



Proceedings of the Seventh Congress  
of the European Society for Research  
in Mathematics Education



**Editors**

Marta Pytlak, Tim Rowland, Ewa Swoboda

University of Rzeszów, Poland

## **Editors**

Marta Pytlak, University of Rzeszów, Poland

Tim Rowland, University of Cambridge, UK

Ewa Swoboda University of Rzeszów, Poland

## **Editorial Board**

Paul Andrews, Carmen Batanero, Claire Berg, Rolf Biehler, Angelika Bikner-Ahsbals, Laurinda Brown, María C. Cañadas, Susana Carreira, Maria Luiza Cestari, Sarah Crafter, Thérèse Dooley, Viviane Durand-Guerrier, Andreas Eichler, Ingvald Erfjord, Pier Luigi Ferrari, Anne Berit Fuglestad, Athanasios Gagatsis, Uwe Gellert, Alejandro S. González-Martin, Núria Gorgorió, Sava Grozdev, Ghislaine Gueudet, Kirsti Hemmi, Marja van den Heuvel-Panhuizen, Jeremy Hodgen, Paola Iannone, Eva Jablonka, Niels Jahnke, Uffe Thomas Jankvist, Gabriele Kaiser, Ivy Kidron, Götz Krummheuer, Alain Kuzniak, Jan van Maanen, Kate Mackrell, Mirko Maracci, Pietro Di Martino, Snezana Lawrence, Roza Leikin, Thomas Lingefjärd, Nicolina Malara, Emma Mamede, Carlo Marchini, Maria Meletiou-Mavrotheris, Alain Mercier, John Monaghan, Elena Nardi, Reinhard Oldenburg, Alexandre Pais, Marilena Pantziara, Efi Paparistodemou, Bettina Pedemonte, Richard Philippe, Demetra Pitta-Pantazi, Despina Potari, Dave Pratt, Susanne Prediger, Luis Radford, Leonor Santos, Wolfgang Schloeglmann, Gérard Sensevy, Florence Mihaela Singer, Nad'a Stehlikova, Andreas Ulovec, Paola Valero, Konstantinos Tatsis, Joke Torbeyns, Jana Trgalová, Konstantinos Tzanakis, Fay Turner, Geoff Wake, Kjersti Wæge, Hans-Georg Weigand, Carl Winsløw.

The proceedings are published by University of Rzeszów, Poland  
on behalf of the European Society for Research in Mathematics

**ISBN 978-83-7338-683-9**

© Copyright 2011 left to the authors



## TABLE OF CONTENTS

<b><u>General Introduction</u></b>	1
<i>Ferdinando Arzarello, Tim Rowland</i>	
<b><u>Editotrial Introduction</u></b>	5
<i>Tim Rowland, Ewa Swoboda, Marta Pytlak</i>	
<b>PLENARY LECTURES</b>	
<b>Plenary lecture 1</b>	
<b><u>Research into Pre-service elementary teacher education courses</u></b>	9
<i>Anna Sierpińska</i>	
<b>Plenary lecture 2</b>	
<b><u>The structure and dynamics of affect in mathematical thinking and learning</u></b>	34
<i>Markku S. Hanula</i>	
<b>Plenary lecture 3</b>	
<b><u>Proving and proof as an educational task</u></b>	61
<i>Maria Alessandra Mariotti</i>	
<b>RESEARCH PAPERS</b>	
<b>WORKING GROUP 1</b>	
<b><u>Introduction to the papers of WG 1: Argumentation and proof</u></b>	93
<i>Viviane Durand-Guerrier, Kirsti Hemmi, Niels Jahnke, Bettina Pedemonte</i>	
<b><u>Meta-cognitive unity in indirect proofs</u></b>	99
<i>Ferdinando Arzarello, Cristina Sabena</i>	
<b><u>Abduction in generating conjectures in dynamic geometry through maintaining dragging</u></b>	110
<i>Anna Baccaglioni-Frank</i>	
<b><u>Argumentation and proof: discussing a “successful” classroom discussion</u></b>	120
<i>Paolo Boero</i>	
<b><u>Mathematical proving on secondary school level I: supporting student understanding through different types of proof. A video analysis</u></b>	131
<i>Esther Brunner, Kurt Reusser, Christine Pauli</i>	
<b><u>Everyday argumentation and knowlegde construction in mathematical tasks</u></b>	141
<i>Julia Cramer</i>	
<b><u>Analyzing the proving activity of a group of three students</u></b>	151
<i>Patricia Perry, Óscar Molina, Leonor Camargo, Carmen Samper</i>	
<b><u>On the role of looking back at proving processes in school mathematics: focusing on argumentation</u></b>	161
<i>Yosuke Tsujiyama</i>	

<a href="#"><u>Making the discovery function of proof visible for upper secondary school students</u></a>	172
<i>Kirsti Hemmi, Clas Löfwall</i>	
<a href="#"><u>Conjecturing and proving in AlNuset</u></a>	181
<i>Bettina Pedemonte</i>	
<a href="#"><u>A schema to analyse students' Proof evaluations</u></a>	192
<i>Kirsten Pfeiffer</i>	
<a href="#"><u>The appearance of algorithms in curricula a new opportunity to deal with proof?</u></a>	202
<i>Modeste Simon, Ouvrier-Buffer Cécile</i>	
<a href="#"><u>Proof: a game for pedants?</u></a>	213
<i>Joanna Mamona-Downs, Martin Downs</i>	
<a href="#"><u>Two beautiful proofs of Pick's theorem</u></a>	223
<i>Manya Raman, Lars-Daniel Öhman</i>	
<a href="#"><u>Multimodal derivation and proof in algebra</u></a>	233
<i>Reinert A. Rinvold, Andreas Lorange</i>	
<a href="#"><u>The view of mathematics and argumentation</u></a>	243
<i>Antti Viholainen</i>	
<a href="#"><u>Upper secondary school teachers' views of proof and the relevance of proof in teaching mathematics</u></a>	253
<i>Emelie Reuterswärd, Kirsti Hemmi</i>	
<a href="#"><u>Development of beginning skills in proving and proof-writing by elementary school students</u></a>	263
<i>Stéphane Cyr</i>	
<a href="#"><u>Designing interconnecting problems that support development of concepts and reasoning</u></a>	273
<i>Margo Kondratieva</i>	

## **WORKING GROUP 2**

<a href="#"><u>Introduction to the Papers of WG 2: Teaching and learning of number system and arithmetics</u></a>	283
<i>Susanne Prediger, Nad'a Stehlikova, Joke Torbeyns, Marja van den Heuvel-Panhuizen</i>	
<a href="#"><u>Cognitive styles and their relation to number sense and algebraic reasoning</u></a>	287
<i>Marilena Chrysostomou, Chara Tsingi, Eleni Cleanthous</i>	
<a href="#"><u>Danish number names and number concepts</u></a>	297
<i>Lisser Rye Ejersbo, Morten Mistfeld</i>	
<a href="#"><u>Strategies and procedures: what relationship with the development of number sense of students?</u></a>	307
<i>Elvira Ferreira, Lurdes Serrazina</i>	
<a href="#"><u>Preschool children's understanding of equality: opting for a narrow or a broad interpretation?</u></a>	316
<i>Anatoli Kouropatov, Dina Tirosh</i>	
<a href="#"><u>Using the double number line to model multiplication</u></a>	326
<i>Dietmar Küchemann, Margaret Brown, Jeremy Hodgen</i>	

<a href="#"><u>Proportion in mathematics textbooks in upper secondary school</u></a>	336
<i>Anna Lundberg</i>	
<a href="#"><u>Teaching arithmetic for the needs of the society</u></a>	346
<i>Hartwig Meissner</i>	
<a href="#"><u>Analysing children's learning in arithmetic through collaborative group work</u></a>	356
<i>Carol Murphy</i>	
<a href="#"><u>A theoretical model for understanding fractions at elementary school</u></a>	366
<i>Aristkolis Nikolaou, Demetra Pitta-Pantazi</i>	
<a href="#"><u>Between algebra and geometry: the dual nature of the number line</u></a>	376
<i>Ildikó Pelczer, Cristian Voica, Florence Mihaela Singer</i>	
<a href="#"><u>Special education students' ability in solving subtraction problems up to 100 by addition</u></a>	386
<i>Marjolijn Peltenburg, Marja van den Heuvel Panhuizen</i>	
<a href="#"><u>Mental calculation strategies for addition and subtraction in the set of rational numbers</u></a>	396
<i>Sebastian Rezat</i>	
 <b>WORKING GROUP 3</b>	
<a href="#"><u>Introduction to the papers of WG 3: Algebraic thinking</u></a>	406
<i>María C. Cañadas, Thérèse Dooley, Jeremy Hodgen, Reinhard Oldenburg</i>	
<a href="#"><u>Can they "see" the equality?</u></a>	410
<i>Vassiliki Alexandrou-Leonidou and George N. Philippou</i>	
<a href="#"><u>Patterning problems: sixth graders' ability to generalize</u></a>	420
<i>Ana Barbosa</i>	
<a href="#"><u>The role of technology in developing principles of symbolical algebra</u></a>	429
<i>Giampaolo Chiappini</i>	
<a href="#"><u>Secondary school students' perception of best help generalising strategies</u></a>	440
<i>Boon Liang Chua, Celia</i>	
<a href="#"><u>Using epistemic actions to trace the development of algebraic reasoning in a primary classroom</u></a>	450
<i>Thérèse Dooley</i>	
<a href="#"><u>Graphical representation and generalization in sequences problems</u></a>	460
<i>María C. Cañadas, Encarnación Castro, Enrique Castro</i>	
<a href="#"><u>The entrance to algebraic discourse: informal meta-arithmetic as the first step toward formal school algebra</u></a>	470
<i>Shai Caspi, Anna Sfard</i>	
<a href="#"><u>Students' reasoning in quadratic equations with one unknown</u></a>	479
<i>M. Gözde Didiş, Sinem Baş, A. Kürşat Erbaş</i>	
<a href="#"><u>Investigating the influence of student's previous knowledge on their concept of variables - an analysis tool considering teaching reality</u></a>	490
<i>Sandra Gerhard</i>	
<a href="#"><u>What is algebraic activity? Consideration of 9-10 year olds learning to solve linear equations</u></a>	500
<i>Dave Hewitt</i>	

<a href="#"><u>The role of discursive artefacts in making the structure of an algebraic expression emerge</u></a>	511
<i>Laura Maffei, Maria Alessandra Mariotti</i>	
<a href="#"><u>Algebraic thinking of grade 8 students in solving word problems with a spreadsheet</u></a>	521
<i>Sandra Nobre, Nélia Amado, Susana Carreira, João Pedro da Ponte</i>	
<a href="#"><u>Algebraic reasonings among primary school 4th grade pupils</u></a>	532
<i>Marta Pytlak</i>	
<b>WORKING GROUP 4</b>	
<a href="#"><u>Introduction to the papers of WG 4: Geometry teaching and learning</u></a>	542
<i>Philippe Richard, Athanasios Gagatsis, Sava Grozdev</i>	
<a href="#"><u>Innovative early teaching of isometries</u></a>	547
<i>Carlo Marchini, Paola Vighi</i>	
<a href="#"><u>Static and dynamic approach to forming: the concept of rotation</u></a>	558
<i>Edyta Jagoda, Ewa Swoboda</i>	
<a href="#"><u>Elementary students' transformational geometry abilities and cognitive style</u></a>	568
<i>Xenia Xistouri, Demetra Pitta-Pantazi</i>	
<a href="#"><u>Geometrical transformations as viewed by prospective teachers</u></a>	578
<i>Xhevdet Thaqi, Joaquin Giménez, Nuria Rosich</i>	
<a href="#"><u>Preservice teachers and the learning of geometry</u></a>	588
<i>Lina Fonseca, Elisabete Cunha</i>	
<a href="#"><u>Towards a comprehensive theoretical model of students' geometrical figure understanding and its relation with proof</u></a>	598
<i>Eleni Deliyianni, Athanasios Gagatsis, Annita Monoyiou, Paraskevi Michael, Panayiota Kalogirou, Alain Kuzniak</i>	
<a href="#"><u>Secondary Students behavior in proof tasks: understanding and the influence of the geometrical figure</u></a>	608
<i>Athanasios Gagatsis, Paraskevi Michael, Eleni Deliyianni, Annita Monoyiou, Alain Kuzniak</i>	
<a href="#"><u>Relations between geometrical paradigms and van Hiele levels</u></a>	618
<i>Annette Braconne-Michoux</i>	
<a href="#"><u>Geometry as propaedeutic to model building – a reflection on secondary school teachers' beliefs</u></a>	628
<i>Boris Girnat</i>	
<a href="#"><u>Geometric work at the end of compulsory education</u></a>	638
<i>Alain Kuzniak</i>	
<a href="#"><u>Language in the geometry classroom</u></a>	649
<i>Caroline Bulf, Anne-Cécile Mathé, Joris Mithalal</i>	
<a href="#"><u>Proofs and refutations in lower secondary school geometry</u></a>	660
<i>Taro Fujita, Keith Jones, Susumu Kunimune, Hiroyuki Kumakura, Shinichiro Matsumoto</i>	

<a href="#"><u>Identifying the structure of regular and semiregular solids – a comparative study between different forms of representation</u></a>	670
<i>Jürgen Steinwandell and Matthias Ludwig</i>	
<a href="#"><u>Generating shapes in a dynamic environment</u></a>	681
<i>Sue Forsythe</i>	
<a href="#"><u>Integrating number, algebra, and geometry with interactive geometry software</u></a>	691
<i>Kate Mackrell</i>	
 <b>WORKING GROUP 5</b>	
<a href="#"><u>Introduction to the papers of WG 5: Stochastic thinking</u></a>	701
<i>Dave Pratt</i>	
<a href="#"><u>Designing pedagogic opportunities for statistical thinking within inquiry based science</u></a>	705
<i>Janet Ainley, Tina Jarvis and Frankie McKeon</i>	
<a href="#"><u>Preservice primary school teachers' intuitive use of representations in uncertain situations</u></a>	715
<i>Chiara Andrà</i>	
<a href="#"><u>Relating graph semiotic complexity to graph comprehension in statistical graphs produced by prospective teachers</u></a>	725
<i>Pedro Arteaga, Carmen Batanero</i>	
<a href="#"><u>The challenges of teaching statistics in secondary vocational education</u></a>	735
<i>Arthur Bakker, Monica Wijer, Sanne Akkerman</i>	
<a href="#"><u>Children's emergent inferential reasoning about samples in an inquiry-based environment</u></a>	745
<i>Dani Ben-Zvi, Katie Makar, Arthur Bakker, Keren Aridor</i>	
<a href="#"><u>Investigating relative likelihood comparisons of multinomial, contextual sequences</u></a>	755
<i>Egan J Chernoff</i>	
<a href="#"><u>Prospective teachers' common and specialized knowledge in a probability task</u></a>	766
<i>J. Miguel Contreras, Carmen Batanero, Carmen Díaz, José A. Fernandes</i>	
<a href="#"><u>Investigating secondary teachers' statistical understandings</u></a>	776
<i>Helen M. Doerr, Bridgette Jacob</i>	
<a href="#"><u>Mental models of basic statistical concepts</u></a>	787
<i>Andreas Eichler, Markus Vogel</i>	
<a href="#"><u>Instructional representations in the teaching of statistical graphs</u></a>	797
<i>Maria Teresa González Astudillo, Jesús Enrique Pinto Sosa</i>	
<a href="#"><u>Assessing difficulties of conditional probability problems</u></a>	807
<i>M. Pedro Huerta, Fernando Cerdán, M<sup>a</sup> Ángeles Lonjedo, Patricia Edo</i>	
<a href="#"><u>Using a rasc partial credit model to analyze the responses of Brazilian undergraduate students to a statistics questionnaire</u></a>	818
<i>Verônica Y. Kataoka, Claudia Borim da Silva, Claudette Vendramini, Irene Cazorla</i>	
<a href="#"><u>Attitudes of teachers towards statistics: a preliminary study with Portuguese teachers</u></a>	828
<i>José Alexandre Martins, Maria Manuel Nascimento, Assumpta Estrada</i>	
<a href="#"><u>Developing an online community of teaching practitioners: a case study</u></a>	838
<i>Maria Meletiou-Mavrotheris, Efi Paparistodemou</i>	

<a href="#"><u>Risk taking and probabilistic thinking in preschoolers</u></a>	848
<i>Zoi Nikiforidou, Jenny Pange</i>	
<a href="#"><u>Influential aspects in middle school students' understanding of statistics variation</u></a>	853
<i>Antonio Orta, Ernesto Sánchez</i>	
<a href="#"><u>Carrying out, modelling and simulating random experiments in the classroom</u></a>	864
<i>Michel Henry, Bernard Parzysz</i>	
<a href="#"><u>Risk-based decision-making by mathematics and science teachers</u></a>	875
<i>Dave Pratt, Ralph Levinson, Phillip Ken, Cristina Yogui</i>	
<a href="#"><u>Individual pathways in the development of students' conceptions of patterns of chance</u></a>	885
<i>Susanne Predige, Susanne Schnell</i>	
<a href="#"><u>The role of relevant knowledge and cognitive ability in gambler fallacy</u></a>	896
<i>Caterina Primi &amp; Francesca Chiesi</i>	
<a href="#"><u>Connecting experimental and theoretical perspectives</u></a>	905
<i>Theodosia Prodromou</i>	
<a href="#"><u>Implementing a more coherent statistics curriculum</u></a>	915
<i>Anneke Verschut and Arthur Bakker</i>	
 <b>WORKING GROUP 6</b>	
<a href="#"><u>Introduction to the papers of WG 6: Application and modelling</u></a>	925
<i>Gabriele Kaiser, Susana Carreira, Thomas Lingefjärd, Geoff Wake</i>	
<a href="#"><u>Are integrated thinkers better able to intervene adaptively? – a case study in a mathematical modelling environment</u></a>	927
<i>Rita Borromeo Ferri and Werner Blum</i>	
<a href="#"><u>A modelling approach to developing an understanding of average rate of change</u></a>	937
<i>Helen M. Doerr and AnnMarie H. O'Neil</i>	
<a href="#"><u>An investigation of mathematical modelling in the Swedish national course test<sup>47</sup> in mathematics</u></a>	947
<i>Peter Frejd</i>	
<a href="#"><u>Modelling problems and digital tools in German centralised examinations</u></a>	957
<i>Gilbert Greefrath</i>	
<a href="#"><u>Analysis of the problem solving process and the use of representations while handling complex mathematical story problems in primary school</u></a>	964
<i>Johannes Groß, Katharina Hohn, Siebel Tellil, Renate Rasch, Wolfgang Schnotz</i>	
<a href="#"><u>Application and the identity of mathematics</u></a>	973
<i>Kasper Bjering Søby Jensen</i>	
<a href="#"><u>Students constructing modelling tasks to peers</u></a>	982
<i>Thomas Lingefjärd</i>	
<a href="#"><u>Modelling as a big idea in mathematics with significance for classroom instruction – how do pre-service teachers see it?</u></a>	990
<i>Hans-Stefan Siller, Sebastian Kuntze, Stephen Lerman, Christiane Vogl</i>	

<a href="#"><u>Modelling in an integrated mathematics and science curriculum: bridging the divide</u></a> <i>Geoff Wake</i>	1000
<a href="#"><u>Exploring the solving process of groups solving realistic Fermi problem from the perspective of the anthropological theory of didactics</u></a> <i>Jonas Bergman Årlebäck</i>	1010
<a href="#"><u>Hypotheses and assumptions by modeling – a case study</u></a> <i>Roxana Grigoraş</i>	1020
<b>WORKING GROUP 7</b>	
<a href="#"><u>Introduction to the papers of WG 7: Mathematical potential, creativity and talent</u></a> <i>Roza Leikin, Demetra Pitta-Pantazi, Florence Mihaela Singer, Andreas Ulovec</i>	1030
<a href="#"><u>Integrating theories in the promotion of critical thinking in mathematics classrooms</u></a> <i>Einav Aizikovitsh-Udi, Miriam Amit</i>	1034
<a href="#"><u>High attaining versus (highly) gifted pupils in mathematics: a theoretical concept and an empirical Survey</u></a> <i>Matthias Brandl</i>	1044
<a href="#"><u>Does mathematical creativity differentiate mathematical ability?</u></a> <i>Maria Kattou, Katerina Kontoyianni, Demetra Pitta-Pantazi, Constantinos Christou</i>	1056
<a href="#"><u>Unraveling mathematical giftedness</u></a> <i>Katerina Kontoyianni, Maria Kattou, Demetra Pitta-Pantazi, Constantinos Christou</i>	1066
<a href="#"><u>Questioning assumptions that limit the learning of fractions: The story of two fifth graders</u></a> <i>Andreas O. Kyriakides</i>	1075
<a href="#"><u>Mathematical creativity of 8th and 10th grade students.</u></a> <i>Roza Leikin, Yona Kloss</i>	1084
<a href="#"><u>Employing multiple-solution-tasks for the development of mathematical creativity: two comparative studies</u></a> <i>Roza Leikin, Anat Levav-Waynberg, Raisa Guberman</i>	1094
<a href="#"><u>Mathematical creativity in elementary school: Is it individual or collective?</u></a> <i>Esther Levenson</i>	1104
<a href="#"><u>Developing creative mathematical activities: method transfer and hypotheses' formulation</u></a> <i>Božena Maj</i>	1115
<a href="#"><u>Didactical vs. mathematical modelling of the notion competence in mathematics education: case of 9-10-year old pupils' problem solving</u></a> <i>Bernard Sarrazy, Jarmila Novotná</i>	1125
<a href="#"><u>Problem posing and modification as a criterion of mathematical creativity</u></a> <i>Florence Mihaela Singer, Ildikó Pelczer, Cristian Voica</i>	1133
<a href="#"><u>Creativity in three-dimensional geometry: How can an interactive 3d-geometry software environment enhance it?</u></a> <i>Paraskevi Sophocleous, Demetra Pitta-Pantazi</i>	1143
<a href="#"><u>Mathematical challenging tasks in elementary grades</u></a> <i>Isabel Vale, Teresa Pimentel</i>	1154

## WORKING GROUP 8

<a href="#"><u>Introduction to the papers of WG 8: Affect and mathematical thinking</u></a>	1165
<i>Marilena Pantziara, Pietro Di Martino, Kjersti Wæge, Wolfgang Schloeglmann</i>	
<a href="#"><u>Self – about using representations while solving geometrical problems</u></a>	1167
<i>Areti Panaoura, Eleni Deliyianni, Athanasios Gagatsis and Iliada Elia</i>	
<a href="#"><u>A comparative study of Norwegian and English secondary students' attitude towards mathematics</u></a>	1179
<i>Birgit Pepin</i>	
<a href="#"><u>The pupils' voice in creating a mathematically resilient community of learners</u></a>	1189
<i>Clare Lee, Sue Johnston-Wilder</i>	
<a href="#"><u>"You understand him, yet you don't understand me?!" – On learning mathematics as an interplay of mathematizing and identifying</u></a>	1199
<i>Einat Heyd-Metzuyanim, Anna Sfard</i>	
<a href="#"><u>An intervention on students' problem-solving beliefs</u></a>	1209
<i>Gabriel Stylianides, Andreas Stylianides</i>	
<a href="#"><u>A reversal theory perspective on disaffection using two examples</u></a>	1219
<i>Gareth Lewis</i>	
<a href="#"><u>Students' dispositions to study further mathematics in higher education: the effect of students' mathematics self-efficacy</u></a>	1229
<i>Irene Kleanthous, Julian Williams</i>	
<a href="#"><u>An examination of the connections between self discrepancies' and effort, enjoyment and grades in mathematics</u></a>	1239
<i>Laura Tuohilampi</i>	
<a href="#"><u>The impact of context and culture on the construction of personal meaning</u></a>	1249
<i>Maïke Vollstedt</i>	
<a href="#"><u>The effect of a teacher education program on affect: the case of Teresa and PFCM</u></a>	1259
<i>Maria Pezzia and Pietro Di Martino</i>	
<a href="#"><u>Fear of failure in mathematics. What are the sources?</u></a>	1269
<i>Marilena Pantziara and George Philippou</i>	

## WORKING GROUP 9

<a href="#"><u>Introduction to the papers of WG 9: Language and mathematics</u></a>	1279
<i>Maria Luiza Cestari</i>	
<a href="#"><u>Infinite and unbounded sets: a pragmatic perspective</u></a>	1282
<i>Cristina Bardelle</i>	
<a href="#"><u>Mathematical joint construction at elementary grade – a reconstruction of collaborative problem solving in dyads</u></a>	1292
<i>Birgit Brandt, Gyde Höck</i>	
<a href="#"><u>The concept of equivalence in a socially constructed language in a primary school class</u></a>	1302
<i>Cristina Coppola, Monica Mollo, Tiziana Pacelli</i>	



<a href="#"><u>Contention in mathematical discourse in small Groups in elementary school teaching</u></a>	1313
<i>Andrea Gellert</i>	
<a href="#"><u>Turn-taking in the mathematics classroom</u></a>	1325
<i>Jenni Ingram, Mary Briggs and Peter Johnston-Wilder</i>	
<a href="#"><u>Communicating experience of 3D space: Mathematical and Everyday Discourse</u></a>	1335
<i>Candia Morgan, Jehad Alshwaikh</i>	
<a href="#"><u>A working model for Improving Mathematics teaching and learning for bilingual students</u></a>	1346
<i>Máire Ní Ríordáin</i>	
<a href="#"><u>Revoicing in processes of collective mathematical argumentation among students</u></a>	1356
<i>Núria Planas, Laura Morera</i>	
<a href="#"><u>Epistemological and semiotic issues related to the concept of symmetry</u></a>	1366
<i>Frode Rønning</i>	
<a href="#"><u>Language as a shaping identity tool: The case of in-service Greek teachers</u></a>	1376
<i>Konstantinos Tatsis</i>	

## **WORKING GROUP 10**

<a href="#"><u>Introduction to the papers of WG 10:</u></a>	
<a href="#"><u>Discussing diversity in mathematics education from social, cultural and political perspectives</u></a>	1386
<i>Paola Valero, Sarah Crafter, Uwe Gellert, Núria Gorgorió</i>	
<a href="#"><u>Agency in mathematics education</u></a>	1389
<i>Annica Andersson, Eva Norén</i>	
<a href="#"><u>Interplays between context and students' achievement of agency</u></a>	1399
<i>Annica Andersson</i>	
<a href="#"><u>Climate change and mathematics education: making the invisible visible</u></a>	1409
<i>Richard Barwell, Christine Suurtamm</i>	
<a href="#"><u>Teachers discussions about parental use of implicit and explicit mathematics in the home</u></a>	1419
<i>Sarah Crafter, Guida de Abreu</i>	
<a href="#"><u>Students perceptions about the relevance of mathematics in an Ethiopian preparatory school</u></a>	1430
<i>Andualem Tamiru Gebremichael, Simon Goodchild and Olav Nygaard</i>	
<a href="#"><u>Differential access to vertical discourse</u></a>	
<a href="#"><u>– Managing diversity in a secondary mathematics classroom</u></a>	1440
<i>Uwe Gellert, Hauke Straehler-Pohl</i>	
<a href="#"><u>Mathematics teachers' social representations and identities made available to immigrant students</u></a>	1450
<i>Núria Gorgorió, Montserrat Prat</i>	
<a href="#"><u>Social functions of school mathematics</u></a>	1460
<i>David Kollosche</i>	
<a href="#"><u>Becoming disadvantaged: public discourse around national testing</u></a>	1470
<i>Troels Lange, Tamsin Meaney</i>	

<a href="#"><u>Parent-child interactions on primary school-related mathematics</u></a>	1481
<i>Richard Newton, Guida de Abreu</i>	
<a href="#"><u>Socio-cultural roots of the attribution process in family mathematics education</u></a>	1491
<i>Javier Díez-Palomar, Sandra Torras Ortin</i>	
<a href="#"><u>Between school and company: a field of tension?</u></a>	1501
<i>Toril Eskeland Rangnes</i>	
<a href="#"><u>Ethnomathematics in European Context</u></a>	1511
<i>Charoula Stathopoulou, Karen Françoi, Darlinda Moreira</i>	
<a href="#"><u>Doctoral programs in mathematics education: current status and future pathways for Turkey</u></a>	1521
<i>Behiye Ubuz, Erdinç Çakıroğlu, Ayhan Kürşat Erbaş</i>	
<a href="#"><u>Connecting the notion of foreground in critical mathematics education with the theory of habitus</u></a>	1531
<i>Tine Wedege</i>	

## **WORKING GROUP 11**

<a href="#"><u>Introduction to the papers of WG 11: Comparative studies in mathematics education</u></a>	1541
<i>Eva Jablonka, Paul Andrews</i>	
<a href="#"><u>The case of calculus: Comparative look at task representation in textbooks</u></a>	1545
<i>Emmanuel Adu-tutu Bofah, Markku Hannula</i>	
<a href="#"><u>The teaching of linear equations: Comparing effective teachers from three high achieving European countries</u></a>	1555
<i>Paul Andrews</i>	
<a href="#"><u>Exploratory data analysis of a european teacher training course on modelling</u></a>	1565
<i>Richard Cabassut, Jean-Paul Villette</i>	
<a href="#"><u>Comparison of item performance in a Norwegian study using U.S. developed mathematical knowledge for teaching measures</u></a>	1575
<i>Arne Jakobsen, Janne Fauskanger, Reidar Mosvold and Raymond Bjuland</i>	
<a href="#"><u>Belgian and Turkish pre-service primary school mathematics teachers' metaphorical thinking about mathematics</u></a>	1585
<i>Çiğdem Kiliç</i>	
<a href="#"><u>Problem solving and open problems in teachers' training in the French and Mexican modes</u></a>	1594
<i>Alain Kuzniak, Bernard Parzysz, Manuel Santos-Trigo, Laurent Vivier</i>	
<a href="#"><u>What kind of teaching in different types of classes?</u></a>	1605
<i>Céline Maréchal</i>	
<a href="#"><u>Comparing the construction of mathematical knowledge between low-achieving and high-achieving students – a case study</u></a>	1614
<i>Ingolf Schäfer, Alexandra Winkler</i>	
<a href="#"><u>Conceptual metaphors and “grundvorstellungen”: a case of convergence?</u></a>	1625
<i>Jorge Soto-Andrade, Pamela Reyes-Santander</i>	

## WORKING GROUP 12

<a href="#"><u>Introduction to the papers of WG 12: History in mathematics education</u></a>	1636
<i>Uffe Thomas Jankvist, Snezana Lawrence, Constantinos Tzanakis, Jan van Maanen</i>	
<a href="#"><u>Uses of history in mathematics education: development of learning strategies and historical awareness</u></a>	1640
<i>Tinne Hoff Kjeldsen</i>	
<a href="#"><u>Classifying the arguments and methodological schemes for integrating history in mathematics education</u></a>	1650
<i>Constantinos Tzanakis, Yannis Thomaidis</i>	
<a href="#"><u>The development of attitudes and beliefs questionnaire towards using history of mathematics in mathematics education</u></a>	1660
<i>Mustafa Alpaslan, Mine Isiksal, Cigdem Haser</i>	
<a href="#"><u>Implementing ‘modern math’ in iceland –Informing parents and the public</u></a>	1670
<i>Kristín Bjarnadóttir</i>	
<a href="#"><u>Voices from the field: incorporating history of mathematics in Teaching</u></a>	1680
<i>Kathleen M. Clark</i>	
<a href="#"><u>Designing teaching modules on the history, application, and philosophy of mathematics</u></a>	1690
<i>Uffe Thomas Jankvist</i>	
<a href="#"><u>Uses of history in mathematics education: development of learning strategies and historical awareness</u></a>	1700
<i>Tinne Hoff Kjeldsen</i>	
<a href="#"><u>Establishing the ‘meter’ as citizens of French National assembly during French Revolution</u></a>	1710
<i>Panayota Kotarinou, Charoula Stathopoulou, Anna Chronaki</i>	
<a href="#"><u>Lessons from early 17th century for current mathematics curriculum design</u></a>	1720
<i>Jenneke Krüger</i>	
<a href="#"><u>How much meaning can we construct around geometric constructions?</u></a>	1730
<i>Snezana Lawrence and Peter Ransom</i>	
<a href="#"><u>Identity of mathematics educators. The Portuguese case (1981 - 1990)</u></a>	1740
<i>José Manuel Matos</i>	
<a href="#"><u>The Teaching of Mathematics in Portugal in the 18th century – The creation of the 1st faculty of Mathematics in the world</u></a>	1750
<i>Catarina Motaa, Maria Elfrida Ralhab, Maria Fernanda Estrada</i>	
<a href="#"><u>Using students’ journals to explore their affective engagement in a module on the history of mathematics</u></a>	1760
<i>Maurice OReilly</i>	
<a href="#"><u>A cross-curricular approach using history in the mathematics classroom with students aged 11-16</u></a>	1770
<i>Peter Ransom</i>	

## WORKING GROUP 13

<a href="#"><u>Introduction to the papers of WG 13: Early years mathematics</u></a>	1780
<i>Ingvald_Erfjord, Ema Mamede, Götz Krummheuer</i>	
<a href="#"><u>Issues on children's ideas of fractions when quotient interpretation is used</u></a>	1782
<i>Ema Mamede, Manuela Oliveira</i>	
<a href="#"><u>Finger-symbol-sets and multi-touch for a better understanding of numbers and operations</u></a>	1792
<i>Silke Ladel, Ulrich Kortenkamp</i>	
<a href="#"><u>Similar but different - investigating the use of MKT in a Norwegian kindergarten setting</u></a>	1802
<i>Reidar Mosvold, Raymond Bjuland, Janne Fauskanger, Arne Jakobsen</i>	
<a href="#"><u>Kindergartners' perspective taking abilities</u></a>	1812
<i>Aaltje Berendina Aaten, Marja van den Heuvel-Panhuizen, Iliada Elia</i>	
<a href="#"><u>The linguistic coding of Mathematical Supports</u></a>	1822
<i>Anna-Marietha Hümmel</i>	
<a href="#"><u>Analysis of mathematical solutions of 7 year old pupils when solving an arithmetic problem on distribution</u></a>	1832
<i>Edelmira Badillo, Vicenç Font, Mequè Edo, Núria Planas</i>	
<a href="#"><u>Kindergartners' use of gestures in the generation and communication of spatial thinking</u></a>	1842
<i>Iliada Elia, Athanasios Gagatsis, Paraskevi Michael, Alexia Georgiou, Marja van den Heuvel-Panhuizen</i>	
<a href="#"><u>How do children's classification appear in free play? A case study</u></a>	1852
<i>Vigdis Flottorp</i>	
<a href="#"><u>ERSTMAL-Fast (early Steps in Mathematics Learning - Family Study)</u></a>	1861
<i>Ergi Acar</i>	
<a href="#"><u>Changing mathematical practice of kindergarten teachers.</u></a>	
<a href="#"><u>Co-learning in a developmental research project</u></a>	1872
<i>Per Sigurd Hundeland, Martin Carlsen and Ingvald Erfjord</i>	
<a href="#"><u>"Looking for tricks": a natural strategy. early forerunner of algebraic thinking</u></a>	1882
<i>Maria Mellone</i>	
<a href="#"><u>Can you take half? Kindergarten children's responses</u></a>	1891
<i>Dina Tirosh, Pessia Tsamir, Michal Tabach, Esther Levenson, and Ruthi Barkai</i>	
<a href="#"><u>Investigating geometric knowledge and self-efficacy among abused and neglected kindergarten children</u></a>	1902
<i>Pessia Tsamir, Dina Tirosh, Esther Levenson, Michal Tabach, and Ruthi Barkai</i>	
<a href="#"><u>Playing and learning in early mathematics education – Modelling a Complex relationship</u></a>	1912
<i>Stephanie Schuler</i>	

## WORKING GROUP 14

<a href="#"><u>Introduction to the papers of WG 14: University mathematics education</u></a>	1923
<i>Elena Nardi, Alejandro S. González-Martin, Ghislaine Gueudet, Paola Iannone, Carl Winsl�w</i>	
<a href="#"><u>Why abstract algebra for pre-service primary school teachers</u></a>	1928
<i>Eleni Agathocleous</i>	
<a href="#"><u>‘Applicationism’ as the dominant epistemology at university level</u></a>	1938
<i>Berta Barquero, Marianna Bosch, Josep Gasc�n</i>	
<a href="#"><u>Designing alternatice undergraduate delivery: oil and massage</u></a>	1949
<i>Bill Barton</i>	
<a href="#"><u>Why do Students go to Lectures?</u></a>	1960
<i>Christer Bergsten</i>	
<a href="#"><u>Designing and evaluating blended learning bridging courses in mathematics</u></a>	1971
<i>Biehler, R.; Fischer, P.R; Hochmuth, R.; Wassong, Th.</i>	
<a href="#"><u>Employing potentialities and limitations of electronic environments: The case of derivative</u></a>	1981
<i>Irene Biza, Victor Giraldo</i>	
<a href="#"><u>The changing profile of third level service mathematics in Ireland (1997-2010)</u></a>	1992
<i>Fiona Faulkner, Ailish Hannigan, Olivia Gill</i>	
<a href="#"><u>Using CAS based work to ease the transition from calculus to real analysis</u></a>	2002
<i>Erika Gy�ngy�si, Jan Philip Solovej, Carl Winsl�w</i>	
<a href="#"><u>Undergraduate students’ use of deductive arguments to solve “prove that...” tasks</u></a>	2012
<i>Paola Iannone, Matthew Inglis</i>	
<a href="#"><u>How we teach mathematics: discourses on/in university teaching</u></a>	2022
<i>Barbara Jaworski and Janette Matthews</i>	
<a href="#"><u>What affects retention of core calculus concepts among university students? A study of different teaching approaches in Croatia and Denmark</u></a>	2033
<i>Ljerka Juki�, Bettina Dahl (Soendergaard)</i>	
<a href="#"><u>University students linking limits, derivatives, integrals and continuity</u></a>	2043
<i>Kristina Juter</i>	
<a href="#"><u>‘Driving noticing’ yet ‘risking precision’: university mathematicians’ pedagogical perspectives on verbalisation in mathematics</u></a>	2053
<i>Elena Nardi</i>	
<a href="#"><u>Threshold concepts: a framework for research in University mathematics education</u></a>	2063
<i>Kerstin Pettersson</i>	
<a href="#"><u>Challenges with visualization: the concept of integral with undergraduate students</u></a>	2073
<i>Blanca Souto Rubio, In�s Maria G�mez-Chac�n</i>	
<a href="#"><u>The secondary-tertiary transition: a clash between two mathematical discourses</u></a>	2083
<i>Erika Stadler</i>	

<a href="#"><u>Students' conceptions of functions at the transition between secondary school and university</u></a>	2093
<i>Fabrice Vandebrouck</i>	
<a href="#"><u>Discourses of functions – university mathematics teaching through a commognitive lens</u></a>	2103
<i>Olov Viirman</i>	
<a href="#"><u>Secondary-tertiary transition and evolutions of didactic contract: the example of duality in linear algebra</u></a>	2113
<i>Martine De Vleeschouwer, Ghislaine Gueudet</i>	
<a href="#"><u>A didactic survey of the Main characteristics of Lagrange's Theorem in mathematics and in economics</u></a>	2123
<i>Sebastian Xhonneux, Valérie Henry</i>	
<a href="#"><u>A questionnaire for surveying mathematics self-efficacy expectations of prospective teachers</u></a>	2134
<i>Marc Zimmermann, Christine Bescherer, Christian Spannagel</i>	
<b>WORKING GROUP 15</b>	
<a href="#"><u>Introduction to the papers of WG 15: Technologies and resources in mathematics education</u></a>	2144
<i>Jana Trgalová, Anne Berit Fuglestad, Mirko Maracci, Hans-Georg Weigand</i>	
<a href="#"><u>Mathematics student teachers' pedagogical use of technologies – different taxonomies in the classroom</u></a>	2148
<i>Nélia Amado</i>	
<a href="#"><u>Symbolic generalization in a computer intensive environment: the case of Amy</u></a>	2158
<i>Michal Tabach</i>	
<a href="#"><u>Teachers' and students' first experience of a curriculum material with TI-Nspire technology</u></a>	2168
<i>Per-Eskil Persson</i>	
<a href="#"><u>Teachers transforming resources into orchestrations</u></a>	2178
<i>Paul Drijvers</i>	
<a href="#"><u>The use of mathematics software in university mathematics teaching</u></a>	2188
<i>Iris Attorps, Kjell Björk, Mirko Radic</i>	
<a href="#"><u>Calculators as digital resources</u></a>	2198
<i>Gilles Aldon</i>	
<a href="#"><u>Transitions between micro-contexts of mathematical practices</u></a>	2208
<i>Vasilis Tsitsos, Charoula Stathopoulou</i>	
<a href="#"><u>Function concept and transformations of functions: the role of the graphic calculator</u></a>	2218
<i>Madalena Consciência, Hélia Oliveira</i>	
<a href="#"><u>A survey of technology use: the rise of interactive whiteboards and the mymaths website</u></a>	2228
<i>Nicola Bretscher</i>	
<a href="#"><u>Graphic calculator use in primary schools: an example of an instrumental approach</u></a>	2238
<i>Per Storfossen</i>	
<a href="#"><u>An online games as a learning environment for early algebraic problem solving by upper primary school students</u></a>	2248
<i>Angeliki Kolovou, Marja van den Heuvel-Panhuizen</i>	

<a href="#"><u>Framing a problem solving approach based on the use of computational tools to develop mathematical thinking</u></a>	2258
<i>Manuel Santos-Trigo, Matías Camacho-Machín</i>	
<a href="#"><u>Analysing teachers' classroom practice when new technologies are in use</u></a>	2268
<i>Mary Genevieve Billington</i>	
<a href="#"><u>Implementation of a multi-touch environment supporting finger symbol sets</u></a>	2278
<i>Silke Ladel, Ulrich Kortenkamp</i>	
<a href="#"><u>Researching technological, pedagogical and mathematical undergraduate primary teachers' knowledge (TpaCK)</u></a>	2288
<i>Spyros Doukakis, Maria Chionidou-Moskofoglou, Dimitrios Zibidis</i>	
<a href="#"><u>The co-construction of a mathematical and a didactical instrument</u></a>	2298
<i>Mariam Haspekian</i>	
<a href="#"><u>Working with teachers: collaboration in a community around innovative software</u></a>	2308
<i>Jean-Baptiste Lagrange</i>	
<a href="#"><u>Extending the technology acceptance model to assess secondary school teachers' intention to use cabri in geometry teaching</u></a>	2318
<i>Marios Pittalis &amp; Constantinos Christou</i>	
<a href="#"><u>Challenges teachers face with integrating ICT with an inquiry approach in mathematics</u></a>	2328
<i>Anne Berit Fuglestad</i>	
<a href="#"><u>Technologies and tools in teaching mathematics to visually impaired students</u></a>	2339
<i>Iveta Kohanová</i>	
<a href="#"><u>A study on mathematics teachers' use of textbooks in instructional process</u></a>	2349
<i>Meriç Özgeldi, Erdinç Çakıroğlu</i>	
<a href="#"><u>Collective design of an online math textbook: when Individual and collective documentation works meet</u></a>	2356
<i>Hussein Sabra, Luc Trouche</i>	
<a href="#"><u>Developing a competence model for working with symbolic calculators</u></a>	2367
<i>Hans-Georg Weigand</i>	

## **WORKING GROUP 16**

<a href="#"><u>Introduction to the papers of WG 16:</u></a>	
<a href="#"><u>Different theoretical perspectives and approaches in research in mathematics education</u></a>	2377
<i>Ivy Kidron, Angelika Bikner-Ahsbals, John Monaghan, Luis Radford, Gérard Sensevy</i>	
<a href="#"><u>Research praxeologies and networking theories</u></a>	2381
<i>M. Artigue, M. Bosch, J. Gascón</i>	
<a href="#"><u>Exploring fragmentation in mathematics education research</u></a>	2391
<i>Ayshea Craig</i>	
<a href="#"><u>Complementing and integrating theoretical tools: a case study concerning poor learners</u></a>	2401
<i>Nadia Douek</i>	
<a href="#"><u>Mathematical objects through the lens of three different theoretical perspectives</u></a>	2411
<i>Vicenç Font, Uldarico Malaspina, Joaquin Giménez, Miguel R. Wilhelmi</i>	

<a href="#"><u>Using different sociocultural perspectives in mathematics teaching developmental research</u></a>	2421
<i>Simon Goodchild</i>	
<a href="#"><u>The possibility of a perspectival research on mathematics learning</u></a>	2431
<i>Tim Jay</i>	
<a href="#"><u>Recovering mathematical awareness by linguistic analysis of variable substitution</u></a>	2441
<i>Kaenders, R.H., Kvasz, L., Weiss-Pidstrygach, Y.</i>	
<a href="#"><u>How a general epistemic need leads to a need for a new construct</u></a>	
<a href="#"><u>A case of networking two theoretical approaches</u></a>	2451
<i>Ivy Kidrona, Angelika Bikner-Ahsbahs, Tommy Dreyfus</i>	
<a href="#"><u>Mathematics learning through the lenses of cultural historical activity theory and the theory of knowledge objectification</u></a>	2462
<i>Lionel N. LaCroix</i>	
<a href="#"><u>Using practical epistemology analysis to study the teacher's and students' joint action in the mathematics classroom</u></a>	2472
<i>Florence Ligozat, Per-Olof Wickman, Karim Hamza</i>	
<a href="#"><u>Discerning in and between theories in mathematics education</u></a>	2482
<i>John Mason</i>	
<a href="#"><u>Theoretical genesis of an informal meta-theory to develop a way of talking about mathematics and science education and to connect European and North American literature</u></a>	2493
<i>John Monaghan</i>	
<a href="#"><u>Meaning of mathematical objects: a comparison between semiotic perspectives</u></a>	2503
<i>Giorgio Santi</i>	
<a href="#"><u>Modeling external representations as mediators of meaning in the mathematics classroom</u></a>	2513
<i>Håkan Sollervall</i>	
<a href="#"><u>Combining theories to analyze classroom discourse: a method to study learning process</u></a>	2524
<i>Michal Tabach Talli Nachlieli</i>	
<b>WORKING GROUP 17</b>	
<a href="#"><u>Introduction to the papers of WG 17:</u></a>	
<a href="#"><u>From a study of teaching practices to issues in teacher education</u></a>	2533
<i>Leonor Santos, Claire Berg, Laurinda Brown, Nicolina Malara, Despina Potari, Fay Turner</i>	
<a href="#"><u>Primary school teachers' practices using a same manual written by didactician</u></a>	2539
<i>Sara Arditì</i>	
<a href="#"><u>A study of a problem solving oriented lesson structure in mathematics in Japan</u></a>	2549
<i>Yukiko Asami-Johansson</i>	
<a href="#"><u>Lesson study as a process for professional development: working with teachers to effect significant and sustained changes in practice</u></a>	2559
<i>Jenni Back, Marie Joubert</i>	
<a href="#"><u>An attempt at defining teachers' mathematics through research on mathematics at work</u></a>	2569
<i>Nadine Bednarz, Jérôme Proulx</i>	



<a href="#"><u>Adopting an inquiry approach to teaching practice: the case of a primary school teacher</u></a> <i>Claire Vaugelade Berg</i>	2580
<a href="#"><u>Mathematical investigations in the classroom: a context for the development of professional knowledge of mathematics teachers</u></a> <i>Ana Paula Canavarro, Mónica Patrício</i>	2590
<a href="#"><u>Why do some french teachers propose «problèmes ouverts» in mathematics to their pupils in primary school ?</u></a> <i>Christine Choquet</i>	2600
<a href="#"><u>The need to make ‘boundary objects’ meaningful: a learning outcome from lesson s tudy research</u></a> <i>Dolores Corcoran</i>	2609
<a href="#"><u>Analysis of the teacher’s role in an approach to algebra as a tool for thinking: problems pointed out during laboratorial activities with perspective teachers</u></a> <i>Annalisa Cusi, Nicolina A. Malara</i>	2619
<a href="#"><u>The impact of teaching mental calculation strategies to primary PGCE students</u></a> <i>Sue Davis</i>	2630
<a href="#"><u>Re-defining HCK to approach transition</u></a> <i>Sáinz Fernández, Lourdes Figueiras, Jordi Deulofeu, Mario Martínez</i>	2640
<a href="#"><u>Conceptions and practices of mathematical communication</u></a> <i>António Guerreir, Lurdes Serrazina</i>	2650
<a href="#"><u>Lesson study in Teacher Education: a tool to establish a learning community</u></a> <i>Guðný Helga Gunnarsdótti, Guðbjörg Pálsdóttir</i>	2660
<a href="#"><u>The use of the empty number line to develop a programme of mental mathematics for primary trainee teachers</u></a> <i>Gwen Ineson</i>	2670
<a href="#"><u>Readings of the mathematical meaning shaped in the classroom: exploiting different lenses</u></a> <i>Maria Kaldrimidou, Haralambos Sakonidis, Marianna Tzekaki</i>	2680
<a href="#"><u>The nature of preservice teachers’ pedagogical content knowledge</u></a> <i>Hulya Kilic</i>	2690
<a href="#"><u>Literacy in mathematics – a challenge for teachers in their work with pupils</u></a> <i>Bodil Kleve</i>	2697
<a href="#"><u>Helping in-service teachers analyze and construct mathematical tasks according to their cognitive demand</u></a> <i>Eugenia Koleza, Christos Markopoulos, Stella Nika</i>	2707
<a href="#"><u>Professional knowledge related to Big Ideas in Mathematics – an empirical study with pre-service teachers</u></a> <i>Sebastian Kuntze, Stephen Lerman, Bernard Murphy, Elke Kurz-Milcke, Hans-Stefan Siller, Peter Winbourne</i>	2717
<a href="#"><u>The role of video-based experiences in the teacher education of pre-service mathematics teachers</u></a> <i>Miriam Listo, Olivia Gill</i>	2727

<a href="#"><u>Multicommented transcripts methodology as an educational tool for teachers involved in constructive didactical projects in early algebra</u></a>	2737
<i>Nicolina A. Malara, Giancarlo Navarra</i>	
<a href="#"><u>Laboratory activities in teacher training</u></a>	2746
<i>Francesca Martignone</i>	
<a href="#"><u>Plannig teaching activity within a continuous training program</u></a>	2756
<i>Cristina Martin, Leonor Santos</i>	
<a href="#"><u>Mathematics problem solving professional learning through collaborative action research</u></a>	2766
<i>Joyce Mgombelo, Kamini Jaipal-Jamani</i>	
<a href="#"><u>A study of the differences between the surface and the deep structures of math lessons</u></a>	2777
<i>Edyta Nowińska</i>	
<a href="#"><u>Teachers managing the curriculum in the context of the mathematics' subject group</u></a>	2787
<i>Cláudia Canha Nunes, João Pedro da Ponte</i>	
<a href="#"><u>Prospective mathematics teachers' noticing of classroom practice through critical events</u></a>	2798
<i>Despina Potari, Giorgos Psychari, Eirini Koulets, Maria Diamantis</i>	
<a href="#"><u>Curricular changes in preparation of future teachers – financial mathematics course</u></a>	2808
<i>Michaela Regecová, Mária Slavičková</i>	
<a href="#"><u>Knowing mathematics as a teacher</u></a>	2817
<i>C. Miguel Ribeiro; José Carrillo</i>	
<a href="#"><u>Secondary mathematics teachers' content knowledge: the case of Heidi</u></a>	2827
<i>Tim Rowland, Libby Jared, Anne Thwaites</i>	
<a href="#"><u>Pre-service teacers learning to assess mathematical competencies</u></a>	2838
<i>Norma Rubio, Vicenç Font, Joaquim Giménez, Uldarico Malaspina</i>	
<a href="#"><u>Moving beyond an evaluative teaching mode: the case of diana</u></a>	2848
<i>Rosa Antónia Tomás Ferreira</i>	
<a href="#"><u>Preservice elementary teachers' geometry content knowledge in methods course</u></a>	2858
<i>Fatma Aslan-Tutak</i>	
<a href="#"><u>How to promote sustainable professional development?</u></a>	2868
<i>Stefan Zehetmeier, Konrad Krainer</i>	
<a href="#"><u>Concepts from mathematics education research as a trigger for mathematics teachers' reflections</u></a>	2878
<i>Mario Sánchez</i>	
<a href="#"><u>Teacher competences prerequisite to natural differentiation</u></a>	2888
<i>Marie Tichá, Alena Hošpesová</i>	
<a href="#"><u>Differences in the propositional knowledge and the knowledge in practice of beginning primary school teachers</u></a>	2898
<i>Fay Turner</i>	

## POSTERS COMMUNICATIONS

### Posters WG02

[Didactical environments “stepping” and “staircase”](#) 2911

*Nad’a Stehliková, Milan Hejný, Darina Jirotková*

[Understanding the infiniti sets of numbers](#) 2913

*Cristian Voica, Florence Mihaela Singer*

### Posters WG03

[Exploring patterns and algebraic thinking](#) 2915

*António Borralho, Elsa Barbosa*

[Prospective teachers doing modeling activities and interpreting students work](#) 2917

*Neusa Branco*

[Differentiated learning routes for school algebra using online database systems](#) 2919

*Julia Pilet*

[The teaching and learning of parameters in families of functions](#) 2921

*Manuel Saraiva*

### Posters WG05

[Learning risk in socio-scientific context](#) 2923

*Hasan Akyuzlu*

[Future elementary and kindergarten teachers’ knowledge of statistics and of its didactics](#) 2925

*Raquel Santos*

[Comparing attitudes towards mathematics and statistics of k-10 students:  
preliminary results](#) 2927

*Assumpta Estrada, Ana Serradó*

### Posters WG06

[Teacing differential equations with modelling](#) 2929

*Guerrero-Ortiz, C., Camacho-Machín, M.*

### Poster WG08

[Beliefs of the usefulness of mathematics and mathematics self-beliefs as  
important factors for mathematics attitudes](#) 2931

*Peter Vankus*

### Posters WG09

[On the meaning of multiplication for different sets of numbers in a context of visualization](#) 2933

*Raquel Barrera*

[Gesture and visual-spatial thinking](#) 2935

*Conceição Costa, José Manuel Matos*

<a href="#"><u>The evolution of school mathematics discourse as seen through the lens of GCSE examinations</u></a>	2937
<i>Candia Morgan, Anna Sfard</i>	
<a href="#"><u>Self-regulation of students in Mathematics and oral communication in classroom</u></a>	2939
<i>Sílvia Semana, Leonor Santos</i>	
<a href="#"><u>Teachers' endorsed and enacted narratives to promote mathematical communication</u></a>	2941
<i>Marie Bergholm</i>	
<a href="#"><u>Communication – a/the key to mathematics</u></a>	2943
<i>Birgit Gustafsson</i>	
<b>Posters WG10</b>	
<a href="#"><u>Family mathematics involvement: drawing from sociological point of view</u></a>	2945
<i>Javier Díez-Palomar, Sandra Torras-Ortín</i>	
<a href="#"><u>Learning mathematics: thoughts and interpretations of students with foreign backgrounds</u></a>	2947
<i>Petra Svensson</i>	
<a href="#"><u>The emergence of cultural mathematics: an ethnomathematical approach in the context of classroom</u></a>	2949
<i>Joana Latas, Darlinda Moreira</i>	
<a href="#"><u>Parental involvement in children's achievement: An exploratory study with French 2nd graders in Mathematics</u></a>	2951
<i>Ana Lobo de Mesquita</i>	
<b>Posters WG11</b>	
<a href="#"><u>Students' self-regulation, self-efficacy and mathematical competence in OECD's PISA</u></a>	2953
<i>David Pepper</i>	
<a href="#"><u>Contrasting prospective teacher education and student teaching in England and Slovakia</u></a>	2955
<i>Ján Šunderlík, Soňa Čeretková</i>	
<b>Posters WG12</b>	
<a href="#"><u>Developing a modern mathematics pedagogical content knowledge: the case of telescola in Portugal in the middle 1960'</u></a>	2957
<i>Mária Correia de Almeida, José Manuel Matos</i>	
<a href="#"><u>A study on the fundamental concept of 'measure' and its history</u></a>	2959
<i>Ana Amaral; Alexandra Gomes; Elfrida Ralha</i>	
<a href="#"><u>The Project of Modernization of the Mathematical Initiation in Primary School as curriculum development (1965-1973)</u></a>	2961
<i>Rui Candeias</i>	
<a href="#"><u>Who can understand the gifted students?</u></a>	
<a href="#"><u>A lesson plan based on history to enhance the gifted students' learning</u></a>	2963
<i>Ersin İlhan</i>	
<a href="#"><u>Teacher training at Pedro Nunes normal secondary school (1956-1969)</u></a>	2965
<i>José Manuel Matos, Teresa Maria Monteiro</i>	

### Posters WG13

[The pedagogical consequences of a laissez Faire individualistic society](#) 2967  
*Judy Sayers*

[ICT supported learning of mathematics in kindergarten](#) 2969  
*Martin Carlsen, Per Sigurd Hundeland, Ingvald Erfjord*

### Posters WG14

[The design and implementation of mathematical tasks to promote advanced mathematical thinking](#) 2971  
*Sinéad Breen, Ann O'Shea*

[Critical multicultural instruction for undergraduate mathematical thinking courses](#) 2973  
*Irene M. Duranczyk*

[Problem-based learning as a methodology of studying the didactic knowledge of derivatives in undergraduate courses in mathematics for economists](#) 2975  
*Moreno, M.M.; García, L., Azcárate, C.*

[Some meanings of the derivative of a function](#) 2977  
*Perdomo-Díaza, J., Camacho-Machina, M. and Santos-Trigob, M.*

[Transition secondary-tertiary level education via Math-Bridge](#) 2979  
*Juliana Zsidó, Viviane Durand-Guerrier*

### Posters WG15

[Chase for a bullet – use of ICT for developing students' functional thinking](#) 2981  
*Antonín Jančařík, Jarmila Novotná, Alena Pelantová*

[The challenge of developing a European course for supporting teachers' use ICT](#) 2983  
*Michèle Artigue, Claire Cazes, Françoise Héroult, Gilles Marbeuf, Fabrice Vandebrouck*

[Remedial scenarios for online and blended-learning bridging courses](#) 2985  
*Rolf Biehler, Pascal R. Fischer, Reinhard Hochmuth, Thomas Wassong*

[I2geo.net – a platform for sharing dynamic geometry resources all over Europe](#) 2987  
*Jana Trgalova, Ulrich Kortenkamp, Ana Paula Jahn, Paul Libbrecht, Christian Mercat  
Tomas Recio, Sophie Soury-Lavergne*

[Mobile technology in mathematics courses for teacher students](#) 2989  
*Iveta Kohanová*

[Pepimep Project: online database system and differentiated learning routes for school algebra](#) 2991  
*Françoise Chenevotot, Brigitte Grugeon-Allys*

[Extending the mathematics textbooks analysis: questions of language and ICT](#) 2993  
*Carlos Alberto Batista Carvalho, José Manuel Leonardo de Freitas*

[Intensive use of ICT in pre-service primary teachers' professional training in mathematics: Impact on teaching practices](#) 2995  
*Jean Baptiste Lagrange, Alexandre Becart*

## Posters WG16

- [Theoretical framework to analyse the problem solving process while handling complex mathematical story problems in primary school](#) 2997  
*Johannes Groß, Katharina Hohn, Sibel Telli, Renate Rasch, Wolfgang Schnotz*
- [The complexity of advanced mathematical thinking at the non-university level](#) 2999  
*Miguel Silva, António Domingos*
- [Networking theories: the ‘kom project’ and ‘adding it up’ through the lens of a learning situation](#) 3001  
*Yvonne Liljekvistand Jorryt van Bommel*
- [A successful combination of COP and CHAT to understand prospective primary mathematics teachers’ learning?](#) 3003  
*Kicki Skog*
- [Connecting theories to think of plane geometry teaching from elementary to middle school](#) 3005  
*Marie-Jeanne Perrin-Glorian*

## Posters WG17

- [Didactical analysis and citizenship with prospective mathematics teachers](#) 3007  
*Yuly M. Vanegas M., Joaquín Giménez, Vicenç Font*
- [How to teach mathematical knowledge for teaching](#) 3009  
*Jorryt van Bommel*
- [Analysing exams mathematical questions](#) 3011  
*Mário José Miranda Ceia*
- [The practices of prospective teachers in South African and Canadian mathematical literacy teacher education programs: What works and what does not?](#) 3013  
*Joany Fransman, Joyce Mgombel, Marthie Van der Walt*
- [Teachers’ use of graphing calculators in high school mathematics classroom – the influence of teachers’ professional knowledge](#) 3015  
*Helena Rocha*
- [Deeper mathematical understanding through teacher and teaching assistant collaboration](#) 3017  
*Paul Spencer, Julie-Ann Edwards*

# **PROCEEDINGS OF THE SEVENTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION: GENERAL INTRODUCTION**

This volume contains the Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (ERME), which took place 9-13 February 2011, at Rzeszów in Poland. ERME came into being at its first congress in Osnabrueck, Germany, in 1998. Thereafter, CERME congresses have taken place every two years since CERME2 in 2001.

CERME places great emphasis on promoting participation by all who attend the congress, with a deliberate shift from the succession of parallel research report presentations that characterise most international scientific conferences. Thus, the vast majority of CERME time is devoted to discussion and debate within thematic Working Groups (WGs), facilitated by a team of leaders for each WG.

These leaders give a great deal of their time in organising the peer review for their WG before the congress, the working sessions at the congress itself, and in coordinating and editing those parts of these proceedings related to their WG. CERME participants must commit themselves to membership of just one such Group, and to 6 or 7 sessions of 90-120 minutes *working* with the Group.

The number of WGs increased to 17 at CERME7, and the number of participants in each was around 25-30 on average, including about 4 WG leaders. The WG themes were as follows:

- WG1: argumentation and proof;
- WG2: teaching and learning of number systems and arithmetic;
- WG3: algebraic thinking;
- WG4: geometry teaching and learning;
- WG5: stochastic thinking;
- WG6: applications and modelling;
- WG7: mathematical potential, creativity and talent;
- WG8: affect and mathematical thinking;
- WG9: mathematics and language;
- WG10: diversity and mathematics education;
- WG11: comparative studies in mathematics education;
- WG12: history in mathematics education;
- WG13: early years mathematics;
- WG14: university mathematics education;

## Introduction

WG15: technologies and resources in mathematics education;

WG16: different theoretical perspectives and approaches in research in mathematics education;

WG17: from a study of teaching practices to issues in teacher education.

Research paper and poster proposals were submitted to a *Group*, and were then subject to peer review within the WG and a decision about acceptance by the WG leaders. This structure results in significantly devolved and distributed responsibility for the organisation of the congress, and – hopefully – a sense of belonging to their chosen WG, for all participants. All accepted papers were posted on the CERME website before the congress, and each participant was expected to read all of those related to their WG – typically 15 or more – *in advance* of the congress.

At the congress itself, it was then possible to devote *all* of the time allotted to each paper to discussion, rather than the usual monologue-presentation. In this way, opportunities for interaction and participation are maximised. Following the congress, the authors of the accepted papers had the opportunity to make further revisions in response to the feedback and discussion within their WG. The form in which they appear in these proceedings reflects this continuous process of improvement, assisted by critical support within the WG.

The success of the ERME movement can be measured, in part, by the numbers of participants and presentations at recent meetings. In Rzeszów, 453 participants came from 50 countries, and were involved in 17 Working groups, coordinated by 75 leaders. 293 research papers were accepted for presentation and discussion at the congress, as well as 69 posters. Taken to the next stage, 279 papers and 49 poster-communications appear in these proceedings. Interest in CERME continues to grow, with participants coming from 17 countries beyond Europe (such as Canada, US, Brazil, Singapore, Australia ...) this time.

The ERME ‘spirit’ of communication, cooperation and collaboration is explicitly enshrined in its aims and vision. *Inclusion* is central to its ethos, and the way that the WGs organise their activity. At the same time, ERME must promote and support scientific *Quality* if it is to be useful to its members and credible on the international stage. At times the two goals, inclusion and quality, seem to pull in different directions, creating tension and sometimes dissatisfaction when, for example, a research paper is not accepted for presentation, despite formative feedback and revision, and the author is then unable to access funds to attend the congress. Thus, by upholding a notion of necessary ‘standards’ of scientific quality, someone is effectively denied participation in the congress.

This tension has recently been addressed in collaborative research by Barbara Jaworski (UK), João Pedro da Ponte (Portugal) and Maria Alessandra Mariotti (Italy), which is reported in Jaworski et al. (2011). A paper related to this research was made available to CERME7 participants in advance of the congress.



## Introduction

In addition to the WG activities, the sense of belonging to the *whole* congress was fostered in a number of plenary scientific activities, and in a varied social and cultural program. The opening ceremony, held in the splendid surroundings of the Rzeszów Philharmonic Concert Hall, included a plenary address by Anna Sierpiska, on her recent research into elementary mathematics methods courses in preservice teacher education. Sierpiska maintains close links with mathematicians and mathematics educators in her native Poland. Two other plenary talks later in the conference recognised the work of two CERME WGs across several congresses, and were given by researchers who had led these WGs until recently. Thus, the address of Markku Hannula (Finland) was on the Structure and Dynamics of Affect in Mathematical Thinking and Learning; Maria Alessandra Mariotti's title was Proof as an Educational Task.

Three papers corresponding to these plenary addresses are published later in these proceedings. Another notable event in the plenary program was a session introducing the current work of the Education Committee of the European Mathematical Society, in particular the project to itemise 'solid findings' in mathematics education. This session included fruitful discussion about the nature and warrants of such findings, and their relationship to the work of ERME and its members.

The vision and ethos of ERME is also distinctive in its support for 'young' researchers in the field of mathematics education, notably those undertaking doctoral research or in post-doctoral positions. Every CERME conference is preceded by two half-days of YERME ('young-ERME') discussion groups and workshops.

