Special symplectic connections and Poisson geometry

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Abstract

By a special symplectic connection we mean a torsion free connection which is either the Levi-Civita connection of a Bochner-Kähler metric of arbitrary signature, a Bochner-bi-Lagrangian connection, a connection of Ricci type or a connection with special symplectic holonomy. A symplectic manifold or orbifold with such a connection is called special symplectic.

We show that any special symplectic connection can be constructed using symplectic realizations of quadratic deformations of a certain linear Poisson structures. Moreover, we show that these Poisson structures cannot be symplectically integrated by a Hausdorff groupoid.

As a consequence, we obtain a canonical principal line bundle over any special symplectic manifold or orbifold, and we deduce numerous global consequences.

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1 Introduction

Let (M, ω) be a symplectic manifold. A *symplectic connection* is, by definition, a torsion free connection on TM such that the symplectic form ω is parallel.

Given (M, ω), there are many symplectic connections on M. In fact, the space of symplectic connections is an affine space whose linear part is given by the sections in $S^3(TM)$. Thus, in order to investigate 'meaningful' symplectic connections, we have to impose further conditions.

A typical way to restrict a connection is to put certain conditions on its curvature. For example, if we decompose the curvature of a symplectic connection into its irreducible summands under the action of the symplectic group, then it decomposes into the Ricci curvature and the space of Ricci flat curvature maps. If the Ricci flat component of the curvature vanishes, then the connection is said to be of *Ricci type*, and such connections have been investigated e.g. in [BC], [CGR], [CGHR], [BC2], [CGS].

Another condition which one may put on a connection is the restriction of its *holonomy group*. Indeed, we say that a symplectic connection has *special symplectic holonomy* if its holonomy is contained in a proper absolutely irreducible subgroup of the symplectic group. These connections have been investigated e.g. in [Br1], [CMS], [MS], [S1], [S2], [S3].

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Thirdly, if the symplectic manifold is *(pseudo-)Kähler*, i.e. symplectic and (pseudo-)Riemannian at the same time, then one may again require that some part of its curvature vanishes. Namely, the curvature decomposes into the Ricci curvature and the *Bochner curvature* ([Bo]), and a metric is called *Bochner-Kähler* if its Bochner curvature vanishes. Similarly, if the manifold is equipped with a *bi-Lagrangian* structure, i.e. two complementary Lagrangian distributions, then again the curvature of a symplectic connection for which both distributions are parallel decomposes into the Ricci curvature and the Bochner curvature, and such a connection is called *Bochner-bi-Lagrangian* if its Bochner curvature vanishes. For results on Bochner-Kähler metrics and Bochner-bi-Lagrangian connections, see [Br2] and [K] and the references cited therein.

The interest in these various conditions on symplectic connections has been motivated for quite distinct reasons. Connections of *Ricci type* were of interest since they satisfy a certain variational principle and hence are critical points of a functional on the moduli space of symplectic connections ([BC]), and furthermore the canonical almost complex structure on the twistor space induced by a symplectic connection is integrable iff the connection is of Ricci type ([BR], [V2]). The second class of examples was investigated in the context of the classification of irreducible holonomy groups ([MS]), and the Bochner-Kähler and Bochner-bi-Lagrangian structures were of interest in complex geometry and in the study of a certain class of differential equations ([Br2]).

Note that we can consider all of these conditions also in the complex case, i.e. for complex manifolds with a holomorphic symplectic form and a holomorphic connection. In this report, we shall treat all of these structures simultaneously.

Definition. *Let (M, ω) be a (real or complex) symplectic manifold of dimension ≥ 4 with a (real or complex) symplectic connection ∇, i.e. a torsion free connection for which ω is parallel. We say that ∇ is a* special symplectic connection *if one of the following holds:*

1. *∇ is of Ricci type in the sense of [BC].*
2. *∇ has special symplectic holonomy, i.e. its holonomy is contained in an absolutely irreducible proper subgroup of the symplectic group.*
3. *∇ is the Levi-Civita connection of a pseudo-Riemannian Bochner-Kähler metric.*
4. *∇ is Bochner-bi-Lagrangian in the sense of [Br2].*

At first, it may seem entirely unmotivated to collect all these structures in one definition, but we shall provide ample justification for doing so. In fact, there is a beautiful link between special symplectic connections and parabolic contact geometry. Namely, we consider a (real or complex) simple Lie group G with Lie algebra $\mathfrak{g}$. We say that $\mathfrak{g}$ is 2-*gradable*, if $\mathfrak{g}$ contains the root space of a long root.

We associate to each (real or complex) simple 2-gradable Lie algebra $\mathfrak{g}$ one of the curvature conditions of a special symplectic connection. This establishes a one-to-one correspondence. In fact, the Lie algebras of type A_n correspond to the Bochner-Kähler and Bochner-bi-Lagrangian connections, the Lie algebras of type C_n correspond to the connections of Ricci type, and each 2-gradable Lie algebra of one of the remaining types can be associated to a special symplectic holonomy group.

There is a by now classical method to construct torsion free connections on an H-structure, where $\mathrm{H} \subset \mathrm{Aut}(V)$ is a (real or complex) matrix group with corresponding matrix subalgebra $\mathfrak{h} \subset \mathfrak{gl}(V)$.

This method relies on the existence of deformations of a linear Poisson structure ([CMS], [MS]). Connections which are constructed by this method are called *Poisson type connections*.

As we shall show, all special symplectic connections are locally equivalent to a Poisson type connection. In fact, there is a one-parameter family of Poisson structures Λ^c with $c \in \mathbb{F}$ on $\mathfrak{h}^* \oplus V^*$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, which are used to construct these connections. But in order to deduce *global* properties, we need to decide whether or not these Poisson structures are completely integrable.

Instead of regarding the Λ^c as a one-parametrer family of Poisson structures on $\mathfrak{h}^* \oplus V^*$, we may regard it as a single Poisson structure Λ on the larger space $Q := \mathfrak{h}^* \oplus V^* \oplus \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, such that the embeddings $(\mathfrak{h}^* \oplus V^*, \Lambda^c) \hookrightarrow Q_c := (\mathfrak{h}^* \oplus V^* + c, \Lambda) \subset (Q, \Lambda)$ are Poisson maps.

The main results which we present here are the following.

Theorem A: *The Poisson structure Λ on $Q := \mathfrak{h}^* \oplus V^* \oplus \mathbb{F}$ is integrable by a symplectic groupoid $\Gamma \rightrightarrows Q$, and there is an inclusion $Q \hookrightarrow \mathfrak{g}^*$ and $\Gamma \hookrightarrow T^*\mathrm{G}$, where G is a simple Lie group with Lie algebra $\mathfrak{g}$, such that Γ becomes a sub-groupoid of $T^*\mathrm{G}$.*

In contrast, the Poisson structures $(\mathfrak{h}^* \oplus V^*, \Lambda^c)$ are not integrable by a Hausdorff groupoid for any $c \in \mathbb{F}$.

In fact, $(\mathfrak{h}^* \oplus V^*, \Lambda^c)$ can be realized by Γ^c/τ^c, where $\Gamma^c \subset \Gamma$ is a Lie subgroupoid and $\tau^c \subset \Gamma^c$ is a one dimensional groupoid acting on Γ^c. However, this action is irregular, hence the quotient is not Hausdorff.

This global description allows us to give some results on the global structure of the special symplectic connections. Namely, we have the following

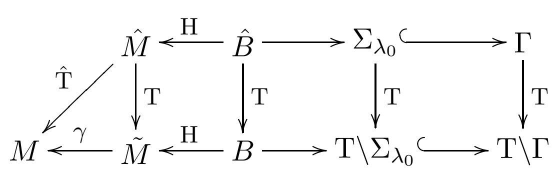
Theorem B: *Let (M, ω) be a (real or complex) symplectic manifold with a special symplectic connection of class C^4 associated to the simple Lie algebra $\mathfrak{g}$, and $\dim M \geq 4$.*

1. *The connection on M is of Poisson type, associated to a quadratic Poisson structure Λ^c on $\mathfrak{h}^* \oplus V^*$, which depends on a constant $c \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.*

2. *There is a principal $\hat{\mathrm{T}}$-bundle $\hat{M} \to M$, where $\hat{\mathrm{T}}$ is a (real or complex) one dimensional Lie group which is not necessarily connected, and this bundle carries a connection whose curvature equals -2ω.*

3. *Let $\mathrm{T} \subset \hat{\mathrm{T}}$ be the identity component, and let $\gamma := \hat{\mathrm{T}}/\mathrm{T}$. Also, let $\tilde{M} := \hat{M}/\mathrm{T}$ be equipped with the special symplectic connection for which the regular covering $\tilde{M} \to M = \tilde{M}/\gamma$ is connection preserving. Moreover, let $B \to M$ be the H-structure of the connection and $\hat{B} \to \hat{M}$ be the pull back of this H-structure.*

   Then there is an $\lambda_0 \in Q \subset \mathfrak{g}^*$ and a T-equivariant local diffeomorphism $\hat{B} \to \Sigma_{\lambda_0} \subset \Gamma$ where $\Sigma_{\lambda_0} := r^{-1}(\lambda_0) \subset \Gamma$, and where $l, r : \Gamma \rightrightarrows Q$ is the symplectic groupoid from Theorem A, such that the following diagram commutes.



As consequences of these results, we obtain the following

Corollary C: *All special symplectic connections of C^4-regularity are analytic, and the local moduli space of these connections is finite dimensional, in the sense that the germ of the connection at one point up to 4th order determines the connection entirely.*

Also, the Lie algebra $\mathfrak{s}$ of vector fields on M whose flow preserves the connection is isomorphic to $stab(\lambda_0)/(\mathbb{F}\lambda_0)$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, where $\lambda_0 \in Q \subset \mathfrak{g}^*$ is the element from Theorem B, and where $stab(\lambda_0) = \{x \in \mathfrak{g} \mid ad^*_x(\lambda_0) = 0\}$. In particular, $\dim \mathfrak{s} \geq \mathrm{rk}(\mathfrak{g}) - 1$ with equality implying that $\mathfrak{s}$ is abelian.

Thus, one may regard $M_{\lambda_0} := \mathrm{T}\backslash\Sigma_{\lambda_0}/\mathrm{H}$ as the maximal analytic continuation of the cover $\tilde{M}$. Needless to say, M_{λ_0} may be neither Hausdorff nor locally Euclidean. Also, if $\lambda_0' \in Q \subset \mathfrak{g}^*$ is another element, then the corresponding connections on M_{λ_0} and $M_{\lambda_0'}$ are equivalent iff λ_0 and λ_0' are G-conjugate. Hence there is a one-to-one correspondence between the maximal analytic continuations of special symplectic connections associated to $\mathfrak{g}$ and coadjoint orbits in $\mathfrak{g}^*$ which intersect $Q \subset \mathfrak{g}^*$.

