Connections with Exotic Holonomy

Lorenz Johannes Schwachhöfer
Department of Mathematics
Washington University, St. Louis

March 25, 1993

Abstract

Berger [Ber] partially classified the possible irreducible holonomy representations of torsion free connections on the tangent bundle of a manifold. However, it was shown by Bryant [Br2] that Berger's list is incomplete. Connections whose holonomy is not contained on Berger's list are called *exotic*.

We investigate a certain 4-dimensional exotic holonomy representation of $Sl(2, \mathbf{R})$. We show that connections with this holonomy are never complete and do not exist on compact manifolds. We give explicit descriptions of these connections on an open dense set and compute their groups of symmetry.

1 Introduction

Let M^n be a smooth connected n-dimensional manifold. Let $\mathcal{P}(M)$ denote the set of piecewise smooth paths $\gamma : [0, 1] \to M$, and for $x \in M$, let $\mathcal{L}_x(M) \subseteq \mathcal{P}(M)$ denote the set of x-based loops, i.e. paths for which $\gamma(0) = \gamma(1) = x$.

Let ∇ be a torsion free affine connection on the tangent bundle of M. For each $\gamma \in \mathcal{P}(M)$, the connection ∇ defines a linear isomorphism $P_\gamma : T_{\gamma(0)}M \to T_{\gamma(1)}M$, called *parallel translation along* γ. For each $x \in M$, we define the *holonomy group of* ∇ *at* x to be $H_x := \{P_\gamma \mid \gamma \in \mathcal{L}_x\} \subseteq Gl(T_xM)$.

It is well known that H_x is a Lie subgroup of $Gl(T_xM)$, and that for any $\gamma \in \mathcal{P}(M)$, P_γ induces an isomorphism of $T_{\gamma(0)}M$ with $T_{\gamma(1)}M$ which identifies $H_{\gamma(0)}$ with $H_{\gamma(1)}$ [KN].

Choose an $x_0 \in M$ and an isomorphism $i : T_{x_0}M \to \mathbf{R}^n$. Then, because M is connected, the conjugacy class of the subgroup $H \subseteq Gl(n, \mathbf{R})$ which corresponds under i to $H_{x_0} \subseteq Gl(T_{x_0}M)$ is independent of the choice of x_0 or i. By abuse of language, we speak of H as the *holonomy group* and of the Lie algebra $\mathfrak{h}$ of H as the *holonomy algebra of* (M,∇).

The following is a basic question in the theory:
Which (conjugacy classes of) subgroups $H \subseteq Gl(n, \mathbf{R})$ can occur as the holonomy of some torsion free connection ∇ on some n-manifold M?

The condition of torsion freeness makes this problem non-trivial. In fact, it is not hard to see that any representation of a connected Lie group can be realized as the holonomy of some connection (with torsion) on some manifold.

A necessary condition on the holonomy algebra of a torsion free connection was derived by M. Berger [Ber] in his thesis as follows.

Let V be a vector space and define for a given Lie algebra $\mathfrak{g} \subseteq \mathfrak{g}l(V)$:

$$\mathbf{K}(\mathfrak{g}) := \left\{ \phi : \mathbf{\Lambda^2}(V) \to \mathfrak{g} \;\middle|\; \phi \text{ linear}, \sum_{\sigma \in A_3} \phi(u_{\sigma(1)}, u_{\sigma(2)})u_{\sigma(3)} = 0 \text{ for all } u_1, u_2, u_3 \in V \right\},$$

and

$$\mathbf{K^1}(\mathfrak{g}) := \left\{ \psi : V \to \mathbf{K}(\mathfrak{g}) \mid \psi \text{ linear}, \sum_{\sigma \in A_3} \psi(u_{\sigma(1)})(u_{\sigma(2)}, u_{\sigma(3)}) = 0 \text{ for all } u_1, u_2, u_3 \in V \right\}.$$

Given (M, ∇) as above and $x_0 \in M$, the *curvature tensor of* ∇ *at* x_0, i.e. the map $R_{x_0} : \mathbf{\Lambda}^2(T_{x_0}M) \to \mathfrak{gl}(T_{x_0}M)$ defined by $R_{x_0}(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$, is known to have its values in the holonomy algebra $\mathfrak{h}_{x_0} \subseteq \mathfrak{gl}(T_{x_0}M)$, and to satisfy the *first and second Bianchi identities.* This is equivalent to saying

$$R_{x_0} \in \mathbf{K}(\mathfrak{h}_{x_0}),$$

and

$$\nabla R_{x_0} \in \mathbf{K^1}(\mathfrak{h}_{x_0}).$$

In this notation, Berger's criterion is:

> If $\mathfrak{g}^{'} \subset \mathfrak{g}$ is a *proper* sub-algebra, and $\mathbf{K}(\mathfrak{g}^{'}) = \mathbf{K}(\mathfrak{g})$, then $\mathfrak{g}$ *cannot* be the holonomy algebra of any torsion free connection on any n-manifold M.

This criterion is a consequence of the *Ambrose-Singer Holonomy Theorem*, which states that the holonomy algebra $\mathfrak{h}_{x_0}$ is generated by the image of the curvature map R_{x_0} and its parallel translations [KN, II.8.1].

The study of locally symmetric connections, i.e. connections with $\nabla R = 0$, can be reduced to certain problems in the theory of Lie algebras. We therefore wish to exclude this case from our discussion. A second necessary condition for $\mathfrak{g}$ to be the holonomy of a torsion free connection which is *not locally symmetric* is therefore

$$\mathbf{K^1}(\mathfrak{g}) \neq 0.$$

These two criteria are also referred to as *Berger's first and second criterion.* Using these, Berger [Ber] was able to partially classify the possible Lie algebras of holonomy groups of torsion free connections which are not locally symmetric. His classification falls into three parts:

- The first part classifies all possible *Riemannian* holonomies, i.e. connections with holonomy group $H \subseteq O(n, \mathbf{R})$. It turns out that the possible holonomy groups of non-symmetric connections are those subgroups which act transitively on the unit sphere in $\mathbf{R}^n$. In fact, it is by now well known which elements in Berger's list actually *do* occur as holonomies of Riemannian metrics. We mention in this context the work of Simons [S], Calabi [C], Alekseevskii [A] and Bryant [Br1].

- The second part classifies all possible irreducible *pseudo-Riemannian* holonomies, i.e. connections with holonomy group $H \subseteq O(p,q)$. In this case, the question whether or not these candidates actually *do* occur as holonomies has been resolved except for the group $SO^*(2n) \subseteq Gl(4n, \mathbf{R})$ for $n \geq 3$.

- The third part classifies the possible irreducibly acting holonomy groups of *affine torsion free connections*, i.e. holonomy groups which *do not* leave invariant any non-degenerate symmetric bilinear form. These connections are the least understood. In fact, Berger's list in this case is incomplete, conceivably omitting a finite number of possibilities. The holonomies which are *not* contained in Berger's list are referred to as *exotic holonomies.*

R.Bryant [Br2] showed that exotic holonomies do, in fact, exist. He investigated the irreducible representations of $Sl(2, \mathbf{R})$ which can be described as follows:

For $n \in \mathbf{N}$, let $V_n := \{$*homogeneous polynomials in* x *and* y *of degree* $n\}$ which is an $n+1$-dimensional vector space. There is an $Sl(2,\mathbf{R})$-action on V_n induced by the transposed action of $Sl(2,\mathbf{R})$ on $\mathbf{R}^2$, i.e. if $p \in V_n$ and $A \in Sl(2,\mathbf{R})$ then

$$(A \cdot p)(x,y) := p(u,v) \quad \text{with} \quad (u,v) = (x,y)A.$$

It is well known that this action is irreducible for every n and moreover that - up to equivalence - this is the only irreducible $n+1$-dimensional representation of $Sl(2,\mathbf{R})$. [BD]

Let $H_n \subseteq Gl(V_n)$ be the image of this representation and let $\mathfrak{h}_n \subseteq \mathfrak{gl}(V_n)$ be the Lie algebra of H_n. Bryant showed that $\mathfrak{h}_n$ does *not* satisfy Berger's first and second criterion if $n \geq 4$.

For $n = 3$, however, he proved the existence of torsion free connections on 4-manifolds whose holonomy group is H_3, even though this group does *not* appear on Berger's list. We shall refer to these connections as *H_3-connections.*

Note that for n odd, there is no nondegenerate symmetric bilinear form which is invariant under the $Sl(2,\mathbf{R})$-action on V_n. There is, however, an invariant 2-form and hence we conclude that any H_3-connection must admit a *parallel symplectic form.*

A diffeomorphism $\phi : M \to M$ preserving the connection will be called a *symmetry of* ∇.

It turns out that locally there are very few examples of H_3-connections. In fact, the local classification given by Bryant can be summarized as follows:

- There is one example of a *homogeneous H_3-connection* whose symmetry group is *five-dimensional.*
- There is a finite set of H_3-connections with a *three-dimensional* symmetry group.
- There is a 1-parameter family of H_3-connections with a *one-dimensional* symmetry group.

H_3-connections with a one-dimensional symmetry group will be called *regular,* all others will be called *singular H_3-connections.*

Bryant's classification is obtained by the methods of Exterior Differential Systems. This approach, however, makes a concrete description of these connections very difficult. In this paper, we will describe the H_3-connections more explicitly and will also investigate their global behavior.

- The *homogeneous H_3-connection* can be described *globally.*
- The *singular H_3-connections* can be described on the dense open subsets on which the action of their symmetry groups is *locally free.*
- For the *regular H_3-connections* we can only give a description on some open subset which will not be dense in general.

A global description of the homogeneous H_3-connection was given in [Br2]. However, our description includes the symmetry group and presents the connection in a somewhat more explicit way. In fact, we will compute the Lie algebras of the symmetry groups of *all* H_3-connections.

Furthermore, in the case of some *singular* H_3-connections we can conclude that the parallel symplectic form must be *exact.*

Following this introduction, we will first give the descriptions of the singular H_3-connections (section 2).

In section 3, we derive the structure equations for H_3-connections, following closely [Br2].

As a first *global* result we will show in section 4 the

Theorem 4.1 *H_3-connections are never complete.*

In section 5, we then solve the structure equations for the *singular* H_3-connections under the generic assumption that the group action of the symmetry group is locally free, as well as the structure equations for the *regular* H_3-connections on some open subset. This will show that the examples given in section 2 form in fact a complete list of singular H_3-connections under these restrictions.

Finally, in section 6 we shall be concerned with H_3-connections on *compact* manifolds, and we will prove

Theorem 6.1 *There are no H_3-connections on compact manifolds.*

Note that Theorem 6.1 is not a consequence of Theorem 4.1 since for affine connections compact manifolds are not necessarily complete.

The main part of this paper represents the authors thesis and he wishes to thank his advisor Wolfgang Ziller for his support.

2 Examples of Singular H_3-connections

We first introduce some notational conventions. Recall that V_3 is the vector space of homogeneous polynomials of degree 3 in the variables x and y, and $Sl(2,\mathbf{R})$ acts on V_3 by the action induced by the transposed action of $Sl(2,\mathbf{R})$ on $\mathbf{R}^2 = span\{x,y\}$.

We let $\rho_3 : Sl(2,\mathbf{R}) \to H_3 \subseteq Gl(V_3)$ and $(\rho_3)_* : \mathfrak{sl}(2,\mathbf{R}) \to \mathfrak{h}_3 \subseteq \mathfrak{gl}(V_3)$ denote the representation homomorphisms, and define a basis $\{E_1, E_2, E_3\}$ of $\mathfrak{h}_3$ by $E_i := (\rho_3)_*(\tilde{E}_i)$ where the basis $\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3\}$ of $\mathfrak{sl}(2,\mathbf{R})$ is given by

$$\begin{pmatrix} a & c \\ b & -a \end{pmatrix} = a\tilde{E}_1 + b\tilde{E}_2 + c\tilde{E}_3.$$

Furthermore, we let $\{e_0, \ldots, e_3\}$ with $e_i = x^{3-i}y^i$ be a basis of V_3, and $\{\underline{e}_i \mid 0 \le i \le 3\}$ be the standard basis of $\mathbf{R}^4$.

We fix once and for all a linear isomorphism

$$\begin{aligned} \lambda : \quad V_3 &\to \mathbf{R}^4 \\ e_i &\mapsto \underline{e}_i. \end{aligned}$$

Then $\underline{\mathfrak{h}}_3 := \lambda \mathfrak{h}_3 \lambda^{-1} \subseteq \mathfrak{gl}(4,\mathbf{R})$ has $\{\underline{E}_1, \underline{E}_2, \underline{E}_3\}$ with $\underline{E}_i = \lambda E_i \lambda^{-1}$ as a basis, and one sees that

$$r_1\underline{E}_1 + r_2\underline{E}_2 + r_3\underline{E}_3 = \begin{pmatrix} 3r_1 & r_3 & & \\ 3r_2 & r_1 & 2r_3 & \\ & 2r_2 & -r_1 & 3r_3 \\ & & r_2 & -3r_1 \end{pmatrix}.$$

We shall from now on use λ to identify $\mathfrak{h}_3$ with $\underline{\mathfrak{h}}_3$.

Recall that if M is a manifold with an H_3-connection ∇ then a diffeomorphism $\phi : M \to M$ preserving the connection ∇, i.e. satisfying

$$\phi_*\left(\nabla_X Y\right) = \nabla_{\phi_*(X)}\phi_*(Y) \tag{1}$$

for all vector fields X and Y on M, will be called a *symmetry of* ∇. We shall call a vector field S on M an *infinitesimal symmetry* iff $\mathfrak{L}_S\nabla = 0$, i.e.

$$[S, \nabla_X Y] = \nabla_{[S,X]}Y + \nabla_X[S,Y] \tag{2}$$

for all vector fields X and Y on M. Note that the flow along an infinitesimal symmetry is a 1-parameter family of (local) symmetries.

In order to describe a connection on a manifold M, we shall in each case give a *frame* on M and the *connection form* θ w.r.t. this frame which takes values in $\mathfrak{h}_3$. This form is the pullback of the connection form on the *frame bundle* of M under the section given by the frame.

Let $X_0, \ldots, X_3$ be the given frame and let $\omega_0, \ldots, \omega_3$ denote the dual coframe. We define the V_3-valued 1-form ω by

$$\omega = \sum_i \omega_i \, e_i,$$

which establishes an isomorphism between T_pM and V_3 for all $p \in M$.

Then we can describe the *covariant derivative* associated to the connection by

$$\omega(\nabla_{X_i} X_j) := -\theta(X_i) \cdot \omega(X_j) = -\theta(X_i) \cdot e_j. \tag{3}$$

The connection being torsion free is equivalent to the condition that

$$\theta(X_i) \cdot e_j - \theta(X_j) \cdot e_i = d\omega(X_i, X_j) \quad \text{for all } i, j. \tag{4}$$

The holonomy algebra of these connections is contained in $\mathfrak{h}_3$ by the *Ambrose-Singer-Holonomy Theorem* mentioned earlier. In fact, we will show (Corollary 3.2) that the holonomy algebra of any such connection is actually *equal* to $\mathfrak{h}_3$, provided the connection is *not flat*.

