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Twistor solution of the holonomy problem

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1. Introduction. Holonomy group is one of the most informative characteristics of an affine connection. The problem of classification of holonomy groups has a long history which starts in 1920s with the works of Cartan [9, 10] where he used this notion to classify locally symmetric Riemannian manifolds. In 1955, Berger [2] showed that the list of irreducibly acting matrix Lie groups which can, in principle, occur as the holonomy of a torsion-free affine connection must be very restrictive. This is in a sharp contrast to the result of Hano and Ozeki [13] which says that there is no interesting holonomy classification in the class of arbitrary affine connections — *any* closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ can be realized as the holonomy of an affine connection (with torsion, in general).

Berger presented his classification list of all possible candidates to irreducible holonomies[1] in two parts — the first part was claimed to contain all possible groups which preserve a non-degenerate symmetric bilinear form, while the second part was conjectured to contain all the rest, *up to a finite number of missing terms* which Bryant [6] suggested to call the *exotic holonomies*.

2. Main result. The classification of all metric holonomies has been recently completed [7][2]. The resulting Table 1 is a culmination of efforts of many people to show that most entries of Berger's original metric list do occur as holonomies of Levi-Civita connections and that just a few of them are superious (see, e.g., [1, 4, 5, 6, 7, 17] and the references cited therein).

[1] From now on by a holonomy group we always understand the irreducibly acting holonomy of a *torsion-free* affine connection which is *not* locally symmetric. The second assumption is motivated by the fact that, due to Cartan [10] and Berger [3], the list of locally symmetric affine spaces is completely known.

[2] Note that we have added the 4-dimensional representation of $\mathrm{SO}(2,\mathbb{R})\mathrm{SL}(2,\mathbb{R})$ which got lost in [7] and several other publications. The moduli space of Levi-Cevita connections with this holonomy is non-empty and finite-dimensional [11].

Table 1 Complete list of metric holonomies

Group G	Representation space	Group G	Representation space
$\mathrm{SO}(p,q)$	$\mathbb{R}^{p+q}$, $(p+q)\geqslant 2$	G_2	$\mathbb{R}^7$
$\mathrm{SO}(n,\mathbb{C})$	$\mathbb{R}^{2n}$, $n\geqslant 2$	G_2'	$\mathbb{R}^7$
$\mathrm{U}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q)\geqslant 2$	$\mathrm{G}_2^{\mathbb{C}}$	$\mathbb{R}^{14}$
$\mathrm{SU}(p,q)$	$\mathbb{R}^{2(p+q)}$, $(p+q)\geqslant 2$	$\mathrm{Spin}(7)$	$\mathbb{R}^8$
$\mathrm{Sp}(p,q)\cdot\mathrm{Sp}(1)$	$\mathbb{R}^{4(p+q)}$, $(p+q)\geqslant 2$	$\mathrm{Spin}(4,3)$	$\mathbb{R}^8$
$\mathrm{Sp}(p,q)$	$\mathbb{R}^{4(p+q)}$, $(p+q)\geqslant 2$	$\mathrm{Spin}(7,\mathbb{C})$	$\mathbb{R}^{16}$
$\mathrm{Sp}(n,\mathbb{R})\cdot\mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^{4n}$, $n\geqslant 2$	$\mathrm{SL}(2,\mathbb{R})\mathrm{SO}(2,\mathbb{R})$	$\mathbb{R}^4$
$\mathrm{Sp}(n,\mathbb{C})\cdot\mathrm{SL}(2,\mathbb{C})$	$\mathbb{R}^{8n}$, $n\geqslant 2$		

Berger's second list of non-metric holonomies, refined and extended, is given in Tables 1 and 2. The 4-dimensional representations of $\mathrm{T}_{\mathbb{R}}\cdot\mathrm{SL}(2,\mathbb{R})$, $\mathrm{T}_{\mathbb{C}}\cdot\mathrm{SL}(2,\mathbb{C})$, $\mathbb{R}^*\cdot\mathrm{SO}(2)\cdot\mathrm{SL}(2,\mathbb{R})$ and $\mathbb{C}^*\cdot\mathrm{SU}(2)$, and the fundamental representations of various real forms of $\mathrm{T}_{\mathbb{C}}\cdot\mathrm{Spin}(10,\mathbb{C})$ and $\mathrm{T}_{\mathbb{C}}\cdot\mathrm{E}_6^{\mathbb{C}}$ have been added to the list of non-metric holonomies by Bryant [6, 7]. He also conjectured that the 4-dimensional representations of $H_\lambda\cdot\mathrm{SU}(2)$ and $H_\lambda\cdot\mathrm{SL}(2,\mathbb{R})$ may occur as holonomies. The *infinite* series $\mathrm{SL}(2,\mathbb{R})\cdot\mathrm{SO}(p,q)$ and $\mathrm{SL}(2,\mathbb{C})\cdot\mathrm{SO}(n,\mathbb{C})$ as well as exceptional representations E_7^5, E_7^7, $\mathrm{E}_7^{\mathbb{C}}$ have been found by Chi et al [11, 12]. Finally, the representations the groups