While the analyticity of the connection and the determinedness by the 4th order germ at a point has been known in the Bochner-Kähler and Bochner-bi-Lagrangian case ([Br2][1]) and for connections with special symplectic holonomies (e.g. [CMS], [MS]), it was entirely unclear what the maximal analytic continuations of these structures look like and in which cases they are *regular*. This question is now completely answered in principle.

Also, the inequality $\dim \mathfrak{s} \geq \mathrm{rk}(\mathfrak{g}) - 1$ was known for the Bochner cases ([Br2]), whereas for the special symplectic holonomies, it was only known that $\mathfrak{s} \neq 0$ ([S3]).

Following this introduction, we shall briefly review standard results from the theory of simple Lie algebras and develop the algebraic formulas needed to describe the curvature conditions for special symplectic connections in a unifying way.

In the third section, the core of this report, we shall first review some standard notions of Poisson geometry and symplectic groupoids, and then deduce the above results.

We are grateful to R.Bryant for having made us aware of the reference [Br2] and the notion of Bochner-Kähler and Bochner-bi-Lagrangian structures, and for valuable comments on the link to parabolic contact geometry. Also, it is a pleasure to thank P.Bieliavski and S.Gutt for many stimulating conversations.

2 Special symplectic representations

This section establishes the algebraic formalism of special symplectic connections. We shall not give any proofs here, but instead refer the interested reader to [CS] for more details.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and let $\mathrm{G}_{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Choose a Cartan decomposition of $\mathfrak{g}$ and fix a long root α and an element $0 \neq x \in \mathfrak{g}_\alpha$. Then the orbit of x under the adjoint action of $\mathrm{G}_{\mathbb{C}}$ is called the *root cone of* $\mathfrak{g}_{\mathbb{C}}$. Evidently, the root cone is well defined, independently of the choice of Cartan decomposition. Elements of the root cone are called *maximal root elements.*

[1] The C^4-regularity of the connection is equivalent to the C^5-regularity of the Bochner-Kähler metric.

Definition 2.1 *Let $\mathfrak{g}$ be a simple real or complex Lie algebra. We say that $\mathfrak{g}$ is* 2-gradable *if either $\mathfrak{g}$ is complex, or $\mathfrak{g}$ is real and contains a maximal root element of the simple complex Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$.*

We shall justify this terminology in (1) below. If $\mathfrak{g}$ is 2-gradable and G is a Lie group with Lie algebra $\mathfrak{g}$, then we write

$$\hat{\mathcal{C}} := \mathrm{Ad}_G x \subset \mathfrak{g},$$

where $x \in \mathfrak{g}$ is a maximal root element. Given $x \in \hat{\mathcal{C}}$, there is a $y \in \hat{\mathcal{C}}$ with $B(x, y) \neq 0$, and we can choose a Cartan decomposition of $\mathfrak{g}$ such that $x \in \mathfrak{g}_{\alpha_0}$ and $y \in \mathfrak{g}_{-\alpha_0}$, where α_0 is a long root.

Let $H_{\alpha_0} \in [\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}]$ be the unique element with $\alpha_0(H_{\alpha_0}) = 2$, so that $\mathfrak{g}$ contains the Lie subalgebra $\mathfrak{sl}_{\alpha_0} := span < \mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}, H_{\alpha_0} >$ which is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $\mathrm{ad}(H_{\alpha_0})|_{\mathfrak{g}_\beta} = \langle \beta, \alpha_0 \rangle \, Id_{\mathfrak{g}_\beta}$, where $\langle \beta, \alpha_0 \rangle$ is the Cartan number, and since $\alpha_0 \in \Delta$ is a long root, the eigenvalues of $\mathrm{ad}(H_{\alpha_0})$ are the possible Cartan numbers $\{0, \pm 1, \pm 2\}$, so that we get the eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}, \quad \text{and} \quad [\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}, \tag{1}$$

where $\mathfrak{g}^i = \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = i\}} \mathfrak{g}_\beta$ for $i \neq 0$ and $\mathfrak{g}^0 = \mathfrak{t} \oplus \bigoplus_{\{\beta \in \Delta | \langle \beta, \alpha_0 \rangle = 0\}} \mathfrak{g}_\beta$. In particular, $\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm \alpha_0}$, and $\mathfrak{g}^0 = \mathbb{F} H_{\alpha_0} \oplus \mathfrak{h}$, where the Lie algebra $\mathfrak{h}$ is characterized by $[\mathfrak{h}, \mathfrak{sl}_{\alpha_0}] = 0$. Observe that $\mathfrak{g}^0$ and hence $\mathfrak{h}$ are reductive. Thus, as a Lie algebra,

$$\mathfrak{g}^{ev} := \mathfrak{g}^{-2} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{2} \cong \mathfrak{sl}_{\alpha_0} \oplus \mathfrak{h} \quad \text{and} \quad \mathfrak{g}^{odd} := \mathfrak{g}^{-1} \oplus \mathfrak{g}^{1} \cong \mathbb{F}^2 \otimes V \quad \text{as a } \mathfrak{g}^{ev}\text{-module},$$

where $\mathfrak{h}$ acts effectively on V. Identifying $\mathfrak{h}$ with its image under this representation, we may regard it as a subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$, and hence we have the decomposition

$$\mathfrak{g} = \mathfrak{g}^{ev} \oplus \mathfrak{g}^{odd} \cong (\mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}) \oplus (\mathbb{F}^2 \otimes V), \tag{2}$$

where this notation indicates the representation $\mathrm{ad} : \mathfrak{g}^{ev} \to \mathrm{End}(\mathfrak{g}^{odd})$.

We fix a non-zero $\mathbb{F}$-bilinear area form $a \in \Lambda^2(\mathbb{F}^2)^*$. There is a canonical $\mathfrak{sl}(2, \mathbb{F})$-equivariant isomorphim

$$S^2(\mathbb{F}^2) \longrightarrow \mathfrak{sl}(2, \mathbb{F}), \quad (ef) \cdot g := a(e, g) f + a(f, g) e \ \text{ for all } e, f, g \in \mathbb{F}^2, \tag{3}$$

and under this isomorphism, the Lie bracket on $\mathfrak{sl}(2, \mathbb{F})$ is given by

$$[ef, gh] = a(e, g) fh + a(e, h) fg + a(f, g) eh + a(f, h) eg.$$

Thus, if we fix a basis $e_+, e_- \in \mathbb{F}^2$ with $a(e_+, e_-) = 1$, then we have the identifications

$$H_{\alpha_0} = -e_+ e_-, \quad \mathfrak{g}^{\pm 2} = \mathbb{F} e_\pm^2, \quad \mathfrak{g}^{\pm 1} = e_\pm \otimes V.$$

Proposition 2.2 *[CS] Let $\mathfrak{g}$ be a 2-gradable simple Lie algebra, and consider the decompositions (1) and (2). Then there is an $\mathfrak{h}$-invariant symplectic form $\omega \in \Lambda^2 V^*$ and an $\mathfrak{h}$-equivariant product $\circ : S^2(V) \to \mathfrak{h}$ such that*

$$[\ ,\ ] : \Lambda^2(\mathfrak{g}^{odd}) \longrightarrow \mathfrak{g}^{ev} \cong \mathfrak{sl}(2, \mathbb{F}) \oplus \mathfrak{h}$$

is given as

$$[e \otimes x, f \otimes y] = \omega(x, y) ef + a(e, f) x \circ y \quad \textit{for } e, f \in \mathbb{F}^2 \textit{ and } x, y \in V, \tag{4}$$

using the identification $S^2(\mathbb{F}^2) \cong \mathfrak{sl}(2,\mathbb{F}) \subset \mathfrak{g}^{ev}$ *from (3). Moreover, there is a multiple* $(\ ,\ )$ *of the Killing form* B *on* $\mathfrak{g}$ *such that for all* $x, y, z \in V$ *and* $h \in \mathfrak{h}$*, we have*

$$\begin{aligned} (h, x \circ y) &= \omega(hx, y) = \omega(hy, x) \\ (x \circ y)z - (x \circ z)y &= 2\ \omega(y,z)x - \omega(x,y)z + \omega(x,z)y. \end{aligned} \tag{5}$$

In general, given a (real or complex) symplectic vector space (V, ω), i.e. $\omega \in \Lambda^2 V^*$ is non-degenerate, we define the *symplectic group* $\mathrm{Sp}(V,\omega)$ and the *symplectic Lie algebra* $\mathfrak{sp}(V,\omega)$ by

$$\mathrm{Sp}(V,\omega) := \{g \in \mathrm{Aut}(V) \mid \omega(gx, gy) = \omega(x,y) \text{ for all } x, y \in V\},$$

$$\mathfrak{sp}(V,\omega) := \{h \in \mathrm{End}(V) \mid \omega(hx, y) + \omega(x, hy) = 0 \text{ for all } x, y \in V\}.$$

Then $\mathrm{Sp}(V,\omega)$ is a Lie group with Lie algebra $\mathfrak{sp}(V,\omega)$.

Definition 2.3 *Let* (V,ω) *be a symplectic vector space over* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, and let* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *be a subalgebra for which there exists an* $\mathfrak{h}$*-equivariant map* $\circ : S^2(V) \to \mathfrak{h}$ *and an* $\mathrm{ad}_{\mathfrak{h}}$*-invariant inner product* $(\ ,\ )$ *for which the identities (5) hold. Then we call* $\mathfrak{h}$ *a* special symplectic subalgebra. *Moreover, we call the connected subgroup* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ *with Lie algebra* $\mathfrak{h}$ *a* special symplectic subgroup.

Thus, by Proposition 2.2, each (real or complex) 2-gradable simple Lie algebra yields a (real or complex) special symplectic subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$. The converse is also true. Namely, we have

Proposition 2.4 *[CS] Let* (V,ω) *be a symplectic vector space over* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$*, and let* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *be a special symplectic subalgebra. Then there exists a unique* 2*-gradable simple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{F}$*, which admits the decompositions (1) and (2), and the Lie bracket of* $\mathfrak{g}$ *is given by (4).*

From this proposition, we obtain a complete classification of special symplectic subalgebras by considering all complex simple Lie algebras and their 2-gradable real forms ([OV]).

Corollary 2.5 *Table 1 yields the complete list of special symplectic subgroups* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$*.*

Definition 2.6 *Let* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *be a special symplectic Lie algebra, and let* $\mathfrak{g}$ *be the (unique) simple Lie algebra from Proposition 2.4. Then we say that* $\mathfrak{h}$ *is* associated to $\mathfrak{g}$*. Let* G *be a connected Lie group with Lie algebra* $\mathfrak{g}$*. Then we say that the special symplectic group* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ *is* associated to G.

This description yields the following result.