At CERME7 these were led by professors Paolo Boero, Pessia Tsamir, Dina Tirosh, Barbara Jaworski, João Pedro da Ponte and Heinz Steinbring. The Society is fortunate to have such experts willing to give freely of their time for the benefit of the future leaders of ERME.

We extend sincere thanks to Ewa Swoboda and all the local organisers for their hard work, and to the University of Rzeszów for making us so welcome at the first CERME to be held in Eastern Europe. The next CERME will take place in February 2013, in Antalya, Turkey.

### *Information on-line*

The CERME website was at <http://www.cerme7.univ.rzeszow.pl/>

These proceedings can be accessed online from <http://www.erne.unito.it/>

**Reference**

Jaworski, B., Da Ponte, J.P. & Mariotti, M.A. (2011) The CERME spirit: issues of quality and inclusion in an innovative conference style. In B. Atweh et al. (Eds.) *Mapping equity and quality in mathematics education* (pp. 457-477). Dordrecht: Springer.

Ferdinando Arzarello

Università di Torino, Italia

President, European Society for Research in Mathematics Education

Tim Rowland

University of Cambridge, UK

Chair, CERME7 Scientific Program Committee

# PROCEEDINGS OF THE SEVENTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION: EDITORIAL INTRODUCTION

In this brief introduction to these proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (CERME7), we set out to explain the nature and the organisation of the contents. Individual contributions can be accessed directly from hyperlinks on the contents page.

Following the general introduction by the ERME President and the CERME7 Chair, the first section-proper of the CERME7 Proceedings consists of three major papers, by Anna Sierpinska, Markku Hannula and Maria Alessandra Mariotti. These three distinguished scholars were invited by the scientific program committee to give **plenary addresses** to the congress. Each of them responded to the request for a written paper, recording and elaborating on the content of their presentations. Following discussions with the editorial panel, these papers are published here.

The second section of the proceedings is the written record of the activity which lies at the heart of every CERME congress. It is in 17 parts, corresponding to the 17 **Working Groups** (WGs) at CERME7. Each part begins with an introduction, an overview of the work of the WG by the team that led it, followed by those papers accepted for presentation *and* publication by the WG leaders, following peer review before and discussion during the congress. The papers are ordered alphabetically, by first author, although thematic groupings are sometimes suggested in the WG-leaders' introductions. We extend our thanks to, and acknowledge the editorial contribution of, the leaders of each WG in their scrutiny of the papers accepted for publication from their Group.

The third section of the proceedings is an innovation in the dissemination of the content of posters accepted for presentation and discussion at CERME. In response to a request from some WG leaders, the ERME Board agreed that posters accepted for CERME7 could be 'published' in the proceedings. This was realised by inviting the relevant poster-authors to submit a two-page communication of the content of their poster. These **poster-communications**, also grouped by Working Group, appear in this final section.

Since CERME3 in 2003, Proceedings of CERME congresses have been produced electronically, usually in CD format. They can be also be accessed online from <http://www.erne.unito.it/>

Tim Rowland, University of Cambridge, UK

Ewa Swoboda and Marta Pytlak, University of Rzeszów, Poland

Editors, Proceedings of CERME7.

## Introduction

# Plenary lectures



# RESEARCH INTO PRE-SERVICE ELEMENTARY TEACHER EDUCATION COURSES

Anna Sierpinska  
Concordia University, Montreal, Canada

*Considerable effort has been put into mathematics education research towards the development of knowledge that could serve as a basis for a professionalization of the work of mathematics teachers. In recent years, a similar effort has been expended with regard to the work of the mathematics teacher educator. Three years ago, I joined this current by engaging in a research project on the “Teaching Mathematics” (TM) courses for future elementary school teachers with Helena P. Osana, a researcher in educational psychology and an experienced elementary mathematics teacher educator. The goal of this research, ultimately, is to make public, communicable and open to critical analysis the personal experience of university professors teaching such courses. The more modest and immediate goal of our small project is to describe and make sense of TM courses in six Canadian universities, three Francophone and three Anglophone. We observed classes, studied course descriptions, and conducted long interviews with instructors; we also interviewed students. In the talk, I present some results of our research so far, focusing on a framework for analysing the TM courses that we have started developing in our research. This framework might be useful for other researchers wishing to contribute to professionalization of elementary mathematics teacher educators’ work.*

## HOW IT STARTED

It all started with a meeting of two singularities: Helena P. Osana, the single mathematics educator in the Education Department, and myself, one of two mathematics educators in the Mathematics & Statistics department of our university. We started talking about possibilities of collaboration, such as teaching courses in the other department, and doing research together. Helen could certainly teach a course in educational psychology in the *Master in the Teaching of Mathematics* program in the math department. But what could I teach? The Education Department has only two courses in mathematics education, both in the undergraduate program leading to certification of elementary school teachers. Moreover, Helen’s research has been in the area of elementary education, whereas my research so far, was on secondary or tertiary mathematics teaching and learning. So, if I wanted to engage in collaboration with Helen – and I did – I had to learn something about her area of practice and research.

I was curious about her elementary mathematics teacher preparation courses, thinking that I could maybe teach one in the future. There were plans, in fact, to add one more course on mathematics teaching to the two already in place in the

elementary teacher preparation program. To prepare, I started reading about mathematics teacher education. There was no shortage of sources. This is a huge area. There is a specialized journal, *Journal of Mathematics Teacher Education*, and one of the recent ICMI Studies was devoted to the area (Even & Ball, 2009). I was finding interesting ideas for activities for teachers and information about teacher education programs around the world. All this was, however, not enough for me to make decisions in designing a whole course for future teachers. Of course, I could just start by teaching one of the courses already in place by taking Helen's course outline and her lecture notes and materials and teach according to those. But I don't think I would be able to do that properly without seeing her course as a result of a rational choice. To understand it as a result of a choice, I had to see alternative ways of conceiving such course. I wanted to see what other instructors of elementary mathematics teacher preparation courses in Canada are doing. And this is how I embarked, with Helen, on the research of which I will be showing some snapshots in this talk.

## THE DATA

We visited elementary mathematics teacher preparation courses in six universities in three Canadian provinces: Québec, Nova Scotia and New Brunswick. Three universities were Francophone and three Anglophone. We labelled the French universities FU1, FU1, FU3; AU1, AU2, AU3 are labels for the English universities. I will call all these courses "Teaching Mathematics" courses (TM), although they had different names in different universities. The courses we visited were part of 120 credit Bachelor of Education programs (Elementary Education option). At all the English language universities, 6 of the 120 credits had to be obtained in two 3-credit TM courses, each lasting about 13 weeks. FU1 and FU3 had four 3-credit compulsory TM courses. At FU2, located in New Brunswick, beside the two 3-credit TM courses, students preparing to teach in grades 5-8 had to take four mathematics courses, including one-variable Calculus. In New Brunswick, elementary education counts 8 grades, not 6 as in other provinces. There were no compulsory mathematics courses for elementary teachers at the other universities, although at FU1, there was a mathematics "placement test", and students who failed it had to take a remedial mathematics course based on secondary school material.

We interviewed instructors and students, observed classes, collected various documents such as course outlines, lecture notes, assignments, tests and examinations. We labelled the instructor of the course at AU1 "Professor Aone"; similarly, Atwo, Athree, Efune, Efdeux, Eftrois were labels for instructors of courses at the other universities. "Professor" marks the fact that these instructors were full time faculty members in charge of these courses. For the sake of brevity, I will omit the title "Professor" when referring to the instructors. Most of them designed their courses from scratch and had been developing them for several years. An exception was Atwo who was in her second year of teaching at the time we met her, and she



was teaching her course along a design inherited from her predecessor, who was retiring. There was also a somewhat exceptional situation at AU3, where, at the time of our visit, Athree did not teach the TM course, and was replaced by “Ms. West”. Ms. West was a part-time instructor at the university, and a pedagogical consultant for mathematics teaching on the Elementary School Board in the area. She did not use Athree’s course design but developed her own. We interviewed both Ms. West and Athree about the way they designed and ran the TM course, but could visit only Ms. West’s class.

## **FOCAL ASPECTS OF THE STUDY**

As we collected and analysed data, we tried to identify, in each course: the content structure; the tasks given to students; the formats of teacher-student interaction (see Bruner, 1985; Sierpinska, 1997); the instructor’s reasons for his or her choices about the previous three aspects; and students’ perceptions of the course. Based on our analysis of these aspects, we attempted a description of the *nature of knowledge* that was emerging in the course.

## **SOME SNAPSHOTS OF THE RESULTS SO FAR**

This is a study in progress, so what I can offer is only a glimpse of partial results obtained so far. I will talk about the following aspects of the TM courses: (a) course content components, (b) differences in course content and the weighting of course components, (c) our analysis of tasks in TM courses and in particular, the epistemic actions they are designed to generate, (d) the formats of interaction between TM instructors and students as another way to capture the nature of epistemic actions emerging from TM courses, and (e) student perceptions of the TM courses.

### **Content Components**

The content of the courses seemed to have the following components, in various proportions:

- Mathematical Knowledge for Teaching (MKT)
- Psychology of Mathematics Learning (PML)
- Teacher’s Didactic Actions (TDA)
- Reflection on teacher preparation (R)
- Ideology (I)

The name *Mathematical Knowledge for Teaching* sounds like Ball’s MKT model of a knowledge base for mathematics teaching (Ball, Thames, & Phelps, 2008), but it refers to only a subset of this model, namely *Common Content Knowledge* and *Specialized Content Knowledge*. Ball’s MKT encompasses our MKT, PML and TDA combined. Our PML is close to Ball’s *Knowledge of Content and Students*, and TDA, to Ball’s *Knowledge of Content and Teaching*. We chose to separate MKT

from PML and TDA because syllabi of the TM courses sometimes do that too. At AU2, the first TM course focuses on MKT only: it is intended to be a review of elementary school mathematics.

Our MKT, PML, and TDA categories are different from Ball's model in that they refer not to what teachers *should* know but to what TM instructors *choose to teach* in their courses. PML may contain any content related to how elementary school children learn mathematics (e.g. Piagetian theories of learning mathematics, the van Hiele model, or "principles" of learning mathematics advanced by mathematics educators), and TDA – any content related to teacher's didactic actions (planning a lesson; designing an assessment, etc.). I will not discuss the PML and TDA components in general terms; some examples of PML and TDA content will be given later in this talk. Here, I will dwell a bit more on MKT, focusing on the aspect that has been stressed also in Ball's model, namely that mathematical knowledge for teaching is very different from the mathematician's mathematics.

As Freudenthal stressed long ago (1986), MKT for elementary school is neither a part nor a "didactic transposition" of scholarly mathematics (Chevallard, 1985). One immediately obvious difference is the kind of quantities they work with: mostly abstract numbers for the mathematician, and mostly (concrete) quantities of units (of lengths, areas, time, weight, money) in the elementary school, and abstract numbers figure as "multipliers" (or scalars) that can scale the quantities (e.g., one can double a quantity, or halve it). Vergnaud's notion of "measure spaces" models rather well the quantitative structures of MKT (Vergnaud, 1983).

The mathematician conceives of operations on numbers as functions; in MKT, an arithmetic operation is conceived of via problem situations that call for this operation. For example, in a recent textbook for elementary school teachers (Sowder, Sowder, & Nickerson, 2010), the chapter on "Understanding Whole Number Operations" has a list of "Problem situations that call for multiplication" (p. 63). The "problem situations" in the list are named and described as "ways of thinking about", "views," or "models" of multiplication or division. The categories are labelled as: 1. Repeated addition; 2. Area/array model; 3. Operator view, and 4. Fundamental counting principle. A mathematician may be baffled by the categories because some sentences in their descriptions look like definitions, some like computational procedures, and others like statements of problems. Some conditions appear unnecessary and some categories, redundant. This confusion is one of the consequences of a different understanding of the notion of "model" by the two parties. An expression such as "models of multiplication" used in reference to a "context that calls for multiplication" is surprising for the mathematician who would find it more natural to say, "multiplication is a (mathematical) model for this context" than "this context is a model for multiplication." The mathematician looks at a problem situation and tries to find or construct a mathematical theory that would represent it. MKT works in the opposite direction: Given a piece of mathematical

theory (e.g., the operation of multiplication), look for situations that could be described using this piece of theory, so that, in teaching, the theory acquires some sense for the students, and which can be used to practice the theory and assess students' knowledge of it. MKT contains a categorization of situations that can be described using the piece of theory. The categorization then serves to design exercises, curricula, and textbooks.

The existence of different “models” for multiplication in MKT leads to more differences with the mathematician's mathematics. In particular, while for the mathematician, multiplication in real numbers is commutative, for MKT, it is not necessarily. It depends on the model of multiplication the class is presently working on. In the repeated addition model of multiplication, which applies to “problem situations”, where one has to find “the sum when a whole number of like quantities is combined” (Sowder et al., 2010, p. 59), in the US textbooks for children, an expression such as  $2 \times 3$  means  $3 + 3$ , and not  $2 + 2 + 2$ . The first number is understood as the multiplier; the second – as the quantity multiplied. From the point of view of this model, the expression  $2.5 \times 3$  “cannot be interpreted” (ibid.)

The *Reflection on teacher preparation* component has no counterparts in Ball's model, but we had to add it to the aspects we looked at in the courses because it played an important part in some of them. When Aone and Athree were starting to develop their courses, *Reflection* could take the form of only an informal classroom discussion, but, over the years, it would gradually occupy a larger space. For about 3-4 years now, Athree has not been preparing a detailed course syllabus before the course starts. He begins his course by developing the syllabus together with his students. He also designs the assignment and assessment techniques for the course with them. Athree claims that by devolving part of the responsibility for the course content to his students, they understand better the goals of the course and what is expected of them in the assignments. This was Athree's solution to the recurrent problem he faced in his courses of students asking him “Why do we need to know all that?” and not producing what he expected in the assignments, despite increasingly detailed and lengthy instructions and explanations. Aone has not turned to such radical solutions, but she believes her students need a preview of the kind of knowledge she is going to give them a glimpse of in her TM course and that they will be responsible for developing later in their professional careers in schools. Thus for quite some years now, she would start her course with one long written assignment, where students are asked to reflect on what they think an elementary mathematics teacher should know, and then revisit their thoughts after reading an article on teacher knowledge (usually Ball, Hill, & Bass, 2005).

We use the word *Ideology* in the sense of “visionary theorizing” (see Endnote 1). Some instructors promote a “vision” of mathematics learning and teaching. This vision may or may not be explicitly linked to a theory, but it is not a theory. In a theory, statements about teaching and learning are hypotheses or results of empirical

## Plenary lecture

research. An ideology makes statements about how teaching and learning *should be*. Thus, a theoretical statement such as, “a child learns by adapting her cognitive structures to deal with problems arising in her own environment” may be turned, in an ideology, into “an invitation to teach mathematics from a problem-solving perspective” (promoted, in particular, by the NCTM Standards), and “placing the child in the centre”, as in a textbook used, at the beginning of his work as a TM course instructor, by Athree:

The vision of mathematics learning presented in this book places the student at its centre. The living context of each student is important to the learning scene and it is out of that culture that ideas for proposing mathematical problems should emerge. (Cathcart, Pothier, Vance, & Bezuk, 2000, p. xi)

At the beginning of his career, Athree was strongly promoting “constructivism” as the preferred vision of learning for teaching. In his second year of teaching, for example, students’ responses to a task of planning a take-home mathematics teaching activity were evaluated based on, among a few other aspects, “the extent to which your planning reflects consideration of constructivist learning theory” (Athree, Course Outline 2000). Now, he leaves his students the choice of the learning theory on which they want to build their own visions of teaching and learning, and even according to which they want to be assessed in the course. He expects them, however, to be conscious of the choice they make and aware of its consequences, and suggests relevant articles to read. However, there can be no escape from ideology in a TM course. If it is not explicit, it is conveyed by the instructor’s choice of classroom activities and comments, however neutral he or she may try to appear. Education, in general, is value-laden.

### **Course Differences: Weights and Content of Course Components**

Courses differ in the weight they assign to the content components. When we examined the evolution of a course developed by an instructor over a longer period of time, changes in the weight assigned to different components were one of the more salient observations. In the case of AU1 and AU3, we noticed a drastic change in the weight of MKT from very small in the first year to quite large in the second year. This was the result of the realization, by both instructors, that future elementary teachers know even less mathematics than they had been warned to expect, and TM instructors must review with them the most basic ideas.

We measured the weight of a content component by analysing a sample of tasks instructors assigned the students: we looked at the content area of the knowledge each task called for and counted the relative frequency of tasks corresponding to the given content area. An example of such analysis will be given later. We considered this measure to adequately represent the weights of the components because there was usually very little lecturing in the courses. Classes were organized around “activities”, which we were able to decompose into tasks.

## Plenary lecture

With respect to course content components, the categories of MKT, TDA, PML, etc. are quite broad and their detailed content varies widely from one TM course to another. Even using the same textbook for elementary mathematics teachers, such as the quite popular textbook by Van de Walle (1998), does not guarantee many commonalities in the content of the courses. Some instructors choose to “cover” all fundamental elementary mathematics topics; others pick only a few. Some insist on knowing the precise mathematical terminology and definitions for mathematical concepts; others are much more relaxed about it. Ms. West at AU3, in particular, would correct any student’s departure from the conventional mathematical terminology. In the class we observed, a student called a regular hexagon, an “equilateral hexagon”. She laughed at the expression, and corrected it: “regular!”. Professor Athree said that he would not correct this expression since it is quite logical: a generalization of the term “equilateral triangle” to a hexagon with all sides equal.

Some instructors insist also on knowing terminologies developed in mathematics education, educational psychology or cognitive psychology in relation with chosen theories and models of teaching and/or learning. We started calling these theories and models “structuring frameworks”. One of the frameworks is a repertoire of problem-solving strategies, included in all textbooks for elementary teachers we had a chance to look at. The textbook used by Professor Athree in the first few years of his teaching the TM course, for example, presented strategies such as “Draw a picture or a diagram”; “Find a pattern”; “Solve a simpler problem”, etc. (Cathcart, Pothier, Vance, & Bezuk, 2000). Professor Athree, however, did not require his students to know and use the exact names of these strategies. Professor Aone, on the other hand, does, and she tests her students on this knowledge. On a test, students may be given a solution of a problem and the question would be, “Name the strategy used in solving the problem”. At FU2 (and only at this university), the Piagetian theory of developmental stages was an important structuring framework. One of major assignments in the course was to design and implement a diagnostic interview with children between 4 and 12 years of age to assess their developmental stage in relation to a particular mathematical concept (number, chance, space). To prepare for this task, students had to read Copeland (1984, 1974) and Kamii (1985, 1989).

The course at AU1 appeared to be the strictest in the use of structuring frameworks. In fact, at AU1, in the first TM course, there was at least one explicit structuring framework for each content area, and students had to know the categories well since they were used and required on tests and the written final examination. The MKT component of the first of the two TM courses contained the following structuring frameworks (and only those):

- Mathematical principles of counting, based on Ginsburg (1989), e.g. “One-to-one Principle” (each member of the set must be counted once and only once)
- Place value and numeration systems, based on a chapter from Burris (2005)

## Plenary lecture

- Arithmetic operations on whole numbers and their properties (Burris, 2005)
- Types of problems involving each of the four arithmetic operations, based on Carpenter et al. (1999) – written from the perspective of Cognitively Guided Instruction (CGI).

Structuring frameworks for the PML component were:

- Psychological principles of counting, based on Ginsburg (1989); e.g. different results after having counted a set twice do not usually bother pre-schoolers; most pre-schoolers have difficulty touching each thing in a set once and only once when counting
- General problem solving strategies, based on Reys et al. (2008)
- Children's problem solving strategies, based on the CGI book by Carpenter et al. (1999)

For TDA, the frameworks were:

- Teaching counting based on mathematical and psychological counting principles (Ginsburg, 1989)
- Flexible interviewing techniques, based on Ginsburg, Jacobs and Lopez (1998)
- Effective learning environments, based on Hiebert et al. (1997).

Hiebert et al. (1997) provided also a structuring framework for the Ideology component, and, as already mentioned, a structuring framework for the Reflection component was Ball et al.'s (2005) model of mathematical knowledge for teaching.

## Tasks for future elementary mathematics teachers

Tasks in mathematics teacher education have been at the centre of attention in the area for some years now. In 2007, the *Journal of Mathematics Teacher Education* produced a Special Issue on tasks in 2007 (Jaworski, 2007). A recent ICMI Study on mathematics teacher education was also interested in tasks (Clarke, Grevholm, & Millman, 2009). The question of such tasks has puzzled me for years, long before I embarked on this research, and the hope of getting some insight into those tasks was a great incentive to engage in it. What can future teachers be asked to do in a course besides solving mathematical problems? How can we ask them to do problems about the teaching of mathematics if they have never taught before, and particularly because TM courses are often given before they have their practicum? Aren't the problems of teaching mathematics always very deeply situated in the actual, day-to-day practice of teaching?

*What can the TM instructor do to engage his or her students in tasks that are meaningful for teachers' work in the absence of real life elementary classroom experience?*

We saw several ways that instructors deal with this issue in the courses we observed:

## Plenary lecture

- organizing *authentic* educational interactions with real children, with future teachers in the role of educators; e.g. tutors, animators of an extra-curricular activity, researchers;
- *simulating* an elementary mathematics classroom with future teachers in the role of elementary school-children and the TM instructor as the teacher, or some TM students in the role of teachers and others in the role of their students;
- engaging future teachers in a teacher's "*private*" tasks, i.e. tasks that are mostly done outside of the classroom.

We saw the "*authentic*" technique applied at AU2 and FU2. One of the assignments at AU2 was for future teachers to do at least two weeks of volunteer tutoring in math in a community centre, where children came after school to get some help with their homework. Future teachers had to write a report about their experience as part of their coursework. Interviewing the future teachers after their volunteer tutoring, we were amazed at how enthusiastic they were about the tutoring experience, even those who were extremely critical about the TM course in all our previous interviews. Those future teachers, who up to that point had talked only of how miserable and unsure they were about being able to pass the exam in the TM course, suddenly started talking from the position of teachers concerned about children's difficulties in math and the challenge of helping them. This was difficult for them, and yet they were saying they "had a wonderful time".

At FU2, future teachers had to interact with real children in two assignments. One of the assignments was the Piagetian diagnostic interview task already mentioned above. The other, was an "enrichment project". Students had to invent problems for a "math trail" activity to be run in one of the buildings of the university, and then actually organize the event in collaboration with a school and animate it, accompanying the children along the trail. The math problems had to deal with objects actually found in the building. They were therefore intended to be "situational" or contextual problems, whose importance is stressed in Canadian mathematics curricula at both the elementary and secondary levels (see, e.g. Ministère de l'Éducation, des Loisirs et du Sport, Gouvernement du Québec, n.d., or New Brunswick, Department of Education, Curriculum Development Branch, 2011).

All the activities mentioned here as examples of the "*authentic*" technique are authentic teaching tasks only in the sense that they involve real children. They do not belong to an ordinary teacher's day-to-day practice.

The "simulation" technique with future teachers playing the part of elementary school children solving mathematical problems was used to some extent in all courses. In particular, in the first TM course at AU2, future teachers spent most of the time in a "lab", solving, in small groups, problems that could be given to elementary school children (although they were perhaps more "fun" and involved the use of a much greater variety of manipulatives and hands-on activities than is

## Plenary lecture

common in most schools). The lab was intended to provide future teachers with an experience of a model learning environment for teaching mathematics in elementary school. There was no instructor-guided reflection on those activities from the point of view of a teacher at AU2, however. Such reflection was attempted in all the other courses, with greater or lesser use of the structuring frameworks. I give an example from Ms. West's class at AU3.

In one of the activities we observed in Ms. West's class, students were given a bagful of shapes, cut out by Ms. West herself, from a plastic foam sheet. The shapes were of a large variety of kinds, not only those with known names such as circles, squares, triangles and rectangles; some looked like apples, others like eyes; there were concave shapes, convex shapes, rectilinear shapes, curvilinear shapes, etc. The task for students was the following: "The group chooses one shape and places it at the centre of the table. One person chooses a shape that is like the shape in the centre, and explains how it is alike. The rest of the people at the table must choose shapes that have the same properties." The task's name was "The sorting task". After 15 minutes, Ms. West asked the students: "What is the purpose of this activity?" Students would venture some one-word responses, sounding tentative. I overheard, "Categories?" This was obviously not what Ms. West expected; so, without making any comments about these responses, she asked a question that indicated better her expectations: "What does it tell me about the children?" I hear a student responding, again with hesitation in her voice: "How they think?" Now Ms. West looked more satisfied, but she felt obliged to make explicit her expected answer herself: "It is an informal assessment of their van Hiele level of understanding. At Level 0, a student would say, 'it is a triangle because it looks like a triangle'. At level 1, the student would say, 'it is a triangle because it has three sides'." In Ms. West's class, the simulation phase of an activity always dealt with the MKT content area (students were learning elementary school mathematics); the reflection phase, in this example, brought up PML content: the van Hiele model of understanding in geometry.

This technique was also used by Athree, who, in a discussion about Ms. West's class, said:

Something I would do exactly the same [as Ms. West], I think, is this operating on two levels simultaneously. I am teaching mathematics and I am teaching elementary school mathematics because they don't know it. And I am trying to teach it that way that I would like for them to be teaching it because I think that that is the most effective way to be teaching it. And that I can simultaneously comment on that as I go along. Whether that shifting of levels is in fact something the students are comfortable with and can follow, whether they can reflect on their activity while they are engaged in that activity, is ... Somebody could do some research on that. (Athree, interview)

In yet another type of simulation, Athree engaged his students in co-teaching with him. The technique was also used by Efune. Athree used it for several years in his



## Plenary lecture

TM courses in a task called “Teachers of the week”. The task was formulated as follows:

You and your teaching group will share responsibility (with the instructor) for the teaching of one component of one class. Prior to the class you will meet with the instructor to plan the lesson for that week.... All classes will conclude with a 20 minute discussion of the class: how it was taught, usefulness of the content, etc....

The “lesson” would consist of an activity to be done in small groups. Members of the teachers of the week group would present the activity, distribute any handouts, if needed, and then circulate among the tables, interacting with students, giving hints or prompts to reflect; the lesson would close with a plenary discussion of the results of the activity. Athree used this technique for a couple of years and then abandoned it. He said it was taking up a lot of his and his students’ time, and it was not obvious the students were gaining much from this experience for their future teaching in schools.

*Tasks related to the teacher’s “private” activities were common in the courses.*

In particular, at AU1, there were no other tasks. The closest the future teachers were brought into contact with the reality of teaching was through watching and analyzing videos of teachers in the classroom and children solving problems [examples of videos used at AU1 can be viewed at the website of Annenberg Media (1997, 2000); videos of children solving problems were taken from the DVD accompanying the CGI publication by Carpenter et al. (1999)].

The private tasks technique might seem a rather poor substitute for real life experience of teaching. Yet the following tasks on which Ball’s (2008) model of mathematical knowledge for teaching is built could be, in principle, engaged with outside of direct interactions with children: 1. do the mathematical work that the teacher assigns to students; 2. plan the teaching of a thematic unit and each of its lessons; 3. explain a mathematical idea, concept or procedure in a manner appropriate to a given educational level and curricular context; 4. design questions, exercises and problems for various purposes; 5. design own teaching materials, which involves choosing or developing appropriate representations; 6. evaluate teaching materials designed by others (in particular, textbook materials); 7. assess students’ learning and understanding of mathematics not only in terms of correct/incorrect but also in terms of identifying mathematical sources of students’ errors (as opposed to psychological sources, such as lack of attention); 8. find a way to help a student to correct an error he or she does in a systematic way; 9. find some “strategic examples” and /or representations to highlight a feature misunderstood by the student.

Therefore, the private tasks seem important to mathematics teacher educators, not as a substitute for real teaching, but on their own merits. Future teachers do not always appreciate the relevance of these tasks for teaching practice, however. We heard criticisms pointing in this direction from students in all universities. Admittedly, if

done outside of the school context, tasks 1-9 are not *authentic*. Instructors only take students through a “dry run” of such tasks, without the prospect of implementing the results of the private work in the public arena of classroom teaching. The tasks are imposed by the instructor of the TM course rather than arising naturally from interactions with children and colleagues.

*If TM courses cannot engage students with knowledge that a practitioner actually uses in teaching, with what kind of knowledge do future teachers engage in those courses?*

To obtain a deeper insight into the ‘epistemic nature’ of knowledge that tasks in the TM courses called for, we needed to analyze them in more detail. The idea was to select a sample of tasks in a TM course, examine the actions related to mathematical or teaching practices that were involved in solving each task, and then calculate the distribution of the actions. This method is similar to that used by Sánchez and García (2009) in their analysis of tasks in TM courses. These authors used a grid of analysis that identified elements related to, on the one hand, “mathematical practices” (defining, justifying, modelling, symbolizing, etc.), and, on the other, “mathematics teacher’s systems of activity” (e.g., the organization of the mathematical content for teaching; management of the mathematical contents and discourse in the classroom; analysis and interpretation of students’ mathematical thinking, which correspond to some of the private teacher tasks we enumerated above).

To derive some information about the nature of actions involved in solving each task, we needed a way to analyze each task in detail. We decided to use the *praxeology framework* developed by Chevallard in his Anthropological Theory of the Didactic (1999). Praxeology models a practice by the tasks it sets out to accomplish; the techniques normally used to accomplish the tasks; and two levels of discourse, one developed to objectify, communicate, and justify the techniques (“technology”), and the other, to justify the choice of the technology as one theoretical choice among others (“theory”). Based on this framework, we used a *grid of analysis* made of the following rubrics: Task, Content Area addressed in the task (MKT, PML, TDA, etc.); Technique; Technology; Theory; Type of Action (epistemic, pragmatic, mathematical, didactic, sub-type of such action, etc.). We also identified the institutional status of the task (e.g., small group activity in class; quiz question; midterm examination question; final examination question; take-home project). Below are a few examples of tasks, and Table 1 illustrates how these tasks were coded.

Task 1. (AU1, Quiz question) True or false? Justify. Statement: “Children must grasp the One-to-one Principle before they can effectively apply the Stable Order Principle”

Task 2. (AU1, Final examination question) Name the problem type: “Martha has 12 Polly dolls. She has 5 more Polly dolls than Sam. How many dolls Sam has?” (Expected answer: Addition-subtraction problem / Compare problem / Compared set unknown)

## Plenary lecture

Task 3. (AU1, Final examination question) “When we use the standard algorithm to compute  $454 \times 3$ , we write the following [3 is written below 454; a horizontal line below 3, sign of multiplication on the left of 3, 1362 below the line; small 1 above the leftmost 4 and small 1 above the 5 in 454]. Show how you would model this computation with Base 10 blocks. Make sure to be careful to explicitly point out the connection between the little ‘1’s in the algorithm and your manipulations with the blocks.”

Task 4. (AU3, Mrs. West, in-class activity) “Van Hiele Guided exploration of a square”. Envelopes with squares of different sizes are distributed, one envelope per table. Each student takes one square. Ms. West: “Jot on a piece of paper everything you know about a square”. When students finished writing, Ms. West gives further instructions, and asks consecutive questions to guide students in their exploration; she either repeats a student’s answer that she approves of, or states the expected answer herself: “Draw a line underneath your writing. (...) We are now going to be generating a vocabulary. (...) Fold your square on one diagonal. Underline ‘diagonal’. What do you get? (...) What happens when you fold the square along a diagonal? The diagonal is...? (...) The diagonal’s a line of symmetry (...) We are building our knowledge about the square. What does the diagonal do to the angle? (...) What’s the word for it? Bisects. The diagonal is bisecting the angle. Now fold the square on the other diagonal. Measure your diagonals with a ruler. (...)”. At the end of the activity, students are asked to fill out a table with names of shapes (Square, Rectangle, Rhombus, Parallelogram, Trapezoid, Kite) as headings of the columns, and a selection of properties as headings of the rows: Number of pairs of congruent sides; Number of pairs of parallel sides; Diagonals are congruent (Y/N); Diagonals bisect each other (Y/N); Has half-turn symmetry (Y/N); Number of lines of reflective symmetry.

Task	Content area	Technique	Technology	Theory	Type of action
Task 1, True/False, Justify, statement about counting principles (AU1, Quiz)	PML	Recall the meaning of technical terms, figure out relations between them	Mathematical and Psychological Counting Principles; empirical cases of children’s behaviour (Ginsburg)	No theory: Ginsburg’s “Psychological Counting Principles” treated as facts, not theoretical and hence hypothetical knowledge.	Epistemic actions: Reason based on recalled definitions of technical terms and knowledge of empirical facts
Task 2, Name the type of problem (AU1, Final exam)	MKT	Recall the list of categories of types of problems; identify the category that fits the example	Classification of addition / subtraction problems according to Carpenter et al. (CGI)	No theory: CGI categories treated as facts, not theoretical knowledge	Epistemic action: Identify an object (i.e. recognize it and name)
Task 3, Explain a standard algorithm (AU1, Final exam)	TDA	Use Base 10 blocs to model regrouping	Base 10 blocs, the notion of regrouping in positional numeration systems	No formal theory	Epistemic action: Represent a mathematical operation using material objects
Task 4, Van Hiele Guided	MKT	Recall and recognize properties	Properties of polygons; concepts of	No theory. Euclidean	Epistemic actions: identify

## Plenary lecture

exploration (AU3, Ms. West)		of the square and other polygons	diagonal, angle, bisecting segments, perpendicular, parallel, half-turn symmetry, reflective symmetry	geometry treated as empirical knowledge about idealization of material objects	properties of classes of shapes (functioning at the van Hiele level of “Analysis”)
--------------------------------	--	-------------------------------------	--	--	---

**Table 1. Examples of our coding of tasks used in the TM courses.**

### *Some examples of results of our detailed analysis of tasks*

In Ms. West’s 3-hour class on geometry, future teachers were engaged in 23 tasks in the first 2 hours of the class. Of these, 13 tasks focused on MKT; 4 tasks (short questions asked by Ms. West in the ‘reflection’ phase of each activity) focused on TDA only; 2 tasks on PML only; and 4 tasks included both a TDA and PML reflection. In the MKT tasks, the techniques expected and/or hinted at were to represent a mathematical situation with material objects; observe; experiment; state one’s observations; and use known mathematical ‘facts’ to derive new information about the properties of the studied objects. The ‘technology’ part of the praxeology consisted of mathematical terminology (restricted in this class to names of geometric shapes); properties such as ‘the sum of angles in a triangle is 180 degrees’, or ‘angles at vertices in a regular polygon are congruent’. These properties were treated as ‘facts’, not as hypothetical, theoretical activities. The level of geometric thinking expected of students did not go beyond the van Hiele level of “Analysis” and never touched upon the levels of Abstraction, Deduction and Rigor. In tasks related to TDA and/or PML, the techniques consisted mostly in recalling didactic or psychological principles and guessing which one the teacher had intended to illustrate with the mathematical activity she proposed in the simulation phase. Technology in both TDA and PML areas was restricted to a loose collection of ‘principles’ such as, in TDA, “always relate the math to the real world”; “use varied examples so as not to instructionally disable the children”, or, in PML, “children build categories based on typical examples, not definitions”; “recognizing is easier than naming”. The van Hiele model was expected to be known to the future teachers, but, in the class, they were only expected to recognize its application in Ms. West’s organization of the activities. They might, if they chose to, use it themselves in planning a geometric activity intended for children, when doing the homework assignment titled ‘Unit plan’. There were no written limited-time tests or examinations in the TM courses at AU3.

There were such tests in all other universities. In the first of Aone’s two TM courses, there was a 30-minute Quiz, a 2-hour midterm test, and a 3-hour final examination. We analyzed the 51 tasks involved in all these three examinations and the “Reflection” assignment given at the beginning of the course, where students had to read Ball et al.’s paper on mathematical knowledge for teaching. In the Fall 2010 version of the course, the distribution of the tasks among the content areas of MKT,

## Plenary lecture

PML, TDA and Reflection was 17, 17, 11 and 6 respectively. In the first year of Aone's teaching, MKT tasks held a much smaller weight.

The techniques expected to be used in these tasks (but not explicitly taught) could be described as: use a checklist; recall a definition; apply principles or properties; and use common sense to find relations. The technology made available to students was very elaborate: it was made of the many structuring frameworks mentioned above. The frameworks were not presented as results of theoretical choices, rather, they started from explicit assumptions about mathematics, didactics and psychology of mathematical learning.

Examining the actions involved in tasks belonging to each content area, we could identify the following types:

MKT: Produce, discuss, assess a representation of a number; Assess the use of a standard algorithm; Identify the property of an operation used in solving a problem; Identify the type of problem; Identify the category of a mathematical object; Solve an elementary school mathematics problem.

PML: Identify the child's strategy in solving a problem; Name the advantages of a strategy used in the solution of a problem; Produce an example of use of a strategy; Model and assess a child's solution in a more formal mathematical symbolism; Find relations among the principles of counting and research findings; Discuss the relative difficulty of one problem over another.

TDA: Identify the didactic objective of a task; Produce a problem of a given type; Describe a didactic action of a given type; Explain a standard algorithm to (hypothetical) children of a given age; Assess a teacher's didactic actions (observed in a video or described in words); Assess the counting skills of a child (observed in a video or described in words).

Reflection: Express your beliefs about (a) the reasons why knowing mathematics is important for an elementary mathematics teacher; (b) ways a good teacher uses mathematics in the classroom; Read a research article; Identify a new kind of knowledge, namely the mathematical knowledge for teaching, and distinguish it from the 'plain old math'.

In 18 out of the 51 tasks (7 in MKT, 7 in PML, 2 in TDA and 2 in Reflection tasks), students were required to justify their answers. Taking this into account, abstracting from the content areas and generalizing the actions involved in the types of tasks, we found the following distribution of actions among the 51 tasks: Justify (18 tasks); Identify the type of an object (17); Produce an example of a type of object (9); Discuss / reflect on (8); Assess correctness of (7); Model child's work (3); Solve a mathematical problem (2); Find relations among mathematical statements (2); Explain something to a hypothetical child (2); Explain the sources of a mistake (1).

Solving a mathematical problem was given little weight in Aone's written tests: only two out of the 51 tasks required this action. Problem solving was given more space

in the class sessions, where students sometimes spent almost 2 hours on solving mathematical problems, e.g. an “Adventures of Jasper Woodbury” series problem (Cognition and Technology Group at Vanderbilt, 1997). Students were not tested on solving school mathematics problems, however. Aone believes that, if they were, most would probably fail the course.

As we can see from the distribution, the action of “Identifying” (i.e. recognizing and naming) was very frequently expected in the examination tasks in Aone’s course. This action is not usually ranked high on a ladder of levels of thinking. In particular, it belongs to the lowest level of “Visualization” in the van Hiele model. Ms. West also insisted on the action of identifying, mainly mathematical objects: in the geometry class, she expected future teachers to recognize geometric shapes and name them correctly. Athree, on the other hand, said he had no use for identify tasks in his courses. He would not insist on strict observation of institutionalized terminologies for mathematical objects and he did not teach classifications of types of problems or problem solving strategies. He said his students were free to communicate their reasoning using a mixture of “official” terms and ad hoc names and representations.

Athree insists on reasoning. So does Aone, with the stress on the action of justifying in her examination tasks. The justification is expected to be systemic, i.e. based on a system of concepts, properties and principles from the areas of MKT, TDA and PML. It was also written. The systemic and written character of Aone’s justification tasks contrasts with such tasks in Ms. West’s course, where students were asked to justify their claims, but they could do so only if they wanted (questions were asked to the whole class and only some students responded to them) and they could do so orally, without having to use complete sentences organized into a consistent discourse. Aone aims at teaching her students systemic knowledge. Ms. West is teaching her students that there exists a systemic knowledge out there, but she is not making it necessary for them to engage in using this knowledge in their own thinking.

### **Formats of interaction**

Looking at *actions* involved in tasks was one source of our reflection on the epistemic nature of knowledge emerging in the TM courses. Modeling *formats of interaction* between the instructor and the future teachers were another. As mentioned, Ms. West interacted with future teachers on two levels: as a model elementary school teacher with her students (with future teachers in the role of students), and as a teacher educator. Each episode related to one mathematical problem concludes with a presentation of one or more solutions that Ms. West considers correct. We could say that there were three main formats of interaction in her classroom: the ‘Pretend-we-are-in-an-elementary-classroom’ format, the ‘Guided-reflection-on-the-math-activity’ format, and the ‘Here-are-the-correct-solutions’ format.

## Plenary lecture

In Aone's class, we could identify two main formats: 'Make-a-mess-clean-it-up' used during the phase of small group work, and 'Interactive-lecture' used during the whole class discussion phase. In the 'Make-a-mess-clean-it-up' interactions, Aone provokes cognitive conflicts in her students to make them aware of what they don't know. She uses the structuring frameworks made explicit in her course to help students resolve the conflicts and learn more content, or see connections among pieces of content. She provokes her students to pose "why?" questions and uses this as an opportunity to teach the fundamental concepts of the structuring frameworks.

Based on the interview with Athree, we could infer that he was fond of what we labelled as 'Keep-thinking', and 'That's-an-interesting-question' formats. When, after a phase of small group work on a mathematical problem, a student asks him, "But what's the answer, sir?", Athree's answer is "Keep thinking". Athree collects, compares and discusses his students' solutions, but does not tell them whose answer is correct or what the correct answer is. In the interview, Athree said:

I have absolutely no problem with leaving a problem without an answer. ... Usually, the main reason why I am interested in the problem is to talk about the process. And usually you talk about the process without ever coming to an end. It's really not my goal in the course that they know the answer to a particular problem. (Athree, interview)

When he was asked, in the interview, if he would revisit the problem more formally later, he said, 'No'. When asked then, 'Suppose students are adding fractions by adding numerators, and adding the denominators? Would you still not react?', he said:

Well, in that case I would probably take it as an opportunity to say that there are several ways of adding fractions. They are appropriate in different contexts. It depends if you are adding parts of the same whole or parts of different wholes. (Athree, interview)

The 'That's-an-interesting-question' format was identified based on the following Athree's description of his seemingly disorganized way of teaching, which exasperated his students, and was one of the reasons why they liked Ms. West's way of teaching better:

Part of my teaching is that if an opportunity comes up, some abstraction in a weird and wonderful way, then, oh! Okay, the topic for today was supposed to be 3D geometry, but forget about that, we are doing fractions instead. (Athree, interview)

Based on this kind of analyses of the instructors' interactions with future teachers, and the actions involved in the tasks, we can say that there were at least these types of knowledge emerging in the TM courses:

*Craft knowledge*: knowing how to accomplish a type of task (Grimmett & MacKinnon, 1992); (Hiebert, Gallimore, & Stigler, 2002); in Chevallard's praxeology terms, this knowledge refers to the practical block: task + technique.

*Téchnê*, in Aristotle's sense of craft knowledge grounded in theoretical understanding [see *Metaphysics*, Book I, Part 1 (Aristotle, written 350 B.C.)]; in Chevallard's terms, this knowledge corresponds to task + technique + technology, or to knowing how to accomplish a task and knowing to justify the technique using a system of concepts and principles.

*Craft knowledge with a theoretical attitude*: knowing how to accomplish a type of task, knowing that there probably exist other ways, and that any justification of the solution is hypothetical, not certain. This type of knowledge is not easily described in terms of praxeology, although perhaps the closest would be: task + technique + theory-awareness.

Ms. West appeared to aim mainly at some limited *Craft* knowledge: knowing how to solve elementary mathematics problems and knowing how to set up a classroom activity to teach a given topic at a given grade level. This is, of course, a very limited portion of what teachers' craft knowledge may become after several years of practice (Grimmett & MacKinnon, 1992).

Aone aimed at *Téchnê*, also limited to certain teacher's "private" tasks: knowing how to choose tasks or sequences of tasks for different teaching and learning purposes and evaluate children's reactions to them, based on "structuring frameworks" such as knowledge of types of tasks, typical children's mistakes and strategies, or characteristics of effective learning environments.

Athree seemed to aim at *Craft knowledge with a theoretical attitude*. In Athree's course, no technology or theory was explicitly and systematically taught. However, mathematical concepts were not treated as facts but as theoretical, hypothetical entities. Didactic decisions were not justified by didactic or psychological principles accepted as true. They were based on experience and hypothetical thinking; e.g. 'If we choose to do X, what could be the consequences for achieving our assumed goals Y and Z, according to our past experience?' Future teachers were expected to gain this experience by planning lessons, implementing their plans in simulated teaching activities with their peers as students, reflecting on the implementation and having their teaching evaluated by others. Since 2008, Athree engaged his students in co-designing the whole TM course with him.

### **Interviews with students**

As researchers and theoreticians we may value *Téchnê* and the *Craft* knowledge with theoretical attitude higher than *Craft* knowledge, especially if the latter is focused on only one component of the knowledge we think is absolutely necessary for teaching. But Ms. West's students were considerably more satisfied than Athree's students, and happier than Aone's students. Ms. West's students said in an interview:

She gives such ... great ideas and examples of how to get that across to students. (...) And her love of math really comes through, too. Which makes us want to love math, too.



## Plenary lecture

(...) You go into every class excited. There's something new going on and there's always fun things to do and you don't realize that you actually are learning.

Athree had an explanation why future teachers like Ms. West better: "They like her better, because she is energetic, well-organized, and positive, while I am perceived as phlegmatic, disorganized and cynical".

Will Ms. West's students be happy also when they start teaching? They feel confident. They love math. They've seen a model of teaching that they find attractive, exciting even. They do not feel the pressure to be at the forefront of a "constructivist revolution" when they start teaching. Did Ms. West's students express any fears? Yes, they did. They were afraid of not being able to deal with the noisy reality of the elementary classroom; they were being acquainted with the "normal" curriculum, but not prepared to adapt it to special cases: e.g., to "special needs" children. But they voiced these fears only when prompted to say what they did not like about the course. And after responding, they still said, "Given the time allotted to the course, it is relatively perfect".

Yet, when we interviewed the other instructors, asking them, if this is what makes future teachers happy, whether they would teach like Ms. West, they all said *No*. A large variety of reasons was quoted, but the most common was that there was no guarantee that Ms. West was achieving even the limited Craft knowledge she was aiming at, since this knowledge was never debated, put to the test, or even confronted with alternative ways of knowing and doing things.

## DISCUSSION AND CONCLUSIONS

The idea of constructing a knowledge base for teaching was advocated by Shulman (1986, 1987), and adapted for teaching mathematics by Ball (2008). In the Abstract, we suggested that the same idea could be applied to teaching future elementary school teachers, and that contribution to the construction of a knowledge base for teacher educators is the goal of our research on TM courses.

Not all researchers on teacher education would agree with this statement. The very idea of a "knowledge base for teaching" has been strongly criticized, particularly by researchers working in Schön's tradition of "the reflective practitioner" (Schön, 1987). It has been criticized for promoting the "theory-into-practice" approach to teacher education, based on the "epistemology of propositional knowledge... predicated on the assumption that teacher candidates are novice consumers, not expert producers, of knowledge" (Cochran-Smith & Lytle, 1999, quoted in Bullock, 2011, p. 26). Critics claim that "theory" in the "theory-into-practice" approach to teacher education refers to a mere "rhetoric of conclusions" (a phrase coined by Schwab, 1971, quoted by Clandinin & Connelly, 1995, p. 9), or "codified outcomes of inquiry" stripped of their origins in the inquiry process, and "packaged for teachers in textbooks, curriculum materials, and professional-development workshops" (ibid.).

## Plenary lecture

This knowledge seems to correspond to the “technology” component in Chevallard’s praxeology model: it is technology without theory. Aone’s structuring frameworks such as “Effective learning environments”, “Psychological Principles of Counting” etc., presented in her course without the stories of the research processes that led to their formulation as plausible hypotheses based on explicit assumptions, would be examples of such “rhetoric of conclusions”. But Aone did not teach her students the technology only. She taught her students a praxeology of which the technology was one of the elements: she aimed at *téchnê*. Bullock’s description of teacher education programs based on “purely propositional approach to understanding how teacher candidates construct professional knowledge” which he claims is underlying Shulman’s model (2011, pp. 26-7) is a caricature that distorts and hides the complexity of the knowledge that emerges in Aone’s course, and probably in any TM course:

Teacher education programs usually require teacher candidates to complete a certain amount of coursework before having a practicum experience. The assumption underpinning this design is that coursework can begin to *transmit the knowledge base for teaching* to teacher candidates. The practicum experience is then an opportunity for teacher candidates to practice *applying the knowledge* gained from both undergraduate course work (subject matter knowledge) and professional studies courses (pedagogical content knowledge). (Bullock, 2011, pp. 26-7) (my emphasis)

The knowledge base for teaching TM courses need not be reduced to a technology abstracted from the praxeology within which it was constructed. Nor need it be transmitted by means of “rhetoric of conclusions” in a crash course for newly hired faculty expected to teach such courses at the university. There is no risk of this happening any time soon, anyway, because a body of “codified outcomes of inquiry” on TM courses does not exist. All we have, so far, are stories of individual TM instructors’ experience, and information about the various organizations of teacher preparation programs around the world. Stories of TM instructors’ professional experience, somewhat structured and interpreted using a network of metaphors or other tools for noticing, recognizing and naming phenomena in teaching/learning situations, would be a research outcome quite in line with Schön’s tradition, and would satisfy such vehement opponents of Shulman’s take on teacher education as Connelly and Clandinin (1995, 1996) or Bullock (2011).

That is what we could do in our research and stop there. I am not sure we would be satisfied with it, however. We think it would be useful for a TM instructor to know not only what choices other TM have made in designing their courses but also why and what consequences a given choice is likely to have for his or her goals in the course (e.g., the kind of knowledge he or she would like to see emerge in the course). We also believe that TM instructors have good reasons to ask themselves questions such as “What is known about effective teaching? What do teachers know? What knowledge is essential for teaching? Who produces knowledge about teaching?” TM

instructors ask these questions because they have a course to design and run, and the responsibility of assessing, at the end of it, that certified teachers coming out of the program will not have to be too ashamed of themselves in front of their students when they start teaching mathematics. Yet, these questions have been quoted by Clandinin and Connelly (1996, p. 24) as not the type of questions that they, working in the “narrative tradition of describing and interpreting teachers’ professional knowledge”, would ask to obtain “valid, reliable knowledge... making possible better educated teachers” (ibid.).

Thus our goals put us on an epistemological position that is contradictory with the narrative tradition. In Fernstermacher’s terms, our position is grounded in “the *epistemic status* sense of knowledge”, whereas the narrative tradition understands knowledge in the “*grouping* [categorical] sense”, meaning that the professional knowledge of teachers is produced “in the course of acting on experience... generat[ing] ideas, conceptions, images or perspectives when performing as teachers” (Fernstermacher, 1994, p. 31).

TM instructors appreciate the value of knowledge in the grouping sense, and we have seen how they have tried to provide future teachers with at least a substitute of the authentic teaching experience. Some of them would welcome the possibility of combining their courses with the practicum, thus getting a chance to engage future teachers more with experiential knowledge. But they have no influence over the organization of the teacher education programs of which TM courses constitute a minuscule part (6 to 12 credits out of 120). There is no guarantee that a teacher candidate will even have a chance to teach mathematics during the practicum, since they are being trained to be “generalists”, not mathematics specialists. All the instructor can do is engage future teachers in “reflection-BEFORE-action” (rather than “reflection-in-action”, (Schön, 1987)). I mentioned several techniques the instructors use in their courses to obtain this engagement, without putting students in real-life educational interactions with children. One was watching videos of classrooms and commenting or analyzing didactic and mathematical behaviors of the actors. There are other ways, of course. Videos of live teacher-student interactions in real classrooms can be replaced by animated movies of classroom interactions (Chazan & Herbst, 2011). Reading and reflecting on narratives of individuals’ practicum experience or practicing teachers’ narratives such as found in Clandinin and Connelly (1995) could be another technique.

In the just cited book, the narratives are analyzed and commented upon by the authors through the lens of a definite structuring framework – the framework of teachers’ *professional knowledge landscapes* – which, if used in a TM course, would probably be “transmitted” as a “technology without theory” condemned by Schwab (1971) as “rhetoric of conclusions”. Narrative approaches thus do not eliminate propositional knowledge. I don’t see how one could avoid presenting future teachers with unquestioned frameworks if one wants them to notice certain things in a

narrative. Future teachers have no personal experience of teaching with which these representations of practice could resonate. The instructor can only hint at the hypothetical character of the frameworks but may find it inappropriate to embed them in long-winded stories of processes of inquiry leading to them. TM instructors have varying reasons for avoiding narrative engagements, such as a lack of time or, like Aone, the fear of confusing future elementary teachers, who are notoriously very unsure about their knowledge (Marchand, 2010)).

Therefore, the landscape – to use Connelly and Clandinin’s metaphor – of TM instructors’ professional knowledge is constrained so that it is difficult, if not impossible, to totally avoid a transmission of propositional knowledge.

On the plane of research on TM’s instructors’ professional knowledge, therefore, it is also difficult to avoid asking how this knowledge addresses the four fundamental questions that the narrative tradition has issue with. As already mentioned, we do not think it would be enough to recount – as I have mostly done in this talk – individual TM instructors’ stories about their struggle with these questions, noticing similarities and differences and grouping tasks into types and epistemic actions. We would like to obtain more knowledge in the epistemic sense about TM instructors’ practices. This would require an effort of “instrumented synthesis” of individual instructors’ stories, but, at this point of our research, we do not know what the instrument we are looking for could look like. We don’t think there are any ready-made answers to our problem. We have to keep thinking.

## ACKNOWLEDGEMENTS

1. The research reported in this paper was supported by a Social Sciences and Humanities Research Council of Canada grant no. 410-2008-28981.
2. I wish to thank all those – students and instructors – who kindly agreed to participate in the study, which was conducted under the leadership of Helena P. Osana.

## Note

1. <http://www.merriam-webster.com/dictionary/ideology>

## REFERENCES

- Annenberg Media. (1997). *Teaching Math: A video library, K-4*. Retrieved April 1, 2011 from Annenberg Learner: <http://www.learner.org/resources/series32.html>
- Annenberg Media. (2000). *Looking at learning again, part 2*. Retrieved April 1, 2011 from Annenberg Learner: <http://www.learner.org/resources/series114.html>
- Aristotle. (written 350 B.C.). *Metaphysics*. Retrieved April 1, 2011, from The Internet Classics Website: <http://classics.mit.edu/Aristotle/metaphysics.1.i.html>

## Plenary lecture

- Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14-17, 21-22, 43-46.
- Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*(59), 389-407.
- Bruner, J. (1985). The role of interaction formats in language acquisition. In J. Forgas, *Language and social situations* (pp. 31-46). New York: Springer-Verlag.
- Bullock, S. M. (2011). *Inside teacher education. Challenging prior views of teaching and learning*. Rotterdam / Boston / Taipei: Sense Publishers.
- Burris, A. (2005). *Understanding the math you teach: Content and methods for prekindergarten through grade 4*. Upper Saddle River, NJ: Pearson.
- Carpenter, T. P., Empson, S. B. , Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.
- Cathcart, G., Pothier, Y., Vance, J., & Bezuk, N. (2000). *Learning mathematics in elementary and middle schools. Third edition*. Toronto: Pearson.
- Chazan, D., & Herbst, P. (2011). Challenges of particularity and generality in depicting and discussing teaching. *For the Learning of Mathematics*, 31(1), 9-13.
- Chevallard, Y. (1985). *La transposition didactique du savoir savant au savoir enseigné*. Grenoble: La Pensée Sauvage éditions.
- Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques* 19(2), 221-266.
- Clandinin, D., & Connelly, F. M. (1995). *Teachers' professional knowledge landscapes*. New York: Teachers College Press.
- Clandinin, D., & Connelly, F. M. (1996). Teachers' professional knowledge landscapes: Teacher stories - stories of teachers - school stories - stories of schools. *Educational Researcher*, 24(3), 24-30.
- Clarke, B., Grevholm, B., & Millman, R. (2009). *Tasks in primary mathematics teacher education. Purpose, use and exemplars*. New York, NY: Springer.
- Cochran-Smith, M., & Lytle, S. L. (1999). relationships of knowledge and practice: teacher learning in communities. *Review of Research in Education*, 24, 249-305.
- Cognition and Technology Group at Vanderbilt. (1997). *The Jasper Project: Lessons in curriculum, instruction, assessment and professional development*. New York: Routledge.
- Connelly, F., & Clandinin, D. J. (1995). Teachers' professional knowledge landscapes: Secret, sacred, and cover stories. In D. Clandinin, & F. M. Connelly,

## Plenary lecture

- Teachers' professional knowledge landscapes* (pp. 3-15). New York and London: Teachers College Press.
- Copeland, R. (1974). *Diagnostic and learning activities in mathematics for children*. New York: Macmillan.
- Copeland, R. (1984). *How children learn mathematics: teaching implications of Piaget's research*. New York: Macmillan.
- Even, R., & Ball, D. (2009). *The professional education and development of teachers of mathematics: The 15th ICMI Study*. New York: Springer.
- Fernstermacher, G. (1994). The knower and the known: The nature of knowledge in research on teaching. *Review of Research in Education*, 20, 3-56.
- Freudenthal, H. (1986). review of Chevallard, Y., 'La Transposition Didactique'. *Educational Studies in Mathematics*, 17(4), 323-327.
- Ginsburg, H. (1989). *Children's arithmetic: How they learn it and how you teach it*. Austin, TX: Pro-Ed.
- Ginsburg, H., Jacobs, S. F., & Lopez, L. S. (1998). *The teacher's guide to flexible interviewing in the classroom: Learning what children know about math*. Don Mills, ONT: Allyn & Bacon.
- Grimmett, P., & MacKinnon, A. M. (1992). Craft knowledge and the education of teachers. *Review of Research in Education*, 18, 385-456.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., et al. (1997). *Making sense: teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, 31(5), 3-15.
- Jaworski, B. (2007). Tasks: a fitting end to an era. *Journal of Mathematics Teacher Education*, 10(4-6), 201-204.
- Kamii, C. (1985). *Young children continue to reinvent mathematics. Implications of Piaget's theory*. New York, NY: Teacher's College Press.
- Kamii, C. (1989). *Young children continue to reinvent mathematics - 2nd grade. Implications of Piaget's theory*. New York, NY: Teacher's College Press.
- Marchand, P. (2010). Formation initiale des maîtres au primaire et en adaptation scolaire et sociale: quelle formation mathématique? In J. Proulx, & L. Gattuso, *Formation des enseignants en mathématiques: tendances et perspectives actuelles* (pp. 11-30). Sherbrooke, QC: Editions du CRP.

## Plenary lecture

- Ministère de l'Éducation, des Loisirs et du Sport, Gouvernement de Québec. (n.d.). Retrieved March 27, 2011, from [http://www.mels.gouv.qc.ca/dfgj/dp/programme\\_de\\_formation/secondaire/pdf/qep2004/qepsecfirstcycle.pdf](http://www.mels.gouv.qc.ca/dfgj/dp/programme_de_formation/secondaire/pdf/qep2004/qepsecfirstcycle.pdf)
- New Brunswick, Department of Education, Curriculum Development Branch. (2011). *Foundation for the Atlantic Canada Mathematics Curriculum*. Retrieved March 27, 2011, from Government of New Brunswick, Department of Education and Early Childhood Development: <http://www.gnb.ca/0000/publications/curric/apecmath.pdf>
- Reys, R., Lindquist, M., Lambdin, D. V., & Smith, N. L. . (2008). *Helping children learn mathematics*. Toronto: Wiley.
- Sánchez, V., & García, M. (2009). Tasks for primary student teachers: a task of mathematics teacher educators. In B. Clarke, B. Grevholm, & R. Millman, *Tasks in primary mathematics teacher education. Purpose, use and exemplars* (pp. 37-56). New York, NY: Springer.
- Schön, D. (1987). *Educating the reflective practitioner: Toward a new design for teaching and learning in the professions*. San Francisco: Jossey-Bass Publications.
- Schwab, J. (1971). The practical: Arts of eclectic. *School Review*, 79(4), 493-542.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Sierpinska, A. (1997). Formats of interaction and model readers. *For the Learning of Mathematics* 17(2), 3-12.
- Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing mathematics for elementary school teachers. Instructor's edition*. New York, NY: W.H. Freeman and Company.
- Van de Walle, J. (1998). *Elementary and middle school mathematics. Teaching developmentally. Third edition*. New York: Longman.
- Vergnaud, G. (1983). Multiplicative structures. In R. Lesh, & M. Landau, *Acquisition of mathematics concepts and processes* (pp. 128-175). Orlando, FL: Academic Press, Inc.

# THE STRUCTURE AND DYNAMICS OF AFFECT IN MATHEMATICAL THINKING AND LEARNING

Markku S. Hannula

University of Helsinki

*In this paper, I will review the development of research on affect in mathematics education since 1990s and forecast some directions for future development. One trend of development has been the elaboration of the theoretical foundation for affect. I will suggest that a useful description of the affective domain can be based on distinctions on three dimensions: 1) cognitive, motivational and emotional aspects of the affect, 2) rapidly changing affective states vs. relatively stable affective traits, and 3) the social, the psychological and the physiological nature of affect. Another direction of development has been to explore the structural nature of affect empirically. I will review some instruments that have been developed to measure different dimensions of beliefs, motivation and emotional traits. Moreover, I will look at some empirical results concerning how the different dimensions are related to each other and how they develop over time.*

## INTRODUCTION

Mathematics is typically considered as the most objective and logical of academic disciplines. Yet, emotions, attitudes and motivation play an important role in contemporary research on mathematics education. From a practical perspective, mathematics-related attitudes and motivation are perhaps the most important determinants of mathematics attainment, because they determine how much people choose to study mathematics after it becomes optional. Compared with motivation to study mathematics, it is irrelevant how talented the student is for mathematics. Moving beyond the individual, there is consensus in educational policy in most countries that society has a need for mathematically educated persons in scientific and technical fields.

Affective issues are important in mathematics education also from a scientific perspective. When we investigate creativity, genuine problem solving, proofs and other higher-level cognitive processes, we see that cognition is intrinsically intertwined with emotions. However, we do not yet understand these processes well enough.

Research on mathematics-related affect has been a topic of interest at least since the 1970s. However, the reason for such interest has varied across decades and this has been also reflected in the variety of theoretical and methodological approaches to the topic. McLeod's (1992) review and reconceptualization of research has been extremely influential in this field of study. According to Harzing's 'Publish or Perish' software, (Harzing, 2011), it has received over 600 citations so far. The review provides an excellent overview of the state of art in the early 1990s.



## Plenary lecture

Moreover, McLeod's conceptualization of the research area as well as his suggestions for future research have had a major influence on research on mathematics-related affect for the last two decades.

In this article, I will first review McLeod's seminal paper, then we will summarize the main criticisms of his approach and most important new ideas and findings since 1990s, with a special focus on work done within CERME.

As a synthesis of these reviews, this paper will suggest a new theoretical foundation for research on mathematics-related affect. The most important notions in this new framework are 1. a distinction between trait and state-aspects of affect; 2. perceiving emotions, cognition and motivation in a synergistic relationship; and 3. the identification of biological, psychological and social levels of affect.

Based on this conceptual framework, we will elaborate some structural properties of mathematics-related affect and review a number of instruments developed to measure affect towards mathematics. In a more speculative tone, we will use the theoretical framework and previous research to suggest some dynamic relationships between different dimensions of affect. Finally, the paper will look for some future venues for research on mathematics-related affect.

### **History pre 1990s**

In this section of the article, I will provide an overview of the main conclusions that McLeod made in his article as a summary of research on mathematics-related affect prior to 1992. In the era of behaviourism, affect had been largely neglected, being considered an imaginary construct. As an exception, social psychology measured attitudes through a large number of quantitative survey studies. These studies had identified differences between countries and genders and an overall tendency for students' affective relation with mathematics to decline over the school years.

Within research on mathematical beliefs, McLeod had identified four main objects of these beliefs: mathematics as a discipline, self, mathematics teaching (and learning), and social context. In his conceptualization, beliefs about self were related to metacognition and self-regulation and included also motivation. As an important aspect of beliefs about context, McLeod had recognized Cobb, Yackel and Wood's (1989) work on social norms in the (mathematics) classroom.

One of McLeod's main concerns with respect to previous research was the lack of a decent theoretical framework. Research on mathematics-related affect was based on local theories, such as self-efficacy, mathematics anxiety, attribution theories, and aesthetics. However, from a holistic point of view, the local theories used confusing and ambiguous terminology.

McLeod identified cognitive science and cognitive psychology as promising new trends for mathematics-related affect in the 1990s. Cognitive science had accepted the importance of affect. More specifically, cognitive scientists had acknowledged

the significance of affect (interest, confidence) in metacognition. However, at that time, most cognitive scientists avoided affect in their theorizations in order to avoid some of the complexity.

According to McLeod, cognitive psychology emphasized more theoretical issues than cognitive science. Cognitive psychologists used qualitative methods in order to construct an empirically well-founded theoretical framework for the role of affect in mathematical thinking and learning. Their elaborations emphasized beliefs and emotions as central concepts.

As a summary for his review on the literature, McLeod (1992) suggested that research on mathematics related affect should use a “combination of techniques”, i.e. mixed methods. For a theoretical framework he suggested using Mandler’s (1989) theory, which provided not only a sufficient conceptualization for mathematics-related affect, but also for dynamics of affect in mathematical thinking and learning. In this framework beliefs were seen as an element that influenced the initiation of emotions. Moreover, repeated emotional reactions were seen as the origin of attitudes. Regarding the origin of beliefs, McLeod considered both social context (culture) and individual experiences to contribute to their formation.

