.26) is an $\mathcal{H}^*$-subalgebra of $\mathcal{A}^*$, and, conversely, it must be contained in any $\mathcal{H}^*$-subalgebra of $\mathcal{A}^*$, showing that (11.26) indeed defines the small algebra. The finite dimensionality of $\mathcal{A}^k_{\mathrm{small}}$ follows from induction on k. As $\mathcal{A}^*_{\mathrm{small}}$ surjects to $\mathcal{Q}^*_{\mathrm{small}}$, it follows that $\mathcal{Q}^*_{\mathrm{small}}$ is of finite type as well; on the other hand, as the Poincaré- pairing on $\mathcal{Q}^*_{\mathrm{small}}$ is non-degenerate, it follows that $\mathcal{Q}^k_{\mathrm{small}} = 0$ for $k > n$, whence $\mathcal{Q}^*_{\mathrm{small}}$ is finite dimensional. □

As an important consequence of this, we obtain the following result.

Theorem 11.4.4 ([5, Cor. 3.5]) *Any simply connected Poincaré DGCA of Hodge type is weakly equivalent to a finite dimensional DGCA with a non-degenerate Poincaré pairing.*

Proof By definition, the restriction of the Poincaré pairing on $\mathcal{A}^*$ to $\mathcal{A}^*_{\mathrm{small}}$ is a Poincaré pairing on $\mathcal{A}^*_{\mathrm{small}}$ with the Hodge type decomposition given in (11.26). In particular, the inclusion $\mathcal{A}^*_{\mathrm{small}} \hookrightarrow \mathcal{A}^*$ is a quasi-isomorphism. Moreover, Corollary 11.3.8 implies that the canonical projection to the quotient $\mathcal{Q}^*_{\mathrm{small}} = \mathcal{A}^*_{\mathrm{small}}/(\mathcal{A}^*_{\mathrm{small}})_\perp$ is a quasi-isomorphism as well, whence the maps

$$\mathcal{A}^* \longleftarrow \mathcal{A}^*_{\mathrm{small}} \longrightarrow \mathcal{Q}^*_{\mathrm{small}}$$

are quasi-isomorphisms, showing that $\mathcal{A}^*$ is weakly equivalent to $\mathcal{Q}^*_{\mathrm{small}}$. □

Let us now use the description of $\mathcal{A}^*_{\mathrm{small}}$ and $\mathcal{Q}^*_{\mathrm{small}}$ in the case where $\mathcal{A}^*$ is $(r-1)$-connected, $r > 1$, i.e., $H^k(\mathcal{A}^*) = 0$ for $k = 1, \ldots, r-1$.

In this case, it follows from the recursion formula in Proposition 11.4.3 that

$$\mathcal{A}^l_{\text{small}} = \begin{cases} \mathcal{H}^l, & l = 0, \dots, 2r-2, \\ \mathcal{H}^{2r-1} \oplus d^-(\mathcal{H}^r \cdot \mathcal{H}^r), & l = 2r-1. \end{cases} \tag{11.27}$$

Thus, since $\dim \mathcal{Q}^{n-l}_{\text{small}} = \dim \mathcal{Q}^l_{\text{small}} \leq \dim \mathcal{A}^l_{\text{small}}$ and $\mathcal{Q}^l_{\text{small}} \supset \mathcal{H}^l$, it follows that

$$\begin{aligned} \mathcal{Q}^l_{\text{small}} &= \mathcal{H}^l, \qquad l = 0, \dots, 2r-2, \quad l = n-2r+2, \dots, n, \\ \mathcal{Q}^{2r-1}_{\text{small}} &= \mathcal{H}^{2r-1} \oplus d^-(\mathcal{H}^r \cdot \mathcal{H}^r), \end{aligned} \tag{11.28}$$

and we obtain the following result.

Corollary 11.4.5 ([5, Cor. 3.11]) *Let $\mathcal{A}^*$ be an $(r-1)$-connected $(r > 1)$ Poincaré DGCA of Hodge type of degree n. Then $\mathcal{A}^*$ is weakly equivalent to a finite dimensional non-degenerate Poincaré DGCA $\mathcal{Q}^*_{\text{small}}$ for which the differential $d : \mathcal{Q}^{k-1}_{\text{small}} \to \mathcal{Q}^k_{\text{small}}$ is possibly nonzero only for $2r \leq k \leq n-2r+1$. In particular,*

1 if $n \leq 4r-2$, then $\mathcal{A}^*$ is formal;
2 if $n = 4r-1$, then $d : \mathcal{Q}^{k-1}_{\text{small}} \to \mathcal{Q}^k_{\text{small}}$ vanishes for $k \neq 2r$.

Remark 11.4.6 1. The first statement of Corollary 11.4.5 has been shown by Miller in [7] using the Quillen's functor. In particular, it implies that any closed simply connected manifold of dimension ≤ 6 is formal.