Proposition 2.7 *[CS] Let* $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ *be a special symplectic Lie algebra and* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ *be the corresponding special Lie subgroup. Then* $\mathrm{H} \subset \mathrm{Sp}(V,\omega)$ *is closed and reductive, and*

$$\mathfrak{h} = \{h \in \mathfrak{sp}(V,\omega) \mid [h, x \circ y] = (hx) \circ y + x \circ (hy) \ \textit{for all}\ x, y \in V\}.$$

Table 1: Special symplectic subgroups

Notation: $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

	Type of Δ	G	H	V
(i)	A_k, $k \geq 2$	$\mathrm{SL}(n+2,\mathbb{F})$, $n \geq 1$	$\mathrm{GL}(n,\mathbb{F})$	$W \oplus W^*$ with $W \cong \mathbb{F}^n$
(ii)		$\mathrm{SU}(p+1,q+1)$, $p+q \geq 1$	$\mathrm{U}(p,q)$	$\mathbb{C}^{p+q}$
(iii)	C_k, $k \geq 2$	$\mathrm{Sp}(n+1,\mathbb{F})$	$\mathrm{Sp}(n,\mathbb{F})$	$\mathbb{F}^{2n}$
(iv)	B_k, D_{k+1}, $k \geq 3$	$\mathrm{SO}(n+4,\mathbb{C})$, $n \geq 3$	$\mathrm{SL}(2,\mathbb{C}) \cdot \mathrm{SO}(n,\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$
(v)		$\mathrm{SO}(p+2,q+2)$, $p+q \geq 3$	$\mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{SO}(p,q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$
(vi)		$\mathrm{SO}(n+2,\mathbb{H})$, $n \geq 2$	$\mathrm{Sp}(1) \cdot \mathrm{SO}(n,\mathbb{H})$	$\mathbb{H}^n$
(vii)	G_2	G_2', $\mathrm{G}_2^{\mathbb{C}}$	$\mathrm{SL}(2,\mathbb{F})$	$S^3(\mathbb{F}^2)$
(viii)	F_4	$\mathrm{F}_4^{(1)}$, $\mathrm{F}_4^{\mathbb{C}}$	$\mathrm{Sp}(3,\mathbb{F})$	$\mathbb{F}^{14} \subset \Lambda^3\mathbb{F}^6$
(ix)	E_6	$\mathrm{E}_6^{\mathbb{F}}$	$\mathrm{SL}(6,\mathbb{F})$	$\Lambda^3\mathbb{F}^6$
(x)		$\mathrm{E}_6^{(2)}$	$\mathrm{SU}(1,5)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(xi)		$\mathrm{E}_6^{(3)}$	$\mathrm{SU}(3,3)$	$\mathbb{R}^{20} \subset \Lambda^3\mathbb{C}^6$
(xii)	E_7	$\mathrm{E}_7^{\mathbb{C}}$	$\mathrm{Spin}(12,\mathbb{C})$	$\Delta^{\mathbb{C}} \cong \mathbb{C}^{32}$
(xiii)		$\mathrm{E}_7^{(5)}$	$\mathrm{Spin}(6,6)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xiv)		$\mathrm{E}_7^{(6)}$	$\mathrm{Spin}(6,\mathbb{H})$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xv)		$\mathrm{E}_7^{(7)}$	$\mathrm{Spin}(2,10)$	$\mathbb{R}^{32} \subset \Delta^{\mathbb{C}}$
(xvi)	E_8	$\mathrm{E}_8^{\mathbb{C}}$	$\mathrm{E}_7^{\mathbb{C}}$	$\mathbb{C}^{56}$
(xvii)		$\mathrm{E}_8^{(8)}$	$\mathrm{E}_7^{(5)}$	$\mathbb{R}^{56}$
(xviii)		$\mathrm{E}_8^{(9)}$	$\mathrm{E}_7^{(7)}$	$\mathbb{R}^{56}$

In general, for a given Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ we define the space of *formal curvature maps* as

$$K(\mathfrak{h}) := \{R \in \Lambda^2 V^* \otimes \mathfrak{h} \mid R(x,y)z + R(y,z)x + R(z,x)y = 0 \text{ for all } x,y,z \in V\}. \qquad (6)$$

This terminology is due to the fact that the curvature map of a torsion free connection always satisfies the first Bianchi identity, i.e. is contained in $K(\mathfrak{h})$ for an appropriate $\mathfrak{h}$. $K(\mathfrak{h})$ is an H-module in an obvious way.

There is a map $Ric : K(\mathfrak{h}) \to V^* \otimes V^*$, given by $Ric(R)(x,y) := tr(R(x,\_)y)$ for all $R \in K(\mathfrak{h})$ and $x,y \in V$. Note that $Ric(R)(x,y) - Ric(R)(y,x) = -trR(x,y)$. Thus, if $\mathfrak{h} \subset \mathfrak{sl}(n,\mathbb{F})$, then $Ric(R) \in S^2(V^*)$.

We now obtain the following results for the cases where $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ is a special symplectic subalgebra.

Theorem 2.8 *[CS] Let $\mathfrak{h} \subset \mathfrak{sp}(V,\omega)$ be a special symplectic subalgebra where $\dim V \geq 4$. Then there is an* H*-equivariant injective map $\mathfrak{h} \to K(\mathfrak{h})$, given by*

$$h \longmapsto R_h, \quad \text{where} \quad R_h(x,y) := 2\,\omega(x,y)h + x \circ (hy) - y \circ (hx). \qquad (7)$$

In fact, $Ric(R_h) = 0$ iff $h = 0$. In particular, $\mathcal{R}_{\mathfrak{h}} := \{R_h \mid h \in \mathfrak{h}\} \cong \mathfrak{h}$. Moreover, if we define

$$\mathcal{R}_{\mathfrak{h}}^{(1)} := \{\psi \in V^* \otimes \mathcal{R}_{\mathfrak{h}} \mid \psi(x)(y,z) + \psi(y)(z,x) + \psi(z)(x,y) = 0 \;\; \text{for all } x,y,z \in V\},$$

then as an H*-module, $\mathcal{R}_{\mathfrak{h}}^{(1)} \cong V$ with an explicit isomorphism given by*

$$u \longmapsto \psi_u, \quad \text{where} \quad \psi_u(x) := R_{u \circ x} \in \mathcal{R}_{\mathfrak{h}} \quad \text{for all } u,x \in V.$$

The definition of $\mathcal{R}_{\mathfrak{h}}^{(1)}$ is motivated by the *second Bianchi identity* of the covariant derivative of the curvature tensor of a torsion free connection whose curvature lies in $\mathcal{R}_{\mathfrak{h}}$.

For a special symplectic subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$, we can decompose its curvature space as an $\mathfrak{h}$-module into

$$K(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}} \oplus \mathcal{W}_{\mathfrak{h}}, \qquad \text{where} \qquad \mathcal{R}_{\mathfrak{h}} = \{R_h \mid h \in \mathfrak{h}\} \qquad \text{and} \qquad \mathcal{W}_{\mathfrak{h}} = \ker(Ric). \tag{8}$$

The curvature spaces $K(\mathfrak{h})$ have been calculated. Summarizing, we have the following

Theorem 2.9 *Let* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *be a special symplectic subgroup with Lie algebra* $\mathfrak{h} \subset \mathfrak{sp}(V, \omega)$ *listed in Table 1. Then*

1. *For the representations corresponding to* (i) *and* (ii), *we have* $\mathcal{W}_{\mathfrak{h}} = 0$ *if* $n = 1$ $(p + q = 1$, *respectively) and* $\mathcal{W}_{\mathfrak{h}} \neq 0$ *if* $n \geq 2$ $(p + q \geq 2$, *respectively) [Br2].*
2. *For the representations corresponding to* (iii), *we have* $\mathcal{W}_{\mathfrak{h}} = 0$ *for* $n = 1$ *whereas* $\mathcal{W}_{\mathfrak{h}} \neq 0$ *for* $n \geq 2$ *[BC].*
3. *For the representations corresponding to entries* (iv) – $(xviii)$, *we have* $K(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}}$ *and hence* $\mathcal{W}_{\mathfrak{h}} = 0$ *[MS].*

Definition 2.10 *Let* (M, ω) *be a symplectic manifold with a symplectic connection* ∇, *i.e. a torsion free connection for which* ω *is parallel. We say that* ∇ *is a* special symplectic connection associated to the (simple) Lie group G *if there is a special symplectic subgroup* $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ *associated to* G *in the sense of Definition 2.6 such that the curvature of* ∇ *is contained in* $\mathcal{R}_{\mathfrak{h}}$ *(cf. (7) and (8)).*

Definition 2.10 coincides with the definition of special symplectic connections from the introduction. Namely, note that by the Ambrose-Singer holonomy theorem, the (restricted) holonomy of a special symplectic connection is evidently contained in $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$, so that we have an H-reduction $B \to M$ of the frame bundle of M which is compatible with the connection.

If $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ is one of the subgroups (i) or (ii), then either there are two complementary parallel Lagrangian foliations (case (i)), or the connection is the Levi-Civita connection of a pseudo-Kähler metric (case (ii)). In either case, the condition that the curvature lies in $\mathcal{R}_{\mathfrak{h}}$ is equivalent to the vanishing of the *Bochner curvature*, and such connections have been called *Bochner-bi-Lagrangian* in the first and *Bochner-Kähler* in the second case. For a detailed study of these connections, see [Br2].

If $\mathrm{H} = \mathrm{Sp}(V, \omega)$ as in (iii), then the condition that the curvature lies in $\mathcal{R}_{\mathfrak{h}}$ is equivalent to saying that the connection is a (real or holomorphic) *symplectic connection of Ricci type* in the sense of [BC].

Finally, if $\mathrm{H} \subset \mathrm{Sp}(V, \omega)$ is one of the subgroups $(iv) - (xviii)$ in Table 1, then, by Theorem 2.9, *any* torsion free connection on such an H-structure must be special. In fact, these subgroups H are precisely the absolutely irreducible proper subgroups of the symplectic group which can occur as the holonomy of a torsion free connection (cf. [MS], [S1], [S3]).

It shall be the aim of the following sections to study special symplectic connections using the general algebraic setup established here rather than dealing with each of the geometric structures separately.

3 Poisson geometry and torsion free connections

3.1 Subgroupoids and induced Poisson structures

Let us briefly recall some of the basic notions of Poisson geometry. For a more detailed exposition, see e.g. [V1]. A *Poisson manifold* is a manifold P together with a bivector field $\Lambda \in \Gamma(\Lambda^2 TP)$ such that the Schouten bracket $[\Lambda, \Lambda] = 0$ vanishes. Note that Λ induces a bracket on functions

$$\{f, g\} := \langle df \wedge dg, \Lambda \rangle ,$$

called the *Poisson bracket*, and this equipps $C^\infty(P)$ with a Lie algebra structure. Moreover, contraction with Λ induces a bundle map

$$\# : T^*P \longrightarrow TP,$$

and the image of this map is called the *characteristic distribution* of P, denoted by $\mathcal{C}_p := \# T_p^*P \subset T_pP$. While the rank of $\mathcal{C}_p$ will in general depend on p, there is through every $p \in P$ a *characteristic leaf*, i.e. a submanifold $\Sigma \ni p$ with $T_q\Sigma = \mathcal{C}_q$ for all $q \in \Sigma$. The restriction of Λ to $\mathcal{C}_p$ is non-degenerate, thus each characteristic leaf carries a canonical symplectic form.