As we shall see in section 3, to every H_3-connection we can associate a pair of homogeneous polynomials $a \in V_2$ and $b \in V_3$ as well as a constant $c \in \mathbf{R}$. For future reference we shall give these polynomials, called the *structure polynomials*, and this constant in each particular case.

The description of these examples will be motivated in section 5.

2.1 Example I : The homogeneous case (type Σ_0^0)

Let $G = ASl(2, \mathbf{R}) = \left\{ \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & x \\ & & y \\ \hline 0 & 0 & 1 \end{array} \right) \middle| A \in Sl(2, \mathbf{R}), x, y \in \mathbf{R} \right\}$ be the group of *unimodular affine motions* of $\mathbf{R}^2$ and let $\mathfrak{g}$ be the Lie algebra of G. Let A_{ij} stand for the 3×3-matrix with (i, j)-th entry 1 and all other entries 0. Then we define a basis of the Lie algebra $\mathfrak{g}$ of G by

$$\begin{aligned} Z_0 &= -\frac{9}{2} A_{23} \\ Z_1 &= -\frac{3}{4} A_{13} + 3A_{21} \\ Z_2 &= \frac{3}{2}(A_{11} - A_{22}) \\ Z_3 &= -\frac{9}{4} A_{12} \\ Y &= A_{13} + 2A_{21} \end{aligned}$$

and the subgroup $H \leq G$ as $H = \{exp(tY) \mid t \in \mathbf{R}\}$. Furthermore, we decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

with $\mathfrak{h} = span(Y)$ and $\mathfrak{m} = span(Z_0, \ldots, Z_3)$.

One checks that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, i.e. the homogeneous space G/H is *reductive*. Moreover, if we define

$$\begin{aligned} \imath : \quad \mathfrak{m} &\to V_3 \\ Z_i &\mapsto e_i \end{aligned}$$

then $ad(Y) = (\imath)^{-1} \circ E_3 \circ \imath$.

We wish to define an H_3-connection on G/H such that the canonical action of G on G/H is an action by *symmetries*. To do this, we need to define a map

$$\lambda : \mathfrak{m} \longrightarrow \mathfrak{h}_3 \subseteq \mathfrak{g}l(V_3)$$

such that

$$\lambda([Y, Z_i]) = [E_3, \lambda(Z_i)] \quad \text{for} \quad i = 0, \ldots, 3$$

and

$$\lambda(Z_i) \cdot e_j - \lambda(Z_j) \cdot e_i = \imath([Z_i, Z_j]_{\mathfrak{m}})$$

with the isomorphism $\imath$ defined above [KN, X 2.1,2.3,4.2]. The covariant derivatives are then defined by

$$\imath(\nabla_{Z_i} Z_j) = \lambda(Z_i) \cdot e_j,$$

where T_pM is identified with $\mathfrak{m}$.

One then checks that the following map satisfies these two conditions and thus defines an H_3-connection on M.

$$\lambda(Z_0) = 0, \quad \lambda(Z_1) = \tfrac{1}{2}E_3, \quad \lambda(Z_2) = -\tfrac{1}{2}E_1, \quad \lambda(Z_3) = -\tfrac{3}{2}E_2.$$

The structure polynomials and the structure constant in this case are

$$a = x^2, \quad b = \tfrac{1}{3}x^3, \quad \text{and} \quad c = 0.$$

Clearly, G acts transitively on the set of parabolas in $\mathbf{R}^2$. Also, one sees that H is the group of unimodular affine motions which leave the standard parabola $y = x^2$ invariant. Therefore, we can naturally identify $M = G/H$ with the *space of parabolas in* $\mathbf{R}^2$.

Let $\omega_0, \ldots, \omega_3, \theta$ be the coframe on G dual to the vector fields $Z_0, \ldots, Z_3, Y$. Let $\tau_i := \omega_i \wedge \ldots \wedge \omega_3$ for $0 \leq i \leq 3$. τ_i is invariant under the isotropy representation of H, and therefore there are induced forms on G/H which we also denote by $\tau_0, \ldots, \tau_3$.

One checks that these forms satisfy the *Frobenius condition*

$$d\tau_i = \alpha_i \wedge \tau_i \;\; \text{for some 1-form } \alpha_i,$$

and hence the flag of distributions $\mathcal{D}_0 \subseteq \ldots \subseteq \mathcal{D}_3$ on the tangent space of G/H given by $\tau_i(\mathcal{D}_i) = 0$ is integrable [EDS, II.1.1].

Let $L : ASl(2, \mathbf{R}) \to Sl(2, \mathbf{R})$ be the homomorphism which maps an affine motion to its *linear part*. Then we can define an *angle function*

$$\begin{aligned} \theta : \quad G/H &\longrightarrow S^1 \\ gH &\longmapsto \frac{L(g) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}{||L(g) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}||}. \end{aligned}$$

This function assigns to each parabola its direction, and it is straightforward to check that θ is a *fibration* over S^1 whose fibers are the maximal connected integral leaves (m.c.i.l.'s) of $\mathcal{D}_3$, and that the fibers are *totally geodesic*. Also, since these fibers are all parabolas with a given direction it is easy to see that they are diffeomorphic to $\mathbf{R}^3$.

Now let us investigate the level sets of θ. By homogeneity all level sets are equivalent. Thus, let $Y_3 := \{(y = e^a \; x^2 + b \; x + c) \mid a, b, c \in \mathbf{R}\}$. Then the function $r_3 : Y_3 \to \mathbf{R}$ which assigns the *slope* to a parabola, i.e. $r_3\ (y = e^a\ x^2 + b\ x + c) = a$, satisfies $dr_3 = \omega_2|_{Y_3}$. Again, one can check that r_3 is a *fibration* of Y_3 over $\mathbf{R}$ whose fibers are *totally geodesic* and diffeomorphic to $\mathbf{R}^2$.

Next, let Y_2 be a level set of r_3, e.g. $Y_2 := r_3^{-1}(0) = \{(y = x^2 + b\ x + c) \mid b, c \in \mathbf{R}\}$, i.e. a set of parabolas with given direction and slope. We compute the connection on Y_2 and obtain that Y_2 is *flat*. Note that a curve γ in Y_2 can be described by the curve of the vertices of $\gamma(t)$. Then we compute that - up to parametrization - the vertices of a *geodesic* in Y_2 either move along a parabola of slope -2 or along a vertical line.

Also, the restriction $\omega_1|_{Y_2}$ is *closed*, hence there exists a function $r_2 : Y_2 \to \mathbf{R}$ such that $dr_2 = \omega_1|_{Y_2}$. One finds that the level sets of r_2 are the parabolas in Y_2 whose vertices lie on a fixed vertical line. By the previous, these level sets are *geodesics*.

Finally, if Y_1 is a level set of r_2, i.e. a geodesic, we find some parametrization $r_1 : Y_1 \to \mathbf{R}$ such that $dr_1 = \omega_0|_{Y_1}$.

Therefore, on every m.c.i.l. Y_i of $\mathcal{D}_i,\ i \leq 3$ we have a function $r_i : Y_i \to \mathbf{R}$ satisfying $dr_i = \omega_{i-1}|_{Y_i}$, and all these functions are *totally geodesic fibrations* of Y_i.

2.2 Example II : Types $\Sigma_0^{\pm}$ and Σ_c^1

Let $c \in \mathbf{R}$ be a given constant and define the Lie algebra $\mathfrak{g}_c^{\pm} = span(Z_1, Z_2, Z_3)$ with the bracket relations

$$\begin{aligned}
[Z_1, Z_2] &= Z_3 \\
[Z_1, Z_3] &= -\frac{c}{8}\, Z_2 \\
[Z_2, Z_3] &= \pm 2c\, Z_1 .
\end{aligned}$$

One computes that $\mathfrak{g}_0^{\pm} \cong \mathfrak{n}_3$, the Lie algebra of the 3-dimensional Heisenberg group N_3, $\mathfrak{g}_c^{+} \cong \mathfrak{su}(2)$ if $c > 0$ and $\mathfrak{g}_c^{\pm} \cong \mathfrak{sl}(2, \mathbf{R})$ in the remaining cases. We let $G_c^{\pm}$ be a Lie group corresponding to $\mathfrak{g}_c^{\pm}$ such that $G_0^{\pm} = N_3$, $G_c^{\pm} = Sl(2, \mathbf{R})$ or $G_c^{+} = SU(2)$ if $c > 0$.

Let $M_c^{\pm} := \mathbf{R}^+ \times G_c^{\pm}$ and let $t_0 : M_c^{\pm} \to \mathbf{R}^+$ be the projection onto the first factor. Regarding $M_c^{\pm}$ as a subset of the Lie group $\mathbf{R} \times G_c^{\pm}$ we can define left invariant vector fields $Z_0, \ldots, Z_3$ such that

for $1 \leq i, j \leq 3$, $[Z_i, Z_j]$ is given by the equations of $\mathfrak{g}_c^{\pm}$,

$[Z_0, Z_i] = 0$ for $i = 1, 2, 3$,

$Z_0(t_0) \equiv 1$, and $Z_i(t_0) \equiv 0$, $i = 1, 2, 3$.

We then define a *frame* on $M_c^{\pm}$ by

$$\begin{aligned}
X_0 &= \frac{(6t_0^2 \mp c)}{2t_0}\, Z_1 + 2\, Z_3 \\
X_1 &= -\frac{3}{4}(2t_0^2 \pm c)\, Z_0 + Z_2 \\
X_2 &= \pm \frac{1}{t_0}\, Z_1 \\
X_3 &= \pm \frac{3}{2}\, Z_0
\end{aligned}$$

Now we give a *connection* w.r.t. this frame by the $\mathfrak{h}_3$-valued 1-form

$$\theta_c^{\pm} = \theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3$$

where

$$\begin{aligned}
\theta_1 &= && t_0\,\omega_1, \\
\theta_2 &= -c\,\frac{2t_0^2 \pm c}{8t_0}\,\omega_0 && + \frac{4t_0^2 \pm c}{4t_0}\,\omega_2, \quad \text{and} \\
\theta_3 &= \frac{6t_0^2 \pm c}{4t_0}\,\omega_0 && \mp \frac{1}{2t_0}\,\omega_2.
\end{aligned}$$

The covariant derivative is defined by (3). Moreover, one checks that (4) is satisfied, i.e. ∇ is torsion free. Also, ∇ is not flat, thus by Corollary 3.2 we conclude that ∇ is an H_3-connection.

The natural left action of $G_c^\pm$ on $M_c^\pm$ is an action by *symmetries* as it leaves all Z_i and the function t_0 invariant, hence preserves the frame $X_0, \dots X_3$ and therefore satisfies condition (1).

The structure polynomials are

$$a = \pm x^2 + \tfrac{1}{2}\,(2t_0^2 \pm c)\,y^2, \quad \text{and} \quad b = t_0\,y\,(\mp x^2 + \frac{2t_0^2 \pm c}{6}\,y^2)$$

We shall see later that $M_{c_1}^{\varepsilon_1}$ and $M_{c_2}^{\varepsilon_2}$ with $\varepsilon_i \in \{\pm\}$ are equivalent iff $\varepsilon_1 = \varepsilon_2$ and $sign(c_1) = sign(c_2)$. Thus the $M_c^\pm$ give 6 different examples of H_3-connections.

2.3 Example III : Type Σ_c^2

Let $k \in \mathbf{R}\backslash\{0\}$ be a given constant and define the Lie algebra $\mathfrak{g}_k^\pm = span(Z_1, Z_2, Z_3)$ with the bracket relations

$$\begin{aligned}
[Z_1, Z_2] &= Z_3 \\
[Z_1, Z_3] &= \mp Z_2 \\
[Z_2, Z_3] &= \mp 3k Z_1.
\end{aligned}$$

One computes that $\mathfrak{g}_k^+ \cong \mathfrak{su}(2)$ if $k > 0$ and $\mathfrak{g}_k^\pm \cong \mathfrak{sl}(2, \mathbf{R})$ in the remaining cases. We let $G_k^\pm$ be a Lie group corresponding to $\mathfrak{g}_k^\pm$ such that $G_k^\pm = Sl(2, \mathbf{R})$ or $G_k^+ = SU(2)$.

Let $M_k^\pm := \mathbf{R}^+ \times G_k^\pm$ and let $t_0 : M_k^\pm \to \mathbf{R}^+$ be the projection onto the first factor. Define the vector fields $Z_0, \dots Z_3$ on $M_k^\pm$ as in the previous section, i.e. the relations of $\mathfrak{g}_k^\pm$ determine $[Z_i, Z_j]$ for $1 \le i, j \le 3$, $Z_0(t_0) \equiv 1$, $Z_i(t_0) \equiv 0$, and $[Z_0, Z_i] = 0$ for $i = 1, 2, 3$.

We then define a *frame* on $M_k^\pm$ by

$$\begin{aligned}
X_0 &= && -\frac{9}{2t_0}\,(kt_0^2 \pm 1)\,Z_1 && && - 6\,Z_3 \\
X_1 &= -\frac{1}{2}(3kt_0^2 \mp 1)\,Z_0 && && + 2\,Z_2 && \\
X_2 &= && -\frac{3}{2t_0}\,Z_1 && && \\
X_3 &= \frac{3}{2}\,Z_0 && && &&
\end{aligned}$$

Now we give a *connection* w.r.t. this frame by the $\mathfrak{h}_3$-valued 1-form

$$\theta_k^\pm = \theta_1 E_1 + \theta_2 E_2 + \theta_3 E_3$$

where

$$\begin{aligned}
\theta_1 &= && k\,t_0\,\omega_1, \\
\theta_2 &= -\frac{3}{2t_0}\,(\pm 3kt_0^2 - 1)\,\omega_0 && + \frac{1}{2t_0}\,(2kt_0^2 \pm 1)\,\omega_2, \quad \text{and} \\
\theta_3 &= \frac{3}{2t_0}\,(kt_0^2 \mp 1)\,\omega_0 && -\frac{1}{2t_0}\,\omega_2.
\end{aligned}$$

Again, the covariant derivative is defined by (3). Moreover, one checks that (4) is satisfied, i.e. ∇ is torsion free. Also, ∇ is not flat for $k \neq 0$, thus by Corollary 3.2 we conclude that ∇ is an H_3-connection.

The natural left action of $G_k^\pm$ on $M_k^\pm$ is an action by *symmetries* for similar reasons as in the previous section.