$$\mathrm{Sp}(3,\mathbb{R}),\ \mathrm{Sp}(3,\mathbb{C}),\ \mathrm{SL}(6,\mathbb{R}),\ \mathrm{SL}(6,\mathbb{C}),\ \mathrm{Spin}(2,10),\ \mathrm{Spin}(6,6),\ \mathrm{Spin}(12,\mathbb{C})$$

have been added to the list by Merkulov and Schwachhöfer [15] who showed that the moduli space of torsion-free affine connections with these holonomies is non-empty and finite dimensional. The point is that these representations are the last ones which are missing from the original Berger lists:

Main Theorem *If G is an irreducible representation of a reductive Lie group which occurs as the holonomy of a non locally symmetric torsion-free affine connection, then G is one of the entries in Tables 1-3.*

Moreover, due to the efforts of a number of people over the last 40 years all entries of Tables 1-3 are known to occur as holonomies except the 4-dimensional representations of $H_\lambda\cdot\mathrm{SU}(2)$ and $H_\lambda\cdot\mathrm{SL}(2,\mathbb{R})$ which, as candidates to holonomies, have been suggested by Bryant [7].

3. Twistor theory of holonomy groups. Let V be a vector space and $\mathfrak{g}$ an irreducible Lie subalgebra of $\mathfrak{gl}(V)\simeq V\otimes V^*$. In the holonomy group context,

Table 2 Berger's list of non-metric holonomies

group G	representation V	restrictions
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^n$	$n \geqslant 2$
	$\odot^2\mathbb{R}^n \simeq \mathbb{R}^{n(n+1)/2}$	$n \geqslant 3$
	$\Lambda^2\mathbb{R}^n \simeq \mathbb{R}^{n(n-1)/2}$	$n \geqslant 5$
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^n \simeq \mathbb{R}^{2n}$	$n \geqslant 1$
$\mathrm{T}^*_{\mathbb{C}} \cdot \mathrm{SL}(n,\mathbb{C})$	$\odot^2\mathbb{C}^n \simeq \mathbb{R}^{n(n+1)}$	$n \geqslant 3$
	$\Lambda^2\mathbb{C}^n \simeq \mathbb{R}^{n(n-1)}$	$n \geqslant 5$
$\mathbb{R}^* \cdot \mathrm{SL}(n,\mathbb{C})$	$\{A \in M_n(\mathbb{C}) : \bar{A} = A^t\} \simeq \mathbb{R}^{n^2}$	$n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(n,\mathbb{H})$	$\mathbb{H}^n \simeq \mathbb{R}^{4n}$	$n \geqslant 1$
	$\{A \in M_n(\mathbb{H}) : A^* = -A^t\} \simeq \mathbb{R}^{n(2n+1)}$	$n \geqslant 2$
	$\{A \in M_n(\mathbb{H}) : A^* = A^t\} \simeq \mathbb{R}^{n(2n-1)}$	$n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{Sp}(n,\mathbb{R})$	$\mathbb{R}^{2n}$	$n \geqslant 2$
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{Sp}(n,\mathbb{C})$	$\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$	$n \geqslant 2$
$\mathbb{R}^* \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^{p+q}$	$p+q \geqslant 3$
$\mathrm{T}^*_{\mathbb{C}} \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^n \simeq \mathbb{R}^{2n}$	$n \geqslant 3$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(m,\mathbb{R}) \cdot \mathrm{SL}(n,\mathbb{R})$	$\mathbb{R}^{mn}$	$m > n \geqslant 2$ or $m \geqslant n > 2$
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(m,\mathbb{C}) \cdot \mathrm{SL}(n,\mathbb{C})$	$\mathbb{C}^m \otimes \mathbb{C}^n \simeq \mathbb{R}^{2mn}$	$m > n \geqslant$ or $m \geqslant n > 2$
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(m,\mathbb{H}) \cdot \mathrm{SL}(n,\mathbb{H})$	$\mathbb{R}^{16mn}$	$m > n \geqslant 1$ or $m \geqslant n > 1$
$\mathrm{SU}(2) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{R}^2 \otimes \mathbb{R}^{4n} \simeq \mathbb{R}^{8n}$	$n \geqslant 2$