2. A DGCA which is weakly equivalent to a DGCA whose differential vanishes in all but one degree is called *almost formal*. That is, Corollary 11.4.5 (2) shows that an $(r-1)$-connected $(r > 1)$ Poincaré DGCA of Hodge type of degree $n = 4r-1$ is almost formal.

For instance, the case $r := 2$ implies that any simply connected closed 7-manifold is almost formal; this may be compared with [2, Thm. 4.10], which states that closed G_2-manifolds are almost formal.

3. Let M be a closed n-manifold admitting a Riemannian metric of non-negative Ricci curvature (or, more general, such that all harmonic 1-forms are parallel). Using the Cheeger–Gromoll splitting theorem, one can show that $\Omega^*(M)$ is weakly equivalent to $\mathcal{A}^* \otimes \Lambda^*(H^1(M))$, where $\mathcal{A}^*$ is a simply connected Poincaré DGCA of degree $n - b_1(M)$ [5, Prop. 5.3].

Therefore, if $n - b_1(M) \leq 6$, then, by Corollary 11.4.5(1), $\mathcal{A}^*$ is formal and hence $\mathcal{A}^* \otimes \Lambda^*(H^1(M))$ is formal, whence so is M. In particular, this applies to G_2-manifolds with holonomy properly contained in G_2, as these are Ricci-flat and have $b_1(M) > 0$. This generalizes [2, Thm. 4.10] in the non-simply connected case as well.

11.5 Tight DGCAs of Highly Connected DGCAs

In this section, we shall assume throughout that H^* is an $(r-1)$-connected $(r > 1)$ Poincaré GCA of degree $n \leq 5r - 3$. Motivated by the definition of the small quotient algebra, we introduce the notion of a *tight Poincaré DGCA* (see Definition 11.5.1). We shall give an explicit construction of a tight representative in each weak equivalence class of such algebras.

Definition 11.5.1 A Poincaré DGCA $\mathcal{Q}^*$ is called *tight* if it is of Hodge type with a non-degenerate Poincaré pairing $\langle -, - \rangle$, and if there is no proper quasi-isomorphically embedded sub-DGCA $\hat{\mathcal{Q}}^* \hookrightarrow \mathcal{Q}^*$.

In order to describe the construction, consider a graded vector space

$$\left(\mathcal{V}^* = \bigoplus_{k=2r}^{n+1-2r} \mathcal{V}^k, \langle\!\langle -, - \rangle\!\rangle \right) \tag{11.29}$$

with a non-degenerate graded bilinear form $\langle\!\langle -, - \rangle\!\rangle$ of degree $n+1$.

Let $\mathcal{B}^* := (\mathcal{V}^*)^\vee[n]$, and define $\mathcal{Q}^* := \mathcal{V}^* \oplus H^* \oplus \mathcal{B}^*$ as a vector space. We extend the Poincaré pairing $\langle -, - \rangle$ on $\mathcal{H}^* \cong H^*$ to a non-degenerate pairing of degree n on $\mathcal{Q}^*$ by

$$\begin{aligned} \langle \mathcal{H}^*, \mathcal{V}^* \oplus \mathcal{B}^* \rangle \quad &= \langle \mathcal{V}^*, \mathcal{V}^* \rangle = \langle \mathcal{B}^*, \mathcal{B}^* \rangle = 0, \\ \langle \mathcal{B}^*, \mathcal{V}^* \rangle, \quad &\text{the evaluation map.} \end{aligned} \tag{11.30}$$

Since both $\langle -, - \rangle$ and $\langle\!\langle -, - \rangle\!\rangle$ are non-degenerate of degree n and $n+1$, respectively, it follows that there is an isomorphism $d^- : \mathcal{V}^* \to \mathcal{B}^*[1]$ satisfying (11.9), and we denote its inverse by $d : \mathcal{B}^* \to \mathcal{V}^*[-1]$.

Extending d and d^- to all of $\mathcal{Q}^*$ by requiring them to vanish on $\mathcal{V}^* \oplus \mathcal{H}^*$ and on $\mathcal{H}^* \oplus \mathcal{B}^*$, respectively, it follows that $d^2 = 0$ and $(d^-)^2 = 0$, and that $\mathcal{V}^* = d\mathcal{Q}^*$, $\mathcal{B}^* = d^- \mathcal{Q}^*$. That is, we have the decomposition

$$\mathcal{Q}^* = \mathcal{V}^* \oplus \mathcal{H}^* \oplus \mathcal{B}^*, \qquad \mathcal{V}^* = d\mathcal{Q}^*, \qquad \mathcal{B}^* = d^- \mathcal{Q}^*. \tag{11.31}$$

We denote the identification $\mathcal{H}^* \leftrightarrow H^*$ by $h \leftrightarrow [h]$; in particular, $[h_1] \cdot [h_2] \in H^* \cong \mathcal{H}^*$ refers to the multiplication in H^*.