The structure polynomials and the structure constant are

$$a = (kt_0 y)^2 + k\,(x^2 \pm y^2), \quad b = \tfrac{1}{3} k t_0 y\,((kt_0 y)^2 - 3k\,(x^2 \pm y^2)) \quad \text{and} \quad c = \pm 6k^2.$$

We shall see later that $M_{k_1}^{\varepsilon_1}$ and $M_{k_2}^{\varepsilon_2}$ with $\varepsilon_i \in \{\pm\}$ are equivalent iff $\varepsilon_1 = \varepsilon_2$ and $sign(k_1) = sign(k_2)$. Thus the $M_k^\pm$ give 4 different examples of H_3-connections, and we will also show that they are indeed different from the connections given in the previous section.

3 The Structure Equations

In this section we will mainly recall the results of Bryant [Br2] on H_3-connections and introduce a notation convenient for our purposes.

First, we shall define the bilinear parings

$$< \, , \, >_p : V_n \otimes V_m \to V_{n+m-2p}$$

by

$$< u, v >_p = \frac{1}{p!} \sum_{k=0}^{p} (-1)^k \binom{p}{k} \frac{\partial^p u}{\partial^k x \partial^{p-k} y} \frac{\partial^p v}{\partial^{p-k} x \partial^k y} \qquad \text{for} \qquad u \in V_n, v \in V_m.$$

It can be shown that these pairing are $Sl(2, \mathbf{R})$-equivariant and therefore are the projections onto the summands of the Clebsch-Gordan formula.

Now we explicitly describe the spaces

$$\mathbf{K}(\mathfrak{g}) := \left\{ \phi : \mathbf{\Lambda^2}(V) \to \mathfrak{g} \;\middle|\; \phi \text{ linear}, \sum_{\sigma \in A_3} \phi(u_{\sigma(1)}, u_{\sigma(2)}) u_{\sigma(3)} = 0 \text{ for all } u_1, u_2, u_3 \in V \right\},$$

and

$$\mathbf{K^1}(\mathfrak{g}) := \left\{ \psi : V \to \mathbf{K}(\mathfrak{g}) \;\middle|\; \psi \text{ linear}, \sum_{\sigma \in A_3} \psi(u_{\sigma(1)})(u_{\sigma(2)}, u_{\sigma(3)}) = 0 \text{ for all } u_1, u_2, u_3 \in V \right\}.$$

By straightforward computation we get as a basis for $\mathbf{K}(\mathfrak{h}_3)$ the maps

$\phi_0 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$	$\phi_1 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$	$\phi_2 : \mathbf{\Lambda^2}(V_3) \to \mathfrak{h}_3$
$\phi_0(e_0, e_1) = 0$	$\phi_1(e_0, e_1) = 0$	$\phi_2(e_0, e_1) = -6E_3$
$\phi_0(e_0, e_2) = 0$	$\phi_1(e_0, e_2) = 3E_3$	$\phi_2(e_0, e_2) = 3E_1$
$\phi_0(e_0, e_3) = 9E_3$	$\phi_1(e_0, e_3) = -9E_1$	$\phi_2(e_0, e_3) = -9E_2$
$\phi_0(e_1, e_2) = -5E_3$	$\phi_1(e_1, e_2) = E_1$	$\phi_2(e_1, e_2) = 5E_2$
$\phi_0(e_1, e_3) = 3E_1$	$\phi_1(e_1, e_3) = -3E_2$	$\phi_2(e_1, e_3) = 0$
$\phi_0(e_2, e_3) = 6E_2$	$\phi_1(e_2, e_3) = 0$	$\phi_2(e_2, e_3) = 0$

Also, we compute

$$\mathbf{K^1}(\mathfrak{h}_3) := \{\psi_{b_0,\ldots,b_3} \mid b_0, \ldots, b_3 \in \mathbf{R}\},$$

with

$$\psi_{b_0,\ldots,b_3}(e_i) := -(3-i) b_{i+1}\, \phi_0 + (3-2i) b_i\, \phi_1 + i b_{i-1}\, \phi_2, 0 \le i \le 3.$$

As $Sl(2,\mathbf{R})$ acts both on V_3 and on $\mathfrak{h}_3$ via the adjoint representation, there is an induced action of $Sl(2,\mathbf{R})$ on $Hom(\Lambda^2 V_3, \mathfrak{h}_3)$ and $\mathbf{K}(\mathfrak{h}_3)$ is easily seen to be invariant under this action. Moreover, the basis ϕ_0, ϕ_1, ϕ_2 was chosen such that the map

$$\imath_{\mathbf{K}} : \begin{array}{ll} \mathbf{K}(\mathfrak{h}_3) & \rightarrow V_2 \\ \phi_i & \mapsto x^{2-i} y^i \end{array}$$

is an $Sl(2,\mathbf{R})$-equivariant isomorphism.

Furthermore, $\mathbf{K^1}(\mathfrak{h}_3)$ can be seen to be invariant under the $Sl(2,\mathbf{R})$-action on $Hom(V_3, \mathbf{K}(\mathfrak{h}_3))$, and the map

$$\imath_{\mathbf{K^1}} : \begin{array}{ll} \mathbf{K^1} & \longrightarrow V_3 \\ \psi_{b_0,b_1,b_2,b_3} & \longmapsto b_3\, x^3 - 3b_2\, x^2 y + 3b_1\, xy^2 - b_0\, y^3 \end{array}$$

is an $Sl(2,\mathbf{R})$-equivariant isomorphism.

The following Lemma is very simple but will be useful later on.

Lemma 3.1 *Every $\phi \in \mathbf{K}(\mathfrak{h}_3)\backslash 0$ is surjective.*

Proof: Let $\phi = a_0\, \phi_0 + a_1\, \phi_1 + a_2\, \phi_2$ and suppose $a_0 \neq 0$. Then $\phi(e_2, e_3) = -6a_0\, E_2$, thus $E_2 \in im(\phi)$. Also, $\phi(e_1, e_3) = -3a_0\, E_1 + 3a_1\, E_2$, thus $E_1 \in im(\phi)$. Finally, $\phi(e_0, e_3) = -9a_0\, E_3 + 9a_1\, E_1 + 9a_2\, E_2$, thus $E_3 \in im(\phi)$. Therefore, ϕ is surjective if $a_0 \neq 0$. Similar arguments show that ϕ is surjective whenever $\phi \neq 0$. **q.e.d.**

Corollary 3.2 *If the holonomy group of a torsion free connection ∇ on a manifold M is contained in H_3 then either the holonomy group of ∇ is equal to H_3 or ∇ is flat.*

Proof: Suppose ∇ is not flat. Then there is a $p \in M$ at which the curvature map $\Omega_p : \Lambda^2 T_p M \rightarrow \mathfrak{h}_3$ is not 0. By Lemma 3.1, Ω_p is surjective and thus by the *Ambrose-Singer-Holonomy Theorem*, the holonomy group of ∇ is equal to H_3. **q.e.d.**

Suppose M is equipped with a torsion free H_3-connection ∇. Let $\pi : \mathfrak{F} \rightarrow M$ denote the total frame bundle of M which is a principal $Gl(4,\mathbf{R})$-bundle. On $\mathfrak{F}$, the tautological 1-form ω takes values in $\mathbf{R}^4$, the connection 1-form θ takes values in $\mathfrak{gl}(4,\mathbf{R})$ and both are $Gl(4,\mathbf{R})$-equivariant [KN].

Then there exists a reduction F of $\mathfrak{F}$ whose structure group is isomorphic to $Sl(2,\mathbf{R})$ [KN, II.7.1] and such that $\theta|_F$ takes values in $\mathfrak{h}_3 \cong \underline{\mathfrak{h}}_3 \subseteq \mathfrak{gl}(4,\mathbf{R})$. Such a reduction will be called an $\mathfrak{h}_3$*-reduction* of $\mathfrak{F}$.

By abuse of notation we will denote the restrictions $\omega|_F, \theta|_F$ and $\pi|_F$ also by ω, θ and π resp.

Of course, $\mathfrak{h}_3$-reductions are not unique. In fact, one sees that F, F' are $\mathfrak{h}_3$-reductions iff there is some $g \in Norm(\underline{\mathfrak{h}}_3) \subseteq Gl(4,\mathbf{R})$ with $F' = L_g(F)$, where L_g denotes the principal action of g on $\mathfrak{F}$. Now one computes that

$$Norm(\mathfrak{h}_3) = H_3 \times N$$

with

$$N = \left\{ \left(\begin{array}{cccc} t & & & \\ & \varepsilon t & & \\ & & t & \\ & & & \varepsilon t \end{array} \right) \middle| \, t \in \mathbf{R}\backslash\{0\}, \varepsilon = \pm 1 \right\}.$$

It follows that any two $\mathfrak{h}_3$-reductions F, F' satisfy $F = L_g(F')$ for some $g \in N$, and two such structures are called *homothetic* to each other.

Clearly, the property of being homothetic forms an *equivalence relation* on the set of $\mathfrak{h}_3$-reductions of $\mathfrak{F}$, and it is not hard to see that there is in fact a 1-1 correspondence between

H_3-connections and homothety classes of $\mathfrak{h}_3$-reductions with a torsion free connection [Br2, 2.1].

We now derive the *structure equations* for H_3-connections on a manifold M.

Let M be as above and fix an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$. On F, we have the 1-form $\omega+\theta$ with values in $\mathbf{R}^4 \oplus \underline{\mathfrak{h}}_3 \cong V_3 \oplus \mathfrak{h}_3$ and may hence regard $\omega+\theta$ as an $Sl(2,\mathbf{R})$-equivariant $V_3 \oplus \mathfrak{h}_3$-valued 1-form. It is well known that this form gives a *coframe* on F, i.e. the real-valued 1-forms $\omega_0, \ldots, \omega_3, \theta_1, \theta_2, \theta_3$ given by

$$\omega(X) = \sum_i \omega_i(X)\, e_i \quad \text{and} \quad \theta(X) = \sum_i \theta_i(X)\, E_i \quad \text{for all} \quad X \in TF$$

are linearly independent.

The frame of F dual to the coframe $\omega_0, \ldots, \omega_3, \theta_1, \theta_2, \theta_3$ will be denoted by $X_0, \ldots, X_3$, Y_1, Y_2, Y_3 and will be called the *canonical frame* on F.

A tangent vector $X \in TF$ will be called *vertical* if $\omega(X) = 0$ and *horizontal* if $\theta(X) = 0$. Thus, X_i is *horizontal* and Y_i is *vertical* for all i.

The curvature 2-form Ω on F takes values in $\mathfrak{h}_3$ and vanishes in vertical directions which means that for every $u \in F$, there is a unique linear map $\phi_u : \Lambda^2 V_3 \to \mathfrak{h}_3$ such that

$$\Omega(X,Y) = \phi_u(\omega(X), \omega(Y)) \quad \text{for all} \quad X, Y \in T_u F.$$

Furthermore, the *first Bianchi identity* implies that $\phi_u \in \mathbf{K}(\mathfrak{h}_3)$, and this defines an $Sl(2,\mathbf{R})$-equivariant map

$$\begin{array}{rll} a: & F & \to V_2 \\ & u & \mapsto 2\, \imath_{\mathbf{K}}(\phi_u). \end{array}$$

We then define the real-valued functions a_0, a_1, a_2 on F by the equation

$$a(u) = \sum_i a_i(u)\, x^{2-i} y^i \quad \text{or, equivalently,} \quad \phi_u = \frac{1}{2} \sum_i a_i(u)\, \phi_i$$

Now we can describe the *structure equations* on F.

The *first structure equation* - using that ∇ is torsion free - yields

$$d\omega = -\theta \wedge \omega \tag{5}$$

or, equivalently,

$$\begin{array}{rcl} d\omega_0 & = & -3\theta_1 \wedge \omega_0 - \theta_3 \wedge \omega_1 \\ d\omega_1 & = & -3\theta_2 \wedge \omega_0 - \theta_1 \wedge \omega_1 - 2\theta_3 \wedge \omega_2 \\ d\omega_2 & = & -2\theta_2 \wedge \omega_1 + \theta_1 \wedge \omega_2 - 3\theta_3 \wedge \omega_3 \\ d\omega_3 & = & -\theta_2 \wedge \omega_2 + 3\theta_1 \wedge \omega_3 \end{array} \tag{6}$$

The *second structure equation* is

$$d\theta = -\theta \wedge \theta + \Omega \tag{7}$$

or, equivalently,

$$\begin{array}{rcl} d\theta_1 & = & \theta_2 \wedge \theta_3 + \frac{3}{2} a_2\, \omega_0 \wedge \omega_2 - \frac{9}{2} a_1\, \omega_0 \wedge \omega_3 + \frac{1}{2} a_1\, \omega_1 \wedge \omega_2 + \frac{3}{2} a_0\, \omega_1 \wedge \omega_3 \\ d\theta_2 & = & 2\,\theta_1 \wedge \theta_2 - \frac{9}{2} a_2\, \omega_0 \wedge \omega_3 + \frac{5}{2} a_2\, \omega_1 \wedge \omega_2 - \frac{3}{2} a_1\, \omega_1 \wedge \omega_3 + 3a_0\, \omega_2 \wedge \omega_3 \\ d\theta_3 & = & -2\,\theta_1 \wedge \theta_3 - 3a_2\, \omega_0 \wedge \omega_1 + \frac{3}{2} a_1\, \omega_0 \wedge \omega_2 + \frac{9}{2} a_0\, \omega_0 \wedge \omega_3 - \frac{5}{2} a_0\, \omega_1 \wedge \omega_2 \end{array} \tag{8}$$

These equations determine the brackets of the canonical vector fields using the relation

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]) \quad \text{for any 1-form} \quad \alpha.$$

For $u \in F$, we define a linear map $\psi_u : V_3 \to Hom(\Lambda^2 V_3, \mathfrak{h}_3)$ by the equation

$$\psi_u(e_i)(\omega(Y),\omega(Z)) = (\mathfrak{L}_{X_i}\Omega)(Y,Z) - \phi_u(\theta(Y)\cdot e_i, \omega(Z)) - \phi_u(\omega(Y), \theta(Z)\cdot e_i)$$

for all $Y, Z \in T_uF$.

To see that this is well defined note that the right hand side is tensorial in Y and Z and vanishes if either Y or Z are vertical. Moreover, the first Bianchi identity implies that $\psi_u(e_i) \in \mathbf{K}(\mathfrak{h}_3)$, and finally, the second Bianchi identity implies that $\psi_u \in \mathbf{K^1}(\mathfrak{h}_3)$. Therefore, we get a map

$$\begin{array}{rll} b: & F & \to V_3 \\ & u & \mapsto \dfrac{2}{3}\, \imath_{\mathbf{K^1}}(\psi_u), \end{array}$$

and define the real-valued functions $b_0, \ldots, b_3$ on F by the equation

$$b(u) = b_3(u)\, x^3 - 3b_2(u)\, x^2 y + 3b_1(u)\, xy^2 - b_0(u)\, y^3$$

or, equivalently,

$$\psi_u = \psi_{\frac{2}{3}b_0(u),\ldots,\frac{2}{3}b_3(u)}.$$

Remark: The constants used in the definition of the maps a and b will enable us to be consistent with [Br2] who uses a slightly different description of these maps.