NOTATIONS: $\mathrm{T}_{\mathbb{F}}$ denotes any connected Lie subgroup of $\mathbb{F}^*$,
$\mathrm{T}^*_{\mathbb{F}}$ denotes any non-trivial connected Lie subgroup of $\mathbb{F}^*$,
$M_n(\mathbb{F})$ denotes the algebra of $n \times n$ matrices with entries in $\mathbb{F}$.

one is interested in the following three $\mathfrak{g}$-modules:

(i) $\mathfrak{g}^{(1)} := (\mathfrak{g} \otimes V^*) \cap (V \otimes \odot^2 V^*)$,

(ii) the *curvature space* $K(\mathfrak{g}) := \ker i_1$, where i_1 is the composition

$$i_1 : \mathfrak{g} \otimes \Lambda^2 V^* \longrightarrow V \otimes V^* \otimes \Lambda^2 V^* \longrightarrow V \otimes \Lambda^3 V^*,$$

(iii) the *2nd curvature space* $K^1(\mathfrak{g}) := \ker i_2$, where i_2 is the composition

$$i_2 : K(\mathfrak{g}) \otimes V^* \longrightarrow \mathfrak{g} \otimes \Lambda^2 V^* \longrightarrow \mathfrak{g} \otimes \Lambda^3 V^*.$$

Note that if ∂ is the composition

$$\mathfrak{g}^{(1)} \otimes V^* \to \mathfrak{g} \otimes V^* \otimes V^* \to \mathfrak{g} \otimes \Lambda^2 V^*$$

Table 3 List of exotic holonomies

group G	representation V	restrictions
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{Spin}(5,5)$	$\mathbb{R}^{16}$	
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{Spin}(1,9)$	$\mathbb{R}^{16}$	
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{Spin}(10,\mathbb{C})$	$\mathbb{C}^{16} \simeq \mathbb{R}^{32}$	
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{E}_6^1$	$\mathbb{R}^{27}$	
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{E}_6^4$	$\mathbb{R}^{27}$	
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{E}_6^{\mathbb{C}}$	$\mathbb{C}^{27} \simeq \mathbb{R}^{54}$	
$\mathrm{T}_{\mathbb{R}} \cdot \mathrm{SL}(2,\mathbb{R})$	$\odot^3\mathbb{R}^2 \simeq \mathbb{R}^4$	
$\mathrm{T}_{\mathbb{C}} \cdot \mathrm{SL}(2,\mathbb{C})$	$\odot^3\mathbb{C}^2 \simeq \mathbb{R}^8$	
$\mathbb{R}^* \cdot \mathrm{SO}(2) \cdot \mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^2 \otimes \mathbb{R}^2 \simeq \mathbb{R}^4$	
$\mathbb{C}^* \cdot \mathrm{SU}(2)$	$\mathbb{C}^2 \simeq \mathbb{R}^4$	
$H_\lambda \cdot \mathrm{SU}(2)$	$\mathbb{R}^4$	
$H_\lambda \cdot \mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}^4$	
$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q} \simeq \mathbb{R}^{2p+2q}$	$p+q>2$
$\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n \simeq \mathbb{R}^{4n}$	$n \geqslant 3$
E_7^5	$\mathbb{R}^{56}$	
E_7^7	$\mathbb{R}^{56}$	
$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{R}^{112} \simeq \mathbb{C}^{56}$	
$\mathrm{Sp}(3,\mathbb{R})$	$\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$	
$\mathrm{Sp}(3,\mathbb{C})$	$\mathbb{R}^{28} \simeq \mathbb{C}^{14} \subset \Lambda^3\mathbb{C}^6$	
$\mathrm{SL}(6,\mathbb{R})$	$\mathbb{R}^{20} \simeq \Lambda^3\mathbb{R}^6$	
$\mathrm{SL}(6,\mathbb{C})$	$\mathbb{R}^{40} \simeq \Lambda^3\mathbb{C}^6$	
$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32}$	
$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32}$	
$\mathrm{Spin}(12,\mathbb{C})$	$\mathbb{R}^{64} \simeq \mathbb{C}^{32}$	

NOTATIONS: $\mathrm{T}_{\mathbb{F}}$ denotes any connected Lie subgroup of $\mathbb{F}^*$,

$$H_\lambda = \left\{ e^{\lambda t} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}, \quad \lambda > 0.$$

then $\partial(\mathfrak{g}^{(1)} \otimes V^*) \subseteq K(\mathfrak{g})$.

