CONNECTIONS WITH EXOTIC HOLONOMY

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In Dankbarkeit meinen Eltern

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ABSTRACT

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In 1955, Berger [Ber] partially classified the possible irreducible holonomy representations of torsion free connections on the tangent bundle of a manifold. However, it was shown by Bryant [Br2] that Berger's list is incomplete. Connections whose holonomy is not contained on Berger's list are called *exotic*.

We investigate examples of a certain 4-dimensional exotic holonomy representation of $Sl(2,\mathbb{R})$. We show that connections with this holonomy are never complete, give explicit descriptions of these connections on an open dense set and compute their group of symmetry. Finally, we give strong restrictions for their existence on compact manifolds.

Was im Endlichen gilt,

gilt im Unendlichen schon lange.

Thomas Horsch

Contents

1 Introduction

Let M^n be a smooth connected n-dimensional manifold. Let $\mathcal{P}(M)$ denote the set of piecewise smooth paths $\gamma : [0,1] \to M$, and for $x \in M$, let $\mathcal{L}_x(M) \subseteq \mathcal{P}(M)$ denote the set of x-based loops, i.e. paths for which $\gamma(0) = \gamma(1) = x$.

Let ∇ be a torsion free affine connection on the tangent bundle of M. For each $\gamma \in \mathcal{P}(M)$, the connection ∇ defines a linear isomorphism $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$, called *parallel translation along* γ. For each $x \in M$, we define the *holonomy group of* ∇ *at* x to be $H_x := \{P_\gamma \mid \gamma \in \mathcal{L}_x\} \subseteq Gl(T_xM)$.

It is well known that H_x is a Lie subgroup of $Gl(T_xM)$, and that for any $\gamma \in \mathcal{P}(M)$, P_γ induces an isomorphism of $T_{\gamma(0)}M$ with $T_{\gamma(1)}M$ which identifies $H_{\gamma(0)}$ with $H_{\gamma(1)}$ [KN].

Choose an $x_0 \in M$ and an isomorphism $i : T_{x_0}M \to \mathbb{R}^n$. Then, because M is connected, the conjugacy class of the subgroup $H \subseteq Gl(n,\mathbb{R})$ which corresponds under i to $H_{x_0} \subseteq Gl(T_{x_0}M)$ is independent of the choice of x_0 or i. By abuse of language, we speak of H as the *holonomy group* and of the Lie algebra $\mathfrak{h}$ of H as the *holonomy algebra of* (M,∇).

The following is a basic question in the theory:

Which (conjugacy classes of) subgroups $H \subseteq Gl(n,\mathbb{R})$ can occur as the holonomy of some torsion free connection ∇ on some n-manifold M?

Note that the condition of torsion freeness makes this problem non-trivial. In fact, it is not hard to see that any representation of a connected Lie group can be realized as the holonomy of some connection (with torsion) on an arbitrary manifold.

A necessary condition on the holonomy algebra of a torsion free connection was derived by M. Berger [Ber] in his thesis as follows.

Let V be a vector space and define for a given Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$:

$$\mathbf{K}(\mathfrak{g}) := \left\{ \phi : \mathbf{\Lambda^2}(V) \to \mathfrak{g} \;\Big|\; \phi \text{ linear}, \sum_{\sigma \in A_3} \phi(u_{\sigma(1)}, u_{\sigma(2)}) u_{\sigma(3)} = 0 \text{ for all } u_i \in V \right\},$$

and

$$\mathbf{K^1}(\mathfrak{g}) := \left\{ \psi : V \to \mathbf{K}(\mathfrak{g}) \mid \psi \text{ linear}, \sum_{\sigma \in A_3} \psi(u_{\sigma(1)})(u_{\sigma(2)}, u_{\sigma(3)}) = 0 \text{ for all } u_i \in V \right\}.$$

Given (M, ∇) as above and $x_0 \in M$, the *curvature tensor of* ∇ *at* x_0, i.e. the map $R_{x_0} : \mathbf{\Lambda}^2(T_{x_0}M) \to \mathfrak{gl}(T_{x_0}M)$ defined by $R_{x_0}(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$, is known to have its values in the holonomy algebra $\mathfrak{h}_{x_0} \subseteq \mathfrak{gl}(T_{x_0}M)$, and to satisfy the *first and second Bianchi identities.* This is equivalent to saying

$$R_{x_0} \in \mathbf{K}(\mathfrak{h}_{x_0}),$$

and

$$\nabla R_{x_0} \in \mathbf{K^1}(\mathfrak{h}_{x_0}).$$

In this notation, Berger's criterion is:

> If $\mathfrak{g}' \subset \mathfrak{g}$ is a *proper* sub-algebra, and $\mathbf{K}(\mathfrak{g}') = \mathbf{K}(\mathfrak{g})$, then $\mathfrak{g}$ *cannot* be the holonomy algebra of any torsion free connection on any n-manifold M.

This criterion is a consequence of the *Ambrose-Singer Holonomy Theorem*, which states that the holonomy algebra $\mathfrak{h}_{x_0}$ is generated by the image of the curvature map R_{x_0} and its parallel translations [KN, II.8.1].

The study of locally symmetric connections, i.e. connections with $\nabla R = 0$, can be reduced to certain problems in the theory of Lie algebras. We therefore wish to exclude this case from our discussion. A second necessary condition for $\mathfrak{g}$ to be the holonomy of a torsion free connection which is *not locally symmetric* is therefore

$$\mathbf{K^1}(\mathfrak{g}) \neq 0.$$

These two criteria are also referred to as *Berger's first and second criterion*. Using these, Berger [Ber] was able to partially classify the possible Lie algebras of holonomy groups of torsion free connections which are not locally symmetric. His classification falls into three parts:

- The first part classifies all possible *Riemannian* holonomies, i.e. connections with holonomy group $H \subseteq O(n, \mathbb{R})$. It turns out that the possible holonomy groups of non-symmetric connections are those subgroups which act transitively on the unit sphere in $\mathbb{R}^n$. In fact, it is by now well known which elements in Berger's list actually *do* occur as holonomies of Riemannian metrics. We mention in this context the work of Simons [S], Calabi [C], Alekseevskii [A] and Bryant [Br1].

- The second part classifies all possible irreducible *pseudo-Riemannian* holonomies, i.e. connections with holonomy group $H \subseteq O(p, q)$. In this case, the question whether or not these candidates actually *do* occur as holonomies has been resolved except for the group $SO^*(2n) \subseteq Gl(4n, \mathbb{R})$ for $n \geq 3$.

- The third part classifies the possible irreducibly acting holonomy groups of *affine torsion free connections*, i.e. holonomy groups which *do not* leave invariant any non-degenerate symmetric bilinear form. These connections are the least understood. In fact, Berger's list in this case is incomplete, conceivably omitting a finite number of possibilities. The holonomies which are *not* contained in Berger's list are referred to as *exotic holonomies.*

R.Bryant [Br2] showed that exotic holonomies do, in fact, exist. He investigated the irreducible representations of $Sl(2,\mathbb{R})$ which can be described as follows:

For $n \in \mathbb{N}$, let $V_n := \{\textit{homogeneous polynomials in } x \textit{ and } y \textit{ of degree } n\}$ which is an $n+1$-dimensional vector space. There is an $Sl(2,\mathbb{R})$-action on V_n induced by the transposed action of $Sl(2,\mathbb{R})$ on $\mathbb{R}^2$, i.e. if $p \in V_n$ and $A \in Sl(2,\mathbb{R})$ then

$$(A \cdot p)(x,y) := p(u,v) \quad \text{with} \quad (u,v) = (x,y)A.$$

It is well known that this action is irreducible for every n and moreover that - up to equivalence - this is the only irreducible $n+1$-dimensional representation of $Sl(2,\mathbb{R})$. [BD]

Let $H_n \subseteq Gl(V_n)$ be the image of this representation and let $\mathfrak{h}_n \subseteq \mathfrak{gl}(V_n)$ be the Lie algebra of H_n. Bryant showed that $\mathfrak{h}_n$ does *not* satisfy Berger's first and second criterion if $n \geq 4$.

For $n = 3$, however, he proved the existence of torsion free connections on 4-manifolds whose holonomy group is H_3, even though this group does *not* appear on Berger's list. We shall refer to these connections as H_3*-connections.*

A diffeomorphism $\phi : M \to M$ preserving the connection will be called a *symmetry of* ∇.

It turns out that locally there are very few examples of H_3-connections. In fact, the local classification given by Bryant can be summarized as follows:

- There is one example of a *homogeneous H_3-connection* whose symmetry group is *five-dimensional*.
- There is a finite set of H_3-connections with a *three-dimensional* symmetry group.
- There is a 1-parameter family of H_3-connections with a *one-dimensional* symmetry group.

This classification is obtained by the methods of Exterior Differential Systems. This approach, however, makes a concrete description of these connections very difficult. In this thesis, we will describe the H_3-connections more explicitly and will also investigate their global behavior.

- The *homogeneous H_3-connection* can be described *globally*.
- The *singular H_3-connections* can be described on the dense open subsets on which the action of their symmetry groups is *locally free*.
- For the *regular H_3-connections* we can only give a description on some open subset which will not be dense in general.

In particular, we will compute the Lie algebras of the symmetry groups of all H_3-connections.

Following this introduction, we will first give the descriptions of the singular H_3-connections (chapter 2).

In chapter 3, we derive the structure equations for H_3-connections, following closely [Br2].

As a first *global* result we will show in chapter 4 the

Theorem 4.1 *H_3-connections are never complete.*

In chapter 5, we then solve the structure equations for the *singular* H_3-connections under the generic assumption that the group action of the symmetry group is locally free, as well as the structure equations for the *regular* H_3-connections on some open subset. This will show that the examples given in chapter 2 form in fact a complete list of singular H_3-connections under these restrictions.

Finally, in chapter 6 we shall be concerned with H_3-connections on *compact* manifolds, and we will prove

Theorem 6.1 *Let M be a compact 4-manifold with an H_3-connection. Then M is locally homogeneous.*

In fact, we will prove that any H_3-connection on a compact manifold is given as a *biquotient* of a certain Lie group (Corollary 6.8). We suspect that these biquotients do not exists, i.e. that there are no H_3-connections on compact manifolds.

2 Examples of Singular H_3-connections

We first introduce some notational conventions. Recall that V_3 is the vector space of homogeneous polynomials of degree 3 in the variables x and y, and $Sl(2,\mathbb{R})$ acts on V_3 by the action induced by the transposed action of $Sl(2,\mathbb{R})$ on $\mathbb{R}^2 = span\{x,y\}$.

We let $\rho_3 : Sl(2,\mathbb{R}) \to H_3 \subseteq Gl(V_3)$ and $(\rho_3)_* : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{h}_3 \subseteq \mathfrak{gl}(V_3)$ denote the representation homomorphisms, and define a basis $\{E_1, E_2, E_3\}$ of $\mathfrak{h}_3$ by $E_i := (\rho_3)_*(\tilde{E}_i)$ where the basis $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$\begin{pmatrix} a & c \\ b & -a \end{pmatrix} = a\tilde{E}_1 + b\tilde{E}_2 + c\tilde{E}_3.$$

Furthermore, we let $\{e_0, \ldots, e_3\}$ with $e_i = x^{3-i}y^i$ be a basis of V_3, and $\{\underline{e}_i \mid 0 \leq i \leq 3\}$ be the standard basis of $\mathbb{R}^4$.

We fix once and for all a linear isomorphism

$$\begin{aligned} \lambda : \quad V_3 &\to \mathbb{R}^4 \\ e_i &\mapsto \underline{e}_i. \end{aligned}$$

Then $\underline{\mathfrak{h}}_3 := \lambda \mathfrak{h}_3 \lambda^{-1} \subseteq \mathfrak{gl}(4,\mathbb{R})$ has $\{\underline{E}_1, \underline{E}_2, \underline{E}_3\}$ with $\underline{E}_i = \lambda E_i \lambda^{-1}$ as a basis, and one sees that

$$r_1\underline{E}_1 + r_2\underline{E}_2 + r_3\underline{E}_3 = \begin{pmatrix} 3r_1 & r_3 & & \\ 3r_2 & r_1 & 2r_3 & \\ & 2r_2 & -r_1 & 3r_3 \\ & & r_2 & -3r_1 \end{pmatrix}.$$

We shall from now on use λ to identify $\mathfrak{h}_3$ with $\underline{\mathfrak{h}}_3$.

Recall that if M is a manifold with an H_3-connection ∇ then a diffeomorphism $\phi : M \to M$ preserving the connection ∇, i.e. satisfying $\phi_* (\nabla_X Y) = \nabla_{\phi_*(X)} \phi_*(Y)$ for all vector fields X and Y on M, will be called a *symmetry of* ∇.

In order to describe a connection on a manifold M, we shall in each case give a *frame* on M and the *connection form* θ w.r.t. this frame which takes values in $\mathfrak{h}_3$. This form is the pullback of the connection form on the *frame bundle* of M under the section given by the frame.

Let $X_0, \ldots, X_3$ be the given frame and let $\omega_0, \ldots, \omega_3$ denote the dual coframe. We define the V_3-valued 1-form ω by

$$\omega = \sum_i \omega_i \, e_i,$$

which establishes an isomorphism between T_pM and V_3 for all $p \in M$.

Then we can describe the *covariant derivative* associated to the connection by

$$\omega(\nabla_{X_i} X_j) := -\theta(X_i) \cdot \omega(X_j) = -\theta(X_i) \cdot e_j. \tag{1}$$

The connection being torsion free is equivalent to the condition that

$$\theta(X_i) \cdot e_j - \theta(X_j) \cdot e_i = d\omega(X_i, X_j) \quad \text{for all } i, j. \tag{2}$$

The holonomy algebra of these connections is contained in $\mathfrak{h}_3$ by the *Ambrose-Singer-Holonomy Theorem* mentioned earlier. In fact, we will show (Corollary 3.2) that the holonomy algebra of any such connection is actually *equal* to $\mathfrak{h}_3$, provided the connection is *not flat*.

As we shall see in chapter 3, to every H_3-connection we can associate a pair of homogeneous polynomials $a \in V_2$ and $b \in V_3$ as well as a constant $c \in \mathbb{R}$. For future reference we shall give these polynomials, called the *structure polynomials*, and this constant in each particular case.

The description of these examples will be motivated in chapter 5.

2.1 Example I : The homogeneous case (type Σ_0^0)

Let $G = ASl(2,\mathbb{R}) = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & x \\ & & y \\ \hline 0 & 0 & 1 \end{array} \right) \middle| A \in Sl(2,\mathbb{R}), x, y \in \mathbb{R} \right\}$ be the group of *unimodular affine motions* of $\mathbb{R}^2$ and let $\mathfrak{g}$ be the Lie algebra of G. Let A_{ij} stand for the 3×3-matrix with (i,j)-th entry 1 and all other entries 0. Then we define a basis of the Lie algebra $\mathfrak{g}$ of G by

$$\begin{aligned} Z_0 &= -9A_{23} \\ Z_1 &= -\frac{3}{2}A_{13} + 6A_{21} \\ Z_2 &= 3(A_{11} - A_{22}) \\ Z_3 &= -\frac{9}{2}A_{12} \\ Y &= A_{13} + 2A_{21} \end{aligned}$$

and the subgroup $H \leq G$ as $H = \{exp(tY) \mid t \in \mathbb{R}\}$. Furthermore, we decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

with $\mathfrak{h} = span(Y)$ and $\mathfrak{m} = span(Z_0, \ldots, Z_3)$.