Given a graded map $\bar{\xi} : S^2(H^*) \cong S^2(\mathcal{H}^*) \to \mathcal{V}^*$, we define a product on $\mathcal{Q}^*$ by

$$k_1 \cdot k_2 = \langle k_1, k_2 \rangle \mathrm{vol}_n \qquad \text{for } k_i \in \mathcal{V}^* \oplus \mathcal{B}^*, \tag{11.32}$$
$$\mathcal{V}^* \cdot \mathcal{H}^l = 0 \qquad \text{for } l > 0, \tag{11.33}$$
$$h_1 \cdot h_2 = [h_1] \cdot [h_2] + \bar{\xi}(h_1, h_2) \qquad \text{for } h_i \in \mathcal{H}^*, \tag{11.34}$$
$$\mathcal{B}^* \cdot \mathcal{H}^l \subset \mathcal{H}^* \qquad \text{for } l > 0, \tag{11.35}$$
$$\langle d^- k \cdot h_1, h_2 \rangle = \langle\!\langle k, \bar{\xi}(h_1, h_2) \rangle\!\rangle \qquad \text{for } k \in \mathcal{V}^*,\ h_i \in \mathcal{H}^*. \tag{11.36}$$

In (11.32), $\mathrm{vol}_n \in \mathcal{H}^n$ denotes the (unique) element for which $\int \mathrm{vol}_n = 1$.

Observe that (11.32) implies

$$d\mathcal{Q}^* \cdot (\mathcal{Q}^*)_d = \mathcal{V}^* \cdot (\mathcal{V}^* \oplus \mathcal{H}^*) = \mathcal{B}^* \cdot \mathcal{B}^* = 0. \tag{11.37}$$

It is now straightforward to verify that, with this product, $\mathcal{Q}^*$ becomes a Poincaré DGCA with Hodge type decomposition (11.31) and the non-degenerate Poincaré pairing $\langle -, - \rangle$, and $\bar{\xi} = \xi_{\mathcal{H}^*}$ is the cocycle of the harmonic subspace $\mathcal{H}^*$ defined in (11.14).

Lemma 11.5.2 *Let $\mathcal{Q}^*$ be a Poincaré DGCA of Hodge type with a non-degenerate Poincaré pairing $\langle -, - \rangle$, and suppose that $\mathcal{Q}^k \cong \mathcal{H}^k$ for all $k \leq 2r-2$. Then $\mathcal{V}^* := d\mathcal{Q}^*$ is of the form (11.29) with $\langle\!\langle -, - \rangle\!\rangle$ from (11.9), and there is a Hodge type decomposition (11.31) such that the product structure on $\mathcal{Q}^*$ is given by (11.32)–(11.36), where $\bar{\xi} := \xi_{\mathcal{H}^*} : S^2(H^*) \to \mathcal{V}^*$ is the cocycle map defined in (11.14).*

Proof Pick a Hodge type decomposition (11.31) of $\mathcal{Q}^*$. It is immediate from (11.9) that the pairing $\langle\!\langle -, - \rangle\!\rangle$ on $\mathcal{V}^* = d\mathcal{Q}^*$ is non-degenerate. Furthermore, $\mathcal{V}^k = d\mathcal{Q}^{k-1}$, whence $\mathcal{V}^k = 0$ for $k \leq 2r - 1$. The non-degenericity of $\langle\!\langle -, - \rangle\!\rangle$ implies that $\mathcal{V}^k \cong (\mathcal{V}^{n+1-k})^\vee$, so that, in particular, $\mathcal{V}^k = 0$ for $k \geq n + 2 - 2r$, i.e., $\mathcal{V}^*$ is of the form (11.29).

The non-degenericity of $\langle -, - \rangle$ and the orthogonality relations (11.12) imply that $\mathcal{B}^* \cong (\mathcal{V}^*)^\vee[n]$, and the Poincaré pairing $\langle -, - \rangle$ corresponds to the pairing from (11.31) under this identification. In particular, $\mathcal{Q}^k \cong (\mathcal{Q}^{n-k})^\vee$, so that

$$\mathcal{Q}^k \cong \mathcal{H}^k \cong (\mathcal{H}^{n-k})^\vee, \qquad \text{for } k \leq 2r - 2 \text{ and } k \geq n - 2r + 2. \tag{11.38}$$

Since $\mathcal{V}^* \oplus \mathcal{B}^*$ has only elements of degree $\geq 2r-1$, and by assumption $2(2r-1) \geq n - (r-1)$, it follows that

$$(\mathcal{V}^* \oplus \mathcal{B}^*) \cdot (\mathcal{V}^* \oplus \mathcal{B}^*) \subset \bigoplus_{k=2(2r-1)}^{n} \mathcal{Q}^k = \mathcal{Q}^n = \mathbb{F}\mathrm{vol}_n,$$

where the latter follows from (11.38) and $\mathcal{H}^k = 0$ for $k = 1, \dots, r-1$. Thus, (11.32) follows from (11.6).