Next, we compute the exterior derivatives of the functions a_0, a_1, a_2. For this, let X_i, X_j and X_k be horizontal vector fields of the canonical frame on F. Then, at some point $u \in F$,

$$\begin{aligned} X_i(\Omega(X_j,X_k)) &= (\mathfrak{L}_{X_i}\Omega)(X_j,X_k) + \Omega([X_i,X_j],X_k) + \Omega(X_j,[X_i,X_k]) \\ &= \psi_u(e_i)\,(e_j,e_k) \\ &= \frac{2}{3}(-(3-i)b_{i+1}(u)\,\phi_0 + (3-2i)b_i\,\phi_1 + ib_{i-1}(u)\,\phi_2)\,(e_j,e_k). \end{aligned}$$

Here we used the fact that the Lie bracket of two canonical *horizontal* vector fields is always *vertical* since the connection is torsion free.

On the other hand,

$$X_i(\Omega(X_j,X_k)) = X_i\left(\frac{1}{2}\sum_{r=0}^{2} a_r(u)\,\phi_r\,(e_j,e_k)\right) = \frac{1}{2}\left(\sum_{r=0}^{2} X_i(a_r)\,\phi_r\right)\,(e_j,e_k).$$

Setting these two equal, we get the *horizontal* derivatives of a. The *vertical* derivatives follow from the $Sl(2,\mathbf{R})$-equivariance of a, and we obtain

$$\begin{aligned} da_0 &= -3\sum_{i=0}^{3} (3-i)\,b_{i+1}\,\omega_i - 2a_0\,\theta_1 - a_1\,\theta_3 \\ da_1 &= 3\sum_{i=0}^{3} (3-2i)\,b_i\,\omega_i - 2a_0\,\theta_2 - 2a_2\,\theta_3 \\ da_2 &= 3\sum_{i=0}^{3} i\,b_{i-1}\,\omega_i + 2a_2\,\theta_1 - a_1\,\theta_2 \end{aligned} \qquad (9)$$

Taking exterior derivatives of these equations and solving for db_i, we can compute that there exists some function c such that

$$\begin{array}{rlllllll}
db_0 = & & \frac{1}{2}\,a_2^2\,\omega_1 - & \frac{1}{2}\,a_1a_2\,\omega_2 + & b_{03}\,\omega_3 + & 3\,b_0\,\theta_1 + & 3\,b_1\,\theta_2 & \\
db_1 = & -\frac{1}{2}\,a_2^2\,\omega_0 & & -\quad b_{12}\,\omega_2 - & \frac{1}{2}\,a_0a_1\,\omega_3 + & b_1\,\theta_1 + & 2\,b_2\,\theta_2 + & b_0\,\theta_3 \\
db_2 = & \frac{1}{2}\,a_1a_2\,\omega_0 + & b_{12}\,\omega_1 & + & \frac{1}{2}\,a_0^2\,\omega_3 - & b_2\,\theta_1 + & b_3\,\theta_2 + & 2\,b_1\,\theta_3 \\
db_3 = & -\,b_{03}\,\omega_0 + & \frac{1}{2}\,a_0a_1\,\omega_1 - & \frac{1}{2}\,a_0^2\,\omega_2 & - & 3\,b_3\,\theta_1 & + & 3\,b_2\,\theta_3
\end{array} \tag{10}$$

with

$$b_{03} = \frac{1}{4}(3(a_1^2 - 2a_0a_2) + 2c) \quad \text{and} \quad b_{12} = \frac{1}{12}(a_1^2 - 10a_0a_2 + 2c)$$

Taking exterior derivatives once again, we find that

$$dc = 0, \tag{11}$$

i.e. c is a *constant.*

Definition 3.3 *Let $\pi : F \to M$ be a principal* $\mathrm{Sl}(2, \mathbf{R})$*-bundle over a 4-dimensional manifold M. Suppose there exist 1-forms ω and θ on F with values in V_3 and $\mathfrak{h}_3$ resp. and functions a and b with values in V_2, V_3 resp. such that the structure equations (5) - (11) are satisfied for some constant c. Moreover, assume that $\omega(ker(\pi_*)) \equiv 0$ and $\theta((\tilde{E}_i)^*) = E_i$ where $\{\tilde{E}_i\}$ is the basis of* $\mathfrak{sl}(2, \mathbf{R})$ *described earlier and $(\tilde{E}_i)^*$ denotes the fundamental vector field corresponding to $\tilde{E}_i$ [KN, Vol I p. 51]. Then $(\pi, F, M, \omega, \theta, a, b, c)$ is called a solution structure over M.*

By the previous we know that any $\mathfrak{h}_3$-reduction of an H_3-connection on M gives rise to a solution structure over M.

Now suppose that $F' = L_g(F)$ with $g \in N$ is an $\mathfrak{h}_3$-reduction homothetic to F. Then if we let $\tilde{a} = L_g \circ a, \tilde{b} = L_g \circ b$ and $\tilde{c} = L_g \circ c$ where L_g denotes the action of g on V_2, V_3 resp., then $(F', (L_{g^{-1}})^*(\omega), (L_{g^{-1}})^*(\theta), \tilde{a}, \tilde{b}, \tilde{c})$ is also a solution structure.

In terms of the associated real valued functions we see that if

$$g = \begin{pmatrix} t & & & \\ & \varepsilon t & & \\ & & t & \\ & & & \varepsilon t \end{pmatrix} \in N$$

then

$$\begin{array}{llll}
\tilde{a}_0 = t^2 a_0, & \tilde{a}_1 = \varepsilon t^2 a_1, & \tilde{a}_2 = t^2 a_3, & \\
\tilde{b}_0 = \varepsilon t^3 b_0, & \tilde{b}_1 = t^3 b_1, & \tilde{b}_2 = \varepsilon t^3 b_2, & \tilde{b}_3 = t^3 b_0 \\
\tilde{c} = t^4 c & & &
\end{array} \tag{12}$$

Definition 3.4 *Two solution structures $(\pi, F, M, \omega, \theta, a, b, c)$ and $(\tilde{\pi}, \tilde{F}, M, \tilde{\omega}, \tilde{\theta}, \tilde{a}, \tilde{b}, \tilde{c})$ over the same manifold M are called homothetic if there exists a bundle isomorphism $L : F \to \tilde{F}$ such that $\omega = L^*(\tilde{\omega}), \theta = L^*(\tilde{\theta})$, and moreover the real valued functions associated to $a, b, \tilde{a}, \tilde{b}$ satisfy (12) for some $t \neq 0$, $\varepsilon = \pm$.*

Clearly, homothety is an equivalence relation of solution structures and homothetic $\mathfrak{h}_3$-reductions give rise to homothetic solution structures. Therefore, by our discussion earlier, to any H_3-connection on a manifold M we can associate a homothety class of solution structures over M.

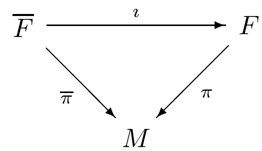
We will now show that this correspondence between H_3-connections and homothety classes of solution structures over a manifold M is in fact bijective. In particular, we need to show the *sufficiency* of the structure equations (5) - (11).

Proposition 3.5 *Let M be a 4-dimensional manifold and let $\pi : \mathfrak{F} \to M$ denote the total frame bundle of M. Suppose there exists a solution structure $(\overline{\pi}, \overline{F}, M, \overline{\omega}, \overline{\theta}, \overline{a}, \overline{b}, c)$ over M.*

Then there exists a unique H_3-connection ∇ on M and a unique embedding $\imath : \overline{F} \hookrightarrow \mathfrak{F}$ such that

1) $F := \imath(\overline{F})$ is an $\mathfrak{h}_3$-reduction of $\mathfrak{F}$,

2) the diagram



commutes and

3) $\imath^*(\omega + \theta) = \overline{\omega} + \overline{\theta}$, where ω and θ denote the restrictions of the tautological form and the connection form to F.

Proof: Let $\overline{F}, \overline{\omega}, \overline{\theta}, \overline{a}$ and $\overline{b}$ be as above. Define the real-valued 1-forms and the real-valued functions $\overline{\omega}_0, \ldots \overline{\omega}_3, \overline{\theta}_1, \ldots, \overline{\theta}_3, \overline{a}_0, \overline{a}_1, \overline{a}_2, \overline{b}_0, \ldots, \overline{b}_3$ as before, and let $\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ be the frame on $\overline{F}$ dual to the coframe $\overline{\omega} + \overline{\theta}$.

By hypothesis, $\overline{\pi}_*((X_0)_u, \ldots, (X_3)_u)$ is a basis of $T_{\pi(u)}M$ for all $u \in \overline{F}$, hence a point in $\mathfrak{F}$. Therefore, we define the map

$$\begin{aligned} \imath : \quad \overline{F} &\to \mathfrak{F} \\ u &\mapsto \overline{\pi}_*((X_0)_u, \ldots, (X_3)_u) \end{aligned}$$

By construction, we have $\imath^*(\omega) = \overline{\omega}$. Moreover, the structure equations and the condition $\theta((\tilde{E}_i)^*) = E_i$ imply that $\overline{\omega}$ and $\overline{\theta}$ are $Sl(2, \mathbf{R})$-equivariant w.r.t. the actions of $Sl(2, \mathbf{R})$ on V_3 and $\mathfrak{h}_3$ induced by ρ_3. This means that

- $\imath$ is an embedding whose image F is an $\mathfrak{h}_3$-reduction of $\mathfrak{F}$ w.r.t. the connection defined by θ.
- if we define on $F := \imath(\overline{F})$ the $\mathfrak{h}_3$-valued 1-form $\theta' := (\imath^{-1})^*(\overline{\theta})$ then there exists a unique $\mathfrak{gl}(4, \mathbf{R})$-valued $Gl(4, \mathbf{R})$-equivariant connection 1-form θ on $\mathfrak{F}$ such that $\theta' = \theta|_F$,

It is then straightforward that the map $\imath$ satisfies the desired properties and is uniquely determined by them. **q.e.d.**

From the construction above it is clear that the images of the embeddings $\imath_1$ and $\imath_2$ associated to two *homothetic* solution structures $(\overline{\pi}_i, \overline{F}_i, M, \overline{\omega}_i, \overline{\theta}_i, \overline{a}_i, \overline{b}_i, c_i)$ over M are in fact *homothetic* $\mathfrak{h}_3$-structures. In particular, the connection ∇ on M only depends on the *homothety class* of the solution structure.

We summarize the discussion so far, including Corollary 3.2, in the

Corollary 3.6 *There is a 1-1 correspondence between H_3-connections on a 4-manifold M and homothety classes of solution structures over M for which a, b do not vanish identically.*

For the rest of this section, let F be an $\mathfrak{h}_3$-reduction of an H_3-connection ∇ on a connected 4-manifold M. If we let $V := V_2 \oplus V_3$, we get a map

$$\begin{aligned} K : \quad F &\to V \\ u &\mapsto a(u) + b(u) \end{aligned}$$

The structure equations (5) - (11) imply that w.r.t. the canonical frame $X_0, \ldots, X_3, Y_0, Y_1, Y_2$ on F and the functions a_0, a_1, a_2 and $b_0, \ldots, b_3$ on V we can write the differential of K

$$K_* : TF \to TV$$

as

$$K_* = \begin{pmatrix} -9b_1 & -6b_2 & -3b_3 & 0 & -2a_0 & 0 & -a_1 \\ 9b_0 & 3b_1 & -3b_2 & -9b_3 & 0 & -2a_0 & -2a_2 \\ 0 & 3b_0 & 6b_1 & 9b_2 & 2a_2 & -a_1 & 0 \\ 0 & \frac{1}{2}a_2^2 & -\frac{1}{2}a_1a_2 & b_{03} & 3b_0 & 3b_1 & 0 \\ -a_2^2 & 0 & -b_{12} & -\frac{1}{2}a_0a_1 & b_1 & 2b_2 & b_0 \\ \frac{1}{2}a_1a_2 & b_{21} & 0 & \frac{1}{2}a_0^2 & -b_2 & b_3 & 2b_1 \\ -b_{30} & \frac{1}{2}a_0a_1 & -\frac{1}{2}a_0^2 & 0 & -3b_3 & 0 & 3b_2 \end{pmatrix}.$$

We compute that $det(K_*) \equiv 0$, i.e. $rank(K_*) \leq 6$. Let L be the cofactor matrix of K_*, consisting of the 6×6 minors of K_*. L has the property that

$$LK_* = K_*L = 0.$$

and L has *rank* at most 1. Thus, there are polynomials $r_0, \ldots, r_6, s_0, \ldots, s_6$ in the variables $a_0, a_1, a_2, b_0, \ldots, b_3$ such that

$$L = \begin{pmatrix} r_0 \\ \vdots \\ r_6 \end{pmatrix} (s_0, \ldots, s_6).$$

Furthermore, if we define the polynomial

$$R_c = \frac{1}{96}(< p_c, p_c >_2 \ + 2 < a, q^2 >_2)$$

with

$$p_c = (2c - \ < a, a >_2)a \ - \ < b, b >_2 \quad \text{and} \quad q = < a, b >_2,$$

then we compute that

$$s_i = \frac{\partial R_c}{\partial a_i} \quad \text{for} \quad 0 \leq i \leq 2, \quad \text{and} \quad s_i = \frac{\partial R_c}{\partial b_i} \quad \text{for} \quad 3 \leq i \leq 6,$$

and

$$r_i = -\frac{1}{3}s_{i+3} \quad \text{for} \quad 0 \leq i \leq 3, \quad r_4 = s_1, \quad r_5 = -s_0 \quad \text{and} \quad r_6 = s_2.$$

This means that

- $d(R_c \circ K) \equiv 0$, hence K maps F into some *level set* of R_c,
- for $u \in F$, $rank(K_*(u)) = 6$ iff $L(u) \neq 0$ iff $(r_0(u), \ldots, r_6(u)) \neq 0$ iff $(s_0(u), \ldots, s_6(u)) \neq 0$ iff $K(u)$ is a *regular point* of R_c.