One checks that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, i.e. the homogeneous space G/H is *reductive*. Moreover, if we define

$$\begin{array}{rll} \imath : & \mathfrak{m} & \to V_3 \\ & Z_i & \mapsto e_i \end{array}$$

then $ad(Y) = (\imath)^{-1} \circ E_3 \circ \imath$.

We wish to define an H_3-connection on G/H such that the canonical action of G on G/H is an action by *symmetries*. To do this, we need to define a map

$$\lambda : \mathfrak{m} \to \mathfrak{h}_3 \subseteq \mathfrak{gl}(V_3)$$

such that

$$\lambda([Y, Z_i]) = [E_3, \lambda(Z_i)] \quad \text{for} \quad i = 0, \dots, 3$$

and

$$\lambda(Z_i) \cdot e_j - \lambda(Z_j) \cdot e_i = \imath([Z_i, Z_j]_{\mathfrak{m}})$$

with the isomorphism $\imath$ defined above [KN, X 2.1,2.3,4.2]. The covariant derivatives are then defined by

$$\imath(\nabla_{Z_i} Z_j) = \lambda(Z_i) \cdot e_j,$$

where T_pM is identified with $\mathfrak{m}$.

One then checks that the following map satisfies these two conditions and thus defines an H_3-connection on M.

$$\lambda(Z_0) = 0, \quad \lambda(Z_1) = E_3, \quad \lambda(Z_2) = -E_1, \quad \lambda(Z_3) = -3E_2.$$

The structure polynomials and the structure constant in this case are

$$a = -2x^2, \quad b = 4x^3, \quad \text{and} \quad c = 0.$$

It is easily seen that G acts transitively on the set of parabolas in $\mathbb{R}^2$. Also, note that H is the group of unimodular affine motions which leave the standard parabola $y = x^2$ invariant. Therefore, we can naturally identify $M = G/H$ with the *space of parabolas in* $\mathbb{R}^2$.

Let $\omega_0, \ldots, \omega_3, \theta$ be the coframe on G dual to the vector fields $Z_0, \ldots, Z_3, Y$. Let $\tau_i := \omega_i \wedge \ldots \wedge \omega_3$ for $0 \leq i \leq 3$. It is easily seen that τ_i is invariant under the isotropy representation of H, and therefore there are induced forms on G/H which we also denote by $\tau_0, \ldots, \tau_3$.

One checks easily that these forms satisfy the *Frobenius condition*

$$d\tau_i = \alpha_i \wedge \tau_i \quad \text{for some 1-form } \alpha_i,$$

and hence the flag of distributions $\mathcal{D}_0 \subseteq \ldots \subseteq \mathcal{D}_3$ on the tangent space of G/H given by $\tau_i(\mathcal{D}_i) = 0$ is integrable [EDS, II.1.1].

Let $L : ASl(2, \mathbb{R}) \to Sl(2, \mathbb{R})$ be the homomorphism which maps an affine motion to its *linear part*. Then we can define an *angle function*

$$\begin{array}{rccc} \theta : & G/H & \longrightarrow & S^1 \\ & gH & \longmapsto & \dfrac{L(g) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{||L(g) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}||}. \end{array}$$

This function assigns to each parabola its direction, and it is easily checked that θ is a *fibration* over S^1 whose fibers are the maximal connected integral leaves (m.c.i.l.'s) of $\mathcal{D}_3$, and that the fibers are *totally geodesic*. Also, since these fibers are all parabolas with a given direction it is easy to see that they are diffeomorphic to $\mathbb{R}^3$.

Now let us investigate the level sets of θ. By homogeneity all level sets are equivalent. Thus, let $Y_3 := \{(y = e^a \, x^2 + b \, x + c) \mid a, b, c \in \mathbb{R}\}$. Then the function $r_3 : Y_3 \to \mathbb{R}$ which assigns the *slope* to a parabola, i.e. $r_3 \, (y = e^a \, x^2 + b \, x + c) = a$, satisfies $dr_3 = \omega_2|_{Y_3}$. Again, one can check that r_3 is a *fibration* of Y_3 over $\mathbb{R}$ whose fibers are *totally geodesic* and diffeomorphic to $\mathbb{R}^2$.

Next, let Y_2 be a level set of r_3, e.g. $Y_2 := r_3^{-1}(0) = \{(y = x^2 + b\,x + c) \mid b, c \in \mathbb{R}\}$, i.e. a set of parabolas with given direction and slope. We compute the connection on Y_2 and obtain that Y_2 is *flat*. Note that a curve γ in Y_2 can be described by the curve of the vertices of $\gamma(t)$. Then we compute that - up to parametrization - the vertices of a *geodesic* in Y_2 either move along a parabola of slope -2 or along a vertical line.

Also, note that the restriction $\omega_1|_{Y_2}$ is *closed*, hence there exists a function $r_2 : Y_2 \to \mathbb{R}$ such that $dr_2 = \omega_1|_{Y_2}$. One finds that the level sets of r_2 are the parabolas in Y_2 whose vertices lie on a fixed vertical line. By the previous, these level sets are *geodesics*.

Finally, if Y_1 is a level set of r_2, i.e. a geodesic, we find some parametrization $r_1 : Y_1 \to \mathbb{R}$ such that $dr_1 = \omega_0|_{Y_1}$.

Therefore, on every m.c.i.l. Y_i of $\mathcal{D}_i$, $i \leq 3$ we have a function $r_i : Y_i \to \mathbb{R}$ satisfying $dr_i = \omega_{i-1}|_{Y_i}$, and all these functions are *totally geodesic fibrations* of Y_i.

2.2 Example II : Types $\Sigma_0^{\pm}$ and Σ_c^1

Let $c \in \mathbb{R}$ be a given constant and define the Lie algebra $\mathfrak{g}_c^{\pm} = span(Z_1, Z_2, Z_3)$ with the bracket relations

$$\begin{aligned} [Z_1, Z_2] &= Z_3 \\ [Z_1, Z_3] &= -\frac{c}{8} Z_2 \\ [Z_2, Z_3] &= \mp 4c Z_1. \end{aligned}$$

It is easily seen that $\mathfrak{g}_0^{\pm} \cong \mathfrak{n}_3$, the Lie algebra of the 3-dimensional Heisenberg group N_3, $\mathfrak{g}_c^- \cong \mathfrak{su}(2)$ if $c > 0$ and $\mathfrak{g}_c^\pm \cong \mathfrak{sl}(2,\mathbb{R})$ in the remaining cases. We let $G_c^\pm$ be a Lie group corresponding to $\mathfrak{g}_c^\pm$ such that $G_0^\pm = N_3$, $G_c^\pm = Sl(2,\mathbb{R})$ or $G_c^- = SU(2)$ if $c > 0$.

Let $M_c^\pm := \mathbb{R}^+ \times G_c^\pm$ and let $t_0 : M_c^\pm \to \mathbb{R}^+$ be the projection onto the first factor. On $G_c^\pm$, we have the left invariant vector fields Z_1, Z_2, Z_3 corresponding to the basis of $\mathfrak{g}_c^\pm$. There exists a vector field Z_0 on $M_c^\pm$ satisfying $Z_0(t_0) \equiv 1$ and $[Z_0, Z_i] = 0$ for $i = 1, 2, 3$.

We then define a *frame* on $M_c^\pm$ by

$$\begin{aligned}
X_0 &= && -\frac{(12t_0^2 \pm c)}{2t_0} Z_1 && +2Z_3 \\
X_1 &= \frac{3}{4}(4t_0^2 \mp c) Z_0 && && +Z_2 \\
X_2 &= && \pm\frac{1}{t_0} Z_1 && \\
X_3 &= \pm\frac{3}{2} Z_0 && &&
\end{aligned}$$

Now we give a *connection* w.r.t. this frame by the $\mathfrak{h}_3$-valued 1-form

$$\theta_c^\pm = \theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3$$

where

$$\begin{aligned}
\theta_1 &= && -2t_0\, \omega_1, && \\
\theta_2 &= c\,\frac{4t_0^2 \mp c}{8t_0}\,\omega_0 && && -\frac{8t_0^2 \mp c}{4t_0}\,\omega_2, \quad \text{and} \\
\theta_3 &= -\frac{12t_0^2 \mp c}{4t_0}\,\omega_0 && && \mp\frac{1}{2t_0}\,\omega_2.
\end{aligned}$$

The covariant derivative is defined by (1). Moreover, one checks that (2) is satisfied, i.e. ∇ is torsion free. Also, ∇ is not flat, thus by Corollary 3.2 we conclude that ∇ is an H_3-connection.

Note that the natural left action of $G_c^\pm$ on $M_c^\pm$ is an action by *symmetries* since it fixes the level sets of t_0 and therefore preserves $\theta_c^\pm$.

The structure polynomials are

$$a = \pm x^2 - \tfrac{1}{2}\left(4t_0^2 \mp c\right) y^2, \quad \text{and} \quad b = -t_0 y \left(\pm 6x^2 + \left(4t_0^2 \mp c\right) y^2\right)$$

We shall see later that $M_{c_1}^{\varepsilon_1}$ and $M_{c_2}^{\varepsilon_2}$ with $\varepsilon_i \in \{\pm\}$ are equivalent iff $\varepsilon_1 = \varepsilon_2$ and $sign(c_1) = sign(c_2)$. Thus the $M_c^{\pm}$ give 6 different examples of H_3-connections.

2.3 Example III : Type Σ_c^2

Let $k \in \mathbb{R}\backslash\{0\}$ be a given constant and define the Lie algebra $\mathfrak{g}_k^{\pm} = span(Z_1, Z_2, Z_3)$ with the bracket relations

$$\begin{aligned} [Z_1, Z_2] &= \quad Z_3 \\ [Z_1, Z_3] &= \quad \mp 9 Z_2 \\ [Z_2, Z_3] &= \quad \pm 24 k Z_1. \end{aligned}$$

It is easily seen that $\mathfrak{g}_k^{+} \cong \mathfrak{su}(2)$ if $k > 0$ and $\mathfrak{g}_k^{\pm} \cong \mathfrak{sl}(2, \mathbb{R})$ in the remaining cases. We let $G_k^{\pm}$ be a Lie group corresponding to $\mathfrak{g}_k^{\pm}$ such that $G_k^{\pm} = Sl(2, \mathbb{R})$ or $G_k^{+} = SU(2)$.

Let $M_k^{\pm} := \mathbb{R}^+ \times G_k^{\pm}$ and let $t_0 : M_k^{\pm} \to \mathbb{R}^+$ be the projection onto the first factor. On $G_k^{\pm}$, we have the left invariant vector fields Z_1, Z_2, Z_3 corresponding to the basis of $\mathfrak{g}_k^{\pm}$. There exists a vector field Z_0 on $M_k^{\pm}$ satisfying $Z_0(t_0) \equiv 1$ and $[Z_0, Z_i] = 0$ for $i = 1, 2, 3$.

We then define a *frame* on $M_k^{\pm}$ by

$$\begin{aligned} X_0 &= \qquad -\frac{3}{2t_0}(2kt_0^2 \mp 1) Z_1 \qquad - Z_3 \\ X_1 &= -\frac{1}{2}(6kt_0^2 \pm 1) Z_0 \qquad + Z_2 \\ X_2 &= \qquad \frac{1}{2t_0} Z_1 \\ X_3 &= -\frac{3}{2} Z_0 \end{aligned}$$

Now we give a *connection* w.r.t. this frame by the $\mathfrak{h}_3$-valued 1-form

$$\theta_k^{\pm} = \theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3$$

where

$$
\begin{aligned}
\theta_1 &= && 2kt_0\,\omega_1, \\
\theta_2 &= -\frac{3}{2t_0}(\pm 6kt_0^2+1)\,\omega_0 && +\frac{1}{2t_0}(4kt_0^2 \mp 1)\,\omega_2, \quad \text{and} \\
\theta_3 &= \frac{3}{2t_0}(2kt_0^2 \pm 1)\,\omega_0 && +\frac{1}{2t_0}\,\omega_2.
\end{aligned}
$$

Again, the covariant derivative is defined by (1). Moreover, one checks that (2) is satisfied, i.e. ∇ is torsion free. Also, ∇ is not flat for $k \neq 0$, thus by Corollary 3.2 we conclude that ∇ is an H_3-connection.

Note that the natural left action of $G_k^{\pm}$ on $M_k^{\pm}$ is an action by *symmetries* since it fixes the level sets of t_0 and therefore preserves $\theta_k^{\pm}$.

The structure polynomials and the structure constant are

$$
a = k\,(x^2 \pm y^2) - 2\,(kt_0 y)^2, \quad b = 2kt_0 y\,(3k\,(x^2 \pm y^2) + 2\,(kt_0 y)^2) \quad \text{and} \quad c = \pm 6k^2.
$$

We shall see later that $M_{k_1}^{\varepsilon_1}$ and $M_{k_2}^{\varepsilon_2}$ with $\varepsilon_i \in \{\pm\}$ are equivalent iff $\varepsilon_1 = \varepsilon_2$ and $sign(k_1) = sign(k_2)$. Thus the $M_k^{\pm}$ give 4 different examples of H_3-connections, and we will also show that they are indeed different from the connections given in the previous section.

3 The Structure Equations

In this section we will mainly recall the results of Bryant [Br2] on H_3-connections and introduce a notation convenient for our purposes.

First, we shall define the bilinear parings

$$< \, , \, >_p : V_n \otimes V_m \to V_{n+m-2p}$$

by

$$< u, v >_p = \frac{1}{p!} \sum_{k=0}^{p} (-1)^k \begin{pmatrix} p \\ k \end{pmatrix} \frac{\partial^p u}{\partial^k x \partial^{p-k} y} \frac{\partial^p v}{\partial^{p-k} x \partial^k y} \qquad \text{for} \qquad u \in V_n, v \in V_m.$$

It can be shown that these pairing are $Sl(2, \mathbb{R})$-equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula.