By (11.29) it follows that

$$\mathcal{H}^l \cdot (\mathcal{V}^* \oplus \mathcal{B}^*) \subset \bigoplus_{k=2r+l-1}^{n} \mathcal{Q}^k.$$

If $l > 0$ then $\mathcal{H}^l \neq 0$ only if $l \geq r$, in which case $2r + l - 1 \geq 3r - 1 \geq n - (2r-2)$, so that (11.38) implies $\mathcal{Q}^k = \mathcal{H}^k$ for all $k \geq 2r + l - 1$. That is, $\mathcal{H}^l \cdot (\mathcal{V}^* \oplus \mathcal{B}^*) \subset \mathcal{H}^*$, showing (11.35). Also, $\mathcal{K}^* \subset (\mathcal{Q}^*)_d$ is an ideal, so that $\mathcal{H}^l \cdot \mathcal{K}^* \subset \mathcal{K}^* \cap \mathcal{H}^* = 0$ which shows (11.33).

Equation (11.34) follows immediately from the definition of $\bar{\xi} = \xi_{\mathcal{H}^*}$ in (11.14), and to show (11.36), let $k \in \mathcal{V}^*$ and $h_1, h_2 \in \mathcal{H}^*$. Then

$$\langle d^- k \cdot h_1, h_2 \rangle \overset{(11.8)}{=} \langle d^- k, h_1 \cdot h_2 \rangle \overset{(11.34)}{=} \langle d^- k, \bar{\xi}([h_1],[h_2]) \rangle = \langle\langle k, \bar{\xi}([h_1],[h_2]) \rangle\rangle.$$

□

Lemma 11.5.3 *There is a one-to-one correspondence of Poincaré DGCAs $\mathcal{Q}^*$ of Hodge type with $\mathcal{Q}^k \cong \mathcal{H}^k$ for all $k \leq 2r-2$ and a non-degenerate Poincaré pairing $\langle -, - \rangle$, and linear maps $\xi : \mathcal{K}^* \to d\mathcal{Q}^*$. This correspondence is given by the restriction of the cocycle map $\xi_{\mathcal{H}^*} : S^2(H^*) \to d\mathcal{Q}^*$ from (11.14) to $\mathcal{K}^* \subset S^2(H^*)$.*

Proof If $\mathcal{Q}^*$ is as requested, Lemma 11.5.2 implies that $\mathcal{Q}^*$ has a Hodge type decomposition (11.31), and the product is determined by (11.32)–(11.36) for the cocycle map $\bar{\xi} := \xi_{\mathcal{H}^*}$.

Let $\hat{\mathcal{H}}^* \subset \mathcal{Q}^*_d$ be another harmonic subspace which is hence of the form (11.15) for some linear map $\beta : H^* \to \mathcal{V}^*$. Since $\mathcal{Q}^k \cong \mathcal{H}^k$, and hence $d^- \mathcal{V}^{k+1} = 0$ for $k \leq 2r-2$ and $d^- : \mathcal{V}^{k+1} \to \mathcal{B}^k$ is an isomorphism, it follows that $\mathcal{V}^k = 0$ for $k < 2r$, whence $|\beta(h)| \geq 2r$. Thus, if $h_1, h_2 \in H^*$ are not 1 and hence of degree $\geq r$, it follows that $|h_1 \cdot \beta(h_2)| \geq 3r > n + 1 - 2r$, whence by (11.29) it follows that $h_1 \cdot \beta(h_2) = 0$ and likewise $\beta(h_1) \cdot h_2 = 0$. Also, $\beta(h_1) \cdot \beta(h_2) \in \mathcal{V}^* \cdot \mathcal{V}^* = 0$. That is,

$$\xi_{\hat{\mathcal{H}}^*}([h_1],[h_2]) = \xi_{\mathcal{H}^*}([h_1],[h_2]) - \beta([h_1] \cdot [h_2]) \tag{11.39}$$

by (11.17). If we now define $\hat{\mathcal{B}}^*$ by (11.16), then we obtain the Hodge type decompositon

$$\mathcal{Q}^* = \mathcal{V}^* \oplus \hat{\mathcal{H}}^* \oplus \hat{\mathcal{B}}^*, \tag{11.40}$$

and the DGCA structure on $\mathcal{Q}^*$ is again determined by (11.32)–(11.36), replacing $\mathcal{H}^*$ by $\hat{\mathcal{H}}^*$ and $\mathcal{B}^*$ by $\hat{\mathcal{B}}^*$ and $\bar{\xi} := \xi_{\mathcal{H}^*}$ by $\bar{\xi}' := \xi_{\hat{\mathcal{H}}^*}$ from (11.40), respectively.