### **Work that builds on McLeod’s paper**

McLeod’s work has had a significant influence on later research and his foundation has proved to be fruitful for accumulating evidence. Much of the research has adopted a simplified version of McLeod’s framework where emotions, attitudes and beliefs are located on a continuum where beliefs are seen to be the most cognitive and stable, emotions as least cognitive and stable, and attitudes in the middle for both dimensions. Fewer studies have elaborated the dynamic nature between beliefs, emotions and attitudes.

One specific field of research on mathematics-related affect that has accumulated very strong evidence over the years is the role of gender. Studies have produced very consistent results that indicate that, across age and performance levels, female students tend to have lower self-confidence in mathematics than male students (e.g. Hannula, Maijala, Pehkonen & Nurmi, 2005; Leder, 1995). Lower self-confidence among female students has been found even at the level of individual tasks, in the case of both correct and incorrect answers (Hannula, Maijala, Pehkonen & Soro, 2002). Related to low self-confidence, female students also suffer mathematics anxiety more often than male students (Frost, Hyde & Fennema, 1994; Hembree, 1990). These results in affect provide an explanation to why female students usually choose not to study optional mathematics, especially when we consider that female students may have higher performance levels in arts and social sciences. Lower self-efficacy is also likely to explain why female students rely on school-taught solution methods and avoid non-standard or their own solution methods that include an element of risk.

There is no reason to believe that the low level of female students' self-efficacy beliefs is a natural and permanent characteristic of the female sex. The research has accumulated evidence for the hypothesis that female students' lack of confidence in mathematics is consistent with their teachers' beliefs (Li, 1999; Soro, 2002; Sumpter, 2009) and that teachers' typical interaction patterns with male and female students may thus attribute to the generation of gender differences. Mathematics teachers tend to believe that their male students often have hidden talent, but due to being lazy and careless they underperform, while female students tend to reach their performance due to diligence and hard work even if they are not very talented. Such teacher beliefs are assumed to lead to different feedback to male and female students and thus to contribute to the observed gender differences in self-efficacy beliefs.

A large number of studies have explored the relationship between mathematical affect and achievement. Ma and Kishor (Ma & Kishor, 1997a, b; Ma, 1999) have summarised much of that research in two meta-analyses. In one of these studies, they synthesised 113 survey studies of the relationship between attitude towards (=liking) mathematics and achievement in mathematics. The causal direction of the relationship was from attitude to achievement. Although the correlations were weak in the overall sample, they were stronger throughout grades 7 to 12, and in studies that had done separate analysis of male and female subjects. (Ma & Kishor, 1997a).

In another meta-analysis, Ma and Kishor (1997b) summarised 143 original studies on the effect of mathematics-related beliefs on achievement in mathematics. They found out that students' self-concept, family support and perception of mathematics as a male domain were positively related to their achievement in mathematics. Moreover, correlations between self-concept and achievement were stronger in studies that had done separate analysis of male and female subjects.

However, there has been criticism of studies that do not use a longitudinal design (see Ma & Xu, 2004). Minato and Kamada (1996) reviewed eight studies that had used a cross-lagged panel correlation technique (a longitudinal design) in order to synthesize findings on the causal relationship between attitude towards mathematics and achievement in mathematics. In most of the studies, there was no predominance of either attitude or achievement. However, in the few instances that predominance was found, the causal direction was from attitude to achievement. However, Ma and Xu (2004) found a contrasting result with a larger and more representative sample. According to their study, the dominant causal relationship is from achievement to attitude.

Research on mathematics-related emotions has been less active than research on attitudes and beliefs. However, several studies have confirmed that experts control their emotions better than novices (e.g. Allen & Garifio, 2007; Schoenfeld, 1985). There has been also some attempt to include physiological measures, e.g. heart rate (Isoda & Nakagoshi, 2000), skin conductivity, and muscle tension, to research on

emotions. In the case of anxiety, physiological measure and questionnaire data did not match (Gentry & Underhill, 1987; Dew, Galassi, & Galassi, 1984)

### Main criticisms of McLeod's framework

Although McLeod's (1992) framework has dominated research in mathematics-related affect since 1990s, criticism has accumulated since then, regarding terminology, blind spots of the conceptual framework and the overall foundation of the framework. Regarding the terminology, researchers on mathematics-related affect had not succeeded in unifying their language. When Furinghetti and Pehkonen (2002) collected a virtual panel of these researchers to evaluate a number of definitions that these same researchers had suggested for the concepts *attitude*, *belief* and *conception* in their papers, the researchers could not agree on any of the their definitions.

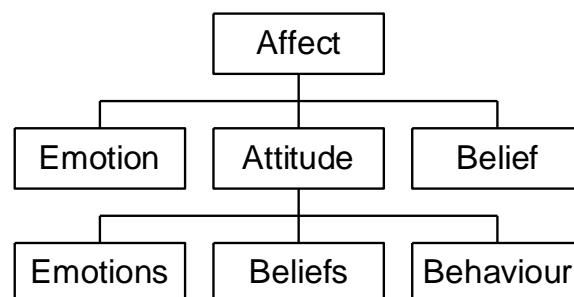
The problem has been raised repeatedly over the years. For example, in the PME 2004 research forum on affect, Gerald Goldin (2004) said:

We do not now have a precise, shared language for describing the affective domain, within a theoretical framework that permits its systematic study (p. 109).

And this perspective has been repeated also more recently:

theories not yet well-developed, terminology used differently and ambiguously, and varying instruments, some untested, make the literature difficult to interpret, and leave researchers open to criticism. (Chretcheley, 2008, p. 147)

Probably the most problematic concept in McLeod's framework is attitudes. Within mathematics attitude research, attitudes have typically been defined as consisting of cognitive (beliefs), affective (emotions) and conative (behaviour) dimensions (e.g. Hart, 1989). If we try to combine the tripartite framework with McLeod's, we see that attitude is at the same time a parent and a sibling to emotions and beliefs.



**Figure 1. An unsuccessful attempt to combine McLeod's (1992) framework for affect with Hart's (1989) tripartite framework for attitude.**

In fact, researchers have provided a variety of definitions of the concept 'attitude'. In addition to the tripartite definition above, some have used a bi-dimensional definition (emotions, beliefs) or simply identified attitude as positive or negative

degree of affect. Many researchers have given only an implicit definition through the instrument they have chosen to use. (Di Martino & Zan, 2010).

There are also more fundamental problems with the concept. Attitude is an **observer's** category which involves several psychological structures (Ruffel, Mason & Allen, 1998; Hannula, 2002a). Hence, it does not really help us to understand the inner world of the person, whose attitude we have observed. In fact, teachers often use the concept attitude as 'a claim of surrender', *i.e.* as an attribution to student's uncontrollable failure (Di Martino & Zan, 2009).

In McLeod's (1992) framework, emotions are considered to be unstable (at least less stable than beliefs or attitudes). This view has been criticized, because people can have very stable patterns for emotional arousal across similar situations (e.g. anxiety, frustration, Aha!) How would this differ from beliefs that appear only in appropriate context (e.g. self-efficacy beliefs in a test situation)? We shall return to this issue below, when we discuss the state and trait aspects of affect.

McLeod's framework did not pay enough attention to the variety of different emotions. Although McLeod (1992) had pointed out the importance of Buxton's (1981) study *Do you panic about maths* and Mason, Burton and Stacey's (1982) studies on the Aha!-experience, in his framework attitude was seen through positive/negative duality. This view misses important distinctions. For example, boredom, anger and shame are all negative emotions, but very different from each other. The three emotions appear in different situations and they have quite different impacts on behaviour. In similar way elation, pride and relief are very different positive emotions

Moreover, more recent research in mathematics education has highlighted affective concepts that are (or seem to be) missing from McLeod's conceptualization, such as values (DeBellis & Goldin, 1997, Bishop, 2001), motivation (Hannula, 2006b; Yates, 2000) and identity (Beijaard, Meijer & Verloop, 2004; Frade, Roesken & Hannula, 2010, Sfard & Prusak, 2005).

Also the 'social turn' in mathematics education (Lerman, 2000) has highlighted the problems of the primarily individual focus of McLeod's framework. Yackel and Cobb (1996) have elaborated the relationship between individual beliefs and the norms that are their social parallel. Moreover, the social view speaks about identity instead of beliefs or attitudes.

### **New findings and ideas since McLeod**

Since McLeod published his paper in 1992, there have been important advances outside mathematics education research that have opened up new opportunities for research in mathematics-related affect.

Within educational psychology, motivation research has developed several theoretical approaches (for an overview see Murphy & Alexander, 2000). One of the

most influential is achievement goal theory (Pintrich & Schunk, 2002). It is a sociocognitive theory, which focuses on students' self-set goals in achievement situations, and is interested in the student's reasons for engage with learning tasks (Middleton, Kaplan, & Midgley, 2004). Different terminologies are used for two main goal orientations: A student who focuses on learning the task is said to have a mastery, task, or learning orientation, while students whose primary interest is to impress others with their performance are said to have a performance, ego, or ability orientation, but there are only nuanced differences between the different terminologies (Ames 1992; Järvelä, 1996; Pintrich & Schunk, 2002).

Different researchers have found rather comparable positive relationships between mastery goal orientation and achievement (Friedel, Cortina, Turner, & Midgley 2007; Midgley *et al.*, 1998). Results concerning performance goal orientation and achievement have been less consistent. Some have identified negative learning behaviour, while other results indicate performance orientation to lead to positive learning behaviour and achievement (Freeman, 2004; Midgley *et al.*, 1998). This has led to a need to differentiate between so-called performance-approach and performance-avoidance goal orientations (Elliot & Harackiewicz, 1996). More recent results have indicated that students may be influenced by several goal orientations simultaneously, and the emphasis of each orientation is influenced by the situation (e.g. Vollmeyer & Rheinberg, 2000).

Affect has also been given a significant role in elaborations of self-regulation. In mathematics education, the role of affect in self-regulation has been elaborated, for example, by Malmivuori (2001, 2006) and Hannula (2006b). Also the concept meta-affect (DeBellis & Goldin, 1997) relates to self-regulation.

The embodied view of cognition (e.g. Maturana, 1988; Brown & Reid, 2006; Núñez, 2006) and advances on the neurophysiology of affect (Damasio, 1994, LeDoux, 1998) have radically challenged the earlier view on human mind. Most notably this research has highlighted the indispensable role of emotions in human reason.

Increased computational capacity has allowed the use of more advanced quantitative methods, e.g. Meta-analysis and Structural Equation Models. These methods have made it possible to test more complex hypotheses with data from large scale surveys. For example, Williams and Williams (2009) used

...a structural equation model in which the mutual influence of self-efficacy and performance in mathematics is represented as a feedback loop. This model was estimated in each of 33 nations on the basis of data on the mathematics self-efficacy and mathematics achievement of 15-year-olds. The model was a good fit to the data in 30 nations and was supportive of reciprocal determinism in 24 of these, suggesting a fundamental psychological process that transcends national and cultural boundaries. (Williams & Williams, 2009, p. 453)

## Issues raised in previous CERME meetings

Next, I will review the main issues of the Working Group (WG) on affect in CERME conferences 3, 4, 5, and 6, between 2003 and 2009). This review is very much also a personal journey, because I was a WG coordinator at CERME 3 (2003) and chair of the Affect WG for the three following CERME conferences. What I will present later in this article as a new (meta)theoretical framework for the study on affect is deeply influenced by the discussions in the CERME working group on affect.

Much of the discussions in the affect group in each of these CERME have focused on the conceptual framework and terminology. This has increased our awareness of being specific about the concepts that we use. We have realized that it is not sufficient to give definitions of the concepts that are being used in a particular study, but we have to explicate their relations to the other dimensions of affect research as well. A significant step forward was the graphic representation of the conceptual field that Peter Op 't Eynde drew for the final presentation session of the affect group at CERME 5 (Figure 2). Firstly, the figure identifies three main conceptual categories: cognition, motivation and affect, and their partial overlapping.

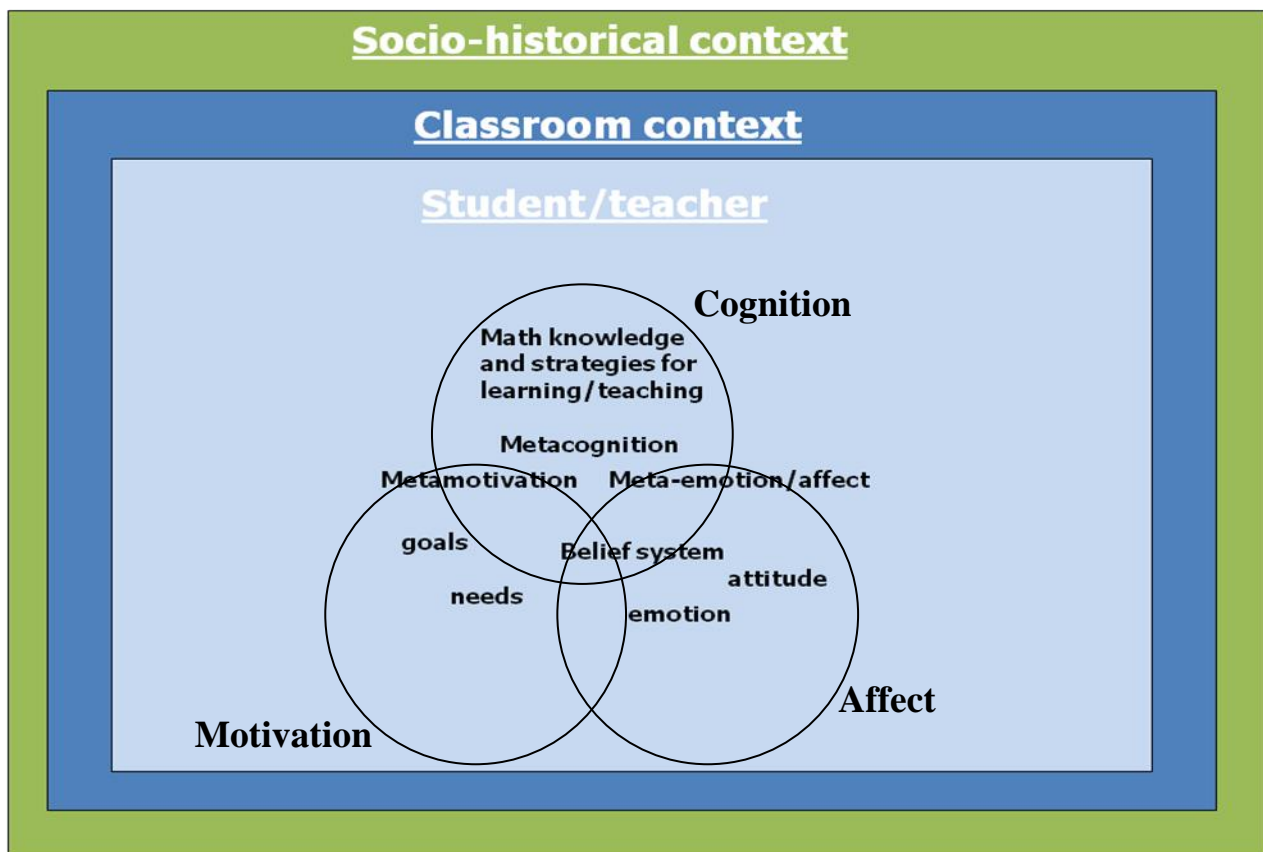


Figure 2. A graphic representation of the different dimensions of mathematics-related affect and their relationships, presented at CERME 5 (Hannula, Op 't Eynde, Schlöglmann & Wedege, 2007, p. 204).

## Plenary lecture

The figure also positions several of the frequently-used concepts in relation to each other and in relation with the three overarching concepts cognition, motivation and affect. Thirdly, the figure identifies the local classroom context and the socio-historical context where the individual student's or teacher's affect is being formed and is developing. Since CERME 5 we have always returned to this figure as an overall framework where we locate individual studies and their relations.

The dynamic nature of affect is another recurrent issue through the history of CERME conferences. How do emotions and other less stable affects interact with cognitive processes? How are relatively stable affects (e.g. beliefs, attitudes and motivation) formed in the first place? How do they develop over time? How can we change these through interventions? As an example of the complexity of the dynamics of affect, let us consider the four different aspects of stability that we identified at CERME 6 (Hannula, M. S., Pantziara, M., Wæge, K., & Schlöglmann, 2009):

- 1) The *state* and *trait* aspects of affect. This will be discussed in more detail below.
- 2) Resistance to change. How strongly are individual beliefs and attitudes held? Which of them change 'naturally' over time? How difficult it is to influence them?
- 3) Robustness of constructs. Are the affective constructs held in a similar way across age levels? For example, some dimensions can not be reliably measured in younger populations. Does this suggest that these constructs are not developed until an older age? Would some of the constructs disappear in older populations? Are some affective constructs formed in a specific socio-historical context (e.g. the belief of mathematics as a male domain) and are they similar across the different cultures? Would some constructs that are relevant now disappear in the historical development of the society?
- 4) Relative stability in relation to other persons. Several studies have observed an overall decline in mathematics-related affect over the school years (e.g. Eccles, Adler, Futterman, Goff, Kaczala, Meece *et al.* 1985; Hannula, Maijala & Pehkonen, 2004). However, do those who have the most positive relationship with mathematics in the early years continue to have more positive relationship with mathematics than their peers who had a more negative affect to begin with?

The refinement of the conceptual framework and the focus on the dynamic nature of affect have highlighted the need to explore the structural properties of affect.

A third issue that we have discussed repeatedly in CERME is methodology. I will not go into detail here, but there has been a trend towards mixed methods that combine quantitative and qualitative methods, as well as a trend to adopt the more complex computational tools that have become available.



## A NEW (META)THEORETICAL FOUNDATION

After this lengthy review of old theories, findings and their criticisms, we will finally begin our constructive efforts. The framework that I will construct below could be seen as a metatheoretical foundation for research in mathematics-related affect. Because of its broadness I do not expect this framework to be used as such in any single empirical study. However, this framework helps to identify similarities and differences between studies in this field, and it is probably useful for relating a variety of theories to each other. Moreover, this framework is likely to help in generating a shared language for researchers to communicate their findings and theories.

The framework will be based on distinctions on three dimensions: 1. cognitive, motivational and emotional aspects of the affect; 2. rapidly changing affective states vs. relatively stable affective traits; and 3. the social, the psychological and the physiological nature of affect.

### The need to integrate cognition, motivation and emotion

Historically, psychologists have adopted three components to describe human learning: cognition, motivation, and emotion [...]. Yet, theorists and researchers have tended to study these processes separately, attempting to artificially untangle them rather than exploring their synergistic relations in the complexity of real life activities. (Meyer & Turner, 2002, p. 107)

My personal journey into mathematics-related affect started from beliefs research under the supervision of Erkki Pehkonen (Hannula, Malmivuori & Pehkonen, 1996). Trying to understand the formation of beliefs, I realized the importance of emotional experiences (Hannula, 2002a, 2003). When building an understanding of emotions, their relationship to personal goals became unavoidable (Hannula, 2002b, 2006b). Hence, my personal journey led me to attempt an integration of cognition, emotions and motivation. (Hannula, 2004, 2006a)

Of course, I have been influenced by the work of colleagues. If we accept values to be an aspect of motivation, all these three categories are present already in the tetrahedral framework of DeBellis and Goldin (1997), where the vertices are emotions, attitudes, beliefs and values. Another important influence has been Schoenfeld's framework for teacher decision making, where the key components are knowledge, goals and beliefs (Schoenfeld, 1998). In Schoenfeld's terminology, 'beliefs' is a broad category which includes also emotional aspects. Goals, on the other hand, are clearly a motivational concept.

In my framework, the *cognitive* domain includes knowledge, beliefs and memories, i.e. those mental representations to which it makes sense to attribute a truth value (cf. Goldin 2002). For example, "I have solved this task before", "Mathematics is useful" and "I can solve this task using an equation" belong in this category. With some hesitation I have also classified other cognitive schemata (e.g. scripts and

concepts) in this category as their applicability can be seen as a sort of truth value. In the category of *emotions* belong joy, pride, sadness, frustration, anxiety and other feelings, moods and emotional reactions. The third category, *motivation*, is perhaps the most difficult to define. The most important characteristic of this category is that motivation reflects personal preferences and explains choices. The distinction from the cognitive aspect is that preferences are subjective and it is not possible to attribute truth value or applicability to them. Motivation varies from very local preferences (“This would be a perfect moment for a cappuccino”) to a variety of different levels of goals (“I want to solve this task”, “I want my peers to think that I am clever”) and very global needs such as needs for nutrition and social belonging. Although the basic needs seem very universal, there are individual differences to how much importance they are given in different situations.

Some of the distinctions between the categories may seem arbitrary. However, in this integrative approach, the point is not simply to acknowledge all three aspects. The purpose is to go beyond that, and focus on the synergistic relations between the three. The distinctions between these categories make more sense when one focuses on their function in human thinking and behaviour. From the functional point of view, cognition codes the personal information about self and the environment (e.g. “I don’t believe that I can solve any of my homework”), motivation gives direction for behaviour through giving preference to some self-environment relationships over some others (e.g. “I want to show my parents that I at least try”). Success or failure in motivation-directed behaviour (e.g. solving a mathematics task) is reflected in emotions (e.g. relief). These emotions, in turn, can influence cognition through shifting the focus of attention (e.g. from ego to task), which may modify motivation (e.g. from showing effort to solving the task).

### **The state and trait of mathematics related affect**

I was not successful in my effort to track down the origin of the distinction between trait and state aspects of psychological constructs. However, the first instance where I was able to find it (Bergmann 1955), considered the terms ‘state’ and ‘trait’ idiomatic. The distinction between trait and state (Spielberger, 1966) was an important step forward in anxiety research. More recently, Dweck (2002) has claimed that a similar distinction (traits and processes) is also relevant in motivational systems and that these are formed at an early age.

The state and trait aspects of affect towards mathematics have been implicitly present in most of the research done. However, these two temporal aspects have seldom been addressed explicitly. The rapidly changing affective state has typically been addressed as an element in problem solving. For example, the continuously fluctuating emotions and beliefs may influence the critical choices that determine whether the problem will be solved or not. These affects are situational and contextual. However, there is also a stable pattern in how an individual feels and thinks in these different contexts and situations, i.e. an affective trait. For example, in

any classroom situation high achievers are likely to have more positive expectations and affect than low achievers. This distinction has been reflected in different theoretical frameworks. In McLeod's (1992) conceptualisation, beliefs were considered more stable than attitudes, which were more stable than emotions. Also in Goldin's (2002) framework there is a distinction between local and global (more stable) affect. However, both McLeod and Goldin attribute stability to beliefs and rapid changes to emotions.

Unlike McLeod (1992) and Goldin (2002) I see that beliefs have both a state aspect and a trait aspect. While a student may have a belief trait that he is not very good with mathematical tasks, his belief state regarding a specific task evolves as he reads the task and begins to solve it. In a similar fashion, although the emotions of a student may fluctuate and change rapidly during problem solving, students also have very stable patterns of emotional reactions. By this we mean that each individual has typical emotional reactions to typical situations in the mathematics classroom. This is most clear in cases of mathematics anxiety, where worry and fear are typical reactions to many learning situations. This trait aspect of emotions can also be called emotional disposition, and it is also the core of different conceptualizations of attitude.

Also the third element in the present framework, motivation, has a state aspect and a trait aspect. The trait aspect of motivation is related to the overall values the person attributes to mathematics and to the general motivational orientations for learning. However, these traits are not sufficient to understand personal choices during problem solving or learning processes. During such processes one sets more local goals, that represent the motivational state at that time.

The trait and state aspects of cognition, motivation and emotion are summarized in Table 1.

	<b>Cognition</b>	<b>Motivation</b>	<b>Emotion</b>
<b>State</b>	Thoughts in mind	Active goals	Emotional state
<b>Trait</b>	Concepts, facts, scripts, etc.	Needs, values, desires, motivational orientations	Emotional dispositions (attitude)

**Table 1. The state and trait aspects of cognition, motivation and emotion.**

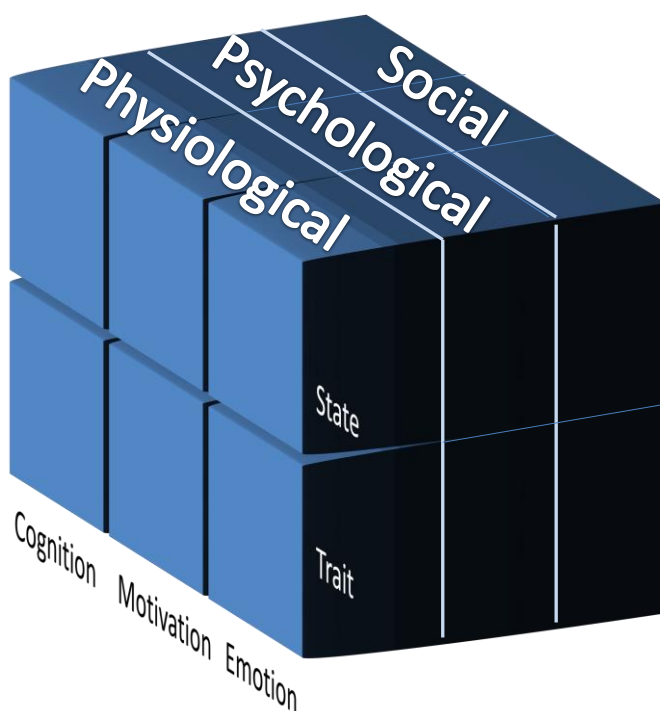
### **How about the social turn and the embodied cognition?**

In order to guide you towards the social and biological dimensions of affect, I will now elaborate a little on emotions. Firstly, recall that we consider emotions here in their full spectrum. Joy, fear, sadness, curiosity, shame etc. have each their specific

characteristics in the dynamics of affect, and they can not be reduced on a single dimension.

Within theories of emotion we can distinguish three different traditions. The first tradition is the Darwinian tradition, which sees emotions as evolutionary functional means of body regulation and social coordination. This view highlights the biological basis of emotions but also acknowledges their social importance. The Freudian tradition conceptualises affects as derivatives of the instinctual drives and emphasizes the unconscious origin and regulation of emotions. (Taylor, Bagby & Parker 1997, p. 11 & p. 14)

The cognitive approach (e.g. Lazarus, 1991; Power & Dalgleish, 1997) argues that emotions are primarily psychological phenomena, generated and guided by cognitive appraisals. Nowadays, there is general agreement that emotions consist of three processes: physiological processes that regulate the body, subjective experience that regulates behaviour, and expressive processes that regulate social coordination (Buck, 1999; Power & Dalgleish, 1997; Schwarz & Skrunik, 2003; Taylor, Bagby & Parker, 1997).



**Figure 3. Identifying the social and psychological dimensions to state and trait of cognition, motivation and emotion.**

Within the cognitive domain it is easy to identify similar distinctions between subjective experience and expressive behaviour (typically talking). Moreover, social theories of learning e.g. *social interactionism* (Bauersfeld, Krummheuer & Voigt, 1988), *emergent perspective* (Cobb & Yackel, 1996) and *didactic contract*,

## Plenary lecture

(Brousseau, 1997) have highlighted the social construction of meaning. There is also a level of physio-cognitive processes, such as gestures as part of thinking (e.g. Núñez, 2006). With respect to motivation, it is possible to make a distinction between physiological, psychological and social needs (Nuttin 1984). On the other hand, the norms that social groups negotiate and form (Yackel & Cobb, 1996; Cobb & Yackel, 1996; Partanen, 2011) can be seen as shared goals, which are another social aspect of motivation.

It should be noted here that the social aspect can be further elaborated. There is the classroom microculture of teacher-student interactions, but also the more institutionalized school culture and broader socio-cultural situation where schooling takes place (Cobb & Yackel, 1996, Partanen 2011).

As a summary, we can add a third dimension to our framework on affect (Figure 3). Furthermore, we can identify the typical state and trait type constructs for psychological, psychological and social level of the affective domain (Table 2).

	<b>Physiological</b>	<b>Psychological</b>	<b>Social</b>
Affect as a <b>state</b>	Neural activation, physiological adaptation.	Feelings, emotions, thoughts, meanings, goals	Social interaction, communication, classroom ‘weather’
Affect as a <b>trait</b>	Brain structure, neural connections.	Attitudes, values, beliefs, motivational orientations	Norms, social structures, classroom climate

**Table 2. Physiological, psychological and social aspects of affective states and traits**

The physiological trait refers, for example, to the structural properties of the brain that are slow to change. On the other hand, the physiological state refers to the dynamic activity in the brain and the hormonal adaptation of the body (e.g. adrenaline rush). With regard to the state and trait of the social dimension of affect, we can perceive the social traits as the overall social ‘climate’ of a group, which includes the norms and the discourse that the group has adapted to, and the social relationships that they have formed. Some of the norms and structures are institutionalized, e.g. the rules of a school and the educational legislation. The state aspect of this social dimension is the ‘weather’ in the class, which is constituted by the verbal and non-verbal communication in the group.

### **The structure of students’ view of mathematics**

The structural nature of affect has been emphasized repeatedly in mathematics education research. One of McLeod’s (1992) recommendations for future research on

mathematics-related affect was to build a theoretical framework that would allow exploration of the relationships that explain the dynamics of affect. In one of the early elaborations of the systematic nature of affect, Green (1971) characterized beliefs systems in terms of three dimensions: quasi-logicalness (primary vs. derived beliefs), psychological centrality (core beliefs vs. peripheral beliefs), and cluster structure (beliefs are held in clusters around specific situations and contexts, more or less isolated from each other) (c.f. Furinghetti & Pehkonen, 2002).

Before delving deeper into the structural properties of affect, I now review a number of survey instruments. Each instrument's subscales have been classified into beliefs, attitudes (emotional traits) and motivations (including values) (Table 3).

<b>Instru- ment</b>	<b>Beliefs (about)</b>				<b>Emotion trait</b>	<b>Motivation</b>
	mathe- matics	sel f	teaching / learning	social context		(including values)
MAS		1	1	3	1	3
BMPS	0 - 1	1	1			1 -2
AtMI		1			1	2
MRBQr		1	4 -5	1	1	1 -2
VoM	1	2	1	1	1	1
PALS		2		11	1	6
MEW		2	2		1	6
AEQ-M					8	

**Table 3. A review of the dimensions of mathematics-related affect measured by some instruments.**

Key: MAS = Mathematics Attitude Scales (Fennema-Sherman, 1976), BMPS = Beliefs about Mathematical Problem Solving (Kloosterman & Stage, 1992), AtMI = Attitude towards Mathematics Inventory (Tapia & Marsh, 2004), MRBQr = Mathematics- Related Belief Questionnaire, refined (original by Op 't Eynde and De Corte, 2003, refined by Diego-Mantecón, Andrews and Peter Op 't Eynde, 2007), VoM = View of Mathematics (Hannula, Kaasila, Laine & Pehkonen, 2006; Roesken, Hannula & Pehkonen, 2011), PALS = Patterns of Adaptive Learning Systems (Midgley *et al.* 2000), MEW = The Motivation and Engagement Wheel (Martin, 2001), AEQ-M = the Academic Emotions Questionnaire-Mathematics (Pekrun, Goetz & Frenzel, 2005).

Such survey instruments focus typically on individual traits. However, some also provide a window on the social dimension. For example, PALS includes several measures on the respondent's perception of the motivational orientation emphasized by parents, and peers. Because of the incoherent use of language in this research domain, my classification is based on the original items of the scales. Attribution of

some of the components was problematic, and rather than emphasizing my subjective doubtful interpretation for those components, their classifications were left uncertain (e.g. BMPS has one component that could be interpreted either as a belief about mathematics or as a motivation for studying mathematics, see Table 3). If classification was very speculative, I omitted the scale from this table.

Most instruments have their specific focus. For example, MRBQr (Diego-Mantecón, Andrews & Op 't Eynde, 2007) has four to five scales on teaching and learning, while no other instrument has more than two scales in this category. In a similar way, PALS (Midgley *et al.* 2000) has a focus on social context and motivation, MEW (Martin, 2001) on motivation and AEQ-M (Pekrun, Goetz & Frenzel, 2005) on different dimensions of emotional traits. On the other hand, most instruments are fairly broad, covering several categories. However, broad instruments have only simplified measures on some of the categories. For example, while VoM (Hannula, Kaasila *et al.*, 2006; Roesken, Hannula & Pehkonen, 2011) has a subscale in each of the categories, it does not have more than two subscales in any of them.

Empirical research on mathematics-related beliefs has identified an overall pattern, where positive beliefs are related to each other and with positive emotions (e.g. Hannula, Kaasila *et al.*, 2006; Roesken *et al.*, 2011).

Although it is generally assumed that there is a relationship between mathematics-related motivation and beliefs, the theories about their relationships are fairly recent (Op 't Eynde, De Corte, & Verschaffel, 2006; Hannula, 2006b). Research has identified a positive relation between mastery orientation and attitudes, effort, competence beliefs (Seo, 2000, Hannula & Laakso, 2011; Kaldo & Hannula, manuscript) and positive emotions (Kumar, Gheen, & Kaplan, 2002; Midgely *et al.*, 1998; Pekrun, Elliot, & Maier, 2006).

### **The development and dynamics of affect**

In this section, I will elaborate my personal understanding of the interplay of different aspects of affect through an example.

Let us consider a situation, where a teacher is about to meet a new class for the first time. Before entering the classroom, the teacher has his preconceptions about the class, based on his previous encounters with students of the same age in the same school. He is also in a certain physiological state, perhaps hungry or tired (this state is influenced by his physiological traits). The teacher's physiological state and preconceptions influence what kind of emotions, goals and expectations he has before entering the classroom. In a similar manner, the students in the class have their emotions, goals and expectations. Moreover, as a group, they have a set of social norms they adhere to (especially if they have studied together for a long time).

When the teacher enters the room, his state of mind will influence how he will interpret what he sees and hears. However, the environment (e.g. the lighting and acoustics, how is the entrance located in the class) will also influence his

impressions. When the teacher and the class begin to interact, they also negotiate the teacher role in this class. This negotiation is basically between the social norms of the class and the individual goals of the teacher, but it takes place through the communication between the teacher and his students. Through this communication the teacher and students influence each others' state of mind (emotions, thoughts, goals etc.) Moreover, they may end up changing the environment (e.g. seating of the students). This negotiation will influence the social norms of the class, the psychological traits of the teacher and to some extent also the negotiators' physiological traits.

After the lesson, both teacher and his students will engage in some reflection on the first lesson. This reflection is influenced by the cultural-historical background they are in and it may lead to further changes in the social norms of the class and the psychological and physiological traits of the teacher. The process will continue in their future lessons.

What I want to highlight here, is the interplay between the individual and the social and between the trait and state. Through the process of communication the individual state interacts with the social state. The individual is forming the social through communication and, through this very same communication, the social is forming the individual. The continuously dynamic psychological state of the individual is always influenced by three factors: the previous state, the traits and the situational information. The situational information and the previous state always determine which (if any) of the traits is activated. On the other hand, the activation and possible reflection of these different states is gradually changing the traits. The process is analogous for physiological and social traits.

## **POSSIBLE NEW DIRECTIONS**

In the future, research on mathematics related affect is likely to make advances in the directions that have not yet been explored sufficiently. For example, quantitative studies on the structural and dynamic aspects of affect have so far relied on linear models. As an example of a study of non-linear relationships, Tuohilampi (2011) has explored how the discrepancy between ideal and real self (how we would like to be versus how we are) might be related to the student's enjoyment of mathematics. The preliminary analysis suggests that students enjoy mathematics most, when their real self is on a slightly lower level or on the same level as their ideal self.

Another future dimension for quantitative studies is to identify to what extent we can attribute variation of different affective measures to individual and social levels. The two origins of beliefs (individual experiences and cultural influences) were pointed out already by McLeod (1992). Researchers who emphasize the individual nature of affect tend to rely on psychological theories and in-depth interviews or surveys. On the other hand, those who emphasize the social construction of affect rely more on social theories and they focus on observing the emergence of affect in social



scenarios. Both of these approaches are valuable, but we should also evaluate the importance of these two origins for each affective variable. For example, is the lower self-efficacy of girls compared to males so culturally determined that teachers can have only a small effect on it? Or could this gender difference disappear in classes that have a gender sensitive teacher?

There are studies that attribute the origin of different affective variables on social and individual levels. For example, results from a Finnish high school indicate that students in the same class tend to have similar effort, enjoyment of mathematics and evaluation of the teacher, while their self-efficacy beliefs are more varied. One aspect of their self-efficacy beliefs is significantly influenced by gender while another aspect of it is mainly influenced by their achievement in mathematics (Hannula, 2009, 2010).

Another under-explored direction is the identification of different affective profiles for example through cluster analysis. In our study, we identified six affective clusters among teacher education students at the beginning of their mathematics education studies and we found this clustering to predict some of the differences in their progress in mathematics over the course (Hannula, Kaasila *et al.*, 2006).

Thirdly, there is shortage of good longitudinal studies. Due to the difficulties in longitudinal data collection, analysis is often based either on relatively small samples or a secondary analysis of data collected for another purpose. For example, Ma and Xu (2004) used data from another study and had to rely on only three items on mathematics usefulness as a measure of mathematics attitude in their attempt to analyse the causal ordering between attitude toward mathematics and achievement in mathematics. As the authors point out themselves, more reliable measures of confidence and enjoyment in mathematics might have produced different results.

Another direction that might produce interesting new results is the use of brain imaging techniques. For example, a recent study has identified that activity in the amygdala during an Aha! experience is a strong predictor of which solutions will remain in long-term memory (Ludmer, Dudai & Rubin, 2011).

The main purpose of this article has been to provide an overview of research on mathematics related affect that would enable bridging between existing theoretical frameworks. Empirical studies can well be based on local theories. However, if we wish to gain an overall understanding of human mathematical thinking and behaviour we need to engage into a dialogue across a number of local theories. I am well aware that my attempt at this synthesis is biased towards the individual level and psychological theories. People with a background in social theories might find different distinctions more relevant and useful for analysing and relating the variety of research in mathematics-related affect. I also expect and welcome other efforts at synthesising theories of affect.

## References

- Allen, B. D., & Carifio J. (2007). Mathematical sophistication and differentiated emotions during mathematical problem solving. *Journal of Mathematics and Statistics* 3 (4), 163-167
- Ames, C. (1992). Classrooms: goals, goal structures, and student motivation. *Journal of Educational Psychology* 84 (3), 261–271.
- Bauersfeld, H., Krummheuer, G., & Voigt, J. (1988). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner, & A. Vermandel (Eds.), *Foundations and Methodology of the Discipline of Mathematics Education*, pp. 174-188. Antwerpen, Belgium: Proceedings of the TME conference.
- Beijaard, D., Meijer, P., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher Education*, 20, 107-128.
- Bergmann, G. (1955). Dispositional properties and dispositions. *Philosophical Studies* 6 (5), 77 – 80.
- Bishop, A. J. (2001). What values do you teach when you teach mathematics? In P. Gates (Ed.), *Issues in Mathematics Teaching*, pp. 93-104. London: Routledge Falmer.
- Boaler, J. (1997). Reclaiming school mathematics: The girls fight back. *Gender and Education* 9, 285-305.
- Brousseau, G. (1997). *Theory of didactical situations*. Edited and translated by N. Balacheff, M. Cooper, R. Sutherland & V. Warfield. Dordrecht, The Netherlands: Kluwer.
- Brown, L., & Reid, D. A. (2006). Embodied cognition: Somatic markers, purposes and emotional orientations. *Educational Studies in Mathematics* 63, 179–192. DOI: 10.1007/s10649-006-9027-3
- Buck, R. (1999). The Biological Affects: A Typology. *Psychological Review* 106 (2), 301-336.
- Buxton, L. (1981). *Do you panic about maths?* London: Heinemann.
- Chapman, J. W., Tunmer, W. E., & Prochnow, J. E. (2000). Early reading-related skills and performance, reading self-concept and the development of academic self-concept: A longitudinal study. *Journal of Educational Psychology* 67, 145-152.
- Cobb, P., & Yackel, E. (1996). Constructivist, emergent and sociocultural perspectives in the context of developmental research. *Educational Psychologist* 31 (3/4), 175-190.

## Plenary lecture

- Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts during mathematical problem solving. In D. B. McLeod, V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective*, pp. 117-148. NY: Springer.
- Cretchley, P. C. (2008). Advancing research into affective factors in mathematics learning: Clarifying key factors, terminology and measurement. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia), pp. 147 – 153. Brisbane: MERGA.
- Damasio, A. R. (1994). *Descartes Error: Emotion, Reason and the Human Brain*, New York: G. P. Putnam's Sons
- DeBellis, V.A., & Goldin, G.A. (1997). The affective domain in mathematical problemsolving. In E. Pehkonen (Ed.), *Proceedings of the PME 21*, Vol. 2, pp. 209-216. Finland: University of Helsinki.
- Dew, K. M. H., Galassi, J. P., & Galassi, M. D. (1984). Math anxiety: Relation with situational test anxiety, performance, physiological arousal and math avoidance behaviour. *Journal of Counseling Psychology* 31, 580-583.
- Di Martino, P., & Zan, R. (2010). 'Me and maths': towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teachers Education*, 13(1), 27-48.
- Diego-Mantecón, J., Andrews, P., & Op 't Eynde, P. (2007) Refining the mathematics related beliefs questionnaire, . In D. Pitta – Pantazi & G. Philippou (Eds.) *European Research in Mathematics Education V, Proceedings of the CERME 5*, Larnaca, Cyprus 22 - 26 February 2007, pp. 229-238. Department of Education: University of Cyprus.
- Dweck, C.S. (2002) The development of ability conceptions. In A. Wigfield and J. S. Eccles (Eds.), *Development of Achievement Motivation*, pp. 57–88. London: Academic Press.
- Eccles, J. S., Adler, T. F., Futterman, R., Goff, S. B., Kaczala, C. M., Meece, J. L., *et al.* (1985). Selfperceptions, task perceptions, socializing influences, and decision to enroll in mathematics. In S. F. Chipman, L. R. Brush, & D. M. Wilson (Eds.), *Women and mathematics: Balancing the equation* (pp. 95–121). Hillsdale, NJ: Erlbaum.
- Elliot, A. J., & Harackiewicz, J. M. (1996). Approach and avoidance achievement goals and intrinsic motivation: A mediational analysis. *Journal of Personality and Social Psychology* 70 (3), 461–475.
- Fennema, E., & Sherman, J.A. (1976). Fennema-Sherman mathematics attitudes scales. *JSAS Catalog of Selected Documents in Psychology*, 6, 31 (Ms. No. 1225).

## Plenary lecture

- Frade, C., Roesken, B., & Hannula, M. S. (2010). Identity and affect in the context of teachers' professional development. In M. M. F. Pinto & T. F. Kawasaki (Eds.) *Proceedings of the PME 34*, Vol. 1, pp. 247 – 249. Belo Horizonte, Brazil: PME.
- Fennema, E., & Sherman, J.A. 1976. Fennema-Sherman mathematics attitudes scales. *JSAS Catalog of Selected Documents in Psychology* 6, 31 (Ms. No. 1225).
- Freeman, K. E. (2004). The significance of motivational culture in schools serving african american adolescents: a goal theory approach. In P. R. Pintrich & M. L. Maehr (Eds.) *Advances in motivation and achievement. Volume 13. Motivating students, improving schools: The legacy of Carol Midgley*, pp. 65–95. The Netherlands: Elsevier Jai.
- Friedel, J. M., Cortina, K. S., Turner, J. C., & Midgley, C. (2007). Achievement goals, efficacy beliefs and coping strategies in mathematics: The roles of perceived parent and teacher goal emphases. *Contemporary Educational Psychology* 32, 434–458.
- Frost, L. A., Hyde, J. S., & Fennema, E. (1994). Gender, mathematics performance, and mathematics related attitudes and affect: a meta-analytic synthesis. *International Journal of Educational Research* 21 (4), 373–385.
- Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G. C. Leder, E. Pehkonen & G. Törner (Eds.) *Beliefs: A Hidden Variable in Mathematics Education*, pp. 39 – 58. Dordrecht, The Netherlands: Kluwer.
- Gentry, W. M., & Underhill, R. (1987). A comparison of two palliative methods of intervention for the treatment of mathematics anxiety among female college students. In J. C. Bergeron, N Hershcovics, & C. Kieran (Eds.) *Proceedings of the PME 11*, Vol 1, pp. 99-105. Montreal: University of Montreal.
- Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning* 2 (3), 209-219.
- Goldin, G. (2002). Affect, meta-affect, and mathematical belief structures. In G. Leder, E. Pehkonen and G. Törner (Eds.) *Beliefs: A Hidden Variable in Mathematics Education?* pp. 59–72. Dordrecht, The Netherlands: Kluwer.
- Goldin, G. (2004). Characteristics of affect as a system of representation. In M. J. Høines & A. B. Fuglestad (Eds.) *Proc. of the 28th conference of the IGPME*. Vol 1, pp. 109-114. Bergen University Collage.
- Green, T. F. (1971). *The activities of teaching*. New York: McGraw-Hill.
- Hannula, M.S. (2002a). Attitude towards mathematics: emotions, expectations and values. *Educational Studies in Mathematics* 49 (1), 25 – 46
- Hannula, M. S. (2002b). Goal regulation: needs, beliefs, and emotions. In A.D. Cockburn & E. Nardi (Eds), *Proceedings of the PME 26*. Vol 3, pp. 73-80. Norwich, UK.

### Plenary lecture

- Hannula, M. S. (2003). Fictionalising experiences – experiencing through fiction. *For the Learning on Mathematics* 23 (3), 33 – 39
- Hannula, M. S. (2004). *Affect in mathematical thinking and learning*. Turku, Finland: Annales universitatis Turkuensis B 273.
- Hannula, M. S. (2006a). Affect in mathematical thinking and learning: Towards integration of emotion, motivation, and cognition. In J. Maaß & W. Schlöglmann (Eds.) *New mathematics education research and practice*, pp. 209-232. Rotterdam: Sense.
- Hannula, M. S. (2006b). Motivation in Mathematics: Goals reflected in emotions. *Educational Studies in Mathematics* 63, 165-178
- Hannula, M. S. (2009). Upper secondary students' mathematical beliefs and their teachers' teaching beliefs. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.) *Proceedings of the PME 33, Thessaloniki - Greece, July 19-24, 2009*. Vol 3, pp. 129 – 136. Thessaloniki: International Group for the Psychology of Mathematics Education.
- Hannula, M. S. (2010). The effect of achievement, gender and classroom context on upper secondary students' mathematical beliefs. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of CERME 6, January 28th-February 1st 2009, Lyon France*, pp. 34 – 43.
- Hannula, M. S., Kaasila, R., Laine, A., & Pehkonen, E. (2006). The structure of student teacher's view of mathematics at the beginning of their studies. In M. Bosch (Ed.), *Proceedings of the CERME 4. Sant Feliu de Guíxols, Spain - 17 - 21 February 2005*, pp. 205 - 214. Fundemi IQS - Universitat Ramon Llull: CERME.
- Hannula, M. S., & Laakso, J. (2011). The structure of mathematics related beliefs, attitudes and motivation among Finnish grade 4 and grade 8 students. To appear in the Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education. Ankara, Turkey: PME.
- Hannula, M.S., Maijala, H., & Pehkonen, E. (2004). Development of understanding and self-confidence in mathematics; grades 5-8. In M. J. Høines & A. B. Fuglestad (Eds.) *Proceedings of the PME 28*. Vol. 3, 17 – 24. Bergen University College.
- Hannula, M. S., Maijala, H., Pehkonen, E., & Nurmi, A. (2005). Gender comparisons of pupils' self-confidence in mathematics learning. *Nordic Studies in Mathematics Education* 10 (3-4), 29 – 42
- Hannula, M.S. , Maijala, H., Pehkonen, E., & Soro, R. (2002). Taking a step to Infinity. In S. Lehti & K. Merenluoto (eds.) *Third European Symposium on Conceptual Change, A process approach to conceptual change*, June 26-28.2002, Turku, Finland. pp. 195-200

## Plenary lecture

- Hannula, M. S.; Malmivuori, M. L., & Pehkonen, E. (1996). Development of pupils' mathematical beliefs: a description of a research project. In E. Pehkonen (ed.) *Current State of Research on Mathematical Beliefs III; Proceedings of the MAVI-3 Workshop August 23-26, 1996*. Research Report 170. Department of Teacher Education. University of Helsinki.
- Hannula, M.S., Op 't Eynde, P., Schlöglmann, W & Wedege, T. (2007). Affect and mathematical thinking. In D. Pitta – Pantazi & G. Philippou (Eds.) *European Research in Mathematics Education V, Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education, Larnaca, Cyprus 22 - 26 February 2007*, pp. 202 – 208. Department of Education: University of Cyprus.
- Hannula, M. S., Pantziara, M., Wæge, K., & Schlöglmann, W. (2009). Introduction: Multimethod approaches to the multidimensional affect in mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.) *Proceedings of CERME 6, January 28th-February 1st 2009, Lyon France*, pp. 28 – 33. INRP 2010
- Hart, L. (1989). Describing the affective domain: saying what we mean. In McLeod, D., & Adams, V. (eds.) *Affect and Mathematical Problem Solving*, pp. 37-45. New York: Springer Verlag.
- Harzing, A.W. 2011 Publish or Perish, version 3.0, available at [www.harzing.com/pop.htm](http://www.harzing.com/pop.htm)
- Hembree, R. (1990). The Nature, Effects, and Relief of Mathematics Anxiety. *Journal of Research in Mathematics Education*, 21, 33-46
- Isoda, M., & Nakagoshi, A. (2000). A case study of student emotional change using changing heart rate in problem posing and solving Japanese classroom in mathematics. In T. Nakahara & M. Koyama (Eds.) *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education Vol.3* pp. 87–94. Hiroshima, Japan: PME.
- Järvelä, S. (1996). *Cognitive apprenticeship model in a complex technology-based learning environment: Socioemotional processes in learning interaction*. Publications in Education N:o 30. University of Joensuu.
- Kaldo, I., & Hannula, M. S. An unpublished manuscript. Structure of students' view of mathematics in Estonia at university level.
- Kloosterman, P., & Stage, F. K. (1992). Measuring beliefs about mathematical problem solving. *School Science and Mathematics* 92 (3), 109 – 115.
- Kumar, R., Gheen, M. H., & Kaplan, A. (2002). Goal Structures in the Learning Environment and Students' Disaffection From Learning and Schooling. In C. Midgley (Ed.) *Goals, Goal Structures, and Patterns of Adaptive Learning*, pp. 143–173. Mahwah NJ: Lawrence Erlbaum.

## Plenary lecture

- Lazarus, R. S. (1991). *Emotion and Adaptation*. NY: Oxford University Press.
- Leder, G. (1995). Equity inside the mathematics classroom: Fact or artifact? In W.G. Secada, E. Fennema & L. B. Adajian (Eds.) *New directions for equity in mathematics education*. Cambridge University Press.
- LeDoux, J. (1998). *The Emotional Brain*. Phoenix, Orion Books Ltd.
- Lehman, B., D'Mello, S. K., & Person, N. (2008). All alone with your emotions: an analysis of student emotions during effortful problem solving activities. Supplementary proceedings of the workshop on emotional and cognitive issues in ITS (WECITS).
- Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*, pp. 19–44. Westport, CN: Ablex.
- Li, Q. (1999). Teachers' beliefs and gender differences in mathematics: A review. *Educational Research* 41(1), 63-76.
- Linnanmäki, K. (2002). *Matematikprestationer och självuppfattning, en uppföljningsstudie i relation till skolspråk och kön*. [Mathematical proficiency and self-concept; a longitudinal study with an attention to language and gender]. Turku: Åbo Akademi University Press.
- Ludmer, R., Dudai, Y., Rubin, N. (2011). Uncovering camouflage: Amygdala activation predicts long-term memory of induced perceptual insight. *Neuron* 69 (5), 1002-1014.
- Ma, X. (1999). A meta-analysis of the relationship between anxiety toward mathematics and achievement in mathematics. *Journal for Research in Mathematics Education*, 30, 520-541.
- Ma, X., & Kishor, N. (1997a). Assessing the relationship between attitude toward mathematics and achievement in mathematics: A meta-analysis. *Journal for Research in Mathematics Education* 28 (1), 26-47.
- Ma, X., & Kishor, N. (1997b). Attitude toward self, social factors, and achievement in mathematics: A meta-analytic review. *Educational Psychology Review*, 9, 89–120.
- Ma, X & Xu, J. (2004). Determining the Causal Ordering between Attitude toward Mathematics and Achievement in Mathematics. *American Journal of Education* 110 (May), 256 – 280
- Malmivuori, M.L. (2001). *The dynamics of affect, cognition, and social environment in the regulation of personal learning processes: The case of mathematics*. Research report 172. Helsinki: Helsinki University Press.

### Plenary lecture

- Malmivuori, M.-L. (2006). Affect and self-regulation. *Educational Studies in Mathematics*, 63(2), 149–164.
- Martin, A. J. (2001) The Student Motivation Scale: a tool for measuring and enhancing motivation. *Australian Journal of Guidance and Counselling* 11, 1–20.
- Mason, J., Burton, L., & Stacey, K. (1982). Thinking mathematically. London: Addison-Wesley.
- Maturana, H. R. (1988). Reality: The search for objectivity or the quest for a compelling argument. *The Irish Journal of Psychology* 9(1), 25–82.
- McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In D.A.Grouws (Ed.) *Handbook of Research on Mathematics Learning and Teaching*, pp. 575-596. New York: MacMillan.
- Mandler, G. (1989). Affect and learning: Causes and consequences of emotional interactions, In D. B. McLeod & V. M. Adams (Eds.). *Affect and mathematical problem solving: A new perspective*, pp. 3-19. New York: Springer.
- Meyer, D. K., & Turner, J. C. (2002). Discovering emotion in classroom motivation research. *Educational Psychologist* 37 (2), 107 – 114.
- Minato, S., & Kamada, T. (1996). Results on research studies on Causal predominance between achievement and attitude in junior high school mathematics of Japan. *Journal for Research in Mathematics Education* 27, 96-99
- Midgley, C., Maehr, M. L., Hruda, L. Z., Anderman, E., Anderman, L. Freeman, K. E., ... Urdan, T. (2000). *Manual of the Patterns of Adaptive Learning Scales*. University of Michigan.
- Midgley, C., Kaplan, A., Middleton, M., Maehr, M. L., Urdan, T., Anderman, L. H., ... Roeser, R. (1998). The Development and Validation of Scales Assessing Students' Achievement Goal Orientations. *Contemporary Educational Psychology* 23, 113–131.
- Middleton, M. J., Kaplan, A., & Midgley, C. (2004). The change in middle school students' achievement goals in mathematics over time. *Social Psychology of Education* 7, 289–311.
- Murphy, P. K., & Alexander, P. A. (2000). A motivated Exploration of Motivation Terminology. *Contemporary educational psychology* 25, 3–53.
- Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.) 18 unconventional essays on the nature of mathematics (pp. 160–181). New York: Springer.



- Op 't Eynde, P., De Corte, E., & Verschaffel, L. (2006). Accepting emotional complexity: A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics* 63, 193–207.
- Op 't Eynde, P., & De Corte, E. (2003). Students' mathematics-related belief systems: design and analysis of a questionnaire. Paper presented to the symposium, the relationship between students' epistemological beliefs, cognition and learning, at the annual meeting of the American Educational Research Association, April 21-25, Chicago.
- Partanen, A. M. 2011. *Challenging the school mathematics culture: An investigative small-group approach; Ethnographic teacher research on social and sociomathematical norms*. Acta Universitatis Lapponiensis 206. Rovaniemi, Finland: University of Lapland, Department of Education.
- Pehkonen, E. (1994). On differences in pupil's conceptions about mathematics teaching. *The Mathematics Educator*, 5 (1), 3-10
- Pekrun, R., Elliot, A. J., & Maier, M. A. (2006). Achievement Goals and Discrete Achievement Emotions: A Theoretical Model and Prospective Test. *Journal of Educational Psychology*, 98 (3), 583–597.
- Pekrun, R., Goetz, T., & Frenzel, A.C. (2005). *Academic Emotions Questionnaire - Mathematics (AEQ-M) - User's Manual*. University of Munich: Department of Psychology.
- Pintrich, P. R., & Schunk, D. H. (2002). *Motivation in Education: theory, research, and applications*. 2nd ed. Upper Saddle River, NJ: Pearson Education.
- Power, M., & Dalgleish, T. (1997). *Cognition and Emotion; From order to disorder*. UK: Psychology Press.
- Roesken, B., Hannula, M., & Pehkonen, E. (2011). Dimensions of students' view of themselves as learners of mathematics. *ZDM – International Journal of Mathematics Education*, 1-10. Doi: 10.1007/s11858-011-0315-8
- Ruffell, M., Mason, J., & Allen, B. (1998). Studying attitude to mathematics. *Educational Studies in Mathematics* 35, 1–18.
- Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. San Diego: Academic Press.
- Schoenfeld, A. H. (1998). Toward a theory of Teaching-In-Context. *Issues in Education* 4 (1), 1-94.
- Seo, D. (2000). The Motivational Construct in Mathematics Learning Using Structural Equation Modeling: The Korean Elementary School Math Class. Paper presented at the Head Start National Research Conference (Washington, DC, June 28 – July 1, 2000.) Retrived January 15, 2011 from <http://www.eric.ed.gov/PDFS/ED448057.pdf>

## Plenary lecture

- Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14-22.
- Soro, R. (2002). *Opettajien uskomukset tytöistä, pojista ja tasa-arvosta matematiikassa*. [Teachers' beliefs about girls, boys, and equity in mathematics.] Annales universitatis Turkuensis C 191. Turku, Finland: University of Turku.
- Spielberger, C. D. (1966). Theory and research on anxiety. In: C. D. Spielberger (Ed.), *Anxiety and behavior* (pp. 3-20). New York: Academic Press.
- Sumpter, L. (2009). *On aspects of mathematical reasoning; Affect and gender*. Umeå university.
- Tapia, M., & Marsh II, G. E. (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly*, 8, 16-21.
- Taylor, G. J., Bagby, R. M., & Parker, J. D. A. (1997). *Disorders of affect regulation: Alexithymia in medical and psychiatric illness*. Cambridge, England: Cambridge University Press.
- Tuohilampi, L. (2011). An examination of the connections between self discrepancies' and effort, enjoyment and grades in mathematics. To appear in the *Proceedings of 7<sup>th</sup> CERME-conference*, February 2011.
- Vollmeyer, R., & Rheinberg, F. (2000). Does motivation affect performance via persistence? *Learning and Instruction* 10, 293–309.
- Williams, T., & Williams, K. (2009). Self-Efficacy and Performance in Mathematics: Reciprocal Determinism in 33 Nations. *Journal of Educational Psychology* 102 (2), 453-466 doi:10.1037/a0017271
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* 27 (4), 458-477.
- Yates, S. M. (2000). Student optimism, pessimism, motivation and achievement in mathematics: A longitudinal study. In T. Nakahara and M. Koyama (Eds.), *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, pp. 297–304. Hiroshima, Japan. PME

# PROVING AND PROOF AS AN EDUCATIONAL TASK

Maria Alessandra Mariotti  
Università di Siena, Italy

*Situated in the context of current research, my talk presents a point of view on proof as an educational issue. This point of view intends to combine three perspectives: epistemological, what we mean by proof; cognitive, which are the main difficulties that students meet; and didactical, what kind of didactic interventions can be proposed, among those that have been experimented and showed their effectiveness. The examples that will be presented are drawn from the current literature, but also from research studies which I have been directly involved in recently. In particular, I will discuss some results coming from long-term teaching experiments related to the use of computer-based artefacts as tools of semiotic mediation that teachers can exploit to develop the mathematical meaning of proof.*

## INTRODUCTION

It is for me a great honour and a great pleasure to give this lecture, and I thank the Program committee for their invitation.

The theme of my lecture has been that of one of the Working Groups at CERME since the very beginning, and it has been one of the themes of my research for many years. However, for a long time Proof and proving has been a theme of debate, not only in the community of mathematics educators, but also in the community of mathematicians (see for instance, Thurstone, 1994; Hanna, 1989)

Proof represents a very special case relative to other topics, like Geometry or Algebra. Debate among researchers has been very passionate, reflecting sometimes great divergences. This makes this theme fascinating but also shows its complexity, mainly with respect to the objective of outlining didactical implications that can be useful in school practice. In this respect, I found the contribution in the book by Reid and Knipping (2010) to be valuable, though I hardly recognize my work in their interpretation. However this is part of the complexity of the problem: often we take for granted that we share much more than we actually have in common, and only recently have we become aware of the need of making our epistemological assumptions (stance) explicit in our research work. Actually this is crucial in order to foster mutual understanding, but it is also needed in order to find a way of overcoming divergence of opinions, and to make our research results deliverable to an extended scientific community including teachers and curriculum developers.

Thus I will start with a brief introduction addressing some epistemological issues that I feel I must share with you, before entering into the core of my talk, when I will address the issue of proving and proof from a didactical perspective, and specifically I will present the analysis of different research studies concerning teaching approaches to proving and proof.

## EPISTEMOLOGICAL ISSUES

I will start by posing a question:

Is there a shared meaning of ‘mathematical proof’ among researchers in mathematics education?

This question opens a recent paper by Nicolas Balacheff (2008) explicitly posing the issue of the existence of a shared meaning of the term ‘proof’ and consequently of any other term or expression related to it.

Currently the situation of our field of research is quite confusing, with profound differences in the ways to understand what is a mathematical proof within a teaching-learning *problématique* but differences which remain unstated. (Balacheff, 2008, p. 501)

As clearly argued by Balacheff in his paper, different epistemological positions affect mathematics education research on proof and in particular didactic /pedagogical approaches to proof at school. For this reason it is crucial to be aware of this problem and to make an effort to clarify one’s own different assumptions.

Though it may appear to be a simplification, I claim that when approaching the issue of proof there are two opposing positions that may be referred to two different epistemological, and maybe cultural, perspectives.

On the one hand proof is meant as an idiosyncratic issue, strictly dependent on the individual that produces it. In this respect a wide range of cases can be observed and described. An example of a study consistent with this perspective is offered by the research work developed by Harel & Sowder (1998), and the classification (taxonomy) provided by these authors is an exemplar of the results coming from these type of studies. Such a classification is very accurate, and differentiates a large number of students’ behaviours. However such different behaviours (proof schemes) share a common ‘definition’ of proof that is obviously quite general. The authors formulate an explicit definition of the process of proving, a process of which a proof can be considered a product.

Proving is the process employed by an individual [...] to remove or create doubts about the truth of an assertion. The process of proving includes two sub-processes: ascertaining and persuading. (Harel & Sowder, 1998, p. 241)

As the authors explain, ascertaining is the process an individual employs to remove her or his own doubts, whilst persuading is the process employed to remove others’ doubts.

According to this definition, the term *proof* is largely referred to any kind of argument, and “need not connote mathematical proof” (Harel & Sowder, 1996, p. 60)

On the other hand, other researchers take a completely different point of view and hold/claim that proof has a peculiarity of its own, and that such a peculiarity constitutes a key element of mathematics as a theoretical discipline. Proof is meant

as the product of the process of theoretical validation that is considered a specific and indispensable component of Mathematics. Taking such a perspective, that for good reasons we can call ‘formal’ (Arzarello, 2007), means to consider a proof independent on any interpretation and factual verification of the statement /statements that are involved. This is the stance, among others, of Raymond Duval (1991; 1992-93), who clearly states the specificity of proof, contrasting it with argumentation. Duval claims the need to clearly distinguish between argumentation and proof, stressing the epistemological and cognitive gap existing between these two processes, in spite of their similarities. The core of Duval’s argument concerns the difference between the semantic level, where the epistemic value of a statement is fundamental for its acceptability, and the theoretical level where the only matter is checking the theoretical validity of a statement: i.e., checking the validity of the logic dependence of a statement with respect to the axioms, definitions and theorems, without taking into account the epistemic value that can be attributed to any statement involved.

The epistemological and cognitive analysis carried out by Duval makes him warn us that the main issue from an educational perspective is exactly in the proximity of the two processes and in the danger of mistaking one for the other. Thus there is need of a careful distinction between them, stressing the theoretical nature of proof and recognising argumentation as a possible obstacle to the development of a sense of proof.

Of course this radical position leaves place for a wide range of intermediate possibilities and variations in conceiving the relationship between argumentation and proof. However it focuses on specific features of mathematical proof that characterize it, and for this very reason can hardly be neglected without serious/great loss for mathematics education. (Mariotti, 2006)

These same characteristics that inspired the following statement where the educational value of proof was clearly stated.

The concept of proof is one concerning which the pupil should have a growing and increasing understanding. It is a concept which not only pervades his work in mathematics but is also involved in all situations where conclusions are to be reached and decision to be made. Mathematics has the unique contribution to make in the development of this concept [...] (Fawcett, 1938, p.120, quoted by Reid & Knipping)

Coming back to the analysis carried out by Duval, we can remark that the distinction /opposition between argumentation and proof is based on the distinction between the semantic plane and the theoretical plane. While proof is confined on the theoretical plane and does not require any reference to the interpretation of any of the statements involved, it is on the semantic plane that an argumentation derives the epistemic value of the statement in focus. Based on the interpretation of the different statements and on their semantic relationships argumentation it is strictly connected

with the issue of understanding ‘why’ (Dreyfus & Hadas, 1996), in particular to an explication function that seems to be also a proper goal of proof (Hanna, 1989a) From this perspective the following question arises:

What is the role of theoretical derivation in respect to understanding? If we take a theoretical perspective, what about the explication function of proof ?