Now we explicitly describe the spaces

$$\mathbf{K}(\mathfrak{g}) := \left\{ \phi : \mathbf{\Lambda^2}(V) \to \mathfrak{g} \;\middle|\; \phi \text{ linear}, \sum_{\sigma \in A_3} \phi(u_{\sigma(1)}, u_{\sigma(2)}) u_{\sigma(3)} = 0 \text{ for all } u_i \in V \right\},$$

and

$$\mathbf{K^1}(\mathfrak{g}) := \left\{ \psi : V \to \mathbf{K}(\mathfrak{g}) \mid \psi \text{ linear}, \sum_{\sigma \in A_3} \psi(u_{\sigma(1)})(u_{\sigma(2)}, u_{\sigma(3)}) = 0 \text{ for all } u_i \in V \right\}.$$

By straightforward computation we get as a basis for $\mathbf{K}(\mathfrak{h}_3)$ the maps

$\phi_0 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$	$\phi_1 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$	$\phi_2 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$
$\phi_0(e_0, e_1) = 0$	$\phi_1(e_0, e_1) = 0$	$\phi_2(e_0, e_1) = 6E_3$
$\phi_0(e_0, e_2) = 0$	$\phi_1(e_0, e_2) = -3E_3$	$\phi_2(e_0, e_2) = -3E_1$
$\phi_0(e_0, e_3) = -9E_3$	$\phi_1(e_0, e_3) = 9E_1$	$\phi_2(e_0, e_3) = 9E_2$
$\phi_0(e_1, e_2) = 5E_3$	$\phi_1(e_1, e_2) = -E_1$	$\phi_2(e_1, e_2) = -5E_2$
$\phi_0(e_1, e_3) = -3E_1$	$\phi_1(e_1, e_3) = 3E_2$	$\phi_2(e_1, e_3) = 0$
$\phi_0(e_2, e_3) = -6E_2$	$\phi_1(e_2, e_3) = 0$	$\phi_2(e_2, e_3) = 0$

Also, we compute

$$\mathbf{K^1}(\mathfrak{h}_3) := \{\psi_{b_0,\dots,b_3} \mid b_0,\dots,b_3 \in \mathbb{R}\},$$

with

$$\psi_{b_0,\dots,b_3}(e_i) := (3-i)b_{i+1}\,\phi_0 - (3-2i)b_i\,\phi_1 - ib_{i-1}\,\phi_2, 0 \le i \le 3.$$

As $Sl(2,\mathbb{R})$ acts both on V_3 and on $\mathfrak{h}_3$ via the adjoint representation, there is an induced action of $Sl(2,\mathbb{R})$ on $Hom(\Lambda^2 V_3, \mathfrak{h}_3)$ and $\mathbf{K}(\mathfrak{h}_3)$ is easily seen to be invariant under this action. Moreover, the basis ϕ_0, ϕ_1, ϕ_2 was chosen such that the map

$$\imath_{\mathbf{K}} : \begin{array}{cl} \mathbf{K}(\mathfrak{h}_3) & \to V_2 \\ \phi_i & \mapsto x^{2-i}y^i \end{array}$$

is an $Sl(2,\mathbb{R})$-equivariant isomorphism.

Furthermore, $\mathbf{K^1}(\mathfrak{h}_3)$ can be seen to be invariant under the $Sl(2,\mathbb{R})$-action on $Hom(V_3, \mathbf{K}(\mathfrak{h}_3))$, and the map

$$\imath_{\mathbf{K^1}} : \begin{array}{cl} \mathbf{K^1} & \longrightarrow V_3 \\ \psi_{b_0,b_1,b_2,b_3} & \longmapsto b_3\,x^3 - 3b_2\,x^2y + 3b_1\,xy^2 - b_0\,y^3 \end{array}$$

is an $Sl(2,\mathbb{R})$-equivariant isomorphism.

The following Lemma is very simple but will be useful later on.

Lemma 3.1 *Every $\phi \in \mathbf{K}(\mathfrak{h}_3)\backslash 0$ is surjective.*

Proof: Let $\phi = a_0\,\phi_0 + a_1\,\phi_1 + a_2\,\phi_2$ and suppose $a_0 \neq 0$. Then $\phi(e_2, e_3) = -6a_0\,E_2$, thus $E_2 \in im(\phi)$. Also, $\phi(e_1, e_3) = -3a_0\,E_1 + 3a_1\,E_2$, thus $E_1 \in im(\phi)$. Finally, $\phi(e_0, e_3) = -9a_0\,E_3 + 9a_1\,E_1 + 9a_2\,E_2$, thus $E_3 \in im(\phi)$. Therefore, ϕ is surjective if $a_0 \neq 0$. Similar arguments show that ϕ is surjective whenever $\phi \neq 0$. **q.e.d.**

Corollary 3.2 *If the holonomy group of a torsion free connection ∇ on a manifold M is contained in H_3 then either the holonomy group of ∇ is equal to H_3 or ∇ is flat.*

PROOF: Suppose ∇ is not flat. Then there is a $p \in M$ at which the curvature map $\Omega_p : \Lambda^2 T_pM \to \mathfrak{h}_3$ is not 0. By Lemma 3.1, Ω_p is surjective and thus by the *Ambrose-Singer-Holonomy Theorem*, the holonomy group of ∇ is equal to H_3. **q.e.d.**

Suppose M is equipped with a torsion free H_3-connection ∇. Let $\pi : \mathfrak{F} \to M$ denote the total frame bundle of M which is a principal $Gl(4, \mathbb{R})$-bundle. On $\mathfrak{F}$, the tautological 1-form ω takes values in $\mathbb{R}^4$, the connection 1-form θ takes values in $\mathfrak{gl}(4, \mathbb{R})$ and both are $Gl(4, \mathbb{R})$-equivariant [KN].

Then there exists a reduction F of $\mathfrak{F}$ whose structure group is isomorphic to $Sl(2, \mathbb{R})$ [KN, II.7.1] and such that $\theta|_F$ takes values in $\mathfrak{h}_3 \cong \underline{\mathfrak{h}}_3 \subseteq \mathfrak{gl}(4, \mathbb{R})$. Such a reduction will be called an $\mathfrak{h}_3$*-reduction* of $\mathfrak{F}$.

By abuse of notation we will denote the restrictions $\omega|_F, \theta|_F$ and $\pi|_F$ also by ω, θ and π resp.

Of course, $\mathfrak{h}_3$-reductions are not unique. In fact, one sees easily that F, F' are $\mathfrak{h}_3$-reductions iff there is some $g \in Norm(\underline{\mathfrak{h}}_3) \subseteq Gl(4, \mathbb{R})$ with $F' = L_g(F)$, where L_g denotes the principal action of g on $\mathfrak{F}$. Now one computes that

$$Norm(\mathfrak{h}_3) = H_3 \times N$$

with

$$N = \left\{ \left. \begin{pmatrix} t & & & \\ & \varepsilon t & & \\ & & t & \\ & & & \varepsilon t \end{pmatrix} \right| t \in \mathbb{R}\backslash\{0\}, \varepsilon = \pm 1 \right\}.$$

It follows that any two $\mathfrak{h}_3$-reductions F, F' satisfy $F = L_g(F')$ for some $g \in N$, and two such structures are called *homothetic* to each other.

Clearly, the property of being homothetic forms an *equivalence relation* on the set of $\mathfrak{h}_3$-reductions of $\mathfrak{F}$, and it is not hard to see that there is in fact a 1-1 correspondence between H_3-connections and homothety classes of $\mathfrak{h}_3$-reductions with a torsion free connection [Br2, 2.1].

We now derive the *structure equations* for H_3-connections on a manifold M.

Let M be as above and fix an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$. On F, we have the 1-form $\omega + \theta$ with values in $\mathbb{R}^4 \oplus \underline{\mathfrak{h}}_3 \cong V_3 \oplus \mathfrak{h}_3$ and may hence regard $\omega + \theta$ as an $Sl(2,\mathbb{R})$-equivariant $V_3 \oplus \mathfrak{h}_3$-valued 1-form. It is well known that this form gives a *coframe* on F, i.e. the real-valued 1-forms $\omega_0, \ldots, \omega_3, \theta_1, \theta_2, \theta_3$ given by

$$\omega(X) = \sum_i \omega_i(X)\, e_i \quad \text{and} \quad \theta(X) = \sum_i \theta_i(X)\, E_i \quad \text{for all} \quad X \in TF$$

are linearly independent.

The frame of F dual to the coframe $\omega_0, \ldots, \omega_3, \theta_1, \theta_2, \theta_3$ will be denoted by $X_0, \ldots, X_3, Y_1, Y_2, Y_3$ and will be called the *canonical frame* on F.

A tangent vector $X \in TF$ will be called *vertical* if $\omega(X) = 0$ and *horizontal* if $\theta(X) = 0$. Thus, X_i is *horizontal* and Y_i is *vertical* for all i.

The curvature 2-form Ω on F takes values in $\mathfrak{h}_3$ and vanishes in vertical directions which means that for every $u \in F$, there is a unique linear map $\phi_u : \Lambda^2 V_3 \to \mathfrak{h}_3$ such that

$$\Omega(X, Y) = \phi_u(\omega(X), \omega(Y)) \quad \text{for all} \quad X, Y \in T_u F.$$

Furthermore, the *first Bianchi identity* implies that $\phi_u \in \mathbf{K}(\mathfrak{h}_3)$, and this defines an $Sl(2,\mathbb{R})$-equivariant map

$$\begin{aligned} a: \quad F &\to V_2 \\ u &\mapsto \imath_{\mathbf{K}}(\phi_u). \end{aligned}$$

We then define the real-valued functions a_0, a_1, a_2 on F by the equation

$$\phi_u = \sum_i a_i(u)\,\phi_i \quad \text{or, equivalently,} \quad a(u) = \sum_i a_i(u)\,x^{2-i}y^i$$

Now we can describe the *structure equations* on F.

The *first structure equation* - using that ∇ is torsion free - yields

$$d\omega = -\theta \wedge \omega \tag{3}$$

or, equivalently,

$$\begin{aligned} d\omega_0 &= -3\theta_1 \wedge \omega_0 - \theta_3 \wedge \omega_1 \\ d\omega_1 &= -3\theta_2 \wedge \omega_0 - \theta_1 \wedge \omega_1 - 2\theta_3 \wedge \omega_2 \\ d\omega_2 &= -2\theta_2 \wedge \omega_1 + \theta_1 \wedge \omega_2 - 3\theta_3 \wedge \omega_3 \\ d\omega_3 &= -\theta_2 \wedge \omega_2 + 3\theta_1 \wedge \omega_3 \end{aligned} \tag{4}$$

The *second structure equation* is

$$d\theta = -\theta \wedge \theta + \Omega \tag{5}$$

or, equivalently,

$$\begin{aligned} d\theta_1 &= \theta_2 \wedge \theta_3 - 3a_2\,\omega_0 \wedge \omega_2 + 9a_1\,\omega_0 \wedge \omega_3 - a_1\,\omega_1 \wedge \omega_2 - 3a_0\,\omega_1 \wedge \omega_3 \\ d\theta_2 &= 2\,\theta_1 \wedge \theta_2 + 9a_2\,\omega_0 \wedge \omega_3 - 5a_2\,\omega_1 \wedge \omega_2 + 3a_1\,\omega_1 \wedge \omega_3 - 6a_0\,\omega_2 \wedge \omega_3 \\ d\theta_3 &= -2\,\theta_1 \wedge \theta_3 + 6a_2\,\omega_0 \wedge \omega_1 - 3a_1\,\omega_0 \wedge \omega_2 - 9a_0\,\omega_0 \wedge \omega_3 + 5a_0\,\omega_1 \wedge \omega_2 \end{aligned} \tag{6}$$

Note that these equations determine the brackets of the canonical vector fields using the relation

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \quad \text{for any 1-form} \quad \alpha.$$

For $u \in F$, we define a linear map $\psi_u : V_3 \to Hom(\Lambda^2 V_3, \mathfrak{h}_3)$ by the equation

$$\psi_u(e_i)(\omega(Y),\omega(Z)) = (\mathfrak{L}_{X_i}\Omega)(Y,Z) - \phi_u(\theta(Y)\cdot e_i, \omega(Z)) - \phi_u(\omega(Y), \theta(Z)\cdot e_i)$$

for all $Y, Z \in T_uF$.

To see that this is well defined note that the right hand side is tensorial in Y and Z and vanishes if either Y or Z are vertical. Moreover, the first Bianchi identity implies that $\psi_u(e_i) \in \mathbf{K}(\mathfrak{h}_3)$, and finally, the second Bianchi identity implies that $\psi_u \in \mathbf{K^1}(\mathfrak{h}_3)$. Therefore, we get a map

$$\begin{array}{rll} b: & F & \to V_3 \\ & u & \mapsto \imath_{\mathbf{K^1}}(\psi_u), \end{array}$$

and define the real-valued functions $b_0, \ldots, b_3$ on F by the equation

$$\psi_u = \psi_{b_0(u),\ldots,b_3(u)}$$

or, equivalently,

$$b(u) = b_3(u)\ x^3 - 3b_2(u)\ x^2y + 3b_1(u)\ xy^2 - b_0(u)\ y^3.$$

Next, we compute the exterior derivatives of the functions a_0, a_1, a_2. For this, let X_i, X_j and X_k be horizontal vector fields of the canonical frame on F. Then, at some point $u \in F$,

$$\begin{aligned} X_i(\Omega(X_j,X_k)) &= (\mathfrak{L}_{X_i}\Omega)(X_j,X_k) + \Omega([X_i,X_j],X_k) + \Omega(X_j,[X_i,X_k]) \\ &= \psi_u(e_i)\ (e_j,e_k) \\ &= ((3-i)b_{i+1}(u)\ \phi_0 - (3-2i)b_i\ \phi_1 - ib_{i-1}(u)\ \phi_2)\ (e_j,e_k). \end{aligned}$$

Here we used the fact that the Lie bracket of two canonical *horizontal* vector fields is always *vertical* since the connection is torsion free.

On the other hand,

$$X_i(\Omega(X_j, X_k)) = X_i(\sum_{r=0}^{2} a_r(u)\ \phi_r\ (e_j, e_k)) = (\sum_{r=0}^{2} X_i(a_r)\ \phi_r)\ (e_j, e_k).$$

Setting these two equal, we get the *horizontal* derivatives of a. The *vertical* derivatives follow from the $Sl(2,\mathbb{R})$-equivariance of a, and we obtain

$$\begin{aligned} da_0 &= \sum_{i=0}^{3} (3-i)b_{i+1}\,\omega_i - 2a_0\,\theta_1 - a_1\,\theta_3 \\ da_1 &= -\sum_{i=0}^{3} (3-2i)b_i\,\omega_i - 2a_0\,\theta_2 - 2a_2\,\theta_3 \\ da_2 &= -\sum_{i=0}^{3} ib_{i-1}\,\omega_i + 2a_2\,\theta_1 - a_1\,\theta_2 \end{aligned} \tag{7}$$

Taking exterior derivatives of these equations and solving for db_i, we can compute that there exists some function c such that

$$\begin{array}{rcl} db_0 &=& 3(a_2^2\omega_1 - a_1a_2\omega_2 + b_{03}\omega_3 + b_0\theta_1 + b_1\theta_2) \\ db_1 &=& -3a_2^2\omega_0 - b_{12}\omega_2 - 3a_0a_1\omega_3 + b_1\theta_1 + 2b_2\theta_2 + b_0\theta_3 \\ db_2 &=& 3a_1a_2\omega_0 + b_{12}\omega_1 + 3a_0^2\omega_3 - b_2\theta_1 + b_3\theta_2 + 2b_1\theta_3 \\ db_3 &=& -3(b_{03}\omega_0 - a_0a_1\omega_1 + a_0^2\omega_2 + b_3\theta_1 - b_2\theta_3) \end{array} \tag{8}$$

with

$$b_{03} = 3(\frac{a_1^2}{2} - a_0a_2) + c \quad \text{and} \quad b_{12} = \frac{a_1^2}{2} - 5a_0a_2 + c$$

Taking exterior derivatives once again, we find that

$$dc = 0, \tag{9}$$

i.e. c is a *constant*.

Definition 3.3 *Let $\pi : F \to M$ be a principal* $\mathrm{Sl}(2,\mathbb{R})$*-bundle over a* 4*-dimensional manifold* M*. Suppose there exist 1-forms* ω *and* θ *on* F *with values in* V_3 *and* $\mathfrak{h}_3$ *resp. and functions* a *and* b *with values in* V_2, V_3 *resp. such that the structure equations (3) - (9) are satisfied for some constant* c*. Moreover, assume that* $\omega(ker(\pi_*)) \equiv 0$ *and* $\theta((\tilde{E}_i)^*) = E_i$ *where* $\{\tilde{E}_i\}$ *is the basis of* $\mathfrak{sl}(2,\mathbb{R})$ *described earlier and* $(\tilde{E}_i)^*$ *denotes the fundamental vector field corresponding to* $\tilde{E}_i$ *[KN, Vol I p. 51]. Then* $(\pi, F, M, \omega, \theta, a, b, c)$ *is called a solution structure over* M*.*

By the previous we know that any $\mathfrak{h}_3$-reduction of an H_3-connection on M gives rise to a solution structure over M.