The geometric meaning of $\mathfrak{g}^{(1)}$ is that if there exists a (local) torsion-free affine connection ∇ on a manifold M with holonomy algebra $\mathfrak{g}$ then, for any (local) function $\Gamma : M \to \mathfrak{g}^{(1)}$, the affine connection $\nabla + \Gamma$ is again torsion-free and has holonomy algebra $\mathfrak{g}$; put another way, $\mathfrak{g}^{(1)}$ measures non-uniqueness of torsion-free affine connections with holonomy $\mathfrak{g}$ on a fixed manifold.

The meaning of $K(\mathfrak{g})$ and $K^1(\mathfrak{g})$ is that the curvature tensor of a torsion-free affine connection ∇ on a manifold M with holonomy in $\mathfrak{g}$ has the curvature tensor at each $x \in M$ isomorphic to an element of $K(\mathfrak{g})$ while the non-vanishing of $K^1(\mathfrak{g})$ is a necessary condition for ∇ *not* to be locally symmetric.

Therefore, $\mathfrak{g}$ can be a candidate to the holonomy algebra of a torsion-free affine connection only if $K(\mathfrak{g}) \neq 0$. Then the question is how to compute $K(\mathfrak{g})$?

With any real irreducible representation of a real reductive Lie algebra one may associate an irreducible complex representation of a complex reductive Lie algebra. Since all the above $\mathfrak{g}$-modules behave reasonably well under this association, we may assume from now on that V is a finite dimensional complex vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is an irreducible representation of a complex reductive Lie algebra. Clearly, $G = \exp(\mathfrak{g})$ acts irreducibly in V^* via the dual representation. Let $\tilde{X}$ be the G-orbit of a highest weight vector in $V^* \setminus 0$. Then the quotient $X := \tilde{X}/\mathbb{C}^*$ is a compact complex homogeneous-rational manifold canonically embedded into $\mathbb{P}(V^*)$, and there is a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \hookrightarrow & V^* \setminus 0 \\ \downarrow & & \downarrow \\ X & \hookrightarrow & \mathbb{P}(V^*) \end{array}$$

In fact, $X = G_s/P$, where G_s is the semisimple part of G and P is the parabolic subgroup of G_s leaving a highest weight vector in V^* invariant up to a scale. Let L be the restriction of the hyperplane section bundle $O(1)$ on $\mathbb{P}(V^*)$ to the submanifold X. Clearly, L is an ample homogeneous line bundle on X. We call (X, L) the *Borel-Weil data* associated with $(\mathfrak{g}, V)$.

According to Borel-Weil, the representation space V can be easily reconstructed from (X, L) as $V = \mathrm{H}^0(X, L)$. What about $\mathfrak{g}$? The Lie algebra of the Lie group of all global biholomorphisms of the line bundle L which commute with the projection $L \to X$ is isomorphic to $H^0(X, L \otimes (J^1L)^*)$ — a central extension of the Lie algebra $\mathrm{H}^0(X, TX)$. Whence, as a complex Lie algebra, $\mathrm{H}^0(X, L \otimes (J^1L)^*)$ has a natural complex irreducible representation in $\mathrm{H}^0(X, L) = V$; with very few (and well studied in the holonomy context) exceptions, this representation is isomorphic, up to a central extension, to the original $\mathfrak{g}$.

Remarkably enough, the basic $\mathfrak{g}$-modules defined above fit nicely the Borel-Weil paradigm as well. The resulting twistor formulae were among our basic instruments in solving the holonomy problem.

Proposition 0.1 *For a compact complex homogeneous-rational manifold X and an ample line bundle $L \to X$, there is an isomorphism*

$$\mathfrak{g}^{(1)} = H^0\left(X, L \otimes \odot^2 N^*\right),$$

and an exact sequence of $\mathfrak{g}$-modules,

$$0 \longrightarrow \frac{K(\mathfrak{g})}{\partial(\mathfrak{g}^{(1)} \otimes V^*)} \longrightarrow H^1\left(X, L \otimes \odot^3 N^*\right) \longrightarrow H^1\left(X, L \otimes \odot^2 N^*\right) \otimes V^*,$$

where $\mathfrak{g}$ is $H^0(X, L \otimes N^*)$ represented in $V = H^0(X, L)$.