Different answers are possible, however it seems reasonable to agree on the fact that the sense of mathematical proof should include elements related to the semantic plane. This becomes more evident if we consider the process of proving not only in relation to the process of evaluating the acceptability of a proof of a given statement, but also in relation to the process of producing a new theorem; i.e. producing a conjecture and then its proof. Considering the practice of mathematicians, it seems difficult to conceive producing a conjecture, and in particular producing the links of logical implication between two statements, without referring to the meanings involved in the formulations of such statements and in the relationships between them.

to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim ... that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge. (Lolli, 1999)

Meaning and epistemic values, given by the interlocutors to the statements involved, have no theoretical status, nevertheless it seems nearly impossible to think of a ‘practice of theorems’ –the mathematicians’ practice of producing theorems by formulating and proving conjectures - without any reference to meanings.

In this respect I do not see any conflict between the two perspectives presented above; rather I think that each perspective focuses on different aspects which are both fundamental for mathematics education. And taking the risk of oversimplification, I would summarize the didactic issue of proof in terms of resolving the potential conflict between the two main functions of proof, i.e. validating within a theory and explaining with respect to interlocutors’ conceptions. This means finding a way, a pedagogical/ didactical way, of developing the complex relationship between argumentation, with its goal of convincing, and proof, with its goal of theoretically validating. Finally, it means taking into account the differences between argumentation and proof, and exploiting their deep link rather than ignoring it.

Addressing the complexity of this issue sometimes requires the researcher to assume a position that may appear too rigid or oversimplified. Such simplifications might be reasonable for methodological reasons, but it remains necessary for the researcher (and the teacher) not to miss the complexity of the *problematique*.

In order to better clarify this point I will present two theoretical constructs that are consistent with the previous statement, and in my view can be of help to frame the design and the analysis of research studies on proof; at least, I would like to propose

them because I found them personally very useful in the research studies that I carried out recent years and are still in progress.

The first construct is the notion of Theorem that we introduced some years ago (Mariotti et al., 1997) At the beginning there was a need to provide a shared frame to describe commonalities between different research approaches to proof among some Italian research groups. Later, this notion has been used and further elaborated in other studies (see for instance its use in the analysis of the proof by contradiction (Antonini & Mariotti, 2008) The second construct is that of Cognitive Unity as developed by Pedemonte (2002), starting from the original definition introduced by Boero et al. (1996) In the following sections I am going to give a brief description of both such constructs.

### A DEFINITION OF THEOREM

The definition of Theorem that we are going to introduce is motivated by the following remark. The current literature, but also textbooks, commonly refers to the educational issue of proof, and in so doing proof is considered in itself, without any reference to the other key elements without which the idea of proof evaporates. As a matter of fact, as briefly discussed above it is impossible to grasp the sense of a *mathematical proof*, specifically when we want to differentiate it from argumentation, without linking it to the two other elements involved: a *statement* and overall a *theory*.

In other words, we speak of a proof when there is a statement to which it provides a support for validity, but also when there is a theoretical frame within which such a support makes sense. Leaving statement and theory implicit is certainly comprehensible if we take the perspective of mathematics experts, but it may become a serious flaw if we take an educational perspective and we consider the students' point of view.

What may have become an automatic and unconscious implicit reference for experts, cannot be expected to be the same for novices. Grasping the sense of the mathematical proof in terms of *theoretical validation* may be difficult and not spontaneous. Certainly, where students are concerned, the peculiarity of such a way of thinking cannot be taken for granted and its complexity cannot be ignored. In particular, one may expect serious consequences from the confusion between an *absolute* and a *theoretically situated* truth, corresponding to the two main functions of proof - explication and validation - (for a full discussion see Mariotti, 2006)

Hence, taking an educational perspective and in order to express the contribution of each component involved, the following characterization of Mathematical Theorem was introduced, where a proof is conceived as part of a system of elements:

The existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules

## Plenary lecture

are so intimately interrelated so that what characterises a Mathematical Theorem is the system of statement, proof and theory. (Mariotti et al. 1997, 1, p. 182)

Traditionally in school practice, the reference to the Theory within which a proof makes sense is neglected, and, except for the case of Geometry, the theoretical context in which theorems are proved normally remains implicit. Consider, for instance, Calculus courses and textbooks: this is one of the few cases when proofs are provided, yet the axiomatic reference system is largely left implicit, and a careful analysis shows that most of the proofs would be better classified as argumentations.

Coming back to the definition of Mathematical Theorem, it is important to remark that what is shortly referred to as Theory, has a twofold component. On the one hand, Axioms, Definitions and already proved Theorems constitute the means of supporting the single steps of a proof; on the other hand, meta-theoretical rules assure the reliability of the specific way to accomplish this support - that is to say, how the Axioms and Theorems belonging to a Theory can be used to validate a new statement.

Actually, as clearly pointed out by Sierpinska, acting at a meta-theoretical level constitutes the very essence of a theoretical perspective

[T]heoretical thinking is not about techniques or procedure for well-defined actions, [...] theoretical thinking is reflective in that it does not take such techniques for granted but considers them always open to questioning and change. [...] Theoretical thinking asks not only, Is this statement true? but also What is the validity of our methods of verifying that it is true? Thus theoretical thinking always takes a distance towards its own results. [...] theoretical thinking is thinking where thought and its object belong to distinct planes of action. (Sierpinska, 2005, pp. 121-23)

In the school context, the complexity of this meta-theoretical level seems to be ignored. It is commonly taken for granted that students' ways of reasoning are spontaneously adaptable to the sophisticated functioning of a theoretical system. Thus not much is said about it, and in particular deduction rules and their functioning in the development of a Theory are rarely made explicit<sup>1</sup>.

There are at least two aspects of acting at a meta level that need to be made explicit. One consists in the acceptability of some specific deductive means, the other in the fact that no other means, except those explicitly shared, is acceptable. If these two aspects are left implicit, it may happen that students have no access to any control on their arguments. In this case the control remains completely in the hands of the teacher, resulting for students in a general feeling of confusion, uncertainty and lack of understanding.

## THE NOTION OF COGNITIVE UNITY

In the context of a long-term teaching experiment (Boero et al., 1995a), interesting results came to light concerning students' production of conjectures (Boero et al.,



1996; 2007) and the argumentation accompanying it. The teaching experiments were based on the solution of an open-ended problem, requiring both a conjecture and its proof. Clear evidence was found of different kinds of argumentative processes appearing in the solution. Further investigations demonstrated that when the phase of producing a conjecture had shown a rich production of arguments that aimed to support or reject a specific statement, it was possible to recognize an essential continuity between these arguments and the final proof; such continuity was referred to as Cognitive Unity.

During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices.

During the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain. (Boero et al., 1996, p. 113)

If the first results supported a sort of “continuity” (as the authors called it), further investigation brought evidence of a possible gap between the arguments supporting the production of a conjecture and the proof validating it.

This construct has been widely used in analysing students’ solutions to different types of open problems. These studies led to a refinement of the construct and to the introduction of the distinction between *referential* and *structural* Cognitive Unity (Pedemonte, 2002) Further elaborations of this construct have been recently developed (Boero et al., 2010)

## **PROOF AS AN EDUCATIONAL TASK**

Different research projects at different age levels in different countries have designed and implemented possible approaches to proof. Naturally, this has been done assuming different epistemological perspectives and different cultural contexts with respect to proof.

Results of survey studies and research work focused on students’ conceptions of proof, on difficulties that students may encounter, motivated and, at the same time, inspired the development of innovation projects aimed more or less directly at improving students learning about proof. There are two main types of such studies: first, those that more generally address the design of activities aimed at fostering students’ insight into what proof is for fostering their performances in proving tasks; and secondly, those that explicitly address the design of teaching sequences aimed at introducing students to proof, and developing a sense of proof not only for recognizing but also for producing a proof.

Because of the time limitations of this talk, I will deal only with the first types of studies, and focus on approaches to students’ first experiences of proof.

## DESIGN OF TEACHING SEQUENCES TO INTRODUCE STUDENTS TO PROOF

In the light of the analysis carried out in the previous sections, we can synthetically summarize the educational issue concerning students' introduction to proof as follows:

teaching proofs and Theorems should have a double aim: on the one hand that of making students understand what mathematical proof is, and on the other hand that of making students capable of producing it.

Rephrasing what Balacheff (1982) wrote some year ago, proof should be considered both as an object, by which I mean a product that must fulfil the epistemic and communicative requirements of a mathematics community, and also as a process, that is to say the answer to a problem solving stance, intentionally aimed at producing a proof as product.

Consistent with this educational issue, the design of teaching sequences has taken the direction of searching for specific contexts to make possible:

1. fostering the emergence of a culture of explaining (a culture of Why questions); and
2. fostering the development of a theoretical perspective (a culture of Theorems)

In the following sections I will present examples of such contexts, trying to explain in what sense they may fulfil these two requirements.



**Figure 1 Pupils experiencing sun shadows**

### Approaching Theorems in the field of experience of sun shadow

The first example concerns a research project aimed at analysing mental processes underlying the production and proof of conjectures in mathematics. The project was part of a long-term teaching experiment carried out by the research team directed by Paolo Boero, and involved several classes at different school levels (Boero et al., 1995a, 1995b, 2007) Within this project the idea of context is filtered by the notion of

‘field of experience’, explained as follows:

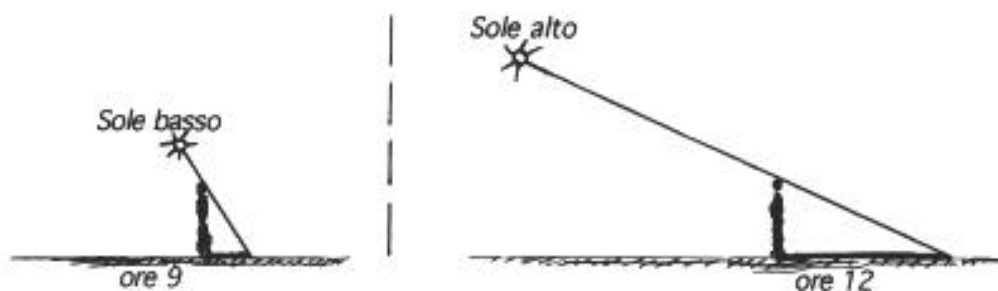
Field of experience: a system of three evolutive components: (external context, student's internal context and teacher's internal context) referred to a sector of human culture, which teacher and students can recognize and consider as unitary and homogeneous. (Boero et al., 1995a, pag.153)

The importance given in the project to the cultural value of Mathematics raised the issue of introducing pupils to theoretical thinking as well as that of developing pupils' competences in argumentation. Argumentative competences were considered a requirement for accessing theoretical thinking, in addition to their function as a means of control over mathematical techniques. But first and foremost, argumentative competences were considered a necessary condition for accessing a *cultural* discourse. (Boero et al. 1995a) In this respect it is important to highlight how the introduction to Theorems was part of a broader educational project aiming at developing concepts and ways of thinking belonging to mathematics, and generally speaking to our culture.

The field of experience of sun shadows was chosen in the project. This choice was motivated by the fact that it offers the possibility of producing, in open problem solving situations, conjectures which are meaningful from a space geometry point of view. Specific activities were designed for introducing students to a theoretical perspective, that is to introduce students to the culture of Theorems, as the designers express it, integrated in the teaching sequence concerning geometrical concepts emerging from making sense and produce conjectures in a real situation. The evolution of the field of experience is based on modelling shared experiences (see Figure 1) related to sun shadows. (Douek, 1999; Boero, 2002)

Conjectures are not immediate, and often there is a great variety of them; for this very reason, 'why' questions arise naturally. However, direct validation is mostly not possible, and any argument may be accompanied by drawings, although it can hardly be substituted by drawings. Because of the difficulties and the ambiguity in representing 3D spatial properties by drawing, drawings may not be convincing.

A first modelling approach leads the class to formulate a set of shared principles that constitute what we call a 'Germ Theory' (Bartolini Bussi et al. 1999)



**Figure 2 The triangle of shadow illustrating the first principle**

This particular Germ Theory is constituted by the following two principles:

1. The length of the shadow depends on the height of the sun.
2. Two vertical sticks have parallel shadows.

The activities which then follow are characterized by posing open-ended problems concerning experiences with sun shadows. As already remarked, the context offers

rich possibilities of posing open problems where uncertainty, stressed by the difficulty and often the impossibility of a direct verification, is the main motivation triggering the production of arguments supporting one's own conjectures.

Thus, conjectures ask for supporting arguments, and the acceptability of such arguments is related to the shared principles. The aim is that of developing the sense of proof by connecting the construction of a theory to the production of arguments that make sense within it. In other words, according to the definition of Theorem given above, the educational aim consists in developing a sense of Theorems through relating three elements: a statement, a proof and a Theory within which the proof makes sense.

It is not surprising that in this experimental context the notion of Cognitive Unity, described above, emerged and became a designing principle. In the following section I briefly outline a specific example (for a full discussion see Boero et al., 2007)

### **The “two sticks” Theorem**

Consider the following problem proposed to the class as individual work or work in pairs, as students prefer.

Problem. In recent years we observed that the shadows of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can shadows be parallel? Sometimes? When? Always? Never? Formulate your conjecture as a general statement.

Some thin, long sticks and three polystyrene platforms were provided in order to support the dynamic exploration process of the problem situation.

The problem refers to a familiar situation – a stick standing and producing a certain shadow – it proposes to add a new element – the oblique stick – and asks students to analyse the effect of this variation. It is also explicitly asks students to formulate the conjecture as a ‘general statement’, while the initial reference to the parallelism of the shadows of vertical sticks implicitly frames the arguments supporting the conjectures, and eventually makes sense of asking for a ‘proof’. Let us consider the following text produced by one of the pupils involved in the teaching experiment.

Simone (8th grade)

If we took into consideration two sticks, of which one is vertical, the shadows will be parallel when the two sticks are seen parallel by the sun. If we suppose that the person is in the position of the sun and looks at the sticks, by going around the sticks we can observe that the sticks are parallel in a certain position and the shadows are also parallel since the difference in position of the two sticks cannot be seen from that position. Thinking about the shadow space we can say that the non-vertical stick seems to be within the shadow space. Let's imagine an imaginary vertical stick representing the oblique one, in line with the sun's rays and the same stick, the oblique one cannot be seen so it seems to be vertical, forming parallel shadows. The shadows can be parallel if the

## Plenary lecture

sun is situated along the direction of the oblique stick [with a gesture he indicates the vertical plane containing the oblique stick].

This text is a good exemplar of the kinds of proofs that can be generated by students, in which the arguments are intermingled with subsequent different formulations of the conjecture statement. In spite of the reference to experience, and in particular to the fictive dynamic experience of “a person in the position of the sun”, the arguments also clearly show reference to the shared principles; thus it is possible to speak of a proof and then of a Theorem in a consistent way with respect to the definition given above.

However, it is clear that the interest and the care invested in developing such a theoretical perspective is deeply rooted in a semantic field: arguments are strictly related to their meanings in the context of sun shadow. They make sense in response to uncertainty, but base their acceptability on being elaborations (consequences) of the shared principles. It is clear, in this respect, how the aim of the Project designers is achieved: starting from an organization of the arguments based on their semantic content, to develop an organization consistent with the expectation and the requirements of the Theory, passing through the awareness of a shared need of communication and acceptability.

The specificity of the approach to Theorems presented above is highlighted by comparing it with others. In this respect we mention the distance between this approach and that of Duval (1991) The distance corresponds to that between two different epistemological perspectives. On the one hand an epistemological approach that recognizes a proof as primarily a derivation within a theory and aims at ‘realizing dissociation between content and operative status of propositions’ (op. cit., p. 223); on the other hand an epistemological approach that recognizes a proof as the construction of a validation in respect to a complex system of reasons, within which the reasons of the theory are only one part, and an epistemological and didactical approach that intends to maintain the priority of propositional content and reasoning with regard to such content.

This last claim is shared by other researchers, as the following passage shows.

In particular proof cannot be taught or learned without taking into consideration the relationships of mathematics to reality (Hanna and Janke 1996, p. 902)

Hence it is interesting to compare the Sun shadow Project approach with another approach that shares with it the choice of using the modelling of physical experience as a source of Geometry theory construction.

### **Modelling and theory: the approach of the theoretical physicist**

When Geometry theory sprouts from a modelling process the relationship between Theorems and the verification of facts cannot be neglected. As we observed above, this problem was intentionally partially avoided in the activities proposed in the Sun

shadow Project. In fact, after an intense experience of sun shadow phenomena from which shared principles emerged, crucial activities were centred on the solution of problems where direct verification was impeded by specific circumstances (for instance, clouds hide the sun or students are asked to make a conjecture sitting in the classroom so that the situation at stake is out of their direct control) forcing the students to develop arguments based on stated principles to support conjectures.

Nevertheless the delicate issue concerning the relationship between Geometry and reality can hardly be neglected. Starting from remarking how measuring and empirical verification have been discredited in common school practice, some authors (Hanna & Jahnke, 2007; Jahnke, 2007) claim the need to confront that issue, taking what they call the point of view of a *theoretical physicist*. From such a perspective Jahnke (op. cit., 2009) discusses the relationship between theory and practice, and in particular he discusses the status of geometrical problems and their solutions in respect to empirical verification, asking for consistency in treating empirical data in relation to Geometry and in relation to Physics. The didactical implication is the following.

Therefore, in teaching beginners an intellectually honest way is to take side by the physicist and to say that the angle sum theorem is true because of empirical measurements. Only at a later stage, one should expose the idea of a purely mathematical theory separated from reality. (Jahnke, 2005, p. 430)

There are similarities between the approach proposed by Jahnke and that proposed in the Sun shadow Project: both consider physical phenomena and their experience as a key element in posing meaningful problems that may generate uncertainty and consequently a culture of ‘why’ questions, and to produce conjectures and their validations according to stated hypotheses. However the issue of the relationship between argumentation and proof, specifically between empirical verification and theoretical validation is approached differently. In the Sun shadow Project, that issue is not explicitly addressed; rather, the implicit relationship between argumentation and proof is exploited, in an attempt to take advantage of spontaneous argumentation processes in order to make sense of mathematical proof. According to Jahnke’s approach the relationship between empirical and theoretical validation has to be explicitly taken into account. The need to state assumptions in relation to answering a ‘why’ question, and the need for a reasonable empirical base for these assumptions is part of the “theoretical physicist” approach that is held to be indispensable in the development of a sense of proof.

All this is related to a specific epistemological perspective that considers the assessment of a theory to be, in the end, a *pragmatic decision*. The ultimate reason for accepting a law is not of a logical, but of a *pragmatic nature*.

[...] a mathematical proof deriving a statement (natural law) of the theory does not provide absolute certainty to this statement, but it will considerably *enhance* its certainty

## Plenary lecture

since the corroboration does not only come from an isolated set of direct measurements but from all measurements related to the theory. Such a theoretical network of statements and measurements connected by mathematical proofs is the safest form of knowledge at our disposal, though this does not change its, in principle, preliminary character. (op cit, p. 430)

Of course, the discussion on such a position is open. The ultimate autonomy of Mathematics might be claimed, nevertheless we must admit that the position of Jahnke is fairly reasonable in respect to the educational objective of introducing students to Theorems within the context of common physical experience.

### **LOOKING FOR CONTEXTS FOSTERING STUDENTS' INTRODUCTION TO PROOF**

Looking for contexts for introducing students to proof, some authors pointed out the specificity of a Dynamic Geometry System (DGS) in offering powerful resources for designing situations for enhancing students' learning of proof (see for instance the Special Issue of ESM edited by Jones, Gutiérrez & Mariotti (2001) and specifically, Laborde's contribution to this Special Issue)

[...] the findings concerning the failure to teach proofs, the recognition of the multiple aspects of proving, and the existence of DG tools, lead naturally to the design of investigative situations in which DG tools may foster these multiple aspects. (Hadas et al., 2000, p. 130)

Open-ended problems are generally recognized as key elements, because of their potential in raising uncertainty and conflicts, and sometimes surprise - all ingredients that may trigger the need for explaining and validating, and eventually producing arguments. I will return to the specific issue of open-ended problems in the discussion of the last example. However other components have to be taken into account in the design; Hadas and colleagues outline a set of *design principles*, and the combination of these principles leads to designing tasks that afford "productive argumentative talk" (Hadas et al., 2000) In particular, besides promoting a collaborative situation, the need of "providing tools for raising and checking hypotheses" is suggested. In the case of a DGS, some of the negative aspects related to empirical verification may be overcome because of the specificity of the system of validation internal to the virtual environment. Such a validation system has a number of positive qualities: it can be explicitly shared in the classroom and it seems easily acceptable because it asks users to conform to the functioning of an artificial environment. Moreover validation by dragging is independent on the authority of the teacher and the aim of pleasing her, and it can be directly activated by any student, raising uncertainty, and sometimes curiosity. It is capable of triggering 'why' questions that open the way to producing arguments and explanations. For this reason, and other characteristics, to which we will return in the following, a DGS offers a rich potential for a didactical approach to proof.

## **FIELD OF EXPERIENCE CENTRED ON THE USE OF ARTEFACTS. THE CASE OF CABRI CONSTRUCTION**

The second example that I present concerns a long-term teaching experiment centred on the use of a particular DGS, Cabri Géomètre, and aiming at introducing students to proof at the upper secondary level.

As mentioned above, the main objective was that of exploiting the didactical potential offered by a DGS, and in particular by the specific system of validation based on the use of dragging. An epistemological analysis of the use of Cabri highlighted the deep relationship between producing figures in Cabri that are stable under dragging, and solving Construction problems within Euclidean Geometry. On the base of this relationship we designed a sequence of activities which aimed to initiate and develop a culture of Theorems within what we can call the field of experience of Cabri Construction (Mariotti, 2000) In order to better explain the principles of the design, I need to clarify some elements of the theoretical framework within which the didactical potential of Cabri has been described, and its utilization in the classroom activities explained. The theoretical frame is that of the Theory of Semiotic Mediation (TSM) as it was elaborated in Bartolini Bussi & Mariotti (2008) Taking quite a broad perspective, TSM aims to describe the use of artefacts<sup>2</sup> as a means to foster the teaching-learning process.

### **The theory of semiotic mediation**

The Theory of Semiotic Mediation (TSM) (Bartolini Bussi and Mariotti, 2008; Mariotti, 2009; Mariotti, 2010; Mariotti & Maracci, 2010) combines a semiotic and an educational perspective and elaborates on Vygotskij's notion of semiotic mediation (Vygotskij, 1978), considering the crucial role of *human mediation* (Kozulin, 2003, p.19) in the teaching-learning process.

Interpreting the teaching-learning process from a semiotic perspective means recognizing the central role of signs<sup>3</sup> and describing the teaching-learning process as a process of evolution (transformation in a given direction) of signs. In particular, TSM focuses on the production of signs, as they originate in the use of an artefact, in relation to personal meanings emerging from it, and on the process of transformation of such signs through social interaction; such a transformation may be applied to specific mathematical knowledge and the signs related to it, thus the evolution of signs may be considered as an indication of a change in the relationship between the subject and mathematical knowledge, and eventually as an evidence of learning.

When using an artefact for accomplishing a task, students' personal meanings emerge. Such meanings can be related to mathematical meanings, but establishing such a relationship is not a spontaneous process for students. On the contrary it can be an explicit educational aim for the teacher, who can *intentionally* orient her/his own actions towards promoting the evolution from personal towards mathematical meanings. Evolution may occur through social interaction, so that in a mathematics



class context, signs<sup>4</sup> produced by students and expressing the relationship between the artefact and the task in which it is used, can evolve into signs expressing the relationship between the artefact and mathematical knowledge (Figure 1)

The organization and fostering of this semiotic process has been the focus of our studies, based on long term teaching experiments (see for instance, Mariotti et al, 1997; Bartolini Bussi, 1996; Bartolini Bussi et al, 1998; Mariotti, 2000; Mariotti & Cerulli, 2001) from which the theoretical framework originated and developed, around two key elements: the notion of the semiotic potential of an artefact and the notion of a didactic cycle (for a full discussion see Bartolini Bussi & Mariotti, 2008)

As said above, the use of an artefact to accomplish a particular task may evoke (Hoyles, 1993) specific mathematical knowledge. In fact, going beyond the immediate sense of its use, experts – mathematicians, and in particular teachers – may recognize mathematical notions in solving a specific problem with the artefact. For example, positional notation and the polynomial notation of numbers may be evoked by an abacus and by its use in counting or addition; similarly, as we will see in the following, constructing a stable figure in a Dynamic Geometry System may evoke classic ‘ruler and compasses’ Geometry. In the TSM framework, the following definition aims to make explicit the twofold relation that may link an artefact to both the sphere of individuals and that of culture, to personal meanings and mathematical meanings. At the same time such a definition aims to draw attention to the need for a clear distinction between meanings related to the use of an artefact, in particular those related to the individual accomplishing a task, and meanings related to mathematical content as cultural attainment.

A double relationship may occur between an artefact and on the one hand the personal meanings emerging from its use to accomplish a task (instrumented activity), and on the other hand the mathematical meanings evoked by its use and recognizable as Mathematics by an expert.

We will call this double semiotic link the *semiotic potential* of an artefact. (Bartolini Bussi & Mariotti, 2008, p. 754)

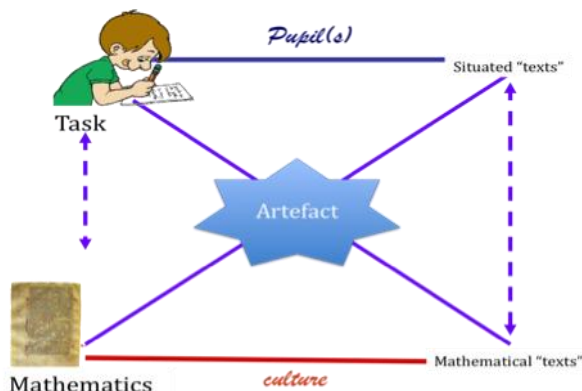


Figure 3 The TMS model for the teaching learning process

The notion of semiotic potential captures the idea that an artefact may be used not only by the student to accomplish a task but also as a vehicle for learning, in other words, it can be used by the teacher as a *tool of semiotic mediation*. Once its use is introduced in the classroom activities the teacher may exploit its semiotic potential to foster students' mathematical learning. The analysis of the semiotic potential is to be considered an a priori phase that

constitutes the core of the teaching sequence design. In the case of classic ancient tools one can take advantage of history, finding there inspiration (Bartolini Bussi, 2001) As far as new technological tools are concerned, the instrumental approach (Rabardel, 1995) as developed by some authors (Trouche, 2005; Artigue, 2002; Lagrange, 2000) offers a good frame for an a priori analysis of the use of the artefact with different tasks and provides the basis for a cognitive and epistemological analysis that contributes to the identification of potential meanings that might emerge during students' activities and the related mathematical meanings.

Exploiting the semiotic potential of the artefact involves for the expert (for instance, the teacher) an awareness of its potential both in terms of evoked mathematical meanings and emergent personal meanings during the activity in the classroom. On the one hand, the teacher must contrive didactic situations where students face tasks that are expected to mobilize specific schemes of utilization and consequently, situations in which they are expected to generate personal meanings. On the other hand, the teacher needs to orchestrate social interactions with the aim of making personal meanings, which have emerged during the artefact-centred activities, develop into the mathematical meanings that constitute the teaching objectives.

Within the frame of the TSM, I will describe how a DGS offers a good context for approaching proof, in particular how some of its elements may function in the hands of the teacher as tools of semiotic mediation to develop mathematical meanings related to proof: specifically, as discussed above, meanings related to the notion of Theorem as a system of statement, proof and theory.

### **Geometrical construction in a DGS**

Let us start from the core of the analysis that lies in the relationship, immediately evoked in the mind of any mathematician, between Cabri figures and geometrical constructions. Such relationship can be elaborated from the point of view of semiotic mediation through both an epistemological and a cognitive analysis, leading to an outline of the semiotic potential of the artefact 'Cabri' with respect to the meaning of Theorem.

In Euclidean Geometry, traditionally referred to as 'ruler and compasses geometry', construction problems have a central position. The theoretical nature of a geometrical construction is clearly stated, and that is in spite of their apparent practical objective, i.e. the drawing which can be produced on a sheet of paper following the solution procedure (Mariotti, 2007) As Vinner clearly points out:

The ancient Greek undertook a challenge which in a way represents some of the most typical features of pure mathematics as an abstract discipline. It is not related to any practical need. (Vinner, 1999, p. 77)

Actually, the use of ruler and compasses generates a set of axioms defining the theoretical system of Euclid's Elements, thus any construction problem to be solved

within Euclidean Geometry – let's say a geometrical construction - leads to a Theorem that validates the construction procedure that solves it.

The appearance of DGSs has triggered a new revival in geometrical constructions. The use of virtual tools can simulate use of the ruler and compass of classic Geometry: lines and circles intersecting each other, over centuries drawn on sand, papyrus or paper, are now reproduced on a computer screen. Any DGS offers something new to the classic world of paper and pencil figures: screen drawings can be acted upon, using the 'dragging modality'. This modality allows the transformation of screen figures, changing the starting points of the construction, but maintaining all the properties defined by the constructing procedure. As a consequence, the stability of the characterizing properties of the drawn figure in respect to dragging, constitutes the natural/standard test of correctness for any construction task in a DGS .

Moreover, consider a DGS like Cabri (Laborde & Bellemain, 1995): the elements of any Cabri figure are related according to the hierarchy of properties determined by its construction procedure. Such a hierarchy of properties corresponds to a relationship of logical dependence between them, while a sub-set of the tools available in a Cabri Menu can be related to its correspondent set of construction tools in Euclidean Geometry (Laborde & Laborde, 1991) This correspondence allows control by "dragging" to be put into a relationship with "theorems and definition" within the system of Euclidean Geometry (Mariotti, 2000; Jones, 2000)<sup>5</sup>.

In summary, as far as the Cabri tools are concerned, a double relationship is recognizable. On the one hand, Cabri tools are related to the construction task that can be solved through their use, resulting in the appearance of a Cabri figure, and to the stability of such a figure by dragging; on the other hand, specific Cabri tools can be related to the geometrical axioms and theorems that can be used to validate the solution of the corresponding construction problem within the Geometry theory.

Hence, a semiotic potential of the Cabri environment is recognizable, residing in the twofold relation that it has with the meaning of the Cabri figure as it emerges from the use of its virtual drawing tools for solving construction problems controlled by the dragging test, and the theoretical meaning of geometrical construction as it is defined within Euclidean Geometry in relation to a given set of axioms.

Exploiting the semiotic potential of the artefact Cabri became the key pedagogical assumption inspiring the design of a teaching sequence that, according to the structure of a Didactic Cycle, consisted of activities involving the use of the artefact and semiotic activities aimed at individual and social elaboration of signs (for details see Mariotti, 2000; 2001)

The use of the artefact was centred on a *construction task* requiring students:

## Plenary lecture

- to produce a specific Cabri figure that should be stable by dragging, and write the description of the procedure used to obtain it;
- to produce a validation of the ‘correctness’ of the construction realized.

The Construction task consisted of two requests, the first corresponding to acting with the artefact, the second to producing a written text referring to such actions. Note that producing a written text consists of both describing and commenting on the procedure carried out. The request of *validating* the solution made sense with respect to the Cabri environment, corresponding to the need for explaining and gaining insight into the reason why the figure on the screen passes the dragging test. The process which began in the students’ semiotic production proceeded through social interactions orchestrated by the teacher in true Mathematics Discussions (Bartolini Bussi, 1998; Mariotti, 2001) Whereas, at the very beginning, the term *construction* made sense only in relation to using particular Cabri tools and to passing the dragging test, later on the meaning slowly evolved, acquiring the theoretical meaning of Geometrical construction.

Such an evolution could be accomplished, under the guidance of the teacher exploiting the correspondence between Euclidean Axioms and specific Cabri tools and their modes of use. Starting from an empty menu, the choice of the appropriate tools to start with was discussed as well the correspondence with a set of *Construction axioms* constituting the first core of the Geometry Theory which any validation can refer to. Then, as long as new constructions were produced, the corresponding Theorems were validated and added to the theory. Students could experience and participate in two parallel processes of evolution: on the one hand, the enlargement of the Cabri menu, and on the other hand the corresponding development of a Geometry Theory. We can claim that students were introduced to the Theorems culture, because students not only produced new statements and their proofs, but they also had the opportunity of becoming aware of the theory within which proofs made sense, and of how such a theory was developing.

Results of several long-term teaching experiments attest the emergence of intermediate meanings, rooted in the semantic field of the artefact, and their evolution into mathematical meanings, consistent with Euclidean Geometry. In particular, the experience in the classroom highlighted how meanings emerging from the use of tools in the constrained world of Cabri were effective in developing and interlacing the sense of proof and the sense of theory. The use of any single tool mediates the meaning of the application of an axiom or a theorem, while the idea of the Cabri menu, that is the set of available tools, mediates the meaning of theory. Conventionality and the evolutionary nature of a theory clearly emerged during collective discussions where students experienced both establishing and developing a Geometry Theory through exploiting the possibility of personalizing the menu by selecting the tools to be used. I have written extensively in the past about these

experiments (Mariotti, 2000, 2001, 2007, 2009), and now I will move on to the last example.

### **More about the semiotic potential of a DGS: Dragging as a semiotic mediator of conditionality**

For my last example I will elaborate a bit more on the potential offered by a DGS, and in particular on the semiotic potential of dragging modes in respect to the mathematical notion of conjecture. The significance of this discussion stems from a shared opinion, already mentioned above, about the fundamental role that open problems and conjecturing activities have in developing the sense of proof, fostering a productive relationship between ‘spontaneous’ argumentation processes and theoretical validation (Arsac & Mante, 1983; Arsac, 1992; Hadas et al. 2000; Pedemonte, 2002)

As I have already said, different contexts afford open-ended questions in different ways, offering different potentialities for both posing and solving open problems. What I want to discuss concerns a very peculiar aspect of Dynamic Geometry, and from this perspective I intend to compare the two contexts considered above, the sun shadow and the DGS contexts.

The studies carried out by Boero and his colleagues focused on different aspects of experiencing sun shadows, highlighting the crucial role played by the dynamic character of the phenomena under investigation. In the case of open problems, when it is requested to produce a conjecture, dynamicity seemed to foster transformational mental processes (Simon, 1996; Harel & Sawder, 1998) providing key elements to produce conditional statements. The conditional statement seemed to emerge as “crystallization” of a dynamic exploration. This crystallization isolates a specific moment, and a position, when the occurrence of one fact has the occurrence of another fact as consequence. In Mathematics, the formulation of a conjecture expresses such a crystallization in a conditional statement (Boero et al., 1999; Boero et al., 2007) Because of the dynamic nature of experiences in a DGS it seems reasonable to address the issue of the role of dragging modes in conjecture production.

Thus, in the broader perspective of the solution of open-ended problems, let us consider the specific request of formulating a conjecture concerning a specific configuration in a DGS. Actually, the term open-ended problem has often been used in the mathematics education literature (see for instance Arsac & Mante, 1983; Silver, 1995) Usually, it refers to a task that asks a question without revealing or suggesting the expected answer. In the specific case when the solver is explicitly requested to formulate a conjecture, we speak of conjecture open problems. This is a very common case in Geometry, when within a specific situation, corresponding to a well-defined geometrical configuration, the solver is asked to make a conjecture that usually assumes the form of a conditional statement between possible properties of

the given configuration. When the solution is sought within a DGS, like Cabri, the solver is expected to explore the configuration dynamically. This means that the solver has to interpret perceptual data coming from observing the screen whilst dragging the figure, and to transform them into a geometrical statement. In other words, the process of exploration is productive if the solver is able to *transform perceptual data into a conditional relationship between geometrical properties*

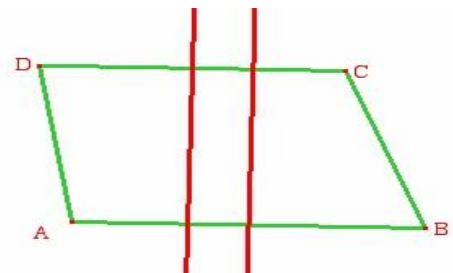
The seminal work of Arzarello, Olivero and colleagues (Arzarello et al., 1998, 2002; Olivero, 2001, 2002) as well that of other researchers (Hölzl, 1996; Leung & Lopez-Real, 2002 2006; Leung, 2008; Healy 2000), has shown the potentialities of different dragging modalities in supporting the conjecturing process. Starting from their results, I want to discuss the potential offered by a DGS not only in supporting the conjecturing processes but also in mediating the mathematical meaning of *conjecture* and specifically of *conditional statement* in the Geometry context. Specifically, in the frame of the TSM, I will outline the semiotic potential of particular modalities of dragging with respect to the notion of conjecture as a conditional statement. Dragging modalities can be considered as specific artefacts used to solve an open problem, and meanings emerging from this use may be related to the mathematical meanings of conditional statements that express the logical dependency between a *premise* and a *conclusion*.

### Exploring in a DGS: invariants by dragging

When dragging is activated on a specific figure obtained by a construction, it provokes a phenomenon usually described as the movement of the figure. Such a movement is perceived because of the contrast between what is changing and what is not changing, because some properties of the figure are maintained by dragging and others are not. The notion of *invariant* by dragging naturally emerges as a property, or a set of properties, that are preserved during a dragging action, but for our discussion, it is useful to look more carefully at this phenomenon.

When a figure is acted upon, the properties stated by the primitives used in the construction are maintained, but it happens that other properties will appear as invariant too, specifically all those properties that are a consequence of the construction properties within the theory of Euclidean Geometry (Laborde & Sträßer, 1990) Take for instance the following configuration:

ABCD is a quadrilateral in which D is chosen on the line parallel to AB through C, and the perpendicular bisectors of AB and CD are constructed.



By dragging the Cabri figure, the constructed properties are preserved - the parallelism between AB and CD, the perpendicularity between the

Figure 4: The result of the construction described in the example above.

each side and its bisector – but also the parallelism between the two bisectors clearly emerges as invariant. This can lead us to formulate the following conjecture: “if two sides of the quadrilateral are parallel, then the corresponding perpendicular bisectors are parallel.”

Actually, the fact that a specific relationship between invariants is preserved corresponds to the general validity of a logical implication between properties of a geometrical figure. Because of their simultaneity it may be difficult to control the hierarchy induced on the different invariants by the construction. Nevertheless, the asymmetry between the two kinds of invariants has a counterpart in an asymmetry in the movement of different elements of the figure. Actually, only basic points – those from which the construction originates - can be selected and dragged, thus two different types of movement occur that it is worth distinguishing and analysing carefully for our purpose.

*Direct motion*, that is the variation of an element in the plane under the direct control of the mouse, and *indirect motion*, that is the variation of any other element as a consequence of direct motion.

During a dynamical exploration, the possibility of discriminating between the two movements allows one to ‘feel’ motion dependency. Hence, the solver can distinguish between *direct invariants* and *indirect invariants* and interpret their dynamic relationship in terms of logical consequences between geometrical properties, eventually expressing it as a conditional statement between a premise and a conclusion.

The distinction between direct and indirect movement leads us to reconsider the results coming from previous studies on conjecturing in a DGS, and in particular to reconsider one specific modality of dragging, previously described as *Dummy locus dragging* (or *Lieu muet dragging*) (Arzarello et al. 1998, 2002; Olivero, 2002) This involves acting on the mouse with the intention of maintaining a specific property, i.e. realizing a constrained movement of the original figure *in order* to make a specific property become ‘invariant’. Such a new type of invariant that we can call an ***Indirectly Induced Invariant*** will correspond to a consequence of all the properties given by the construction plus the new hypothesis corresponding to the constrained dragging. In other words, exploring via this modality that we call Maintaining Dragging (MD) (Baccaglini-Frank, 2010) corresponds to what mathematicians would describe as exploring “under which condition ... a certain property occurs”.

According to the previous analysis using MD to solve a conjecture problem, it is possible to clearly distinguish between the premise and the conclusion of the conditional statement that is the outcome of the exploration as the student can carry it out. The student can directly and intentionally control such a distinction during the exploration: the conclusion will be the property that the solver decides to maintain, the premise will be the property corresponding to the constrained movement.

## Plenary lecture

In summary, as far as dragging and conjecturing is concerned there are two main modalities characterized by the intention with which the solver selects and searches different types of invariants:

- free dragging: looking for indirect invariants as consequences of the direct invariants; and
- constrained dragging: looking for possible construction invariants that may cause a specific indirect intentional invariant to happen.

The simultaneity of invariants combined with the control of the different status of each kind of invariant is the counterpart of the logical dependency between a premise, corresponding to the direct invariants, and a conclusion, corresponding to the indirect invariants (induced either intentionally or unintentionally)

Taking the perspective of semiotic mediation, we claim that the different dragging modalities, together with the different types of invariants that originate from their use in solving a conjecture-production task, offer a rich semiotic potential in respect of the notion of conjecture as a conditional statement between a premise and a conclusion. The semiotic potential is recognizable in the following relationships between:

- *direct and indirect invariants* and respectively *premise and conclusion* of a *conditional statement*; and
- the *dynamic sensation of dependence between the two types of invariants* and the geometrical meaning of *logical dependence between premise and conclusion*.

The analysis of process of exploration that can be expected when using the MD has been the focus of a recent study carried out at the upper secondary level (Baccaglini-Frank, 2010) The experiment consisted of a teaching phase in which students were introduced to the different modalities of dragging and an observation phase in which students were interviewed in pairs during the solution of conjecture open problems.

Results from the study show how different meanings related to the notion of conjecture may emerge. The different kinds of invariant can be characterized by reference to their specific status in the activity of exploration; their specific characteristics make them clearly recognizable by the students and can be used by the teacher to exploit the semiotic potential of dragging and specifically of MD. Such results, and in particular a model of the process of conjecture-generation based on the use of MD, has been presented and discussed in several papers to which I refer for a detailed discussion (Baccaglini-Frank, 2010, in press; Baccaglini-Frank & Mariotti, 2009, 2010; Baccaglini-Frank et al., 2009)



## CONCLUSIONS

We started with some considerations about different epistemological approaches to proving and proof, aiming at clarifying their impact on didactical approaches to proof in the classroom. The educational aim was summarized as resolving the potential conflict between the two main functions of proof, i.e. validating within a theory and explaining with respect to interlocutors' conceptions. This means finding a way, a pedagogical/ didactical way, of developing the complex relationship between argumentation, with its goal of convincing, and proof, with its goal of theoretically validating. Finally, it means taking into account the differences between argumentation and proof, and exploiting their deep link rather than ignoring it.

The approaches to proof that were discussed above can be considered consistent with that aim, though the way they were implemented in the classroom present differences; in particular, differences in the choice of context and the way its potential was exploited.

Differences can be referred to specific theoretical frameworks that inspire the design of the scenario of classroom activities. This is for instance the case of the choice of a DGS in the examples given above. In that case the potential of Cabri, as generally acknowledged in current literature, is elaborated via the filter of the TSM and described in terms of semiotic potential giving a clear explanation of the specific links between acting in Cabri and the meanings that may emerge in such use, and the mathematical meanings that may be recognized and may constitute the educational goal of the instructional intervention.

The complexity of the educational aim certainly requires long term interventions, and this requires long term teaching experiments and research projects that are very demanding, both from the point of view of the methodological design and the implementation in the classroom. Nevertheless, further research is needed, and I hope that this contribution will offer a stimulus in this direction.

## NOTES

1. An exception is that of mathematical induction, which is explicitly treated, and to which a specific training is devoted. But, very rarely, is mathematical induction presented in comparison to other modalities of proving, which are commonly considered natural and spontaneous ways of reasoning.
2. Following the distinction introduced by Rabardel (1995) we use the term artefact in order to distinguish between the tool itself and the specific way of using that tool in order to accomplish a specific task.
3. According to the semiotic approach developed by other researchers (Radford, 2003; Arzarello, 2006) and inspired by Pierce, we use the term sign consistently with the idea of the indissoluble relationship between signified and signifier, and the idea that meaning originate in the intricate interplay of signs (Bartolini Bussi and Mariotti, 2008); for a thoughtful discussion see also (Sfard, 2000, p. 42 and following)
4. The distinction between personal meanings and mathematical meanings may remind of Brousseau's distinction between knowing (*connaissance*) and knowledge (*savoir*) (1997) Even if they are not in antithesis the two perspectives

## Plenary lecture

cannot be reduced one to the other: the former stresses the semiotic dimension of the teaching-learning processes which is in the shadow in the latter.

5. Actually a DGS provides a larger set of tools, including for instance "measure of an angle", "rotation of an angle" and the like. That implies that the whole set of possible constructions do not coincide with that attainable only with ruler and compass, see (Stylianides & Stylianides, 2005) for a full discussion.

## REFERENCES

- Arsac, G. & Mante, M. (1983) Des "problème ouverts" dans nos classes du premier cycle, *Petit x*, 2, 5-33.
- Arsac G. (1992) *Initiation au raisonnement au college*. Presse Universitaire de Lyon
- Artigue, M. (2002) Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal for Computers in Mathematical Learning*, 7(3), 245–274.
- Arzarello, F. (2006) Semiosis as a multimodal process. *Relime* (special issue), 267–299.
- Arzarello F., Micheletti C., Olivero F., Robutti O. (1998) A model for analysing the transition to formal proofs in Geometry. In A. Olivier & K. Newstead (Eds) *Proceedings of PME-XXII* (2, 24-3) Stellenbosch, South Africa.
- Arzarello, F. (2007) The proof in the 20th century: from Hilbert to automatic theorem proving. In P. Boero (Ed.), *Theorems in School from History and Epistemology to Cognitive and Educational Issues* (pp. 43-64) Rotterdam: Sense Publishers.
- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002) A cognitive analysis of dragging practices in Cabri environments, *Zentralblatt fur Didaktik der Mathematik/International Reviews on Mathematical Education*, 34(3), 66-72.
- Baccaglioni-Frank, A. (2010) [Conjecturing in Dynamic Geometry: A Model for Conjecture-generation through Maintaining Dragging](#). *Doctoral dissertation*, University of New Hampshire, Durham, NH. Published by ProQuest.
- Baccaglioni-Frank, A., & Mariotti, M.A. (2009) Conjecturing and proving in dynamic geometry: The elaboration of some research hypotheses. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello, *Proceedings of CERME 6*, Lyon, France. [www.inrp.fr/editions/editions-electroniques/cerme6/](http://www.inrp.fr/editions/editions-electroniques/cerme6/)
- Baccaglioni-Frank, A., Mariotti, M. A., & Antonini, S. (2009) Different perceptions of invariants and generality of proof in dynamic geometry. In M. Tzekaki & H. Sakonidis (Eds.), *Proceedings of the PM33* (2, 89-96), Thessaloniki, Greece.
- Baccaglioni-Frank, A., & Mariotti, M.A. (2010) [Generating Conjectures in Dynamic Geometry: the Maintaining Dragging Model](#). *International Journal of Computers for Mathematical Learning*, 15(3), 225-253.

## Plenary lecture

- Bartolini Bussi, M. G., Boni, Ferri, F & Garuti, R. (1999) Early approach to theoretical thinking: gears in primary school. *Educational Studies in Mathematics*, 39, 1-3, 67-87
- Bartolini Bussi, M. G., & Mariotti, M. A. (2008), Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh, (eds.) *Handbook of International Research in Mathematics Education, second revised edition* (pp. 746-805) Lawrence Erlbaum, Mahwah, NJ.
- Balacheff, N. (2008) The role of the researcher's epistemology in mathematics education: an essay on the case of proof. *ZDM. The International Journal on Mathematics Education*, 40, 501–512.
- Bartolini Bussi, M.G. (1996) Mathematical Discussion and perspective Drawings in Primary School. *Education Studies in Mathematics*, 31, 11-41.
- Bartolini Bussi, M. (2001) The geometry of drawing instruments: arguments for a didactical use of real and virtual copies. *Cubo Matematica Educacional*, 3(2), 27–54.
- Bartolini Bussi M. G., Boni, M., Ferri, F. & Garuti, R. (1999) Early Approach To Theoretical Thinking: Gears in Primary School. *Educational Studies in Mathematics*, 39(1-3), 67-87.
- Boero, P. (2002) Geometric signs and students' verbal reports: The case of the geometric model of sunshadows. In A. D. Cockburn & E. Nardi (Eds.) *Proceedings of PME-XXVI*, (pp.129-136) Norwich, England.
- Boero, P. (Ed.) (2007) *Theorems in school: From history, epistemology and cognition to classroom practice*. Rotterdam, The Netherlands: Sense Publishers.
- Boero, P., Dapuetto, C., Ferrari, P., Ferrero, E., Garuti, R., Lemut, E., Parenti, L. & Scali, E. (1995a) Aspects of the mathematics-culture relationship in mathematics teaching-learning in compulsory school. In L. Meira & D. Carraher (Ed.) *Proceedings of PME-XIX* ( 1, 151–166) Recife, Brazil.
- Boero, P., Garuti, R., Lemut, E., Gazzolo, T. & Llado', C. (1995b) Some Aspects of the Construction of the Geometrical Conception of the Phenomenon of Sunshadows. . In L. Meira & D. Carraher (Eds.) *Proceedings of PME-XIX* (3, 3-10) Recife, Brazil.
- Boero, P. & Garuti, R. (1994) Approaching rational geometry: from physical relationships to conditional statements. In J.P. Da Ponte & J. P. Matos (Eds.) *Proceedings of PME-XVIII* (2, 96-103) Lisbon, Portugal.
- Boero, P.; Garuti, R.; Lemut, E. & Mariotti, M.A. (1996) Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems'. In L. Luig & A. Gutierrez (Eds. ) *Proceedings of PME-XX* (2, 113-120) Valencia, Spain.

- Boero, P., Garuti, R., & Lemut, E. (2007) Approaching theorems in grade VIII: Some mental processes underlying producing and proving conjectures, and conditions suitable to enhance them. In P. Boero (Ed.) *Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 249-264) Rotterdam, The Netherlands: Sense Publishers.
- Brousseau, G. (1997) *Theory of didactical situations in Mathematics*, Dordrecht: Kluwer Academic Publisher.
- Douek, N. (1999) Argumentation and conceptualisation in context. *Educational Studies in Mathematics*, 39, 89-110.
- de Villiers M. (1998) An alternative approach to proof in dynamic geometry. In R. Lehrer & D. Chazan (Eds.) *New directions in teaching and learning geometry* (pp. 369-393) Lawrence Erlbaum.
- Dreyfus, T. & Hadas, N. (1996) Proof as answer to the question why, *Zentralblatt für Didaktik der Mathematik/International Reviews on Mathematical Education* 28(1), 1–5.
- Duval, R. (1991) Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics*, 22(3), 233-263.
- Duval, R. (1992-93) Argumenter, démontrer, expliquer: continuité ou rupture cognitive?, *Petit x*, 31, 37-61.
- Fawcett, H. (1938) *The nature of proof*. The National Council of Teachers of Mathematics Thirteenth Yearbook. New York: Bureau of Publications of Teachers College, Columbia University.
- Ferrari, P.L. (Ed.) (1999) *Proceedings of the School on Epistemology of Mathematics and Didactics*, Levico Terme, 8-12 February 1999.
- Hadas, N., Hershkowitz, R., & Schwarz, B. (2000) The role of contradiction and uncertainty in promoting the need to prove in Dynamic Geometry environments. *Educational Studies in Mathematics*, 44 (1–3), 127–150.
- Hanna G. (1989) More than formal proof. *For the learning of mathematics* 9(1), 20-25
- Hanna, G. (1989a) Proofs that prove and proofs that explain. In *Proceedings of PME 13* (pp. 45–51) Paris, France.
- Hanna, G. (1995) Challenges to the importance of proof, *For the Learning of Mathematics*, 15(3), 42–49.
- Hanna, G. & Jahnke, H. N. (2007) Proving and Modelling. In W. Blum, P. L. Galbraith, H.W. Henn and M. Niss (ed.), *Applications and Modelling in Mathematics Education. The 14th ICMI Study* (pp. 145-152) Dordrecht: Springer.
- Harel, G. & Sowder, L. (1996) Classifying processes of proving. In L. Luig & A. Gutierrez (Eds. ) *Proceedings of PME-XX* (3, 59-65) Valencia, Spain.

### Plenary lecture

- Harel G., Sowder L (1998) Students' proof schemes: Results from exploratory studies. In A. Schonfeld, J. Kaput & E. Dubinsky E. (Eds.) *Research in collegiate mathematics education III*. (Issues in Mathematics Education, Vol. 7, pp. 234-282) American Mathematical Society.
- Healy, L. (2000) Identifying and explaining geometric relationship: interactions with robust and soft Cabri constructions. In Tadao Nakahara & Masataka Koyama (Eds.) *Proceedings of PME-XXIV* (1, 103-117) Hiroshima, Japan.
- Hölzl, R. (1996) How does "dragging" affect the learning of geometry. *International Journal of Computer for Mathematical Learning*, 1(2), 169-187
- Hölzl R., Hoyles C. & Noss R. (1994) Geometrical relationships and dependencies in Cabri, *Micromath* 10 (3), 8-11.
- Hoyles, C. (1993) Microworlds/Schoolworlds: the transformation of an innovation. In Keitel, C. & Ruthven K. (Eds. ), *Learning from computers: Mathematics education and technology*, (pp. 203–221) Berlin: Springer- Verlag.
- Keitel, C. & Ruthven K. (Eds) (1993) *Learning from Computers: mathematics education and technology* (NATO ASI series, Vol. 12) Berlin: Springer-Verlag.
- Kozulin, A. (2003) Psychological tools and mediated learning. In A. Kozulin, B. Gindis, V.S. Ageyev, & S.M. Miller (Eds.), *Vygotskij's Educational Theory in Cultural Context* (15 -38) Cambridge University Press.
- Jahnke, H. N. (2005) A genetic approach to proof. In M. Bosch (Ed.), *Proceedings of the CERME 4*, Sant Feliu de Guíxols, Spain. <http://www.erne.unito.it/CERME4/>
- Jahnke, H. N. (2007) Proof and hypotheses. *ZDM. The International Journal on Mathematics Education*, 39(1-2), 79-86.
- Jahnke, H.N. (2009) Proof and empirical sciences. In F-L Lin, F-J Hsieh, G. Hanna & M. de Villier (Eds.) *ICMI Study 19* (1, 238-243) Taiwan: Taipei.
- Jones, K. (2000) Providing a foundation for a deductive reasoning: students' interpretation when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, 44 (1-2), 55-85.
- Laborde, C. (2000) Dynamic geometry environment as a source of rich learning context for the complex activity of proving. *Educational Studies in Mathematics*, 44 (1-2), 151-61.
- Laborde, J.-M. & Bellemain, F. (1995) Cabri-géomètre II and Cabri-géomètre II plus [computer program]. Dallas, USA: Texas instruments and Grenoble. France: Cabrilog.
- Laborde, C. & Laborde, J.M. (1991) Problem solving in geometry: From microworlds to intelligent computer environments. In In J. P. Ponte, J. F.Matos &

## Plenary lecture

- D. Fernandes (Eds.), *Mathematical problem solving and new information technologies* ( 177-192) NATO AS1 Series F, New York: Springer-Verlag.
- Leung, A. (2008) Dragging in a Dynamic Geometry Environment through the Lens of Variation. *International Journal of Computers for Mathematical Learning*, 13 (2), 135-157.
- Leung, A. & Lopez-Real, F. (2002) Theorem Justification and Acquisition in Dynamic Geometry: a Case of Proof by Contradiction. *International Journal of Computers for Mathematical Learning*, 7, 145-165.
- Lopez-Real, F. & Leung, A. (2006) Dragging as a conceptual tool in dynamic geometry, *International Journal of Mathematical Education in Science and Technology*, 37(6), 665- 679.
- Lagrange, J-B 2000 L'intégration des instruments informatiques dans l'enseignement: Une approche par les techniques. *Educational Studies in Mathematics*, 43 (1), 1-30.
- Lolli, G. (1999) Truth and Proofs, a short course in the epistemology of mathematics. *School on Epistemology and Didactics, Levico Terme, 8-12 February 1999* (mimeographed)
- Mariotti M.A. (2000) Introduction to proof: the mediation of a dynamic software environment. *Educational Studies in Mathematics*, 44, (1-2), 25 – 53
- Mariotti M. A. (2001) Justifying and proving in the Cabri environment. *International Journal of Computer for Mathematical Learning*, 6 (3), 257-81
- Mariotti M.A. (2006) Proof and proving in mathematics education. A. Gutiérrez & P. Boero (eds) *Handbook of Research on the Psychology of Mathematics Education* (pp. 173-204) Sense Publishers, Rotterdam, The Netherlands.
- Mariotti, M.A. (2007) Geometrical proof: the mediation of a microworld. In P. Boero (Ed.) *Theorems in school: From history epistemology and cognition to classroom practice* (pp. 285-304) Sense Publishers, Rotterdam, The Netherlands
- Mariotti M.A. (2010) Proofs, Semiotics and Artefacts of Information Technologies. In G. Hanna et al. (eds.), *Explanation and Proof in Mathematics:Philosophical and Educational Perspectives* (pp. 169-190 ) Springer.
- Mariotti M.A., Bartolini Bussi, M., Boero P., Ferri F. & Garuti R. (1997) Approaching geometry theorems in contexts: from history and epistemology to cognition. In E. Pehkonen (Ed.) *Proceedings of PME-XXI*, (1, 180-95) Lathi, Finland.
- Mariotti , M.A. & Cerulli, C. (2001) Semiotic mediation for algebra teaching and learning. In M. van den Heuvel-Panhuizen (Ed.) *Proceedings of PME-XXV* (3-343-350) Utrecht, The Netherlands.

- Mariotti M.A. & Maracci M (2010) Un artefact comme outils de médiation sémiotique: une ressource pour l'enseignant. In: G. Gueudet & L. Trouche. *Ressources vives. Le travail documentaire des professeurs en mathématiques* (pp. 91-107) Rennes: Presses Universitaires de Rennes et INRP
- Morselli, F. & Boero, P. (2009) Habermas' Construct of Rational Behaviour as a Comprehensive Frame for Research on the Teaching and Learning of Proof. In F-L Lin, F-J Hsieh, G. Hanna & M. de Villier (Eds.) *ICMI Study 19 Conference Proceedings* (2, 100-05) Taiwan: Taipei.
- Pedemonte, B. (2002) *Etude didactique et cognitive des rapports de l'argumentation et de la démonstration en mathématiques*, (Unpublished) Thèse de Doctorat, Université Joseph Fourier, Grenoble. <http://tel.archives-ouvertes.fr/tel-00004579/>
- Rabardel, P. (1995) *Les hommes et les technologies – Approche cognitive des instruments contemporains*. Paris: A. Colin Editions.
- Reid D. A. & Knipping C., 2010 *Proof in Mathematics Education: Research, Learning and Teaching*, Sense Publisher, Rotterdam, The Netherlands.
- Thurston W.P. 1994 On proof and progress in Mathematics. *Bullettin of the Mathematical Society*, 30 (2), 161-177
- Sfard, A. (2000) Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds), *Symbolizing and Communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design* (pp. 37-98) Mahwah, NJ: Erlbaum
- Sierpinska, A.: 2005, On practical and theoretical thinking. In M. H. G. Hoffmann, J. Lenhard, F. Seeger (Eds.) *Activity and Sign – Grounding Mathematics Education. Festschrift for Michael Otte* (pp.117-135) New York: Springer.
- Stylianides, G.J. & Stylianides A. J. (2005) Validation of solutions of construction problems in dynamic geometry environments. *International Journal of Computers for Mathematical Learning*, 10, 31–47
- Trouche, L. (2005) Construction et conduite des instruments dans des apprentissages mathématiques: Nécessité des orchestrations. *Recherches en Didactique des Mathématiques*, 25(1), 91–138.
- Simon, M. A. (1996 ) Beyond Inductive and Deductive Reasoning: The Search for a Sense of Knowing. *Educational Studies in Mathematics*, 30(2), 197-209
- Vinner, S. (1999) The Possible and the Impossible, *ZDM. The International Journal on Mathematics Education*, 99/2, 77.
- Vygotskij, L. S. (1978) *Mind in Society. The Development of Higher Psychological Processes*, Harvard University Press

## Plenary lecture



# Working Groups: Research Papers



# **INTRODUCTION TO THE PAPERS OF WG01: ARGUMENTATION AND PROOF**

Viviane DURAND-GUERRIER, Université Montpellier 2, France

Kirsti HEMMI, Mälardalen University, Sweden

Niels JAHNKE, Universität Duisburg-Essen, Deutschland

Bettina PEDEMONTE, Istituto per le Tecnologie Didattiche - CNR Genova, Italy

This chapter collects the contributions discussed during the working sessions by the thirty-one participants from fourteen countries of the WG1 «Argumentation and proof» at CERME7 in Rzeszow (Poland). Eighteen papers from eleven countries have been presented and discussed. Each presentation was followed by a reaction from a participant presenting the key issues and posing questions to the author(s). The papers are presented under five themes: epistemological and cognitive issues, means for analysing proving activities, transparency, educational implication of views on mathematics, on the relevance of proof on teaching mathematics.. We will conclude this introduction by some challenging perspectives emerging from our discussion.

## **EPISTEMOLOGICAL AND COGNITIVE ISSUES**

Cognitive and meta-cognitive issues were specifically developed in three papers by the Italian participants, in line and deepening the research in this area since 1980.

The paper by Paolo Boero and the one by Ferdinando Arzarello and Cristina Sabena draw explicitly on the integration of the models of Toulmin and Habermas that was presented in CERME 6. Boero presents some tools for analyzing a « successful » classroom discussion at a university course in primary school teacher education, with the aim of developing awareness about the « rules » of argumentation and proof. Arzarello and Sabena consider the construction of “meta-cognitive unity” in order to give reason of success and difficulties in indirect proofs, using Peirce's account of abduction for their description of the cognitive processes involved in indirect proofs. The paper by Anna Baccaglini-Frank focuses on an elaboration of conjectures in dynamic systems. The author emphasises two different types of abduction associated to two different ways of generating conjectures that arise using a particular modality of dragging in dynamic geometry. The hypothesis of the author is that conjectures suggested by the use of the dragging function will lead to cognitive rupture, while conjectures suggested by mathematical considerations will lead to cognitive unity.

## **MEANS FOR ANALYSING PROVING ACTIVITIES**

An important issue in the research on proof and proving is the possibility to have access to the process in which students are involved when they elaborate a proof. The four papers in this section focus on this aspect, in various manners and with

different populations. Esther Brunner, Kurt Reusser and Christine Pauli present an analysis instrument developed to describe content-related aspects of the proving process. This instrument has been used to analyse videos concerning a mathematical problem solving session in 32 classes of the 8th or 9th year of the three school types in Germany and Switzerland. The main result of this study is the dominance of formal deductive proof among the classes whatever the type of school, while experimental or generic proof could be relevant for the proposed problem. In his paper, Yosuke Tsujiyama focuses on the role of looking-back at proof-planning processes in students' proving activity, that are not explicit in proof as a product. This is illustrated with an example in geometry with Japanese 8th grade students. Patricia Perry, Óscar Molina, Leonor Camargo, and Carmen Samper analyse the proving activity of a group of three university students solving a geometrical problem in a dynamic geometric system. Solving the problem requires formulation of a conjecture and justification in a theoretical system. For their analyses, the authors refer to the integration of Toulmin's and Habermas' models mentioned above to elaborate components of a successful performance. The activities of the students were video taped and transcribed. Their analyses indicate that students' proving activities are close to the considered components. Julia Cramer develops a methodology combining Toulmin's scheme and a collection of topical schemes, and an epistemic action model in order to shed light on the relations between argumentation and knowledge construction. In her paper she presents some preliminary results obtained from the work of grade 10 German students and grade 11 Israeli students who were submitted three tasks in the frame of a research project on "effective knowledge construction in interest-dense situations". In the presented cases, everyday argumentation was the starting point to develop mathematical conjectures.

### **TRANSPARENCY**

Through the notion of "transparency" the three papers presented in this section show the importance to make more or less invisible for students some important aspects concerning proof.

Kirsti Hemmi and Clas Löfwall present an ongoing research aiming to develop and test some tasks that could enhance students' understanding of the discovery function of proof. The authors consider proof as an essential artefact (i.e. tools that mediate knowledge between the social and the individual) in mathematical practice. The students seemed to catch the idea of the function of discovery, although they did not manage to construct the proof. The paper provides evidence of the interest of exploring these questions in further research. Kirsten Pfeiffer presents a scheme that she has developed to describe and explore students' proof evaluation performance; following Hemmi, she considers proof as an intellectual artefact in mathematical practice. The author uses this scheme for analysing an experiment consisting of interviews with eight first year university students. Students were asked to evaluate

six different proofs of the same statement in a familiar domain. The results suggest the relevance of this scheme to investigate students' knowledge and skill in proof and proving. In her paper, Bettina Pedemonte explores the development of algebraic proof in a dynamic system (AlNuSet). The main hypothesis of the author is that activities in this environment favour the “transparency” of the proof, due to the fact that in this system, the transformation of an expression is not the result of a calculation, but is carried out by explicit algebraic rules and axioms.

## **EDUCATIONAL IMPLICATION OF VIEWS ON MATHEMATICS**

In their paper, Simon Modeste and Cecile Ouvrier-Bufferet consider the opportunity offered by the appearance of algorithms in the curricula in France to study the links between algorithms and proof. They have developed an epistemological analysis and a study of “how researchers know the algorithms” through interviews, taking into account both tool and object aspects. Their results support the relevance of algorithms for studying proof, but the authors point out that little is known about algorithms as objects. The authors intend to use this epistemological model of analysis in further research with students. Joanna Mamona-Downs and Martin Downs discuss in which respect mathematicians as teachers at tertiary level are able to convey the interest of proof. Relying on the difference between a credited argument and a proof, they hypothesise that in certain circumstances the insistence on proof could appear to students as a game for pedants. The examples that are provided concern examples where mathematical modelling takes place. Considering that beauty should take a place in curricula, which is the case in some countries, but not in others, Manya Raman and Lars-Daniel Öhman investigate in what ways mathematical proofs could be perceived as beautiful by working mathematicians and discuss possible educational implications on insight into the nature of beauty in mathematics. Their analysis relies on two proofs of Pick's theorem that gives a simple formula for calculating the area of a lattice polygon. Relying on the work of Etchmendy concerning Hyperproof, and on the legitimacy of visual or diagrammatic proof through possible formalisation in  $\wp$ -logic, Reinert A. Rinvold and Andreas Lorange investigate the interest of developing multimodal proof, including actions and gestures in mathematics education, with a case study conducted with pairs of teacher training students, involving number patterns represented by visual and physical figures. The authors conclude that this kind of activity has a potential for learning proof. Antti Viholainen examines the effects of the view of mathematics on how the role of argumentation and proof is seen. He considers mainly two opposite views: if mathematics is seen as an axiomatic system, formal argument will be required, while if mathematics is seen as a thinking and learning process, informal arguments play an important role.