Now suppose that $F' = L_g(F)$ with $g \in N$ is an $\mathfrak{h}_3$-reduction homothetic to F. Then if we let $\tilde{a} = L_g \circ a, \tilde{b} = L_g \circ b$ and $\tilde{c} = L_g \circ c$ where L_g denotes the action of g on V_2, V_3 resp., then $(F', (L_{g^{-1}})^*(\omega), (L_{g^{-1}})^*(\theta), \tilde{a}, \tilde{b}, \tilde{c})$ is also a solution structure.

In terms of the associated real valued functions we see that if

$$g = \begin{pmatrix} t & & & \\ & \varepsilon t & & \\ & & t & \\ & & & \varepsilon t \end{pmatrix} \in N$$

then

$$\begin{array}{llll} \tilde{a}_0 = t^2 a_0, & \tilde{a}_1 = \varepsilon t^2 a_1, & \tilde{a}_2 = t^2 a_3, & \\ \tilde{b}_0 = \varepsilon t^3 b_0, & \tilde{b}_1 = t^3 b_1, & \tilde{b}_2 = \varepsilon t^3 b_2, & \tilde{b}_3 = t^3 b_0 \\ \tilde{c} = t^4 c & & & \end{array} \tag{10}$$

Definition 3.4 *Two solution structures* $(\pi, F, M, \omega, \theta, a, b, c)$ *and* $(\tilde{\pi}, \tilde{F}, M, \tilde{\omega}, \tilde{\theta}, \tilde{a}, \tilde{b}, \tilde{c})$ *over the same manifold* M *are called homothetic if there exists a bundle isomorphism* $L : F \to \tilde{F}$ *such that* $\omega = L^*(\tilde{\omega}), \theta = L^*(\tilde{\theta})$*, and moreover the real valued functions associated to* $a, b, \tilde{a}, \tilde{b}$ *satisfy (10) for some* $t \neq 0$*,* $\varepsilon = \pm$*.*

Clearly, homothety is an equivalence relation of solution structures and homothetic $\mathfrak{h}_3$-reductions give rise to homothetic solution structures. Therefore, by our discussion earlier, to any H_3-connection on a manifold M we can associate a homothety class of solution structures over M.

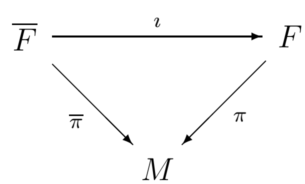
We will now show that this correspondence between H_3-connections and homothety classes of solution structures over a manifold M is in fact bijective. In particular, we need to show the *sufficiency* of the structure equations (3) - (9).

Proposition 3.5 *Let M be a 4-dimensional manifold and let $\pi : \mathfrak{F} \to M$ denote the total frame bundle of M. Suppose there exists a solution structure $(\overline{\pi}, \overline{F}, M, \overline{\omega}, \overline{\theta}, \overline{a}, \overline{b}, c)$ over M.*

Then there exists a unique H_3-connection ∇ on M and a unique embedding $\imath : \overline{F} \hookrightarrow \mathfrak{F}$ such that

1) $F := \imath(\overline{F})$ is an $\mathfrak{h}_3$-reduction of $\mathfrak{F}$,

2) the diagram



commutes and

3) $\imath^*(\omega + \theta) = \overline{\omega} + \overline{\theta}$, where ω and θ denote the restrictions of the tautological form and the connection form to F.

Proof: Let $\overline{F}, \overline{\omega}, \overline{\theta}, \overline{a}$ and $\overline{b}$ be as above. Define the real-valued 1-forms and the real-valued functions $\overline{\omega}_0, \ldots \overline{\omega}_3, \overline{\theta}_1, \ldots, \overline{\theta}_3, \overline{a}_0, \overline{a}_1, \overline{a}_2, \overline{b}_0, \ldots, \overline{b}_3$ as before, and let $\overline{X}_0, \ldots, \overline{X}_3$, $\overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ be the frame on $\overline{F}$ dual to the coframe $\overline{\omega} + \overline{\theta}$.

By hypothesis, $\overline{\pi}_*((X_0)_u, \ldots, (X_3)_u)$ is a basis of $T_{\pi(u)}M$ for all $u \in \overline{F}$, hence a point in $\mathfrak{F}$. Therefore, we define the map

$$\begin{aligned} \imath : \ \overline{F} &\to \mathfrak{F} \\ u &\mapsto \overline{\pi}_*((X_0)_u, \ldots, (X_3)_u) \end{aligned}$$

By construction, we have $\imath^*(\omega) = \overline{\omega}$. Moreover, the structure equations and the condition $\theta((\tilde{E}_i)^*) = E_i$ imply easily that $\overline{\omega}$ and $\overline{\theta}$ are $Sl(2,\mathbb{R})$-equivariant w.r.t. the actions of $Sl(2,\mathbb{R})$ on V_3 and $\mathfrak{h}_3$ induced by ρ_3. This means that

- $\imath$ is an embedding whose image F is an $\mathfrak{h}_3$-reduction of $\mathfrak{F}$ w.r.t. the connection defined by θ.
- if we define on $F := \imath(\overline{F})$ the $\mathfrak{h}_3$-valued 1-form $\theta' := (\imath^{-1})^*(\overline{\theta})$ then there exists a unique $\mathfrak{gl}(4,\mathbb{R})$-valued $Gl(4,\mathbb{R})$-equivariant connection 1-form θ on $\mathfrak{F}$ such that $\theta' = \theta|_F$,

It is then straightforward that the map $\imath$ satisfies the desired properties and is uniquely determined by them. **q.e.d.**

From the construction above it is easily seen that the images of the embeddings $\imath_1$ and $\imath_2$ associated to two *homothetic* solution structures $(\overline{\pi}_i, \overline{F}_i, M, \overline{\omega}_i, \overline{\theta}_i, \overline{a}_i, \overline{b}_i, c_i)$ over M are in fact *homothetic* $\mathfrak{h}_3$-structures. In particular, the connection ∇ on M only depends on the *homothety class* of the solution structure.

We summarize the discussion so far, including Corollary 3.2, in the

Corollary 3.6 *There is a 1-1 correspondence between H_3-connections on a 4-manifold M and homothety classes of solution structures over M for which a, b do not vanish identically.*

For the rest of this section, let F be an $\mathfrak{h}_3$-reduction of an H_3-connection ∇ on a connected 4-manifold M. If we let $V := V_2 \oplus V_3$, we get a map

$$\begin{aligned} K: \quad F &\to V \\ u &\mapsto a(u) + b(u) \end{aligned}$$

The structure equations (3) - (9) imply that w.r.t. the functions a_0, a_1, a_2 and $b_0, \ldots, b_3$ on V and the canonical frame $X_0, \ldots, X_3, Y_0, Y_1, Y_2$ on F and we can write the differential of K

$$K_* : TF \to TV$$

as

$$K_* = \begin{pmatrix} 3b_1 & 2b_2 & b_3 & 0 & -2a_0 & 0 & -a_1 \\ -3b_0 & -b_1 & b_2 & 3b_3 & 0 & -2a_0 & -2a_2 \\ 0 & -b_0 & -2b_1 & -3b_2 & 2a_2 & -a_1 & 0 \\ 0 & 3a_2^2 & -3a_1a_2 & 3b_{03} & 3b_0 & 3b_1 & 0 \\ -3a_2^2 & 0 & -b_{12} & -3a_0a_1 & b_1 & 2b_2 & b_0 \\ 3a_1a_2 & b_{21} & 0 & 3a_0^2 & -b_2 & b_3 & 2b_1 \\ -3b_{30} & 3a_0a_1 & -3a_0^2 & 0 & -3b_3 & 0 & 3b_2 \end{pmatrix}.$$

We compute that $det(K_*) \equiv 0$, i.e. $rank(K_*) \leq 6$. Let L be the cofactor matrix of K_*, consisting of the 6×6 minors of K_*. L has the property that

$$LK_* = K_*L = 0.$$

and L has *rank* at most 1. Thus, there are polynomials $r_0, \ldots, r_6, s_0, \ldots, s_6$ in the variables $a_0, a_1, a_2, b_0, \ldots, b_3$ such that

$$L = \begin{pmatrix} r_0 \\ \vdots \\ r_6 \end{pmatrix} (s_0, \ldots, s_6).$$

Furthermore, if we define the polynomial

$$R_c = \frac{1}{6912}(< p_c, p_c >_2 \; - 36 < a, q_c^2 >_2)$$

with

$$p_c = 18(4c - \; < a, a >_2)a \; + \; < b, b >_2 \quad \text{and} \quad q_c = < a, b >_2,$$

then we compute that

$$s_i = \frac{\partial R_c}{\partial a_i} \quad \text{for} \quad 0 \leq i \leq 2, \quad \text{and} \quad s_i = \frac{\partial R_c}{\partial b_i} \quad \text{for} \quad 3 \leq i \leq 6,$$

and

$$r_i = s_{i+3} \quad \text{for} \quad 0 \leq i \leq 3, \quad r_4 = s_1, \quad r_5 = -s_0 \quad \text{and} \quad r_6 = s_2.$$

This means that

- $d(R_c \circ K) \equiv 0$, hence K maps F into some *level set* of R_c,

- for $u \in F$, $rank(K_*(u)) = 6$ iff $L(u) \neq 0$ iff $(r_0(u), \ldots, r_6(u)) \neq 0$ iff $(s_0(u), \ldots, s_6(u)) \neq 0$ iff $K(u)$ is a *regular point* of R_c.

We now describe the set $\Sigma_c \subseteq V$ of *critical points* of R_c. Using the $Sl(2,\mathbb{R})$-invariance of Σ_c, we compute that

- for $c \neq 0$, $\Sigma_c = \Sigma_c^1 \sqcup \Sigma_c^2$, where

$$\Sigma_c^1 = \{(a,b) \in V \mid p_c = q_c = 0\}$$

and

$$\Sigma_c^2 = \{(v - 2u^2, 2u\,(3v + 2u^2)) \in V \mid u \in V_1, v \in V_2 \text{ with } < v, v >_2 = \frac{4}{3}\,c\}.$$

- for $c = 0$, $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$, where

$$\Sigma_0^{\pm} = \{(\pm v^2 - 2u^2, 2u\,(\pm 3v^2 + 2u^2)) \in V \mid u, v \in V_1\}$$

We also define

$$\Sigma_0^0 := \Sigma_0^+ \cap \Sigma_0^- = \{(-2u^2, 4u^3) \in V \mid u \in V_1\}.$$

Remark: The reader who is familiar with [Br2] might wonder why the descriptions of the sets Σ_c and the function R_c there involves different constants than here. This is due to the fact that the isomorphisms $\imath_{\mathbf{K}}$ and $\imath_{\mathbf{K^1}}$ are not determined canonically but can be multiplied by constants. In fact, if we compose K with the map $\jmath : V \to V$, $\jmath(a,b) := (-\sqrt[3]{18}a, b)$, then we will get the same constants as in [Br2]. However, to avoid roots in our formulas we will stick to the map K as defined.

We wish to determine the topology of the Σ's. To do this we introduce the $Sl(2,\mathbb{R})$-equivariant maps

$$\begin{aligned} \phi^2_c: \quad V_1 \times V_{2,c} &\longrightarrow \Sigma^2_c \\ (u,v) &\longmapsto (v - 2u^2, 2u\,(3v + 2u^2)), \end{aligned}$$

$$\begin{aligned} \phi^{\pm}_0: \quad V_1 \times V_1 &\longrightarrow \Sigma^{\pm}_0 \\ (u,v) &\longmapsto (\pm v^2 - 2u^2, 2u\,(\pm 3v^2 + 2u^2)), \end{aligned}$$

$$\begin{aligned} \phi^0_0: \quad V_1 &\longrightarrow \Sigma^0_0 \\ u &\longmapsto (-2u^2, 4u^3), \end{aligned}$$

where $V_{2,c} := \{v \in V_2 \mid\ < v, v >_2 = \frac{4}{3}\,c\}$.

One then computes that ϕ^2_c is a *diffeomorphism* for all c.

$\phi^{\pm}_0$ is a *branched double cover:* in fact, $\phi^{+}_0|_{V_1 \times (V_1 \setminus \{0\})}$ is a *double cover* of $\Sigma^{+}_0 \setminus \Sigma^0_0$ whose non-trivial deck transformation is given by $(u,v) \mapsto (u,-v)$.

Finally, $\phi^0_0|_{V_1 \setminus \{0\}}$ is a diffeomorphism onto $\Sigma^0_0 \setminus \{0\}$.

Note that Σ^1_c is smooth by the implicit function theorem. However, the number of its connected components is unclear.

For Σ^2_c, we conclude from the above that it is smooth 4-dimensional and has one or two connected components depending on the sign of c. In the case $c > 0$ we let $V_{2,c}^{\pm} = \{\pm(u^2 + v^2) | u, v \in V_1\}$ and let $\Sigma_c^{2,\pm} := \phi_c^2(V_1 \times V_{2,c}^{\pm})$ be the connected components of Σ_c^2.

$\Sigma_0 \backslash \Sigma_0^0$ is 4-dimensional and smooth and has two connected components, namely $\Sigma_0^{\pm} \backslash \Sigma_0^0$.

The set $\Sigma_0^0 \backslash \{0\}$ is a smooth 2-dimensional manifold.

Moreover, for the ranks of K_* we get:

- If $c \neq 0$ then $rank(K_*(u)) = 4$ for all $u \in F$ with $K(u) \in \Sigma_c$.

- If $c = 0$ then

    - $rank(K_*(u)) = 4$ for all $u \in F$ with $K(u) \in \Sigma_0 \backslash \Sigma_0^0$,

    - $rank(K_*(u)) = 2$ for all $u \in F$ with $K(u) \in \Sigma_0^0 \backslash \{0\}$,

    - $rank(K_*(u)) = 0$ for all $u \in F$ with $K(u) = 0$.