We call a submanifold $Q \subset P$ *cosymplectic* if the intersection of Q with each characteristic leaf is transversal, and the restriction of the symplectic form to the intersection $T_pQ \cap \mathcal{C}_p$ is non-degenerate. Equivalently, $Q \subset P$ is a cosymplectic submanifold if

$$T^*P|_Q = W \oplus TQ^\perp, \tag{9}$$

where $W := \{\alpha \in T^*P|_Q \mid \#\alpha \in TQ\} \subset T^*P$, and $TQ^\perp \subset T^*P$ is the annihilator of TQ.

We consider the restriction map $\imath : T^*P \to T^*Q$ and note that by (9) the restriction $\jmath := \imath|_W : W \to T^*Q$ is an isomorphism if $Q \subset P$ is cosymplectic. Then the bivector field $\Lambda_Q \in \Gamma(\Lambda^2 TQ)$ given by

$$\langle \Lambda_Q, u \wedge v \rangle := \left\langle \Lambda, \jmath^{-1}(u) \wedge \jmath^{-1}(v) \right\rangle \tag{10}$$

for all $u, v \in T^*Q$ defines a Poisson structure on Q, called the *coinduced Poisson structre* ([X]).

Every symplectic manifold (S, ω) is a Poisson manifold in a canonical way, the Poisson bracket being defined by the equation

$$\{f, g\} := -\omega(X_f, X_g),$$

where X_f, X_g are the Hamiltonian vector fields corresponding to f and g, respectively. In this case, we have also the identity

$$[X_f, X_g] = X_{\{f,g\}}.$$

Given a Poisson manifold P, a *symplectic realization* of P is a submersion $\pi : S \to P$, where S is a symplectic manifold, such that $\pi^* : C^\infty(P) \to C^\infty(S)$ is a Lie algebra homomorphism w.r.t. the Poisson brackets on $C^\infty(P)$ and $C^\infty(S)$, respectively. Any Poisson manifold P has at least *local* symplectic realizations, i.e. P can be covered by open sets which admit a symplectic realization ([W]).

If $\pi : S \to P$ is a symplectic realization of the Poisson manifold P then, in order to avoid confusion, we denote the Hamiltonian vector fields on S by ξ_h where $h \in C^\infty(S)$, while we denote the Hamiltonian vector fields on P by ζ_f for $f \in C^\infty(P)$. With this, we have for all $f, g \in C^\infty(P)$

$$d\pi(\xi_{\pi^*(f)}) = \zeta_f \quad \text{and} \quad \begin{array}{rcl} [\xi_{\pi^*(f)}, \xi_{\pi^*(g)}] & = & \xi_{\{\pi^*(f), \pi^*(g)\}_S} \\ & = & \xi_{\pi^*(\{f,g\})}. \end{array} \tag{11}$$

This implies that the distribution Ξ on S given by

$$\Xi_s = \{(\xi_{\pi^*(f)})_s \mid \ f \in C^\infty(P)\} \ \text{ for all } s \in S \tag{12}$$

is integrable. Evidently, $(\xi_{\pi^*(f)})_s$ only depends on $df_{\pi(s)}$, and since π is a submersion, the map $d\pi^* : T^*_{\pi(s)}P \to T^*_s S$ is injective. Thus, the canonical map

$$\begin{array}{rccc} \Theta: & \Xi_s & \longrightarrow & T^*_{\pi(s)}P \\ & (\xi_{\pi^* f})_s & \longmapsto & df_{\pi(s)} \end{array} \tag{13}$$

is a linear isomorphism and hence, Ξ has constant rank equal to the dimension of P. Moreover, if $F \subset S$ is an integral leaf of Ξ then by (11), there is a symplectic leaf $\Sigma \subset P$ such that $\pi : F \to \Sigma$ is a submersion.

Next, let us briefly recall the notion of a *groupoid*. A groupoid is a tuple $(\Gamma, \Gamma_0, l, r, m, i)$ of a set Γ together with two maps $l, r : \Gamma \rightrightarrows \Gamma_0$ where $\Gamma_0 \subset \Gamma$ is called the *set of units*, and a multiplication map $m : \Gamma^{(2)} \to \Gamma$ where $\Gamma^{(2)} := \{(x,y) \in \Gamma \times \Gamma \mid r(x) = l(y)\} \subset \Gamma \times \Gamma$, and an inversion $i : \Gamma \to \Gamma$, such that when writing $xy := m(x,y)$ and $x^{-1} := i(x)$ then the following axioms are satisfied for all $x, y \in \Gamma$:

1. $l(x)x = xr(x) = x$.
2. If one of $(xy)z$ and $x(yz)$ is defined then so is the other, and they are equal.
3. $xx^{-1} = l(x)$ and $x^{-1}x = r(x)$; in particular, these products are always defined.

We say that $(\Gamma, \Gamma_0, l, r, m, i)$ is a *symplectic groupoid* if (Γ, ω) is a symplectic manifold, $\Gamma_0 \subset \Gamma$ is a submanifold, $l, r : \Gamma \rightrightarrows \Gamma_0$ are submersions, m and i are differentiable and the graph of m is a Lagrangian submanifold of $(\Gamma \times \Gamma \times \Gamma, \omega \times \omega \times -\omega)$.

One can show that in this case Γ_0 carries a unique Poisson structure Λ such that $l : (\Gamma, \omega) \to (\Gamma_0, \Lambda)$ and $r : (\Gamma, \omega) \to (\Gamma_0, -\Lambda)$ are symplectic realizations, which implies that

$$dl(\#dl^*(\alpha)) = \#(\alpha) \qquad \text{and} \qquad dr(\#dr^*(\alpha)) = -\#(\alpha) \tag{14}$$

for all $\alpha \in T^*\Gamma_0$. For a symplectic groupoid, one has also the property that ([V1])

$$ker(dl) = \#(dr^*(T^*\Gamma_0)) \qquad \text{and} \qquad ker(dr) = \#(dl^*(T^*\Gamma_0)). \tag{15}$$

Definition 3.1 *A Poisson manifold P is called* integrable *if there is a symplectic groupoid Γ such that $P \cong \Gamma_0$.*

Proposition 3.2 *Let Γ be a symplectic groupoid with set of units Γ_0, and let $Q \subset \Gamma_0$ be a cosymplectic submanifold. Then $\Gamma' := l^{-1}(Q) \cap r^{-1}(Q) \subset \Gamma$ is a symplectic subgroupoid, i.e. $\Gamma' \subset \Gamma$ is a regular submanifold such that $\omega|_{\Gamma'}$ is non-degenerate and the restrictions $l|_{\Gamma'}, r|_{\Gamma'}$ are submersions onto Q. Thus, (Γ', Q, l, r, m, i) is itself a symplectic groupoid. Moreover, the restrictions $l : (\Gamma', \omega) \to (Q, \Lambda_Q)$ and $r : (\Gamma', \omega) \to (Q, -\Lambda_Q)$ are symplectic realizations, where Λ_Q is the coinduced Poisson structure from (10).*

Thus, a cosymplectic submanifold of an integrable Poisson manifold is itself integrable.

Proof. Let $p \in \Gamma'$. Then by (9), (14) and (15) we have that $dr(\# dr^*(T_{r(p)}Q^\perp)) = \# T_{r(p)}Q^\perp = W^\perp$ is transversal to $T_{r(p)}Q$, whereas $dl(\# dr^*(T_{r(p)}Q^\perp)) = 0$. Using the analogous identities when interchanging l and r, it follows that the image of $d(l \times r)_p : T_p\Gamma \to T_{l(p)}\Gamma_0 \times T_{r(p)}\Gamma_0$ contains a subspace complementary to $T_{l(p)}Q \times T_{r(p)}Q$ so that $\Gamma' = (l \times r)^{-1}(Q \times Q) \subset \Gamma$ is a regular submanifold with $\dim \Gamma' = 2 \dim Q$ and the restrictions $l, r : \Gamma' \rightrightarrows Q$ are submersions.

Next, we show that the restriction $\omega|_{\Gamma'}$ is non-degenerate. For this, let $\xi_0 \in T_p\Gamma'$ be such that $\omega(\xi_0, T_p\Gamma') = 0$. Then

$$0 = \omega(\xi_0, \#(dl^*(W_{l(p)}))) = -dl^*(W_{l(p)})(\xi_0) = -W_{l(p)}(dl(\xi_0)),$$

and since $dl(\xi_0) \in T_{l(p)}Q$, (9) implies that $dl(\xi_0) = 0$. Thus, by (15), $\xi_0 = \#(dr^*(\alpha))$ for some $\alpha \in T^*_{r(p)}\Gamma_0$. Therefore,

$$0 = \omega(\xi_0, T_p\Gamma') = \omega(\#(dr^*(\alpha)), T_p\Gamma') = dr^*(\alpha)(T_p\Gamma') = \alpha(dr(T_p\Gamma')) = \alpha(T_{r(p)}Q),$$

where the last equation follows since the restriction $r : \Gamma' \to Q$ is a submersion. Thus, $\alpha \in T_{r(p)}Q^\perp$.

On the other hand, by (14), $\#\alpha = -dr(\#(dr^*(\alpha))) = -dr(\xi_0) \in \mathrm{T}_{r(p)}Q$ so that $\alpha \in W_{r(p)}$, and from (9), we now conclude that $\alpha = 0$ and hence $\xi_0 = 0$.

Finally, for $u \in T^*_{r(p)}\Gamma_0$ we have the identities

$$\begin{aligned} dr(\#(dr^*(u)|_{\Gamma'})) &= dr(\#(dr^*(\jmath^{-1}(u))|_{\Gamma'})) && \text{since } u - \jmath^{-1}(u) \in TQ^\perp \text{ and } dr^*(TQ^\perp)|_{\Gamma'} = 0 \\ &= dr(\#(dr^*(\jmath^{-1}(u)))) && \text{since } \#(dr^*(\jmath^{-1}(u))) \in T\Gamma' \text{ by (14)} \\ &= -\#(\jmath^{-1}(u)) && \text{by (14)} \\ &= -(u \lrcorner \Lambda_Q)|_{TQ} && \text{by (10).} \end{aligned}$$

Thus, $r : (\Gamma', \omega) \to (Q, -\Lambda_Q)$ and, analogously, $l : (\Gamma', \omega) \to (Q, \Lambda_Q)$ are symplectic realizations. ∎

3.2 Construction of torsion free connections via Poisson geometry

We now turn to the construction of torsion free connections via Poisson structures. Let V be a finite dimensional vector space and let $\mathrm{H} \subset \mathrm{Aut}(V)$ be any connected closed Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathrm{End}(V)$. As before, we consider the spaces of formal curvature maps $K(\mathfrak{h})$ and of formal curvature derivatives $K^1(\mathfrak{h})$.