## **ON THE RELEVANCE OF PROOF ON TEACHING MATHEMATICS**

In this session papers analyse the role of proof in teaching mathematics Emelie

## Working Group 1

Reuterswärd and Kirsti Hemmi report some results from a case study on five Swedish upper secondary school teachers concerning their view on proof, particularly concerning the role and relevance of proof in teaching mathematics. The data consist of interview transcripts and protocols of observation of lessons. Two main different views appear: "proof as something for all students" and "proof as not necessarily for all students". This study was motivated by the appearance in the new Swedish curriculum of a more explicit focus on proof. The results open for the necessity of studying how these reforms are implemented and what effect they have on different student groups. In some countries like Canada, France or Italy, the relevance of teaching proof at secondary school level is well admitted, and the possibility of developing reasoning skills at elementary school is considered. The paper from Stephan Cyr presents the results of an experiment with elementary school students aged 11-12 showing the possibility of developing ability to reason deductively and validate geometric statements by using geometrical properties rather than measurement, which is an important issue for students in order to become aware of the difference between practical and theoretical geometry. Margo Kondratieva takes into account that development of reasoning skills and formation of concepts is a life-long process, considering that at each level of concept development, reasoning behaviour and degree of rigor depends on the level of concept maturity. In her paper, she illustrates the process of designing interconnecting problems on an example from Euclidian geometry, claiming that the course of formalisation of reasoning affects the conceptualisation process related to the object of the problem.

## CHALLENGING PERSPECTIVES

During this session, we discussed on challenging perspectives about argumentation and proof.. A first issue concerns the use, evolution, elaboration or integration of theoretical constructs introduced at the previous CERME (e.g. cognitive unity; Toulmin's model, transparency background), and discussion of new theoretical frames (e.g. Habermas model for rational behaviour in proving). Closely related are the epistemological and cognitive issues concerning different varieties of indirect proof, and the specificity of proof in accordance with different mathematical fields, in particular according with the prominence or not of axiomatisation, and the effect on the nature of proof of the use of technological environment. Another important issue concerns the cognitive development with three main topics that were discussed during the session and for which further research are needed: the effect on teaching and learning proof, on students' performance in mathematics - the way students really prove and learn to prove - the designing activity fostering argumentation and proof skills along the curriculum from kindergarten to university. We also discussed the logical aspects of proof, the way of taking into account everyday logic competencies in class and of considering the role of semantics aspects and the place for logical matters in the teaching and learning of proof and proving. The implication for teaching is a crucial question addressed to researchers and mathematics educators; it

appears necessary to share relevant problems for fostering the learning of proof all along the curriculum, and on the way to prepare prospective teachers to teach proof and to work with teachers to implement design relying on research results, taking into account the difference of cultural contexts and curricula in the different countries. This may constitute a program for our next up-coming meeting in CERME8.

## **Papers**

### **A: Epistemological and cognitive issue**

Meta-cognitive unity in indirect proofs, *Ferdinando Arzarello, Cristina Sabena*

Abduction in generating conjectures in dynamic geometry through “maintaining dragging”, *Anna Baccaglini-Frank*

Argumentation and proof: discussing a “successful” classroom discussion, *Paolo Boero*

### **B: Means for analysing proving activities**

Mathematical Proving on Secondary School level I: Supporting Student Understanding through different types of Proof. A Video Analysis, *Esther Brunner, Kurt Reusser, Christine Pauli*

Everyday argumentation and knowledge construction in mathematical tasks, *Julia Cramer*

Analyzing the proving activity of a group of three students, *Patricia Perry, Óscar Molina, Leonor Camargo and Carmen Samper*

On the role of looking back at proving processes in school mathematics: focusing on argumentation, *Yosuke Tsujiyama*

### **C: Transparency**

Making the discovery function of proof visible for upper secondary school students, *Kirsti Hemmi, Clas Löfwall*

Conjecturing and proving in AlNuset, *Bettina Pedemonte*

A schema to analyse students' Proof evaluations, *Kirsten Pfeiffer*

### **D: Educational implication**

The appearance of algorithms in curricula a new opportunity to deal with proof?, *Modeste Simon, Ouvrier-Buffet Cécile*

Proof: a game for pedants?, *Joanna Mamona-Downs, Martin Downs*

Two Beautiful Proofs of Pick's theorem, *Manya Raman, Lars-Daniel Öhman*

Multimodal derivation and proof in algebra, *Reinert A. Rinvold, Andreas Lorange*

## Working Group 1

The view of mathematics and argumentation, *Antti Viholainen*

### **E: On the relevance of proof on teaching mathematics**

Upper secondary school teachers' views of proof and the relevance of proof in teaching mathematics, *Emelie Reuterswärd, Kirsti Hemmi*

Development of beginning skills in proving and proof – writing by elementary school students, *Stéphane Cyr*

Designing interconnecting problems that support development of Concepts and reasoning, *Margo Kondratieva*

## **NOTES**

Participants of the working group « Argumentation and proof »:

Ferdinando ARZARELLO – Marina ASCARI – BACCAGLINI-FRANK Anna – Paolo BOERO – Esther BRUNNER – Julia CRAMER – Stéphane CYR – Martin DOWNS – Viviane DURAND-GUERRIER – Kirsti HEMMI – Veronica HOYOS – Hans Niels JAHNKE – Margo KONDRATIEVA – Clas LÖFWALL – Andreas LORANGE – Joanna MAMONA DOWNS – Maria Alessandra MARIOTTI – Cécile OUVRIER-BUFFET – Antoni PARDALA – Bettina PEDEMONTE – Kirsten PFEIFFER – Manya RAMAN – Emelie REUTERSWÄRD – Reinert A. RINVOLD – Cristina SABENA – Carmen SAMPER – Modeste SIMON – Ian STEVENSON – Yosuke TSUJIYAMA – Anti VIHOLAINEN – ZEROMSKA Anna.



# META-COGNITIVE UNITY IN INDIRECT PROOFS

Ferdinando Arzarello, Cristina Sabena

Dipartimento di Matematica, Università di Torino, Italia

*In this paper we focus on the cognitive aspects of indirect argumentation and proving processes. Drawing on the Habermas model of rational behaviour in three components (epistemic, teleological, and communicative) and on the notion of cognitive unity as developed by Boero and his collaborators, we distinguish between two levels of argumentation, a ‘ground level’ and a ‘meta-level’. On the base of a case study in early Calculus context (secondary school), we introduce the notion of ‘meta-cognitive unity’, which may give reason of success and difficulties in indirect proving processes. Furthermore, we use Peirce’s account of abduction to shed light into some cognitive processes behind the production of indirect proofs.*

Key-words: indirect proof, meta-cognitive unity, argumentation processes, diagrammatic reasoning, abduction.

## INTRODUCTION AND THEORETICAL FRAMEWORK

Proving is one of the main activities in mathematics, and the investigation on proving processes has been one of the major subjects of research in mathematics education for thirty years at least (Balacheff, 1987; Hanna, 1989; Duval, 1991; Mariotti & Antonini, 2008; Boero, Douek, Morselli, & Pedemonte, 2010).

Many of these studies have focused on the nature of the relationships between conjecturing and proving, and have put forward different perspectives. Some of them point out the differences between argumentation and proving processes (Balacheff, 1987; Duval, 1991). According to others, “in order to bring about a smooth approach to theorems in school, it is necessary to consider the connections between conjecturing and proving, in spite of the undeniable differences between the two processes” (Garuti, Boero, Lemut, & Mariotti, 1996, p. 113). The possibility of a cognitive continuity between the phases of conjecture production and proof construction has been analysed with the notion of *cognitive unity* (*ibid.*). Within this perspective, Pedemonte (2007) has further distinguished between

- “the *referential system*, made up of the representation system (the language, heuristics, and drawings) and the knowledge system (conceptions and theorems) of argumentation and proof (Pedemonte, 2005). The analysis of *cognitive unity* takes into account the referential system.
- the *structure* intended to allow logical cognitive connection between statements (deduction, abduction, and induction structures) (Pedemonte, 2007).

There is continuity in the referential system between argumentation and proof if some expressions, drawings, or theorems used in the proof have been used in the argumentation supporting the conjecture. There is structural continuity between argumentation and proof

## Working Group 1

when inferences in argumentation and proof are connected through the same structure (abduction, induction, or deduction).”

(Boero, Douek, Morselli, & Pedemonte, 2010, p. 183, emphasis as in the original).

Recently Boero and his collaborators (Boero, Douek, Morselli, & Pedemonte, 2010) have integrated the cognitive unity analysis with Habermas’ elaboration of *rational behaviour in discursive practices*. Adapting the three components of rational behaviour according to Habermas (teleologic, epistemic, and communicative) to the discursive practice of proving, they have identified:

“A) an **epistemic aspect**, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of “theorem” by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);

B) a **teleological aspect**, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product;

C) a **communicative aspect**, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.”

(*ibid.*, p. 188)

In this model, the expert’s behaviour in proving processes can be described in terms of (more or less) conscious constraints upon the three components of rationality: “constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules” (*ibid.*, p. 192). As the authors point out, such constraints work at *two levels* of argumentation:

- a level (that we call *ground-level*) inherent in the specific nature of the three components of rational behaviour in proving;
- a *meta-level*, “inherent in the awareness of the constraints on the three components” (*ibid.*, p. 192).

In our research, we focus on the cognitive aspects involved in argumentation and proving processes, and use a Peircean *semiotic lens* to analyse them (see for instance Arzarello & Sabena, in print). Considering the description given by Peirce (Peirce, 1931-1958) of *diagrammatic reasoning* as a three-step process—constructing a representation, experimenting with it, observing the results—we can observe that the three components of Habermas and of Peirce have deep analogies (though not a one-to-one correspondence): the construction of a representation may be guided mainly by a teleological rationality, whereas the experimentation and observation with it assume an epistemic value.

In this paper we present the first results of an ongoing study on particular cases of argumentation and proving processes, i.e. those related to *indirect proofs*, namely proofs by contraposition or proof by contradiction.

Analysing indirect proofs and argumentations both from a mathematical and a cognitive point of view, Antonini and Mariotti (2008) provide a model to interpret students' difficulties. Essentially, the model splits any indirect proof of a sentence  $S$  (principal statement) in a pair  $(s, m)$ , where  $s$  is a direct proof (within a theory  $T$ , for example Euclidean Geometry) of a secondary statement  $S^*$  and  $m$  is a meta-proof (within a meta-theory  $MT$ , generally coinciding with classical logic) of the statement  $S^* \rightarrow S$ . An example given by the authors is the following. Consider the (principal) statement  $S$ : 'Let  $a$  and  $b$  be two real numbers. If  $ab = 0$  then  $a = 0$  or  $b = 0$ '; and the following indirect proof: 'Assume that  $ab = 0$ ,  $a \neq 0$ , and  $b \neq 0$ . Since  $a \neq 0$  and  $b \neq 0$  one can divide both sides of the equality  $ab = 0$  by  $a$  and by  $b$ , obtaining  $1 = 0$ '. In this proof, the secondary statement  $S^*$  is: 'let  $a$  and  $b$  be two real numbers; if  $ab = 0$ ,  $a \neq 0$ , and  $b \neq 0$  then  $1 = 0$ '. A direct proof is given. The hypothesis of this new statement is the negation of the original statement and the thesis is a false proposition (" $1 = 0$ ").

Through this model, Antonini and Mariotti point out that the main difficulties for students who face indirect proofs consist in switching from  $s$  to  $m$ . On the contrary the difficulties seem less strong for statements that require a proof by contraposition, that is to prove  $\neg B \rightarrow \neg A$  (secondary statement) in order to prove  $A \rightarrow B$  (principal statement).

Let us observe that the meta-proof  $m$  does not coincide with the meta-level we considered above; rather, it is at the meta-mathematical level (based on logic). Integrating the two models, we can say that switching from  $s$  to  $m$  requires a well-established epistemic and teleological rationality in the students. To better disentangle this issue, we make the hypothesis regarding the importance of a *meta-cognitive unity* in argumentation and proving processes.

## META-COGNITIVE UNITY

The distinction between the ground level and the meta-level drawn from the Boero *et al.* (2010) model may be very useful to investigate the proving processes related to indirect proof. Basing on such a distinction, we introduce the notion of *meta-cognitive unity*, as *a cognitive unity between the two levels of argumentation described above, specifically between the teleological component at the meta-level and the epistemic component at the ground-level*.

Differently from the structural and referential cognitive unity, which focuses on two diachronic moments in the discursive activities of students (namely the argumentation and the proving phases), the meta-cognitive unity refers to a *synchronic integration between ground- and meta- levels of argumentations*.

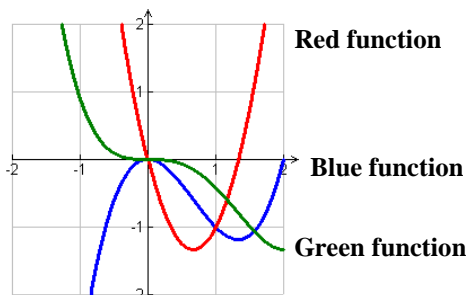
Our hypothesis is that the existence of such a meta-cognitive unity is an important condition for producing indirect proofs. In other words, a missing integration between the two levels of argumentations can block the students' proving processes, or produce cognitive breaks as those described in the literature on cognitive unity mentioned above. Furthermore, our ongoing data analysis suggests that meta-cognitive unity may entail also some kind of cognitive unity at the ground level (e.g. structural or referential).

We shall illustrate our claims through an emblematic example, in which meta-cognitive unity is accompanied by structural-cognitive unity, and develops through what we call the *logic of not* (see also Arzarello & Sabena, in print).

## AN EXAMPLE

To illustrate the meta-cognitive unity and the logic of not, we discuss the case of Simone, related to the following Calculus problem in the graphical register.

*The drawing [reported in Fig. 1] shows the graphs of: a function  $f$ , its derivative, one of its antiderivatives. Identify the graph of each function, and justify your answer.*



**Fig. 1** The graphs of the problem

4) TAVOLANDO UNA FUNZIONE "ROSSA"  
HO SCELTO TRA LE ALTRE DUE  
UNA SUA POSSIBILE PRIMITIVA;  
HO NOTATO CHE NEGLI PUNTI DI  
"ROSSA" TOCCA IL PIANO DELLE ASCISSE, QUINDI HA ORDINATA  
0, E PERCIÒ UNA SUA PRIMITIVA DOVREBBE AVERE IN  $x=0$   
PENDENZA NULLA, MA SIA LA FUNZIONE "VERDE" SIA QUELLA "BLU"  
HANNO PENDENZA  $\neq 0$ , QUINDI HO SCELTO CHE LA FUNZIONE  
ROSSA HA UN PUNTO DI MINIMO, QUINDI HO SCELTO FRA LE  
ALTRE DUE FUNZIONI QUELLA CHE PRESENTAVA UN PUNTO DI TANGENTE  
E C'È SOLO LA FUNZIONE "BLU" CHE L'HA, PER  
ACCENTUAMI HO VISTO CHE QUANDO LA FUNZIONE  
ROSSA TORNA A TOCCARE IL PIANO DELLE ASCISSE SOLO  
LA FUNZIONE "BLU" HA  $p=0$  PER CUI LA FUNZIONE "ROSSA" È UNA  
DERIVATA DELLA FUNZIONE "BLU". ABDOPO HO CONFRONTATO LA  
FUNZIONE "ROSSA" CON QUELLA "VERDE": MA, LA FUNZIONE "VERDE"  
NON PUÒ ESSERE UNA DERIVATA DI QUELLA "ROSSA", PERCHÉ  
NELLA PRIMA PARTE, DOVE LA FUNZIONE "ROSSA" DECRESCe, UNA  
SUA DERIVATA DOVREBBE AVERE SEGNO NEGATIVO, MA LA FUNZIONE  
VERDE HA SEGNO POSITIVO, PERCIÒ LA FUNZIONE "ROSSA" È  
SICURAMENTE  $f'(x)$  E DI CONSEGUENZA LA SUA PRIMITIVA (E  
QUELLA "BLU") È  $f(x)$  E LA FUNZIONE "VERDE" È LA PRIMITIVA  
DI  $f(x)$ , QUINDI  $F(x)$ .

**Fig. 2** Simone's protocol

The problem was given to grade 9 students in a scientifically oriented school (5 hours of mathematics per week). It worked as an assessment task at the end of a teaching sequence on the relationships between a function graph and the graphs of its derivative and of one of its primitives (introduced as 'antiderivatives', i.e. the function graphs whose derivative is the given graph). Though it may appear of simple combinatorial nature, the task indeed shows to be difficult for the students, who by "didactic contract" are asked to provide articulated arguments for their answers. A main source of difficulty consists in involving both direct and inverse problems mixed together. The antiderivative function has in fact been introduced as the inverse of the derivative function.

## Working Group 1

Simone solves the problem correctly, as we can see from his written production (shown in Fig. 2, and translated below). In doing so, he is able to develop a strategy that may be compared to the one of a chemist, who in the laboratory has to detect the nature of some substance. He knows that the substance must be of three different categories (a, b, c) and uses suitable reagents to accomplish his task. For example, he knows that if a substance reacts in a certain way to a certain reagent, it may be of type a or b but *not* c, and so on. This kind of strategy may be referred to the notion of *abduction* (for a discussion of abduction in mathematics education, see for instance Arzarello et al., 1998; and Hoffmann, 2005). According to Peirce (1931-1958, 2.623) abduction is a form of reasoning in which a *Case* is drawn from a *Rule* and a *Result*. It is well known his example about beans:

<i>Rule:</i>	All the beans from this bag are white
<i>Result:</i>	These beans are white
<i>Case:</i>	These beans are from this bag

As such, abduction is different from deduction, which would have the form: the Result is drawn from the Rule and the Case; and it is obviously different from an induction, which has the form: a Rule is drawn from a Case and many Results. Of course the conclusion of an abduction holds only with a certain probability; in fact Polya (1954) calls *heuristic syllogism* this form of reasoning. In our example of the chemist: if, as a *Rule*, the substance S makes blue the reagent r and if the *Result* of the experiment shows that the unknown substance X makes blue the reagent r, the *Case* of the abduction is that  $X=S$ .

Let us see how Simone develops abductive arguments in order to solve the task. We report the translation in English of the protocol shown in Fig. 2, parcelled and numbered for the sake of analysis:

- 0: Starting from the “red” function
- 1: I looked for a possible primitive among the other two:
- 2: I noticed that in the point  $x = 0$  the “red” function touches the plane of abscissas, so it has ordinate = 0;
- 3: and therefore any of its primitives should have in  $x = 0$  null slope,
- 4: but both the “green” function and the “blue” function have slope = 0;
- 5: so I saw that the red function has a point of minimum,
- 6: and I looked among the other two functions for the one with a point of inflection
- 7: and only the “blue” function has it;
- 8: to check [this] I saw that
- 9: when the “red” function comes to touch the plane of abscissas again,
- 10: only the “blue” function has  $s[\text{slope}]=0$ ,

## Working Group 1

11: therefore the “red” function is a derivative of the “blue” function.

Simone first describes two phases of his inquiry (Part A: lines 1-4; Part B: lines 5-8). In both parts he develops what we could call an abductive attitude, namely how he has been looking for Results that allow him to state a Case because of a Rule. More precisely, Simone starts with  $f = \text{red function}$  (line 0), probably because it is the simplest graph, and wonders whether he can apply an abductive argument to the blue or to the green function. In both Parts the Rule is: “any primitive of  $f$  has property  $Q$ ”; the Result is: “a specific function  $h$  has property  $Q$ ”; the Case is “ $h$  is a primitive of  $f$ ”.

In Part A, the Rule is in line 3, the Result is in lines 2 and 4, while the Case is contained implicitly in line 5, which states that the first inquiry has not been successful and starts a new inquiry. In Part B we have a new abductive process with a new Rule (implicitly contained in lines 5 and 6), a new Result (line 7), a judgment about the validity of the abduction (line 7) and a Case, which is not made explicit, but is implicitly stated in line 7.

Afterwards Simone checks his hypothesis (Part C: lines 8-11): he is successful with a fresh abductive argument. Recalling the metaphor with the chemist experiment, Simone has been able first to find a reagent that discriminates between the substances he is analysing, and then to confirm his hypothesis with a further discriminating experiment; that is, within the abductive frame, he has been able first to produce an hypothesis through an abduction and then to corroborate the hypothesis through a further abduction. The experiments of the chemists are here the practices with the graphs of functions. Such practices with graphs are examples of diagrammatic reasoning, according to the definition of Peirce: “by experimenting upon the diagram and observing the results thereof, it is possible to discover unnoticed and hidden relations among the parts” (Peirce, 1931-58, 3.363: quoted in Hoffmann, 2005, p. 48). Hence in Parts A-B-C Simone has produced and checked the Case of line 11.

Using Habermas model, some of Simone’s sentences can be considered *teleological* and at the *meta-level*, since they address the successive actions of Simone and his control of what is happening. On the other hand, other sentences show an *epistemic* character at *ground-level*, since they regard the specific mathematical notions and representations involved in the task. Coding the sentences of the protocol as  $a \Rightarrow b$  to indicate that the sentence  $a$  is at the meta-level and controls the sentence  $b$  at the ground-level, it appears that the teleological component at the meta-level *intertwines* with the epistemic component at the ground level:

0, 1, 3  $\Rightarrow$  2, 4, 5

6, 8  $\Rightarrow$  7, 9, 10, 11

We interpret such intertwining as an index of *meta-cognitive unity* in Simone's argumentation process. Besides, the presence of a meta-cognitive unity is signalled by the fact that the epistemic sentences can be understood only at the light of the teleological ones guiding them, and vice-versa. Even more than this: the sentences at the ground level have an epistemic component, e.g. they are logically linked each other, *because* of the influence of those at the meta-level. Let now see how this complex form of cognitive unity allows Simone to manage the problem using an argument by contraposition. It is the form of reasoning that logicians call "modus tollens" (from "A implies B" to "not B implies not A"). We have called such a process "the logic of not" (see also Arzarello & Sabena, in print). Let us explain it through what is written in Part D (lines 12-15):

12: Then I compared the "red" with the "green" function:

13: but, the "green" function cannot be a derivative of the "red" one,

14. a: because in the first part,

b: when the "red" function is decreasing,

c: its derivative should have a negative sign,

15: but the "green" function has a positive sign.

Here the structure of the sentence is more complex than before: Simone is thinking to a possible abductive argument, like the ones used before:

- |  |   |          |
|--|---|----------|
| (a) Rule: "any derivative of a decreasing function is negative" (lines 14) | } | (ARG. 1) |
| (b) Result: "the function h is negative";                                  |   |          |
| (c) Case: "the function h is the derivative of f"                          |   |          |

But the argument is a refutation of this virtual abduction (line 13); namely it has the form of the following syllogism:

- |  |   |          |
|--|---|----------|
| (a) Major premise:   | } | (ARG. 2) |
| "any derivative of a decreasing function is negative" (lines 14)                                     |   |          |
| (b') Minor premise: "the "green" function has a positive sign" (line 15)                             |   |          |
| (c') Consequence: "the "green" function cannot be a derivative of an increasing function" (line 13). |   |          |

In terms of the structure of the virtual abduction ARG 1, it has the form: *(a) and not (b); hence not (c)*. It is crucial here to observe that also the refutation of the usual Deduction (Rule, Case; hence Result) has the same structure, because of the contrapositive of an implication ("A implies B" is equivalent to "not B implies not A"); namely: *(a) and not (b); hence not (c)* is the same as *(a') and (b'); hence (c')*. In other words, the refutation of a virtual argument drawn through an abduction coincides with the refutation of a virtual argument drawn through a deduction. Simone produces in a very natural way this form of deductive argument within an

abductive modality. This is remarkable from an epistemological point of view: whereas the abductive approach appears very natural for students in conjecturing phases (Arzarello et al., 1998), there is often a cognitive break with the deductive approach of the proving phase (Pedemonte, 2007 analyses it in terms of “structural discontinuity”). In fact, the transition from an abductive to a deductive modality requires a sort of “somersault”, namely an inversion in the way things are seen and structured in the argument (the Case- and the Result-functions in the argument are exchanged). Such an inversion is not present in case of refutation of an abduction, insofar it coincides with the refutation of a deduction (expressed in a syllogistic form). Of course a greater cognitive load is required to manage the refutation of an abduction compared with that required to develop a simple direct abduction. But the coincidence between abduction and deduction in case they are refuted allows avoiding the somersault. In our case study, Simone has been able to lighten the cognitive load of the task through a transition to a new epistemological status of his statements.

Moreover, Simone is doing mental experiments with the graphs and observing their results. The ground level at which Simone epistemic arguments are drawn is very concrete; possibly this makes it possible for him to develop at the meta-level the teleological arguments we have underlined above, which support him in producing a correct proof by contraposition. Lines 16-18 below show the conclusion of Simone’s argument:

16: Therefore the “red” function is surely  $f'(x)$

17: and consequently its primitive (the blue one) is  $f(x)$

18: and the “green” function is the primitive of  $f(x)$ , thus  $F(x)$ .

We shall now consider an example from the literature, in which the indirect argumentation process does not lead to a correct proof by contraposition. The example shows a typical difficulty, in which the students do not realise any meta-cognitive unity and shift from the problems they should solve to another problem, which allows them to skip the difficulties of the indirect proof. It is taken from Antonini & Mariotti (2008). A well known problem is considered in pencil and paper environment: “*Can two bisectors in a triangle form an angle of  $90^\circ$ ?*”. The students “formulate the conjecture that the angle...cannot be a right angle...*The argumentation produced can be summarized as follows: if the angle is right then the sum of two angles of the triangles is  $180^\circ$ , then the triangle becomes a quadrilateral.* After this argumentation, no proofs are generated by the students” (*ibid.*, p. 410). As stated in the paper, the shift to the quadrilateral can be considered an antidote to an “absurd world”: “the theory is changed according to the validity of the theorems he [the student] knows” (*ibid.*).

Here are the excerpts of the interview taken from that paper:



## Working Group 1

61 P: As far as  $90^\circ$ , it would be necessary that both K and H are  $90^\circ$ , then  $K/2 = 45$ ,  $H/2 = 45 \dots 180^\circ - 90^\circ$  and  $90^\circ$ .

62 I: In fact, it is sufficient that the sum is  $90^\circ$ , that  $K/2 + H/2$  is  $90^\circ$ .

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that  $K + H$  is ... a square [...]

65 R: It surely should be a square, or a parallelogram

66 P:  $(K - H)/2$  would mean that [...]  $K + H$  is  $180^\circ \dots$

67 R: It would be impossible. Exactly, I would have with these two angles already  $180^\circ$ , that surely it is not a triangle. [...]

71 R: We can exclude that [the angle] is  $\square/2$  [right] because it would become a quadrilateral.

The excerpts show that no teleological aspects are present in the students at the meta-level: on the contrary they are completely embedded in the situation at the ground (epistemic) level and their geometrical knowledge pushes them to shift from a triangle to a quadrilateral (# 64, 67, 71).

## DISCUSSION

We presented an illustrative example of how a *meta-cognitive unity* (i.e. a unity between a teleological control at meta-level and an epistemic knowledge at ground level) may contribute to the production of indirect arguments and proofs.

In the example, the graphical component is the core of the tasks, and it allows the student a visual approach to the problem, what in Peirce's words can be called 'diagrammatic reasoning'. This "concrete" component possibly lightens the cognitive load of the task, and facilitates the integration between the two levels to produce a meta-cognitive unity. Referring to the protocol, we see that the student is able to reduce the different cases to simple pass/not-pass 'experiments', like a chemist who checks the nature of an unknown substance, and we interpret his ability as an outcome of his meta-cognitive unity, which integrates the teleological meta-level control and the ground-level knowledge. As a consequence, the situation becomes a 'heuristic device' similar to cognitive mechanisms used naturally in everyday life, as pointed out by Freudenthal (1973, p. 629). The meta-cognitive unity construct brings about a unitary analysis of many students' difficulties as pointed out in the literature. We have also given a short example in which the absence of a teleological rationality at meta-level prevents the students from correctly carry out an indirect proof.

As far as concerns didactic consequences, our approach suggests a possible way for teaching indirect proofs, namely in making explicit the teleological dimension that can be developed in any reasoning made by the students. In particular it should be important to cultivate the idea of the rationality of impossible worlds related to the indirect arguments discussing with the students the intertwining between the

teleological control and the epistemic knowledge starting from concrete examples produced by the same students. In a sense this was already suggested by Freudenthal, who wrote: “If the teacher would tell the student what is an indirect proof, he is advised not to contrive examples but to catch a student performing an indirect proof and let him understand consciously what he did unconsciously” (*ibid.*).

In fact in the teaching of both indirect and direct proofs, generally there is more attention to the communicative and possibly to the epistemic component, while the teleological one is not made explicit. This can start a sort of comedy of errors with students, who think that the communicative component is the more relevant and produce what Harel (2007) calls ‘ritual schemes’, which are not useful for understanding and possibly produce proofs, especially indirect ones.

## REFERENCES

- Antonini, S. & Mariotti, M.A. (2008). Indirect proof: What is specific to this way of proving? *ZDM Mathematics Education*, 40, 401-412.
- Arzarello, F., & Sabena, C. (in print). Semiotic and theoretic control in argumentation and proof activities. *ESM Special Issue on Semiotics*.
- Arzarello, F., Micheletti, C., Olivero, F., Paola, D., & Robutti, O. (1998). A model for analysing the transition to formal proofs in geometry. *Proceedings of PME 22* (vol. 2, pp. 24-31), Stellenbosh, South Africa.
- Balacheff, N. (1987). Processus de preuves et situations de validation. *Educational Studies in Mathematics*, 18(2), 147-176.
- Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: a contribution to theoretical perspectives and their classroom implementation. *Proceedings of PME 34* (vol. 1, pp. 179-204), Belo Horizonte, Brazil.
- Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics*, 22(3), 233-261.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel.
- Garuti, R., Boero, P., Lemut, E., & Mariotti, M. A. (1996). Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems *Proceedings of PME 20* (vol. 2, pp. 113-120). Valencia, Spain.
- Hanna, G. (1989). More than formal proof. *For the Learning of Mathematics*, 9(1), 20-23.
- Harel, G. (2007). Students’ proof schemes revisited. In P. Boero (Ed.), *Theorems in school: from history, epistemology and cognition to classroom practice* (pp. 65–78). Rotterdam: Sense Publishers.

### Working Group 1

- Hoffmann, M. H. G. (2005). Signs as means of discovery. In: M. H. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and Sign - Grounding Mathematics Education* (pp. 45-56). Berlin/New York: Springer.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23-41.
- Peirce, C. S. (1931-1958). *Collected Papers* (Vol. I-VIII). In C. Hartshorne, P. Weiss, & A. Burks (Eds.). Cambridge, MA: Harvard University Press.
- Polya, G. (1954). *Mathematics And Plausible Reasoning*. Princeton, NJ: Princeton University Press.

# ABDUCTION IN GENERATING CONJECTURES IN DYNAMIC GEOMETRY THROUGH *MAINTAINING DRAGGING*

Anna Baccaglini-Frank

University of Siena (Italy)

*This paper introduces two types of abduction associated to two different ways of generating conjectures that arise from using a particular dragging modality in dynamic geometry. We refer to such dragging modality as maintaining dragging (MD). A first use of MD relies on physical dragging-support and it seems to lead the solver “automatically” to the formulation of a conjecture. In this case the abductive reasoning seems to occur at a meta-level and to be concealed within the MD-instrument. On the other hand, some of our data shed light onto a different way of generating conjectures which is rooted in use of MD but is “freed” from the physical dragging-support. In this case abductive reasoning seems to occur at the level of the dynamic exploration.*

**Key words:** abduction, conjecture-generation, dragging, dynamic geometry, instrument, instrumented abduction, maintaining dragging (MD), psychological tool

## INTRODUCTION

This paper reports on some findings of a study on conjecture-generation in a dynamic geometry system (DGS). The study<sup>1</sup> blossomed from results of Italian research that provided a classification of various dragging modalities used by solvers during explorations of open problem activities (Arzarello, Olivero, Paola, Robutti, 2002). This classification describes different ways of dragging points on the screen as a conjecture is elaborated and tested, and the solver’s control shifts from “ascending” to “descending” (Arzarello et al., 1998). Such transition was described as occurring in correspondence to use of *dummy locus dragging*, that is moving a base-point so that the drawing *keeps* a discovered property, and as being promoted by abduction. Our study aimed at unravelling the relationship between abduction and the use of particular dragging modalities.

Since according to the literature (Olivero, 2002), spontaneous use of dummy locus dragging does not seem to occur frequently, first, we explicitly introduced the participants of the study to four dragging modalities, elaborated from Arzarello et al.’s classification, during two in-class lessons. The modality elaborated from *dummy locus dragging* that we introduced is what we refer to as *maintaining dragging* (MD), and it consists in dragging a base-point of the dynamic figure on the screen trying to maintain some geometrical property of the figure. In other words, performing MD consists in identifying a property that the figure can have and in trying to induce such property as a *soft*<sup>3</sup> *invariant* during dragging.

In order to unravel the relationship between abduction and the use of MD during a process of conjecture-generation, we constructed a model (Baccaglini-Frank, 2010; Baccaglini-Frank & Mariotti, 2010) that provides a cognitive description of the process, as a sequence of (implicit) tasks that the solver seems to address. During the study we tested, refined, and tested again the model, which was used as a tool of analysis in the final round of data analysis. We proceeded through two rounds of 90-minute clinical interviews<sup>2</sup> with pairs of students, each round with different students who had participated to the introductory lessons on dragging. Data was collected in the form of screenshots with audio, video recording, students' Cabri files and work on paper, transcriptions and subtitled videos. The students (a total of 31), between the ages of 15 and 18, were from three Italian high schools and had been using Cabri in the classroom for at least one year prior to their interviews.

### A few theoretical background notions

The study made use of the notions of *abduction* and *instrument* as follows.

*Abduction.* Peirce was the first to introduce the notion of abduction as follows:

...abduction looks at facts and looks for a theory to explain them, but it can only say a “might be”, because it has a probabilistic nature. The general form of an abduction is: a fact A is observed; if C was true, then A would certainly be true; so, it is reasonable to assume C is true (Peirce, 1960, p. 372).

Recently, there has been renewed interest in the concept of abduction, with a number of studies focused on its various uses in mathematics education (see for example Baccaglini-Frank, 2010, chapter 2). For this paper we will refer to the definition introduced above and to Magnani's description of abduction as an *explanatory hypothesis* (2001, pp. 17-18).

*Instrument.* The study considers “dragging” in a DGS after the instrumentation approach (Vérillon & Rabardel, 1995; Rabardel & Samurçay, 2001; Rabardel, 2002), as has been done fruitfully by other researchers (for example, Lopez-Real & Leung, 2006; Leung, 2008; Strässer, 2009). A particular way of dragging, in our case MD, may be considered an *artifact* that can be used to solve a particular *task* (in our case that of formulating a conjecture). When the user has developed particular *utilization schemes* for the artifact, we say that it has become an *instrument* for the user. We will call the utilization schemes developed by the user in relation to particular ways of dragging, “dragging schemes”. In this sense the model we developed can be interpreted as the description of a utilization scheme for MD, with respect to the task of generating a conjecture. From now on we will refer to our model as the *MD-conjecturing model*.

### INSTRUMENTED ABDUCTION

The successive analyses of our data led to the development of a new notion, that of *instrumented abduction*, through which we describe the place and role of abduction

in the process of conjecture-generation we have studied (Baccaglini-Frank & Mariotti, 2010; Baccaglini-Frank, 2010). We introduce this notion by providing excerpts of two solvers' exploration that constitute a paradigmatic example of how MD can be used in the process of conjecture-generation and how the MD-conjecturing model can be used as a tool of analysis.

James and Simon were given the following open-problem activity:

Construct three points  $A$ ,  $B$ , and  $C$  on the screen, the line through  $A$  and  $B$ , and the line through  $A$  and  $C$ . Then construct the parallel line  $l$  to  $AB$  through  $C$ , and the perpendicular line to  $l$  through  $B$ . Call the point of intersection of these last two lines  $D$ . Consider the quadrilateral  $ABCD$ . Make conjectures on the kinds of quadrilaterals can it become, trying to describe all the ways it can become a particular kind of quadrilateral.

The solvers followed the steps that led to the construction of  $ABCD$  (Fig. 1) and soon noticed that it could become a rectangle. Simon was holding the mouse (this is shown in the excerpts below by his name being in bold letters), and followed James' suggestion to use MD to "see what happens" when trying to maintain the property  $ABCD$  *rectangle* while dragging the base-point  $A$ . The solvers have accomplished Task 1 of our model (Baccaglini-Frank & Mariotti, 2010): determining a configuration to be explored by inducing it as a (soft) invariant. In such situation we refer to the selected property

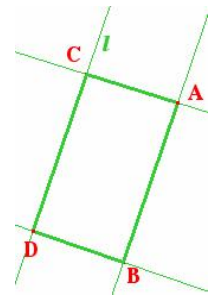


Figure 1

$ABCD$  *rectangle*, as *intentionally induced invariant*. As Simon was focused on performing MD, James' attention seemed to shift to the movement of the dragged-base-point, and he proposed to "do trace" in order to "see if ...[A moves along a "pretty precise curve"]". James seemed to be looking for something that  $A$  can be dragged along in order for  $ABCD$  to remain a rectangle, thus addressing Task 2 of our model (searching for a condition, through MD, that makes the intentionally induced invariant be visually verified, and recognizing a condition in the movement of dragged-base-point along a *path*). This intention seems to indicate that James has conceived an object along which dragging the base point  $A$  will guarantee that the intentionally induced invariant is visually verified. This is what we call a *path*. In order to "understand" what such path might be he suggests activating the trace on  $A$  as Simon performs MD (Fig. 2).

Excerpt 1

- 1 I: and you, James what are you looking at?
- 2 James: That it seems to be a circle...
- 3 **Simon**: I'm not sure if it is a circle...
- 4 James: It's an arc of a circle, I think the curvature suggests that. (...)...
- 10 James: Ok, do half and then more or less you understand it, where it goes through.
- 11 **Simon**: But  $C$  is staying there, so it could be that  $BC$  is...is

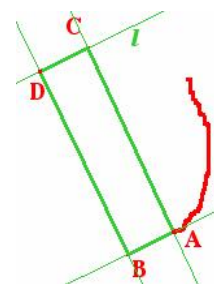


Figure 2

12 James: right! because considering  $BC$  a diameter of a circle...

[They construct the circle and drag  $A$  along it, and then they write the conjecture: " $ABCD$  is a rectangle when  $A$  is on the circle with diameter  $BC$ ."]

In this Excerpt there seems to be the intention of looking for something that leads the solvers to a *geometric description of the path* as "a circle" ([2], [4]) "considering  $BC$  a diameter" ([12]). Recognizing and describing the path can lead to a second invariant, that we call the *invariant observed during dragging*, as a regular movement of the dragged-base-point along such path (in this case "A on the circle with diameter  $BC$ "). Both invariants are perceived within the phenomenological domain of the DGS, where a relationship of "causality" may also be perceived between them. Of course such relationship can be formulated within the domain of Euclidean Geometry as a *conditional link* between geometrical properties corresponding to the invariants, provided that the solver gives an appropriate geometrical interpretation. This can be expressed through a conjecture and checked (Task 3: checking the conditional link between the invariants and verifying it through the dragging test).

Solvers like Simon and James who use MD effectively for generating a conjecture seem to withhold the key for making sense of their findings. This consists in conceiving, within the phenomenology of the DGS, the invariant observed during dragging as a "cause" of the intentionally induced invariant, and then, within the domain of Euclidean geometry, in interpreting such cause as a geometrical "condition" for the intentionally induced invariant, a geometrical property of the figure, to be verified. In other words, the solvers establish a causal relationship between the two invariants generating – as Magnani says (2001) – an *explanatory hypothesis* for the observed phenomenon.

From the data analyzed another characteristic of behaviors like that of Simon and James is the use of MD in an "automatic" way. That is, the solver proceeds through steps that lead *smoothly* to the discovery of invariants and consequently to the generation of a conjecture, with *no apparent abductive ruptures* in the process. So where is abduction when conjecture-generation occurs as described by the MD-conjecturing model? Abduction can be recognized in the expert's *interpretation of the invariant observed during dragging as a cause for the intentionally induced invariant to be visually verified*. Thus, automatic use of MD does not seem to produce explicit abductive arguments during the exploration leading to a conjecture; instead it seems to *condense and subsume* the abductive process. We introduce the new notion of *instrumented abduction* to refer to an abductive inference *supported* by an instrument, like in this case. Here the instrument is the combination of MD (artifact) with the MD scheme (utilization scheme) described in the MD-conjecturing model.

## USE OF MAINTAINING DRAGGING AS A PSYCHOLOGICAL TOOL

Our study was primarily aimed at developing and subsequently testing the MD-conjecturing model. Our final data analyses seemed to confirm the model, however one case opened a window onto a fundamentally different way of generating a conjecture that seems to have roots in use of MD even though no dragging is actually performed. Below we summarize the exploration in which we found such evidence and present an excerpt from it. The solvers, Francesco and Gianni, were given the following task:

Draw a point  $P$  and a line  $r$  through  $P$ . Construct the perpendicular line  $l$  to  $r$  through  $P$ , construct a point  $C$  on it, and construct the circle with center in  $C$  and radius  $CP$ . Construct the symmetric point of  $C$  with respect to  $P$  and call it  $A$ . Draw a point  $D$  on the semi-plane defined by  $r$  that contains  $A$ , and construct the line through  $D$  and  $P$ . Let  $B$  be the second intersection with the circle and the line through  $P$  and  $D$ . Consider the quadrilateral  $ABCD$ . Make conjectures on the kinds of quadrilaterals can it become, trying to describe all the ways it can become a particular kind of quadrilateral.

Francesco and Gianni had effectively used MD to generate conjectures in previous explorations. In this particular exploration they had noticed the potential property  $ABCD$  *parallelogram*. Thus Francesco had chosen a base-point to drag while trying to maintain such property. However Francesco and Gianni seemed to conceive a geometric description of the path that did not coincide with their interpretation of the trace mark left on the screen as Francesco performed MD. This led the solvers to reject the original description and search for a new condition for maintaining  $ABCD$  *parallelogram*. The solvers were not able to reach such condition using MD and they interrupted all forms of dragging. After a moment of silence Gianni started speaking about constructing a circle to drag along, as shown in the following excerpt.

### Excerpt 2

- 1 Gianni: eh, since this is a chord, it's a chord right? We have that this has to be an of another circle, in with center in  $A$ . think if you do, like, center
- 2 Francesco:  $A$ , you say...
- 3 Gianni: symmetric with one, you have to center  $A$ .
- 4 Francesco: uh huh
- 5 Gianni: Do it!
- 6 Francesco: with center  $A$  and radius  $AP$ ?

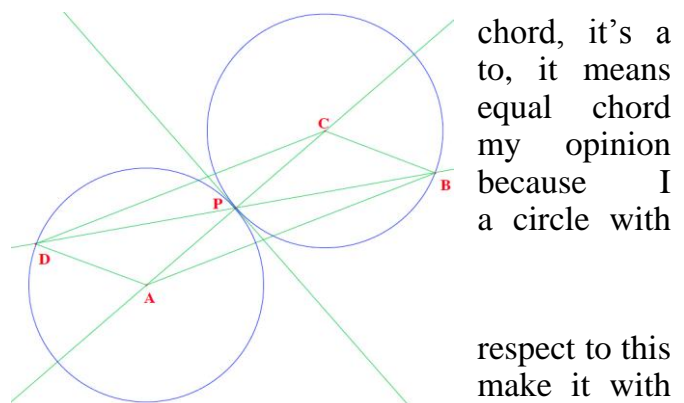


Figure 3



7 Gianni: with center  $A$  and radius  $AP$ . I, I think...

Gianni appears to be trying to solve the problem of finding a way to drag  $D$  in order to maintain the property  $ABCD$  *parallelogram* as if he had to perform MD. However the solvers' inability to perform MD successfully, led to the argumentation above in which the following abductive inference (in Pierce's terms) is evident:

- fact:  $DP=PB$  (recognized as chords [1])
- rule: given symmetric circles with  $DP$  and  $PB$  symmetric chords ([1], [3]), then  $DP=PB$  (as observed)
- abductive hypothesis: there exists a symmetric circle with center in  $A$  and radius  $AP$  ([3]-[7]).

Without further hesitation the solvers formulate their conjecture (Fig.3): " $D$  belongs to the circle centered in  $A$  with radius  $AP$  implies  $ABCD$  parallelogram." We highlight how Gianni applies a way of reasoning, that has roots in his knowledge of the MD scheme, to a substantially different situation. Gianni is trying to find a condition for  $ABCD$  to be a parallelogram, but instead of focusing on the movement of a *point* ( $D$ ) as would have occurred during use of MD, Gianni notices *chords* ( $BP$  and  $PD$ , which he interprets as a chord) and visualizes their symmetric behavior, which leads him to produce an *explicit* abductive argument. In particular now the "rule" *appears*, while in the case of instrumented abduction such rule would have remained implicit in the movement of the dragged base-point and/or the trace mark on the screen.

Taking a Vygotskian perspective (Vygotsky, 1978, p. 52 ff.), the process that was external, supported by the MD-instrument in the case of instrumented abduction, now can be seen as "transformed" into an *internal* process. We can say that the MD-instrument has been *internalized* and it can now be used as a *psychological tool* (Kozulin, 1998) to solve a conjecture-generation problem. Moreover, now we can underline how the intention of searching for a cause that solvers who have appropriated the MD scheme exhibit, resides at a different level, a meta-level (Gollwitzer & Schaal, 1998), with respect to each specific investigation the solvers engage in. Thus, instrumented abduction resides at such meta-level, while the abductive inference in the second case resides at the level of the dynamic exploration.

## A HYPOTHESIS ON PROOF

If we consider conjectures generated in the two different ways described above, the differences between them are not in the *product* of the dynamic exploration, the statement of the conjecture, but in the elements that emerge during the *process* of the exploration. When MD is used "automatically" as in the conjecture-generation process characterized by instrumented abduction, the premise and the conclusion of the statement of the final conjecture seem to be "distant". That is, these conjectures seem to exhibit a "gap" between the premise and the conclusion, because no bridging

## Working Group 1

arguments tend to emerge from the exploration. On the other hand, it seems that when MD is internalized and used as a psychological tool, the produced conjectures are accompanied by arguments that can be used to bridge the premise and the conclusion. To support this hypothesis, we now provide an excerpt (Excerpt 3) containing an oral proof produced by Francesco and Gianni after having reached their conjecture (Excerpt 2).

### Excerpt 3

- Francesco: ah, no! but wait! we know a lot of things here, excuse me, if  $DA$  is equal to  $AP$  which is equal to  $PC$  which is equal to  $CB$ ,  $DAP$  and  $PCB$  are isosceles.
- Gianni: yes... And so the angles, right!
- Francesco: Wait, and so this [pointing to the angle  $ADP$ ]...
- Gianni: the angles over there and down there are..
- Francesco: so, let's say  $ADP$  is equal to  $APD$ , which is equal to
- Gianni: we know that these, these are also opposite at the vertex and so they are *all* equal those angles there. (...)
- Francesco: but, excuse me, if this... if the angles at the base, are equal, also, obviously, the angle at the vertex, uhm, the angle  $DAP$  is equal to  $PCB$  necessarily because of the sum of angles.
- Gianni: Yes, right.
- Francesco: Because it is  $180^\circ$  minus equal angles
- Gianni: okay, so this way we understood that the two triangles are equal.
- Francesco: Exactly.
- Gianni: And so also  $PD$  and  $PB$  are equal.
- Francesco: Okay, so the diagonals divide each other in their midpoints, and therefore  $ABCD$  is a parallelogram.
- Gianni: Yes, right. [Smiling]

In the analysis of the Excerpt 2 we described how Gianni focuses on the two segments  $PB$  and  $PD$ , and interprets them as chords of symmetric circles. This constitutes the key idea (Raman, 2003) in their oral proof summarized as follows:

- the circles are symmetric so  $AD$  is congruent to  $AP$  which is congruent to  $PD$  and to therefore to  $BC$ ;
- the isosceles triangles  $APD$  and  $PBC$  are congruent because they have congruent angles, since the angle  $DPA$  is opposite at its vertex to  $CPB$ ;
- therefore  $PD$  is congruent to  $PB$ ,
- so  $ABCD$  has diagonals that intersect at their midpoints and therefore it is a parallelogram.

The geometrical properties that emerged during the production of the conjecture, now become fundamental ingredients of the solvers' proof. In other words, these geometrical properties seem to help bridge the gap between premise and conclusion of the conjecture. At this point, if we consider conjectures as both the *product* (the statement of the conjecture) and the *process* (the exploration leading to the statement of the conjecture), we can characterize conjectures as those *with a gap* that emerge through automatic use of MD as opposed to those *with bridging elements* that emerge as a product of an internalization of MD. This characterization helps express our hypothesis as follows.

*Hypothesis on proof.* Automatic use of MD seems to generate conjectures with a gap, while use of MD as a psychological tool seems to generate conjectures with bridging elements. Therefore use of MD as a psychological tool may foster the solver's construction of a proof of the statement of his/her conjecture.

## CONCLUDING REMARKS

Through our study we were able to identify two distinct forms of abductive reasoning related to two different ways of generating conjectures that arise from using a particular dragging modality in dynamic geometry. When MD is used automatically through physical dragging, the abductive reasoning seems to reside at a meta-level with respect to the dynamic exploration. This idea is condensed in the notion of instrumented abduction that we introduced. On the other hand, when MD seems to be "freed" from the physical support, and internalized, the abduction seems to occur at the level of the exploration. In this case the conjecture-generation process seems to have the advantage of involving arguments that can be reinvested in a successive proof, like in the case of Francesco and Gianni.

We hypothesize that conjectures generated "automatically" through physical use of MD, that is *conjectures with a gap*, will present *cognitive rupture* with respect to a potential proof since the solver will have no arguments emerging from the conjecturing-process to base his/her proof upon. This seems to be the case because the process of conjecture-generation is supported by the DGS and mostly concealed within it, as is the abductive inference that we refer to as *instrumented abduction*. On the other hand, we hypothesize that if solvers who have appropriated the MD-instrument *also internalize it* transforming it into a psychological tool, or a fruitful "mathematical habit of mind" (Cuoco, 2008) that may be exploited in various mathematical explorations leading to the generation of conjectures, a greater cognitive unity (Pedemonte, 2007) might be fostered. In other words, it may be the case that when the MD-instrument is used as a psychological tool the conjecturing phase is characterized by the emergence of arguments that the solver can set in chain in a deductive way when constructing a proof (Boero et al., 1996). In particular we think this may occur if, as in the case of Francesco and Gianni, abduction in which the rules are taken from the domain of the Theory of Euclidean Geometry is used

during the process of conjecture-generation. An abduction of this sort seems to expose key ideas that can be reinvested in the proof.

The relatively small amount of data analyzed in our study does not allow us to make general statements about the hypothesis on proof we illustrated above. Moreover our study was not focused on investigating internalization of the MD-instrument and its transformation into a psychological tool: the case of Gianni and Francesco was an unexpected isolated instance that suggested new potential insight into how a DGS can be used (or not) in the context of argumentation and proof, opening an alley for future research. For example, as some colleagues have suggested, it would be interesting to investigate *what it takes*, both from learning and teaching perspectives, for the solver to make the cognitive shift we describe, transforming the MD-instrument into a psychological tool.

## NOTES

1. Research study partly funded by PRIN 2007B2M4EK (Instruments and representations in the teaching and learning of mathematics: theory and practice).
2. The activities proposed were open-ended tasks. The interviewer would typically ask the solver to explain an action, to describe what s/he was looking at or trying to accomplish, or to provide clarification or elaboration of a statement s/he made.
3. We use the terminology “soft” and “robust” as introduced by Healy (2000).

## REFERENCES

- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments, *Zentralblatt fur Didaktik der Mathematik/International Reviews on Mathematical Education*, 34(3), 66-72.
- Baccaglini-Frank, A., & Mariotti, M.A. (2009). Conjecturing and Proving in Dynamic Geometry: the Elaboration of some Research Hypotheses. In *Proceedings of CERME 6*, Lyon, France. CD-rom.
- Baccaglini-Frank, A., & Mariotti, M.A. (2010) Generating Conjectures through Dragging in a DGS: the MD-conjecturing Model. *International Journal of Computers for Mathematical Learning*, 15(3), 225-253.
- Baccaglini-Frank, A. (2010). Conjecturing in Dynamic Geometry: A Model for Conjecture-generation through Maintaining Dragging. *Doctoral dissertation*, University of New Hampshire, Durham, NH.
- Boero P., Garuti R., Lemut E. & M.A. Mariotti (1996) Challenging the traditional school approach to theorems: a hypothesis about the cognitive unity of theorems. In *Proceedings of PME20* (Vol. 2, pp. 113-120). Valencia, Spain.
- Cuoco, A. (2008). *Introducing Extensible Tools in Elementary Algebra*. In Algebra and Algebraic Thinking in School Mathematics. 2008 Yearbook of the NCTM, Reston, Va.: NCTM.

## Working Group 1

- Gollwitzer, P. M., & Schaal, B. (1998). Metacognition in Action: The Importance of Implementation Intentions. *Personality and Social Psychological Review*, 2(2), 124-136.
- Healy, L. (2000). Identifying and explaining geometric relationship: interactions with robust and soft Cabri constructions. *Proceedings of the 24th Conference of the IGPME*, (Vol. 1, pp. 103-117) Hiroshima, Japan.
- Kozulin, A. (1998). *Psychological Tools A Sociocultural Approach to Education*. Cambridge, MA: Harvard University Press.
- Leung, A. (2008). Dragging in a Dynamic Geometry Environment Through the Lens of Variation. *International Journal of Computers for Mathematical Learning*, 13 (2), 135-157.
- Lopez-Real, F. & Leung, A. (2006). Dragging as a conceptual tool in dynamic geometry environments. *International Journal of Mathematical Education in Science and Technology*, 37 (6), 665-679.
- Magnani, L. (2001). *Abduction, Reason, and Science. Processes of Discovery and Explanation*. Kluwer Academic/Plenum Publisher.
- Olivero, F. (2002). The Proving Process within a Dynamic Geometry Environment. *PhD Thesis*, University of Bristol.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23-41.
- Peirce C. S. (1960). *Collected Papers II, Elements of Logic*. Harvard: University Press.
- Rabardel, P. (2002). *People And Technology: A cognitive approach to contemporary instruments*. <http://ergoserv.psy.univ-paris8.fr/>.
- Rabardel, P., & Samurçay, R. (2001). From Artifact to Instrument-Mediated Learning. New Challenges to Research on Learning. *International Symposium organized by the Center for Activity Theory and Developmental Work Research*, University of Helsinki, March 21-23.
- Raman, M. (2003). Key Ideas: What are They and How Can They Help Us Understand How People View Proof?, *Educational Studies in Mathematics*, 52, 319-325.
- Strässer, R. (2009). Instruments for learning and teaching mathematics an attempt to theorize about the role of textbooks, computers and other artefacts to teach and learn mathematics. In Tzekaki, M., & Sakonidis, H. (Eds.), *Proceedings of the PME33* (Vol. 1, pp. 67-81). Thessaloniki, Greece: PME.
- Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, 10 (1), 77-101.
- Vygotskij, L. S. (1978). *Mind in Society. The Development of Higher Psychological Processes*, Harvard University Press.

# **ARGUMENTATION AND PROOF: DISCUSSING A "SUCCESSFUL" CLASSROOM DISCUSSION**

Paolo Boero

Dipartimento di Matematica, Università di Genova

*This paper concerns a discussion in a university course of primary school teacher education. The discussion was aimed at developing awareness about the condition of "equally likely possible cases" in the classical definition of probability. During the discussion students engaged several times in reflections about validity of statements, validity of inferences, etc. The aim of this paper is to present some tools to analyze the discussion and hypothesize possible follow-ups, related to the performed analysis and the aim of developing awareness about the "rules" of argumentation and proof.*

Keywords: argumentation; mathematical proof; rationality; meta-mathematics

## **INTRODUCTION**

The development of students' awareness about the "rules" of argumentation and proof in mathematics is one of the main challenges for mathematics education. This statement expresses a conviction largely shared among mathematics educators in the last three decades, in spite of different positions concerning *when* to develop such awareness, *which elements* should awareness consist of, and *how* to deal with them in the classroom (cf Balacheff, 1987; Duval, 1991; Harel, 2008).

Concerning the *how*, we think that the rules of argumentation and proof cannot be taught as a separate subject in the phase of approach to them (obviously it can become a subject for specialized courses in more advanced education). For us the best didactical choice is to exploit *suitable mathematical activities* of argumentation and proof, and develop awareness of the rules according to the *occasions* offered by those activities. But what mathematical activities are *suitable* to offer the expected *occasions*? And *how* to exploit those occasions?

In this paper we will deal only with the *occasions* and the *how* question. After a description of the educational context in which the discussion reported in this paper took place, we will try to frame the analysis of the discussion in order to identify the *occasions* offered in it. We will also discuss *how* to exploit those *occasions*.

## **THE DOCUMENT IN ITS EDUCATIONAL CONTEXT**

In Italy, since 1999 prospective primary school teachers must follow a four-year university preparation, including courses, laboratories and teaching stages (in the future a fifth year will be added). At present, students must follow at least four courses of 30 hours each in the mathematical area, which should integrate the revision of basic mathematical subjects together with didactic considerations, with

## Working Group 1

an eye to national indications for primary school curricula. At the Genova University our courses are strictly co-ordinated. Teaching is organized according to a cyclic structure. In most cases, the starting point of each cycle is an individual problem solving activity; it exploits knowledge supposed as shared by students (sometimes recapitulated by the teacher), and/or new knowledge introduced by the teacher, and/or a document coming from primary school classes. A collective discussion leaded by the teacher follows; it concerns students' individual productions, selected by the teacher. Discussion results in a synthesis, which can provide elements for a further cycle - or constitute an end point for the subject at stake. Some contributions of the teacher during and after the discussions concern the historical origin of the mathematical content dealt with in that moment, and reflections on the relationships mathematics - "reality" and on the ways of reasoning in mathematics. Students' interventions and questions on the above issues are encouraged. Traces of the effects of these choices are in the document.

Students attending our courses come from different kinds of high school; the majority comes from a socio-pedagogical high school, with few hours of mathematics (3 or 2 hours each week, according to the grades); some students come from scientific high school (3-5 weekly hours of mathematics) or technical high school (3-4 hours of mathematics, but no mathematics in the last year). In the year 2004/05 49 students attended the fourth course of 30 hours (I was the teacher); 12 hours were dedicated to a technical and didactical introduction to elementary probability theory. 45 students took part in the activity described below. 15 of them (P-students) had already met elementary probability theory in the high school. Like every year, the starting point of the activity on probability was the following individual problem:

"If we cast two dice and sum the digits on their superior faces, is it more convenient to bet even, odd, or is it indifferent?"

Four solutions were considered for further discussion:

- A) - it is more convenient to bet even, because the possible results are eleven (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) and 6 results out of 11 are even (14 answers similar to this, 6 of them are by students who studied elements of probability in the high school: P-students);
- B) - it is more convenient to bet even, because  $\text{even} + \text{even} = \text{even}$ ,  $\text{odd} + \text{odd} = \text{even}$ ,  $\text{even} + \text{odd} = \text{odd}$ , thus in two cases out of three the result is even (7 answers, 1 from a P-student)
- C) - it is indifferent because  $\text{even} + \text{even} = \text{even}$ ,  $\text{odd} + \text{odd} = \text{even}$ ,  $\text{even} + \text{odd} = \text{odd}$ ,  $\text{odd} + \text{even} = \text{odd}$ , therefore two results out of four are odd and two results are even (11 answers, 3 from P-students);
- D) - it is indifferent because if I take an even digit of the first dice, like 2, the sums with the digits of the other dice (2+1, 2+2, 2+3, 2+4, 2+5, 2+6) are even in three cases and odd

## Working Group 1

in three cases, and the same happens if I take an odd digit of the first dice (8 answers, 3 from P- students).

5 other students had produced other kinds of texts: "(...) I do not remember the solution"- a P-student; "It should be indifferent but I cannot explain why"; "Half of the numbers of each dice are even" (two answers, 1 from a P-student).

The knowledge at stake concerns the classical definition of probability (as the ratio between the number of favorable cases and the number of all possible cases, provided that they are equally likely), particularly the crucial condition of "equally likely cases". Usual approaches to probability in high school (when required in the programs) do not focus enough on this condition; exercises and examples on most texts concern only equally likely cases (without putting this aspect into evidence).

Individual solutions were written on a sheet of paper and distributed to all students; a discussion followed. *The translation has "smoothed" some students' expressions, due to difficulties of finding equivalent English expressions. Only main steps are reported below. The whole texts in Italian and English will be available for discussion.*

Some supporters of solution B quickly recognized to have considered only one of the two possibilities and abandoned it; but we will see that some doubts about this issue still emerged during the discussion. Then:

- 1-S1: I have studied probability at the high school, I have applied the definition of probability as the ratio between the number of favorable cases and the number all possible cases... The possible cases are 11, the even cases are 6, the odd cases are 5, thus the probability of an even sum is  $6/11$ , bigger than  $5/11$ .
- 2-S2: but this is not compatible with the precise reasoning of D. (...)
- 5-S5: we could make an experiment... to cast the dice a lot of times, one hundred, two hundreds, and see which solution is correct!
- 6- S2: we are making a discussion concerning mathematics, we must reach a conclusion by reasoning, not by experiments! Like in geometry, where measurements are not accepted to validate statements! (...)
- 10- S3: the dice are concrete! And if we cast two dice a lot of times we can see who is right (even or indifferent).
- 11- S7: but we are reasoning in our head, by considering different possibilities and reasoning about them. (...)
- 13- T: let us try to see if we can get a shared conclusion by reasoning!
- 14 -S9: I would like to come back to solution C. For me it provides a good justification for the "indifferent" hypothesis
- 15-S10: For me C is too abstract. I prefer the justification D.
- 16-S2: In C do not have the possible cases, we have general, abstract cases... I am convinced that "indifferent" is the correct answer, but (...)



## Working Group 1

- 18-S12: A fits very well what I have studied in the high school: probability of an event is the ratio between the number of favorable cases and the number of all possible cases!
- 19-T: Are you sure? Imagine that you win if the sum is equal to 2. Let us consider the following possible cases: sum 2; and sum different from 2. They are two possible cases, the favorable case is the first one, thus probability of winning is  $\frac{1}{2}$ . Would you accept to bet an important amount of money on 2?
- 20-S11: But in this situation it is evident that the two cases are not.... They are not balanced, the case "different from 2" contains many numbers (like 3, 4, 5, 6, ...), we cannot divide one favorable case by those two unbalanced cases! (...)
- 23-S13: The example provided by the professor is like a counter-example for the use of our definition...
- 24-S2: but what does it happen in our situation? Perhaps, the sums 2, 3, 4, 5, 6 are not well balanced cases!
- 25-S6: There is a contradiction: if we follow A, it is more convenient to bet even; if we follow D, it is indifferent to bet even or odd. Thus some of the premises for A or for D are wrong. (...)
- 28-T: what are the premises? (...)
- 30-S6: If the conclusions are contradictory, it means that the premises are not compatible among them! We discussed it several times!
- 31-S11: I agree with you that one of the premises is right, the other is wrong
- 32-T: Are you sure that one of the two premises is necessarily right?
- 33-S3: Yes
- 34-S6: Not, both premises (though different) might be wrong!
- 35-T: Try to go in depth in the analysis of the premises!
- 36-S2: the most sure premise is in the solution D: the idea is to consider all the sums... (...)
- 40-S4: wait! Now I understand: 2 can be the sum in only one case (1+1), while 3 can be the sum in two cases (1+2 and 2+1)... They are not balanced!
- 41-S1: but I do not agree that 1+2 and 2+1 are different cases... (...)
- 44-S4: now I understand! A is wrong, because the cases (the sums 2, 3, 4, ...) are not well balanced! (*some voices of consensus*)
- 45-T: So, what conclusion can we derive from the above considerations?
- 46-S3: That A is wrong
- 47-S2: That D is right
- 48-S6: I agree that A is wrong, but this does not imply that D is right!
- 49-T: Try to explain why!
- 50-S6: I am still convinced that it is better to bet even, though I have understood that the motivation A is not right.
- 51-S3: But if a A is wrong and D is right, then what follows from D is right!
- 52-S6: We have not yet proved that D is right!