Theorem 3.7 *The differential K_* has constant rank on F, and $rank(K_*) \in \{0, 2, 4, 6\}$.*

We start by proving the following

Lemma 3.8 *Any two points in F can be joined by a piecewise differentiable path γ such that $\gamma'(t) = \pm X_i$ or $\gamma'(t) = \pm Y_i$ for some i, wherever γ' is defined.*

Proof of Lemma: On F, define the equivalence relation $p \sim q$ if $p, q \in F$ can be joined by such a path. Let $\Phi_{X_i}^t, \Phi_{Y_i}^t$ denote the flow along the vector field

X_i, Y_i resp. for time t. Then, given any $p \in F$, we can find some $\varepsilon > 0$ such that $\Phi^t_{X_i}(p), \Phi^t_{Y_i}(p)$ exists for all $t \in (-\varepsilon, \varepsilon)$. Define a map $\Phi : (-\varepsilon, \varepsilon)^7 \to F$ by $\Phi(t_0, \ldots, t_6) := (\Phi^{t_6}_{Y_3} \circ \ldots \circ \Phi^{t_4}_{Y_1} \circ \Phi^{t_3}_{X_3} \circ \ldots \circ \Phi^{t_0}_{X_0})(p)$. Clearly, Φ is differentiable and its derivative $\Phi_*(0, \ldots, 0)$ is invertible. Thus, by the inverse function theorem, we may assume that Φ is a diffeomorphism after shrinking ε if necessary. Now given any $(t_0, \ldots t_6) \in (-\varepsilon, \varepsilon)^7$, there is a polygonal path $\overline{\gamma} : (a, b) \to (-\varepsilon, \varepsilon)^7$ with a subdivision $a = u_0 \leq \ldots \leq u_7 = b$ such that $\overline{\gamma}(a) = 0, \overline{\gamma}(b) = (t_0, \ldots, t_6)$ and $\overline{\gamma}'(u) = \pm \frac{\partial}{\partial t_i}$ for all $u \in (u_{i-1}, u_i)$. Then $\gamma := \Phi \circ \overline{\gamma}$ is a path satisfying the desired condition, showing that $p \sim q$ for all $q \in im(\Phi)$. But since Φ is a diffeomorphism, $im(\Phi)$ is a neighborhood of p, showing that the equivalence classes of $\sim$ are open in F. Since F is connected, there is only one equivalence class which proves the Lemma. **q.e.d.**

Proof of Theorem: By the previous discussion we need to show that

1) either $K(F) \subseteq V \backslash \Sigma_c$ or $K(F) \subseteq \Sigma_c$.

2) if $c = 0$ and $K(F) \subseteq \Sigma_0$ then

- either $K(F) \subseteq \Sigma_0 \backslash \Sigma_0^0$ or
- $K(F) \subseteq \Sigma_0^0 \backslash \{0\}$ or
- $K(F) = \{0\}$.

1) On F, define the vector field $Z := \sum_{i=0}^3 r_i X_i + \sum_{i=0}^2 r_{i+4} Y_i$ with the functions $r_0, \ldots, r_6$ as on page 27. Then clearly, $Z_u \neq 0$ iff $K(u) \in V \backslash \Sigma_c$. Surprisingly, after an enormous calculation it turns out that $[X_i, Z] = [Y_i, Z] = 0$ for all i. Therefore,

if γ is a path in F with $\gamma' = \pm X_i$ or $\gamma' = \pm Y_i$ for some i, then Z vanishes either *nowhere* or *everywhere* along γ. This together with Lemma 3.8 yields that Z vanishes either *nowhere* or *everywhere* on F, in other words, either $K(F) \subseteq V \backslash \Sigma_c$ or $K(F) \subseteq \Sigma_c$.

2) Now suppose that $c = 0$ and $K(F) \subseteq \Sigma_0$.

<u>Claim:</u> Let $\gamma : [0, \varepsilon) \to F$ be a continuous path which is differentiable for $t > 0$ and with $\gamma' = \pm X_i$ or $\gamma' = \pm Y_i$ for some i and $K(\gamma(t)) \in \Sigma_0 \backslash \Sigma_0^0$ for $t > 0$. Then $K(\gamma(0)) \in \Sigma_0 \backslash \Sigma_0^0$.

We will prove this claim only in the case where $\gamma' = X_0$ and $K(\gamma(t)) \in \Sigma_0^+ \backslash \Sigma_0^0$. The other cases are treated similarly.

It is easily seen that there is a *lifting path* $\tilde{\gamma} : [0, \varepsilon) \to V_1 \times V_1$ such that $\phi_0^+ \circ \tilde{\gamma} = K \circ \gamma$ with ϕ_0^+ as defined earlier. Define the functions u_1, u_2, v_1, v_2 by the equation

$$\tilde{\gamma}(t) = (u_1(t)\ x + u_2(t)\ y, v_1(t)\ x + v_2(t)\ y).$$

Note that these functions are continuous everywhere and differentiable for $t > 0$. One then computes that $2a_0^3 + b_3^2 = 2v_1^2\,(6u_1^2 + v_1^2)^2$, and $2a_2^3 + b_0^2 = 2v_2^2\,(6u_2^2 + v_2^2)^2$.

Taking derivatives yields

$$\frac{d}{dt}(2v_2^2\,(6u_2^2 + v_2^2)^2) = \pm X_0(2a_2^3 + b_0^2) = 0,$$

which implies that either v_2 vanishes *identically* or *never*.

Suppose that $v_2 \equiv 0$. Then we get

$$\frac{d}{dt}(2v_1^2\,(6u_1^2+v_1^2)^2) = \pm X_0(2a_0^3+b_3^2) = \mp 144 v_1^2 u_1 u_2^2\,(6u_1^2+v_1^2),$$

i.e.

$$\frac{d}{dt}(v_1\,(6u_1^2+v_1^2)) = \mp 36 v_1 u_1 u_2^2 \tag{11}$$

for all $t > 0$.

Next, observe that $a_2 = -2u_2^2$ and $X_0(a_2) = 0$. Therefore, u_2 is *constant* along γ.

If $u_2 = 0$ then equation (11) implies that either v_1 vanishes *identically* or *never*.

If $u_2 \neq 0$ then $a_1 = -4u_1u_2$ and $X_0(a_1) = 12u_2^3$ implies that $\frac{du_1}{dt} = \mp 3u_2^2$. In this case, equation(11) implies that

$$(2u_1^2+v_1^2)\,\frac{dv_1}{dt} = 0.$$

But as $\frac{du_1}{dt}$ is constant, u_1 vanishes at most at one point, hence we get that

$$\frac{dv_1}{dt} = 0$$

and hence that v_1 either vanishes *identically* or *never*.

By hypothesis, $\gamma(t) \notin \Sigma_0^0$ for $t > 0$, which is equivalent to saying that either $v_1(t) \neq 0$ or $v_2(t) \neq 0$ for $t > 0$. By the above discussion this implies that the same holds at $t = 0$, i.e. $\gamma(0) \notin \Sigma_0^0$, and this proves the claim.

Using the claim it is then clear that any path γ in F as in Lemma 3.8 with $K(\gamma) \subseteq \Sigma_0$ has the property that either $K(\gamma) \subseteq \Sigma_0 \backslash \Sigma_0^0$ or $K(\gamma) \subseteq \Sigma_0^0$. This together with Lemma 3.8 implies that either $K(F) \subseteq \Sigma_0 \backslash \Sigma_0^0$ or $K(F) \subseteq \Sigma_0^0$.

Finally, suppose that $K(F) \subseteq \Sigma_0^0$. One sees easily from the structure equations that $a = b = 0$ at *some* point implies that $a = b = 0$ everywhere, i.e. either $K(F) \subseteq \Sigma_0^0 \backslash \{0\}$ or $K(F) = \{0\}$. This finishes the proof of the Theorem. **q.e.d.**

Of course, if $K(F) = \{0\}$ then the connection form θ is *flat*, i.e. the holonomy group is not equal to H_3.

We can easily check that the following are *independent of the homothety class* of F, and hence by Corollary 3.6 are invariants of the connection:

- the rank of K_*,

- the sign of the constant c,

- for the case that $rank(K_*) = 4$ and $c = 0$, the value of $\varepsilon = \pm$ in $K(F) \subseteq \Sigma_0^\varepsilon$, and in this case the connection is said to be of *type* Σ_0^ε,

- for the case that $rank(K_*) = 4$ and $c \neq 0$, the value of $i \in \{1, 2\}$ in $K(F) \subseteq \Sigma_c^i$, and the connection is then said to be of *type* Σ_c^i,

- for a connection of type Σ_c^2 with $c > 0$, the value of $\varepsilon = \pm$ in $K(F) \subseteq \Sigma_c^{2,\varepsilon}$, and the connection is then said to be of *type* $\Sigma_c^{2,\varepsilon}$.

4 Non-completeness

The purpose of this section is to prove the following

Theorem 4.1 *H_3-connections are never complete.*

Proof: Let M be a manifold with an H_3-connection ∇, and let $(\pi, F, M, \omega, \theta, a, b, c)$ be an associated solution structure over M. By [KN, III.6.3.], the *geodesics* on M are precisely those curves γ in M with the property that if $\tilde{\gamma}$ is a horizontal lift of γ then $\tilde{\gamma}' = \sum_i k_i X_i$ for some *constants* $k_0, \ldots, k_3$.

Now suppose that ∇ is complete. Let $\tilde{\gamma}$ be an integral curve of the vector field X_1 which projects down to the geodesic $\gamma = \pi \circ \tilde{\gamma}$ and therefore by hypothesis is defined on all of $\mathbb{R}$. From the structure equations we get that

$$X_1(a_2) = -b_0 \quad \text{and} \quad X_1(b_0) = 3a_2^2,$$

hence $-3a_2$ satisfies on $\tilde{\gamma}$ the differential equation

$$y'' = y^2. \tag{12}$$

We shall see that the only solution of (12) which is defined on *all* of $\mathbb{R}$ is $y = 0$ and therefore $a_2 = 0$ along $\tilde{\gamma}$.

Since $\tilde{\gamma}$ was an arbitrary integral curve of X_1 we conclude that $a_2 \equiv 0$ on F. But then the structure equations easily imply that all functions $a_0, a_1, a_2, b_0, \ldots, b_3$ vanish identically on F which means that the connection is *flat*, hence not an H_3-connection, and this contradiction will finish the proof.

It remains to show that (12) has no non-trivial solution. Note that any global solution y is *convex* and hence either constant or unbounded. Suppose that y is a global non-constant solution, i.e. $y'(t_0) \neq 0$ for some t_0. Replacing $y(t)$ by $y(-t)$ if necessary, we may assume that $y'(t_0) > 0$ and hence $y'(t) \geq y'(t_0) > 0$ for all $t \geq t_0$.

We get

$$\frac{d}{dt}((y')^2 - \frac{2}{3}\,y^3) = 0,$$

hence

$$(y')^2 = C + \frac{2}{3}\,y^3$$

for some constant C.

As y is unbounded, i.e. $\lim_{t->\infty} y = \infty$, we may assume that $y(t)^3 > 3|C|$ for all $t \geq t_0$ by increasing t_0 if necessary. Then we get for all $t \geq t_0$

$$(y')^2 > \frac{1}{3}\,y^3$$

which implies

$$(y^{-\frac{1}{2}})' < -\frac{1}{2\sqrt{3}}.$$

Thus we get for all $t \geq t_0$

$$y(t)^{-\frac{1}{2}} < C_1 - \frac{1}{2\sqrt{3}}\,t$$

for some constant C_1. But this is a contradiction as the left hand side of this inequality is always *positive* whereas the right hand side is *negative* for large t, and this finishes the proof. **q.e.d.**

5 The Singular H_3-connections

We now will solve the structure equations of the various types of solution structures. As we mentioned in chapter 2, it will be important in each but the homogeneous case to determine the subset $U \subseteq M$ on which the identity component of the symmetry group of ∇ acts *locally free*.

Let Z be a vector field on M such that Φ^t_Z, the flow along Z, is a local symmetry for small time t. Let $\overline{Z}$ be the vector field on F whose flow $\Phi^t_{\overline{Z}}$ is the *differential* of Φ^t_Z. Clearly, this implies that $\overline{Z} \in ker(K_*)$ since $\Phi^t_{\overline{Z}}$ preserves the curvature and hence K. On the other hand, Φ^t_Z is locally free on the set where $Z \neq 0$ or, equivalently, $\overline{Z} \notin ker(\pi_*)$.

Therefore, to determine U we obtain as a *sufficient* condition that U contains all points $p \in M$ satisfying

$$(ker(K_*) \cap ker(\pi_*))|_{T_x F} = 0 \quad \text{for all } x \in \pi^{-1}(p). \tag{13}$$

By Bryant's uniqueness result of H_3-connections it follows that this condition is actually *necessary* and hence determines U [Br2]. In fact, he shows that the flow along *any* vector field in $ker(K_*)$ is a symmetry and therefore the dimension of the symmetry group equals the *corank* of K_*.

5.1 H_3-connections of type Σ^0_0

Let M be a *connected, simply connected* 4-manifold with an H_3-connection ∇ of type Σ^0_0. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbb{R})$-equivariant map $K : F \to \Sigma^0_0 \backslash \{0\} \subseteq V$.

We define functions k_1 and k_2 by $K(u) = \phi_0^0(k_1(u)\, x + k_2(u)\, y)$ where ϕ_0^0 is the map defined on page 29. Then the structure equations imply that

$$dk_1 = -3k_2^2\,\omega_0 + 2k_1k_2\,\omega_1 - k_1^2\,\omega_2 - k_1\,\omega_4 - k_2\,\omega_6$$

$$dk_2 = -k_2^2\,\omega_1 + 2k_1k_2\,\omega_2 - 3k_1^2\,\omega_3 + k_2\,\omega_4 - k_1\,\omega_5.$$

$rank(K_*) = 2$, and therefore K is a *submersion.* Since $Sl(2,\mathbb{R})$ acts transitively on $\Sigma_0^0\backslash\{0\}$ it follows that K is *surjective.* $\overline{F} := K^{-1}(\phi_0^0(x))$ is a *submanifold* of F which intersects all fibers $\pi^{-1}(p)$, $p \in M$, and the intersection with each fiber is *connected;* for $\pi(u_1) = \pi(u_2)$ iff $u_1 = L_h(u_2)$ for some $h \in \overline{H}$ where $\overline{H} := \left\{ \rho_3 \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \middle|\, t \in \mathbb{R} \right\} \subseteq H_3$ which is connected. It follows that $\overline{F}$ is a *principal $\overline{H}$-bundle.* Hence $\pi|_{\overline{F}} : \overline{F} \to M$ is a *homotopy equivalence* and in particular $\overline{F}$ is *connected* and *simply connected.*

On $\overline{F}$, we can now compute that the vector fields

$$\begin{aligned} \underline{Z}_0 &= X_0 \\ \underline{Z}_1 &= X_1 + X_6 \\ \underline{Z}_2 &= X_2 - X_4 \\ \underline{Z}_3 &= X_3 - 3X_5 \\ \underline{Y} &= X_6 \end{aligned}$$

span $ker(K_*)$ and therefore give a *frame* on $\overline{F}$.

Recall the H_3-connection from section 2.1 on $M = G/H$ where

$$G = ASl(2,\mathbb{R}) = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & x \\ & & y \\ \hline 0 & 0 & 1 \end{array} \right) \middle|\, A \in Sl(2,\mathbb{R}), x, y \in \mathbb{R} \right\}$$

is the group of *unimodular affine motions* of $\mathbb{R}^2$. Let μ be the $\mathfrak{g}$-valued 1-form on $\overline{F}$

given by

$$\begin{array}{rcl}\mu(\underline{Z_i}) & = & Z_i \quad \text{for} \quad 0 \leq i \leq 3, \\ \mu(\underline{Y}) & = & Y,\end{array}$$

with the basis $Z_0, \ldots, Z_3, Y$ of $\mathfrak{g}$ defined in section 2.1.

One checks now that μ satisfies the *Maurer-Cartan equation*

$$d\mu + \mu \wedge \mu = 0.$$

Therefore, since $\overline{F}$ is connected and simply connected, the *second Cartan Lemma* applies and we can find an immersion $\jmath : \overline{F} \to G$ such that $\mu = \jmath^*(\mu_G)$ where μ_G denotes the *Maurer-Cartan form* of G.