Conversely, $\mathcal{Q}^*_d = \mathcal{V}^* \oplus \mathcal{H}^*$ is a central extension of H^* by (11.37) whose Hochschild cocycle is given by $\bar{\xi} = \xi_{\mathcal{H}^*}$. Therefore, replacing $\bar{\xi}$ by $\bar{\xi}'$ yields an isomorphic DGCA-structure if and only if $\bar{\xi}' - \bar{\xi}$ is a Hochschild coboundary, implying that $\bar{\xi}' := \xi_{\hat{\mathcal{H}}^*}$ is given in (11.39) for some $\beta : H^* \to \mathcal{V}^*$. That is, two maps $\bar{\xi}'$ and $\bar{\xi}$ in (11.32)–(11.36) yield isomorphic DGCAs if and only if $(\bar{\xi}' - \bar{\xi})(h_1, h_2) = \beta(h_1 \cdot h_2)$ for some $\beta : H^* \to \mathcal{V}^*$, and clearly, this is the case if and only if $\xi := \bar{\xi}_{|\mathcal{K}^*} = \bar{\xi}'_{|\mathcal{K}^*}$; that is, $\mathcal{Q}^*$ is determined up to isomorphism by $\xi : \mathcal{K}^* \to \mathcal{V}^* = d\mathcal{Q}^*$, and, since any such ξ is the restriction of some map $\bar{\xi} : S^2(H^*) \to \mathcal{V}^*$, the assertion follows. □

Lemma 11.5.4 *Let $\mathcal{Q}^*$ be the Poincaré DGCA of Hodge type with $\mathcal{Q}^k \cong \mathcal{H}^k$ for all $k \leq 2r - 2$ and a non-degenerate Poincaré pairing $\langle -, - \rangle$ corresponding to the map $\xi : \mathcal{K}^* \to d\mathcal{Q}^*$ by Lemma 11.5.3. Then $\mathcal{Q}^*$ is tight if and only if ξ is surjective.*

Proof Pick a decomposition $S^2(H^*) = \mathcal{K}^* \oplus \mathcal{N}^*$, and define the map $\bar{\xi} : S^2(H^*) \to \mathcal{V}^*$ by $\bar{\xi}_{|\mathcal{K}^*} = \xi$ and $\bar{\xi}_{|\mathcal{N}^*} = 0$.

According to Lemma 11.5.3, there is a Hodge type decomposition (11.40) of $\mathcal{Q}^*$ such that the product is given by (11.32)–(11.36) for the map $\bar{\xi}$ from above. It follows that

$$\hat{\mathcal{Q}}^* := \bar{\xi}(S^2(H^*)) \oplus \hat{\mathcal{H}}^* \oplus d^-\bar{\xi}(S^2(H^*)) = \xi(\mathcal{K}^*) \oplus \mathcal{H}^* \oplus d^-\xi(\mathcal{K}^*) \subset \mathcal{Q}^*$$

is a sub-DGCA of $\mathcal{Q}^*$ whose inclusion $\hat{\mathcal{Q}}^* \hookrightarrow \mathcal{Q}^*$ is a quasi-isomorphism. If ξ is not surjective, then $\hat{\mathcal{Q}}^* \subsetneq \mathcal{Q}^*$ is a proper quasi-isomorphically embedded sub-DGCA, showing that $\mathcal{Q}^*$ is not tight.

Conversely, suppose that ξ is surjective, and let $\hat{\mathcal{Q}}^* \subset \mathcal{Q}^*$ be a quasi-isomorphically embedded sub-DGCA. Then $\hat{\mathcal{Q}}^*$ contains a harmonic subspace $\mathcal{H}^* \subset \mathcal{Q}^*_d$. We may choose the Hodge type decomposition (11.40) of $\mathcal{Q}^*$ containing $\mathcal{H}^*$ as a factor, whence the product on $\mathcal{Q}^*$ is given by (11.32)–(11.36) for some map $\bar{\xi}$ which extends ξ, so that, in particular, $\bar{\xi}$ is surjective as well. Therefore, (11.34) and $\mathcal{H}^* \subset \mathcal{Q}^*$ imply that $\bar{\xi}(S^2(H^*)) = \mathcal{V}^* \subset \hat{\mathcal{Q}}^*$. Since the inclusion map $\hat{\mathcal{Q}}^* \hookrightarrow \mathcal{Q}^*$ is a quasi-isomorphism, it follows that each $k \in \mathcal{V}^*$ is exact in $\hat{\mathcal{Q}}^*$, implying that $d^-\mathcal{V}^* = \mathcal{B}^* \subset \hat{\mathcal{Q}}^*$, whence $\hat{\mathcal{Q}}^* = \mathcal{Q}^*$. This shows that $\mathcal{Q}^*$ is tight. □