## Working Group 1

- 53-S2: D is the opposite of A... A is wrong, thus D is right!
- 54-S6: No, D is another thing. It is like if I say that Milan is the capital of Italy, and you say that Naples is the capital of Italy: if I prove that Milan is not the capital of Italy, this does not mean that Naples is the capital of Italy! (...)
- 56-T: does D consider all possible cases? (...)
- 60-S9: yes, six cases for each number, it makes 36 cases - all possible cases!
- 61-S2: I see: we could represent it by a 6x6 table
- 62-T: draw it on the blackboard! (*S9 constructs the 6x6 table*)
- 63-S4: now I start to understand - not only D is right, but this allows to get the conclusion that it is indifferent to bet even or odd, (...)
- 65-T: is it OK for everybody? (*consensus*)
- 66-T: what about C? Does it prove that it is the same to bet even or odd? (...)
- 69-T: Can we avoid considering specific couples, like in D? (...)
- 71-T: are you sure that C ensures that all cases are balanced without considering couples? (*long silence*)
- 72-S6: If a dice would have 5 faces...
- 73-S3: But a dice has 6 faces! (...)
- 76-S2: D implies C, D is a justification for C!
- 77-S?: C allows to get the conclusion "indifferent" but is also necessary...
- 78-S?: to get D, C is a consequence of D,
- 79-S?: thus a longer road to prove that it is indifferent to bet even or odd!
- 80-T: Are you sure that C is not an independent way to get the conclusion "indifferent" ?
- 81-S3: It is independent, because a 5-faces dice does not exist!
- 82-S?: In the reality... But we could imagine it!
- 83-S4: C is an independent way to get the conclusion "indifferent" if we can prove in general that the four cases even+odd, odd+even, even+even, odd+odd are balanced...
- 84-T: How we could disprove it?
- 85-S2: To disprove it, it would be necessary to find a counter-example...
- 86-T: Everybody agrees on it? (*silence*)
- 87-T: To invalidate a statement is it necessary to find a counter-example?
- 88-S4: No, it is sufficient ....
- 89-T: Why sufficient and not necessary?
- 90-S4: Because to invalidate a statement we can also make a reasoning ... find a contradiction... without finding counter-examples!
- 91-T: Let us search for a counter-example. Have you any ideas?
- 92-S6: The five-faces dice...
- 93-S3: But it does not exist, and I cannot imagine it
- 94-T: Is it really important to have dice? What is relevant in the reasoning C? Where digits come from, or...
- 95-S2: The fact that half of digits are odd, and half are even!

## Working Group 1

- 97-S6: Thus we could imagine another source of digits...
- 99-S4: For instance, digits taken from two boxes.. each box could contain 5 tickets!
- 100-S2: Let us see with the table:  
(the 5x5 table is constructed; counting results in 13 even sums and 12 odd sums)
- 101-S6: But then it would be easier to imagine boxes with 3 tickets!  
(the 3x3 table is constructed; counting results in 5 even sums and 4 odd sums)

In the economy of the course, if we consider the aim of introducing students into elementary probability theory, the reported discussion can be considered successful not only because crucial aspects of the concept of probability have been focused and clarified, but especially because the subsequent activities put into evidence that the most important "content" knowledge had been learnt by most students. Indeed the individual task "*How to explain to a student that was not there how we got the solution*" was accomplished by 36 students out of 45 with correct and enough precise justifications of the solution "indifferent" (34 students preferred to take the solution D and complete it). More interestingly, after the discussion of some unsatisfactory individual texts another individual task was proposed (the request was to identify in which random situations of a list it was possible to calculate the probability by applying the definition to the given set of cases, and to explain why): 42 students out of 45 were able to identify all the 3 appropriate situations out of the 5 proposed to them, with exhaustive specific justifications for the rejection of the others.

Thus, the "content" aim was achieved. But later I realized (reconsidering the reported discussion) that the potential inherent in some of its "segments" had not been exploited. With the exception of two students (S2 and S6) no trace of the epistemological, logical and meta-mathematical considerations was reported in the individual texts after the discussion. It was like if the attention had shifted from what was required in the task, to the usual presentation of a learned proof (according to the prevailing activity in high school). I must add that during the discussion, the teacher (myself) was aware of the importance of some interventions of the students (this was the reason that induced me to keep the audio-recording till now); what was lacking was not only the time needed to exploit the offered occasions, but also a broad perspective where to situate both "content", and "epistemological", "logical" and "meta-mathematical" aims (elaboration not yet available at that time).

## DISCUSSION OF THE DOCUMENT

In this section I will present some tools to frame the reported discussion, and I will use them to analyze the discussion and prepare possible follow-ups.

### The culture of theorems

By this synthetic expression I will mean both the knowledge needed to master conjecturing and proving and the capacity to use it, with reference to the construct of theorem (Mariotti & al, 1997) as the triad that consists of: the statement of the

theorem; the theory, which the theorem belongs to; and the proof of the theorem, performed within the theory (according to the inference rules and by exploiting the reference knowledge provided by the theory). Knowledge inherent in the culture of theorems concerns the rules of the game of conjecturing and proving. It includes meta-mathematical knowledge about the nature of the acceptable references for the validation of a statement, the role of counter-examples, the logical and textual requirement of a statement and a proof, etc. It includes also more general (logical) rules of arguing. As to the capacity of using knowledge, I will elaborate on it in the following subsections.

This description makes evident the fact that the culture of theorems is part of what we could call "the culture of argumentation". The interest for prospective primary school teachers of the discussion reported in the enclosed document depends on the fact that the development of skills related to argumentation belongs to teachers' professional duties (explicitly stated in the present national programs and national curricula of several states, including Italy). The occasions offered by the document concern in some cases specific requirements of proof, in other cases more general requirements of argumentation; thus prospective teachers can relate reflections on the rules of argumentation to specific issues concerning further mathematics education.

The content of this paper could be rephrased now by saying that it deals with the problem of passing over to students the culture of theorems and of argumentation.

### **Framing argumentation and proof: Toulmin's and Habermas' models at work**

B. Pedemonte in her thesis (2003) and then in Pedemonte (2007) proposed the use of Toulmin's model to study the relationships between argumentation and proof. In Toulmin's model an argument consists of three elements:

(*claim*): the statement of the speaker; (*data*): data justifying claim, (*warrant*): the inference rule, which allows data to be connected to the claim.

In any argument, the first step is expressed by a viewpoint (an assertion, an opinion). In Toulmin's terminology the standpoint is called the claim. The second step consists of providing data to support the claim. The warrant provides the justification for using the data as a support for the claim. The warrant, which can be expressed as a principle or a rule, acts as a bridge between the data and the claim. Three other elements that describe an argument can be taken into account: (*backing*) the support of the rule; (*qualifier*) the strength of the argument; (*rebuttal*) the exception to the rule. The force of the warrant would be weakened if there were exceptions to the rule: in that case conditions of exceptions or rebuttal should be inserted. The claim must then be weakened by means of a qualifier. Backing is required if the authority of the warrant is not accepted straight away.

Following Pedemonte (2007) we can apply this model to analyze specific points of the discussion; for instance, in the first intervention the student's claim (it is more

convenient to bet even) is related to data (the possible sums) through a warrant (the way of evaluating probability as the ratio between the number of favorable cases and the number of all possible cases) that will be weakened by the remark that another reasoning brings to a conclusion different from the claim.

Habermas (2003, ch.2) distinguishes three interrelated components of rational behaviour: the epistemic component (**epistemic rationality**) inherent in the control of the propositions and their chaining, the teleological component (**teleological rationality**) inherent in the conscious choice of tools to achieve the goal of the activity, and the communicative component (**communicational rationality**) inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas' elaboration, in the discursive practice of proving we can identify (see Boero & al, 2010): an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al., 1997); a teleological aspect, inherent in the problem-solving character of proving, and the conscious choices to be made in order to achieve the aim; and a communicative aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture.

Our adapted Habermas model allows to consider some pieces of the transcript under the perspective of rational behavior: for instance, in the last 10 interventions, once students realize that (C) might not stand by itself as a proof that it is indifferent to bet "even" or "odd" without considering (D), the search for an example in which the four cases of (C) are not equally likely is guided by the teacher and performed according to teleological rationality, in dialectic relationship with epistemic rationality.

The complexity of the discussion can be described rather well if we consider the nature of the claims, data, and warrants used by the teacher and the students during the discussion: we could qualify them with the adjectives "mathematical" (like in the case §1), "meta-mathematical" (like in the case §94), "logical" (like in the case §53 - §54), "epistemological" (like in the case §6).

The integrated use of Habermas' and Toulmin's models proposed in Boero & al. (2010) allows us to analyse the reported discussion according to a broader, unified perspective. The discussion can be considered from two points of view: the point of view of the teacher and the point of view of the students. The teacher's intention is to bring students to realize why the condition of equally likely possible cases is necessary to get a way of evaluating probability that makes sense. Thus his teleological rationality consists of interventions that address the students' attention towards data (produced by the students, or provided by him -see §19) that could weaken or confirm their warrants; while his epistemic rationality allows him to take under control the evolution of the argumentation from the point of view of logical

and mathematical validity (see §87), and communicational rationality results in the formulation of interventions that are suitable to reach the students.

From the students' point of view, their teleological rationality is exercised in few occasions and only locally (according to short term aims - see for instance §101) during the discussion, while the path towards the global achievement of the final aim and several steps to approach it are in the hands of the teacher. On the contrary, the students' epistemic rationality is locally at work in an autonomous way in several occasions, both at the mathematical level and at the metamathematical, logical and epistemological level. The task "*How to explain to a student that was not there how we got the solution*" fulfils the aim of re-constructing a mathematical, unitary solution of the problem. Did it allow the students to exercise rationality on the global level in all of its components? The lack of reflexive traces in almost all the texts prevents to answer. It is probable that several students wrote down what they were accustomed to do since the secondary school - a well written presentation of the solution of a task, without exercising awareness of the epistemic and teleological requirements of proving. Thus the task does not fulfill the aims of letting the students recompose the rationality of the guided construction of the solution, and of making them aware of the components of the rational behavior at the epistemological, meta-mathematical and logical level. The students' experience in our previous courses allowed some of them (less than 10 out of 45) to raise or deal with epistemological, meta-mathematical and logical issues during the discussion. Those issues remained as concerns for the students who introduced or discussed them, and in most cases were even lost in their individual texts.

### **The follow-ups**

The analysis of the transcript has put into evidence some important elements of the culture of theorems (and, more generally, of the culture of argumentation), which some students have brought to the fore with their interventions. In order that those elements become shared and conscious acquisition for all students, it is necessary to design suitable didactical situations. Epistemological and meta-mathematical concepts cannot be introduced through definitions (specially when young students are concerned). Thus it is necessary to exploit the *occasions* offered by the discussion to approach those concepts through meaningful examples and situations (that will become "reference situations" for them). Various solutions can be worked out.

In a situation like the reported one, the analysis of the main *motive* of the discussion for the teacher (and the students as well) brings us to exclude the introduction of systematic reflections on epistemological and meta-mathematical issues during the discussion. It would mean to distract the students' (and the teacher's) attention from its important "content" goal. Thus it is necessary to design a-posteriori didactical situations based on the use of the transcript of the discussion. The transcript can play

the role of a permanent mediation tool, useful for addressing attention to elements that remain available to all students for reflection.

According to experiments performed in similar circumstances (see Douek, 2009 and Boero & al, 2010, second part), I think that a possibility might be to choose the final text by S6 (the most rich on the logical and epistemological ground, even if far from being exhaustive!) and ask students to compare it with another "poor" (but satisfactory from the mathematical point of view) text in order to identify the differences, then ask to identify further aspects of the discussion concerning *similar* (epistemological and logical, according to our terminology) aspects.

Another idea might be to present a complete, reasoned re-construction of the problem (where explicit epistemological, logical and meta-mathematical considerations inform and guide the sequence of steps of reasoning needed to achieve the solution), and ask students to identify if, how and when in the transcript (and eventually in students' individual productions) those considerations emerged.

A third idea might be to analyze the first part of the transcript with students, putting into evidence the relevant logical and meta-mathematical aspects, then ask them to complete the analysis of the transcript according to the same categories.

The three proposals are based on the exploitation of written texts; the first two imply a comparison of at least two texts. This choice provides all students with permanent tools for reflection on the issues at stake.

All the proposals need to be followed by a whole class discussion, orchestrated by the teacher, about (some of the) individual productions. According to the students' maturity and the aims of the activity, during the discussion or after it the teacher might introduce some technical terms and explicit logical and meta-mathematical concepts in order to bring students from an initial, informal awareness of the issues at stake to a more mature take in charge of elements of the culture of theorems (and of the culture of argumentation). More importantly, during the discussion the teacher could drive the students' attention towards a re-construction of the whole elaboration of the solution, in order to make them aware of his intentions and the aims of his interventions. A way to contribute to pass over to the students his rationality.

## REFERENCES

- Balacheff, N. (1987). Processus de preuves et situations de validation. *Educational Studies in Mathematics*, 18(2), 147-176.
- Boero, P., Douek, N., Morselli, F. & Pedemonte, B. (2010) Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. Proc. of PME-34 (Vol. 1, pp. 179-209). Belo Horizonte: PME.

### Working Group 1

- Douek, N. (2009). Approaching proof in school: from guided conjecturing and proving to a story of proof construction. *Proceedings of the ICMI Study 19 Conference*, vol. 1, pp. 142-147.
- Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational Studies in Mathematics*, 22(3), 233-261.
- Habermas, J. (2003). *Truth and justification*. Cambridge (MA): MIT Press.
- Harel, G. (2008). DNR Perspective on mathematics curriculum and instruction: focus on proving, part I. *ZDM*, 40, 487–500.
- Mariotti, M. A.; Bartolini Bussi, M.; Boero, P.; Ferri, F. & Garuti, R. (1997). Approaching geometry theorems in contexts. *Proceedings of PME-XXI* (Vol. 1, pp. 180–195). Lahti (FI): PME.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23-41.



# **MATHEMATICAL PROVING ON SECONDARY SCHOOL LEVEL I: SUPPORTING STUDENT UNDERSTANDING THROUGH DIFFERENT TYPES OF PROOF. A VIDEO ANALYSIS**

Esther Brunner, Kurt Reusser, Christine Pauli

University of Teacher Education Thurgau, Kreuzlingen CH, University of Zurich,  
CH

*Within the framework of the Swiss-German study “Instructional Quality, Learning Behaviour and Mathematical Understanding” [1] (Klieme, Pauli & Reusser, 2006, 2009) and by using the example of a purely mathematical problem, it was examined in 32 classes how teachers support the process of proving in classroom instruction from a subject-based and a communicative point of view. For this purpose, an analysis instrument was developed which describes content-related aspects of the problem-solving process as well as the students’ participation. The results clearly indicate that the individual teachers differ in terms of their choice and application of specific types of proof. A special group, however, is constituted by those teachers who prove in multiple ways.*

**Keywords:** Mathematical proving, Mathematics instruction, Secondary school level I, Support of the students, Video analysis

## **INTRODUCTION**

In the context of educational standards (cf. Blum et al., 2006; EDK, 2010; NCTM, 2000), learning to prove and argue has gained new significance and experiences a real renaissance. This change was particularly furthered after the criticism of formal proof and its strictness could be neutralized with respect to public school instruction and was complemented by other concepts like, for example, pre-formal (or operative) proving (e.g. Krauthausen, 2001).

Since mathematical proving is a demanding activity, it requires teachers to support their students in a way which is close to contents and understanding-oriented. And since a mathematical proof is accepted or rejected by the community, argumentation takes place within a discourse. For these reasons, two aspects are crucial to the support of proving: content-related support as well as participation in subject specific discourse. This paper is focused on content-related, subject specific support.

Various empirical studies have shown that rather few students are able to give mathematical reasons for or to prove a given fact (cf. Healy & Hoyles, 1998; Reiss, Klieme & Heinze, 2001). As regards geometrical proofs, Reiss and collaborators (Reiss, Hellmich & Thomas, 2002) found that the varying capabilities of students in terms of proving can be substantially explained by class membership. Against this background it is surprising that only relatively few studies focus on the part of the teacher and his or her support behaviour while proving.

## THEORETICAL FOUNDATIONS

### Proving as an Important and Demanding Mathematical Core Activity

The debate around proving in mathematics instruction is multi-faceted and extensive. Besides questions about the strictness of a proof (cf. Hanna, 1997; Jahnke, 1978) or its significance (cf. Heintz, 2000), there is an interest in the different functions of a proof (cf. Hanna & Jahnke, 1996; Hersh, 1993; de Villiers, 1990), or in the role of communication and the community in proving. The distinctions between process and product, or proof and proving as an activity are likewise objects of the debate. In a genetic conception of learning, the process of proving is important especially to mathematics instruction. It is for this reason that Jahnke (1978), by reverting to Freudenthal's proposition of "local ordering" (1977), recommends that students should also be offered an understanding of sufficient reasons while problem-solving. It is not the completed proof alone or its re-enactment that should dominate mathematics instruction. Various didactic models of teaching proving processes are based on this assumption (cf. Boero, 1999; Reiss, 2002). By starting out by a (constructed) need for proving (cf. Hefendehl-Hebeker & Hussmann, 2003; Reusser, 1984), these models suggest different steps of how to get from a "why"-question to argumentatively supported reasoning. This reasoning, however, must turn out to be conclusive enough also for other persons, and it must be possible to validate it. Owing to this validity claim and external validation, a proof obtains a communicative function as well. Correspondingly, didactic communication and together with it orality are of high importance in classroom instruction.

### Types of Proof

As various authors highlight the importance of sufficient reasons in connection with the strictness of a proof, it becomes clear that in mathematics instruction not only proof in the mathematically strict sense but also its action- and thought-psychological precursors in the sense of more or less "strong" reasoning should be permitted.

There are numerous different classifications of proofs (cf. Balancheff, 1988; Blum & Kirsch, 1989; Leiss & Blum, 2006). The classification of Wittmann and Müller (1988) constitutes the basis for this study. It differentiates three fundamentally different types of proof: 1) formal-deductive proof, 2) experimental proof and 3) content-related illustrative proof.

Formal-deductive proofs are based on the logical deduction of a statement which follows from another statement step by step. They are related to conciseness of formulation which is manifested in as brief a form as possible, i.e. the mathematic formula or the formal language. Because of this, formal-deductive proofs require not only a particular way of thinking but also a particular procedure backed by a specific technical terminology.

In contrast to formal-deductive proofs, experimental proofs do not yield conclusive certainty but remain tied to the respective examples. Yet working directly with examples is fruitful especially for younger and less competent students, and it facilitates argumentations which are bound to examples. Moreover, experimental approaches are suitable for generating a subjective need for proving.

As far as content-related illustrative proofs are concerned, formalism is only of limited significance as well, though in addition to that reasons should not solely be derived by showing plausible examples as this is the case with experimental proofs. Content-related illustrative proofs are rather based on constructions and operations which render it intuitively discernible that they are applicable to a whole class of examples, and that certain conclusions can be drawn (cf. Wittman & Müller, 1988, p. 249). It must be possible to generalise content-related illustrative proofs directly from the given example. This process of generalisation should be as intuitively discernible as possible which is facilitated by making the mathematical structure transparent, mostly on the enactive or iconic level.

These three types of proof each make different demands on the students' competences, and at the same time they offer different approaches – and as a consequence specific, content-related support – depending on the current level of knowledge of the class. A genetic procedure might start out by an experimental or content-related illustrative proof and then advance to a formal-deductive proof, thus inducing an increasing degree of abstraction and symbolization.

### **Research Questions and Aims of the Study**

The aim of the current analysis is to describe teachers' support in phases of mathematical proving in secondary school level I instruction as extensively as possible and to analyse it comparatively. The following two research questions are guiding the study presented below:

- Which types of proof can be observed while the proving and reasoning problem is being worked on?
- Is it possible to distinguish special groups of teachers whose support in proving phases differs clearly from the support of the majority?

### **METHOD**

In order to compare instructional processes and the teachers' acting on the micro-level, it is important to standardise contents. This is realised by the video-based study of mathematics instruction in secondary school level I classes from Germany and Switzerland named "Unterrichtsqualität, Lernverhalten und mathematisches Verständnis" ["Instructional Quality, Learning Behaviour and Mathematical Understanding"] (Klieme, Pauli & Reusser, 2006, 2009) which was jointly conducted by the German Institute for International Educational Research (Deutsches Institut für Pädagogische Forschung, DIPF) and the Institute for

## Working Group 1

Educational Science of the University of Zurich. The aim of the project is to examine components of instructional quality. It ran from 2000 to 2006, and was structured into three phases each of which lasted two years. These three phases had different research focuses: The first phase involved a representative survey of teachers regarding instruction-related, self-related and school environment-related cognitions. The second phase of the project consisted in the video-based recording of two different instructional modules in 20 German classes of the 9<sup>th</sup> and 19 Swiss classes of the 8<sup>th</sup> school year. In the third project phase, an internet-based training programme for the participating teachers was organized.

In total, 5 lessons per teacher on a standardised content were video-taped. Two thematic units, both representing typical learning situations, were recorded as objects of the investigation: on the one hand a unit on the introduction to the Pythagorean Theorem, and on the other hand a unit on solving mathematical word problems picked from a given set. One of these problems was a purely mathematical proving and reasoning task. This paper refers to the implementation of this task in its purely mathematical context with the teachers' support of the students while reasoning and proving being at the centre of the analyses.

### Data Set

The sample consisted of 32 teachers and their classes of the 8<sup>th</sup> or 9<sup>th</sup> school year from the highest track ("Gymnasium") and from the middle track ("Realschule"/"Sekundarschule") of the three school types in Germany and Switzerland.

In order to enable as representative an insight as possible into the everyday working on a reasoning problem, the teachers were not specifically prepared. Several days before the arranged double lesson, they were sent a set of word problems which had to occur in the instructional unit. Apart from that, the teachers were free in their methodological-didactic arrangement.

For the analysis presented in the following, the provided proving and reasoning task was selected. The object of the analysis is the complete tackling of the problem. This was realised in classroom discourse but also in the form of learning support in phases of group work and silent work.

### Mathematical Problem

The task in question is a purely mathematical proving and reasoning problem which opens up different procedural options.

*The sum  $13 + 15 + 17 + 19$  is divisible by 8. Does this hold for any sum of four ensuing uneven numbers?*

The given fact can be both elucidated by means of a numerical example (experimental and/or content-related illustrative proof) and worked out through a formal-deductive procedure. As the formal-deductive procedure is not too

demanding for the age group in focus, the students can actively participate in the line of argument when they are appropriately supported.

### **Instrument**

To be able to describe proving phases with respect to content-connected tackling and understanding-oriented support as well as with respect to opportunities for participation in classroom discourse in a differentiated manner, in a first step a subject-based analysis instrument was developed by means of which the 32 cases were coded. In total, 117 different features of the way of working on the problem were captured per case. These features constituted the starting point for further analyses.

The instrument, specially developed for this analysis, captures features both on the surface structure and on the deep structure of instruction (cf. Reusser, 2005). The three big domains of the instrument refer to 1) the context of working on the problem, 2) the communicative dimension and 3) the content-related dimension. The context of working on the problem captures, among others, the duration and the arrangement pattern of working on the problem. The communicative dimension captures features like the conceptual level, communication of meta-rules and meta-commentaries, and others, as well as the type of didactic communication and the students' participation for every content-related feature. The content-related dimension captures the type of proof chosen by the teacher, the heuristics made use of, the elements of comprehension as actual content-related kernel of the problem, as well as rather peripheral features of working on the problem.

The current paper is restricted to the presentation of results referring to the type of proof chosen by the teacher.

### **Video Analyses**

The 32 instructional units were coded with the help of the analysis instrument (inter-rater reliability: 88–100%). The thus yielded data could then be further processed by means of statistical analyses. The first results, presented below, are based on descriptive statistics, on group comparisons, and on correlative connections between the individual supportive features.

## **RESULTS**

### **Type of Proof**

All of the three different categories of proof occurred, though in a quite varying distribution. Formal-deductive proofs could be found in 65.5% of all cases (21 classes), experimental proofs in 12.5% of the cases (4 classes), and content-related illustrative proofs in 37.5% of the cases (12 classes). Consequently, there are classes in which several categories of proof could be observed. This was true for 9 classes.

In 4 classes, however, no proof at all was (conclusively) completed. The choice of the type of proof does not bear any connection with the type of school.

### **Multiple Proving – A Special Group**

In total, 9 classes could be determined in which two of the three types of proof were implemented. These classes can be described as a special group because the use of different types of proof can be regarded as an elaborated line of argument.

That these classes constitute a special group can be inferred from statistically significant differences. These teachers clearly made more use of heuristic aids in their classes ( $M = 2.33$ ;  $SD = .86$ ;  $N = 9$ ) than those teachers who only realised one type of proof ( $M = 1.53$ ;  $SD = .96$ ;  $N = 19$ ). This difference in the mean is statistically significant ( $t = 2.13$ ;  $df = 26$ ;  $p < .05$ ), and it displays a very strong effect ( $ES_d = .86$ ).

Statistically significant differences between the group of teachers who carry through several types of proof ( $n_1 = 9$ ), the group of teachers who make use of one type of proof ( $n_2 = 19$ ), and the group of teachers who do not implement any proof ( $n_3 = 4$ ) can also be detected with respect to other supportive features in the reasoning process, for example concerning the formal-deductive procedure ( $\chi^2 = 9.55$ ;  $df = 2$ ;  $p < .01$ ), the content-related illustrative procedure ( $\chi^2 = 9.34$ ;  $df = 2$ ;  $p < .01$ ), the criterial assessment of a proof ( $\chi^2 = 10.38$ ;  $df = 2$ ;  $p < .01$ ), and others more which cannot be set out any further here. All differences appear in favour of the group which implements several proofs – i.e. in this group, the supportive aids mentioned above (heuristics) are made use of more often than in the other two groups. The group of teachers who carry through two different types of proof can therefore be characterised as a special group also on the level of individual features of the solving process.

Since the choice of the two different types of proof differs quite obviously within this group, further sub-groups can be differentiated.

*Group A: Experimental and formal-deductive:* Two classes of the middle track (“Sekundarschule”/“Realschule”) belong to this group. Both teachers first carry through an experimental proof and discuss its limits with the class. Afterwards, a formal-deductive proof is worked out in a questioning-developing classroom dialogue. Both cases can be regarded as genetic procedures, because a formal-deductive approach is developed out of an experimental approach and the determination of its weaknesses.

*Group B: Content-related illustrative and formal-deductive:* In six classes there is first a content-related illustrative proof and then a formal-deductive proof. Five of these classes are of the highest track (“Gymnasium”), one is of the middle track (“Sekundarschule”). In one class, the teacher initially asks his students to work out a proving procedure themselves in groups during quite a long period of time.

## Working Group 1

Thereafter, students from two different groups present two different types of proof – first a content-related illustrative proof and then a formal-deductive one – as results of their independent work. Subsequently, these results are discussed and deepened in a classroom dialogue. The other five teachers make use of short phases of independent student work, but they work out the line of argument together with their students by way of a classroom dialogue. The content-related illustrative proof occurs in all classes before the formal-deductive one. In the five classes without presentation of the students' own solutions, the content-related illustrative proof is raised on the next level of generalisation in a further step and the formal-deductive proof is developed. Also this case can be considered as a genetic procedure. It is, however, more structured and more systematic than in group A. In group B, it is not only the weakness of the procedure which is demonstrated by means of the content-related illustrative proof. Moreover, a way of preparatory thinking and a didactic guidance to formal-deductive proving on the basis of the now transparent and understood inherent structure is presented.

*Group C: Experimental and content-related illustrative:* In one class of the highest track ("Gymnasium") an experimental and a content-related illustrative proof occur. This class works out the two types of proof in a questioning-developing classroom dialogue with the teacher collecting the students' suggestions regarding the procedure. In doing so, he first carries through an experimental proof together with his class and then demonstrates that this proof is limited in its validity claim. Afterwards, he applies a content-related illustrative proof to a concrete example. Also in this case we are dealing with a genetic procedure, though in a different way. The experimental approach is discussed with regard to its weaknesses, and thereafter it is complemented by a content-related illustrative proof.

## DISCUSSION

It is not very surprising that in the vast majority of classes a formal-deductive proof is implemented. This can be put down to the fact that strictness of a proof is characteristic of mathematics seen as a science. What is more astonishing, on the other hand, is the fact that the choice of one of the types of proof does not depend on the type of school. Formal-deductive proofs do not occur more often in classes of the highest track ("Gymnasium") than in classes of the middle track ("Sekundarschule"/"Realschule"). This dominance of formal-deductive proof in both types of school indicates that teachers think that proving is necessarily tied up with a strictly formal-deductive procedure. For this reason, the potential of the different types of proof with regard to specific, content-related support is still underused to a great extent.

It is interesting that a special group of teachers can be characterised who differ from the other teachers in their use of multiple proving and thus in various supportive features. As far as this group is concerned, it can be assumed that the instruction with

respect to proving and reasoning displays a clearly higher quality than in the other groups. The reason for this assumption is that different approaches to a given fact are opened up, and that the focus is not one-sidedly on formal-deductive proving, thus preventing a hasty reference to the strictness of the proof. Rather, we are dealing with a genetic procedure which implies a real guidance to formal-deductive proving by means of a supportive arrangement of the argumentation. This is why in the training of mathematics teachers it should be carefully made sure that they do not only know about the strictness of a proof, but also, at the same time, that they are able to regard this formal strictness as an aim of instruction and not as a prerequisite for the implementation of a proof.

However, the chosen type of proof alone does not state anything about the quality of the understanding-oriented support. Rather, it solely describes a predominant practice. In order to investigate the quality of the understanding-oriented support and of the fostering of argumentation there is need for further in-depth analyses which are currently undertaken. It is examined, among other things, whether and under which circumstances the choice of a certain type of proof is functional, to what extent this choice has effects on the students, and which impact the teachers' beliefs have in this respect. Moreover, further content-connected aids by means of which teachers support their students in understanding and arguing are of interest. Finally there is an interest in the way of student participation while arguing and proving. These analyses are under way as well.

## NOTES

[1] We thank the Swiss National Science Foundation (SNSF) for supporting the project.

## REFERENCES

- Balacheff, N. (1991). Benefits and limits of social interaction: The case of teaching mathematical proof. In A. Bishop, S. Mellin-Olsen & J. Van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 175-192). Dordrecht: Kluwer Academic Publisher.
- Blum, W., Drücke-Noe, C., Hartung, R. & Köller, O. (Eds.). (2006). *Bildungsstandards Mathematik: konkret. Sekundarstufe I: Aufgabenbeispiele, Unterrichtsanregungen, Fortbildungsideen*. Berlin: Cornelsen Scriptor.
- Blum, W. & Kirsch, A. (1989). Warum haben nicht-triviale Lösungen von  $f' = f$  keine Nullstellen? Beobachtungen und Bemerkungen zum inhaltlich-anschaulichen Beweisen. In H. Kautschitsch & W. Metzler (Eds.), *Anschauliches Beweisen* (pp. 199-209). Vienna: Hölder-Pichler-Tempsky.
- Boero, P. (1999). Argumentation and mathematical proof: a complex, productive, unavoidable relationship in mathematics and mathematics education. *International Newsletter on the Teaching and Learning of Mathematical Proof*, 7/8. Retrieved



August 2010 form: <http://www-didactique.imag.fr/preuve/Newsletter/990708Theme/990708ThemeUK.html>.

- EDK Generalsekretariat. (2010). *Basisstandards für die Mathematik*. Unterlagen für den Anhörungsprozess. 25. Januar 2010. Bern: EDK.
- Freudenthal, H. (1977). *Mathematik als pädagogische Aufgabe*. Volumes 1 & 2. Stuttgart: Klett.
- Hanna, G. (1997). The ongoing value of proof. *Journal für Mathematikdidaktik*, 18 (2/3), 171-185.
- Healy, L. & Hoyles, C. (1998). *Justifying and Proving in school mathematics. Technical Report on the Nationwide Survey*. London: University of London, Institute of Education.
- Hefendahl-Hebeker, L. & Hussmann, S. (2003). Beweisen – Argumentieren. In T. Leuders (Ed.), *Mathematikdidaktik. Praxishandbuch für die Sekundarstufe I und II* (pp. 93-106). Berlin: Scriptor.
- Heintz, B. (2000). *Die Innenwelt der Mathematik. Zur Kultur und Praxis einer beweisenden Disziplin*. Vienna: Springer.
- Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24 (2), 389-399.
- Jahnke, H. N. (1978). *Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik. Beweisen als didaktisches Problem*. Bielefeld: Institut für Didaktik der Mathematik.
- Klieme, E., Pauli, C. & Reusser, K. (2006). *Dokumentation der Erhebungs- und Auswertungsinstrumente zur schweizerisch-deutschen Videostudie "Unterrichtsqualität, Lernverhalten und mathematisches Verständnis"* (3 Volumes). Materialien zur Bildungsforschung, Band 13-15. Frankfurt a.M.: dipf.
- Klieme, E., Pauli, C. & Reusser, K. (2009). The Pythagoras Study. In T. Janik & T. Seidel (Eds.), *The power of video studies in investigating teaching and learning in the classroom* (pp. 137-160). Münster: Waxmann.
- Krauthausen, G. (2001). "Wann fängt das Beweisen an? Jedenfalls, ehe es einen Namen hat." Zum Image einer fundamentalen Tätigkeit. In W. Weiser & B. Wollring (Eds.), *Beiträge zur Didaktik der Mathematik für die Primarstufe. Festschrift für Siegbert Schmidt* (pp. 99-113). Hamburg: Dr. Kovac.
- Leiss, D. & Blum, W. (2006). Beschreibung zentraler mathematischer Kompetenzen. In W. Blum et al., *Bildungsstandards Mathematik: konkret. Sekundarstufe I: Aufgabenbeispiele, Unterrichtsanregungen, Fortbildungsideen* (pp. 33-50). Berlin: Cornelsen.

## Working Group 1

- NCTM. (Ed.). (2000). *Principles and standards for school mathematics*. Reston: NCTM (National Council of Teachers in Mathematics).
- Reiss, K. (2002). *Argumentieren, Begründen, Beweisen im Mathematikunterricht*. Projektserver SINUS. Bayreuth: Universität.
- Reiss, K., Hellmich, F. & Thomas, J. (2002). Individuelle und schulische Bedingungsfaktoren für Argumentationen und Beweise im Mathematikunterricht. In M. Prenzel & J. Doll (Eds.), *Bildungsqualität von Schule: Schulische und außerschulische Bedingungen mathematischer, naturwissenschaftlicher und überfachlicher Kompetenz*. 45. Beiheft der Zeitschrift für Pädagogik (pp. 51-64). Weinheim: Beltz.
- Reiss, K., Klieme, E. & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education Vol. 4* (pp. 97-104). Utrecht: Utrecht University.
- Reusser, K. (1984). *Problemlösen in wissenschaftstheoretischer Sicht. Problematisches Wissen, Problemformulierung und Problemverständnis*. Dissertation (reprinted 1994). Bern: University.
- Reusser, K. (2005). Problemorientiertes Lernen – Tiefenstruktur, Gestaltungsformen, Wirkung. *Beiträge zur Lehrerbildung*, 23 (2), 159-182.
- Villiers, M. de (1990). The Role and the Function of Proof in Mathematics. *Pythagoras*, 24, 17-24.
- Wittmann, E. C. & Müller, N. (1988). Wann ist ein Beweis ein Beweis? In P. Bender (Ed.), *Mathematikdidaktik – Theorie und Praxis. Festschrift für Heinrich Winter* (pp. 237-258). Berlin: Cornelsen.

# EVERYDAY ARGUMENTATION AND KNOWLEDGE CONSTRUCTION IN MATHEMATICAL TASKS

Julia Cramer

University of Bremen

*The aim of this study is to gain insights into relations between knowledge construction and argumentation. This paper presents a case study showing an analysis that combines different tools: the Toulmin's scheme to reconstruct the argumentation structure, a collection of topical schemes to characterize different types of inferences including everyday inferences and an epistemic action model to describe processes of knowledge construction. Some preliminary results of this case study will illustrate how the combination of these different tools can shed light on relations between argumentation and knowledge construction.*

Key words: Everyday argumentation, knowledge construction, topical schemes.

## INTRODUCTION

The aim of my research is to gain insights into relations between knowledge construction and argumentation. Arguing is an important learning goal for two reasons. Schwarz (2009) distinguishes between learning to argue and argue to learn. Referring to Andriessen et al. he clarifies this distinction:

„'Learning to argue' involves the acquisition of general skills such as justifying, challenging, counterchallenging, or conceding. In contrast 'Arguing to learn' often fits a specific goal fulfilled through argumentation, and in an educational framework, the (implicit) goal is to understand or to construct specific knowledge." (Schwarz 2009, 92)

Some researchers like Krummheuer & Brandt (2001) even assume a constitutive function of arguing for learning mathematics in school: Students learn mathematics by participating in argumentation that means practicing argumentation within social interaction as a collective activity. In addition, learning to argue is stated as a goal in many curricula all over the world (e.g. see NCTM 2000). Nevertheless, students have difficulties to learn how to reason in a deductive way. Research shows that everyday argumentation predetermines their way of reasoning in mathematics. For example, many students infer general rules from just some examples (Martin & Harel 1989), just like they would do in everyday situations, without feeling the need to prove their inference. Furthermore, Galbraith (1981) has shown that students often do not realize that one counterexample disproves a mathematical statement because in everyday life an exception does not mean that a rule is not valid. These findings show that students bring experiences of everyday argumentation with them into math classes interfering with the learning of mathematical reasoning. The question is how to use these everyday experiences as a foundation for mathematical reasoning. Efforts in this direction are complicated by the fact that we do not exactly know how mathematical

argumentation emerges out of everyday argumentation and how this is connected to knowledge construction. This is exactly the point on which I will focus in my study. The leading questions are:

- What elements of everyday argumentation do students use when solving mathematical problems and how does this lead to mathematical argumentation?
- How is knowledge constructed through argumentation? How can the epistemic function of arguing be described?
- How can we describe the relation between argumentation and knowledge construction?
- What components of argumentation processes foster or hinder processes of knowledge construction?

## **ARGUMENTATION**

Schwarz et al. (2003) state that

“constructing knowledge is a never-ending process of marshalling evidence that the chosen belief is (a) supported by the available evidence and (b) more warranted than plausible rival beliefs” (Schwarz et al. 2003, 222).

Following this statement, I assume that arguing is an epistemic action, i.e. an intentional action to gain knowledge. The construction of new knowledge arises from reasoning or checking the validity of claims. Therefore, arguing has two epistemic functions: Constructing new knowledge and/or convincing others of the validity of one’s own hypothesis. The epistemic function of convincing others means: The more people one can convince, the more likely one’s own hypothesis will be. In this sense, an argument is a statement that makes a hypothesis more or less likely.

The Toulmin’s scheme is an appropriate tool to reconstruct the structure and depth of argumentation processes (Krummheuer 1995, Knipping 2004) or to characterize abductive and deductive types of arguments (Pedemonte 2002). But how can we grasp the students’ more intuitive methods of concluding? Looking at philosophy and at rhetoric is worthwhile. A collection of topical schemes has the potential to identify the starting point of how mathematical inferences develop. These two tools, the Toulmin’s scheme and a collection of topical schemes, will be presented in the following paragraphs.

### **The Toulmin’s scheme**

To analyze processes of argumentation, Toulmin (1958) developed a scheme that classifies elements of an argumentation with regard to their function into data, conclusion, warrant, backing and qualifier. The conclusion is the statement that has

to be reasoned. Data are unquestioned facts the conclusion is led back to. Data and conclusion are part of every argumentation. The step of inferring from data to conclusion can be explained by a warrant that sometimes is implicit. To ensure the practicability of a warrant, the warrant can be backed up by additional statements. Qualifiers or exceptions specify the validity of a conclusion.

In processes of argumentation with more than one step, conclusions that are already accepted by (most of) the other participants of the process can turn into new data. If backings, warrants or data are questioned, these elements have to be reasoned in a separate process of argumentation before they can be used in the primal argumentation. In complex processes of argumentation, Krummheuer & Brandt (2001) name these separate processes of argumentation as lines of argumentation.

### **A collection of topical schemes**

Topical thinking means that someone has access to some basic ideas for generating arguments. These basic ideas for concluding methods are called topical schemes. In this sense, topical schemes are facilities to create steps of an argumentation or to ensure persuasively the power of argumentation steps. Ottmers (2007) presents a collection of topical schemes that is divided into two prime classes: everyday logical schemes and convention-based schemes. The everyday logical schemes contain five types: causal-based, comparison-based, contrast-based, classification-based and example-based conclusions. The first four types conclude from general to special statements (schemes of inferential nature), the last type concludes from special cases to general statements (schemes of inductive nature). Everyday logical schemes are redolent of formal logical rules. Causal-based conclusions use causal relations to ensure plausibility. Causal relations are those between cause and effect, between reason and consequence of human activities or between means and end. Comparison-based conclusions relate different parameters. They refer to equality, diversity, or more or less probable cases. Contrast-based conclusions refer to relevance between contrasts. These contrasts can be absolute, relative, or alternative. Classification-based conclusions use relations between parts and the whole issue, between species and genus, or between definition and the defined issue. To come to example-based conclusions, either a set of examples or just a single example is used as a (model/prototype/as evidence) of a general rule. A general example that is appropriate for reasoning in the direction of generalization is called an inductive example. An example that just shows that a rule is valid in this case is called an illustrative example.

Convention-based conclusions do not use any kind of logical structures but concluding methods that are established within a group. Therefore it is not possible to present a complete collection of convention-based concluding schemes. Ottmers presents two topical schemes as examples for this prime class: authority-based conclusions and metaphor-based conclusions. Authority-based concluding means referring to another person (a special group/special institution/etc.) that is accepted

as an authority in the relevant field. Metaphor-based conclusions are similar to example-based or comparison-based conclusions. They relate the conclusion to other similar cases for plausibility. There are two important differences between these concluding methods. Metaphor-based conclusions refer to one example. In contrast, example- or comparison-based conclusions regard an amount of similar cases or examples. Furthermore, the similar cases in this concluding method stem from another field and explain the conclusion in a more metaphorical way [1].

### **Knowledge construction**

Bikner-Ahsbabs (2005) developed an epistemic action model to analyze the epistemic process in interest-dense situations in comparison to other learning situations [2]. This model consists of three epistemic actions that shape epistemic processes. These epistemic actions are gathering mathematical meanings, connecting mathematical meanings and structure-seeing. Gathering means assembling similar mathematical entities; connecting means linking a limited amount of these to other entities; structure-seeing means constructing or reconstructing a mathematical structure that refers to an unlimited number of mathematical entities.

### **METHODOLOGY AND METHODS**

This study is embedded into the research project “*Effective knowledge construction in interest-dense situations*” that investigates processes of in-depth knowledge construction and its background conditions by linking two theories of constructing mathematical knowledge. This project is a joint study between two research teams from Israel and one team from Germany and it is supported by the German-Israeli Foundation for Scientific Research and Development (Grant 946-357.4/2006). Within this project, three different tasks have been developed that offer an opportunity to construct mathematical knowledge. In each country, three pairs of students (grade 10 in Germany, grade 11 in Israel) are video- and audiotaped when solving these tasks in a varying order. There is no teacher, however, in some instances an interviewer takes over the role of a teacher. The audio- and videodata are transcribed and translated into English to exchange them between the research teams. Including some transcripts of the project’s pilot phase, there are currently 11 German transcripts and 12 translated transcripts at my disposal.

The transcripts will be analysed interpretatively in turn-by-turn-analyses and in several steps according to the leading questions above. The kind of student’s involvement and the epistemic process is reconstructed through interpretation on three levels: the locutionary, the illocutionary and the perlocutionary level. The locutionary level consists of what is said. On this level, the construction of mathematical knowledge is reconstructable. The illocutionary level consists of what is said through an utterance (the underlying subtext) and the perlocutionary level contains the people’s intentions and the impact of utterances (Davis 1980).

Reconstructing the illocutionary and the perlocutionary level shows how mathematical knowledge is constructed socially (Arzarello et al. 2009).

The analyses of the argumentation processes and of the epistemic processes will be done in separate steps. First, the epistemic processes will be reconstructed by identifying the epistemic actions gathering, connecting and structure-seeing and some social actions like asking for an explanation, valuing, initiating and contrasting. A diagram will illustrate the processes of knowledge construction as well as the social processes in a nutshell. Secondly, the structure of the argumentation is reconstructed with the aid of Toulmin's scheme, and the kind of warrants the students use is characterized by means of Ottmer's collection of topical schemes. The results will be shown in a diagram as well. In a last step, these diagrams will be compared and contrasted (Prediger, Bikner-Ahsbahs & Arzarello 2008). The comparison of the results of the different analyses will shed light on relations between argumentation and knowledge construction. However, the problem to compare a diagram showing a process (the epistemic actions) with a diagram showing products (argumentation) is not solved yet. Hence, some elements of the diagrams will be developed and modified in the study.

## PRELIMINARY RESULTS

Two high-achieving students (grade 10 of a German Gymnasium), Tim and Matthias, solve a task asking them to interpret the continued fraction  $1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{\dots}}}}$ . The task is

structured into two parts. In the first part, the students are asked to find the first 7 elements of the sequence, reflect on how they computed them, extend the computation to 20 elements and write them as simple as well as decimal fractions. In the second part, the students are asked to make a conjecture on the sequence from the first part and justify their conjecture.

In the case of Tim and Matthias, the transcript can be divided into main three parts according to the task. The first main part of the transcript corresponds with the first part of the task (line 31 – 718). In this part, Tim and Matthias compute the elements of the sequence very accurately always checking that they made no miscalculation. Their accurate and pre-cautious way of solving the task becomes apparent in the following parts as well. In the second main part, Tim and Matthias make conjectures on the sequence (line 719 – 1465); in the third main part (line 1466 – 2630) they try to justify their conjectures with the interviewer directing the students' focus on some aspects. This paper presents the analysis of the second main part of the transcript that can illustrate some interesting relations between argumentation and knowledge construction.

## Working Group 1

When Tim and Matthias are asked to make conjectures on the sequence, they focus on the decimal places. They look at the elements of the sequence from  $f(7)$  on and observe that the amount of same decimal places increases.

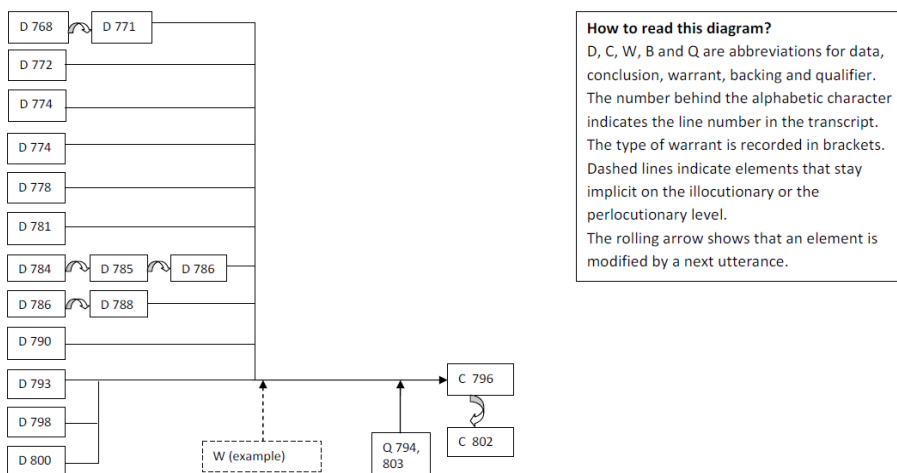
729 Matthias: [...] ,wait here were only three zeros (points at the following on the sheet) ,then three nines ,then three zeros again ,then four nines ,then four zeros ,then four nines again ,so always after [...]

731 /Matthias: always after- ,always after three. ,it becomes ,it becomes one more.

Matthias gathers information of their previous work (three zeros, three nines ...). This information is unquestioned and can be used as data later on. When gathering the information, Matthias uses words like “only”, “then” and “again”. This is a hint that he already connects these data in order to make a conjecture. Matthias infers example-based that after every third element of the sequence the amount of nines respectively zeros behind the decimal point increases by one. Tim starts counting the amount of nines respectively zeros behind the decimal point from the first element of the sequence on. He realizes that the results are not in line with Matthias’ conjecture. But instead of abandoning this conjecture now, they first check if they have made any mistake. They find that they did no miscalculation, and therefore they change the conjecture a bit:

744 Tim: yes at least ,one notices that always (.) ,always three nines ,three zeros ,the amount of nines and zeros is identical ,and then it always switches after three or four.

The whole second main part of the transcript (line 719 – 1465, approximately 30 minutes) consists of making conjectures on the sequence and sharpening the formulation of these conjectures. All the conjectures are associated. They do not contradict, but deal with different aspects of the sequence. For lack of space it is not possible to show all diagrams of the second main part. Some selected parts are presented to illustrate important observations. Figure 1 shows how Tim and Matthias develop a new aspect concerning their conjecture.



**Figure 1: diagram of line 768 - 803**



## Working Group 1

Tim and Matthias start gathering a lot of data. Some data is modified. They connect the data immediately what is shown by words like “then”, “here again”, etc. They realize that  $f(8)$  till  $f(10)$  have two nines or zeros behind the decimal point,  $f(11)$  till  $f(13)$  have three nines or zeros and  $f(14)$  till  $f(17)$  have five nines or zeros behind the decimal point.  $f(8)$  till  $f(10)$  and  $f(11)$  till  $f(14)$  are rows of three elements with the same amount of nines or zeros behind the decimal point.  $f(14)$  till  $f(17)$  is a row of four elements with the same amount of nines or zeros behind the decimal point [3]. From these data, they infer example-based.

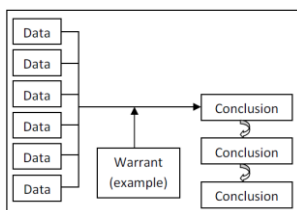
796 Tim: I would say ,ohm- ,as soon as it is a row of three four five whatever ,ohm- ,lets just say a row of five ,o- or no ,lets say a row of four ,and twenty-five is inside of it ,so five squared ,from then on ,are ,five nines ,five times

They sharpen their formulations and even create a new term in order to specify what they mean. The term “space of places” describes a space (some following elements of the sequence) where the amount of nines or zeros behind the decimal point is the same. For instance,  $f(4)$  till  $f(7)$  is a space of places as there is one nine respectively one zero behind the decimal point.  $f(8)$  till  $f(10)$  is the following space of places. With the aid of this term, they develop and clarify their conjectures.

During the whole second main part of the transcript, Tim and Matthias write down their conjectures in a very accurate and careful manner. They always discuss their formulation and do not note it down until both agree on it. Here is what they wrote down as their conjectures:

“The amount of zeros or nines behind the decimal point is the same in a particular space of places. If you go from one to another space of places, the amount of nines or zeros increases by another nine or zero. If there are nines or zeros behind the decimal point, depends on the  $x$ -value. If the  $x$ -value is even, there are nines behind the decimal point and a one in front of it. If the  $x$ -value is odd, there are zeros behind the decimal point and a two in front of it. The length of the space of places changes to  $c$  as soon as the space of places contains  $c^2$ .”

All of these conjectures are inferred example-based. Before a new aspect of a conjecture arises, Tim and Matthias gather a lot of data. In the following, they sharpen and modify this conjecture. A characteristic argumentation line during this part is shown in figure 2.



**Figure 2: characteristic argumentation line in part 2 of the transcript**

Obviously, several data in the form of examples are necessary for example-based inferences. However, such argumentation lines with a high concentration of data do not seem to be characteristic only for example-based inferences. In another case study (see Cramer 2010), the first argumentation lines contained a lot of data as well. The conclusions in this case were inferred causal-based. This could be a hint, that in general a high concentration of data is characteristic for argumentation lines in the beginning. This result corresponds with the finding of interest-dense situations, that gathering phases until a certain saturation level are necessary to provide a basis for further increases of knowledge.

The fact that conjectures arise example-based is obvious as well. It is a heuristic strategy stemming from everyday life and the way how discovery learning works. The question is how to encourage students to justify their conjecture in a mathematical way. Tim and Matthias insist several times that their conclusions are conjectures. They believe that they are valid, but they are not sure. The way they become more and more sure in the second part of the transcript stems from everyday life as well. When they observe that nines and zeros behind the decimal point alternate as well as one and two in front of the decimal point, they take the alternation as an argument for the validity of their conjecture.

1000 Tim: Yes ,that always switches ,thats why it is logical

The implicit argument is based on previous experiences. The students are asked to look for patterns and regularities. Something alternating is a pattern, therefore it is more likely that their conjecture is true. Ottmers does not mention experience-based conclusions. It could be a type of example-based conclusions in the sense that previous experiences are kind of examples. But the related examples are not mentioned explicitly. Therefore it is not possible to check the relevance of these experiences, and for this reason I would describe it as a convention-based scheme. Another example of this concluding scheme appears later on. Tim and Matthias realizes that the space of places from  $f(4)$  till  $f(7)$  has a length of four. This is not in line with their conjecture that the length of the space of places changes to  $c$  as soon as the space of places contains  $c^2$ . Tim would like to solve the problem in this way:

1345 Tim: Yes exc- ,exception ,or then write exceptions prove the rule next to it

This is an everyday proverb that even contradicts with mathematical argumentation. This finding shows that, on the one hand, Tim and Matthias are influenced by everyday argumentation; but on the other hand they show that they are aware of the fact that mathematical argumentation works differently by insisting that their rule has the status of a conjecture. A little episode of the third part of the transcript illustrates this awareness. At the end of the second part they mentioned that the sequence tends towards two. They are now working on a justification. Matthias remembered that  $0,\bar{9} = 1$  and justified this fact authority-based referring to his teacher.

1446 Matthias: So one found out that ohm- ,on(e)- one say ,our teacher told us that ohm- ,one point nine period equals two.

## Working Group 1

Such a convention-based conclusion seems to be insufficient or unsatisfactory, therefore Matthias justifies it classification-based.

1449 Tim: [...] ,because one plus nine ninth is precisely two ,but nine ,one ninth ,is zero point one one one one one

Tim and Matthias use convention-based schemes seldomly. However, when they use these schemes, they give up these argumentations lines quickly. This is a hint that they regard these concluding methods as insufficient or unsatisfactory as convention-based schemes do not lead to deeper understanding why a conclusion is valid.

## CONCLUSIONS

The combination of the Toulmin's scheme and a collection of topical schemes turned out to be an appropriate tool to describe argumentation and to identify elements of everyday argumentation. In this case, everyday argumentation was the starting point to develop mathematical conjectures. Everyday logical schemes can turn into strategies to come to conjectures and to create ideas of how to justify them. The analysis of the episode presented in this paper focuses on this aspect. To find out more about typical argumentation structures and relations between knowledge construction and argumentation, analyses of different episodes have to be compared. A first hint concerning relations between knowledge construction and argumentation is that argumentation lines with a high concentration of data prevail in the beginning of epistemic processes. This corresponds to phases of gathering and combining in terms of the epistemic action model. In structure-seeing phases, argumentation lines occur where warrants and backings are formulated and sharpened. In the future, several case studies will be compared to deliver deeper insights into relations between argumentation and knowledge construction.

## NOTES

1. Ottmers calls these metaphor-based conclusions analogy-based ones. His example for such an analogy-based conclusion makes clear that these analogies are metaphorical. As analogy is understood in a different way in mathematics, I decided to call these conclusions metaphor-based ones.
2. In this study, Bikner-Ahsbahs' epistemic action model is used to describe processes of knowledge construction. The emergence of interest-dense situations will not be analysed. Therefore, I do not enlarge upon this term here. For further information about the theory of interest-dense situations see Bikner-Ahsbahs 2004.
3. To follow the excerpts of the transcript, here are the elements of the sequence presented as decimal fractions:

	Decimal fraction		Decimal fraction		Decimal fraction		Decimal fraction		Decimal fraction
f(0)	1	f(4)	1,909090909	f(8)	1,994152047	f(12)	1,999633834	f(16)	1,999977112
f(1)	3	f(5)	2,047619048	f(9)	2,002932551	f(13)	2,000183117	f(17)	2,000011444
f(2)	1,666666667	f(6)	1,976744186	f(10)	1,998535871	f(14)	1,99990845	f(18)	1,999994278
f(3)	2,2	f(7)	2,011764706	f(11)	2,000732601	f(15)	2,000045777	f(19)	2,000002861

## REFERENCES

Arzarello, F., Bikner-Ahsbahs, A. & Sabena, C. (2009). Complementary networking: Enriching understanding. In: Proceedings of CERME 6, Lyon 2009, ERME, 1545 - 1554.

## Working Group 1

- Bikner-Ahsbahr, A. (2004). Towards the emergence of constructing mathematical meaning. In: Proceedings of the PME 28, Bergen, PME 2004, vol. 2, 119 – 127.
- Cramer, J. (2010). Induktion durch vollständiges Zeigen. Schlussweisen in Argumentationsprozessen. In: Beiträge zum Mathematikunterricht. Vorträge auf der 44. Jahrestagung der Gesellschaft für Didaktik der Mathematik, 229 - 232.
- Davis, S. (1980). Perlocutions. In: Searle, J. R.; Kiefer, F. & Bierwisch, M. (Eds.). Speech act theory and pragmatics. Dordrecht, London: D. Reidel Publishing Company, 37 – 55.
- Galbraith, P. L. (1981). Aspects of proving: A clinical investigation of process. In: Educational Studies in Mathematics, 12, 1 – 29.
- Knipping, C. (2004). Argumentations in proving discourses in mathematics classrooms. In: Törner, G. et al. (Eds.). Developments in Mathematics Education in German-speaking Countries. Selected Papers from the Annual Conference on Didactics of Mathematics, Ludwigsburg, March 5 - 9, 2001, Hildesheim: Franzbecker, 73 – 84.
- Krummheuer, G. (1995). The ethnography of argumentation. In: Cobb, P. & Bauersfeld, H. (Eds.). The emergence of mathematical meaning: Interaction in classroom cultures. Hillsdale, NJ: Lawrence Erlbaum Associates, 229 – 269.
- Krummheuer, G. & Brandt, B. (2001). Paraphrase und Traduktion. Partizipations-theoretische Elemente einer Interaktionstheorie des Mathematiklernens in der Grundschule., Weinheim: Beltz.
- Martin, G. & Harel, G. (1989). Proof frames of preservice elementary teachers. In: Journal for Research in Mathematics Education, 20(1), 41 – 45.
- National Council of Teachers of Mathematics: 2000, Principles and standards for school mathematics. Reston, VA: Author.
- Ottmers, C. (2007). Rhetorik [Rhetoric]. Stuttgart: Metzler.
- Pedemonte, B. (2002). Relation between argumentation and proof in mathematics: cognitive unity or break? In: Proceedings of the CERME 2, Mariánské Lázně 2001, vol. 2, 70 – 80.
- Prediger, S., Bikner-Ahsbahr, A. & Arzarello F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. In: The International Journal on Mathematics Education (ZDM), 40, 165 – 178.
- Schwarz, B., Neumann, Y., Gil, J. & Ilya, M. (2003). Construction of collective and individual knowledge in argumentative activity. In: The Journal of the Learning Sciences, 12(2), 219 – 256.
- Schwarz, B. (2009). Argumentation and Learning. In: Muller Mirza, N. & Perret-Clermont, A.-N. (Eds.). Argumentation and Education. Theoretical Foundations and Practices. Berlin: Springer.
- Toulmin, S. (1958). The uses of arguments. Cambridge: University Press. German translation of Berk, U. (1996). Der Gebrauch von Argumenten. Weinheim: Beltz.

# ANALYZING THE PROVING ACTIVITY OF A GROUP OF THREE STUDENTS

Patricia Perry, Óscar Molina, Leonor Camargo and Carmen Samper

Universidad Pedagógica Nacional, Bogotá, COLOMBIA

*We present an analysis and outline an evaluation of the proving activity of a group of three university level students when solving a geometrical problem whose solution required the formulation of a conjecture and its justification within a specific theoretical system. To carry out the analysis, we used the model presented by Boero, Douek, Morselli and Pedemonte (2010) that centers on the arguments and rational behavior. Our analysis indicates that the student's proving activity is close to the one we used as a reference.*

Key words: evaluation, proving activity, rational behavior, type of argument

## INTRODUCTION

Producing and reading proofs are complex mathematical practices because they require being able to articulate many and diverse, not necessarily routine, actions; therefore mastering them is not an easy or direct process. Understanding this has recently impelled research and reflection of a didactic character of the different methodological approaches used to teach proof and proving in the tertiary level, and to subsequent innovations which could lead to satisfactory results in students' learning (see Selden, 2010).

In this regard, since 2004, we have been engaged in consolidating and implementing a curricular innovation whose primary scenario is the Euclidian geometry course that takes place in the second semester of a pre-service teacher program at Universidad Pedagógica Nacional (Colombia). The innovation aims to deliberately support students' learning to prove and seeks to have students conform an ample idea of what proving activity consists of (Perry, Samper, Camargo, Echeverry, & Molina, 2008). Presently, we are interested in finding different types of evidence that will permit us to evaluate, in a long term, the effectiveness of the curriculum design and development achieved with the innovation. We start by undertaking the evaluation of the students' performance in specific tasks that were assigned in the third semester geometry course.

The purpose of this paper is to analyze and evaluate some excerpts of the proving activity displayed by a group of three students when they solve a given problem without the teacher's intervention. In the task, a conjecture must be found, based on a dynamic geometry exploration, and justified deductively. In order to analyze the students' proving activity, we shall use the integrated model presented in Boero et al. (2010). To evaluate the analyzed activity, we shall consider a list of key actions, that we designed in the light of the model, which we consider conform a successful

performance; naturally, the actions are coherent with the learning goals set in the innovation and the learning experiences promoted by it. Thus, we first contextualize the study indicating what we mean by proving activity and by learning to prove, and we mention three key characteristics of the methodological approach to teaching used in the course. Secondly, we expose our interpretation of Boero and his colleagues' model which will guide our analysis. Thirdly, we describe aspects of the experimental design; we include information about the course the students belonged to, state the problem we proposed, present some components of a successful performance to which we compare the students' proving activity, and give details of the data treatment for the analysis. Fourthly, we analyze the evidence that provides elements to evaluate their proving activity. Finally, we expose the evaluation.

### STUDY CONTEXTUALIZATION

The purpose of our innovation is to support learning to prove. Thus we promote student participation in *proving activity* that is carried out as a means to develop the geometric course content. For us, *proving activity* includes two processes, not necessarily independent or separate. The first process consists of actions that support the production of a conjecture; these actions generally begin with the computer-based exploration of a geometric situation to seek regularities, followed by the formulation of conjectures and the respective verification that the geometric fact enounced is true. Hereafter, the actions of the second process are concentrated on the search and organization of ideas that will become a *proof*. This last term refers to an argument of deductive nature based on a reference theoretical system in which the proven statement can be a theorem (Mariotti, 1997). *Learning to prove* is a process through which students gradually become more able to participate in proving activity in a genuine (i.e., voluntarily assuming their role in achieving the enterprise set in the course), autonomous (i.e., activating their resources to justify their own interventions and to understand those given by other members of the class community), and relevant form (i.e., making related contributions that are useful even if erroneous).

Three characteristics of our methodological approach to teach proof and proving are the key roles of: (i) the student *geometrical problem solutions* as a means to provide elements that contribute to the development of the course content; (ii) the *interaction* between teacher and students or among students to develop the course content and to support individual learning; (iii) the use of a *software of dynamic geometry* (e.g., Cabri) in the feasibility of an autonomous, genuine and relevant student participation; this resource provides them with an environment in which actions such as empirical exploration, communication and validation of statements are propitiated.

## ANALYTICAL TOOL

Boero's et al. (2010) integration of the Toulmin argumentation model and the Habermas rational behavior model highlights some of the elements that must be articulated to face the complexity of proof; we therefore find it useful for our purpose. In what follows, we present our interpretation of their proposal.

**Type of argument:** Accepting that proving activity involves arguments of different nature, describing it requires focusing on the different types of arguments (i.e. inductive, deductive, abductive) that students formulate during the problem solving process. Every argument, according to Toulmin, has three basic components<sup>1</sup>: a statement whose validity is argued by someone (claim), premises that motivate concluding the claim (data) and the statement considered as valid that connects data to claim (warrant). More precisely, the analysis is centered on how the three components are connected, that is, what the structure of the argument is, because our methodological approach induces it and requires it.

**Teleological aspect:** Considering proving activity as a special case of problem solving, an important part of it is focusing on the goal which must be reached, so that the different actions carried out have a clear purpose. Also included in this aspect are the formulation of a plan to reach the goal, the determination of the strategies that can contribute to following it and reaching the goal, and the control of the latter.

**Epistemic aspect:** Considering proof as an object that must satisfy epistemic requirements established by the community of mathematical discourse in which it is being constructed or presented, when describing it and evaluating it, the focus is on if there is or not conscious validation of the statements, taking into account shared premises and legitimate forms of reasoning.

**Communicative aspect:** Considering proving activity as a sociocultural practice, it is natural to take into account the care students have in the way they communicate their arguments, and how conscious they are of the elements, associated to proof, that affect communication.

## EXPERIMENTAL DESIGN

### The problem proposed to the students

In a one and a half hour class session, the students, in groups of three, worked collaboratively on the following problem. As usual, they were asked to hand in a group document that reports: details of the Cabri construction and exploration, the conjecture formulated as the result of empirical exploration, and its proof.

With Cabri, construct a circle with center  $C$  and a fixed point  $P$  in its interior. For which chord  $AB$  of the circle, that contains point  $P$ , is the product  $AP \times PB$  maximum?

The teacher informed the students that they had to work without his intervention. The teacher and the members of the research group acted as non participative observers in some of the student groups, with the intention of registering in video the solution process and intervening, only if necessary, to favor the exposition of ideas by students and thus obtain as complete information of the process as possible.