Moreover, $\jmath(\overline{H}) = H$ and thus we get an *induced immersion* $\imath : \overline{F}/\overline{H} = M \to G/H$.

Let θ denote the connection form on F. Then $\theta(\underline{Z_i}) = \lambda(Z_i)$ for $0 \leq i \leq 3$ and $\theta(Y) = E_3$. Thus, $\imath$ *preserves the connection* and we have the following

Theorem 5.1 *Let M be a connected, simply connected 4-manifold with an H_3-connection ∇ of type Σ_0^0. Then there exists a connection preserving immersion*

$$\imath : M \to G/H$$

with G/H as in section 2.1, i.e. $\imath_*(\nabla_X Y) = \overline{\nabla}_{\imath_*(X)}\imath_*(Y)$ for all vector fields X, Y on M, where $\overline{\nabla}$ denotes the connection on G/H.

5.2 H_3-connections of type $\Sigma_0^\pm$

Let M be a *connected* 4-manifold with an H_3-connection ∇ of type $\Sigma_0^\pm$. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbb{R})$-equivariant map $K : F \to \Sigma_0^\pm \backslash \Sigma_0^0 \subseteq V$.

Let $\overline{F} := K^{-1}(\{\phi_0^{\pm}(v_1, x) \mid v_1 \in V_1\}) \subseteq F$ with $\phi_0^{\pm}$ as defined on page 29. Since $\phi_0^{\pm}$ is a cover and K has constant maximal rank, it follows that $\overline{F}$ is a smooth 5-dimensional submanifold of F. Moreover, the $Sl(2,\mathbb{R})$-equivariance of K implies that $\overline{F}$ is a *reduction* of F with structure group

$$\left\{ \left(\begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right) \middle| \, t \in \mathbb{R} \right\} \subseteq Sl(2,\mathbb{R}).$$

In particular, $\overline{F}$ is *connected*. We then define functions k_1 and k_2 on $\overline{F}$ by the equation

$$K(u) = \phi_0^{\pm}(k_1(u)\, x + k_2(u)\, y,\ x) \quad \text{for all} \quad u \in \overline{F}.$$

Since $rank(K|_{\overline{F}}) = 2$ we conclude that $dk_1 \wedge dk_2 \neq 0$ on $\overline{F}$. Also, k_2 is constant along the fibers of $\overline{F}$ and hence factors through to a function $f : M \to \mathbb{R}$.

Now we compute that condition (13) is satisfied iff $k_2(x) \neq 0$ iff $f(p) \neq 0$ with $p = \pi(x)$. Since $dk_2 \neq 0$ and therefore $df \neq 0$, we conclude that $k_2 \neq 0$ ($f_2 \neq 0$ resp.) on an *open dense* subset of $\overline{F}$ (M resp.).

Note that the solution structures on $f^{-1}(\mathbb{R}^+)$ and on $f^{-1}(\mathbb{R}^-)$ are *homothetic*, i.e. the connections on theses sets are *equivalent*. Thus, we only need to consider the set $U := f^{-1}(\mathbb{R}^+) \subseteq M$, and we let $\overline{U} := \overline{F} \cap \pi^{-1}(U) = k_2^{-1}(\mathbb{R}^+)$.

Let $S := k_1^{-1}(0) \cap \overline{U} \subseteq \overline{F}$. Again, $dk_1 \neq 0$ implies that S is a smooth 4-dimensional submanifold of $\overline{U}$. Moreover, S intersects every fiber $\pi^{-1}(p)$ in exactly one point for every $p \in U$. This means that S is the image of a smooth section $\sigma : U \to \overline{U}$.

Using σ, we can describe the connection on U as follows. Let $W_0, \dots, W_3$ be the frame on U given by σ and let $\tilde{W} := \sigma_*(W_i)$ for all i. Also, let $\omega_0, \dots, \omega_3$ be the dual coframe and define the V_3-valued 1-form $\omega = \sum_i \omega_i e_i$.

Of course, $\tilde{W}_i = \overline{W}_i + r_i Y_1 + s_i Y_2 + t_i Y_3$ for some functions r_i, s_i, t_i where $\overline{W}_i$ denotes the horizontal lift of W_i. One easily computes these functions explicitly using that $\tilde{W}_i$ is tangent to $\overline{F}$ for all i. In particular, this determines the *covariant derivatives* by

$$\omega(\nabla_{W_i} W_j) = -(r_i E_1 + s_i E_2 + t_i E_3) \cdot e_j.$$

Using the torsion freeness of ∇ we get from these the *bracket relations*

$$[W_i, W_j] = \nabla_{W_i} W_j - \nabla_{W_j} W_i.$$

Moreover, we know the derivatives $W_i(f)$ for all i and from there we can compute W_i w.r.t. some local coordinate system one of whose coordinate functions is f.

It then turns out that locally the connection on U is equivalent to the connection on $\Sigma_0^{\pm}$ given in example 2.2. Using the second Cartan Lemma we therefore get the following

Theorem 5.2 *Let M be a connected 4-manifold with an H_3-connection ∇ of type $\Sigma_0^{\pm}$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be its decomposition into connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_0^{\pm}$$

with $M_0^{\pm}$ from section 2.2, where $\tilde{U}_i$ denotes the universal cover of U_i.

We will call an H_3-structure of type $\Sigma_0^\pm$ *global* if the image of the curvature map K is all of $\Sigma_0^\pm$. Suppose we have such a global H_3-connection on M. Then $f : M \to \mathbb{R}$ is surjective and the symmetry group acts locally free on $U = f^{-1}(\mathbb{R}\backslash\{0\})$. Therefore, a *global* structure will contain two (equivalent) copies of $M_0^\pm$, namely the two components of U, glued together along the smooth submanifold $f^{-1}(0)$ of M. We conjecture that $f^{-1}(0)$ is diffeomorphic to $\mathbb{R}^3$ and hence M is diffeomorphic to $\mathbb{R}^4$. Note, however, that global structures need not exist.

5.3 H_3-connections of type Σ_c^1

Let M be a *connected* 4-manifold with an H_3-connection ∇ of type Σ_c^1 where $c \in \mathbb{R}\backslash\{0\}$. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbb{R})$-equivariant map $K : F \to \Sigma_c^1 \subseteq V$.

Again, we wish to determine the set where condition (13) is satisfied. One computes that this is the case iff $< b, b >_2 \neq 0$.

Therefore, we let $\overline{U} := K^{-1}\left(\{(a,b) \in \Sigma_c^1 \mid \; < b, b >_2 \neq 0\}\right) \subseteq F$ and let $U := \pi(\overline{U}) \subseteq M$. It is easy to verify that U ($\overline{U}$ resp.) is open and dense in M (F resp.).

Let $f : M \to \mathbb{R}$ be given by $f \circ \pi = 4c - < a, a >_2$. Clearly, $f|_U \neq 0$, and we decompose $U = U^+ \cup U^-$, $\overline{U} = \overline{U}^+ \cup \overline{U}^-$ by $U^\pm = f^{-1}(\mathbb{R}^\pm)$ and $\overline{U}^\pm = \pi^{-1}(U^\pm)$.

Define a function $t_0 : U^\pm \to \mathbb{R}^+$ by the equation $f = \pm(4t_0)^2$. Then one can compute that - after changing to a homothetic solution if necessary - the $Sl(2,\mathbb{R})$-

orbit of each element $(a,b) \in K(\overline{U}^{\pm})$ contains precisely one pair of the form

$$\left(\pm x^2 - \frac{1}{2}\left(4t_0^2 \mp c\right)y^2, -t_0 y\left(\pm 6x^2 + \left(4t_0^2 \mp c\right)y^2\right)\right).$$

Let $L^{\pm} := \left\{ \left(\pm x^2 - \frac{1}{2}\left(4t_0^2 \mp c\right)y^2, -t_0 y\left(\pm 6x^2 + \left(4t_0^2 \mp c\right)y^2\right)\right) \;\middle|\; t_0 \in \mathbb{R}^+ \right\} \subseteq \Sigma_c^1$. Then one sees easily that $L^{\pm}$ is a smooth 1-dimensional submanifold of V, hence $S^{\pm} := K^{-1}(L^{\pm})$ is a smooth 4-dimensional submanifold of F and is in fact the image of a smooth section $\sigma : U^{\pm} \to \overline{U}^{\pm}$.

Using σ, we do a calculation similar to the one described in the previous section and obtain the connection on $U^{\pm}$. To describe the result we treat the two possibilities for the sign of c separately.

If $c > 0$ then we see that the connection on $U^{\pm}$ is locally equivalent to the connection on $M_c^{\pm}$ described in example 2.2. However, the Lie algebras of the symmetry group of these examples are different, and we conclude that either $f \geq 0$ or $f \leq 0$ on M. We say the connection is of type $\Sigma_c^{1,\pm}$, depending on the sign of f. Thus we get the

Theorem 5.3 *Let M be a connected 4-manifold with an H_3-connection ∇ of type $\Sigma_c^{1,\pm}$ with $c > 0$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_c^{\pm}$$

with $M_c^{\pm}$ from section 2.2, where $\tilde{U}_i$ denotes the universal cover of U_i.

Now assume $c < 0$. In this case one computes that the connection is locally equivalent to the connection on $M_c^{\pm}$ described in example 2.2. Thus we get the

Theorem 5.4 *Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ_c^1 with $c < 0$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_c^+ \sqcup M_c^-$$

with $M_c^\pm$ from section 2.2, where $\tilde{U}_i$ denotes the universal cover of U_i.

The difference between these two results is that in the case $c > 0$ an H_3-connection of type $\Sigma_c^{1,\pm}$ on M is *either* locally equivalent to M_c^+ *or* locally equivalent to M_c^- since the symmetry groups in both cases are different.

In the case $c < 0$, however, it is very well possible that an H_3-connection is locally equivalent to M_c^+ in some neighborhood, but equivalent to M_c^- in some other neighborhood on the same manifold M.

5.4 H_3-connections of type Σ_c^2

Let M be a *connected* 4-manifold with an H_3-connection ∇ of type Σ_c^2 where $c \in \mathbb{R}\backslash\{0\}$. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbb{R})$-equivariant map $K : F \to \Sigma_c^2 \subseteq V$.

Recall the diffeomorphism ϕ_c^2 from page 29. Recall also that $V_{2,c}$ has one or two connected components depending on the sign of c. We say that ∇ is of type $\Sigma_c^{2,\pm}$ if $c > 0$ and $((\phi_c^2)^{-1} \circ K)(F) \subseteq V_1 \times V_{2,c}^\pm$.

$V_{2,c}$ contains a polynomial $k\,(x^2 \pm y^2)$, where " $\pm$ " $= sign(c)$ and $k^2 = \frac{1}{6}|c|$. In the case $c > 0$ the choice of sign of k determines the component $V_{2,c}^\pm$.

If we let $\overline{F}_k := K^{-1}(\{\phi_c^2(v_1,\ k\,(x^2 \pm y^2)) \mid v_1 \in V_1\}) \subseteq F$ then $\overline{F}_k$ is a smooth 5-dimensional submanifold of F and moreover a *reduction* of F with structure group

$$O(2) \subseteq Sl(2,\mathbb{R}) \quad \text{if } c > 0, \text{ and}$$

$$O(1,1) \subseteq Sl(2,\mathbb{R}) \quad \text{if } c < 0.$$

We then define functions r_1 and r_2 on $\overline{F}_k$ by the equation

$$K(u) = \phi_c^2(r_1(u)x + r_2(u)y,\ k\,(x^2 \pm y^2)) \quad \text{for all} \quad u \in \overline{F}_k.$$

Since $rank(K|_{\overline{F}_k}) = 2$ we conclude that $dr_1 \wedge dr_2 \neq 0$ on $\overline{F}_k$. Also, $s := r_1^2 \pm r_2^2$ is constant along the fibers of $\overline{F}_k$ and hence factors through to a function $f : M \to \mathbb{R}$.

We compute that the set $U_k \subseteq M$ on which condition (13) is satisfied is the set on which $f \neq 0$.

Let $\overline{F}_k^0$ denote an $SO(2)$, $O(1,1)_e$-reduction of $\overline{F}_k$ resp. Let $U_k^{\pm} := f^{-1}(\mathbb{R}^{\pm}) \subseteq M$. Then $U_k := U_k^+ \cup U_k^- = f^{-1}(\mathbb{R}\backslash\{\not{1}\}) \subseteq \mathbb{M}$ and we let $\overline{U}_k^{\pm} := \overline{F}_k^0 \cap \pi^{-1}(U_k^{\pm})$.

Since $dr_1 \wedge dr_2 \neq 0$, we conclude that U_k is *open dense* in M and $\overline{U}_k$ is *open dense* in $\overline{F}_0$.

Now we let $S := r_1^{-1}(0) \cap \overline{U}_k \subseteq \overline{F}_0$. Since $dr_1 \neq 0$, we conclude that S is a smooth 4-dimensional submanifold of $\overline{U}_k$. Moreover, S intersects every fiber $\pi^{-1}(p)$ in exactly one point for every $p \in U_k$. This means that S is the image of a smooth section $\sigma : U_k \to \overline{U}_k$.

Using σ, we do a calculation similar to the one described in the previous sections and obtain the connection on U_k. To describe the result we treat the two possibilities for the sign of c separately.

If $c > 0$ then $f \geq 0$ on M, i.e. $U_k = U_k^+$. Define a smooth function $t_0 : U_k \to \mathbb{R}^+$ by the equation $f = (kt_0)^2$. Then one sees that the connection described by the section σ is locally equivalent to the connection on M_k^+ described in example 2.3. Thus we get the

Theorem 5.5 *Let M be a connected 4-manifold with an H_3-connection ∇ of type $\Sigma_c^{2,\pm}$ with $c > 0$ and let k such that sign$(k) = \pm$ and $c = 6k^2$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_k^+$$

with M_k^+ from section 2.3, where $\tilde{U}_i$ denotes the universal cover of U_i.

Now assume $c < 0$. In this case one sees easily that the solution structures of $U_k^\pm$ and $U_{-k}^\mp$ are *homothetic*. Therefore, we only need to investigate U_k^+, i.e. we may assume that $f > 0$. Again, define a smooth function $t_0 : U_k \to \mathbb{R}^+$ by the equation $f = (kt_0)^2$. Then one computes that the connection described by the section σ is locally equivalent to the connection on M_k^- described in example 2.3. Thus we get the

Theorem 5.6 *Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ_c^2 with $c < 0$ and let k such that $c = -6k^2$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U*

into its connected components. Then there exist connection preserving immersions

$$\imath : \tilde{U}_i \to M_k^- \sqcup M_{-k}^-$$

with $M_{\pm k}^-$ from section 2.3, where $\tilde{U}_i$ denotes the universal cover of U_i.

As in the last section, the difference between these two results is that in the case $c > 0$ an H_3-connection of type $\Sigma_c^{2,\pm}$ on M is *either* locally equivalent to M_k^+ *or* locally equivalent to M_{-k}^+ since the image of K is contained in different components of Σ_c^2 in these cases.

In the case $c < 0$, however, it is very well possible that an H_3-connection is locally equivalent to M_k^- in some neighborhood, but equivalent to M_{-k}^- in some other neighborhood on the same manifold M.

5.5 Regular H_3-connections

In this section we will discuss those H_3-connections on M for which the map $K : F \to V$ has maximal rank 6.