Proof of Theorem 11.1.1 Let $\beta = \langle\!\langle -, - \rangle\!\rangle_{\mathcal{K}^*}$ on $\mathcal{K}^*$ be a symmetric bilinear form of degree $n+1$, and define the quotient space $\mathcal{V}^* := \mathcal{K}^*/\mathcal{K}^*_\perp$ with the canonical projection $\xi : \mathcal{K}^* \to \mathcal{V}^*$. Evidently, there is an induced non-degenerate symmetric pairing $\beta_{\mathcal{V}^*}$ on $\mathcal{V}^*$ satisfying

$$\beta_{\mathcal{V}^*}(\xi(k_1), \xi(k_2)) = \beta(k_1, k_2). \tag{11.41}$$

Since H^* is $(r-1)$-connected, it follows that $\mathcal{K}^* \subset S^2(H^*)$ has only elements of degree $\geq 2r$, whence so does $\mathcal{V}^*$. By the non-degenericity of $\beta_{\mathcal{V}^*}$ it follows that $\mathcal{V}^k \cong (\mathcal{V}^{n+1-k})^\vee$ and, in particular, $\mathcal{V}^k = 0$ for $k > n + 1 - 2r$. Therefore, $\mathcal{V}^*$ is of the form (11.29).

Pick an extension $\bar{\xi} : S^2(H^*) \to \mathcal{V}^*$ of ξ and define the algebra $\mathcal{Q}^*_\beta$ by (11.31) with the product given by (11.32)–(11.36). The surjectivity of ξ implies that $\mathcal{Q}^*_\beta$ is a tight DGCA by Lemma 11.5.4, and it is independent of the choice of the extension $\bar{\xi}$ of ξ by Lemma 11.5.3. That is, to a given $\beta \in S^2(\mathcal{K}^*)^\vee$ we have associated a tight DGCA $\mathcal{Q}^*_\beta$ with cohomology H^*.

For the converse, let $\mathcal{Q}^*$ be tight and choose a Hodge type decomposition (11.31). As $\mathcal{Q}^*_{\text{small}} \hookrightarrow \mathcal{Q}^*$ is quasi-isomorphically embedded, it follows that $\mathcal{Q}^*_{\text{small}} = \mathcal{Q}^*$, whence by (11.28), the $(r-1)$-connectedness of $\mathcal{Q}^*$ implies that $\mathcal{Q}^k = \mathcal{H}^k$ for $k \leq 2r - 2$. Thus, Lemma 11.5.2 implies that multiplication in $\mathcal{Q}^*$ is given by (11.32)–(11.36) for some decomposition (11.31) and some cocycle map $\bar{\xi} := \xi_{\mathcal{H}^*} : S^2(H^*) \to \mathcal{V}^*$, and the pairing $\beta_{\mathcal{V}^*} := \langle\langle -, - \rangle\rangle$ on $\mathcal{V}^* = d\mathcal{A}^*$ defined in (11.9) is non- degenerate.

The tightness of $\mathcal{Q}^*$ and Lemma 11.5.4 implies that $\xi := \bar{\xi}_{|\mathcal{K}^*}$ is surjective, whence the pull-back $\beta := \xi^*(\beta_{\mathcal{V}^*}) \in (S^2(\mathcal{K}^*))^\vee$ satisfies (11.41), so that $\mathcal{V}^* = \mathcal{K}^*/\mathcal{K}^*_\perp$. Therefore, $\mathcal{Q}^* = \mathcal{Q}^*_\beta$. □

11.6 The Bianchi–Massey Tensor

In this section, we recall the definition and basic properties of the Bianchi–Massey tensor of a DGCA $\mathcal{A}^*$ introduced by Crowley–Nordström [3], and apply it to the tight DGCAs $\mathcal{Q}^*_\beta$ constructed in Section 11.5.

Let $\mathcal{K}^* \subset S^2(H^*)$ be the kernel of the multiplication map (11.18). We define the space $\mathcal{E}^* \subset S^2(\mathcal{K}^*) \subset S^2(S^2(H^*))$ as the kernel

$$\begin{array}{ccccccc} & & & & S^2(S^2(H^*)) & & \\ & & & \nearrow & & \searrow & \\ 0 \longrightarrow & \mathcal{E}^* & \longrightarrow & S^2(\mathcal{K}^*) & \longrightarrow & & S^4(H^*) \end{array} \tag{11.42}$$

where the bottom row is exact, and the diagonal maps are induced by the inclusion $\mathcal{K}^* \hookrightarrow S^2(H^*)$ and the multiplication map $S^2(S^2(H^*)) \to S^4(H^*)$, respectively. If we wish to emphasize the dependence of $\mathcal{K}^*$ and

$\mathcal{E}^*$ on the cohomology ring $H^* = H^*(\mathcal{A}^*)$, then we shall denote them by $\mathcal{K}^*_{\mathcal{A}^*}$ and $\mathcal{E}^*_{\mathcal{A}^*}$, respectively.