Let $W := \mathfrak{h} \oplus V$. We shall denote elements of $\mathfrak{h}$ and V by $A, B, \ldots$ and $x, y, \ldots$, respectively, and elements of W by $w, w', \ldots$. We may regard W as the semi-direct product of Lie algebras, i.e. we define a Lie algebra structure on W by the equation

$$[A + x, B + y] := [A, B] + A \cdot y - B \cdot x.$$

This induces a Poisson structure on the dual space W^*. Now, we wish to perturb this Poisson structure. For this, we need the

Definition 3.3 *A C^∞-map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ is called* deforming *if*

(i) *ϕ is* H*-equivariant,*

(ii) *for every $p \in \mathfrak{h}^*$, the dual map $(d\phi_p)^* : \Lambda^2 V \to \mathfrak{h}$ is contained in $K(\mathfrak{h})$ (cf. (6)).*

Now, the following important observation is easily proven.

Proposition 3.4 *([CMS], [S3]) Let V, $\mathfrak{h} \subset \mathrm{End}(V)$, W and $K(\mathfrak{h})$ be as above, and let $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ be a deforming map. Let $\Phi := \phi \circ pr$, where $pr : W^* \to \mathfrak{h}^*$ is the natural projection. Then the following bracket on W^* is Poisson:*

$$\{f, g\}(p) := p([A + x, B + y]) + \Phi(p)(x, y). \tag{16}$$

Here, $df_p = A + x$ and $dg_p = B + y$ are the decompositions of $df_p, dg_p \in T_p^* W^* \cong W$.

Note that for $\phi = 0$, we simply obtain the Poisson structure induced by the Lie algebra structure on W.

Consider a Poisson structure on W^* induced by a deforming map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ and let $\pi : S \to U$ be a symplectic realization of an open subset $U \subset W^*$. Let $F \subset S$ be an integral leaf of the distribution Ξ from (12) and $\pi : F \to \Sigma$ be the projection where $\Sigma \subset W^*$ is an integral leaf. Recall the map $\Theta : TF_s \longrightarrow T^*_{\pi(s)} W^*$ from (13). Since W^* is a vector space, we have $T^*_{\pi(s)} W^* \cong W^{**} \cong W$ canonically, so that we may regard Θ as a W-valued 1-form on F. We decompose $\Theta = \eta + \theta$ where η and θ are 1-forms on F with values in $\mathfrak{h}$ and V, respectively.

Proposition 3.5 *([CMS], [S3]) Let V, $\mathfrak{h} \subset \mathrm{End}(V)$ and W be as above, and consider a Poisson structure on W^* induced by a deforming map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$. If $\pi : S \to U$ is a symplectic realization of an open subset $U \subset W^*$, and if $F \subset S$ is a leaf of the distribution Ξ, then the $(\mathfrak{h}^* \oplus V^*)$-valued coframe $\Theta = \theta + \eta$ on F satisfies the equations*

$$\begin{aligned} d\theta &= -\eta \wedge \theta \\ d\eta &= -\eta \wedge \eta - \pi^*(d\Phi) \circ (\theta \wedge \theta). \end{aligned} \tag{17}$$

Here, $d\Phi$ is regarded as a map on U with values in $\Lambda^2 V^* \otimes \mathfrak{h}$.

Proof. Each $w \in W$ may be regarded as a (linear) function on $U \subset W^*$, and we shall write ξ_w instead of $\xi_{\pi^*(w)} \in \mathfrak{X}(F)$. From (11) we have $[\xi_{w_1}, \xi_{w_2}] = \xi_{\{w_1, w_2\}}$ so that by (16) we have

$$\begin{aligned} [\xi_A, \xi_B] &= \xi_{[A,B]} \\ [\xi_A, \xi_x] &= \xi_{A \cdot x} \\ [\xi_x, \xi_y](s) &= \xi_{d\Phi(p)(x,y)} \quad \text{where } p = \pi(s), \end{aligned} \tag{18}$$

where $A, B \in \mathfrak{h}$ and $x, y \in V$. Evidently, this is equivalent to (17). ∎

The first equation in (18) implies that the flow along the vector fields $\{\xi_A \mid A \in \mathfrak{h}\}$ induces a locally free H-action on $F \subset S$. After shrinking F if necessary, we may assume that $M := F/\mathrm{H}$ is a *manifold*. From (17) it follows that there is a unique torsion free connection on M and a unique immersion $\imath : F \hookrightarrow \mathfrak{F}_V$ into the V-valued coframe bundle $\mathfrak{F}_V$ of M such that $\theta = \imath^*(\underline{\theta})$ and $\eta = \imath^*(\underline{\eta})$, where $\underline{\theta}$ and $\underline{\eta}$ are the tautological and the connection 1-form on $\mathfrak{F}_V$, respectively. Clearly, the holonomy of this connection is contained in H and its curvature is represented by $\pi^*(d\Phi)$.

Definition 3.6 *Let $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ be a deforming map. Then a torsion free connection which is obtained from the above construction is called a* Poisson connection induced by ϕ*.*

Let $\mathcal{P}^{(k)}(\mathfrak{h})$ be the *k-th prolongation of* $K(\mathfrak{h}) \subset \Lambda^2 V^* \otimes \mathfrak{h}$, i.e. $\mathcal{P}^{(k)}(\mathfrak{h})$ is given by

$$\mathcal{P}^{(k)}(\mathfrak{h}) = \left(S^{k+1}(\mathfrak{h}) \otimes \Lambda^2 V^*\right) \cap \left(S^k(\mathfrak{h}) \otimes K(\mathfrak{h})\right),$$

where both are regarded as subspaces of $S^k(\mathfrak{h}) \otimes \mathfrak{h} \otimes \Lambda^2 V^*$. Suppose that there is an H-invariant element $\phi_k \in \mathcal{P}^{(k-1)}(\mathfrak{h})$. If we regard ϕ_k as a polynomial map of degree k, $\phi_k : \mathfrak{h}^* \to \Lambda^2 V^*$, then it follows that ϕ_k is deforming. Conversely, given an *analytic* map $\phi : \mathfrak{h}^* \to \Lambda^2 V^*$ with analytic expansion at $0 \in \mathfrak{h}^*$

$$\phi = \phi_0 + \phi_1 + \cdots,$$

then it is straightforward to show that ϕ is deforming iff all ϕ_k are, iff $\phi_k \in \left(\mathcal{P}^{(k-1)}(\mathfrak{h})\right)^H$.

Consider an element $\phi_2 \in \left(\mathcal{P}^{(1)}(\mathfrak{h})\right)^H$. On the one hand, we may regard ϕ_2 as an element of $\mathfrak{h} \otimes K(\mathfrak{h})$, on the other hand, it is easy to verify that also $\phi_2 \in V \otimes K^1(\mathfrak{h}) \subset V \otimes V^* \otimes K(\mathfrak{h})$. Thus, by the natural contractions, ϕ_2 induces H-equivariant linear maps

$$\phi_2' : \mathfrak{h}^* \longrightarrow K(\mathfrak{h}) \qquad \text{and} \qquad \phi_2'' : V^* \longrightarrow K^1(\mathfrak{h}). \tag{19}$$

Theorem 3.7 *([CMS], [S3]) Let* H $\subset$ Aut(V) *be a closed irreducible subgroup with Lie algebra* $\mathfrak{h} \subset \mathrm{End}(V)$*, and suppose that there is an element* $\phi_2 \in \left(\mathcal{P}^{(1)}(\mathfrak{h})\right)^H$*. We denote by* $\mathcal{R} \subset K(\mathfrak{h})$ *and* $\mathcal{R}^{(1)} \subset K^1(\mathfrak{h})$ *the images of the corresponding* H*-equivariant maps* ϕ_2' *and* ϕ_2'' *from* (19)*.*

Then every torsion free affine connection of regularity C^4 *whose curvature is contained in* $\mathcal{R}$ *and whose covariant derivatives of the curvature are contained in* $\mathcal{R}^{(1)}$ *is a Poisson connection induced by a polynomial map*

$$\phi = \phi_2 + \tau,$$

with $\phi_2 \in \mathcal{P}^{(1)}(\mathfrak{h})$ *from above and some* H*-invariant (possibly vanishing)* 2*-form* τ*.*

For the proof, we shall need the following Lemma whose proof we shall not present here.

Lemma 3.8 *([CMS], [S3]) Let* H $\subset$ Aut(V) *be an irreducible representation of a connected, reductive Lie group* H*, and let* $\mathfrak{h} \subset \mathrm{End}(V)$ *be the corresponding Lie algebra. If* $\tau \in V^* \otimes V^*$ *satisfies the condition*

$$\tau(x, A \cdot y) = \tau(y, A \cdot x) \quad \text{for all } x, y \in V \text{ and } A \in \mathfrak{h},$$

then τ *is skew-symmetric and hence an* H*-invariant 2-form.*

Proof of Theorem 3.7. Let $F \subset \mathfrak{F}_V$ be an H-structure on the manifold M where $\mathfrak{F}_V \to M$ is the V-valued coframe bundle of M, and denote the tautological V-valued 1-form on F by θ. Suppose that F is equipped with a torsion free connection, i.e. an $\mathfrak{h}$-valued 1-form η on F. Since the curvature lies in $\mathcal{R}$, the *first and second structure equations* read

$$\begin{aligned} d\theta &= -\eta \wedge \theta \\ d\eta &= -\eta \wedge \eta - 2(\phi_2'(\rho)) \circ (\theta \wedge \theta), \end{aligned} \tag{20}$$

where $\rho : F \to \mathfrak{h}^*$ is an H-equivariant map. Differentiating (20) and using that the covariant derivative of the curvature is contained in $\mathcal{R}^{(1)}$ yields the *third structure equation* for the differential of ρ:

$$d\rho = -\eta \cdot \rho + \jmath(u \otimes \theta), \tag{21}$$

for some H-equivariant map $u : F \to V^*$, where $\jmath : V^* \otimes V \to \mathfrak{h}^*$ is the natural projection. The multiplication in the first term refers to the coadjoint action of $\mathfrak{h}$ on $\mathfrak{h}^*$. In other words, (21) should be read as

$$\begin{aligned}(\xi_A \rho)(B) &= \rho([A, B])\\ (\xi_x \rho)(B) &= u(B \cdot x).\end{aligned}$$