We now describe the set $\Sigma_c \subseteq V$ of *critical points* of R_c. Using the $Sl(2, \mathbf{R})$-invariance of Σ_c, we compute that

- for $c \neq 0$, $\Sigma_c = \Sigma_c^1 \sqcup \Sigma_c^2$, where

$$\Sigma_c^1 = \{(a, b) \in V \mid p_c = q = 0\}$$

and

$$\Sigma_c^2 = \{(v + u^2, \frac{1}{3}\, u\,(u^2 - 3v)) \in V \mid u \in V_1, v \in V_2 \text{ with } < v, v >_2 = \frac{2}{3}\, c\}.$$

- for $c = 0$, $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$, where

$$\Sigma_0^{\pm} = \{(\pm\, v^2 + u^2, \frac{1}{3}\, u\, (\mp 3v^2 + u^2)) \in V \mid u, v \in V_1\}$$

We also define

$$\Sigma_0^0 := \Sigma_0^+ \cap \Sigma_0^- = \{(u^2, \frac{1}{3}\, u^3) \in V \mid u \in V_1\}.$$

We wish to determine the topology of the Σ's. To do this we introduce the $Sl(2, \mathbf{R})$-equivariant maps

$$\begin{array}{rccl} \phi_c^2: & V_1 \times V_{2,c} & \longrightarrow & \Sigma_c^2 \\ & (u, v) & \longmapsto & (v + u^2, \frac{1}{3}\, u\, (u^2 - 3v)), \end{array}$$

$$\begin{array}{rccl} \phi_0^{\pm}: & V_1 \times V_1 & \longrightarrow & \Sigma_0^{\pm} \\ & (u, v) & \longmapsto & (\pm\, v^2 + u^2, \frac{1}{3}\, u\, (\mp 3v^2 + u^2)), \end{array}$$

$$\begin{array}{rccl} \phi_0^0: & V_1 & \longrightarrow & \Sigma_0^0 \\ & u & \longmapsto & (u^2, \frac{1}{3}\, u^3), \end{array}$$

where $V_{2,c} := \{v \in V_2 \mid \; < v, v >_2 = \frac{2}{3}\, c\}$.

One computes that ϕ_c^2 is a *diffeomorphism* for all c.

$\phi_0^{\pm}$ is a *branched double cover:* in fact, let

$$\begin{array}{lll} S^+ := & (\phi_0^+)^{-1}(\Sigma_0^0) = & \{(u, v) \in V_1 \times V_1 \mid v = 0 \text{ or } v^2 = 3u^2\}, \text{ and} \\ S^- := & (\phi_0^-)^{-1}(\Sigma_0^0) = & \{(u, v) \in V_1 \times V_1 \mid v = 0\}. \end{array}$$

Then $\phi_0^{\pm}|_{(V_1 \times V_1)\setminus S^{\pm}}$ is a *double cover* of $\Sigma_0^{\pm}\setminus\Sigma_0^0$ whose non-trivial deck transformation in either case is given by $(u, v) \mapsto (u, -v)$.

Finally, $\phi_0^0|_{V_1\setminus\{0\}}$ is a diffeomorphism onto $\Sigma_0^0\setminus\{0\}$.

Note that Σ_c^1 is smooth by the implicit function theorem. Let $(a, b) \in \Sigma_c^1$ with $< a, a >_2 > 0$. By the $Sl(2, \mathbf{R})$-invariance we may assume that $a = a_0(x^2+y^2)$ for some $a_0 \in \mathbf{R}\setminus\{0\}$. We compute that $b = b_1\, y(3x^2 - y^2) + b_2\, x(3y^2 - x^2)$ with $b_i \in \mathbf{R}$ and that $a_0\,(2a_0^2 - c) = 18(b_1^2 + b_2^2)$. Therefore if $c > 0$, then either $-\sqrt{\frac{c}{2}} \le a_0 < 0$ or $\sqrt{\frac{c}{2}} \le a_0$. Thus, $a(\Sigma_c^1)$ and hence Σ_c^1 is disconnected for $c > 0$.

One can show that these are the only components, i.e. Σ_c^1 is *connected* if $c < 0$ and has two components $\Sigma_c^{1,\pm}$ if $c > 0$ where $\pm$ stands for the sign of a_0 in the notation above.

For Σ_c^2, we conclude from the above that it is smooth 4-dimensional and has one or two connected components depending on the sign of c. In the case $c > 0$ we let $V_{2,c}^{\pm} = V_{2,c} \cap \{\pm(u^2 + v^2) | u, v \in V_1\}$ be the connected components of $V_{2,c}$, and therefore $\Sigma_c^{2,\pm} := \phi_c^2(V_1 \times V_{2,c}^{\pm})$ are the two connected components of Σ_c^2. On the other hand, if $c < 0$ then $V_{2,c}$ and therefore Σ_c^2 are connected.

$\Sigma_0\setminus\Sigma_0^0$ is 4-dimensional and smooth and has two connected components, namely $\Sigma_0^{\pm}\setminus\Sigma_0^0$.

The set $\Sigma_0^0\backslash\{0\}$ is a smooth connected 2-dimensional manifold.

Moreover, for the ranks of K_* we get:

- If $c \neq 0$ then $rank(K_*(u)) = 4$ for all $u \in F$ with $K(u) \in \Sigma_c$.
- If $c = 0$ then
  - $rank(K_*(u)) = 4$ for all $u \in F$ with $K(u) \in \Sigma_0\backslash\Sigma_0^0$,
  - $rank(K_*(u)) = 2$ for all $u \in F$ with $K(u) \in \Sigma_0^0\backslash\{0\}$,
  - $rank(K_*(u)) = 0$ for all $u \in F$ with $K(u) = 0$.

Theorem 3.7 *The differential K_* has constant rank on F, and $rank(K_*) \in \{0, 2, 4, 6\}$.*

We start with the following Lemma whose proof is left to the reader:

Lemma 3.8 *Any two points in F can be joined by a piecewise differentiable path γ such that $\gamma'(t) = \pm X_i$ or $\gamma'(t) = \pm Y_i$ for some i, wherever γ' is defined.*

Proof of Theorem: By the previous discussion we need to show that

1) if $c \neq 0$ then either $K(F) \subseteq V\backslash\Sigma_c$ or $K(F) \subseteq \Sigma_c$.

2) if $c = 0$ then
   - either $K(F) \subseteq V\backslash\Sigma_0$ or
   - $K(F) \subseteq \Sigma_0\backslash\Sigma_0^0$ or
   - $K(F) \subseteq \Sigma_0^0\backslash\{0\}$ or
   - $K(F) = \{0\}$.

We define vector fields $\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3$ on V by

$$\left(\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)^t = K_* \left(\frac{\partial}{\partial a_0}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial b_0}, \ldots, \frac{\partial}{\partial b_3}\right)^t.$$

Let $p = (a_0, a_1, a_2, b_0, \ldots, b_3) \in V$. One computes that

- if $p \in \Sigma_c$ and $c \neq 0$ then $span\left(\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)_p = T_p\Sigma_c$.
- if $p \in \Sigma_0\backslash\Sigma_0^0$ then $span\left(\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)_p = T_p\left(\Sigma_0\backslash\Sigma_0^0\right)$.
- if $p \in \Sigma_0^0\backslash\{0\}$ then $span\left(\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)_p = T_p\left(\Sigma_0^0\backslash\{0\}\right)$.
- if $p = 0$ then $span\left(\overline{X}_0, \ldots, \overline{X}_3, \overline{Y}_1, \overline{Y}_2, \overline{Y}_3\right)_p = \{0\}$.

This implies that an integral curve of $\overline{X}_i$ ($\overline{Y}_i$ resp.) for any i always stays inside the same stratum. Therefore, if $\gamma : (a, b) \to F$ is an integral curve of X_i (Y_i resp.) then the curve $K(\gamma)$ lies completely inside the same stratum. But this means that $rank(K_*)$ remains *constant* along γ. This together with Lemma 3.8 finishes the proof. **q.e.d.**

Of course, if $K(F) = \{0\}$ then the connection form θ is *flat*, i.e. the holonomy group is not equal to H_3.

We check that the following are *independent of the homothety class* of F, and hence by Corollary 3.6 are invariants of the connection:

- the rank of K_*,
- the sign of the constant c,
- for the case that $rank(K_*) = 4$ and $c = 0$, the value of $\varepsilon = \pm$ in $K(F) \subseteq \Sigma_0^\varepsilon$, and in this case the connection is said to be of *type* Σ_0^ε,
- for the case that $rank(K_*) = 4$ and $c \neq 0$, the value of $i \in \{1, 2\}$ in $K(F) \subseteq \Sigma_c^i$, and the connection is then said to be of *type* Σ_c^i,
- for a connection of type Σ_c^i with $c > 0$, the value of $\varepsilon = \pm$ in $K(F) \subseteq \Sigma_c^{i,\varepsilon}$, and the connection is then said to be of *type* $\Sigma_c^{i,\varepsilon}$.

We will refer to the regular parts of the level sets of R_c and to the manifold parts of the sets Σ_c as *strata*.

4 Non-completeness

The purpose of this section is to prove the following

Theorem 4.1 *H_3-connections are never complete.*

PROOF: Let M be a manifold with an H_3-connection ∇, and let $(\pi, F, M, \omega, \theta, a, b, c)$ be an associated solution structure over M. By [KN, III.6.3.], the *geodesics* on M are precisely those curves γ in M with the property that if $\tilde{\gamma}$ is a horizontal lift of γ then $\tilde{\gamma}' = \sum_i k_i X_i$ for some *constants* $k_0, \ldots, k_3$.

Now suppose that ∇ is complete. Let $\tilde{\gamma}$ be an integral curve of the vector field X_1 which projects down to the geodesic $\gamma = \pi \circ \tilde{\gamma}$ and therefore by hypothesis is defined on all of $\mathbf{R}$. From the structure equations we get that

$$X_1(a_2) = 3b_0 \quad \text{and} \quad X_1(b_0) = \tfrac{1}{2} a_2^2,$$

hence $\frac{3}{2} a_2$ satisfies on $\tilde{\gamma}$ the differential equation

$$y'' = y^2. \tag{13}$$

We shall see that the only solution of (13) which is defined on *all* of $\mathbf{R}$ is $y = 0$ and therefore $a_2 = 0$ along $\tilde{\gamma}$.

Since $\tilde{\gamma}$ was an arbitrary integral curve of X_1 we conclude that $a_2 \equiv 0$ on F. But then the structure equations easily imply that all functions $a_0, a_1, a_2, b_0, \ldots, b_3$ vanish identically on F which means that the connection is *flat*, hence not an H_3-connection, and this contradiction will finish the proof.

It remains to show that (13) has no non-trivial solution. Note that any global solution y is *convex* and hence either constant or unbounded. Suppose that y is a global non-constant solution, i.e. $y'(t_0) \neq 0$ for some t_0. Replacing $y(t)$ by $y(-t)$ if necessary, we may assume that $y'(t_0) > 0$ and hence $y'(t) \geq y'(t_0) > 0$ for all $t \geq t_0$.

We get

$$\frac{d}{dt}((y')^2 - \frac{2}{3}\, y^3) = 0,$$

hence

$$(y')^2 = C + \frac{2}{3}\, y^3$$

for some constant C.

As y is unbounded, i.e. $\lim_{t->\infty} y = \infty$, we may assume that $y(t)^3 > 3|C|$ for all $t \geq t_0$ by increasing t_0 if necessary. Then we get for all $t \geq t_0$

$$(y')^2 > \frac{1}{3}\, y^3$$

which implies

$$(y^{-\frac{1}{2}})' < -\frac{1}{2\sqrt{3}}.$$

Thus we get for all $t \geq t_0$

$$y(t)^{-\frac{1}{2}} < C_1 - \frac{1}{2\sqrt{3}}\, t$$

for some constant C_1. But this is a contradiction as the left hand side of this inequality is always *positive* whereas the right hand side is *negative* for large t, and this finishes the proof. **q.e.d.**

5 The Singular H_3-connections

5.1 Immersions of Solution Structures

We will begin with the following

Definition 5.1 *We call $(F, \omega, \theta, a, b, c)$ a pseudo solution structure if*

1) *F is a 7-dimensional connected manifold, $\omega + \theta$ is a coframe on F with values in $V_3 \oplus \mathfrak{h}_3$, a, b are functions on F with values in V_2, V_3 resp. and the structure equations (5) - (11) are satisfied for some constant $c \in \mathbf{R}$.*

2) *there is a locally free* $\mathrm{Sl}(2, \mathbf{R})$*-action on F such that if we define vector fields E_i^* by $(E_i^*)_p := \frac{d}{dt}|_{t=0}(e^{tE_i}(p))$ with the basis $\{E_i\}$ of $\mathfrak{h}_3$ as before, then $\theta(E_i^*) = E_i$ for all i.*

Consider the canonical projection $\pi : F \to {}_{Sl(2,\mathbf{R})}\backslash F =: M$. In general, M will not be a manifold. We shall call a pseudo solution structure *holonomic over M* if the $Sl(2, \mathbf{R})$-action is *globally* free and M is a manifold. In this case, $(\pi, F, M, \omega, \theta, a, b, c)$ is a *solution structure.*

Definition 5.2 *Let $(F, \omega, \theta, a, b, c)$ a pseudo solution structure. A (local) vector field S on F is called a (local) infinitesimal symmetry of F if $\mathfrak{L}_S(\omega) = \mathfrak{L}_S(\theta) = 0$.*

If F is holonomic over M and S is an infinitesimal symmetry of F then $\mathfrak{L}_S(\omega) = 0$ implies that the vector field $\underline{S} := \pi_*(S)$ is well defined, and $\mathfrak{L}_S(\theta) = 0$ implies that $\underline{S}$ is an *infinitesimal symmetry* on M as defined earlier. Conversely, if $\underline{S}$ is an infinitesimal symmetry on M then there is a unique lift to an infinitesimal symmetry S on F such that $\pi_*(S) = \underline{S}$. This justifies the ambiguous use of this term.

We continue by showing the

Proposition 5.3 *The local infinitesimal symmetries on F form a Lie algebra $\mathfrak{g} \subseteq \mathfrak{X}(F)$ whose dimension equals the corank of K_* where $K := a + b : F \to V$. In fact, they span $ker(K_*)$ at each point $p \in F$.*

Proof: It is immediate that infinitesimal symmetries are closed under Lie brackets and hence form a Lie algebra. Next, any $S \in \mathfrak{g}$ vanishes either *everywhere* or *nowhere*. To see this note that along an integral curve γ of X_i (Y_i resp.), $[S, \gamma'] \equiv 0$, hence S vanishes either everywhere or nowhere along γ. Then apply Lemma 3.8.

It follows that the $\mathfrak{g}$ is a *finite dimensional* Lie algebra. Moreover, it is obvious from the structure equations that $K_*(S) = 0$ for all $S \in \mathfrak{g}$, hence $dim(\mathfrak{g}) \leq corank(K_*)$.

It remains to show that any tangent vector $S_p \in ker(K_*)$ at some point $p \in F$ extends to an infinitesimal symmetry. Let γ be a smooth curve s.th. $\gamma'(0) = S_p$ and $K_*(\gamma'(t)) = 0$ for all t. By the *uniqueness result* of solutions to the structure equations [Br2], there exists for every t a local diffeomorphism g_t of some neighborhood of p such that $g_t^*(\theta + \omega) = \theta + \omega$ and $g_t(p) = \gamma(t)$. The curves $t \mapsto g_t(q)$ are smooth for all points q in some neighborhood of p. We define the vector field S by $S_q := \frac{d}{dt}|_{t=0}(g_t(q))$. This vector field extends S_p in a neighborhood and moreover, $g_t = \Phi_S^t$ for all t, hence S is an infinitesimal symmetry. **q.e.d.**

Of course this means that every H_3-connection admits at least one 1-parameter family of (local) symmetries. Moreover, the (local) symmetry group G of M is a finite dimensional Lie group.