Given a Poincaré DGCA $\mathcal{A}^*$ with cohomology ring H^*, let $\mathcal{H}^* = \imath(H^*) \subset \mathcal{A}^*_d$ be a harmonic subspace, where $\imath$ is a right inverse of the projection $\mathcal{A}^*_d \to H^*$. Then multiplication induces maps $m_k : S^k(H^*) \to \mathcal{A}^*_d$, $m_k(x_1, \dots, x_k) := \imath(x_1) \cdots \imath(x_k)$, and, evidently, $\mathcal{K}^* = m_2^{-1}(dA^*)$. Therefore, we may choose a map $\varepsilon : \mathcal{K}^* \to \mathcal{A}^*[1]$ such that

$$m_2(\varphi) = d\varepsilon(\varphi), \qquad \varphi \in \mathcal{K}^*, \tag{11.43}$$

and furthermore we define the map

$$\hat{\varepsilon} : S^2(\mathcal{K}^*) \longrightarrow \mathcal{A}^*[1], \qquad (\varphi \circ \psi) \longmapsto \varepsilon(\varphi) \cdot m_2(\varphi) = \varepsilon(\varphi) \cdot d\varepsilon(\psi). \tag{11.44}$$

Observe that $d\hat{\varepsilon}(\varphi \circ \psi) = m_2(\varphi) \cdot m_2(\psi) = m_4(\varphi \circ \psi)$. That is, for $e \in \mathcal{E}^* \subset S^2(\mathcal{K}^*)$ we have $d\hat{\varepsilon}(e) = 0$, so that projection onto cohomology yields a map

$$\mathcal{BM}_{\mathcal{A}^*} : \mathcal{E}^* \longrightarrow H^*[1], \qquad \mathcal{BM}_{\mathcal{A}^*}(e) := [\hat{\varepsilon}(e)],$$

which is the *Bianchi–Massey tensor of* $\mathcal{A}^*$ (see [3, Def. 1.1]).

Since ε is uniquely determined up to adding closed elements, it follows that $\hat{\varepsilon}$ is well defined up to adding exact elements, whence $\mathcal{BM}_{\mathcal{A}^*}$ is well defined, independently of the choice of ε. Moreover, it is natural in the sense that for a DGCA-morphism $f : \mathcal{A}^* \to \mathcal{B}^*$ there is an induced commutative diagram

$$\begin{array}{ccc} \mathcal{E}^*_{H^*(\mathcal{A}^*)} & \xrightarrow{\mathcal{BM}_{\mathcal{A}^*}} & H^*(\mathcal{A}^*)[1] \\ \downarrow f_* & & \downarrow f_* \\ \mathcal{E}^*_{H^*(\mathcal{B}^*)} & \xrightarrow{\mathcal{BM}_{\mathcal{B}^*}} & H^*(\mathcal{B}^*)[1] \end{array}$$

Here we denote both the cohomology morphism $H^*(\mathcal{A}^*) \to H^*(\mathcal{B}^*)$ induced by f and its extension to the symmetric tensor algebra

$$S^*(H^*(\mathcal{A}^*)) \to S^*(H^*(\mathcal{B}^*))$$

by the same symbol f_*. In particular, if f is a quasi-isomorphism, then f_* canonically identifies the Bianchi–Massey tensors, so that these are invariants of the weak equivalence class of $\mathcal{A}^*$.

Remarkably, in the case in which $\mathcal{A}^*$ is highly connected, $\mathcal{BM}_{\mathcal{A}^*}$ uniquely determines the weak equivalence class of $\mathcal{A}^*$. Namely, Crowley–Nordström showed the following.

Theorem 11.6.1 ([3, Thm. 1.3]) *Let H^* be an $(r-1)$-connected $(r > 1)$ Poincaré GCA of degree $n \leq 5r-3$. Then two DGCAs $\mathcal{A}_i^*$, $i = 1, 2$, with cohomology H^* are weakly equivalent if and only if their Bianchi–Massey tensors*

$$\mathcal{BM}_{\mathcal{A}_i^*} : \mathcal{E}^{n+1} \longrightarrow H^n = \mathbb{F}\mathrm{vol}_n$$

coincide.