Unfortunately, in this case the structure equations involved are so complex that they cannot be solved explicitly on an dense open subset of M. A (not very illuminating) description of the connection on some open subset, however, can be obtained.

Let $U := \{p \in M \mid \; < a, a >_2 < 0, \text{ and } a \text{ does not divide } b \text{ on } \pi^{-1}(p)\}$. For $p \in U$ there is a unique frame $u \in \pi^{-1}(p)$ such that

$$b_3(u) = 1, \quad a(u) = f(u)\, xy \quad \text{with } f(u)^2 = -\tfrac{1}{2} < a(u), a(u) >_2 > 0.$$

Using this section, we can describe the connection w.r.t. some coordinate system on U as follows where the constants c and R_c are given.

Let $x_0, \ldots, x_3$ be local coordinates and define the functions

$$t_0 = \frac{1}{x_0}, \quad t_1 = \frac{x_1}{x_0^2}, \quad t_2 = \frac{x_1^2}{x_0^3} + \frac{x_2}{x_0}, \quad \text{and} \quad t_3 = \frac{x_1^3}{x_0^4} + 3\frac{x_1 x_2}{x_0^2} - x_0^2 + c + \frac{1}{x_0} r$$

where

$$r = \pm 2\sqrt{x_1^3 - x_2^3 + c x_1 x_2 + R_c}.$$

Furthermore, let

$$s_i := \frac{\partial}{\partial x_0}(x_0 t_i) \text{ for all } i.$$

Then we define a frame as follows:

$$\begin{aligned}
X_0 &= 3x_0 t_0 \frac{\partial}{\partial x_0} \\
X_1 &= x_0 t_1 \frac{\partial}{\partial x_0} + t_0 \overline{X}_1 \\
X_2 &= -x_0 t_2 \frac{\partial}{\partial x_0} + 2t_1 \overline{X}_1 + \overline{X}_2 \\
X_3 &= -3x_0 t_3 \frac{\partial}{\partial x_0} + 3t_2 \overline{X}_1 + 3x_0 t_1 \overline{X}_2 + \overline{X}_3
\end{aligned}$$

with

$$\begin{aligned}
\overline{X}_1 &= r \frac{\partial}{\partial x_2} + \frac{\partial a}{\partial x_2} \frac{\partial}{\partial x_3} \\
\overline{X}_2 &= -r \frac{\partial}{\partial x_1} - \frac{\partial a}{\partial x_1} \frac{\partial}{\partial x_3} \\
\overline{X}_3 &= 2(3x_2^2 - c x_1) \frac{\partial}{\partial x_1} + 2(3x_1^2 + c x_2) \frac{\partial}{\partial x_2}
\end{aligned}$$

and where $a = a(x_1, x_2)$ is some function satisfying $\overline{X}_3(a) = 1$.

This frame is defined on $\{(x_0, \ldots, x_3) \mid x_0 \neq 0, x_1^3 - x_2^3 + c\, x_1 x_2 + R_c > 0\}$ for the given constants c and R_c.

The connection form is then given as

$$\theta = \theta_1\, E_1 + \theta_2\, E_2 + \theta_3\, E_3$$

where

$$\begin{aligned}
\theta_1 &= -\sum_{i=0}^{3} s_i\,\omega_i \\
\theta_2 &= -\sum_{i=0}^{3} i\,t_{i-1}\,\omega_i \\
\theta_3 &= \sum_{i=0}^{3} (3-i)\,t_{i+1}\,\omega_i
\end{aligned}$$

and where $\omega_0, \ldots, \omega_3$ denotes the dual basis of $X_0, \ldots, X_3$.

One can check that this connection is indeed torsion free and is a *regular* H_3-connection. Note in particular that the 1-dimensional symmetry group is given as the flow along the vector field $\frac{\partial}{\partial x_3}$.

6 H_3-connections on compact manifolds

Theorem 6.1 *Let M be a compact 4-manifold with an H_3-connection. Then the H_3-connection is of type Σ_0^0, i.e. M is locally homogeneous.*

Proof: Let $\{\pi, F, M, a, b, c\}$ be an associated solution structure. Consider the function

$$\begin{array}{rll} f: & F & \longrightarrow \mathbb{R} \\ & u & \longmapsto discr(a(u)) = -\frac{1}{2} < a(u), a(u) >_2 . \end{array}$$

Since f is constant along the fibers of F there is a unique function $\underline{f} : M \to \mathbb{R}$ such that $\underline{f} \circ \pi = f$.

We shall now use that $\underline{f}$ must have both a maximum and a minimum on M. Therefore, we shall investigate the critical points of $\underline{f}$.

Using the structure equations (3) - (9) we find that

$$df = 2 \begin{pmatrix} b_0 & \dots & b_3 \end{pmatrix} \begin{pmatrix} -3a_1 & 2a_0 & & \\ -6a_2 & -a_1 & 4a_0 & \\ & -4a_2 & a_1 & 6a_0 \\ & & -2a_2 & 3a_1 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_3 \end{pmatrix} .$$

The determinant of the matrix in this equation is $9f^2$. So if $u \in F$ is a critical point of f and $f(u) \neq 0$ then $b(u) = 0$. We wish to compute the *Hessian* of $\underline{f}$ at a critical point $p \in M$ w.r.t. some appropriate frame.

If $\underline{f}(p) > 0$ then there is a frame $u \in \pi^{-1}(p)$ with $a_0(u) = a_2(u) = 0$ and $a_1(u) \neq 0$. We compute the Hessian w.r.t. this frame as

$$\begin{pmatrix} & & & -9a_1(2c + 3a_1^2) \\ & & a_1(2c + a_1^2) & \\ & a_1(2c + a_1^2) & & \\ -9a_1(2c + 3a_1^2) & & & \end{pmatrix} .$$

We see easily that this matrix has some positive eigenvalue regardless of the values of a_1 and c. Therefore, $\underline{f}$ *cannot* have a positive maximum, and we conclude that $\underline{f} \leq 0$.

Let $\mathcal{C} := \{a \in V_2 \mid discr(a) \leq 0\}$. One sees easily that we have $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ with $\mathcal{C}_\pm = \{\pm(v_1^2 + v_2^2) \mid v_1, v_2 \in V_1\}$, and $\mathcal{C}_+ \cap \mathcal{C}_- = \{0\}$. Moreover, $\mathcal{C}_\pm$ is invariant under the $Sl(2, \mathbb{R})$-action on V_2.

Suppose that $\underline{f}(p) = 0$ for some $p \in M$ and $a \neq 0$ on $\pi^{-1}(p)$. Then we can find some frame $u \in \pi^{-1}(p)$ with $a(u) = \pm x^2$. Since u is a critical point of f, hence $df(u) = 0$, we get that $b(u) = \tilde{b}x^3$ for some $\tilde{b} \in \mathbb{R}$. Computing the Hessian of f at u we get

$$\begin{pmatrix} 0 & & & \\ & 0 & & \pm 12c \\ & & \mp 8c & \\ & \pm 12c & & 18(\tilde{b}^2 \pm 2) \end{pmatrix}$$

The characteristic polynomial of this matrix is

$$\lambda(\lambda \pm 8c)(\lambda^2 - 18(\tilde{b}^2 \pm 2)\lambda - 144c^2)$$

The Hessian must be negative semidefinite, i.e. this polynomial cannot have any positive root. It is easily seen that this is satisfied only if $a(u) = -x^2$, $c = 0$ and $\tilde{b}^2 \leq 2$. We conclude that $f(u) = 0$ implies $a(u) \in \mathcal{C}_-$.

Now suppose that $p \in M$ with $\underline{f}(p) < 0$ is a critical point of $\underline{f}$. Then there is a frame $u \in \pi^{-1}(p)$ with $a_1(u) = 0$ and $a_0(u) = a_2(u)$. We compute the Hessian w.r.t.

this frame as

$$\begin{pmatrix} 36a_0^3 & 0 & -12a_0(5a_0^2-c) & 0 \\ 0 & 4a_0(13a_0^2-2c) & 0 & -12a_0(5a_0^2-c) \\ -12a_0(5a_0^2-c) & 0 & 4a_0(13a_0^2-2c) & 0 \\ 0 & -12a_0(5a_0^2-c) & 0 & 36a_0^3 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$r(\lambda)^2, \quad \text{where} \quad r(\lambda) = \lambda^2 - 8a_0(11a_0^2 - c)\,\lambda - 144a_0^2(2a_0^2 - c)(6a_0^2 - c).$$

If $a(u) \in \mathcal{C}_+$, i.e. $a_0(u) > 0$, then r has at least one positive root since either $r(0) < 0$ or $r'(0) < 0$. Therefore, $\underline{f}$ cannot have a negative maximum on $\pi(f^{-1}(\mathcal{C}_+))$. But this set is closed in M and hence compact. We conclude that if $a(F) \cap \mathcal{C}_+ \neq \emptyset$ then $0 \in a(F)$.

Similarly, if $a(u) \in \mathcal{C}_-$, i.e. $a_0(u) < 0$, then r has at least one negative root since either $r(0) < 0$ or $r'(0) > 0$ if $a_0 < 0$. Therefore, $\underline{f}$ cannot have a negative minimum on $\pi(f^{-1}(\mathcal{C}_-))$. But this set is a closed and hence compact subset of M, therefore $\underline{f}$ must have a minimum in this set. We conclude that if $a(u) \in \mathcal{C}_-$ then $f(u) = 0$.

Suppose now that $a(U) \subseteq \mathcal{C}_- \backslash \{0\}$ for some open set $U \subseteq F$. By the above, $K(U) \subseteq \{(-v^2, \tilde{b}v^3) \mid v \in V_1\backslash\{0\}, \tilde{b} \in \mathbb{R}\}$. Since this set is 3-dimensional we get $rank(K) \leq 3$ on U. But then Theorem 3.7 implies that M is of type Σ_0^0. In particular, $a(F) \subseteq \mathcal{C}_- \backslash \{0\}$.

Thus, either M is of type Σ_0^0 or $a(F) \subseteq \mathcal{C}_+$.

But in the latter case, $a(u) = 0$ for some $u \in F$. Since $a(F) \subseteq \mathcal{C}_+$, hence $a_0, a_2 \geq 0$ this means that u is a minimum for both a_0 and a_2, hence $da_0(u) = da_2(u) = 0$

and therefore $b(u) = 0$. Computing the Hessian of a_0 at u yields the matrix

$$\begin{pmatrix} 0 & & -3c & \\ & 2c & & \\ -3c & & 0 & \\ & & & 0 \end{pmatrix}$$

This matrix must be positive semidefinite which is satisfied iff $c = 0$. Thus, $a(u) = b(u) = c = 0$, i.e. $K(u) = 0$, and again Theorem 3.7 implies that the connection is *flat*, violating our assumption. Therefore, we get a contradiction from the assumption $a(F) \subseteq \mathcal{C}_+$ and this shows that M is of type Σ_0^0. **q.e.d.**

Suppose now that M is a *compact* manifold with an H_3-connection and let $\alpha : \tilde{M} \to M$ denote its universal cover. Then by Theorems 5.1 and 6.1 there is a connection preserving immersion $\imath : \tilde{M} \to G/H$.

Recall the description of G/H in 2.1. Since the forms τ_i on G/H described there are G-invariant, there are forms $\underline{\tau}_i$, $0 \leq i \leq 3$ on M such that $\alpha^*(\underline{\tau}_i) = \imath^*(\tau_i) =: \tilde{\tau}_i$. Also, the distributions $\underline{\mathcal{D}}_i$ given by $\underline{\tau}_i(\underline{\mathcal{D}}_i) = 0$ are *integrable* for all i.

Let $\underline{Z}_0, \ldots, \underline{Z}_3$ be a *frame* on M such that $\underline{\mathcal{D}}_i = span(\underline{Z}_0, \ldots, \underline{Z}_{i-1})$ and $\underline{\tau}_i(\underline{Z}_i, \ldots, \underline{Z}_3) = 1$. Such a frame can be easily constructed using an arbitrary Riemannian metric on M. If we denote the dual 1-forms of $\underline{Z}_0, \ldots, \underline{Z}_3$ by $\underline{\omega}_0, \ldots, \underline{\omega}_3$ then we have $\underline{\tau}_i = \underline{\omega}_i \wedge \ldots \wedge \underline{\omega}_3$ for all i.

Passing to the universal cover $\tilde{M}$, let $\tilde{\omega}_i := \alpha^*(\underline{\omega}_i)$, and let $\tilde{Z}_i$ be the lifts of $\underline{Z}_i$. Note that $\tilde{Z}_i$ is *complete* for all i, i.e. the flow along $\tilde{Z}_i$ is defined for all times. This follows from the completeness of the vector fields $\underline{Z}_i$ which are defined on a compact manifold.

Let $\tilde{Y}_i$ be a maximal connected integral leave (m.c.i.l.) of $\tilde{\mathcal{D}}_i$. Since $\imath_*(\tilde{\mathcal{D}}_i) = \mathcal{D}_i$, the image $\imath(\tilde{Y}_i)$ is contained in some m.c.i.l. Y_i of $\mathcal{D}_i$.

Lemma 6.2 *The restriction* $\imath : \tilde{Y}_i \to Y_i$ *is a diffeomorphism for* $i \leq 3$.

Proof: We proceed by induction on i. For $i = 0$, both $\tilde{Y}_i$ and Y_i are *points* and there is nothing to prove.

Suppose the claim is true for some $i \leq 2$ and let $\tilde{Y}_{i+1}$ and Y_{i+1} be m.c.i.l.'s such that $\imath(\tilde{Y}_{i+1}) \subseteq Y_{i+1}$. Recall the function $r_{i+1} : Y_{i+1} \to \mathbb{R}$ defined in section 2.1, and let $s_{i+1} : \tilde{Y}_{i+1} \to \mathbb{R}$ be the composition $s_{i+1} = r_{i+1} \circ \imath$.

Suppose $\imath(p) = \imath(q)$ for some points $p, q \in \tilde{Y}_{i+1}$ and assume for convenience that $s_{i+1}(p) = s_{i+1}(q) = 0$. Let $\gamma : [0,1] \to \tilde{Y}_{i+1}$ be a smooth path joining p and q. Then define a new path σ by

$$\sigma(t) = \Phi_{\tilde{Z}_i}^{-s_{i+1}(\gamma(t))}(\gamma(t)),$$

where Φ_X^t denotes the flow along a vector field X for time t.

Note that $\sigma(t)$ is well defined for all $t \in [0,1]$ by completeness of $\tilde{Z}_i$. Also, since $\tilde{Z}_i(s_{i+1}) \equiv 1$ it follows that $s_{i+1}(\sigma(t)) = 0$, thus σ lies in a m.c.i.l. $\tilde{Y}_i$ of $\tilde{\mathcal{D}}_i$. In particular, $q \in \tilde{Y}_i$, and now the induction hypothesis implies that $p = q$, i.e. $\imath|_{Y_{i+1}}$ is *injective*.

The induction hypothesis and the fact that the m.c.i.l.'s of $\mathcal{D}_{i+1}$ are precisely the level sets of r_{i+1} imply that $\imath(\tilde{Y}_{i+1}) = r_{i+1}^{-1}(\mathcal{I})$ for some open interval $\mathcal{I} \subseteq \mathbb{R}$. However, s_{i+1} is *surjective*. In fact, given $p \in \tilde{Y}_{i+1}$, we have $s_{i+1}(\Phi_{\tilde{Z}_i}^t(p)) = s_{i+1}(p) + t$, and t is arbitrary. Therefore, $\mathcal{I} = \mathbb{R}$ and this shows *surjectivity* of $\imath|_{\tilde{Y}_{i+1}}$, and this finishes the proof. **q.e.d.**

Unfortunately, since $d\omega_3 \neq 0$, we cannot continue the same proof to show global injectivity of $\imath$.