### **The students and the course content**

The students were registered in the third course of the geometry trend of the pre-service program. They had participated of our methodological approach since their first semester. With respect to the geometrical content covered up to the moment of the proposed task, congruency of triangles, parallelism of lines, and quadrilaterals were given a thorough treatment. With respect to similarity of triangles, the definition, the criteria to determine it, and theorems such as Ceva and Menelao were established. The students had experience in proving properties that are deduced from the similarity of two triangles, and in using the similarity to prove other geometrical properties. The existence of chords, diameters and secant lines was discussed from a theoretical point of view. The *Theorem of the interior point of a circle*, which establishes that a line which contains a point in the circle's interior, intersects it in two points, was proved. The special relations between angles and circles had not been studied. Precisely, with the proposed problem these were expected to be introduced. In this article, we concentrate on the analysis of the activity of just one group (henceforth NAF) that, for the purpose of this research, had no special characteristics with respect to the other groups.

### **Components of a successful performance**

The students perform relevant intentional actions towards the final goal or the recognizable sub-goals throughout the solution process (teleological aspect), such as: modeling the situation in Cabri appropriately; exploring by dragging and measuring; detecting the regularity; producing different and relevant types of arguments (inductive, abductive, deductive) in the different phases of the solution process; enriching the figure with an auxiliary construction, if necessary, to favor a search process of key ideas for the proof. Specifically, to solve the proposed problem, the students construct another chord containing point  $P$  to verify that the result obtained with the first chord is also true for the second one or as a mechanism to prove the thesis. They use the two chords to determine two triangles, visualize or conjecture their similarity, establish a path to obtain the equality of the two products involved, and justify such invariance within the theory of similar triangles. They also carry out empirical explorations with Cabri to identify the corresponding angles that are congruent, and to discover the theorem that establishes that inscribed angles that subtend the same arc are congruent, that can be used provisionally as a justification since it is not yet part of their theoretical system.



## Working Group 1

The students perform general actions associated with the requirements of the mathematical discourse of proof related to the production of appropriate arguments in the different phases of the solution process (epistemic aspect): every statement that is part of the arguments must be justified; every justification must come from the theoretical system in which they are working; the representation system used provides information only based on the conventions previously established in class; a warrant can be used to obtain a conclusion only if the conditions required in its antecedent have been established before; in a deductive chain the premises change their operative status (i.e., a premise obtained as a conclusion, in one step, can be data in a posterior step); between two deductive chains that lead from the same premises to the same conclusion, the one that presents a simpler path is preferred. Specifically, they look for possible warrants to validate the congruency between at least two pairs of angles, and recognize that in the theory available to them they can only show the congruency of the vertical angles.

The students communicate their ideas carefully: formulating the conjecture as a conditional statement; reformulating the conjecture, if needed, to facilitate the construction of the proof; using the terminology established in the classroom appropriately; and using the format established in class to expose their final proof.

### **Data treatment for the analysis**

The video of the group's work was transcribed, and the observer's figures and notes were included in the margins, so that reading the transcript permitted following the students' detailed activity comprehensively. The transcription was divided in phases, each one covering an important sub-activity of the complete process. The different types of arguments were identified, typified and outlined, and the interventions analyzed to determine signs of the other three aspects of the integrated model. Due to space limitations, we shall present emblematic episodes that well represent the activity we are evaluating.

It is necessary to make two comments. Firstly, when the data for the study was collected, using the model to analyze the activity was not part of the plan; therefore, no questions were designed to promote student allusions to the epistemic and teleological aspects. Secondly, we are not analyzing finished reports made retrospectively, but student conversation when carrying out the task. Thus the arguments are mostly a collective construction, although, occasionally, the observer's questions impel one of the group members to synthesize the discussion and thus assume the responsibility of exposing the co-constructed argument. This is why we evaluate the group's proving activity and not that of individuals.

**EVIDENCE TO EVALUATE NAF'S PROVING ACTIVITY****NAF detects that the product is constant, writes the conjecture and sketches a way to prove it**

Alejandro reports the result of their first exploration: "I am measuring  $\overline{AP}$  and  $\overline{PB}$  to multiply them and check if the maximum is when it is a diameter, or if it is in some other place [...] The product remains the same always [...] even if the measure of the chords change; the product will be the same rotate it wherever we rotate it."

In this fragment, we see signs of an inductive argument. The students, using Cabri, generate innumerable positions of chord  $\overline{AB}$  together with the respective products  $AP \times PB$  and thereof detect the invariance. The premises that provide evidence (data) to affirm that the product is constant (claim) come from the conditions found in the problem statement and of the numerous cases that are offered by dragging the chord. The warrant is the conjecture that suggests that for any chord that contains  $P$ , the product is constant. On the other hand, in Alejandro's verbalization we find an initial plan to answer the question asked in the problem, plan that he carries out in Cabri as he talks, and that is evidence of the presence of the teleological aspect.

The students become involved in writing the conjecture as a conditional statement. To start with they mention the if-then format as the proper one to express the conjecture. With Fabian's intervention: "Shall we put given or must we construct it?" they evaluate if they can assume the existence of chord  $AB$  as given or if they must include, in their proof, statements and justifications that theoretically validate the construction of the chord. Afterwards, they agree on a first statement: "*Given a circle with center  $C$ , a fixed point  $P$  which belongs to the interior and a given chord  $AB$  which contains  $P$ , then the product  $AP \times BP$  is constant.*" However, Nancy manifests inconformity: "No, look, you know what? ... It's better, given a circle with center  $C$  and a fixed point  $P$  such that  $P$  belongs to the interior of the circle, for **any** chord  $AB$  of circle  $C$  such that  $P$  belongs to  $AB$ , then the product such and such." Alejandro points out that the purpose of changing the word is to bring out the generality of the fact: "...we had said **one**, only one chord; then, it was saying that only one chord  $AB$  exists; now we are saying that for any chord that passes through point  $P$ , then it's going to be constant."

The epistemic aspect appears when they ask themselves if they must justify the existence of chord  $AB$ ; this suggests that they see the difference between a chord that exists, because it is given in the statement's premises, and a chord whose existence is justified theoretically; they are obeying the class norm of justifying the existence of the geometric objects that are being used. We also see the communicative aspect when they write their conjecture; on the one hand, because they know they have to formulate it as a conditional and, on the other hand, because they note that their first formulation is incorrect since it does not express the detected generality, reason why they include the universal quantifier.

When they are rewriting the conjecture, Alejandro asks: “In the proof we can construct another chord, right? To have similar triangles.” and explains, “...because what we need to prove is a ratio.” When the observer asks him the reason for the auxiliary construction, he answers “Because we see that the product will always be the same, right? Then another chord can give us similarity or ratio between this side, the segment that we would create new and this...” Nancy amplifies Alejandro’s idea: “We would have that the ratio of... or that by ratios we get that  $AP$  times  $BP$  is going to be the same for both chords. That way we would confirm that it would be for **any** chord... not only the given one but also the one we compare it with.”

NAF has set the goal: proving that the product is constant for any chord. This leads them to construct two chords that contain  $P$  with the purpose of obtaining similar triangles, to thereof work with ratios that will lead to equal products. This goal motivates an auxiliary construction without which, as we know, it is practically impossible to prove the conjecture. We recognize the teleological aspect in the conversation because they sketch a plan to reach their goal and propose an auxiliary construction as a tool to obtain it.

### **NAF examines how to justify the existence of another chord**

As they start writing the proof, they consider how to justify theoretically the existence of both chords that contain  $P$ , which they have represented in Cabri. They establish that the first chord is given and that they only have to justify the existence of the other chord. Alejandro points out that they must guarantee the existence of a line through  $P$ , maybe motivated by the construction done in Cabri. Nancy mentions that the line must also contain a point of the circle. Fabián says: “To create the line we need the Line Postulate<sup>2</sup> and it requires the two points. What shall we do?”

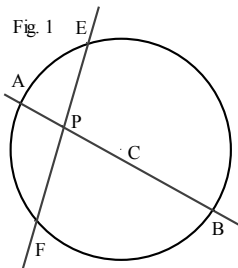
In the summarized interchange, the existence of the other chord containing  $P$  can be seen as the claim of a possible abductive argument that does not take shape because the warrant is not explicit (i.e., the chord is a subset of the line). In contrast, the existence of the line containing  $P$  (and not the given chord) becomes the claim of an abductive argument when Nancy indicates the necessity of having two points (data) and Fabián completes it by mentioning the Line Postulate (warrant). The goal they establish, due to this argument, is to justify the existence of the two points; it guides their next actions. Thus the teleological aspect is present. With respect to the epistemic aspect, it is worth noting that in the first case the warrant is not mentioned while in the second it is.

Trying to justify the existence of two points that determine the line whose existence they want to show, Nancy suggests using the Interior Point of a Circle Theorem, and, due to Alejandro’s petition, she says: “if we have a circle and a point of it and a line, ah! no, but we need a line anyway, that is we have to construct it.” When Nancy discards this possibility, Alejandro proposes a plan: “The best would be to construct

a line by the Line theorem [Postulate] and then we can say that the points intersect the circle.” NAF eventually realizes that neither way is useful for their purpose.

The claim of the argument in the conversation is the existence of two points,  $P$  and a point on the circle; the failed warrants are the Interior Point of a Circle Theorem and the Line Postulate together with its intersection with the circle. The data required is the existence of a line that intersects the circumference in two points. The epistemic aspect is evidenced when Nancy realizes that they do not have the elements of the hypothesis of the theorem they want to use, and that, therefore, they must discard it.

### NAF discards the established path to validate the existence of a pair of congruent angles



The students have two chords that contain  $P$  (Fig. 1). Alejandro declares the congruency of  $\angle FPB$  and  $\angle APE$  because they are vertical angles. Nancy questions: “And, from there, where are we going?” As an answer, Fabián proposes the following plan: “We construct the triangles and then we talk about the angles to talk about similarity.” Alejandro adds that with the Angle-Angle Criteria they would already have similarity. Nancy objects:

“And, where is the other [pair] angle? Okay, we already have these two angles, and the others, where are we going to get them? We need another one.” Once they have the triangles, Fabián discards parallelism as a way to reach their goal: “There are no parallels. Because if we had them, this would facilitate finding alternate interior angles, and we already have the other angle and similarity would be the result.”

We find two abductive arguments. In the first one, the claim is the existence of similar triangles, the warrant the Angle-Angle Similarity Criteria, and the data required the congruency of two pairs of angles, of which one is already guaranteed. In the second one, the claim is the existence of another pair of congruent angles, the warrant is the theorem that guarantees that alternate interior angles between parallel lines are congruent, and the data required is assuring that two lines are parallel.

We point out three issues. Firstly, we can see the control Nancy exerts over the trend of the activity they are developing, sign that she is conscious, on the one hand, of the necessity of not losing sight of what they want to justify and, on the other hand, of the class norm of justifying every statement in the context of the situation they are studying. Nancy’s interventions —the first one of teleological nature and the second of epistemic nature— lead them to formulate a plan or discard a possible path. Secondly, Fabián’s argument, with which he discards parallelism as useful to reach their goal, makes us think that he tacitly assumes that the congruent angles are  $\angle EBA$  and  $\angle FAB$  or  $\angle BEF$  and  $\angle AFE$ , which is not correct. Thirdly, we are surprised that, during a good part of their activity, NAF refers to similar triangles without explicitly establishing the correspondence for the similarity. Maybe they could have established it much sooner than when they actually did if they had allowed

themselves to explore the situation numerically with Cabri (angle measurements, calculating the proportion) as Fabián suggested: “The only thing we can do is measure this one with..., the ones that are going to determine, form similarity and see whether that proportionality remains in the three [angles] and then, with that we can then determine that they [the triangles] are similar.” Nancy responds: “[...] we find measures and that, but how do we find it here... geometrically?”, and Fabián accepts the veiled objection: “With the postulates and that.”

### **NAF determines the proportionality of the measurements of the sides and the congruent angles**

After various failures in trying to find the way to validate the congruency of two angles, without indicating exactly which pair they are referring to, Alejandro turns to the observer: “Help us. How do we relate another angle?” She responds: “Are you sure that the triangles are similar?” Nancy explains why they are similar: “[...] they are similar, not because of the angles but... because we have this angle [referring to the vertical angles] and as Alejandro showed, having  $AP$  times  $PB$  is equal to  $EP$  times  $PF$  [she writes  $AP \times PB = EP \times PF$ ] we can make our proportion [...] Then  $AP$  is to  $PF$  as  $EP$  is to  $PB$  [writes  $\frac{AP}{PF} = \frac{EP}{PB}$ ] and that way we have it.” Fabián asks surprised: “ $EP$  to  $PB$ ? ... This segment is to this segment ... which triangles are you talking about?” Nancy responds: “Of triangles  $FPB$  and  $APE$  because as you superimpose [moves hand] let’s say this one [signals  $\triangle APE$ ] over this one [signals  $\triangle FPB$ ] we have that this [shows  $\overline{PE}$ ] is to this one [points to  $\overline{PB}$ ] as this [shows  $\overline{AP}$ ] is to this [points to  $\overline{PF}$ ].” Alejandro adds: “In the calculator I looked at angle  $PAE$  [...] and angle  $PFB$  and they are congruent.”

To explain why the triangles are in fact similar (claim), Nancy recurs to the Side-Angle-Side Similarity Criteria (warrant) which she does not mention. She takes into account the empirically found fact when exploring the situation: the constant product of the measures of the segments determined by point  $P$  on each chord of the circle (data). Using that fact, she obtains the proportionality of the measurements (intermediate claim). Thus we see that with the purpose of justifying the similarity of triangles  $FPB$  and  $APE$ , Nancy carries out a deductive argument. It must be noted that the students are conscious that such reasoning is not the adequate one for the situation they are tackling because they use what they want to prove.

### **CONCLUSIONS**

As we compare the description of NAF’s activity with the components established for a successful performance, we recognize that NAF sets four sub-goals that lead them to the expected goal: detect regularity; formulate a conjecture in the terms used by the class community; justify the auxiliary construction of a chord; and prove the congruency of two pairs of corresponding angles. Intentional actions to reach the sub-goals are evidenced although proving that the triangles are similar is not

## Working Group 1

proposed with the precision required and desired. Only after more than an hour, does NAF discover which are the corresponding angles that guarantee the similarity, and they never find the relation between the angles and the circle which would permit justifying their congruency. Maybe with this deficiency in their proving activity, many of their actions to search for the justification could be considered as not relevant. Related to the former is the fact that NAF prefers using the theory as resource more than the empirically obtained information; this shows an undesired unbalance between exploratory actions and justification actions. Yet, the abductive arguments that arise show that NAF has enough knowledge of the theme to allow them to make connections that are not incongruent with the situation they are studying; these arguments impulse and guide their actions. Although in the whole process we evidence skill in handling the teleological, epistemic and communicative aspects, they still lack the mastery needed to perform as an expert. Or maybe we cannot expect the students to act as if they were already writing a report of successful arguments, in the course of the proving activity process.

## NOTES

1 In fact, Toulmin's model presents six components of an argument: claim, data, warrant, backing, qualifier and rebuttal.

2 The Line Postulate states that given two points there exists a unique line containing them.

## REFERENCES

- Boero, P., Douek, N., Morselli, F., & Pedemonte, B. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. In M.F.F. Pinto & T.F. Kawasaki (Eds.), *Proceedings of the PME 34* (vol. 1, pp. 179-205). Belo Horizonte, Brazil: PME.
- Mariotti, M.A., Bartolini Bussi, M.G., Boero, P., Ferri, F., & Garuti, R. (1997). Approaching geometry theorems in contexts: From history and epistemology to cognition. In E. Pehkonen (Ed.), *Proceedings of the PME21* (vol. 1, pp. 180-195). Lahti, Finland: PME.
- Perry, P., Samper, C., Camargo, L., Echeverry, A., & Molina, Ó. (2008). Innovación en la enseñanza de la demostración en un curso de geometría para formación inicial de profesores. In *Electronic book of the XVII Simposio Iberoamericano de Enseñanza de las Matemáticas: "Innovando la enseñanza de las matemáticas"*. Toluca, México: Universidad Autónoma del Estado de México.
- Selden, A. (2010). Transitions and Proof and Proving at Tertiary Level. Department of Mathematics, New Mexico State University, Tech Report No. 2010-3. Retrieved from: [http://www.tntech.edu/math/reports//images/stories/math/files/TR\\_2010\\_3.pdf](http://www.tntech.edu/math/reports//images/stories/math/files/TR_2010_3.pdf)

# ON THE ROLE OF LOOKING BACK AT PROVING PROCESSES IN SCHOOL MATHEMATICS: FOCUSING ON ARGUMENTATION

Yosuke Tsujiyama

Graduate School of Comprehensive Human Sciences, University of Tsukuba, Japan

*The purpose of this paper is to theoretically examine the role of looking back at proof-planning processes in school mathematics from the perspective of argumentation. Various research studies have elaborated the significance of proof and proving in school mathematics. A particular focus of this paper is on what students can gain by looking back at proving processes that do not appear explicitly in the presentation of proof. Examples of hypothetical proof-planning processes are chosen to illustrate that two kinds of ideas can occur by looking back at the processes. In particular, the ideas are related to how to identify why the statement is true and how to prove the initial statement and another related statement.*

Keywords: proving, looking back, proof-planning, argumentation, changing condition

## INTRODUCTION

Proof and proving should be central to students' school mathematical experience (Yackel & Hanna, 2003). Many research studies have addressed to the question: what students can/should gain through proof and proving. Several studies among them focus on various functions of proof from a theoretical perspective (de Villiers, 1990; Hanna, 1990) or using empirical evidence (Miyazaki, 2000). In addition to the issue of verification of truth of a statement, these studies have pointed out the need to investigate what roles proof can/should play, and how we can apply such functions.

On the other hand, more recently, Hanna & Barbeau (2008) have focused on another role of proof, inspired by Rav's (1999) philosophical work, as a way to convey mathematical methods, tools, strategies and concepts. Their discussion is based on what has been underestimated in the mathematics education literature when this literature "seem to have dealt primarily with the logical aspects of proof and with the problems encountered in having students follow deductive arguments" (Hanna and Barbeau, 2008, p.347).

As to the discussion on the functions, Hanna & Barbeau emphasize the role of proving as a process, over the role of proof as a product. For example, Hanna and Barbeau describe how students can obtain the technique of 'completing the square' by illustrating how it can appear in proving the formula for the solution of a quadratic equation. Then they mention that this technique "does not stem logically from a previous statement or axiom" but "is a topic-specific move" and useful for their on-going mathematical learning (ibid, p.349).

This study focuses on the educational significance of proving processes. Moreover, the motive of this study is to design teaching material to promote improvement in students' proving abilities and habits and to encourage further mathematical inquiry from the side of the students by reflecting on past proving processes. For example, students will try out various approaches in a process of proving. If they are promoted to reflect the past process to see how well each approach worked or not, they can be expected to obtain typical ways of proving which are useful not only for the statement at hand but also for many other statements which they will meet in the future. Since many students have difficulties in reaching ideas concerning how to establish a proof, acquisition of such ways can be considered valuable. However, the value of promoting students' reflection on the past processes to extract general ways of proving has been underestimated in the existing related literature. Therefore, it is necessary to answer the question: what can students gain through proving processes? This paper examines this question theoretically as a platform for future empirical studies.

## **LOOKING BACK AT PROOF-PLANNING PROCESSES**

### **Literature review and research questions**

To gain something through proving processes, students need at least to reflect and clarify what they have done in the processes. In relation to these activities, Polya (1957/2004) refers to the act of "looking back", which involves not only checking errors or the insufficiency of one's own activities or their outcomes, but also finding new problems or ideas based on the activities or outcomes (pp.14-16).

About the aspects of looking back, checking errors and identifying insufficiency are discussed in Heinze et al. (2008) in terms of "coordination". In addition, the studies on the roles and functions of proof (e.g., de Villiers, 1990; Hanna, 1990) discuss finding new problems or ideas based on a product of proof, although they do not use the term of looking back. Tsujiyama (2010) also analyses how mathematicians from the ancient Greek era and beyond could find, by examining proposed proofs, not only aspects of insufficiency in the proofs but also ideas which were applicable to other related proofs.

These studies focus on the part of proving processes related to a product. In contrast, Hanna & Barbeau (2008) focus on obtaining general ideas based on processes of proving rather than products. To reconcile the perspectives appearing in the literature, this paper focuses on looking back at processes that are not evident in a product.

Among other aspects, this paper especially focuses on proof-planning processes. Polya (1957/2004) listed the following four phases of proving: understanding a problem; devising a plan to establish a proof (this paper refers to this as "proof-planning"); constructing a proof; and looking back. Shimizu (1994) proposed, through a hypothetical process, that proof-planning in particular involves important



ideas that are not apparent from a product. Further, Thurston (1994) states that informal elements occur even when professional mathematicians are engaged in producing a proof and these elements promote their understanding of proof and their on-going work.

Therefore, this paper proposes to examine the following research questions: “by looking back at proof-planning processes, what kinds of new problems or ideas can students obtain?”; “what is the significance of the problems or ideas?”

### **Ways of proof-planning**

Proof-planning is to find deductive connections between the statement and other known theorems, in order to establish a proof of the statement. Nevertheless, in processes of proof-planning, even mathematicians do not necessarily think only in a formal or deductive way. In order to capture students’ activities, we need to widen our awareness of different ways of proof-planning. This paper takes a wide perspective in analysing ways of proof-planning by distinguishing three components: order (i.e. how to proceed from the premise to the conclusion); objective (i.e. the statement to be proved); and manner (e.g. formality).

First, about order, it is extensively pointed out in the literature that not only forward reasoning but also backward reasoning are useful in proof-planning. In recent years, Heinze et al. (2008) have analysed complex processes of constructing multi-step proof from the logical point of view. They focus on how to deal with intermediary conditions/conclusions obtained by backward/forward reasoning; then they point out an essential aspect of proof competence that they call “coordination” (p.445). According to their study, this kind of looking back during a proof-planning process plays a role in choosing the most propitious deduction available.

For the second component, Polya also suggests that changing the statement can be useful in proof-planning. When backward and forward reasoning do not work well, it is important to find special, general or analogous problems and “try to solve first some related problem” (Polya, 1957/2004, p.10) by changing the initial statement. Proof-planning by changing the condition is important for the following two reasons. First, especially in geometry, students tend to spontaneously consider only special cases. For example, when students prove the statement: “diagonals in any parallelogram intersect at their midpoints”, they usually draw diagrams and judge if the statement seems to be true or not. Intentionally or not, through such activities, they may consider special cases such as a rhombus, rectangle or square.

Another reason to consider changing the condition is that it is an effective way to find a useful problem for solving the initial problem in mathematics. Particularly, in regard to specialization, Polya (1954) refers to using “a leading special case”: restricting the initial problem to a simpler case; after, utilizing a solution of the simpler case in order to solve the initial problem (p.25). This kind of proof-planning

is seen in several articles but its value has been underestimated in mathematics education literature.

For example, Hanna and Barbeau (2008), note that students can extract the technique often referred to as ‘completing the square’ through considering how to obtain the formula for the solution of the quadratic equation  $ax^2 + bx + c = 0$ . In the discussion, they refer to an instance where students restrict their attention to an easy case of the equation,  $x^2 = k$  where  $k$  is positive. Then the students consider why it is easy to solve  $x^2 = k$  and realize that the key lies in the absence of the linear term (pp.348-349). Starting with this illustration, Hanna and Barbeau argue that the technique of completing the square that students possibly can obtain through the above process “does not stem logically from a previous statement or axiom” (ibid, p.349). From the focus of this paper, it is remarkable that neither the equation  $x^2 = k$  nor its solution appears explicitly in the presentation of proof. Moreover, in the discussion, Hanna and Barbeau do not analyse or even mention the role of the specialization to  $x^2 = k$ .

The third component deals with ‘manner’. In contrast to a formal manner towards a product of proof, processes of proof-planning have informal aspects. Especially in geometry, students can use a diagram for proof-planning. For example, Douek (1999) considers the references used in proof and proving as involving not only mathematical theory but also diagrams or visual evidence and other kinds of representation. Using empirical evidence she points out that semantically routed arguments including diagrams or numerical examples play an important role in proving.

Thus, this paper focuses on the following three ways of proof-planning: backward/forward reasoning; changing the condition; referring to diagrams.

## **PROOF-PLANNING PROCESSES BASED ON ARGUMENTATION**

Since students do not know how the product of proof should be, they may make errors in directing their proof-planning processes. Thus, in analysing proof-planning processes, it is necessary to consider that the processes may have plausibility. That means, an intermediary condition/conclusion obtained by backward/forward reasoning is not necessarily used in the proof; reasoning obtained under the changed condition is not necessarily applicable for the initial condition; an idea found based on diagrams does not necessarily work generally. Therefore, this paper focuses on the concept of argumentation, that is, a process of making arguments that may include plausibility and of examining already made arguments. An argument is logically connected (but not necessarily deductive or formal) reasoning (Douek, 1999).

In this section, first, we briefly summarize the literature on proof/proving and argumentation. Second, we characterize proof-planning processes according to the nature of argumentation. Third, based on the characterization, we illustrate two kinds

of hypothetical proof-planning processes. This illustration is used to discuss the role of looking back at proving processes in the next section.

### **Proof/proving and argumentation**

Based on mathematicians' accounts of informal elements as made by Lakatos (1976) or Thurston (1994), studies on proof/proving and argumentation mainly analyse how students can (or cannot) construct a proof by utilizing argumentative activities. Several studies point out connections between argumentation and proof (e.g. Douek, 1999), while other studies point out the gap between them (e.g. Balacheff, 1991). Prompted by the findings of these studies, Pedemonte (2007) focuses not only on "content" but also on "structure" and introduces Toulmin's (1958/2003) "layout of arguments" for a methodological tool to analyse relationships between them.

These studies focus not only on processes of making arguments that may include plausible content or structure, but also on processes that can remove the original plausibility. In addition, for the aims of this paper, we need to focus on reasoning that does not necessarily appear in a product, for example reasoning that students use in order to obtain an idea of proof but do not use in order to present a proof. Therefore, this paper re-examines the nature of argumentation and, by synthesizing findings in the above literature, it characterizes proof-planning processes.

### **Characteristics of proof-planning from the viewpoint of argumentation**

#### *Making an argument based on observations concerning the conclusion*

The studies on proof/proving and argumentation mainly analyse contents (e.g. reference to diagrams) and structures (e.g. abductive reasoning) (Pedemonte, 2007). What is common to both is an emphasis on observations concerning a conjecture. For example, Pedemonte mentions the role of "abductive argumentation", which "allows the construction of a claim starting from an observed fact" (Pedemonte, 2007, p.29), and some difficulties related to it. Toulmin (1958/2003) also emphasizes such observations and regards that an assertion that is going to be claimed comes first. Thus, one aspect of argumentation is to make an argument based on observations concerning the conclusion. Even if the proposed statement might be false in general, the foci of the literature are on the processes of justifying a conjectured "fact". Therefore, this aspect of argumentation is only suitable for the case when the conclusion holds.

Thus, the first characteristic of proof-planning processes is: (i) to make an argument for which a claim holds based on observations.

#### *Uncertainty*

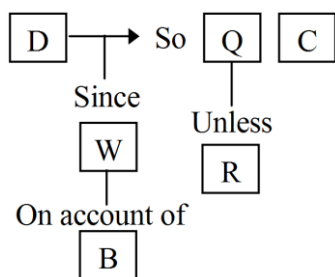
Toulmin's "layout of arguments" consists of six related components of an argument: three elements "Data", "Warrant" and "Claim" which have similar functions to minor premise, major premise and conclusion in syllogism; and three more elements "Rebuttal", and "modal Qualifier" which are used for ordinary arguments (Figure 1

uses abbreviations by their initials). Since the framework can capture logical and plausible aspects at the same time, it is an effective tool for the aim of this paper. In particular modal qualifier functions to show uncertainty of an argument. This element is necessary since we sometimes have to make an argument by limited data or warrants (Toulmin, 1958/2003, pp.93-94).

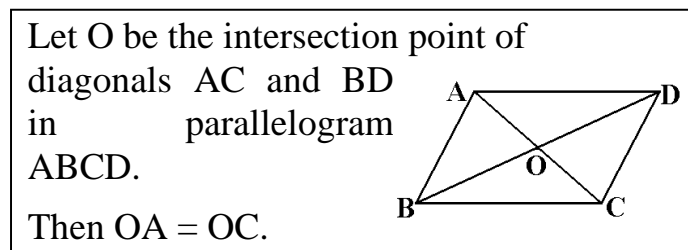
Thus, the second characteristic is: (ii) to make an argument which may include uncertainty.

### *Adding the condition*

Rebuttals represent “conditions of exception” and are used when one notices exceptional cases or conditions under which the data and the warrant cannot implicate



**Figure 1** Layout of arguments  
(Toulmin, 1958/2003)



**Figure 2** Statement  $\alpha'$

the claim (ibid, p.93). Since such cases are special cases, one needs to consider several cases and check if the claim holds or not for each case. This consideration is necessary because even having certain data and warrant sometimes cannot implicate a claim without a suitable connection between them.

Thus, the third characteristic is: (iii) based on checking special cases or conditions under which a claim does not hold, to make an argument that takes in consideration exceptional cases or conditions. In this paper, “unless” in Figure 1 is replaced by “under the condition”, in order to allow usage also in an affirmative form.

### **Hypothetical proof-planning processes**

To illustrate hypothetical proof-planning processes based on the characteristics (i)-(iii) above, we consider Statement  $\alpha$ , “diagonals in any parallelogram intersect at their midpoint”. We neglect possible issues in understanding the statement and assume that students ‘translated’ Statement  $\alpha$  to Statement  $\alpha'$  with the diagram shown in Figure 2 leaving the demonstration that  $OB = OD$  for later.

In Japan, 8<sup>th</sup> graders are expected to be able to tackle proving Statement  $\alpha$ . Students have already proved several statements such as “opposite sides in any parallelogram are equal” or “opposite angles in any parallelogram are equal” based on properties concerning parallel lines, alternate angles and triangle congruency. The key idea in proof-planning of Statement  $\alpha'$  is to find an appropriate pair of congruent triangles.

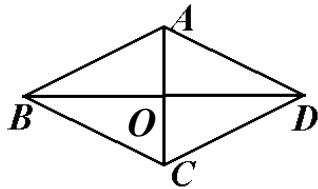
## Working Group 1

If students use backward/forward reasoning successfully, they can find a pair of triangles ADO and CBO or ABO and CDO. However, many of the students seem to have difficulties in processing the reasoning. These difficulties seem to be related to the lack of students' experience of working with such complex diagrams.

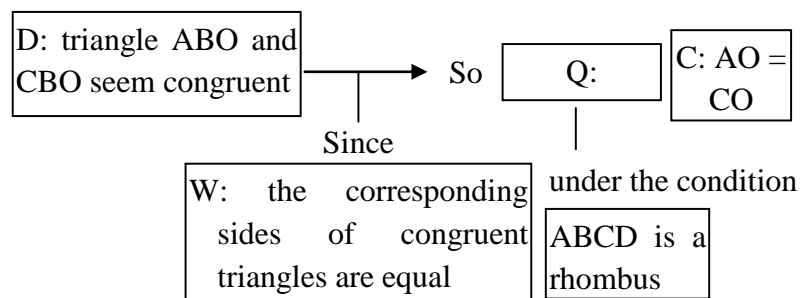
Thus, we assume that students could not find an appropriate pair of triangles; instead they refer to the diagram and try to change the condition. Students will consider several cases, for example rhombus, rectangle, square, trapezoid and so on. The conclusion holds for some cases, but it does not for other cases.

### *On examining cases for which the conclusion holds*

In the case of a rhombus, students can easily see many pairs of triangles that seem congruent by referring to a diagram. If they focus on a pair of triangles ABO and CDO (or ABO and ADO and so on), they can verify that the triangles are congruent by the



**Figure 3** Diagram in the case of a rhombus



**Figure 4** Argument in the case of a rhombus

ASA congruency condition by backward/forward reasoning. On the other hand, if they focus on a pair of triangles ABO and CBO (or ADO and CDO), they will find that none of the congruency conditions SAS, ASA and SSS can be applied.

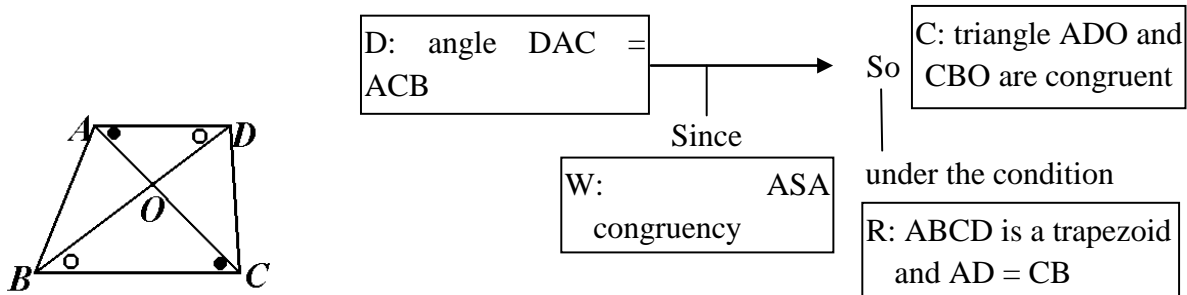
In the latter case, students do not have enough information to make a deduction to show the property “triangles ABO and CBO are congruent” (i.e. they only know that  $AB = CB$ ). At this time, students would be uncertain whether they can deduce the property. However, if they consider based on observations that the property ‘probably’ holds, they can make an argument as shown in Figure 4, due to the characteristics (i) and (ii) of proof-planning.

By changing the condition back to the general case of a parallelogram, students can check whether this argument holds for the case of parallelogram. Then they will confirm that neither triangle ABO and CBO nor ABO and ADO are generally congruent, but they will also feel that triangle ADO and CBO seem to be congruent. Thus they arrive at the key idea in proof-planning of Statement  $\alpha'$ .

### *On examining cases for which the conclusion does not hold*

If students consider a trapezoid, they can easily find, by referring to a diagram, that the conclusion does not generally hold. However, due to the characteristic (iii) of

proof-planning, they can look for an additional condition which makes the conclusion “ $OA = OC$ ” hold. If they consider “ $AB \parallel CD$ ” as such a condition, they will find that this idea does not help. On the other hand, if students notice the definition of a trapezoid and the relationship between parallel lines and alternate angles, they can argue “ $\angle DAC = \angle ACB$ ”, as well as “ $\angle ADB = \angle DBC$ ” (Figure 5).



**Figure 5** Diagram in the case of a trapezoid

**Figure 6** Argument in the case of a trapezoid

In the latter case, they have found out that two angles of the triangles  $ADO$  and  $CBO$  are equal. By backward reasoning, they would find out that, if the condition “ $AD = CB$ ” is added, the conclusion “ $OA = OC$ ” holds by ASA. Due to the characteristic (iii) of proof planning, they can make an argument on the lines indicated in Figure 6.

By changing the condition back to the case of a parallelogram, students will notice that “ $AD = CB$ ” holds by the known proposition “opposite sides in any parallelogram are equal”. Thus, they can deduce that triangle  $ADO$  and  $CBO$  are congruent.

## THE ROLE OF LOOKING BACK AT PROOF-PLANNING PROCESSES

Even if students manage to construct a proof of the initial Statement  $\alpha$ , the presentation of the proof will not show the practices concerning the above processes (e.g. Figure 4 or 6) but only show a sequence of deductive reasoning from the premise to the conclusion of Statement  $\alpha$ . Supposing that students reach a proof by going through one of the above processes, we discuss what kinds of new problems or ideas could occur by looking back at the process that does not shown in the presentation of the proof.

### The sufficient condition of the conclusion

Let us take the case of a trapezoid, for which the conclusion does not hold. By looking back at the proof-planning process related to Figure 5 and 6, students can compare the case of considering a parallelogram to the case of considering a

trapezoid, that is, cases where the conclusion holds and where it does not. Hence, they can discover a sufficient condition that ensures that the conclusion holds, as follows.

The process related to Figure 6 shows that a key condition to implicate the conclusion “ $OA = OC$ ” is “ $AD = CB$ ”. In detail, in addition to “angle  $DAC = ACB$ ” and “angle  $ADB = DBC$ ” (which hold in the case of a trapezoid as well), “ $AD = CB$ ” holds in the case of a parallelogram (which does not hold in the case of trapezoid).

The value of this sufficient condition is that students can obtain an answer to the mathematically important question: “why do diagonals intersect at their midpoints for parallelograms but not for trapezoids”. By looking back at the above process, students can answer: “because for parallelograms, not only opposite sides are parallel but also these sides have equal length”.

What has just been said was made to illustrate the explanatory role of proving processes, inspired by the explanatory role of proof as a product discussed in the literature. To utilize the explanatory role of proof, students have to identify the crucial elements raised from all the deductive connections that appear in the proof. This seems difficult for many 8<sup>th</sup> graders who are not attuned to mathematical logic. In contrast, the process illustrated above develops only for the purpose of proving Statement  $\alpha$ . To utilize the explanatory role of proving, students only have to look back at the process.

### **A statement which was not proved**

We now take the case of a rhombus, for which the conclusion holds. By looking back at the process related to Figure 4, students are able to prove Statement  $\beta$ , i.e. “in any rhombus  $ABCD$  triangles  $ABO$  and  $CBO$  are congruent”, which they considered to be ‘probably’ true but they were not able to deductively show at that time. After proving Statement  $\alpha$ , by using it they can easily prove Statement  $\beta$ .

The value of isolating and proving Statement  $\beta$  lies not only in posing new problems (cf. the discovery function of proof) but also in obtaining ideas of how to change the condition. In the above process, students obtained ideas how to prove Statement  $\alpha$  (that concerns parallelograms) by considering rhombi instead. Reversely, in looking back at the process, they can obtain ideas for proving a new situation concerning rhombi (i.e. Statement  $\beta$ ) by considering parallelograms. From both experiences, students will appreciate that changing the condition is a useful way for proof-planning for both the initial statement and changed statement.

### **CONCLUDING REMARKS**

The analysis made in this paper suggest that there are other roles of looking back at proving processes than those that appear in the related literature such as Hanna and Barbeau (2008) or studies on functions of proof. However, the analysis is not based on actual students’ practices but based on hypothetical illustration. Considering the

fact that many students have difficulties in constructing proof or looking back at a proof, it would be a great effort for them to look back at proving processes. Therefore, there is an especial need for empirical studies on how teachers can promote students' practices of looking back, and on designing situations for which students can appreciate the practice. The analysis also suggests that it can be useful to give students opportunities to compare different eventualities that can occur in proving the same statement.

## ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to Dr. Martin Downs (University of Patras) for his very detailed and significant suggestions on earlier version of this paper. I also would like to thank Dr. Viviane Durand-Guerrier (Université Montpellier 2) and Dr. Joanna Mamona-Downs (University of Patras) for their arrangements and considerations for my situation as well as the reviewers and participants in the WG for their valuable comments on my presentation. This study is supported by research grants from the Japan Society for the Promotion of Science as a Research Fellow.

## REFERENCES

- Balacheff, N. (1991). The benefits and limits of social interaction: The case of mathematical proof. In A. Bishop et al (Eds.), *Mathematical Knowledge: Its Growth through Teaching* (pp.175-192), Netherlands: Kluwer Academic Publishers.
- De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.
- Douek, N. (1999). Argumentative aspects of proving: analysis of some undergraduate mathematics students' performances. In O. Zaslavsky (Ed.), *Proceedings of the 23<sup>rd</sup> Conference of the International Group for the Psychology of Mathematics Education Vol.2* (pp.273-280), Haifa, Israel: University of Haifa.
- Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21(1), 6-13.
- Hanna, G. & Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. *ZDM The International Journal on Mathematics Education*, 40, 345-353.
- Heinze, A., Cheng, Y-H., Ufer, S., Lin, F-L., Reiss, K. (2008). Strategies to foster students' competencies in constructing multi-steps geometric proofs: teaching experiments in Taiwan and Germany. *ZDM The International Journal on Mathematics Education*, 40, 443-453.
- Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.
- Miyazaki, M. (2000). What are essential to apply the "discovery" function of proof in lower secondary mathematics? In T. Nakahara & M. Koyama (Eds.),



## Working Group 1

*Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education Vo.4* (pp.1-8). Hiroshima, Japan: Hiroshima University.

Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66(1), 23-41.

Polya, G. (1954). *Mathematics and Plausible Reasoning Vol.1 Induction and Analogy in Mathematics*. New Jersey: Princeton University Press.

Polya, G. (1957/2004). *How to Solve It: A New Aspect of Mathematical Method (Second Edition)*. New Jersey: Princeton University Press.

Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica*, 7(1), 5-41.

Shimizu, S. (1994). Demonstration. *CRECER vol.6 Geometry and Demonstration* (pp.204-236). Tokyo: Nichibun. [In Japanese]

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, 30(2), 161-177.

Toulmin, S. (1958/2003). *The Uses of Argument (Updated Edition)*. Cambridge: Cambridge University Press.

Tsujiyama, Y. (2010). Argumentative aspects of geometrical proofs: an approach to the notion of proving in school mathematics. In Y. Shimizu, Y. Sekiguchi & K. Hino (Eds.), *Proceedings of the 5th East Asia Regional Conference on Mathematics Education Vol.2* (pp.316-323), Tokyo: National Olympics Memorial Youth Center.

Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds), *A Research Companion to Principles and Standards for School Mathematics* (pp.227-236). Reston, VA: National Council of Teachers of Mathematics.

# MAKING THE DISCOVERY FUNCTION OF PROOF VISIBLE FOR UPPER SECONDARY SCHOOL STUDENTS

Kirsti Hemmi

Clas Löfwall

Mälardalen University, Sweden

Stockholm University, Sweden

*This paper presents an analysis of a teaching experiment with seven high achieving upper secondary school students in Sweden focusing on the de Villiers' discovery function of proof. The aim of the experiment was to test if it is possible for students to get insights to, and use, this function. The data consists of a tape recorded introductory pass, students' group work and the final discussion together with the students. The results show that the students did get some insights about the function. However, it was difficult for the students to construct the original proofs in order to use them to discover new results. The paper also shows that the function of discovery needs to be explored and clarified, as there are different interpretations of it in our field.*

Key words: mathematical proof, discovery function, transparency, upper secondary school mathematics

## INTRODUCTION

Many mathematics educators have explored and discussed the functions of proof in mathematics as science (e.g. conviction, explanation, intellectual challenge) and their relevance for the teaching and learning of proof (e.g. Bell, 1976; de Villiers, 1990; 1999; Hanna 2000; Weber, 2002) Several studies also apply these functions in empirical studies (e.g. Knuth, 2002; Hemmi & Löfwall, 2009). The discovery function was first presented by de Villiers 1990. With this function he refers to discovery/invention of new results by purely deductive manner exploring and analysing a proof.

To the working mathematician proof is therefore not merely a means of a *posteriori* verification, but often also a means of exploration, analysis, discovery and invention. (de Villiers, 1990, p. 21)

New results in his examples often refer to generalisation of the initial statement. The function of discovery as de Villiers defined it has not been so much in the focus of empirical research. Miyazaki (2000) conducted a teaching experiment with tasks specifically designed for students' engagement with activities connected to the discovery function. However, his concept of the function is wider than the one that de Villiers (1990; 1999) presents.

The functions of proof are sometimes interpreted in different ways by researchers. There is also some confusion about the difference between the concept of *function of proof* on the one hand and *approach to proof* on the other hand (i.e. how we work with proofs and proving and how we present mathematics) (c.f. Hemmi, 2010). For

example, Knuth (2002), although referring to de Villiers, connects the function of discovery, not to finding new truths by deductions, but quite the opposite, to inductive ways of finding patterns and making conjectures that may be followed by deductive proofs. Hence, there seems to be a need to scrutinise and clarify also the concept of function in order not to create confusion between different research results and, in the end, their consequences to the teaching practice.

De Villiers (2007) discusses the recent focus on the investigative working manners for example in the mathematics textbooks and points out that they do not promote students' understanding of and skills in using the function of discovery but leads rather to the need for verification, the function that has traditionally been in focus in mathematics education. We do not question that "new" ways to approach proof (investigations-conjectures-proofs) may enhance students' understanding and appreciation of the verification function in another way than just confronting them with complete proofs constructed and verified by others. We also agree with de Villiers (2007) when he points out the importance of the balance between experimentation and deductive thought.

The aim of our ongoing explorative study is to develop and test some tasks that could enhance students' understanding of the discovery function as defined by de Villiers (1990). We also take as our starting point that proof sometimes provides valuable insights into why something is true and that looking back and reflecting on it can enable one to generalise or vary the results in different ways (c.f. de Villiers, 2007). We delimit to look at the function of discovery more narrowly than Miyazaki (2000) in order to be able to scrutiny some details in this function as well as students' learning and understanding of it. The earlier studies have mainly focused on geometrical problems. In our study, we explore this function and students' encounter with it within both algebraic and geometrical contexts.

## THEORETICAL STANDPOINTS

We look at our experiment from the perspective of social practice theory applied to proof and proving in mathematical practice (Hemmi, 2008) and consider proof as an essential *artefact* in mathematical practice. Artefacts are tools that mediate knowledge between the social and the individual. According to the theory there is a balance between how much to focus on artefacts and how much to work with them without a focus on them, called the *condition of transparency*, and a lot of research in our field illuminates this balance in different ways (c.f. Hemmi, 2008). According to the social practice theory, an important part of learning is experiencing meaning in the practice (Wenger, 1998). Understanding the role and functions of proof in mathematical practice could enhance students' experience of meaning in both scrutinising and reflecting on complete proofs and trying to construct their own proofs. Teaching is seen to only offer possibilities for learning and we want to study

what aspects of the object of learning become visible for the students (c.f. Marton & Booth, 1997).

### **What we mean by the discovery function**

The example that de Villiers (1990; 1999) gives in order to enlighten the function of discovery is about a kite where the midpoints of the sides form a rectangle. The perpendicularity of the diagonals is the essential step in the proof and the property of equal adjacent sides is not required. Hence, it was possible to generalise the result to any quadrilateral with perpendicular diagonals. Recently, de Villiers (2007) presents another problem that further enlightens the function of discovery. Miyazaki (2000) extends the meaning of the discovery function to concern not only the generalisations of the original results by using the proof but also finding tacit assumptions, making new mathematical concepts and so on (Miyazaki, 2000, p. 5).

This is a very wide conception about the discovery function and some of the aspects Miyazaki includes in the function (e.g. finding tacit assumptions) have been connected to the function of systematisation for example by de Villiers (1990; 1999).

The aim of our study is to make visible certain aspects of the discovery function for upper secondary students in order to enhance their understanding of them and at the same time enhance their appreciation of proof as a useful tool in mathematical activities. Therefore, we created some special problems the proofs of which could be used for finding new results.

Miyazaki (2000) first let students work with a proving task and analyse the logical structure of their individual proofs carefully. Then, the students received an additional problem where they would study the conditions of the initial problem and find out generalisations. The Swedish students are not as familiar with proving tasks as the Japanese students seem to be according to the results of Miyazaki's study. We also wanted to control the experiment by holding some variables fixed in order to be able to look at more limited aspects of the function than Miyazaki did. Therefore, we focused on the investigations of the initial proofs in order to find out more general statements that the same proof would work for. In Miyazaki's experiment, new proofs were sometimes needed in order to solve the additional problem. In our examples, this is not the case.

The following examples illustrate our view of the discovery function and were used when introducing the students to the topic. We also carefully present the proofs of the examples in order to show the possibilities of using the function in different ways.

#### ***Statement 1***

*In a rectangle the midpoints of the sides are connected. Then one obtains a parallelogram.*

**Proof**

Draw the diagonal in the rectangle. We will use the following theorem in Euclidean geometry:

Let P be a point on the line AB and Q a point on the line AC. Then PQ is parallel to BC if and only if  $AP:PB = AQ:QC$ .

It follows that the line connecting the two midpoints on the same side of the diagonal is parallel to the diagonal:

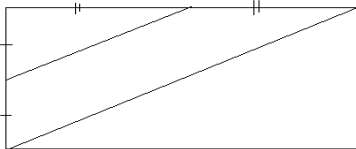


Figure 1

The same is true for the side on the other side of the diagonal. Hence these two lines are parallel. Also, we find that the other two lines are parallel, by considering the second diagonal.

**QED**

Analysing this proof, we see that we have not used the fact that the quadrilateral is a rectangle. The proof goes through as it stands for any quadrilateral. Hence, we get a new true statement:

*Connecting the four midpoints of the sides of an arbitrary quadrilateral, yields a parallelogram.*

In this way we have got a more general result, by realising that the proof for the original statement is valid under weaker assumptions.

It is also possible to obtain new results, by realising that the proof in fact proves more than the original statement. In the next example we will see an application of this. One can combine these two ways to find new truths by both making the assumptions weaker and the conclusion stronger. Formally, this may be illustrated in the following way:

$$A \Rightarrow C \Rightarrow D \Rightarrow B$$

Here  $A \Rightarrow B$  is the original statement and  $C \Rightarrow D$  is the newly discovered statement.

Another way to create new statements is by “generalisation”. The proof is perhaps a special case of a more general proof. In the above example we see that the proof uses a theorem in the special situation where the proportion is 1:1. The proof works equally well if we instead divide the sides outgoing from two opposite corners in the

## Working Group 1

same proportion. Hence, we get the following generalisation of the original statement:

*In a quadrilateral the two sides outgoing from a corner are divided in the same proportion. The same proportion is also used to divide the two sides outgoing from the opposite corner. Connecting these four points gives a parallelogram.*

Next, we illustrate what we mean by the discovery function with an algebraic example.

### **Statement 2**

*If two prime numbers greater than 2 are added, then the result is not a prime number.*

### **Proof**

A prime number greater than 2 cannot be divisible by 2. Hence it is 1 greater than an even positive number. If two such numbers are added, the result is a number which is 2 more than an even positive number. This number is greater than 2 and divisible by 2 and hence it is not a prime number.

### **QED**

Analysing the proof, we see the following structure:

$x, y$  prime numbers  $> 2 \Rightarrow x, y$  odd  $> 1 \Rightarrow x+y$  even  $> 2 \Rightarrow x+y$  not prime

We may hence discover a new truth by both weaken the assumptions and draw a stronger conclusion:

$$x, y \text{ odd } > 1 \Rightarrow x+y \text{ even } > 2$$

Compare with the general picture above,  $A \Rightarrow B$  is replaced by . Finally, we remove the assumption,  $x, y > 1$ , and weaken the conclusion to,  $x+y$  even, to obtain a more aesthetic statement. Hence, we have discovered the following truth by examining the proof.

*The sum of two odd numbers is even.*

Is it possible to find generalisations?

Here is one possible generalisation (with the same proof):

*If the integers  $a$  and  $b$ , when dividing them with the integer  $k$ , have the rests  $r$  and  $s$  and  $r+s$  is divisible by  $k$  then also  $a+b$  is divisible by  $k$ .*

The following two problems were left to the students to work with, in two groups during about one and a half hour.

### **Problem 1**

*Let  $n$  be an integer which is not divisible by 3. Prove that  $n^3 - n$  is divisible by 3.*

### **Problem 2**

## Working Group 1

*Two circles intersect at the origin of an orthogonal coordinate system. The centre of one of the circles is on the  $x$ -axis, while the centre of the other circle is on the  $y$ -axis. The circles intersect at one more point. Prove that they intersect there under right angle.*

### METHODOLOGY

This is the first explorative study where we focus on the function of discovery. We introduced this function to a group of high-achieving upper secondary school students, two girls and five boys. The students were to finish their secondary level studies during the time of the experiment. We chose these students because they had taken some special courses in mathematics, for example in geometry, so we could be sure that they were familiar also with geometrical proofs. In Sweden, not much time is usually spent on geometry in ordinary upper secondary school classes.

During the introduction, we presented the examples above. Clas led the session and engaged also the students in the presentation with appropriate questions. The students did not need to take notes because we handed out the written presentation to them after the introduction. Also their mathematics teacher took part of this session together with the students. Kirsti observed the session and she also videotaped the presentation. The presentation took about 30 minutes.

After the presentation we divided the students into two small groups according to the recommendations of their teacher and they obtained the written introduction with the two tasks that they would work with. We asked the students to read the tasks individually and then together discuss and try to solve them. They could come and ask for help if needed and Clas visited the groups twice during the session in order to offer his help. The group work was tape-recorded.

After the groups had struggled with both problems about one and a half hour we gathered the students together again and asked them to tell us how they had solved the problems. This discussion was also tape-recorded. We kept also the notes that the individual students had made during the group work session in order to use them as a complementary data. We decided to meet after one week and the students could read the material we handed out to them and think about two additional tasks.

During the last meeting we asked the students to tell us if they had obtained some new insights concerning the use of proof and proving. Finally, Clas showed the solution of one of the tasks they had been thinking about at home. This session was tape-recorded as well.

### Data analysis

We watched and listened the videotaped session to find out students insights during the presentation. Then, we listened the tape-recorded group work sessions several times and identified the parts that in various ways enlightened the students' insights concerning the discovery function. We transcribed the relevant parts of the

## Working Group 1

discussions. In parallel, we also studied the notes that the individual students made during the group session.

### **Ethical aspects**

We carefully informed the students and the teacher about the background and the aim of our study and that it was optional to take part of it. We informed also what we would do with the data and that the data would be handled in a way that would protect their anonymity.

### **RESULTS OF THE TEACHING EXPERIMENT**

In the presentation of the Statement 1 above, some students nicely pointed out that one even obtains a rhombus and proved that the sides are equal using the Pythagorean Theorem. However, to be able to use the discovery function, Clas showed them the proof given above. Concerning the algebraic example, one of the students seemed to catch the idea of discovery already and suggested a generalisation of the statement.

The analysis of the group sessions shows that students in both groups seemed to realize that it was a question about finding new results. Group 1 started their work enthusiastically encouraging each other:

S1:            Now, let's find new truths!

They also show that they have understood that it is important to first find a solution to the original problem first.

S2:            First, we have to show that we have a solution.

However, they did not manage to find a proof for the initial statement which was a prerequisite for exploring and deriving deductively new results. Instead, they started to discuss the possibilities of proving something more general.

S1:            Is it possible to take away some demands in some way?

S3:            I wonder if it works for all odd integers.

The second group managed to prove the statement by expressing the number  $n$  with  $t+1$  and  $t+2$  where  $t$  was divisible with three. During the proving process they also noticed that the expression is always even. However, they did not succeed to put their results together and extend the divisibility result from 3 to 6. They did neither notice that they could omit the prerequisite that  $n$  is not divisible by three. In the similar manner as the first group they started to make and test conjectures about possible generalizations without reflecting on their proof.

S4:            But the point is that this is a correct proof and we have to go on, and what was it we would do, we would generalise or specify... why not just put  $n^x - n$  divided by  $x$ ?



## Working Group 1

Then they tested the generality of the statement and arrived at a conjecture that  $n^x - n$  is divisible by  $x$  if  $x$  is a prime. Hence, they did not reflect on the proof they had constructed but went on inductively testing new conjectures inspired by the initial statement.

Some students suggested a proof by induction for the first problem and also worked a bit with it. However, they did not seem to understand the idea of mathematical induction properly.

S5: The proof by induction, would it work here?

S4: We can always do it.

S6: But I don't remember proof by induction.

S4: But exactly like we have started now (testing with the conjecture) we start by the basic... and then, you know, one first proves that it holds for  $n$  and then that it works for  $n + 1$ .

S7: Exactly.

One student also suggested a proof by contradiction and here also we noticed that the student did not really grasp the idea from the logical point of view.

S4: Shall we do this classical that we assume the opposite that  $n$  is divisible by three and then look how it works?

Regarding the geometry task, both groups managed to prove the initial statement and also to generalize the statement (the angles are equal in both intersection points).

When we met the students after one week and posed a question about what they had learned the students answered:

S5: Mm, yes, the key word is I think 'generalise' in these proofs and then it is we have learned, or any way I have learned, I think in another way and it is that if one can extend the proof and use it and if it works with other assumptions.

S4: The way in which I understood this was that one either generalises or specifies that one, as you (Clas) expressed it, strengthens the prerequisites of this proof or that one opens it to see what more cases hold.

Hence, these students state they had got insights in the aspects of discovery function that we aimed with our experiment. Yet, the second extract shows that the student has difficulties to distinguish between assumptions and conclusions. We also noticed that the two problems they would think about at home were too difficult in the light of the analysis of the group work.

## CONCLUSIONS AND DISCUSSION

The students in our study showed in several ways that they caught the idea of the discovery function. However, they had great difficulties to construct and analyse

their own proofs in a way the students in Miyazaki's (2000) study did. Miyazaki stresses that it is important for the teacher to let students first themselves prove the initial statement, in order to successfully apply the function in the teaching of lower secondary school mathematics. Concerning the Swedish students we found out that it could be better to also work with complete proofs in order to engage the students in exploring the very deductions in searching for further discoveries. One of the hindrances for the groups concerning the first problem was the construction of the initial proof.

Our focus was not on the very construction of the proof of the initial statement but on the use of it in order to enlighten the discovery function. If the teacher has designed a situation that aims to enhance students' understanding of certain aspects of proof, it can be better to hold some other aspects constant and vary the ones that are the object of learning (c.f. Marton & Booth, 1997; Hemmi, 2008). Recent international comparisons and national evaluations show that Swedish students are not very strong in algebra (e.g. The Swedish National Agency of Education, 2009; Brandell et al, 2008). This is confirmed by our study. Although the students were considered as high-achieving they had difficulties with some elementary algebra, e.g., finding the factorisation  $n^3 - n = n(n+1)(n-1)$ . The students in our study had also difficulties to cope with some logical aspects involved in proving. Hemmi's (2008) study shows that many university students still struggle with them.

Our aim is to go on conducting a new teaching experiment where we will modify the design of it according to the results from this study. The present study shows that there are a lot of interesting aspects to explore concerning the function of discovery, both theoretically and in applying it in the teaching of mathematics at different levels.

## REFERENCES

- Bell, A. W. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7, 23-40.
- Brandell, G., Hemmi, K., & Thunberg, H. (2008). The widening gap – A Swedish perspective. *Mathematics Education Research Journal. The Special Issue on the Transition from School to University Mathematics*, 20(2), 38–56.
- De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, South Africa*, 23, 17-24.
- De Villiers, M. (1999). The role and function of proof with Sketchpad. In M. de Villiers (Ed.), *Rethinking Proof with Sketchpad* (pp. 3-10). Emeryville, CA, USA: Key Curriculum Press.

## Working Group 1

- De Villiers, M. (2007). An example of the discovery function of proof. *Mathematics in School*, 36(4), pp. 9-11.
- Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, 44, 5–23.
- Hemmi, K. (2008). Students' encounter with proof – the condition of transparency. *ZDM – The International Journal on Mathematics Education. The Special Issue on Proof*, 40, 413-426.
- Hemmi, K. (2010). Three styles characterising mathematicians' pedagogical perspectives on proof. *Educational studies in mathematics*, 75, 271–291.
- Hemmi, K., & Löfwall, C. (2009). *Why do we need proof*. A paper presented at CERME 6 - Sixth Conference of European Research in Mathematics Education. Lyon, France. Jan 28th - Feb 1st, 2009.
- Knuth, E. J. (2002). Teachers' conceptions about proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5(1), 61-88.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. New Jersey: Lawrence Erlbaum Associates.
- Miyazaki, M. (2000). What are essential to apply the “discovery” function of proof in lower secondary mathematics? In T. Nakahara & K. Mastaka (Ed.). *Proceedings of the 24<sup>th</sup> Conference of the International Group for the Psychology of Mathematics education*. Vol. 4. Hiroshima University. Japan.
- The Swedish National Agency for Education. (2008). *Svenska elevers kunskaper i TIMSS Advanced 2008 och 1995. En djupanalys av hur eleverna i gymnasieskolan förstår centrala begrepp inom matematiken. (Swedish students' knowledge and skills in TIMSS Advanced 2008 and 1995. A deep analysis about how upper secondary school students understand central notions in mathematics)*. Available [2010-08-31] at <http://www.skolverket.se/sb/d/3535/a/7757>.
- Weber, K. (2002). Beyond proving and explaining: Proofs that justify the use of definitions and axiomatic structures and proofs that illustrate technique. *For the Learning of Mathematics*, 22(3), 14-17.
- Wenger, E. (1998). *Communities of Practice*. Cambridge: Cambridge University Press.

# CONJECTURING AND PROVING IN ALNUSET

Bettina Pedemonte

DiDiMa srl., Istituto per le Tecnologie Didattiche – CNR Genova

*This report proposes an approach to algebraic proof. It is based on the use of the AlNuSet system, a dynamic, interactive system to enhance the teaching and learning of algebra, numerical sets and functions in secondary school. This work wishes to show in which way AlNuSet can be used in the educational practice to enhance the teaching and learning of algebraic proof. The research hypothesis is that educational activities performed in AlNuSet favour the combination of the aspects of visibility and invisibility in the approach to proof, making it “transparent”.*

Key words: Algebra, AlNuSet, transparency, conjecturing and proving

## PROOF IN ELEMENTARY ALGEBRA

Bibliography on the theme of algebraic proof at secondary school level is sporadic (Healy & Hoyles, 2000). A possible reason is that in the secondary school curricula of many countries, the approach to proof is still taught in the context of traditional geometry (Hanna & Jahnke, 1993). Nevertheless, rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. This kind of proof clearly reveals the influence of algebra (Hanna & Jahnke, 1993).

Furthermore, some recent studies (Pedemonte, 2008) show that Algebra seems to be a good domain to introduce proof. In fact, unlike the geometrical case, in Algebra some difficulties students have in the construction of proof seem not to be present. This has been observed when students solve open problems requiring the construction of a conjecture and the production of a proof. Some Italian researchers (Boero, Garuti, Mariotti 1996, Garuti et al. 1998, Mariotti 2001) showed that open problems are suitable for proof learning because *cognitive unity* between argumentation supporting the conjecture and the construction of the proof can be realised. According to *cognitive unity* hypothesis, the argumentation used to construct a conjecture can be used by students in the construction of proof by organising some of the previously produced arguments in a logical chain. This continuity supports the construction of a proof. However, another kind of continuity, the *structural continuity*, exists between argumentation supporting a conjecture and proof (Pedemonte, 2007). This kind of continuity occurs when argumentation and proof have the same structure (abductive, inductive, deductive). Pedemonte (2007) observed that this continuity can be an obstacle for the construction of a geometrical proof: some students do not construct a proof because they are unable to transform abductive steps of argumentation into deductive ones in the proof. When constructing algebraic proof this obstacle seems not to be present (Pedemonte, 2008). Since algebraic proof is characterised by a strong deductive structure, abductive

steps in the argumentation activity can be useful in linking the meaning of the letters used in the algebraic proof with numbers used in the argumentation. Thus, the approach to proof in Algebraic domain could be more effective than in Geometry.

In school practice, however, algebra is not usually considered as a way of seeing and expressing relationships but as a body of rules and procedures for manipulating symbols. Normally, students can develop manipulations but are not aware of axioms and theorems they are using in performing it. Thus, algebra is taught and learned as a language and emphasis is given to its syntactical aspects. The actual “vision” of Algebra should be modified to introduce it as a domain for proof - algebra should be considered as a theoretical system where techniques used to make manipulation derive from mathematical axioms and rules.

This paper proposes an approach to algebraic proof. It is based on the use of the AlNuSet system which can be used to propose specific tasks requiring the construction of a conjecture and the production of an algebraic proof. AlNuSet was developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (years 12-13 to 16-17). It is constituted by three integrated environments: the *Algebraic Line*, the *Algebraic Manipulator*, and the *Functions*. In this paper we consider two of them: the Algebraic Line (AL) and the Algebraic Manipulator(AM). The AL is an explorative environment to construct conjectures through a motor perceptive approach; the AM is a symbolic calculation environment to produce algebraic proof. The aim of this report is to show how this system can be used to support the teaching and learning of algebraic proof, making proof “transparent”.

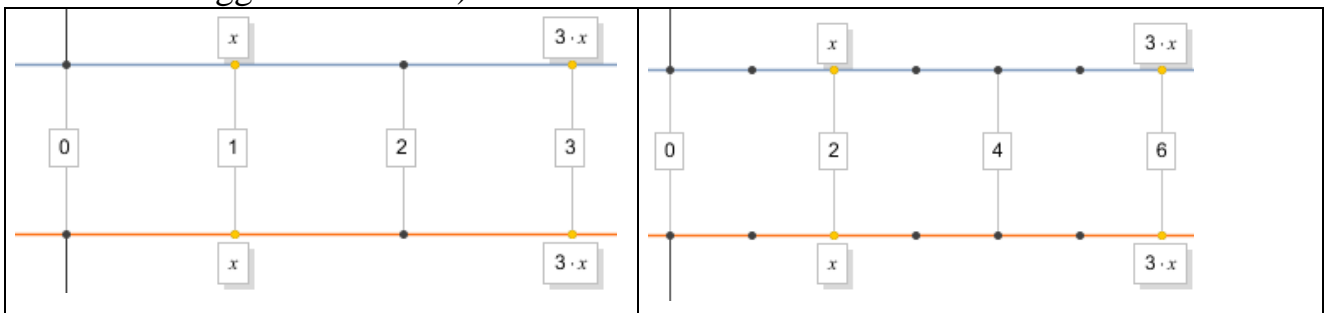
### **PROOF HAS TO BE “TRANSPARENT”**

The role of the proof in the educational practice is not well defined and very often difficulties emerge because some aspects of proof are not explicit for students and they are not well explained by teachers (Hemmi, 2008). Through the notion of “transparency”, in her report Hemmi contributes to solve the dilemma to make more or less invisible for students some important aspects concerning proof. The concept of transparency (Lave and Wenger, 1991) combines two characteristics: visibility and invisibility. Visibility concerns the ways that focus on the significance of proof (construction of the proof, logical structure of proof, its function, etc.). Invisibility is the form of “unproblematic interpretation and integration to the activity” (Hemmi, 2008, p. 414). It concerns the proof as a justification of the solution of a problem without thinking it as a proof. It has been underlined that “*Proof as an artifact needs to be both seen (to be visible) and used and seen through (to be invisible) in order to provide access to mathematical learning*” (Hemmi, p. 425). The lack of visibility in the teaching of proof regards the lack of knowledge about proofs techniques, key ideas and proof strategies.