Definition 5.4 *Let $(F, \omega, \theta, a, b, c)$ be a pseudo-solution structure and let $X_0, \ldots, X_3, Y_1, Y_2, Y_3$ be the canonical vector fields dual to $\omega + \theta$.*

1) *F is called saturated if the map $K : F \to V$ is a principal G-fibration where G denotes the symmetry group of F. This is equivalent to saying that every local symmetry of F extends to a global symmetry.*

2) *F is called maximal if it is saturated and $K(F)$ is an entire stratum.*

The following proposition can be shown using the standard techniques to prove generalizations of the second Cartan lemma (cf. e.g. [Br2]). We will leave the proof to the reader.

Proposition 5.5 *Let $(\pi, F, M, \omega, \theta, a, b, c)$ be a solution structure over some manifold M and suppose that F is saturated. Let $(\pi', F', M', \omega', \theta', a', b', c')$ be another solution structure over a simply connected manifold M' such that $K(F') \subseteq K(F)$. Then there exist immersions $\jmath : F' \to F$ and $\imath : M' \to M$ such that $\jmath^*(\omega' + \theta') = \omega + \theta$, $a' + b' = (a + b) \circ \jmath$ and such that the diagram*

$$\begin{array}{ccc} F' & \xrightarrow{\ \jmath\ } & F \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{\ \imath\ } & M \end{array}$$

commutes. In particular, $\imath$ is connection preserving, i.e. $\imath_*(\nabla'_X Y) = \nabla_{\imath_*(X)} \imath_*(Y)$ for all vector fields X and Y on M', where ∇, ∇' denote the covariant derivatives on M, M' resp.

Therefore, if we have a maximal solution structure then every other simply connected H_3-manifold of the same type can be immersed into M by a connection preserving map. It also illustrates the significance of the examples given in section 2 which will be discussed now.

Let us first determine the subset of an H_3-manifold on which the symmetry group of the connection acts *locally free.*

Lemma 5.6 *Let M be an H_3-manifold and let $U \subseteq M$ be the set on which the local symmetries of M act fixed point free or - equivalently - where the infinitesimal symmetries of M do not vanish. Let $W^{reg} \subseteq V$ be the points in $V = V_2 \oplus V_3$ on which the group* $\mathrm{Sl}(2, \mathbf{R})$ *acts locally free, i.e. W^{reg} is the union of all regular* $\mathrm{Sl}(2, \mathbf{R})$ *orbits of V. Then*

$$U = \pi(K^{-1}(W^{reg})). \tag{14}$$

In particular, U is an open dense subset of M.

Proof: For $p \in M$, the infinitesimal symmetries on M do not vanish at p iff the infinitesimal symmetries on F do not lie in the kernel of π_* at all points $x \in \pi^{-1}(p)$. If we let Y_1, Y_2, Y_3 denote the vertical vector fields of the canonical frame of F, then this is equivalent to saying that $K_*(Y_i)_x \neq 0$ for all $x \in \pi^{-1}(p)$. Finally, the $Sl(2, \mathbf{R})$-equivariance of K implies that this is satisfied iff $K(x) \in W^{reg}$ for all $x \in \pi^{-1}(p)$. Now W^{reg} is open dense in V, hence so is U in M. **q.e.d.**

Recall that for each example in section 2 the structure polynomials a, b as well as the constant c were given. These polynomials gave the restriction of the map $K : F \to V$ to the section of F given by the frame. Therefore, the set $K(F)$ equals the $Sl(2, \mathbf{R})$-orbit of the structure polynomials in each example.

1) In example 2.1, we have the structure polynomials $a = x^2$, $b = \frac{1}{3}x^3$ and $c = 0$. Thus,

$$K(F) = Sl(2, \mathbf{R})(a, b) = \{(u^2, \frac{1}{3}u^3) \mid u \neq 0\} = \Sigma_0^0 \backslash \{0\}.$$

Also, it is not hard to show that the map K is a principal $ASl(2, \mathbf{R})$-fibration and hence F is maximal. Thus, we have a *maximal solution structure of type* Σ_0^0 and from Proposition 5.5 we get

Corollary 5.7 *Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ_0^0 and let $\tilde{M}$ denote its universal cover with the induced connection $\tilde{\nabla}$. Then there exists a connection preserving immersion*

$$\imath : \tilde{M} \to G/H$$

with G/H as in section 2.1, i.e. $\imath_*(\tilde{\nabla}_X Y) = \overline{\nabla}_{\imath_*(X)} \imath_*(Y)$ for all vector fields X, Y on M, where $\overline{\nabla}$ denotes the connection on G/H.

2) Consider the connections on $M_0^{\pm}$ described in section 2.2. The structure polynomials in these cases are $(a, b) = \phi_0^{\pm}(t_0 y, x)$ and hence

$$K(F) = Sl(2, \mathbf{R})(a, b) = \phi_0^{\pm}\{(u, v) \in V_1 \times V_1 \mid \ < u, v >_1 \neq 0.\}$$

which is precisely $\left(\Sigma_0^{\pm} \backslash \Sigma_0^0\right) \cap W^{reg}$. Again, we can show that the map K is a principal G-fibration where $G = N_3$ is the 3-dimensional Heisenberg group which acts via symmetries on $M_0^{\pm}$. Hence F is *saturated.* We will show that there exist *maximal solution structures* of types $\Sigma_0^{\pm}$ and they therefore contain $M_0^{\pm}$ as the dense open subset on which the symmetry group acts locally free.

3) Consider the connections on $M_c^{\pm}$ described in section 2.2 and suppose that $c \neq 0$. The structure polynomials in these cases are

$$(a, b) = (\pm x^2 + \frac{1}{2}\,(2t_0^2 \pm c)\; y^2,\; t_0 y\; (\mp x^2 + \frac{2t_0^2 \pm c}{6} y^2))$$

which is seen to lie in Σ_c^1. We find that

$$\Sigma_c^1 \cap W^{reg} = \{(a, b) \in \Sigma_c^1 \mid \ << b, b >_2, < b, b >_2 >_2 = 0\}.$$

Consider first the case $c > 0$. Then recall that Σ_c^1 has two components, and we check that $M_c^{\pm}$ is of type $\Sigma_c^{1,\pm}$ if $c > 0$. Moreover, in each case $K(F_c^{\pm}) = \Sigma_c^{1,\pm} \backslash W^{reg}$ where $F_c^{\pm}$ is the solution structure associated to $M_c^{\pm}$. We also check that $F_c^{\pm}$ is *saturated* and therefore, as a consequence of Proposition 5.5 we have

Corollary 5.8 *Let M be a connected 4-manifold with an H_3-connection ∇ of type $\Sigma_c^{1,\pm}$ with $c > 0$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_c^{\pm}$$

with $M_c^{\pm}$ from section 2.2, where $\tilde{U}_i$ denotes the universal cover of U_i.

Now assume $c < 0$. In this case, both examples $M_c^{\pm}$ have type Σ_c^1. One checks that $\Sigma_c^1 \backslash W^{reg}$ has two connected components, and they equal $K(F_c^+) \sqcup K(F_c^-)$ with $F_c^{\pm}$ as above. Again, we check that $F_c^{\pm}$ is *saturated*, hence Proposition 5.5 yields

Corollary 5.9 *Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ_c^1 with $c < 0$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_c^+ \sqcup M_c^-$$

with $M_c^{\pm}$ from section 2.2, where $\tilde{U}_i$ denotes the universal cover of U_i.

The difference between these two results is that in the case $c > 0$ an H_3-connection of type $\Sigma_c^{1,\pm}$ on M is *either* locally equivalent to M_c^+ *or* locally equivalent to M_c^- since the connections are of different type.

In the case $c < 0$, however, it is very well possible that an H_3-connection is locally equivalent to M_c^+ in some neighborhood, but equivalent to M_c^- in some other neighborhood on the same manifold M. In fact, this *will* happen if there exists a maximal solution structure of type Σ_c^1.

Also, we can conclude that connections of type $\Sigma_c^{1,+}$ with $c > 0$ have (local) symmetry group $SU(2)$, and connections of type $\Sigma_c^{1,-}$ with $c > 0$ or of type Σ_c^1 with $c < 0$ have (local) symmetry group $Sl(2, \mathbf{R})$.

4) Consider the connections on $M_k^{\pm}$ described in section 2.3 for $k \neq 0$. The structure polynomials in these cases are $(a, b) = (u^2 + v, \frac{1}{3}u(u^2 - 3v))$ where $u = kt_0 y$ and $v = k(x^2 \pm y^2)$. Also, the structure constant $c = \pm 6k^2$. Thus, $M_k^{\pm}$ is of type Σ_c^2. Note that $\Sigma_c^2 \cap W^{reg} = \{(u^2 + v, \frac{1}{3}u(u^2 - 3v)) \mid \; < v, v >_2 = \frac{2}{3}c \,$, and u divides v$\}$.

By an analysis similar to the previous case, we obtain from Proposition 5.5

Corollary 5.10 *Let M be a connected 4-manifold with an H_3-connection ∇ of type $\Sigma_c^{2,\pm}$ with $c > 0$ and let k such that $sign(k) = \pm$ and $c = 6\,k^2$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_k^+$$

with M_k^+ from section 2.3, where $\tilde{U}_i$ denotes the universal cover of U_i.

Corollary 5.11 *Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ_c^2 with $c < 0$ and let k such that $c = -6\,k^2$. Let $U \subseteq M$ be the subset on which the symmetry group of M acts locally free and let $U = \bigsqcup_i U_i$ be the decomposition of U into its connected components. Then there exist connection preserving immersions*

$$\imath : \tilde{U}_i \to M_k^- \sqcup M_{-k}^-$$

with $M_{\pm k}^-$ from section 2.3, where $\tilde{U}_i$ denotes the universal cover of U_i.

Again, the same holds true about the difference of these two results as in the previous case.

Also, we can conclude that connections of type $\Sigma^{2,+}_c$ with $c > 0$ have (local) symmetry group $SU(2)$, and connections of type $\Sigma^{2,-}_c$ with $c > 0$ or of type Σ^2_c with $c < 0$ have (local) symmetry group $Sl(2, \mathbf{R})$.

Of course, these corollaries are not ideal statements. The main question which is only partially answered by them is

For which (connected) strata do maximal pseudo solution structures exist? If they do, are they holonomic?

We already answered this question for H_3-connections of type Σ^0_0 in the affirmative. We will also answer it affirmatively for H_3-connections of types $\Sigma^{\pm}_0$. In the remaining cases, however, the above statements are the best we can do.

5.2 Reductions of H_3-connections

We will now show that the $\mathfrak{h}_3$-reductions of singular H_3-connections admit further reductions. These reductions are equipped with another connection with a 1-dimensional Holonomy group. However, these connections will have torsion.

5.2.1 H_3-connections of type Σ^0_0

Let M be a *connected* 4-manifold with an H_3-connection ∇ of type Σ^0_0. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2, \mathbf{R})$-equivariant map $K : F \to \Sigma^0_0 \backslash \{0\} \subseteq V$.

We define functions k_1 and k_2 by $K(u) = \phi^0_0(k_1(u)\, x + k_2(u)\, y)$ where ϕ^0_0 is the map defined on page 16. Then the structure equations imply that

$$dk_1 = -\frac{3}{2} k_2^2\, \omega_0 + k_1 k_2\, \omega_1 - \frac{1}{2} k_1^2\, \omega_2 - k_1\, \theta_1 - k_2\, \theta_3$$

$$dk_2 = -\frac{1}{2} k_2^2\, \omega_1 + k_1 k_2\, \omega_2 - \frac{3}{2} k_1^2\, \omega_3 + k_2\, \theta_1 - k_1\, \theta_2.$$

$rank(K_*) = 2$, and therefore K is a *submersion*. Since $Sl(2, \mathbf{R})$ acts transitively on $\Sigma^0_0 \backslash \{0\}$ it follows that K is *surjective*. $\overline{F} := K^{-1}(\phi^0_0(x))$ is a 5-dimensional *submanifold* of F which intersects all fibers $\pi^{-1}(p)$, $p \in M$, and the intersection with each fiber is *connected;* for $\pi(u_1) = \pi(u_2)$ iff $u_1 = L_h(u_2)$ for some $h \in \overline{H}$ where $\overline{H} := \left\{ \rho_3 \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \middle| t \in \mathbf{R} \right\} \subseteq H_3$ which is connected. It follows that $\overline{F}$ is a *principal* $\overline{H}$*-bundle.*

Let $\iota : \overline{F} \hookrightarrow F$ be the inclusion map. We compute that the 1-forms $\overline{\omega}_0, \ldots, \overline{\omega}_3, \overline{\theta}$ with $\overline{\omega}_i := \iota^*(\omega_i)$ and $\overline{\theta} := \iota^*(\theta_3 - \frac{1}{2}\, \omega_1)$ yield a *coframe* of $\overline{F}$, satisfying

$$\begin{array}{rcl} d\overline{\omega}_0 & = & -\frac{3}{2}\, \overline{\omega}_0 \wedge \overline{\omega}_2 + \overline{\omega}_1 \wedge \overline{\theta} \\ d\overline{\omega}_1 & = & -\frac{9}{2}\, \overline{\omega}_0 \wedge \overline{\omega}_3 - \frac{3}{2}\, \overline{\omega}_1 \wedge \overline{\omega}_2 + 2\, \overline{\omega}_2 \wedge \overline{\theta} \\ d\overline{\omega}_2 & = & -\frac{9}{2}\, \overline{\omega}_1 \wedge \overline{\omega}_3 + 3\, \overline{\omega}_3 \wedge \overline{\theta} \\ d\overline{\omega}_3 & = & -3\, \overline{\omega}_2 \wedge \overline{\omega}_3 \\ d\overline{\theta} & = & \frac{9}{4}\, (3\, \overline{\omega}_0 \wedge \overline{\omega}_3 - \overline{\omega}_1 \wedge \overline{\omega}_2) \quad . \end{array}$$

Let σ be the (up to multiples) unique *parallel symplectic form* on M. One computes that

$$\begin{aligned}\pi^*(\sigma) &= 3\,\overline{\omega}_0\wedge\overline{\omega}_3-\overline{\omega}_1\wedge\overline{\omega}_2\\ &= \frac{4}{9}\,d\overline{\theta},\end{aligned}$$

and - using that the restriction $\pi:\overline{F}\to M$ is a *homotopy equivalence* - we arrive at the

Proposition 5.12 *Let M be an H_3-manifold of type Σ_0^0, and let σ be the parallel symplectic form on M. Then σ is exact.*

We denote the frame dual to the coframe on $\overline{F}$ by $X_0,\ldots,X_3,Y$. We have $\mathfrak{L}_Y(\theta)=0$, and this implies that θ is the *connection form* of a connection on the principal bundle $\pi:\overline{F}\to M$. Its curvature is given by the symplectic form σ. However, this connection has torsion.