On $\tilde{M}$, we declare two points to be *equivalent* iff they lie in the same m.c.i.l. of $\tilde{\mathcal{D}}_3$. Let $\pi : \tilde{M} \to \tilde{M}/_\sim$ be the canonical projection. Then there is a unique function $\underline{\theta} : \tilde{M}/_\sim \to S^1$ making the diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\imath} & G/H \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \theta} \\ \tilde{M}/_\sim & \xrightarrow{\underline{\theta}} & S^1 \end{array} \tag{14}$$

commute where θ is the *angle function* defined in section 2.1.

Lemma 6.3 *Consider the maps described above. Then*

1) $\pi : \tilde{M} \to \tilde{M}/_\sim$ *is open,*

2) $\underline{\theta} : \tilde{M}/_\sim \to S^1$ *is a local homeomorphism, and*

3) $\tilde{M}/_\sim$ *is Hausdorff.*

PROOF: 1) For this note that for a given open set $U \subseteq \tilde{M}$, we have

$$\pi^{-1}(\pi(U)) = \bigcup_{t_0,\ldots,t_3} \Phi^{t_0}_{\tilde{Z}_0} \circ \ldots \circ \Phi^{t_3}_{\tilde{Z}_3}(U),$$

which is clearly open.

2) Let $p \in \tilde{M}/_\sim$ and let $x \in \tilde{M}$ such that $\pi(x) = p$. Choose a path $\gamma : (-\varepsilon, \varepsilon) \to \tilde{M}$ with $\gamma(0) = x$ which is transverse to the distribution $\mathcal{D}_3$. By the same argument as above we see that $V := \pi(\gamma)$ is an open neighborhood of $p \in \tilde{M}/_\sim$. Moreover, the restriction $\pi : \{\gamma(t)\} \to V$ is an *homeomorphism* being an open and continuous

bijection. Since the restriction $(\theta \circ \imath)|_\gamma$ is an homeomorphism, so is the restriction $\underline{\theta}|_V$, and this shows the claim.

3) Let $p_1, p_2 \in \tilde{M}/_\sim$ be two different points. If $\underline{\theta}(p_1) \neq \underline{\theta}(p_2)$ then these two points can be separated by open neighborhoods, so we assume that $\underline{\theta}(p_1) = \underline{\theta}(p_2)$.

By 2), there are open neighborhoods V_i of p_i such that the restrictions $\underline{\theta}|_{V_i}$ is an homeomorphism onto its image $\mathcal{I}_i \subseteq S^1$. Since singleton sets are closed in $\tilde{M}/_\sim$ we may assume that $p_i \notin V_j$ for $i \neq j$. Furthermore we may assume that $\mathcal{I}_1 = \mathcal{I}_2 =: \mathcal{I}$ after shrinking one of the sets if necessary.

Let $T := \theta^{-1}(\mathcal{I}) \subseteq G/H$ and let $U_i := \pi^{-1}(V_i) \subseteq \tilde{M}$. By Lemma 6.2 and the choice of V_i it follows that the restrictions $\imath_i : U_i \to T$ are *diffeomorphisms* for $i = 1, 2$. We therefore can define a diffeomorphism $\phi : U_1 \to U_2$ as $\phi := \imath_2^{-1} \circ \imath_1$. Note that the intersection $U_1 \cap U_2$ is precisely the set $\{x \in U_1 \mid \phi(x) = x\}$. Thus, $U_1 \cap U_2$ is *closed* in U_1 and we conclude that either $U_1 \cap U_2 = \emptyset$ or $U_1 = U_2$. But $U_1 = U_2$ implies that $V_1 = V_2$ which is impossible. Hence, $U_1 \cap U_2 = \emptyset$ which implies $V_1 \cap V_2 = \emptyset$. **q.e.d.**

Let $\tilde{G}$ be the universal cover of the Lie group $G = ASl(2, \mathbb{R})$ and let $\tilde{H} \subseteq \tilde{G}$ be the 1-parameter subgroup which covers H. Then $\tilde{G}/\tilde{H}$ is the universal cover of G/H.

Corollary 6.4 *Let $\imath : \tilde{M} \to G/H$ be the immersion described above. Then the lift*

$$\tilde{\imath} : \tilde{M} \to \tilde{G}/\tilde{H}$$

is an embedding onto some subset $T := \tilde{\theta}^{-1}(\mathcal{I})$ for some open interval $\mathcal{I} \subseteq \mathbb{R}$, where $\tilde{\theta} : \tilde{G}/\tilde{H} \to \mathbb{R}$ denotes the lift of θ.

PROOF: By Lemma 6.3 we know that $\tilde{M}/_\sim$ is a connected 1-manifold. Moreover as a consequence of the proof of this Lemma we get that π is a *fiber bundle* whose fibers are diffeomorphic to $\mathbb{R}^3$. Therefore, $\tilde{M}/_\sim$ is *simply connected*, i.e. $\tilde{M}/_\sim$ is homeomorphic to $\mathbb{R}$. $\underline{\tilde{\theta}}$ being a *local* homeomorphism then implies that $\underline{\tilde{\theta}}$ is a *global* homeomorphism onto some open interval $\mathcal{I} \subseteq \mathbb{R}$.

Let $T := \tilde{\theta}^{-1}(\mathcal{I}) \subseteq \tilde{G}/\tilde{H}$. Then the lifts of the maps in diagram (14) yield a commutative diagram

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\tilde{\imath}} & T & \xrightarrow{\subseteq} & \tilde{G}/\tilde{H} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\theta}} & & \downarrow{\scriptstyle \tilde{\theta}} \\ \tilde{M}/_\sim & \xrightarrow{\underline{\tilde{\theta}}} & \mathcal{I} & \xrightarrow{\subseteq} & \mathbb{R} \end{array}$$

Now use that $\underline{\tilde{\theta}}$ is a homeomorphism and apply Lemma 6.2. **q.e.d.**

By virtue of this Corollary we can naturally identify $\tilde{M}$ with $T \subseteq \tilde{G}/\tilde{H}$.

We now investigate the *symmetries* on $\tilde{M} \cong T$.

Lemma 6.5 *Every symmetry of T extends to a symmetry of $\tilde{G}/\tilde{H}$.*

PROOF: Recall the definition of the principal $\mathbb{R}$-bundle $\pi : \overline{F} \to \tilde{G}/\tilde{H}$ from section 5.1, and let $\overline{F}_T := \pi^{-1}(T) \subseteq \overline{F}$. Note that the symmetry group $\tilde{G}$ acts *transitively* on $\overline{F}$.

Let $\phi : T \to T$ be a symmetry and let $\overline{\phi} : \overline{F}_T \to \overline{F}_T$ be the map induced by the differential of ϕ. Fix a point $x \in \overline{F}_T$ and let $p := \pi(x) \in T$. There exists a $g \in \tilde{G}$ such that $\overline{\psi}(x) = x$ where $\overline{\psi} := g \circ \overline{\phi}$. This means that for $\psi : T \to \tilde{G}/\tilde{H}$ with $\psi = g \circ \phi$ we have $\psi(p) = p$ and $d\psi_p = Id$. But ψ is a *symmetry*, hence ψ is the *identity map* on T, i.e. $\phi = g^{-1} \in \tilde{G}$. **q.e.d.**

Suppose now that $T \neq \tilde{G}/\tilde{H}$, i.e. $T = \tilde{\theta}^{-1}(\mathcal{I})$ with $\mathcal{I} \neq \mathbb{R}$. Let t_0 be in the boundary of $\mathcal{I}$. By Lemma 6.5 the symmetries of T must leave the set $\tilde{\theta}^{-1}(t_0)$ invariant. If we precompose $\tilde{\imath} : \tilde{M} \to T$ with an appropriate element of $\tilde{G}$ we may assume that $t_0 = \frac{\pi}{2}$.

Let $\tilde{S} \subseteq \tilde{G}$ be the subgroup which fixes $\tilde{\theta}^{-1}(\frac{\pi}{2})$. Let $\alpha : \tilde{G} \to G$ be the covering homomorphism. Note that any element $g \in ker(\alpha) \cong \mathbb{Z}$ acts on $\tilde{G}/\tilde{H}$ such that $\tilde{\theta}(g(x)) = \tilde{\theta}(x) + 2\pi n$ with $n \in \mathbb{Z}$. Thus, $\tilde{S} \cap ker(\alpha) = \{e\}$, i.e. $\alpha|_{\tilde{S}}$ is *injective*.

Let $S := \alpha(\tilde{S}) \subseteq G$. Then clearly S leaves $\theta^{-1}(\frac{\pi}{2})$ invariant, i.e.

$$S = \left\{ \left(\begin{array}{ccc} e^a & x & z \\ & e^{-a} & y \\ & & 1 \end{array} \right) \middle| \, a, x, y, z \in \mathbb{R} \right\}.$$

From this we also conclude that $\tilde{S}$ leaves all level sets $\tilde{\theta}(\frac{(2n+1)\pi}{2}), n \in \mathbb{Z}$ invariant.

The fundamental group of M acts on $\tilde{M} \cong T$ by *symmetries*. Since $\alpha|_{\tilde{S}}$ is an *isomorphism* we conclude that $\pi_1(M)$ is isomorphic to a discrete subgroup $\Gamma \subseteq S$.

Lemma 6.6 *Let $N_3 \subseteq S$ denote the Heisenberg group. Then Γ cannot be contained in N_3.*

PROOF: Suppose $\Gamma \subseteq N_3$. The first step is to show that $L_n := \tilde{\theta}^{-1}(-\frac{(2n+1)\pi}{2}) \not\subseteq \tilde{\theta}(T)$. To see this note that L_n is Γ-invariant, hence Γ acts on L_n with compact quotient. On L_n, we have the *slope function* r_3 which was defined in section 2.1, and it is easy to see that r_3 is invariant under N_3 and hence by hypothesis under Γ. Thus, there is an induced function $\tilde{r}_3 : L_n/\Gamma \to \mathbb{R}$. However, r_3 has no critical point on L_n and hence on L_n/Γ, but on the other hand L_n/Γ is *compact*. This is a contradiction which finishes the first step.

Therefore, $\tilde{\theta}(T)$ is a *bounded interval* of length $< \pi$ and we may assume that $\frac{\pi}{2}$ is an *upper* bound of this interval. Let $C \subseteq T \subseteq \tilde{\theta}^{-1}(-\frac{\pi}{2}, \frac{\pi}{2})$ be a compact subset whose Γ-orbit is T. Note that the restriction of the covering map $\pi|_T$ is a *diffeomorphism*, thus if we define $T_1 := \pi(T) \subseteq \theta^{-1}(-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq G/H$ then the orbit of $C_1 := \pi(C) \subseteq T_1$ must equal T_1. Consider the function

$$\begin{aligned} s: \quad G/H &\longrightarrow \mathbb{R} \\ gH &\longmapsto < L(g) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} > \end{aligned}$$

where as before $L : ASl(2,\mathbb{R}) \to Sl(2,\mathbb{R})$ is the homomorphism assigning the linear part to an affine motion and $< , >$ is the standard inner product on $\mathbb{R}^2$.

Clearly, $s(T_1) = \mathbb{R}^+$ since $\theta(T_1)$ has $\frac{\pi}{2}$ as its supremum. On the other hand, one sees easily that s is invariant under N_3 and hence under Γ. But $s(C_1)$ is *compact*, hence bounded away from 0 which is a contradiction. **q.e.d.**

Lemma 6.7 *Every finitely generated discrete subgroup $\Gamma \subseteq S$ is either contained in N_3 or is cyclic and diagonalizable.*

PROOF: Suppose Γ is *not* contained in N_3, i.e. there is some $g_0 \in \Gamma$,

$$g_0 = \begin{pmatrix} e^a & r & t \\ & e^{-a} & s \\ & & 1 \end{pmatrix}$$

with $a \neq 0$. Let $\Gamma_1 := [\Gamma, \Gamma] \subseteq N_3$. Then Γ_1 is also finitely generated and is a normal subgroup of Γ.

Let $\phi : N_3 \to \mathbb{R}^2$ be the homomorphism given by

$$\phi \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} := \begin{pmatrix} x \\ y \end{pmatrix},$$

and consider the finitely generated subgroup $\Gamma_2 := \phi(\Gamma_1) \subseteq \mathbb{R}^2$.

If $g \in \Gamma_1$ with $\phi(g) = (x, y)$ then one computes $\phi(g_0^n g g_0^{-n}) = (e^{2an}x, e^{-an}y) \in \Gamma_2$. However, there is no finitely generated subgroup of $\mathbb{R}^2$ which contains these elements for all $n \in \mathbb{Z}$ unless $x = y = 0$.

We conclude that $\Gamma_1 \subseteq ker(\phi)$. Using the same argument once again we can conclude that $\Gamma_1 = 0$, i.e. Γ is *abelian*.

Thus Γ is diagonalizable in S, hence we may assume that Γ consists of *diagonal* matrices. Finally, the discreteness of Γ implies that Γ is *cyclic*. **q.e.d.**

We then arrive at the

Corollary 6.8 *Let M be a compact manifold with an H_3-connection. Then*

$$M = \Gamma \backslash \tilde{G} / \tilde{H}$$

for some discrete subgroup $\Gamma \subseteq \tilde{G}$.

PROOF: Suppose $\Gamma \subset S$ is *cyclic* and *diagonal*. Again, let $C \subseteq T$ be a compact subset whose Γ-orbit is T. Since $\tilde{S}$ leaves $T' := \tilde{\theta}^{-1}[-\frac{\pi}{2}, \frac{\pi}{2})$ invariant, the orbit of the compact set $C' := C \cap T'$ must equal $T' \cap \tilde{\imath}(\tilde{M})$. Note that the restriction of the covering map $\pi|_{T'}$ is a *diffeomorphism*, thus if we define $T_1 := \pi(T') \subseteq \theta^{-1}[-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq G/H$ then the orbit of $C_1 := \pi(C') \subseteq T_1$ must equal $T_1 \cap \imath(\tilde{M})$.

There exists a constant $k \in \mathbb{R}$ such that every parabola $p \in C_1$ intersects the set $U := \{(x, y) \mid xy \leq k\} \subseteq \mathbb{R}^2$. But we see easily that U is *invariant* under Γ, hence every parabola in the orbit of C_1 intersects U. But this is impossible since there are parabolas in $T_1 \cap \imath(\tilde{M})$ which *do not* intersect U.

Since the fundamental group of a compact manifold is finitely generated, this together with the previous two Lemmas implies that $\tilde{\imath} : \tilde{M} \to \tilde{G}/\tilde{H}$ is a *diffeomorphism*. The fundamental group $\pi_1(M)$ acts on $\tilde{G}/\tilde{H}$ by *symmetries*. Finally, following the proof of Lemma 6.5 we see that $\tilde{G}$ is the full symmetry group of $\tilde{G}/\tilde{H}$.
q.e.d.

Unfortunately, there are no known examples of such biquotients. Hence the question of existence of compact manifolds with H_3-connections remains open.

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