We say that a bilinear form $\hat{\beta}$ on $S^2(H^*)$ is of *Riemannian type* if it satisfies for all homogeneous $h_i \in H^{|h_i|}$ the symmetry relation

$$\hat{\beta}(h_1h_2, h_3h_4) = -(-1)^{|h_2||h_3|}\hat{\beta}(h_1h_3, h_2h_4). \tag{11.45}$$

Furthermore we say that a bilinear form β on $\mathcal{K}^*$ is of Riemannian type if $\beta = \hat{\beta}_{|\mathcal{K}^*}$ for $\hat{\beta}$ a bilinear form on $S^2(H^*)$ of Riemannian type. This terminology is due to the fact that tensors of Riemannian type satisfy all (graded) symmetries of a Riemannian curvature tensor.

It follows that there is a decomposition $S^2(S^2(H^*))^\vee = \imath(S4(H^*)) \oplus \mathcal{R}(H^*)$, where $\imath : S^4(H^*)^\vee \to S^2(S^2(H^*))^\vee$ is the dual of the multiplication map and $\mathcal{R}(H^*)$ is the space of bilinear forms of Riemannian type.

Proof of Theorem 11.1.2 We need to compute the Bianchi–Massey tensors $\mathcal{BM}_{\mathcal{Q}_\beta^*}$. By construction, the product structure of $\mathcal{Q}_\beta^*$ is defined by (11.32)–(11.36) for some extension $\bar{\xi} : S^2(H^*) \to \mathcal{V}^*$ of the canonical projection $\xi : \mathcal{K}^* \to \mathcal{V}^*$. Thus, when defining the Bianchi–Massey tensor, the map $\varepsilon : \mathcal{K}^* \to \mathcal{Q}_\beta^*[1]$ from (11.43) may be chosen as

$$\varepsilon(k) := d^-\xi(k).$$

Let $e \in \mathcal{E}^* \subset S^2(\mathcal{K}^*)$ and write it as $e = \sum_i k_1^i \circ k_2^i \in \mathcal{E}^* \subset S^2(\mathcal{K}^*)$ with $k_j^i \in \mathcal{K}^*$. Then, by (11.44),

$$\hat{\varepsilon}(e) = \sum_i d^- k_1^i \circ k_2^i = \sum_i \langle\langle k_1^i, k_2^i \rangle\rangle \mathrm{vol}_n = \beta(e)\mathrm{vol}_n,$$

that is, the Bianchi–Massey tensor of $\mathcal{Q}_\beta^*$ is determined by the restriction of β to $\mathcal{E}^* \subset S^2(\mathcal{K}^*)$:

$$\mathcal{BM}_{\mathcal{Q}_\beta^*}(e) = \beta(e)\mathrm{vol}_n \qquad \text{for all } e \in \mathcal{E}^*. \tag{11.46}$$

Since any element in $(\mathcal{E}^*)^\vee$ can be realized as the restriction of some $\beta \in (S^2(\mathcal{K}^*))^\vee$ of Riemannian type, and since by Theorem 11.6.1 the Bianchi–Massey tensor determines the weak equivalence type, the statements follow. ∎

Example 11.6.2 Let H^* be a simply connected Poincaré algebra of Hodge type of degree 7, so that $r = 2$ and $n = 7 = 5r - 3$. As $\mathcal{K}^*$ has elements of degree $\geq 2r = 4$ only, any bilinear form β of degree $n+1 = 8$ on $\mathcal{K}^*$ is a bilinear form on

$$\mathcal{K}^4 = \ker(\cdot : H^2 \otimes H^2 \longrightarrow H^4).$$

In fact, we may assume that $\beta \in S^2(\mathcal{K}^4)^\vee$ is the restriction of an element in $\mathcal{R}^8 \subset S^2(S^2(H^2))^\vee$ of Riemannian type to $\mathcal{K}^4$.

Then $\mathcal{V}^* = \mathcal{V}^4 = \mathcal{K}^4/\mathcal{K}^4_\perp$, where $\mathcal{K}^4_\perp$ is the null space of β, so that

$$\mathcal{Q}^k_\beta = \begin{cases} \mathcal{H}^k, & \text{for } k \neq 3, 4, \\ \mathcal{H}^3 \oplus d^- \mathcal{V}^4, & \text{for } k = 3, \\ \mathcal{H}^4 \oplus \mathcal{V}^4, & \text{for } k = 4, \end{cases}$$

and the algebra structure of $\mathcal{Q}^*_\beta$ is given by (11.32)–(11.36) for some extension $\bar{\xi} : S^2(H^2) \to \mathcal{V}^4$ of the canonical projection $\xi : \mathcal{K}^4 \to \mathcal{V}^4$.

In particular, the de Rham algebra $\Omega^*(M)$ of a closed simply connected 7-manifold M is weakly equivalent to such an DGCA $\mathcal{Q}^*_\beta$.

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