5.2.2 H_3-connections of type $\Sigma_0^\pm$

Let M be a *connected* 4-manifold with an H_3-connection ∇ of type $\Sigma_0^\pm$. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbf{R})$-equivariant map $K:F\to\Sigma_0^\pm\backslash\Sigma_0^0\subseteq V$.

Let $\overline{F}:=K^{-1}(\{\phi_0^\pm(v_1,x)\mid v_1\in V_1\})\subseteq F$ with $\phi_0^\pm$ as defined on page 16. Since $\phi_0^\pm$ is a double cover and K has constant maximal rank, it follows that $\overline{F}$ is a smooth 5-dimensional submanifold of F. Moreover, the $Sl(2,\mathbf{R})$-equivariance of K implies that $\overline{F}$ is a *reduction* of F with structure group

$$\left\{\left(\begin{array}{cc}1&0\\t&1\end{array}\right)\middle|\,t\in\mathbf{R}\right\}\subseteq Sl(2,\mathbf{R}).$$

In particular, $\overline{F}$ is *connected*. We then define functions k_1 and k_2 on $\overline{F}$ by the equation

$$K(u)=\phi_0^\pm(k_1(u)\,x+k_2(u)\,y,\ x)\quad\text{for all}\quad u\in\overline{F}.$$

Recall that $(\phi_0^\pm)^{-1}(\Sigma_0^\pm\backslash\Sigma_0^0)$ is *restricted*. In terms of the functions k_1 and k_2 this means that

$$k_2=0\Longrightarrow s\neq 0\quad\text{where}\quad s:=-3\,k_1^2\pm 1.$$

Of course, for connections of type Σ_0^- this condition is redundant.

Let $\iota:\overline{F}\hookrightarrow F$ be the inclusion map. We compute that the 1-forms $\overline{\omega}_0,\ldots,\overline{\omega}_3,\overline{\theta}$ with $\overline{\omega}_i:=\iota^*(\omega_i)$ and $\overline{\theta}:=\iota^*(\theta_3-3\,k_2\,\omega_0+k_1\,\omega_1)$ yield a *coframe* of $\overline{F}$, satisfying

$$\begin{array}{rcl}d\overline{\omega}_0&=&-6\,k_2\,\overline{\omega}_0\wedge\overline{\omega}_1+3\,k_1\,\overline{\omega}_0\wedge\overline{\omega}_2+\overline{\omega}_1\wedge\overline{\theta}\\ d\overline{\omega}_1&=&-9\,k_2\,\overline{\omega}_0\wedge\overline{\omega}_2+9\,k_1\,\overline{\omega}_0\wedge\overline{\omega}_3+3\,k_1\,\overline{\omega}_1\wedge\overline{\omega}_2+2\,\overline{\omega}_2\wedge\overline{\theta}\\ d\overline{\omega}_2&=&-9\,k_2\,\overline{\omega}_0\wedge\overline{\omega}_3-3\,k_2\,\overline{\omega}_1\wedge\overline{\omega}_2+9\,k_1\,\overline{\omega}_1\wedge\overline{\omega}_3+3\,\overline{\omega}_3\wedge\overline{\theta}\\ d\overline{\omega}_3&=&-3\,k_2\,\overline{\omega}_1\wedge\overline{\omega}_3+6\,k_1\,\overline{\omega}_2\wedge\overline{\omega}_3\\ d\overline{\theta}&=&\pm 3\,(\,3\overline{\omega}_0\wedge\overline{\omega}_3-\overline{\omega}_1\wedge\overline{\omega}_2)\end{array}$$

and the functions k_1 and k_2 satisfy

$$\begin{array}{rcl}dk_1&=&-\frac{9}{2}\,k_2^2\,\overline{\omega}_0+3\,k_1k_2\,\overline{\omega}_1+\frac{1}{2}\,s\,\overline{\omega}_2-k_2\,\overline{\theta}\\ dk_2&=&-\frac{3}{2}\,k_2^2\,\overline{\omega}_1+3\,k_1k_2\,\overline{\omega}_2+\frac{3}{2}\,s\,\overline{\omega}_3\end{array}$$

By a similar argument as in the previous section - using that the restriction $\pi:\overline{F}\to M$ is a *homotopy equivalence* - we obtain the

Proposition 5.13 *Let M be an H_3-manifold of type $\Sigma_0^{\pm}$, and let σ be the parallel symplectic form on M. Then σ is exact.*

Again, we see that θ is the *connection form* of a connection on the principal bundle $\pi : \overline{F} \to M$ whose curvature is given by the symplectic form σ. As before, this connection has torsion.

We will now give *explicit* solutions to the equations given above. We will do this in two different ways.

Let G be the 3-dimensional *Heisenberg group,* and let $\mathfrak{g}$ be the corresponding Lie algebra.

We will first give the solution which corresponds to the H_3-connection on $M_0^{\pm}$ of section 2.2. From the reduction described above, we get a bundle $\pi : \overline{F} \to M_0^{\pm}$ and we have $K(\overline{F}) = \phi_0^{\pm}\{(k_1 x + k_2 y, x) \in V_1 \times V_1 \mid k_2 \neq 0\} =: U_1$. On $U_1 \times G$ we define the $\mathfrak{g}$-valued 1-form $\alpha := p^*(\omega_G)$ where ω_G is the left invariant Maurer-Cartan form on G, and $p : U_1 \times G \to G$ is the projection map. We can decompose α into three *real valued* 1-forms $\alpha_1, \alpha_2, \alpha_3$ satisfying

$$d\alpha_1 = d\alpha_2 = 0 \quad \text{and} \quad d\alpha_3 = d\alpha_1 \wedge d\alpha_2.$$

Then a solution to the structure equations on $U_1 \times G$ can be given by

$$\begin{array}{rl}
\overline{\omega}_0 = & -\dfrac{2\,k_1}{3\,k_2^3}\,dk_2 + \dfrac{1}{k_2}\,\alpha_1 - \dfrac{k_1\,(k_1^2 \pm 1)}{k_2}\,\alpha_2 + \dfrac{3}{2}\,s\,\alpha_3 \\
\overline{\omega}_1 = & -\dfrac{2}{3\,k_2^2}\,dk_2 - (6\,k_1^2 + s)\,\alpha_2 - 9\,k_1 k_2\,\alpha_3 \\
\overline{\omega}_2 = & -3\,k_1 k_2\,\alpha_2 - \dfrac{9}{2}\,k_2^2\,\alpha_3 \\
\overline{\omega}_3 = & -k_2^2\,\alpha_2 \\
\overline{\theta} = & -\dfrac{1}{k_2}\,dk_1 + \dfrac{k_1}{k_2^2}dk_2 - \dfrac{9}{2}\,\alpha_1 \mp 9\,k_2\,\alpha_3
\end{array}$$

G acts on $U_1 \times G$ in a canonical way from the left, and this is an action by symmetries.

We now turn to the second description of a solution.

Let $U_2 := \{k_1\,x + k_2\,y \in V_1 \mid s \neq 0\}$. (Note that for connections of type Σ_0^-, $U_2 = V_1$.) We define the 1-forms $\alpha_1, \alpha_2, \alpha_3$ on $U_2 \times G$ in the same way as before. Then the solution is given by

$$\begin{array}{rl}
\overline{\omega}_0 = & -6\,s\,\alpha_3 \\
\overline{\omega}_1 = & \dfrac{1 \pm 6\,k_1^2 - 3\,k_1^4}{s}\,\alpha_1 - \dfrac{24\,k_1}{s}\,\alpha_2 + 36\,k_1 k_2\,\alpha_3 \\
\overline{\omega}_2 = & \dfrac{2}{s}\,dk_1 - \dfrac{6\,k_1 k_2\,(k_1^2 \mp 1)}{s}\,\alpha_1 - \dfrac{12\,k_2}{s}\,\alpha_2 + 18\,k_2^2\,\alpha_3 \\
\overline{\omega}_3 = & -\dfrac{4\,k_1 k_2}{s^2}\,dk_1 + \dfrac{2}{3s}\,dk_2 + k_2^2\,\alpha_1 \\
\overline{\theta} = & \dfrac{6\,k_1\,(1 \pm k_1^2)}{s}\,\alpha_1 - \dfrac{6\,(9\,k_1^2) \pm 1}{s}\,\alpha_2 \pm 36\,k_2\,\alpha_3
\end{array}$$

Since $s \neq 0$ on U_2 this frame is well defined. Also, the natural left action of G on $U_2 \times G$ is an action by *symmetries*.

Let $P^{\pm} := range(k_1 x + k_2 y)$, i.e. $P^+ = V_1 \backslash \{\pm\sqrt{3}x\}$ and $P^- = V_1$.

If the connection is of type Σ_0^+ then these two solutions turn out to be the *local trivializations* of a principal G-bundle over P^+. Note, however, that the total space of such a bundle is *oriented* by the volume form $\overline{\omega}_0 \wedge \cdots \wedge \overline{\theta}$. One can show that any oriented G-bundle over P^+ must be trivial.

We conclude that there exists a solution to the above equations on $P^{\pm} \times G =: \overline{F}^{\pm}$.

Let us denote the frame dual to $\overline{\omega}_0, \ldots, \overline{\omega}_3, \overline{\theta}$ by $X_0, \ldots X_3, Y$ as before. Then the **R**-action on $\overline{F}^{\pm}$ defined by the flow along the vector field Y is *proper*, i.e. the map

$$\begin{array}{ccc} \overline{F}^{\pm} \times \mathbf{R} & \longrightarrow & \overline{F}^{\pm} \times \overline{F}^{\pm} \\ (u,t) & \longmapsto & (u, \Phi^t_Y) \end{array}$$

is *compact*. This in turn implies that the quotient space $\overline{F}^{\pm}/\mathbf{R} =: M^{\pm}$ is a *manifold* and the natural projection $\pi : \overline{F}^{\pm} \to M^{\pm}$ is a *principal* **R***-fibration.*

It is not hard to see that these solutions determine an H_3-connection on $M^{\pm}$, i.e. we can 'revers' the reduction to $\overline{F}$ described before and get a *solution structure* over $M^{\pm}$ with type $\Sigma^{\pm}_0$ and whose **R**-reduction is $\overline{F}$.

Thus we have *maximal solutions of type* $\Sigma^{\pm}_0$ on $M^{\pm}$. One can show that M^- is diffeomorphic to $\mathbf{R}^4$ since it is the quotient of the flow along a complete vector field on $F^- \approx \mathbf{R}^5$. As for M^+, we can conclude that it has the same homotopy type as 'figure eight'.

Thus, from proposition 5.5 we conclude:

Corollary 5.14 *Let M be a simply connected connected 4-manifold with an H_3-connection ∇ of type $\Sigma^{\pm}_0$ and let $M^{\pm}$ be the manifolds described above. Then there exists a connection preserving immersion*

$$\imath : M \to M^{\pm}.$$

5.2.3 H_3-connections of type Σ^2_c

Let M be a connected 4-manifold with an H_3-connection ∇ of type Σ^2_c where $c \in \mathbf{R}\backslash\{0\}$. This means that we have an $\mathfrak{h}_3$-reduction F of the total frame bundle $\mathfrak{F}$ of M and an $Sl(2,\mathbf{R})$-equivariant map $K : F \to \Sigma^2_c \subseteq V$.

Recall the diffeomorphism ϕ^2_c from page 16. Recall also that $V_{2,c}$ has one or two connected components depending on the sign of c.

$V_{2,c}$ contains a polynomial $k\,(x^2 \pm y^2)$, where "$\pm$" $= sign(c)$ and $k^2 = \frac{1}{6}|c|$. In the case $c > 0$ the choice of sign of k determines the component $V^{\pm}_{2,c}$.

If we let $\overline{F}_k := K^{-1}(\{\phi^2_c(v_1,\ k\,(x^2 \pm y^2)) \mid v_1 \in V_1\}) \subseteq F$ then $\overline{F}_k$ is a smooth 5-dimensional submanifold of F and moreover a *reduction* of F with structure group

$$SO(2) \subseteq Sl(2,\mathbf{R}) \quad \text{if } c > 0, \text{ and}$$

$$SO(1,1) \subseteq Sl(2,\mathbf{R}) \quad \text{if } c < 0.$$

We then define functions r_1 and r_2 on $\overline{F}_k$ by the equation

$$K(u) = \phi^2_c(r_1(u)\,x + r_2(u)\,y,\ k\,(x^2 \pm y^2)) \quad \text{for all} \quad u \in \overline{F}_k.$$

Let $\iota : \overline{F}_k \hookrightarrow F$ be the inclusion map. We compute that the 1-forms $\overline{\omega}_0, \ldots, \overline{\omega}_3, \overline{\theta}$ with $\overline{\omega}_i := \iota^*(\omega_i)$ and $\overline{\theta} := \frac{1}{2}\iota^*(\theta_2 \mp \theta_3 - r_1\,(3\,\omega_3 \pm \omega_1) + r_2\,(\pm 3\,\omega_0 + \omega_2))$ yield a *coframe* of $\overline{F}_k$, satisfying

$$
\begin{array}{lcrcrcrcr}
d\overline{\omega}_0 & = & -6\,r_2\,\overline{\omega}_0\wedge\overline{\omega}_1 & + & 3\,r_1\,\overline{\omega}_0\wedge\overline{\omega}_2 & & & \mp & \overline{\omega}_1\wedge\overline{\theta} \\
d\overline{\omega}_1 & = & -9\,r_2\,\overline{\omega}_0\wedge\overline{\omega}_2 & + & 9\,r_1\,\overline{\omega}_0\wedge\overline{\omega}_3 & + & 3\,r_1\,\overline{\omega}_1\wedge\overline{\omega}_2 & + & (3\,\overline{\omega}_0\mp 2\,\overline{\omega}_2)\wedge\overline{\theta} \\
d\overline{\omega}_2 & = & -9\,r_2\,\overline{\omega}_0\wedge\overline{\omega}_3 & - & 3\,r_2\,\overline{\omega}_1\wedge\overline{\omega}_2 & + & 9\,r_1\,\overline{\omega}_1\wedge\overline{\omega}_3 & + & (2\,\overline{\omega}_1\mp 3\,\overline{\omega}_3)\wedge\overline{\theta} \\
d\overline{\omega}_3 & = & -3\,r_2\,\overline{\omega}_1\wedge\overline{\omega}_3 & + & 6\,r_1\,\overline{\omega}_2\wedge\overline{\omega}_3 & & & + & \overline{\omega}_2\wedge\overline{\theta} \\
d\overline{\theta} & = & \mp 3\,k\,(\,3\overline{\omega}_0\wedge\overline{\omega}_3 & - & \overline{\omega}_1\wedge\overline{\omega}_2) & & & &
\end{array}
$$

and the functions r_1 and r_2 satisfy

$$
\begin{array}{lcl}
dr_1 & = & -\frac{3}{2}\,(3\,r_2^2\mp k)\,\overline{\omega}_0 + 3\,r_1r_2\,\overline{\omega}_1 - \frac{1}{2}\,(3r_1^2-k)\,\overline{\omega}_2 \pm r_2\,\overline{\theta} \\
dr_2 & = & -\frac{1}{2}\,(3r_2^2\mp k)\,\overline{\omega}_1 + 3\,r_1r_2\,\overline{\omega}_2 - \frac{3}{2}\,(3r_1^2-k)\,\overline{\omega}_3 - r_1\overline{\theta}
\end{array}
$$

As before, we get the

Proposition 5.15 *Let M be an H_3-manifold of type Σ_2^c, and let σ be the parallel symplectic form on M.*

1) *If $c > 0$ then the element of $H^2(M, \mathbf{R})$ represented by σ is a multiple of the Euler class of a circle bundle over M.*

2) *If $c < 0$ then σ is exact.*

Proof: The first case follows from the *Gysin sequence* for circle bundles.