Proof. The result follows easily from the exact sequences

$$0 \longrightarrow L \otimes \odot^2 N^* \longrightarrow L \otimes N^* \otimes V^* \longrightarrow L \otimes N^* \otimes \Lambda^2 V^*$$

and

$$0 \longrightarrow L \otimes \odot^3 N^* \longrightarrow L \otimes \odot^2 N^* \otimes V^* \longrightarrow L \otimes N^* \otimes \Lambda^2 V^* \longrightarrow L \otimes \Lambda^3 V^*,$$

where arrows are a combination of a natural monomorphism $N^* \longrightarrow V^* \otimes O_X$ (which holds due to ampliness of L) with the antisymmetrization. $\square$

This proposition is a group-theoretic manifestation of the fact [14] that *any* torsion-free affine connection with irreducibly acting holonomy can, in principle, be constructed by twistor methods. This universal twistor construction can be formulated in the language of G-structures as follows.

Theorem 0.2 *Let X be a generalised flag variety embedded as a Legendre submanifold into a complex contact manifold Y with contact line bundle L such that L_X is very ample on X. Then*

(i) *There exists a complete analytic family $F \hookrightarrow Y \times M$ of compact Legendre submanifolds with moduli space M being an $h^0(X, L_X)$-dimensional complex manifold. For each $t \in M$, the associated Legendre submanifold X_t is isomorphic to X.*

(ii) *The Legendre moduli space M comes equipped with an induced irreducible G-structure, $G_{ind} \to M$, with G isomorphic to the connected component of the identity of the group of all global biholomorphisms $\phi : L_X \to L_X$ which commute with the projection $\pi : L_X \to X$. The Lie algebra of G is isomorphic to $H^0\left(X, L_X \otimes (J^1 L_X)^*\right)$.*

(iii) *If G_{ind} is k-flat, $k \geqslant 0$, then the obstruction for G_{ind} to be $(k+1)$-flat is given by a tensor field on M whose value at each $t \in M$ is represented by a cohomology class $\rho_t^{[k+1]} \in \tilde{H}^1\left(X_t, L_{X_t} \otimes S^{k+2}(J^1 L_{X_t})^*\right)$.*

(iv) *If G_{ind} is 1-flat, then the bundle of all torsion-free connections in G_{ind} has as the typical fiber an affine space modeled on $H^0\left(X, L_X \otimes S^2(J^1 L_X)^*\right)$.*

The geometric meaning of cohomology classes

$$\rho_t^{[k+1]} \in \tilde{H}^1\left(X_t, L_{X_t} \otimes S^{k+2}(J^1 L_{X_t})^*\right)$$

is very simple — they compare to $(k+2)$th order the germ of the Legendre embedding $X_t \hookrightarrow Y$ with the "flat" model, $X_t \hookrightarrow J^1 L_{X_t}$, where the ambient contact manifold is just the total space of the vector bundle $J^1 L_{X_t}$ together with its canonical contact structure and the Legendre submanifold X_t is realised as a zero section of $J^1 L_{X_t} \rightarrow X_t$. Therefore, the cohomology class $\rho_t^{[k]}$ can be called the kth Legendre jet of X_t in Y. Then it is natural to call a Legendre submanifold $X_t \hookrightarrow Y$ *k-flat* if $\rho_t^{[k]} = 0$. With this terminology, the item (iii) of the above Theorem acquires a rather symmetric form: *the induced G-structure on the moduli space M of a complete analytic family of compact Legendre submanifolds is k-flat if and only if the family consists of k-flat Legendre submanifolds.*

A very intriguing aspect of the holonomy list of the Main Theorem is that all holonomy groups share one and the same property — the associated Borel-Weil data (X, L) have X biholomorphic to a compact Hermitian symmetric manifold. Put another way, twistor theory of holonomy groups reveals a surprising pattern in the classification list of holonomy groups which is not visible in the standard $(\mathfrak{g}, V)$-description. The list of compact Hermitian symmetric manifolds is very short, and it is desirable to get an independent proof (explanation) of this phenomenon. Such an explanation may result in much shorter proof of the Main Theorem than the one given in [15].

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