Let us define the map $\mathbf{c} : F \to V^* \otimes V^*$ by

$$\mathbf{c}_p(x, y) := du(\xi_x)(y) - \phi_2(\rho_p, \rho_p, x, y). \tag{22}$$

Differentiation of (21) yields

$$\mathbf{c}_p(x, Ay) = \mathbf{c}_p(y, Ax) \quad \text{for all } x, y \in V \text{ and all } A \in \mathfrak{h}.$$

Then Lemma 3.8 implies that $\mathbf{c}_p \in \Lambda^2 V^*$ is H-invariant. Moreover, differentiation of (22) implies that $\xi_A(\mathbf{c}) = 0$ and $(\xi_x \mathbf{c})(y, z) = (\xi_y \mathbf{c})(x, z)$ for all $A \in \mathfrak{h}$ and $x, y, z \in V$. Since $\mathbf{c}$ is skew-symmetric, it follows that

$$d\mathbf{c} = 0,$$

i.e. $\mathbf{c}_p \equiv \tau \in \Lambda^2 V^*$ is *constant*. Thus, the H-equivariance of u and (22) yield

$$du = -\eta \cdot u + \left(\rho_p^2 \lrcorner \phi_2 + \tau\right) \circ \theta, \tag{23}$$

where $\lrcorner$ refers to the contraction of $\rho_p^2 \in S^2\mathfrak{h}^*$ with $\phi_2 \in S^2\mathfrak{h} \otimes \Lambda^2 V^*$. In other words, (23) should be read as

$$\begin{aligned}(\xi_A u)(y) &= u(A \cdot y)\\ (\xi_x u)_p(y) &= \phi_2(\rho_p, \rho_p, x, y) + \tau(x, y).\end{aligned}$$

Let us now define the Poisson structure on $W^* = \mathfrak{h}^* \oplus V^*$ induced by $\phi := \phi_2 + \tau$, and let $\pi := \rho + u : F \to W^*$. From (21) and (23) it follows that $d\pi(\xi_w) = \eta_w$ for all $w \in W$, and from there it follows that, at least locally, the connection is indeed a Poisson connection induced by ϕ. ∎

Corollary 3.9 *Let* $\mathrm{H} \subset \mathrm{Sp}(V, \Omega)$ *be a special symplectic subgroup associated to the simple Lie group* G. *Then every special symplectic connection associated to* G *is locally equivalent to a Poisson connection induced by the map* $\phi^c : \mathfrak{h}^* \to \Lambda^2 V^*$ *given by*

$$\langle \phi^c(\rho), x \wedge y \rangle = \Omega((\rho^2 + (c - (\rho, \rho)))x, y) \tag{24}$$

for some constant c. *Here, we identify* $\mathfrak{h}$ *and* $\mathfrak{h}^*$ *via the* H*-invariant bilinear form* $(\ ,\ )$ *on* $\mathfrak{h}$ *from (5).*

Proof. For $c \in \mathbb{F}$, we define the H-invariant element $\phi_2^c \in S^2\mathfrak{h} \otimes \Lambda^2 V$ by

$$\phi_2^c(\rho_1, \rho_2, x, y) = \frac{1}{2}\Omega((\rho_1\rho_2 + \rho_2\rho_1)x, y) + (c - (\rho_1, \rho_2))\Omega(x, y).$$

It is straightforward to verify that the induced contraction maps $\phi_2' : \mathfrak{h} \to K(\mathfrak{h})$ and $\phi_2'' : V^* \to K^1(\mathfrak{h})$ from (19) coincide with the maps $h \mapsto R_h$ and $u \mapsto R_{u\circ\_}$ from Theorem 2.8 so that the hypotheses of Theorem 3.7 are satisfied. ∎

While Corollary 3.9 shows that any special symplectic connection is locally equivalent to a Poisson connection induced by ϕ^c for some $c \in \mathbb{F}$, the investigation of the *global* properties of special symplectic connections will depend on the question whether or not the Poisson structures on $\mathfrak{h}^* \oplus V^*$ induced by ϕ^c admit a complete symplectic realization. In fact, rather than regarding ϕ^c as a one parameter family of Poisson structures on $\mathfrak{h}^* \oplus V^*$, we may regard them as a single Poisson structure on $\mathfrak{h}^* \oplus V^* \oplus \mathbb{F}$, given by

$$\begin{aligned}\{h_1 + v_1 + t_1, h_2 + v_2 + t_2\}(\rho, u, c) := \quad & (\rho, [h_1, h_2]) + \Omega(u, h_1 v_2 - h_2 v_1) + c\,\Omega(v_1, v_2) \\ & + \Omega((\rho^2 - (\rho, \rho)) v_1, v_2),\end{aligned} \tag{25}$$

where $h_i, \rho \in \mathfrak{h}$, $v_i, u \in V$, $t_i, c \in \mathbb{F}$ and where we identify $\mathfrak{h}^* \oplus V^* \oplus \mathbb{F}$ with $\mathfrak{h} \oplus V \oplus \mathbb{F}$ via the inner product $(\ ,\ )$ from (5) and the symplectic form Ω. Note that the first three terms of the Poisson structure (25) yield the Lie-Poisson structure of the graded Lie algebra $\mathfrak{h} \oplus V \oplus \mathbb{F}$ where the bracket on V takes values in $\mathbb{F}$ and is given by Ω. Thus, we may regard this Poisson structure as a quadratic deformation of the linear one.

3.3 Two-gradable simple Lie algebras

A particularly well studied class of examples of symplectic groupoids are the *Lie-Poisson-groupoids* [V1]. Namely, let G be a Lie group with Lie algebra $\mathfrak{g}$. Then $T^*\mathrm{G}$ is a symplectic manifold, and there is a symplectic groupoid structure on $(T^*G, \mathfrak{g}^*, l, r, m, i)$, where $\mathfrak{g}^* \cong T_e^*\mathrm{G} \subset T^*\mathrm{G}$ is a submanifold, and where the left and right projections are given by

$$\begin{array}{rccclrcccl} l: & T^*\mathrm{G} & \longrightarrow & \mathfrak{g}^* \cong T_e^*\mathrm{G}, & \qquad & r: & T^*\mathrm{G} & \longrightarrow & \mathfrak{g}^* \cong T_e^*\mathrm{G} \\ & \alpha_g & \longmapsto & dR_g^*(\alpha_g) & & & \alpha_g & \longmapsto & dL_g^*(\alpha_g). \end{array} \tag{26}$$

Moreover, the induced Poisson structure on $\mathfrak{g}^*$ is defined by

$$\{f, g\}(\lambda) := \lambda([df_\lambda, dg_\lambda]), \qquad \text{where } df_\lambda, dg_\lambda \in T_\lambda^*\mathfrak{g}^* \cong \mathfrak{g}.$$

Lemma 3.10 *Let $\mathfrak{g}_+ \subset \mathfrak{g}$ be a subalgebra, and suppose that there is a $\lambda_+ \in \mathfrak{g}_+^*$ such that the map*

$$\mathfrak{g}_+ \longrightarrow \mathfrak{g}_+^*, \qquad x \longmapsto ad_x^*(\lambda_+) \tag{27}$$

is an isomorphism. Then the affine hyperplane

$$Q := \{\lambda \in \mathfrak{g}^* \mid \lambda|_{\mathfrak{g}_+} = \lambda_+\} \subset \mathfrak{g}^*$$

is a cosymplectic submanifold of $\mathfrak{g}^*$ and hence $\Gamma := l^{-1}(Q) \cap r^{-1}(Q) \subset T^*\mathrm{G}$ is a symplectic subgroupoid.

Proof. For $\lambda \in \mathfrak{g}^*$, we have $T_\lambda^*\mathfrak{g}^* \cong \mathfrak{g}$ and $T_\lambda\mathfrak{g}^* \cong \mathfrak{g}^*$. Moreover, $\# : T_\lambda^*\mathfrak{g}^* \to T_\lambda\mathfrak{g}^*$ is given by $x \mapsto ad_x^*\lambda$.

With these identifications, we thus have $T_\lambda Q^\perp = \mathfrak{g}_+$, and $W_\lambda = \{x \in \mathfrak{g} \mid ad_x^*(\lambda)|_{\mathfrak{g}_+} = 0\}$. Thus, for $x \in T_\lambda Q^\perp \cap W_\lambda$ with $\lambda \in Q$, we have $0 = (ad_x\lambda)|_{\mathfrak{g}_+} = ad_x(\lambda|_{\mathfrak{g}_+}) = ad_x\lambda_+$, hence by (27) we conclude that $x = 0$, so that $T_\lambda Q^\perp \cap W_\lambda = 0$, and since these two spaces have complementary dimensions, (9) follows. ∎

Let us now assume that $\mathfrak{g}$ is a simple Lie algebra which contains a maximal root, and use the decompositions (2) and the notation for the Lie bracket as in (4). With this, it is straightforward to verify that the Lie subalgebra and one form given as

$$\mathfrak{g}_+ := \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathbb{F}H_{\alpha_0} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathbb{F}e_+e_-, \qquad \lambda_+(e_+^2) = 1,\ \lambda_+(\mathfrak{g}_1 \oplus \mathbb{F}e_+e_-) = 0$$

satisfies the hypotheses of Lemma 3.10, hence we have the corresponding cosymplectic submanifold

$$Q = \left\{ \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \;\middle|\; \rho \in \mathfrak{h}, u \in V, f \in \mathbb{F} \right\} \subset \mathfrak{g}^* \cong \mathfrak{g}, \tag{28}$$

where we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by the biinvariant inner product $(\ ,\ )$, i.e. by the correspondence $x \mapsto (x, \_)$ for $x \in \mathfrak{g}$. Then we have the following proposition.

Proposition 3.11 *The diffeomorphism*

$$\Phi : Q \longrightarrow \mathfrak{h} \oplus V \oplus \mathbb{F}, \qquad \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \longmapsto \rho + u + (f + (\rho, \rho)) \tag{29}$$

is a Poisson isomorphism, where the cosymplectic submanifold $Q \subset \mathfrak{g}^*$ is equipped with the canonically induced Poisson structure (10), and $\mathfrak{h} \oplus V \oplus \mathbb{F}$ carries the quadratic Poisson structure from (25). In particular, these Poisson structures are integrable by a symplectic groupoid.