In the second case, we may assume that $\overline{F}_k$ has two components by passing to a double cover of M if necessary. The restriction of π to one connected component of $\overline{F}_k$ is then a homotopy equivalence. Finally, one can show that for a double cover $\alpha : \tilde{M} \to M$, the induced map $\alpha^* : H^*(M, \mathbf{R}) \to H^*(\tilde{M}, \mathbf{R})$ is *injective.* **q.e.d.**

Again, we see that θ is the *connection form* of a connection on the principal bundle $\pi : \overline{F} \to M$ whose curvature is given by the symplectic form σ. As before, this connection has torsion.

Remark: Since the description of Σ_c^1 does not give a parametrization by an $Sl(2, \mathbf{R})$-equivariant map as in the other cases, we cannot get a reduction of it as easily. Therefore, it seems more difficult to make any statement about its symplectic form.

Remark: We will now give a hint how we obtained the examples of section 2. Given a stratum Σ, it is not hard to find a smooth curve in $\Sigma \cap W^{reg}$ which intersects every orbit exactly once. These curves are the parametrized structure polynomials (a, b) given in each of the examples. Once we have this curve, say $\alpha(t)$, we consider $S := K^{-1}(\alpha)$ which is a smooth submanifold of F and intersects every fiber of $\pi : F \to M$ at most once. This means that S is the image of a smooth section on some subset $U \subseteq M$. Then we let $\overline{\omega}_i := \imath^*(\omega_i)$ for $0 \le i \le 3$ where $\imath : S \hookrightarrow F$ is the inclusion map. The structure equations yield equations for $d\overline{\omega}_i$ and these can be solved *explicitly.*

5.3 Regular H_3-connections

In this section we will discuss those H_3-connections on M for which the map $K : F \to V$ has maximal rank 6.

Unfortunately, in this case the structure equations involved are so complex that they cannot be solved explicitly on an dense open subset of M. A (not very illuminating) description of the connection on some open subset, however, can be obtained.

Let $U := \{p \in M \mid \ < a, a >_2 < 0, \text{ and } a \text{ does not divide } b \text{ on } \pi^{-1}(p)\}$. For $p \in U$ there is a unique frame $u \in \pi^{-1}(p)$ such that

$$b_3(u) = 1, \quad a(u) = f(u)\, xy \quad \text{with } f(u)^2 = - < a(u), a(u) >_2 > 0.$$

Using this section, we can describe the connection w.r.t. some coordinate system on U as follows where the constants c and R_c are given.

Let $x_0, \ldots, x_3$ be local coordinates and define the functions

$$t_0 = \frac{1}{x_0}, \quad t_1 = \frac{x_1}{x_0^2}, \quad t_2 = \frac{x_1^2}{x_0^3} + \frac{x_2}{x_0}, \quad \text{and} \quad t_3 = \frac{x_1^3}{x_0^4} + 3\frac{x_1 x_2}{x_0^2} - x_0^2 + c + \frac{1}{x_0} r$$

where

$$r = \pm 2\sqrt{x_1^3 - x_2^3 + c x_1 x_2 + R_c}.$$

Furthermore, let

$$s_i := \frac{\partial}{\partial x_0}(x_0 t_i) \text{ for all } i.$$

Then we define a frame as follows:

$$\begin{aligned}
X_0 &= 3x_0 t_0 \frac{\partial}{\partial x_0} \\
X_1 &= x_0 t_1 \frac{\partial}{\partial x_0} + t_0 \overline{X}_1 \\
X_2 &= -x_0 t_2 \frac{\partial}{\partial x_0} + 2t_1 \overline{X}_1 + \overline{X}_2 \\
X_3 &= -3x_0 t_3 \frac{\partial}{\partial x_0} + 3t_2 \overline{X}_1 + 3x_0 t_1 \overline{X}_2 + \overline{X}_3
\end{aligned}$$

with

$$\begin{aligned}
\overline{X}_1 &= r \frac{\partial}{\partial x_2} + \frac{\partial u}{\partial x_2} \frac{\partial}{\partial x_3} \\
\overline{X}_2 &= -r \frac{\partial}{\partial x_1} - \frac{\partial u}{\partial x_1} \frac{\partial}{\partial x_3} \\
\overline{X}_3 &= 2(3x_2^2 - c x_1) \frac{\partial}{\partial x_1} + 2(3x_1^2 + c x_2) \frac{\partial}{\partial x_2}
\end{aligned}$$

and where $u = u(x_1, x_2)$ is some function satisfying $\overline{X}_3(u) \equiv 1$.

This frame is defined on $\{(x_0, \ldots, x_3) \mid x_0 \neq 0, x_1^3 - x_2^3 + c\, x_1 x_2 + R_c > 0\}$ for the given constants c and R_c.

The connection form is then given as

$$\theta = \theta_1\, E_1 + \theta_2\, E_2 + \theta_3\, E_3$$

where

$$\begin{aligned}
\theta_1 &= - \sum_{i=0}^{3} s_i\, \omega_i \\
\theta_2 &= - \sum_{i=0}^{3} i\, t_{i-1}\, \omega_i \\
\theta_3 &= \sum_{i=0}^{3} (3 - i)\, t_{i+1}\, \omega_i
\end{aligned}$$

and where $\omega_0, \ldots, \omega_3$ denotes the dual basis of $X_0, \ldots, X_3$.

One can check that this connection is indeed torsion free and is a *regular* H_3-connection. Note in particular that the 1-dimensional symmetry group is given as the flow along the vector field $\dfrac{\partial}{\partial x_3}$.

6 H_3-connections on compact manifolds

The purpose of this section is to show the

Theorem 6.1 *There are no H_3-connections on compact 4-manifolds.*

PROOF: We will show that any compact 4-manifold M with an H_3-connection must be of type Σ_0^0. This together with Proposition 5.12 will finish the proof since symplectic forms on compact manifolds cannot be exact.

Suppose M is compact and has an H_3-connection, and let $\{\pi, F, M, a, b, c\}$ be an associated solution structure. Consider the function

$$\begin{aligned} f: \quad F &\longrightarrow \mathbf{R} \\ u &\longmapsto < a(u), a(u) >_2 = -discr(a(u)) \end{aligned}$$

Since f is constant along the fibers of F there is a unique function $\underline{f} : M \to \mathbf{R}$ such that $\underline{f} \circ \pi = f$.

We shall now use that $\underline{f}$ must have both a maximum and a minimum on M. Therefore, we shall investigate the critical points of $\underline{f}$.

Using the structure equations (5) - (11) we find that

$$df = 6 \begin{pmatrix} b_0 & \dots & b_3 \end{pmatrix} \begin{pmatrix} -3a_1 & 2a_0 & & \\ -6a_2 & -a_1 & 4a_0 & \\ & -4a_2 & a_1 & 6a_0 \\ & & -2a_2 & 3a_1 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \vdots \\ \omega_3 \end{pmatrix}.$$

The determinant of the matrix in this equation is $9f^2$. So if $u \in F$ is a critical point of f and $f(u) \neq 0$ then $b(u) = 0$. We wish to compute the *Hessian* of $\underline{f}$ at a critical point $p \in M$ w.r.t some appropriate frame.

If $\underline{f}(p) < 0$ at a critical point $p \in M$ then there is a frame $u \in \pi^{-1}(p)$ with $a_0(u) = a_2(u) = 0$ and $a_1(u) \neq 0$. We compute the Hessian w.r.t. this frame as

$$\frac{1}{2} \begin{pmatrix} & & & -9a_1(2c+3a_1^2) \\ & & a_1(2c+a_1^2) & \\ & a_1(2c+a_1^2) & & \\ -9a_1(2c+3a_1^2) & & & \end{pmatrix}.$$

We see easily that this matrix has some negative eigenvalue regardless of the values of a_1 and c. Therefore, $\underline{f}$ *cannot* have a negative minimum, and we conclude that $\underline{f} \geq 0$.

Let $\mathcal{C} := \{a \in V_2 \mid \ < a, a >_2 \geq 0\}$. We find that $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ with $\mathcal{C}_\pm = \{\pm(v_1^2 + v_2^2) \mid v_1, v_2 \in V_1\}$, and $\mathcal{C}_+ \cap \mathcal{C}_- = \{0\}$. Moreover, $\mathcal{C}_\pm$ is invariant under the $Sl(2, \mathbf{R})$-action on V_2.

If $\underline{f}(p) > 0$ at a critical point $p \in M$ then there is a frame $u \in \pi^{-1}(p)$ with $a_1(u) = 0$ and $a_0(u) = a_2(u)$. We compute the Hessian w.r.t. this frame as

$$\begin{pmatrix} 18a_0^3 & 0 & -6a_0(5a_0^2 - c) & 0 \\ 0 & 2a_0(13a_0^2 - 2c) & 0 & -6a_0(5a_0^2 - c) \\ -6a_0(5a_0^2 - c) & 0 & 2a_0(13a_0^2 - 2c) & 0 \\ 0 & -6a_0(5a_0^2 - c) & 0 & 18a_0^3 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$r(\lambda)^2, \quad \text{where} \quad r(\lambda) = \lambda^2 - 4a_0(11a_0^2 - c)\,\lambda - 36a_0^2(2a_0^2 - c)(6a_0^2 - c).$$

If $a(u) \in \mathcal{C}_+\backslash\{0\}$, i.e. $a_0(u) > 0$, then r has at least one positive root since either $r(0) < 0$ or $r'(0) < 0$. Therefore, $\underline{f}$ cannot have a *positive maximum in* $\pi(a^{-1}(\mathcal{C}_+\backslash\{0\}))$.

Similarly, if $a(u) \in \mathcal{C}_-\backslash\{0\}$, i.e. $a_0(u) < 0$, then r has at least one negative root since either $r(0) < 0$ or $r'(0) > 0$. Therefore, $\underline{f}$ cannot have a *positive minimum in* $\pi(f^{-1}(\mathcal{C}_-\backslash\{0\}))$.

Suppose that $\underline{f}(p) = 0$ for some $p \in M$ and $a \neq 0$ on $\pi^{-1}(p)$. Since $f \geq 0$, p is critical. There is a frame $u \in \pi^{-1}(p)$ with $a(u) = \pm x^2$. From $df(u) = 0$ we conclude that $b(u) = \tilde{b}x^3$ for some $\tilde{b} \in \mathbf{R}$. Computing the Hessian of f at u we get

$$\begin{pmatrix} 0 & & & \\ & 0 & & \pm 6c \\ & & \mp 4c & \\ & \pm 6c & & -18(9\tilde{b}^2 \mp 1) \end{pmatrix}$$

The characteristic polynomial of this matrix is

$$\lambda(\lambda \pm 4c)(\lambda^2 + 18(9\tilde{b}^2 \mp 1)\lambda - 36c^2)$$

The Hessian must be positive semidefinite, i.e. this polynomial cannot have any negative root. It is easily seen that this is satisfied only if $a(u) = x^2$, $c = 0$ and $\tilde{b}^2 \leq \frac{1}{9}$.

We conclude that $f(u) = 0$ implies $a(u) \in \mathcal{C}_+$.

Suppose now that $a(U) \subseteq \mathcal{C}_+\backslash\{0\}$ for some open set $U \subsetneq F$. From the above we conclude that $f(U) = 0$. It follows that $K(U) \subseteq \{(v^2, \tilde{b}v^3) \mid v \in V_1\backslash\{0\}, \tilde{b} \in \mathbf{R}\}$. Since this set is 3-dimensional we get $rank(K_*) \leq 3$ on U. But then Theorem 3.7 implies that M is of type Σ_0^0. In particular, $a(F) \subseteq \mathcal{C}_+\backslash\{0\}$.

Thus, either M is of type Σ_0^0 or $a(F) \subseteq \mathcal{C}_-$.

But in the latter case we conclude that $a(u) = 0$ for some $u \in F$. Since $a(F) \subseteq \mathcal{C}_-$, hence $a_0, a_2 \leq 0$, this means that u is a maximum for both a_0 and a_2, hence $da_0(u) = da_2(u) = 0$ and therefore $b(u) = 0$. Computing the Hessian of a_0 at u yields the matrix

$$\frac{1}{2}\begin{pmatrix} 0 & & 3c & \\ & -2c & & \\ 3c & & 0 & \\ & & & 0 \end{pmatrix}$$

This matrix must be negative semidefinite which is satisfied iff $c = 0$. Thus, $a(u) = b(u) = c = 0$, i.e. $K_*(u) = 0$, and again Theorem 3.7 implies that the connection is *flat*, violating our assumption. This contradiction shows that M is of type Σ_0^0, and this finishes the proof. **q.e.d.**

References

[A] D.V. Alekseevskii, *Compact quaternion spaces*, Funktional Anal. i Priložen. 2, 11-20 (1968)

[Ber] M. Berger, *Sur les groupes d'holonomie des variétés à connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France, 83, 279-330 (1955)

[BD] T. Bröcker, T. Dieck, *Representations of compact Lie groups*, Springer-Verlag (1985)

[Br1] R. Bryant, *Metrics with exceptional holonomy*, Ann. Math. 126, 525-576 (1987)

[Br2] R. Bryant, *Two exotic holonomies in Dimension four, Path geometries, and twistor theory*, Proc. Symp. in Pure Math. 53 (1991)

[EDS] R. Bryant, S. Chern, R. Gardner, H. Goldschmidt, P. Griffith, *Exterior Differential Systems*, Springer-Verlag (1991)

[C] E. Calabi, *Métriques kählериennes et fibrés holomorphes*, Ann. Ecol. Norm. Sup. 12, 269-294 (1979)

[KN] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. 1 & 2*, Interscience Publishers (1969)

[S] J. Simons, *On transitivity of holonomy systems*, Ann. Math. 76, 213-234 (1962)