Proof. For $\lambda = \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \in Q$ we have $W_\lambda = \{x \in \mathfrak{g} \mid ad_x^*(\lambda)|_{g_+} = 0\}$, and thus by a straightforward calculation

$$W_\lambda = \mathbb{F}\lambda \oplus \mathfrak{h} \oplus \left\{ e_- \otimes x + e_+ \otimes \rho x + \frac{1}{2}\omega(u, x)e_+^2 \;\middle|\; x \in V \right\},$$

and hence the inversion $\jmath_\lambda^{-1} : T_\lambda^* Q \to W_\lambda$ where $T_\lambda^* Q \cong \mathfrak{g}/\mathfrak{g}^+ \cong \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{h}$ is given by

$$\jmath_\lambda^{-1}(h + e_- \otimes x + ce_-^2) = h + e_- \otimes x + e_+ \otimes \rho x + \frac{1}{2}\omega(u, x)e_+^2 + c(\lambda - \rho),$$

so that the coinduced Poisson structure on Q can be calculated by (10) is then given by

$$\{h + e_- \otimes x + te_-^2, h' + e_- \otimes x' + t'e_-^2\}_Q = (\lambda, [\jmath_\lambda^{-1}(h + e_- \otimes x + te_-^2), \jmath_\lambda^{-1}(h' + e_- \otimes x' + t'e_-^2)]),$$

and using (5) it is now straightforward to verify that this corresponds to the Poisson structure on $\mathfrak{h} \oplus V \oplus \mathbb{F}$ from (25) under the diffeomorphism Φ. ∎

Let us choose the diffeomorphism $l + \pi : T^*\mathrm{G} \to \mathfrak{g}^* \times \mathrm{G}$, where $l : T^*\mathrm{G} \to \mathfrak{g}^*$ is the symplectic realization from (26), and $\pi : T^*\mathrm{G} \to \mathrm{G}$ is the canonical projection. Evidently, under this identification, the subgroupoid $\Gamma := l^{-1}(Q) \cap r^{-1}(Q)$ is given by

$$\Gamma := \{(\lambda, g) \in Q \times \mathrm{G} \mid Ad_g^*\lambda \in Q\} \subset T^*\mathrm{G}.$$

Note that $\mathrm{H} \subset \mathrm{G}$ is the identity component of the subgroup stabilizing Q, i.e.

$$\mathrm{H} = (stab(Q))_0 = (\{g \in \mathrm{G} \mid Ad^*_g(Q) = Q\})_0 \subset \mathrm{G},$$

and evidently, Γ is invariant under the free actions of H on $T^*\mathrm{G}$ induced by either left or right multiplication. Also, for a fixed $\lambda_0 \in Q$, we let

$$S_{\lambda_0} := stab(\lambda_0) = \{g \in \mathrm{G} \mid Ad^*_g(\lambda_0) = \lambda_0\}$$

and

$$\Sigma_{\lambda_0} := l^{-1}(\lambda_0) \cap \Gamma = \{\alpha_g \in T^*_g\mathrm{G} \mid R^*_g\alpha_g = \lambda_0,\ L^*_g\alpha_g \in Q\}. \tag{30}$$

Thus, Σ_{λ_0} is H-invariant if H acts by right multiplication on $T^*\mathrm{G}$, and under the action of S_{λ_0} when S_{λ_0} acts on $T^*\mathrm{G}$ by left multiplication. Evidently, either of these commuting actions is free, whereas the induced product action of $S_{\lambda_0} \times \mathrm{H}$ fails to be free in general.

Proposition 3.12 *For a fixed $\lambda_0 \in Q$, define $\Sigma_{\lambda_0} \subset \Gamma \subset T^*\mathrm{G}$ as in (30). Then $r : \Sigma_{\lambda_0} \to Ad^*_G(\lambda_0) \cap Q$ is a principal S_{λ_0}-bundle.*

Define the function $\rho + u + f : Q \to \mathfrak{h} \oplus V \oplus \mathbb{F}$ as in (28). Then there is an $(\mathfrak{h} \oplus V \oplus \mathbb{F})$-valued one-form $\sigma = \eta + \theta + \kappa$ with $\kappa \in \Omega^1(Q)$, $\theta \in \Omega^1(Q) \otimes V$ and $\eta \in \Omega^1(Q) \otimes \mathfrak{h}$, such that the following equations are satisfied:

$$d\kappa = \omega(\theta \wedge \theta) \tag{31}$$

and

$$\begin{aligned} d\theta + \eta \wedge \theta &= 0 \\ d\eta + [\eta, \eta] &= R_\rho(\theta \wedge \theta) \end{aligned} \qquad \begin{aligned} d\rho + [\eta, \rho] &= u \circ \theta \\ du + \eta \cdot u &= (\rho^2 + f) \cdot \theta \\ df + d(\rho, \rho) &= 0 \end{aligned} \tag{32}$$

with the map $R : \mathfrak{h} \to K(\mathfrak{h})$ from (7). Moreover, for any $\lambda_0 \in Q$, the restriction of σ to Σ_{λ_0} yields an isomorphism $\sigma_p : T^*_p\Sigma_{\lambda_0} \to \mathfrak{h} \oplus V \oplus \mathbb{F}$ for all $p \in \Sigma_{\lambda_0}$, and κ, θ and η are invariant under the action of S_{λ_0}. Finally, κ is also invariant under the action of H*, while θ, η are* H*-equivariant.*

Proof. Since $l, r : \Gamma \rightrightarrows Q$ is a symplectic groupoid, it follows by standard arguments that $r : \Sigma_{\lambda_0} \to Ad^*_G(\lambda_0) \cap Q$ is a principal S_{λ_0}-bundle.

We use the Poisson equivalence $\Phi : Q \to \mathfrak{h} \oplus V \oplus \mathbb{F}$ from (29) to regard Γ as a groupoid over $\mathfrak{h} \oplus V \oplus \mathbb{F}$, where the latter is equipped with the Poisson structure given in (25) with $c = f + (\rho, \rho)$.

We regard each element $a := h + x + t \in \mathfrak{h} \oplus V \oplus \mathbb{F}$ as a constant one form on $\mathfrak{h} \oplus V \oplus \mathbb{F}$, given as $a(h', x', t') := (h, h') + \omega(x, x') + tt'$, and define the vector field on $\Gamma \subset T^*\mathrm{G}$ by

$$(\xi_a)_p := \#(dr^*(\jmath^{-1}_{r(p)}(a))).$$

By (14) and (15), it follows that these vector fields are tangent to Σ_{λ_0} and in fact establish a linear isomorphism $\mathfrak{h} \oplus V \oplus \mathbb{F} \cong T_p\Sigma_{\lambda_0}$. Thus, we can define the form $\sigma \in \Omega^1(\Sigma_{\lambda_0}) \otimes \mathfrak{g}/\mathfrak{g}_+$ by $\sigma(\xi_a) := a$. Evidently, the vector fields ξ_a and hence σ are S_{λ_0}-invariant and H-equivariant, so the statements about the invariance and equivariance follow.

Next, $dr(\xi_a) = \#\jmath_{r(p)}^{-1}(a) = -[\jmath_{r(p)}^{-1}(a), r]$ follows from (14), and this implies the second part of (32). Finally, for the Lie brackets we calculate for $a, a' \in \mathfrak{h} \oplus V \oplus \mathbb{F}$,

$$[\xi_a, \xi_{a'}] = \xi_{\{a,a'\}},$$

and then the asserted structure equations follow immediately from the form of the Poisson bracket given in (25). ∎

3.4 The structure equations of special symplectic connections

By Corollary 3.9, *any* special symplectic connection is locally equivalent to a Poisson connection. This together with the explicit descrition of the preceding section yields the following result.

Proposition 3.13 *Let (M, ω, ∇) be a (real or complex) symplectic manifold of dimension ≥ 4 with a special symplectic conection of regularity C^4, and let $\pi : B \to M$ be the associated* H*-structure on M. Then ∇ is a connection of Poisson type, induced by the Poisson structure on $\mathfrak{h}^* \oplus V^*$ which comes from the deforming map ϕ^c from (24).*

In particular, there are maps $\rho : B \to \mathfrak{h}$, $u : B \to V$ and a constant $c \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, such that the tautological form $\theta \in \Omega^1(B) \otimes V$, the connection form $\eta \in \Omega^1(B) \otimes \mathfrak{h}$ and the functions ρ and u satisfy the structure equations (32) where $f := c - (\rho, \rho)$.

Observe that (32) is equivalent to the equations

$$dq = [q, \sigma] \qquad \text{and} \qquad d\sigma + \frac{1}{2}\sigma \wedge \sigma = \omega(\theta \wedge \theta)q, \tag{33}$$

where $q : B \to Q \subset \mathfrak{g}^*$ is defined as $q := \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2$, and where

$$\sigma_\lambda := e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \frac{1}{2}\omega(u, \theta)e_+^2 \quad \text{with } \lambda = \frac{1}{2}e_-^2 + \rho + e_+ \otimes u + \frac{1}{2}fe_+^2 \in Q. \tag{34}$$

The first equations in (33) implies that the image $q(B) \subset Q$ is contained in a single adjoint orbit as B is connected. Let $\lambda_0 \in Q$ be such that $q(B) \subset Q \cap Ad_G(\lambda_0)$, and consider the principal S_{λ_0}-bundle $r : \Sigma_{\lambda_0} \to Q \cap \mathrm{Ad}_G(\lambda_0)$ from Proposition 3.12. If we define the map

$$p_1 : B_{\hat{S}} := \pi^*(\Sigma_{\lambda_0}) \longrightarrow B, \tag{35}$$

then, since $\Sigma_{\lambda_0} \subset (Q \cap \mathrm{Ad}_G(\lambda_0)) \times \mathrm{G}$, we may regard $B_{\hat{S}}$ as a regular submanifold of $B \times \mathrm{G}$ such that p_1 corresponds to the restriction of the canonical projection onto the first factor.

Note that there is a free right H-action on $B \times \mathrm{G}$, given by $(b, g) \cdot h := (b \cdot h, gh)$, where the multiplication on the first factor denotes the principal action of the bundle $B \to M$. Observe that $B_{\hat{S}} \subset B \times \mathrm{G}$ is invariant under this action, thus the quotient of the H-action yields the principal bundles

$$\begin{array}{ccc} B_{\hat{S}} & \hookrightarrow & B \times \mathrm{G} \\ \downarrow{\scriptstyle \mathrm{H}} & & \downarrow{\scriptstyle \mathrm{H}} \\ M_{\hat{S}} & \hookrightarrow & B \times_H \mathrm{G} \\ & {\scriptstyle \hat{S}} \searrow \quad \swarrow {\scriptstyle \mathrm{G}} & \\ & M & \end{array}$$

where $M_{\hat{S}} := (B_{\hat{S}})/\mathrm{H}$ and where the indices at the arrows represent the structure groups.

Proposition 3.14 *Let $\mu \in \Omega^1(\mathrm{G}) \otimes \mathfrak{g}$ be the left invariant Maurer-Cartan form of* G*, i.e.* $\mu = g^{-1}dg$*. Then the restriction of the* $\mathfrak{g}$*-valued one form*

$$\alpha \in \Omega^1(B \times G) \otimes \mathfrak{g}, \qquad \alpha := \mathrm{Ad}_g(\mu - \sigma)$$

to $B_{\hat{S}} \subset B \times G$ takes values in $\hat{\mathfrak{s}}$ and establishes a connection on the left principal $\hat{S}$-bundle $p_1 : B_{\hat{S}} \to B$ whose curvature is given by

$$d\alpha - \frac{1}{2}[\alpha, \alpha] = -2\pi^*(\omega)\lambda_0. \tag{36}$$

Moreover, $\alpha = p^*(\alpha_0)$ for some $\hat{\mathrm{S}}$-invariant one form $\alpha_0 \in \Omega^1(M_{\hat{S}}) \otimes \hat{\mathfrak{s}}$, where $p : B_{\hat{S}} \to M_{\hat{S}}$ is the principal H*-bundle from above. In particular, $d\alpha_0 - \frac{1}{2}[\alpha_0, \alpha_0] = -2\pi^*(\omega)\lambda_0$ as well.*

The minus sign on the left hand side of (36) is due to the fact that $p_1 : B_{\hat{S}} \to B$ is a *left* principal bundle. Also, note that the submersion $M_{\hat{S}} \to M$ is *not* a (left) principal bundle in general since the action of $\hat{\mathrm{S}}$ on $M_{\hat{S}}$ may fail to be free, though it is locally free and acts transitively on the fibers.

Proof. We differentiate the defining relation

$$B_{\hat{S}} = \{(b, g) \in B \times \mathrm{G} \mid \mathrm{Ad}_g(q(b)) = \lambda_0\}$$

to obtain for $(v, xg) \in T_{(b,g)}B_{\hat{S}}$

$$\begin{aligned} 0 &= [x, \mathrm{Ad}_g q(b)] + \mathrm{Ad}_g(dq_b(v)) \\ &= [x, \mathrm{Ad}_g r(b)] + \mathrm{Ad}_g[q(b), \sigma(v)] \quad \text{by (33)} \\ &= [\lambda_0, -x + \mathrm{Ad}_g \sigma(v)] \qquad \text{since } (b, g) \in B_{\hat{S}} \\ &= -[\lambda_0, \alpha(v, xg)], \end{aligned}$$

whence $\alpha(TB_{\hat{S}}) \subset \hat{\mathfrak{s}}$ for as claimed. The equivariance of α is clear, and the restriction of α to the principal fibers coincides with $\mathrm{Ad}_g\mu$, hence α is a connection.

Now we calculate $d\alpha = \mathrm{Ad}_g[\mu, \mu - \sigma] + \mathrm{Ad}_g(d\mu - d\sigma)$ and $[\alpha, \alpha] = \mathrm{Ad}_g([\mu, \mu] - 2[\mu, \sigma] + [\sigma, \sigma])$. Using that $d\mu + \frac{1}{2}[\mu, \mu] = 0$ by the Maurer-Cartan equation, we conclude that $d\alpha - \frac{1}{2}\alpha \wedge \alpha = -\mathrm{Ad}_g(d\sigma + \frac{1}{2}[\sigma, \sigma]) = -\mathrm{Ad}_g(2\pi^*(\omega)q)$ by (33), and since $\mathrm{Ad}_g q = \lambda_0$ on $B_{\hat{S}}$, (36) follows.

For $h \in \mathrm{H}$ we have $dR_h^*(\alpha_{(bh,gh)}) = dR_h^*(\mathrm{Ad}_{gh}\mu) - dR_h^*(\mathrm{Ad}_{gh}\sigma) = (\mathrm{Ad}_{gh} dR_h^*\mu) - (\mathrm{Ad}_{gh} dR_h^*\sigma) = (\mathrm{Ad}_g dL_h^*\mu) - (\mathrm{Ad}_g(\sigma)) = \alpha$, as μ is left invariant and σ is Ad_H-equivariant. Thus, α is H-invariant, and for $x \in \mathfrak{h}$, we compute $\alpha(b \cdot x, gx) = \mathrm{Ad}_g(\mu(gx) - \sigma(x)) = \mathrm{Ad}_g(x - x) = 0$ since $\sigma(x) = x$ for all $x \in \mathfrak{h}$ by (34), so the final claim follows. ∎

3.5 Proof of the results from the introduction

Proof of Theorem B. The first statement was already shown in Proposition 3.13.

Let $\hat{B} \subset B_{\hat{S}}$ be the holonomy reduction of the connection α from Proposition 3.14, and let $\hat{M} := \hat{B}/\mathrm{H} \subset M_{\hat{S}}$. Then $\hat{B} \to B$ is a principal $\hat{\mathrm{T}}$-bundle where $\hat{\mathrm{T}} \subset \mathrm{S}_{\lambda_0}$ is the holonomy group of α. By the Ambrose-Singer holonomy theorem, $\hat{\mathrm{T}} \subset \mathrm{S}_{\lambda_0}$ is a one dimensional (not necessarily closed) subgroup whose identity component equals $\mathrm{T} = \exp(\mathbb{F}\lambda_0)$ where $\lambda_0 \in \mathfrak{g}^* \cong \mathfrak{g}$ is such that $q(B) \subset Q \cap Ad_G(\lambda_0)$. Moreover, the restrictions of α and α_0 to $\hat{B}$ and $\hat{M}$, respectively, take the form $\alpha = \kappa\lambda_0$ and $\alpha_0 = \kappa_0\lambda_0$ for some $\kappa_0 \in \Omega^1(\hat{M})$ and $\kappa = dp_1^*(\kappa_0)$. In particular, by (31) we have $d\kappa = dp_1^*(d\kappa_0) = -2\pi^*\omega$ with $p_1 : B_{\hat{S}} \to B$ from (35).

Combining all of this with the principal map $B \to M$, we obtain the principal $(\hat{\mathrm{T}} \times \mathrm{H})$-bundle $\hat{B} \to M$ which is equipped with the $(\mathbb{F}\lambda_0 \oplus \mathfrak{h})$-valued connection form $\kappa\lambda_0 + \eta$. Taking the H-quotient implies that $\hat{M} \to M$ is a principal $\hat{\mathrm{T}}$-bundle with connection one form κ_0, and this shows the second part.

Let us now assume that the holonomy $\hat{\mathrm{T}} = \mathrm{T} = \exp(\mathbb{F}\lambda_0)$ of $\hat{M} \to M$ is connected, which can be achieved by replacing M by its regular cover $\hat{M}/\gamma$ where $\gamma := \hat{\mathrm{T}}/\mathrm{T}$. Since $\kappa\lambda_0 = \alpha = \mathrm{Ad}_g(\mu - \sigma)$, it follows by (34) that

$$\mu = \kappa \mathrm{Ad}_{g^{-1}}\lambda_0 + \sigma = \kappa q + e_- \otimes \theta + \eta + e_+ \otimes (\rho\theta) + \frac{1}{2}\omega(u,\theta)e_+^2. \tag{37}$$

Now (37) implies that the restriction of $(q \times Id_G)$ to $\hat{B} \subset B \times \mathrm{G}$ is a local diffeomorphism with an open subset of $\Sigma_{\lambda_0} \subset \Gamma$, and this map is equivariant w.r.t. the action of T on $\Sigma_{\lambda_0} \subset \Gamma \subset \mathfrak{g}^* \times \mathrm{G} \cong T^*\mathrm{G}$ which is given by left multiplication on the second factor. ∎

Note that $\mathrm{T}\backslash\Sigma_{\lambda_0}$ is not necessarily a manifold. Indeed, it *is* a manifold iff $\mathrm{T} \subset \mathrm{G}$ is a *closed* subgroup.

Proof of Theorem A. That $\Gamma \rightrightarrows Q$ is a groupoid realization was already shown.

Suppose that $l, r : \Gamma_c \rightrightarrows \mathfrak{h}^* \oplus V^*$ is a realization by a groupoid. Let $\lambda_0 \in \mathfrak{h}^* \oplus V^*$, and define $B_0 := r^{-1}(\lambda_0) \subset \Gamma_c$ such that $l : B_0 \to l(B_0) \subset \mathfrak{h}^* \oplus V^*$ is a principal bundle whose structure group we denote by S_0. By Proposition 3.5, there is an $(\mathfrak{h} \oplus V)$-valued coframe $\theta + \eta$ on B_0, which satisfies (17), and from the form of ϕ^c, it follows that $q^*(d\Phi) \circ (\theta \wedge \theta) = R_\rho(\theta \wedge \theta)$, so that $\theta + \eta$ satisfies the structure equations (32).

Therefore, as in the proof of Theorem B, there is a covering $B \to B_0$, a principal T-bundle $\hat{B} \to B$ and a T-equivariant immersion $\hat{B} \to \Sigma_{\lambda_0}$ such that

$$\begin{array}{ccccc}
\hat{B} & \xrightarrow{q \times Id_G} & \Sigma_{\lambda_0} & \hookrightarrow & \Gamma \\
\downarrow \mathrm{T} & & \downarrow \mathrm{T} & & \downarrow \mathrm{T} \\
B & \longrightarrow & \mathrm{T}\backslash\Sigma_{\lambda_0} & \hookrightarrow & \mathrm{T}\backslash\Gamma
\end{array}$$

commutes.

Now consider $F_0 := l^{-1}(\lambda_0) \subset \Sigma_{\lambda_0}$ which is acted on simply transitively by S_{λ_0} where $\mathrm{S}_{\lambda_0} \subset \mathrm{G}$ is the stabilizer of $\lambda_0 \in \mathfrak{h}^* \oplus V^* \subset \mathfrak{g}^*$. By hypothesis, the restriction of $q \times Id_G$ to $\tilde{F}_0 := (q \times Id_G)^{-1}(F_0) \subset \hat{B}$ is an equivariant covering which we may hence regard as the covering of Lie groups $\hat{\mathrm{S}} \to \mathrm{S}_{\lambda_0}$. Moreover, the principal action of T on the fiber $\hat{\mathrm{S}} \cong \tilde{F} \subset \hat{B}$ is given by left multiplication, so that it follows that $\mathrm{S}_0 = \mathrm{T}\backslash\hat{\mathrm{S}}_0$, and this must be a *manifold*.

Thus, $\mathrm{T} \subset \hat{\mathrm{S}}$ must be a *closed* Lie subgroup, which imples that $\mathrm{T}_0 := \exp(\mathbb{F}\lambda_0) \subset \mathrm{S}_{\lambda_0}$ is a closed subgroup, and since $\mathrm{S}_{\lambda_0} \subset \mathrm{G}$ is closed, being the stabilizer of λ_0, it follows that $\mathrm{T}_0 \subset \mathrm{G}$ must be a closed subgroup.

But now, it is easy to see that for any $c \in \mathbb{F}$, there is a $\lambda_0 \in \mathfrak{g} \cong \mathfrak{g}^*$ such that $(\lambda_0, \lambda_0) = c$ and $\mathrm{T}_0 = \exp(\mathbb{F}\lambda_0) \subset \mathrm{G}$ is *not* closed, which is a contradiction. ∎

Corollary C now follows immediately from Theorem B and Proposition 3.14.

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