The hypothesis of this study is that AlNuSet can be used in teaching and learning algebraic proofs to make proof more “transparent”. The AL can be used to make “visible” some important mathematical concepts that are usually implicit in the algebraic manipulation (the variable, the dependence of an expression from the variable, the meaning of equation, etc.). The AM of AlNuSet can be used in teaching and learning algebraic proofs to make rules and axioms used “visible” in proof processes and to let theoretical aspects usually implicit in algebraic manipulation emerge. The AL and the AM are briefly presented in the following. A more detailed presentation can be found in other reports (Chiappini, Pedemonte, Robotti, 2008; Pedemonte, Chiappini, 2008).

## THE ALGEBRAIC LINE OF ALNUSET

The AL of AlNuSet is constituted by two lines<sup>1</sup> where it is possible to insert letters and mathematical expressions involving numbers and letters. These expressions can be inserted (or constructed) and represented as points on the line depending on the mobile point of the variable contained in such expressions. Once an expression has been inserted, dragging the  $x$  mobile point, the expression(s) that depend on it move accordingly (i.e. in the figure below the expression  $3x$  moves when  $x$  is dragged on the line).



This dynamic characteristic is very important to allow students experience important algebraic concepts - the dependence of the expression from a variable, the meaning of denotation for an expression, the equivalence among expressions, etc. These aspects are detailed in the following.

## THE ALGEBRAIC MANIPULATOR OF ALNUSET

The AM of AlNuSet is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequalities. Its operative features are based on pattern matching techniques. In the Algebraic Manipulator pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules are explicit for students. They appear as commands on the

<sup>1</sup> The two lines are used to construct expressions through a geometrical model. It is not possible here to explain in which way they work. In this report we use these lines as a unique line.

interface and made active only if they can be applied to the part of expression previously selected. An expression is transformed into another through this set of commands that corresponds to axioms and rules. Students can see the transformation of an expression as the result of the application of a rule on it.

## TEACHING EXPERIMENT

In this section student's resolution processes of some tasks involving the production of a conjecture and the construction of proof in AlNuSet are analysed. They are taken from a set of data collected from an experiment carried out in a class of 22 students of the Second year of Lower Secondary School (12-13 years old). Empirical data were both qualitative and quantitative and were collected according to different methodologies: written tests, observations and recording of students dialogs. Data were transcribed and translated from Italian into English. The main aim of this experiment was to analyse the role of AlNuSet in a teaching experiment centred on algebraic expressions and propositions. The experiment lasted 6 weeks, with sessions of two hours per week. The first session of the teaching experiment focused on algebraic expressions. In this report we present results of this session. Students worked in pairs with AlNuSet under the supervision of the teacher and the researcher. Students had not used AlNuSet previously. The teacher presents the software showing some specific technical features. Then she distributes a paper containing the tasks.

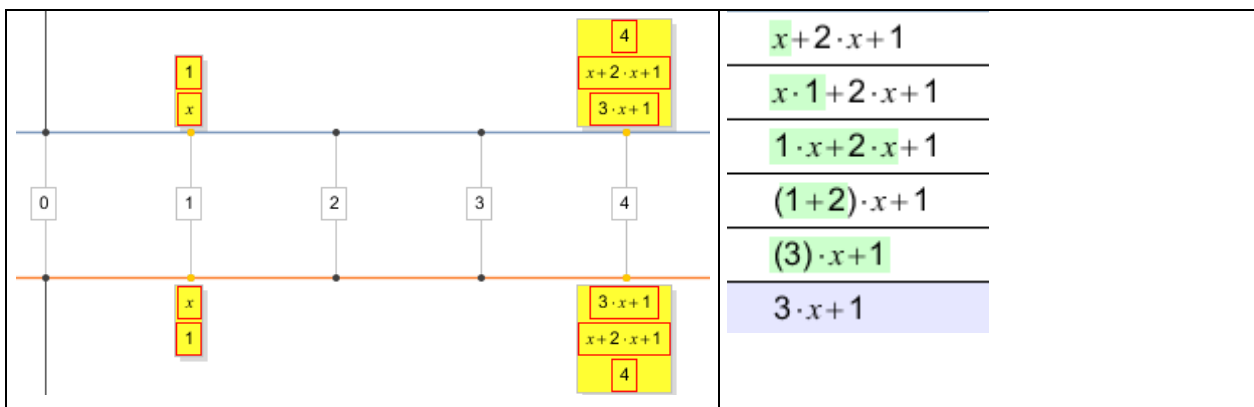
### Tasks

- a) Let  $x$  be an integer number. Write an expression for the triple of  $x$ .  
Represent this expression on the AL. Is your answer correct? Why?  
Write an expression for the consecutive of the triple of  $x$ . Represent it on the AL and verify your answer.  
Consider the expression  $x+2x+1$ . Compare this expression with the previous one.  
Check your answer using the AL and AM of AlNuSet.
- b) Let  $x$  be an integer number. Write an expression that represents the quadruple of  $x$  increased by 3.  
Is there any value of  $x$  such that this expression is 27?
- c) The teacher asks a student to carry out the following computation:  
*Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially.*  
The teacher says:  
*The result is 3*  
The teacher proposes the exercise to two other students changing the number to add.  
At the end she proposes the following exercise:  
If  $x$  is the initial number, and  $a$  is the number to add, write an expression to translate the described computation.  
Represent the expression on the AL.  
What can you observe when you move  $x$ ? What can you observe when you move  $a$ ?  
What is the result of the expression? Represent the result on the AL and check your hypothesis.

## Working Group 1

Tasks a) requires to construct the expression  $3x$  in the AL and verify that this expression represents the triple of  $x$ . Moving  $x$  on the line the expression  $3x$  moves accordingly. Through a perceptive approach students can see that the point associated to the expression  $3x$  assumes values that are multiples of 3. In this way, what the expression  $3x$  denotes is made more explicit. Furthermore, only moving the variable  $x$  it is possible to move  $3x$  allowing students to experience the expression's dependence from the variable  $x$ .

The second part of the task requires to construct the consecutive of the triple of  $x$  and to compare it with the expression  $x+2x+1$ . The aim of this second question is to point out the equivalence between the two expressions from a perceptive point of view and not from a formal one. In the AL the equivalence among expressions is represented by a post-it (see figure below). The two expressions  $3x+1$  and  $x+2x+1$  belong to a same post-it for each value the variable  $x$  assumes on the line.



Students can “experience” the equivalence of the two expressions and then they can prove the equivalence in the AM. In the AL, students make visible the equivalence between the two expressions. The focus here is not to prove the equivalence but to experience it. In the AM proof is made explicit - students are obliged to explicit the rules necessary to transform the first expression into the other. This is not obvious because this transformation in a paper and pen environment is usually not treated as a proof; here proof is in general invisible to students.

### Analysis of tasks a) results

All students are able to answer the first question: they write the expression  $3x$ . However it is interesting to observe that students are not able to justify why  $3x$  is the triple of  $x$ .

Some students construct a table, other students tell the 3 times table. Only when they can move the expression on the AL they are able to explicit that the expression  $3x$  denotes the triple of  $x$  because “*the expression assumes only values that are multiples of 3... it probably takes all multiple of 3*” (Sara).

Another interesting aspect that emerged during the exploration was the dependence of the expression  $3x$  from the variable  $x$ . When students have to move the expression  $3x$ , they fail because they move directly the expression and not the variable  $x$ . Here



## Working Group 1

an example of two students who are not able to move  $3x$ .

Francesca:  $3x$  does not move

Danilo: We are not able to move  $3x$

Teacher: Why isn't it moving?

Francesca: we are trying but with no results...

Teacher: Why are you moving directly  $3x$ ?

Francesca: Because I want to move the expression  $3x$

Teacher: You should not move directly  $3x$  because this expression is dependent on the variable  $x$

*Silence*

Teacher: In which way can you move  $3x$ ?

*Silence*

Teacher: you have to move  $x$

Danilo:  $x$ ??

*Danilo moves  $x$*

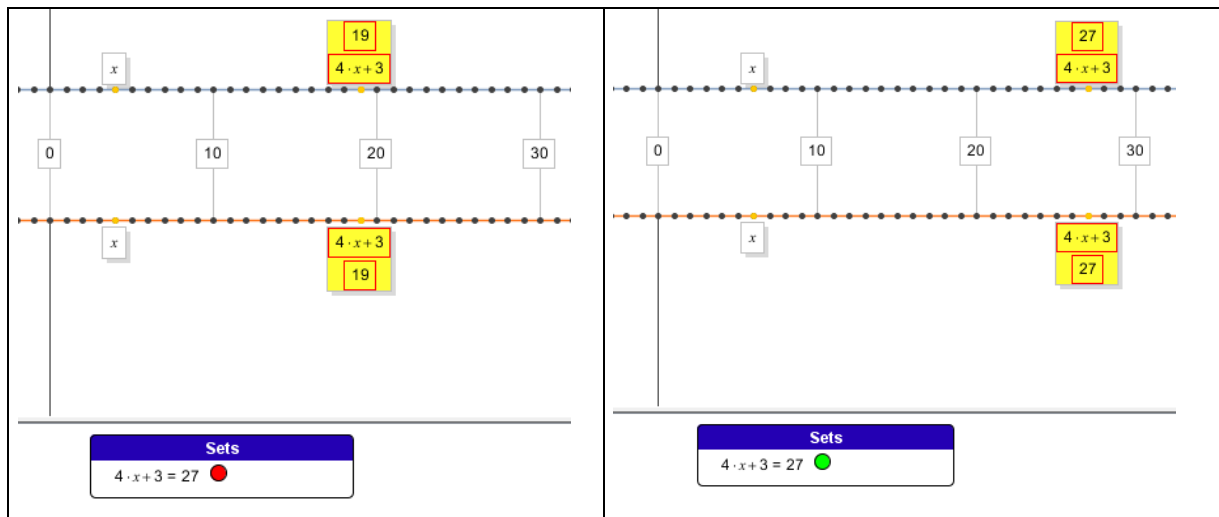
Danilo: Ahh...  $3x$  is dependent on  $x$

Francesca: Ahhh... it was really too difficult for us...

All students write the expression  $3x+1$  and they insert it on the line. Once more some students try to move the expression directly on the line. The intervention of the teacher is necessary to overcome this obstacle. All students are aware that the two expressions  $x+2x+1$  and  $3x+1$  are equal because a calculus rule ( $x+2x$  is equal to  $3x$ ). Proof is invisible here. On the contrary, in AM, the transformation of an expression into the other becomes visible: many students feel frustrated when they have to prove their calculations with AM. The teacher has to guide them specifying properties they are using (distributive property, insertion of the neutral element, etc.). In AM proof is not spontaneous for students. As highlighted by Hemmy, visibility and invisibility of proof interact in the process of learning and both are needed. It was unexpected for the teacher that after the teaching experiment and during the usual lessons students could explicit these properties also in a paper and pen environment or on the blackboard and when not required. For example, they used to say "*we are applying distributive property*" to replace the usual statement "*we are performing a calculation*". Moreover, the work with AL was important for students to make visible the relation between the value of a variable and the value of an algebraic expression or proposition. The following statement was written by a student "*An expression is dependent from a variable, from the letter that is contained in it. If in the expression there are only numbers, then the expression is not dependent from the variable*" (Carlo).

Tasks b) is an implicit requirement to solve an equation. In AlNuSet (as shown in the next figure) it is possible to solve the equation dragging  $x$  to move the expression  $4x+3$  in the point 27. When  $x$  is situated on the point 6, the expression  $4x+3$  is on the point 27 and the little ball associated to the equation is green. On the contrary, when  $x$  is moved on the other values, the little ball is red to show that other points are not a solution for the equation. As a consequence, AL makes available functionalities to solve equations in a non formal way.

## Working Group 1



This feature helps understand the meaning of equation. Students are usually able to solve the equation through the manipulation rules but they cannot say that the solution makes it true if replaced in the equation.

### Analysis of tasks b) results

All students are able to solve this task, and a pair of them recognises that they are solving an equation. However, it is interesting to observe that many solve the equation on the paper before the exploration on the line. Proof is invisible in the paper. In the AL some important mathematical concepts useful to understand the proof may become visible. Students are not able to move directly  $x$  on point 6 to obtain the solution. They try to move directly the expression on point 27, and then they make an exploration moving  $x$ . When they see that the expression  $4x+3$  is situated on point 27 only if the variable  $x$  assumes number 6 as values, they seem to understand that they are solving the equation in a completely different way. The insertion of the equation  $4x+3=27$  and the different color assumed by the corresponding little balls moving the variable on the line, is really effective to construct a justification: “6 is the solution of the equation, for this reason when  $x$  is on point 6 the expression  $4x+3$  is on point 27 and the little ball here is green... when  $x$  is situated on the other values the little ball is red and not green!” (Martina).

The AL is really important to make visible the relationship between the variable and the equation. The teacher observed that some students, even after the end of this teaching experiment, in spontaneous way, replaced in the equation the value of its solution to see if the solving process was correct.

The proof in AM is constructed on the screen by the teacher supported by students.

### Tasks c)

Task c) is a real effective task to understand the different meaning between variable and parameter. Observe that the expression  $\frac{2 \cdot x + a}{2} - x$  is equivalent to the expression  $\frac{a}{2}$ .

## Working Group 1

As consequence, the teacher may always guess the result of the expression because it is not dependent on the value thought by the student.

In AL, the expression  $\frac{2 \cdot x + a}{2} - x$  does not move dragging the variable  $x$ . In this way, students can experience that this expression is not dependent on  $x$ . Students may also observe that the value of this expression is always the half of the value of  $a$  (which is the value that the teacher requested to add). This explains why the teacher can always know the value thought by the students. After the production of this conjecture, students have to prove it.

$\frac{(2 \cdot x + a)}{2} - x$ <hr/> $(2 \cdot x + a) \cdot \frac{1}{2} - x$ <hr/> $2 \cdot x \cdot \frac{1}{2} + a \cdot \frac{1}{2} - x$ <hr/> $x \cdot 2 \cdot \frac{1}{2} + a \cdot \frac{1}{2} - x$ <hr/> $x \cdot 1 + a \cdot \frac{1}{2} - x$ <hr/> $x + a \cdot \frac{1}{2} - x$ <hr/> $x + \frac{a}{2} - x$ <hr/> $\frac{a}{2} + x - x$ <hr/> $\frac{a}{2} + x + -x$ <hr/> $\frac{a}{2} + 0$ <hr/> $\frac{a}{2}$	<p>On the left the required proof.</p> <p>The proof in the AM is more complex than in a paper and pen environment.</p> <p>Even if a lot of effort is required by students to prove the equivalence between these two expressions, the system can make visible the rules and procedures of manipulation supporting the comprehension of proof as part of a theoretical system.</p>
--	---

### Analysis of tasks c) results

When the teacher asks to carry out the following computation, all students are really surprised when she guesses the results of the calculations “*Think a number, double it, add 6, divide the result by two, and subtract from it the number that you thought initially*”. The teacher proposes the same task modifying the number to be added to two other students. She asks to add 4 and then 8.

The students are not able to explain why the teacher is able to guess always the result. The teacher asks to write an expression to translate the computation.

Two pairs of students write the expression  $\frac{2 \cdot x + 6}{2} - x$  but the others write the correct expression.

The teacher asks to insert the correct expression on the AL and move alternatively  $a$  and  $x$  to observe the behaviour of the expression.

In the following a part of the discussion developed in the class.

Teacher: What does it happen when you move  $x$ ?

Alberto: the expression does not move!

Fabrizio: nothing happens!

## Working Group 1

Teacher: are you surprised?

A lot of students: yesss

*They continue to move x*

Teacher: and when you move **a**?

Fabrizio: there are some values that are ok...

Giuseppe: but not all values... sometimes the expression disappears

*Students speak among them to try to explain why sometimes the expression disappears*

Sara: the expression is dependent only on **a**

Teacher: This is an important point! We have seen that if we move **x** this expression does not move accordingly. This expression does not depend on **x**. On the contrary, we have seen that moving **a** the expression has a particular behaviour. Who can account for this behaviour?

Carlo: wait... if **a** is an even number then I can see the expression, but if **a** is an odd number, the expression disappears! I cannot see it!!?

Dylan: The expression is exactly the half of **a**

Teacher: are you sure that it is always the half of **a**?

Federico: yes it is true, it is the half of **a**

Sara: this is why you guessed the results of the expression... It is always the half of the number you required to add..

A lot of students: yess! It is true!!!

Teacher: but why, in your opinion, the expression disappears when **a** is an odd number?

Carlo: because an odd number divided by 2 is a decimal number

Alberto: it is a decimal number

Carlo: You asked to add only even numbers...otherwise it wouldn't have been possible to divide by two

Teacher: Perfect! In a previous case, in the computational task I asked to add 6, then 4 and then 8. They are even numbers. I can calculate the half of an even number. This is why I could guess the result. Is it clear to all?... But what happens if we change the domain?

Danilo: the expression does not disappear

Alberto: it is always the half of **a**, we cannot see it in the Integer numbers because the integers are not decimal numbers!

Teacher: So the expression is always equal to...

All students: the half of **a**

Teacher: so we can write

$$\frac{2 * x + a}{2} - x = \frac{a}{2}$$

*The teacher writes on the blackboard*

Teacher: can you prove it? Try to prove it in the manipulator.

Only three pairs of students were able to complete the proof by themselves. The intervention of the teacher was required for the other students. However, the constructed proof obliged them to be aware of axioms and rules that are used step by step during the transformation of an expression into another. Only at the end of the teaching experiment students are able to use the AM by themselves effectively.

## CONCLUSIONS

In this report we have presented AlNuSet to introduce proof in algebraic domain. The recent paper (Pedemonte, 2009) it was already shown how the AM of AlNuSet makes proof “visible”. In AM the transformation of an expression into another one is not the result of a calculation, but it is carried out by the applications of “explicit” algebraic axioms and rules. Likewise, AL can be used to make “visible” some mathematical concepts (the variable, the parameter, the equation, etc.) that have a crucial role in understanding an algebraic proof. As a consequence, AlNuSet maintains the balance between “visible” and “invisible” in the approach to proof in Algebra, where proof in the ordinary educational practice is usually “invisible”.

## REFERENCES

- Chiappini, G., Pedemonte, B., Robotti, E. (2008) Using AlNuSet to construct the notions of equivalence and equality in algebra. In M. Kendall and B. Samways (eds), *Learning to live in the knowledge Society*, Springer, pp. 345-349.
- Hanna, G., & Jahnke, N. (1993) Proof and application. *Educational Studies in Mathematics*, 24, 421-438.
- Healy, L., Hoyles, C. (2000) A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31(4), 396-428.
- Hemmi, K. (2008) Students’ encounter with proof: the condition of transparency. *The International Journal on Mathematics education*, 40, 413-426.
- Boero, P., Garuti, R., Mariotti M. A. (1996) Some dynamic mental processes underlying producing and proving conjectures. *Proceedings of the PME 20*, vol. 2, (pp. 121 – 128), Valencia.
- Garuti, R.; Boero, P.; Lemut, E. (1998) Cognitive unity of theorems and difficulties of proof. *Proceedings of the PME 22*, vol. 2, pp. 345-352.
- Mariotti, M. A. (2001) Justifying and proving in the Cabri environment. *International Journal of Computer for Mathematical Learning*, Dordrecht: Kluwer.
- Pedemonte, B. (2007) How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23-41.
- Pedemonte, B. (2008) Argumentation and algebraic proof. *ZDM – The International Journal on Mathematics Education*, 40(3), 385-400.
- Pedemonte, B. (2009) The Algebraic Manipulator of AlNuSet: a tool to prove *Proceeding of the CERME 6*, Lyon, France.
- Pedemonte, B., Chiappini G. (2008) ALNUSET: a system for teaching and learning algebra *International Journal of Continuing Engineering Education and Life-Long Learning (IJCEELL)*, Vol. 18, n°5/6, Inderscience Publishers, pp. 627-639.

# A SCHEMA TO ANALYSE STUDENTS' PROOF EVALUATIONS

Kirsten Pfeiffer

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway

*Abstract. In this exploratory study I investigate first year mathematics undergraduates' practice of proof evaluation of alternative mathematical proofs. This paper describes the theoretical background which I chose as a basis for developing a schema to describe and explore students' proof evaluation performances. This schema is illustrated on students' evaluations on one particular example proof. Further I demonstrate insights arising from this study about to what degree the nature and purposes of proofs are visible to the participating students.*

Key words: proof, proof evaluation, proof validation, artefacts, purposes of proofs.

## INTRODUCTION

Interested particularly in the students' transition from school to university, I explore first year students' behaviour and knowledge when validating and evaluating mathematical proofs. Students' proof validation performances have been discussed in the research literature, for example by Selden & Selden (2003) and Alcock & Weber (2005). With Selden and Selden, I call the readings and considerations to determine the correctness of mathematical proofs and the mental processes associated with them *validations of proof*. In a mathematical community the process of accepting a proof involves more than its validation. Validation, the determination of the correctness of an argument, is a significant part of the process of accepting a proof, followed by a more extensive and open-ended process that involves a search for understanding as well as correctness, a desire for clarity and an alertness to the possibility of adaptation or extension. Seeing learning as assessing and participating in the practices of a community, I suggest to widen the context from *proof validation* to the notion of *proof evaluation*. With *proof evaluation* I mean two things: determining whether a proof is correct and establishes the truth of a statement (validation) and also how **good** it is regarding a wider range of features such as clarity, context, sufficiency without excess, insight, convincingness or enhancement of understanding. That is, *proof evaluation* includes assessment of the significance and merits of a proposed proof.

In her doctoral thesis, Hemmi (2006) developed a theoretical framework to describe how students encounter proof when studying mathematics at university level in Sweden. Her theoretical framework combines a sociocultural perspective with Lave and Wenger's (1991) and Wenger's (1998) social practice theories and with theories about proof obtained from the mathematical education research. In my study I adopt parts of Hemmi's theoretical framework and its terminology and combine it with some new ideas for investigating and describing how students validate and evaluate

mathematical arguments. A significant aim of my study is to develop and test a schema to describe and analyse students' proof evaluation skills and habits.

In this paper I present this schema and use it in the interpretation of some excerpts of interview transcripts. The first part of this paper introduces the theoretical framework. I describe Hemmi's view of proof as an artefact in a community of mathematical practice, then explain how I employed her ideas in my development of a schema to describe and explore how students evaluate mathematical proofs. The second part of this paper describes the experiment, in particular one of the tasks I used in interviews held in 2009 with eight first year students. The third part of this article describes the students' behaviour when evaluating one of the proofs proposed in the interviews, using the specialized schema mentioned above. Observations arising from this study provide opportunities for researchers to learn about the students' views of mathematical proofs. In the final part of this paper I outline some of those findings and discuss the value of the suggested framework.

## **THEORETICAL BACKGROUND**

### **1 Students as newcomers in a community of mathematical practice**

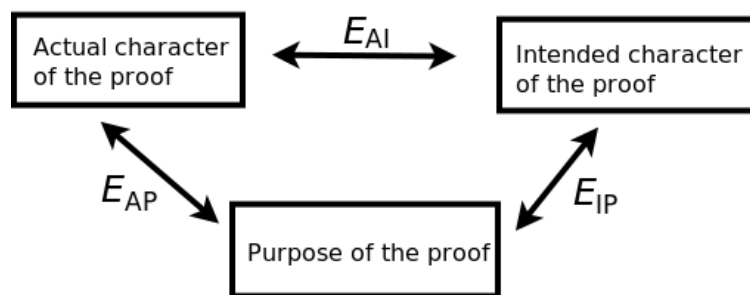
Influenced by Vygotsky's theories, Lave and Wenger (1991) established their notion of *legitimate peripheral participation*, a situated learning theory that argues that knowledge is distributed throughout a *community of practice* and can only be understood with the 'interpretive support' provided by participation in the community of practice itself. I consider the mathematical community, as Hemmi does, as a *community of mathematical practice* and the students as its *newcomers*. A fundamental concept of sociocultural theory is that mental activity is organized through culturally constructed *artefacts*. Becoming knowledgeable or *learning* means increasing membership in the practice which includes the ability to use and understand its artefacts. They provide learners with opportunities to enter a community.

Motivated by Adler (1999) and Hemmi (2006) I argue that proofs can be seen as intellectual *artefacts* in mathematical practice. Adler considers *talk* as an artefact in mathematical learning and Hemmi extends this idea to mathematical *proof*. Recognizing that validating and evaluating proofs are crucial activities in a mathematical community, I investigate novice students' habits when performing these activities. Considering Hilpinen's (2004) philosophical approach towards artefacts, I describe how artefacts can be evaluated in general and specialize this to the practice of proof evaluation throughout the mathematical community and by *newcomers* such as first year students. Observation of novice students' behaviour when validating and evaluating proofs will give us insights into their existing knowledge about the artefact proof.

## 2 Proof - an artefact in the mathematical practice

An *artefact* can be described as an object that has been intentionally made or produced for a certain purpose. The philosopher Hilpinen (2004) describes how artefacts can be evaluated. He distinguishes between the *intended character* of an artefact, its *actual character*, and its *purpose*, and evaluates on the basis of the relationships among those features. In the following sections I apply this philosophical approach to artefacts of the type *proof* and use it to describe how the interview participants evaluate proofs.

**Figure 1** below describes how a *proof* can be evaluated, applying Hilpinen's description of evaluations of artefacts in general. A proof can be evaluated in relating the three features of an artefact, its *intended character*, its *actual character*, and its *purposes*.



**Fig1: Evaluation of the artefact proof**

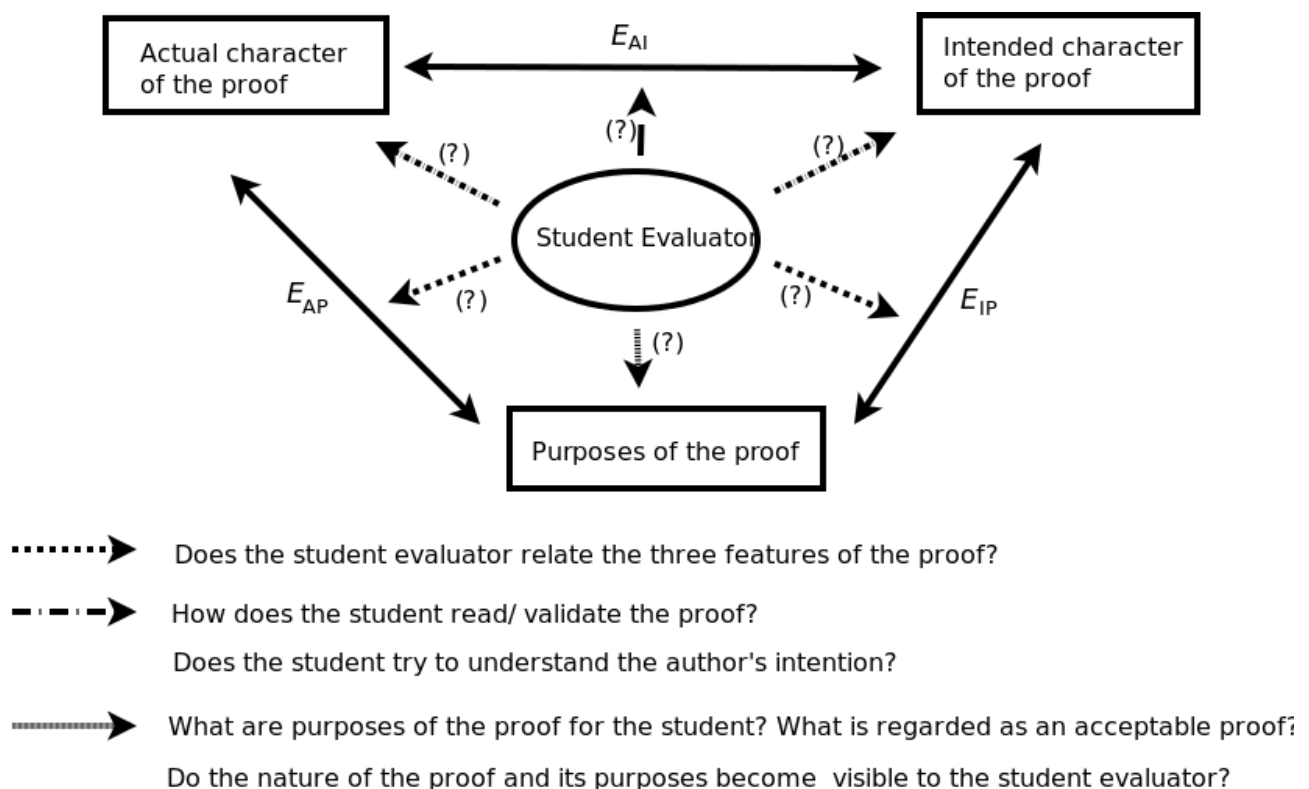
In this graphic the *actual character of the proof* means the actual realization of the author's intention, whereas the *intended character of the proof* designates this intention. *Purposes* (or functions) of proofs [1] have been widely discussed within the mathematical education literature in the last four decades, considering that a broader range of functions of proofs than that of establishing the truth of a statement should be recognised. De Villiers' (1999) suggested model for the functions of proof has been broadly accepted and applied within the mathematical education community. In his model functions of proofs include *verification* (concerned with the truth of a statement), *explanation* (providing insight into why it is true), *systematisation* (the organization of various results into a deductive system of axioms, major concepts and theorems), *discovery* (the discovery or invention of new results), *communication* (the transmission of mathematical knowledge) and *intellectual challenge* (the self-realization derived from constructing a proof). Expansions to this list of functions have been suggested. For example Hanna and Barbeau (2008) claim that the list “stopped short of stating that proof contains techniques and strategies useful for problem solving.” Acknowledging that those purposes of proofs weigh differently, depending on preferences of authors and readers and also on the circumstances of the presentation of a proof, I consider in the context of proof evaluation particular *proofs*, not *proof* in general. In **Figure 1** the



relationships among the three features are labelled  $E_{AI}$ ,  $E_{IP}$  and  $E_{AP}$ , where 'E' symbolizes 'Evaluation of a proof'. [2]

- $E_{AI}$  is concerned with how a proof is a successful realization of the author's intention, e.g. whether all steps of the proof are mathematically correct or whether the proof is clearly structured.
- $E_{IP}$  is concerned with how an intended proof, the author's idea of the proof, is suitable for its purposes. Is the idea appropriate to prove the mathematical statement?
- $E_{AP}$  is concerned with how the author was successful in proving the mathematical statement as claimed, establishing its truth, potentially convincing a mathematical community or regarding other purposes of proofs as suggested above.

In the interpretation of the transcripts of the conducted interviews I focus on the students' proof evaluating habits, in particular on whether and how they reflect on the relationships  $E_{AI}$ ,  $E_{IP}$  and  $E_{AP}$  among the actual and intended character and the purposes of a proof. **Figure 2** below demonstrates how the researcher might learn about the students' views of proofs through observations of their proof evaluation skills and habits.



**Fig2: Research questions: how does the student evaluate a mathematical proof?**

## THE EXPERIMENT

The study is based on a series of tests and interviews conducted with first year honours mathematics students at NUI Galway. In March 2009 interviews were held with eight students. Eighteen students, who had attended a written exercise including

## Working Group 1

an evaluation task in September 2008 as well, were invited to participate in a research project. They were chosen carefully in an effort to cover a wide spread of performances in the written experiment. All eight students who volunteered participated in the project. Each of the interviews took 30 to 45 minutes. Every interview was tape recorded and transcribed. The aim of the interviews was to get a deeper insight into students' opinions about valuable proofs, students' validation and evaluation processes and learning effects during the validation and evaluation processes. The students were presented with two mathematical statements and five or six proposed proofs of each statement and asked to evaluate and rank them. One proof of the first statement was purely visual, one consisted only of a fairly random assortment of examples, one was completely wrong but written in "algebraic" language, one was more general than required, another was written in text. [2] I will now present one of the proposed proofs and reflect on how an experienced reader might evaluate it. An interpretation of the students' evaluations and rankings of this proof during the interviews will demonstrate how the transcripts were used to learn about the students' evaluation habits and their knowledge about mathematical proofs. Finally I will outline the results of the analysis of the entire student evaluations of the six proposed proofs of *Statement I* [3].

**Statement I.** Consider the following statement. The squares of all even numbers are even, and the squares of all odd numbers are odd.

**Anna's answer:**

Even numbers end in 0,2,4,6 or 8.

$0^2 = 0$ ,  $2^2 = 4$ ,  $4^2 = 16$ ,  $6^2 = 36$ ,  $8^2 = 64$ .

When you square them the answer will end in 0, 4 or 6 and is therefore even.

So it's true for even numbers.

Odd numbers end in 1,3,5,7 or 9.

$1^2 = 1$ ,  $3^2 = 9$ ,  $5^2 = 25$ ,  $7^2 = 49$ ,  $9^2 = 81$ .

Squaring them leaves numbers ending with 1,5 or 9, which are also odd.

So it's true for odd numbers.

An experienced evaluator would probably identify that Anna's argument centres on her assertion that the last digit of the square of an integer is determined by the last digit of that integer itself. This assertion is correct. It certainly could be argued that the assertion needs some justification. If the evaluator is prepared to accept Anna's assertion, the actual character of this proof does coincide with the intention and therefore the argument does satisfy condition  $E_{AI}$ . However, Anna's argument does not provide an essential explanation of WHY squaring an integer preserves parity (i.e. oddness or evenness). There is no reason to construct a modulo 10 argument (based on the last digit - the remainder on division by 10) for a problem in modulo 2

arithmetic (the problem is about remainders on division by 2). A reader may well complain that by using 10 cases where two would suffice, this proof misses the key point. The intended character of this proof, involving 10 different cases, is not a good fit to the purpose of explaining why squaring preserves parity. For that reason an experienced evaluator might regard Anna's proof not satisfactory concerning  $E_{IP}$  and  $E_{AP}$ .

### 3 Example: An Interpretation of some transcript excerpts

The coding table below (**Table 2**) provides an overview about the participants' evaluations of Anna's proof; the codes in the bottom line refer to how the students placed Anna's proof in the ranking of all six proposed proofs. The heading line refers to the codes for the students (*Students C and D* were interviewed together). **Table 1** describes the coding scheme.

Satisf	The student regards the answer as satisfying.
NotSatisf	The student regards the answer as <b>not</b> satisfying.
Proof?	The student is not sure whether the proposed approach is a sufficient proof of the statement or not.
NoProof	The student does not regard the answer as proof of the statement.
NotGeneral	The student criticises that the proposed approach is not applicable in general.

**Table 1: Coding Scheme**

<i>A</i>	<i>BC/D</i>	<i>EF</i>	<i>G</i>	<i>H</i>
Proof?	Satisf NotSatisf NoProof NotGeneral NotGeneral	NotSatisfNotSatisf  NotGeneral NotGeneral	Satisf Proof? NotGeneral	Satisf NoProof
first	fourththird	fourthfourth (jointly) (jointly)	fifth	second

**Table 2: Coding Table: Students' evaluation of Anna's proof**

The coding table refers to three groups of comments.

- **Satisf/NoProof:** The student is happy with Anna's answer and considers approvingly that Anna is using examples. The student considers that Anna's answer is not a proof of the statement. “*It's not a proof, but it works*” or “*It's a good answer. (...) There is no kind of proof (...)*” are responses assigned to this group.

- **Satisf/Proof?:** The student likes Anna's answer because Anna “*gives examples*” or the answer “*is different*”. It is not clear from the interview conversation whether the student regards Anna's answer as a sufficient proof of the statement.
- **NotSatisf/NotGeneral:** The student does not accept the answer as a proof of the statement because it “is not general”.

With the research questions listed in the above diagram (**Fig2**) in mind I suggest an interpretation of the students' evaluations of Anna's proof, occasionally using short transcript exemplars to advance certain claims.

#### 4 How do the student evaluators relate the three features of Anna's proof?

Two of the five students whose opinions belong to (Satisf/NoProof) or (Satisf/Proof?) do not seem to focus on relations  $E_{IP}$  or  $E_{AP}$  as there is no evidence to suggest that they are considering the purposes of mathematical proof. Even though the other three of those students (Satisf/NoProof) express the opinion that Anna's answer is not a valid proof of the statement and in particular that the argument is not applicable in general they acknowledge the unusual approach and the internal correctness and rank Anna's proof relatively highly (second or third out of six). Their responses to Anna's answer suggest that  $E_{AI}$  may be more important to them than the relations  $E_{IP}$  or  $E_{AP}$ : “*It's not a proof, but it works.*”(C/D) or “*There is no kind of proof, it's just ---. But it does make sense*”(H). Internal correctness seems to be considered more important by this group of students than the purpose of establishing the general truth of the statement.

The three (NotSatisf/NotGeneral)-students do relate the actual proof not only with the author's intention but with its purposes and therefore do consider relations  $E_{IP}$  and  $E_{AP}$  as well as  $E_{AI}$ . They consider at least one purpose of proof, namely its general applicability, criticize the poor relation between the actual or intended proof and its purposes, and therefore regard Anna's proof as unsatisfactory, which is indicated by their ranking of this proof. These three students seem to regard relations  $E_{IP}$  and/or  $E_{AP}$  as being at least as important as  $E_{AI}$ .

#### 5 Do the student evaluations of Anna's proof indicate what the students consider purposes of proofs?

Five students criticize a lack of general applicability in Anna's proof, which indicates that they consider this as one purpose of mathematical proof. One of the (NotSatisf/NotGeneral)-students (E) finds the level of justification insufficient “*She doesn't prove that 'When you square, the answer will end in 0,4,6...' If she'd proved that, it would be ok.*” Certainly for this student justification of intermediate steps is a necessary ingredient of mathematical proof. This student seems to see Anna's answer as an attempt at a general argument about the last digit that could be improved to a proof. This is similar to how an experienced evaluator is likely to see it, namely as more than a collection of examples. In an experienced evaluator's view the examples that are included in Anna's proof are not intended as examples but as items in an

exhaustive list that covers all cases. The other students who complain that Anna's answer is not general and consists of “*just examples*” interpret this in a different way: *Student F* for example seems to see it as basically the same as another proposed proof which consists of a collection of examples, just “*up to ten numbers*”. *Student B* like *student F* considers Anna's answer as just a selection of examples: “*She took the numbers from 1 to 9, but what about all the other numbers? (...) Nice example, but that's about it.*”

**6 Do the student evaluations of Anna's proof indicate a learning process? Do nature or purposes of mathematical proof become *visible* to the student evaluator?**

*Student G's* reaction indicates that a learning process is initiated by the task. Her first reaction (“*Very cool*”, “*different*”) indicates that she admires the unusual approach (“*I could never think of anything like that, (...) the way she writes it down (...)*”). After careful prompting by the interviewer a reflection process is initiated and the student is getting more and more unsure, until at some point she almost decides that this is not a proof, but is never really sure about this. *Student G's* comments do not show clearly what she considers as a valid or valuable proof, but she certainly thinks about it.

The second proposed proof (*Benny's answer*) consists of a collection of ten examples. Interestingly seven of the eight students commented in the interviews on how they compare Benny's answer to Anna's, even though they weren't asked to do so. Four students regard the answers as very similar. Two students approve the fact that Benny includes examples of negative integers in his answer. Five students, all agreeing that neither answer proves the statement sufficiently, mention that Anna's answer is more like a proof than Benny's. They identify two aspects of proof more present in Anna's than in Benny's answer:

- the description of general patterns: “*She has this --- with the endings*” (C/D), “*In [Anna's answer] there is more thinking in it. She saw this fact, if you square an even number, that there is a 0,2,4,6,8 at the end of each one.*” (E)
- Anna's answer includes some attempts to explain **why** the statement is true. “*She says why the squares are odd, because they end in that. He [Benny] just presumes that they are odd numbers.*” (F)

Considering Benny's answer in comparison to Anna's, some of the students who have interpreted Anna's proof as list of randomly chosen examples when discussing Anna's answer now identify some potential in Anna's answer to provide a general proof: *Student F* states that “*Anna's is more of a proof [than Benny's]. She says why the squares are odd, because they end in that*”. Likewise *Student B* regards Benny's answer as “*more example than proof than Anna's was*”. These changes in some of the students' opinions about Anna's proof indicate a learning effect about proofs

through the comparing process. It seems that some purposes of mathematical proof became *visible* to these students.

## SUMMARY AND FURTHER OBSERVATIONS

Consideration of the interview data with the relationships  $E_{AI}$ ,  $E_{IP}$  and  $E_{AP}$  among the actual and intended character and the purposes of a proof in mind led to the following observations. The students' evaluations of all six proposed proofs of *Statement I* indicate which purposes of proofs they consider relevant. In *Student A's* evaluation the most relevant consideration is how the statement's plausibility is verified and demonstrated by the proposed answer. Consequently she favours answers consisting of examples. Generality of a proof is an important evaluation criterion to most of the students. Five students mention at some point during the interviews that they appreciate considerations and explanations about *why* the statement is true in certain proposed answers. Some of the students take into account whether the proposed proof emphasizes some mathematical contents or general patterns. Some students consider sufficiency without excess in their proof evaluations. Some students appreciate a didactical value in a proof, which includes how well a reader's interest is stimulated or how well both statement and proof are being explained to the reader. The proof idea or method does not seem to play a significant role in the students' evaluations of proofs of the first statement, which is indicated by three observed phenomena. Firstly, a proposed visual approach is liked least considering the ranking of all eight students together. The intrinsic idea behind this approach seems unimportant to the students. Secondly the fact that one of the proposed arguments proves a more general fact than the facts of *Statement I*, is rarely being recognized and not appreciated by the students. The third surprising fact indicating poor appreciation of proof ideas or methods is the relatively high ranking score of an irredeemably wrong approach. While some of the students noticed errors in this proof, none questioned the basic strategy.

Interpretations of oral and written proof evaluation exercises so far suggest that the developed conceptual framework and schema to interpret student-evaluations are beneficial to gain some understanding about students' knowledge and skills about proofs and proving. The schema appears to be in particular useful to identify students' criteria to accept or value a mathematical proof and also to what extent and how first year students consider purposes of mathematical proofs. *Proof evaluation* as an important activity in mathematical practice might carry some potential to provide students with opportunities to enter the practice. The suggested schema to interpret student-evaluations is appropriate to determine whether that is the case, i.e. to what degree *proof evaluation* performances support learning effects. However, the suggested method seems to be less effectual regarding observations about students' proof reading habits. I did not gain a lot of noteworthy information about how the students try to understand a proposed proof.

## Working Group 1

Overall, results of this study indicate that, considering importance and challenges in the teaching and learning of mathematical proof, exploration and practice of incoming students' proof evaluation skills and habits are worthy of further attention.

### NOTES

1. By *function* of proof I mean with De Villiers(1999) “meaning, purpose and usefulness” of proof.
2. Hilpinen (2004) introduced the notation  $(E1) - (E3)$  for the relationships of the three features of artefacts, relating to three aspects of evaluations, where 'E' symbolizes 'Evaluation'.
3. Detailed descriptions of the tasks, interview questions and transcripts can be found in my forthcoming PhD Thesis.

### REFERENCES

- Adler, J. (1999). The dilemma of transparency: seeing and seeing through talk in the mathematics classroom. *Journal for Research in Mathematics Education*, 30(1): 47-64.
- Alcock, L. and Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behaviour*, 24: 125-134.
- De Villiers, M. (1999). The role and function of proof in mathematics with sketchpad. *Rethinking Proof with Sketchpad*, Key Curriculum Press.
- Hanna, G. and Barbeau, E. (2008). Proofs as bearers of mathematical knowledge. *ZDM Mathematics Education*, 40: 345-353.
- Hemmi, K. (2006). *Approaching proof in a community of mathematical practice. Doctoral Dissertation*. Department of Mathematics, Stockholm University.
- Hilpinen, R. (2004). Artifact. *Stanford Encyclopedia of Philosophy*.
- Lave, J. & Wenger, E. (1991). *Situated Learning: Legitimate peripheral participation*. Cambridge University Press, Cambridge.
- Selden, A. and Selden, J. (2003). Validations of proofs written as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34 (1): 4-36.

# THE APPEARANCE OF ALGORITHMS IN CURRICULA A NEW OPPORTUNITY TO DEAL WITH PROOF?

Modeste Simon, Institut Fourier, Université Joseph Fourier, Grenoble.

Ouvrier-Bufferet Cécile, UPEC & Laboratoire André Revuz, Université Paris 7.

*We deal with the concept of algorithm which is taking importance in curricula in many countries. In particular, we develop an epistemological analysis of this concept and discuss its place in the mathematical science and the link it has with proof. This analysis is enriched by the study of “how researchers know the algorithm”. We conclude with implications of the changes in curricula on proof learning.*

Keywords: algorithm, proof learning, interview of researchers, epistemological study.

Discrete mathematics represents a mathematical field which takes a growing importance in our society. In particular, the accessibility of the concepts of this field brings new tracks to teach and learn proof (e.g. Grenier & Payan, 1999; ZDM, 2004). This paper will deal with one of the concept of this field: the algorithm. Indeed, with the omnipresence of computers and technologies in our society, it seems that algorithm will take more and more importance in curricula and it raises questions that impacts on the teaching of mathematics. Moreover, new types of proofs involving computations appear, and with them, algorithmic proofs for instance and the philosophical and epistemological questions of the use of computers to create and/or validate proofs (Hanna, 2007). The 1998 Yearbook of the National Council of Teachers of Mathematics was entirely dedicated to the questions of algorithms and was “an attempt to answer many of [the questions provoked about the place of algorithms in today curricula] and to stimulate other questions that all of us in mathematical education need to consider as we continually adapt school mathematics for the twenty-first century” (NCTM, 1998, p. vii). Recently, the studying of algorithms got into the class of mathematics in the French curriculum of the secondary school. The appearance of the concept “algorithm” in the mathematical curriculum questions the role in mathematics of an object which seems, at first look, to belong more to computer science. Actually, algorithm is first of all, from historical and epistemological points of view, a mathematical concept.

The links between algorithm and proof are not easy to describe and have not been much studied epistemologically. The goals of our study (started in Modeste, Ouvrier-Bufferet & Gravier (2010)) are here twofold: from a mathematically-centered perspective, we want to bring an epistemological analysis of the algorithm which emphasizes its interplay with proof. The way researchers in mathematics and computer science know the concept of algorithm allows a validation of the epistemological model we develop in this paper. From a didactical perspective, we ask how the concept of algorithm can enhance the curriculum, focusing on links



between algorithm and proof (avoiding the ubiquitous computer aspect). We ultimately want to build Research Situations for the Classroom (RSC) (Grenier & Payan, 1998, 1999; Godot & Grenier, 2004) involving algorithm and proof processes. In particular, the design of such situations implies to rely on an epistemological analysis and a study of the practises of researchers (see Knoll & Ouvrier-Buffet, 2006). Then the concluding session highlights the links between the concept of algorithm and the proving process and brings new research tracks in order to design and to analyze situations for the classroom.

## EPISTEMOLOGICAL ASPECTS OF THE CONCEPT OF ALGORITHM

### A common definition

The usual definition of algorithm is presented by Knuth who takes algorithm as an object of study (and also questions the differences between mathematical thinking and algorithmic thinking):

“(...) an algorithm is a set of rules or directions for getting a **specific output** [1] from a **specific input**. The distinguishing feature of an algorithm is that **all vagueness must be eliminated**; the rules must describe operations that are so simple and so well defined that they can be **executed by a machine**. Furthermore, an algorithm must always terminate after a **finite number of steps**.” (Knuth, 1996, p. 59)

This implies that an algorithm solves a specific *problem* *P* by returning in the *output* the answer corresponding to the instance of *P* given in the *input*. It is important to remember that the input and output information are coded and “algorithms deal primarily with the manipulation of symbols that need not represent numbers.” (Knuth, 1996, p. 61)

The previous definition also shows the *effective aspect* of algorithm: no ambiguity must exist in the instructions so that any *operator* – most of the time a computer, and in this case the algorithm can be described by a *program* – gets the same output with the same steps. The *finiteness* cannot be dissociated from the notion of algorithm, as Chabert notes it in his book on the history of the algorithm:

“Today, principally because of the influence of computing, the idea of **finiteness** [1] has entered into the meaning of algorithm as an essential element, distinguishing it from vaguer notions such as process, method or technique. [...] Here we have a finite number of operations, a finite number of input values, but also a finite number of solution procedures, that is that each step should be able to be carried out by a finite process – something which is not possible, for example, in determining the quotient of two incommensurable real numbers. We also refer to an **effective** procedure, that is one that will effectively achieve a result (in a finite time).” (Chabert et al., 1999, p. 2-3)

This finiteness raises questions regarding *complexity*: given an input, how many steps does the algorithm take to answer? How much space does it need to store the involved information? These questions respectively deal with *time complexity* and

*space complexity*. This complexity depends on the size of the input and can be studied from two points of view: the *worst-case complexity*, the maximum complexity of the algorithm for an input of size  $n$  and the *average-case complexity*, the average of the complexity for an input of size  $n$  (usually, the inputs of size  $n$  are considered to have the same probability). Since Knuth, complexity is a specific and fundamental aspect of the algorithm which is absent from the other fields of mathematics. He says (about Bishop's mathematics [2]) that “[it] is constructive, but it does not have all the ingredients of an algorithm because it ignores the “cost” of the constructions” (Knuth, 1996, p. 110). Here, we have detailed three important aspects of algorithm: the link with **problems**, the **effectivity** and the **complexity**.

### Ambiguity of this definition

As examples of algorithms, authors often give Euclid's algorithm for gcd, arithmetic operations on integers, algorithms for sorting or algorithms for shortest paths in a graph... Among all these examples, one strikes us: the method to find the roots of the quadratic equation  $ax^2 + bx + c = 0$  using the discriminant  $b^2 - 4ac$ .

With more details, the algorithm is the following:

Input:  $a, b, c$   
 $\Delta = b^2 - 4ac$   
**if**  $\Delta > 0$  **then** return  $\frac{-b+\sqrt{\Delta}}{2a}, \frac{-b-\sqrt{\Delta}}{2a}$   
**else if**  $\Delta < 0$  **then** return  $\frac{-b+i\sqrt{-\Delta}}{2a}, \frac{-b-i\sqrt{-\Delta}}{2a}$   
**else** return  $\frac{-b}{2a}$

Judging by the definition we gave above, we could say that it is an algorithm. But we find it surprising that such a method was chosen as an illustration of the concept “algorithm”. Indeed, in each case, the algorithm is just a formula. Moreover, from the point of view of the “complexity”, such an algorithm is not interesting, as the complexity is independent of the size of the input (we are not speaking here about the complexity of the arithmetic operations involved in the discriminant, which are for us better examples of algorithms). This example raises the question of the border between algorithmic and non-algorithmic areas and the “usual” definition is ambiguous about this. For our study, for a didactical purpose, it would be useful to distinguish this kind of formula with “real” algorithms.

### A more theoretical definition

In the beginning of the 20<sup>th</sup> century, the quest of foundations for mathematics caused mathematicians to give a more theoretical definition of algorithm.

“The works of Gödel inspired the research of Alonso Church, Stephen Kleene, Alan Turing and Emil Post. These mathematicians attacked Hilbert's Entscheidungsproblem and showed that there were, indeed, undecidable problems, that is mathematical statements for which no procedure exists by which it can be decided if the statement is

true or false. To do this, each of them defined a concept of computability, that is a concept of algorithm.” (Chabert et al., 1999, p. 457)

Two of these concepts should be quoted: the *Turing machine* and the *recursive functions*. This theoretical work leads to a classification of problems depending on whether there exists an algorithm to solve them or not (*undecidable* and *decidable problems*) and if they are “easy” or “hard” problems, which means if they can be solved by a polynomial algorithm or not (we refer here to *P* and *NP-hard problems* [3]). This point of view will constitute a fundamental aspect of algorithm, the **theoretical models**.

### Algorithm and proof

Algorithm and proof interplay in many ways, it will be another important aspect for us. First, an algorithm has to be proved; more precisely, it is necessary to prove its *correctness* (i.e. it gives the expected answer) and its *termination* (i.e. it always stops after a finite number of steps). And, once it is proved, an algorithm can be used as a *step in another proof*. Actually, all the aspects raised previously have a link with proof. In particular, correctness and termination correspond respectively with *problem solving* and *effectivity*. The *complexity aspect* involves proof too, and studying the complexity of an algorithm often needs substantial mathematics. The same is true of *theoretical models*, which only make sense in a proof process. Just like any mathematical object, the algorithm raises questions involving proof. But some of them are specific and only the algorithm raises the mathematical questions previously mentioned. Moreover, the algorithm is not only linked with proof on that way. An algorithm can also be a *tool for proving* a property, and for a given problem, an algorithm will give a *constructive proof* of its resolution (e.g. Euclid's algorithm provides a proof of the existence of the gcd of two integers and an effective way to compute it). Conversely, an algorithm often lies under a constructive proof and it can be interesting to formulate this algorithm clearly. For instance, from any *proof by induction* follow a recursive algorithm.

Algorithms, seen as proofs, allow to deal with two kinds of problems: *existence problems* and *testing a property*.

Recently, a link has been pointed out between proof and algorithm, with the computer-assisted proofs, that is the use of algorithms to build proofs which are much too long to be verified by a human being. For instance the four-color theorem has been proved this way. However, the algorithm has to be proved in order to validate the mathematical result. This new kind of demonstration asks philosophical and epistemological questions about the nature of proof.

### Tool-Object

The aspects of algorithms underlined above can be divided into two parts since they refer to algorithm as a tool or as an object. Looking at the algorithm as an object means studying questions of validity, of complexity and description of algorithms.

Looking at the algorithm as a tool is focusing on the use of algorithms to solve problems. Among the aspects discussed here, the **effective aspect** and **problem aspect** refer to the algorithm as a **tool** whereas the **complexity aspect**, the **theoretical models** and the **link with proof** refer to the algorithm as an **object**.

## MATHEMATICIANS' POINT OF VIEW ON ALGORITHM

This analysis of the algorithm concept is mainly theoretical and it would be interesting to compare it with the reality of mathematics, that is the ongoing research.

- Are mathematicians' representations in accordance with our epistemological study of algorithm?
- How do algorithms interplay in their practice of research?
- Which aspects of algorithm are involved in mathematical research and which ones are not?
- Do researchers refer mostly to the algorithm as a tool or as an object?
- Do these questions depend on their field of research?

Actually, the main point which interests us is the following: validating our epistemological analysis when comparing the descriptions of mathematicians of coming to use and to know algorithm and our epistemological model (in the same way that Burton did). Here, the form of our interviews does not permit to describe the whole conceptions of the researchers in a specific theoretical model. Right now, the trends in our results are enough to use this analysis as a preliminary work in order to build Research Situations for the Classroom (RSC).

### Interviewing researchers

To answer the previous questions, we chose to interview researchers both in applied and fundamental mathematics. We also interviewed researchers from fields at the intersection of mathematics and computer science, like operational research, combinatorics, computational geometry... These researchers have a mathematical activity too, that is a proof activity, but should have a rich and different vision on the algorithm, provided by the links they have with computer science. We interviewed 22 researchers. From their point of view, they belong to the following fields:

	Fundamental	Applied
Mathematics	10	7
Computer Science	5	5

**Table 1: Distribution of the researchers [4]**

In fact, in order to study the practice and the representations of mathematicians, the tool of interview seemed to be the most convenient possibility (in the same way as Burton, 2004). We have met the researchers face-to-face and the interviews were

audio taped. The researchers received a brief questionnaire to provide personal information (name, gender, function, discipline (mathematics, computer science; fundamental and applied mathematics [5]), research subjects, teaching level at university), and an excerpt of the new French curriculum with the appearance of algorithms [6]. We followed a list of questions (see above) in conversational style. These interviews last between 20 and 30 minutes and the part which interest us here makes up around half of this time.

### **The interviews**

In the interviews, we chose to bring up three points:

- 1) What is an algorithm, how can one define it and recognize it, and what examples can be given?
- 2) How is it used in mathematics and in the field of the researcher?
- 3) The place and the role of algorithms in the researchers' work.

The questions were the following:

- Q1: How would you define what an algorithm is?  
Q2: How can one recognize an algorithm?  
Q3: Give examples of algorithms, of non-algorithms. Is the discriminant (presented like above) an algorithm? Where is the border between algorithmic and non-algorithmic areas?  
Q4: What are algorithms useful for? What are their roles in your field?  
Q5: Are there algorithms in your personal research? Where?

The first three questions aim at making the researcher talk about the definition(s) of algorithm. Giving examples, counterexamples and thinking about the discriminant should make the researchers question their own definition of "algorithm". The last two questions aim at making them evoke the role and place of algorithms and more particularly in their field. All the questions were reformulated if necessary, so that there was no misunderstanding. And as it was an open discussion, these points were not been necessarily mentioned in this order.

### **Analysis of the interviews**

In order to study the transcriptions of the interviews, we built an analysis grid based on our epistemological study. The goal was to find in the interviews which aspects were present. The difficulty was to give the grid a good granularity: the purpose was to associate each idea of the researcher to one or more precise aspect of algorithm, but many aspects were often vaguely mentioned. For example, as far as complexity is concerned, most of the time, the researchers just spoke about its importance but did not give details about the different kinds of complexities or about the question of optimality. After different draft versions of an analysis grid, we finally decided on the following:

## Working Group 1

TOOL						OBJECT							
Problem		Effectivity				Complexity		Theoretical models			Proof		
Solve a problem	Input/Output	Any operator and non-ambiguity	Computer or Program	Finiteness		Complexity (Space/Time)	Optimality	Turing Machine and other models	(Un)decidables problems	P/NP	Correctness	Termination	Tool for proving Induction

**Table2: The analysis grid**

## Results

### *Validation of the epistemological analysis*

In the interviews, all the aspects we expected researchers to speak about were mentioned by them. Obviously the researchers did not all talk about every aspect and they did not always give many details. But each aspect has been mentioned enough to confirm our study. We will discuss in details how each aspect has been brought up.

### *Domination of the algorithm as a tool*

The “effective” aspect and the “problem” aspect were mentioned by all the researchers. That means that the “tool” aspect is very important in their representation of the algorithm. The “effectivity” and “problem” aspects mainly appear in the definitions of algorithm researchers gave. The “effectivity” aspect is often linked with the use of computer, and many researchers pointed out the importance of computers for algorithms (and some of them made the confusion between algorithms and programs).

### *Definition of the algorithm*

As the definitions researchers gave very often involve the effective aspect of the algorithm, we can say that their definitions are very close to the “usual definition” we mentioned at the beginning. To illustrate this, here are some definitions of “algorithm” given by the researchers:

“A sequence of instructions which enable from an input to produce an output.”

“A finite chain of steps which can be described, and which allows to compute or find the solution of a given problem.”

“An effective process which allows to achieve a calculus or an automatic deductive task.”

“An automatic method to solve a problem which does not need any human intervention, and which is workable for a machine.”

About the discriminant, most of the researchers answered that, according to their definition, it was an algorithm. Some others felt embarrassed that their definition

## Working Group 1

encompassed the discriminant and explained that it was an algorithm but had no interest:

“It is not a very rich algorithm, not a good example.”

“We can compute it explicitly. It is just applying a formula.”

“We are between the “method” and the “algorithm”. There is no real process. If any time we have a formula we consider it as an algorithm, it reduces the meaning of algorithm.”

“The problem has a set size... for me, in the idea of algorithm, there is an aspect of variable size, there is the complexity behind...”

The problem of the ambiguity of the definition we underlined seems to be shared by some researchers. The “complexity” aspect seems to be closely related to this problem. Most of the researchers must not have noticed this because, as we will see below, the notion of complexity is not of a big importance for them.

### *Presence of the “proof” aspect*

The “proof” aspect, which is for us the most important when looking at the algorithm from a mathematical point of view, has been brought up by about half of the researchers (12 among 22). There is a link with their field of research: indeed, most of the researchers who brought up the proof consider themselves as fundamental researchers (in maths or computer science) whereas the majority of the others consider themselves as applied mathematicians or computer scientists. We can say that the importance of proof activity has a link with fundamental questions. Moreover, among all the parts of the “proof” aspect, the most quoted is that an algorithm is a tool of proof, that is to say that the algorithm is associated with the notion of constructive proof. The notion of proof of an algorithm (correctness or termination) has not been mentioned much, and more precisely it is always the correctness which was quoted.

### *The “complexity” aspect*

The “complexity” aspect was not mentioned by all the researchers, only 11 of them spoke about it. In this case, it seems to be linked with the computer science field of research: among the 10 (self declared) computer scientists, 8 raised the questions regarding the complexity involved by the algorithm. We can infer that the complexity is not really important from the mathematicians' point of view (as the quotation of Knuth about Bishop's mathematics let us think).

### *The theoretical models*

Theoretical models were mentioned by only 8 researchers, not only from fundamental research but mainly from fields at the intersection of mathematics and computer science (computational geometry and topology, operational research, combinatorial optimization, graph theory or cryptology). In fact, theoretical models for algorithms have a very important role in these fields. That must be the reason

why those researchers mentioned them. However, it should be noted that very few researchers from other fields (only two) brought up those theoretical models. It seems that this “recent” aspect of the algorithm (but older than the link with computers!) is not known by the researchers or does not seem important to them. The first possibility is, according to us, the most plausible. As an interviewed mathematician underscored:

“As far as I’m concerned, I’m aware of these questions [about algorithm and theoretical models] because my husband is a computer scientist. But this is not in mathematicians’ culture...”

## CONCLUSION AND PERSPECTIVES

Our epistemological analysis has been validated by the interviews. Hence, we can ask ourselves what this study implies for the teaching of algorithm and the teaching of proof. These questions have their importance at the moment in France, but also widely impact on mathematical education. We saw that if one does not want to teach algorithm as a tool only, but also as an object, it cannot be separated from the “proof” or “complexity” aspects. Learning algorithm seems to be a good way to learn proof, judging by the connections there exists between these two concepts.

Learning from these interviews, it seems that researchers in mathematics and in fields which link mathematics and computer science, do not have a wide view of the algorithm concept. We can say that little is known about this concept. We could explain this by the recent development of the study of algorithm, but this still seems pretty worrying. We can assume that this lack of knowledge about algorithm is shared by teachers of mathematics (at least in France) and their training curriculum has to be questioned.

This study of the algorithm in mathematics should allow us to study curricula and textbooks of mathematics of the secondary in order to know if the “tool” and “object” aspects are involved.

We would also like to study how the algorithm can be handled as an object by pupils and how it can make them enter in a proof process. We already made experimentations about this, at the beginning of university and in the training of primary teachers, and we obtained promising results (the students and pre-service teacher training were able to build several algorithms and their proof, see Modeste, Ouvrier-Buffet, Gravier, 2010). Our goal is to carry on these kinds of experiments in the secondary level. Schuster (2004) has studied combinatorial optimization problems in the secondary and has obtained very positive results about pupils’ skills in manipulating algorithms and proving. We could work on the “Konigsberg’s bridges problem” studied by (Cartier, Moncel, 2008) or on other problems studied by the “Maths à Modeler” team, but from an algorithmic point of view. The analysis of the way one can deal with algorithms and proof in the classroom and the results of



## Working Group 1

such experiments will be the object of a new article, based upon the epistemological model developed in this paper.

### NOTES

1. Bold types added.
2. In this article, Knuth studied books from many mathematicians and notice that, for him, Bishop's mathematics where the most close to the algorithmic thinking. However, even in Bishop's mathematics, Knuth noted that the notion of complexity was absent.
3. The definitions of P and NP-hard problems are not exactly these ones. For more details about theoretical models, one can read: Hopcroft, J.E., Motwani, R., & Ulmann, J.D. (2007). *Introduction to Automata Theory, Languages, and Computation*. Pearson Education.
4. The total number is not 22 because some researchers consider that they have 2 fields of research.
5. The choice of the researcher is not necessary dichotomous. Indeed, the presentation for their choice of the discipline was the following:

Mathematics ☐☐☐☐ Computer science  
Fundamental Research ☐☐☐☐ Applied research

6. The analysis of this excerpt by the researchers and their answers to questions about teaching the algorithm will not be discussed here.

### REFERENCES

- Burton, L. (2004). *Mathematicians as enquirers. Learning about learning mathematics*. Kluwer Academic Publishers.
- Cartier, L., & Moncel, J. (2008). Learners' conceptions in different class situations around Königsberg's bridges problem. *Proceedings of ICME 11*, Mexico. Web available: <http://tsg.icme11.org/document/get/758>
- Chabert, J.-L. et al. (1999). *A History of Algorithms From the Pebble to the Microchip*. Springer.
- Godot, K., & Grenier, D. (2004). Research situations for teaching : a modelization proposal and examples. *Proceedings of ICME 10*, Copenhagen.
- Grenier, D., & Payan, C. (1998). Spécificités de la preuve et de la modélisation en mathématiques discrètes. *RDM* 18/1, 59-100.
- Grenier, D., & Payan, C. (1999). Discrete mathematics in relation to learning and teaching proof and modelling. In Schwank, I. (Ed.), *Proceedings of CERME 1*, Vol. 1, 143-155. Osnabrück.
- Hanna, G. (2007). The ongoing value of proof. In Boero, P. (Ed.), *Theorems in school: from history, epistemology and cognition to classroom practice*, 3-16. Sense Publishers.

## Working Group 1

- Knoll, E., & Ouvrier-Bufferet, C. (2006). Discussing the Challenge of categorising mathematical knowledge in mathematics research situations. *Proceedings of CERME 4*, 332-341.
- Knuth, D.E. (1996). *Selected Papers on Computer Science*. Center for the Study of Language and Information - Lecture Notes.
- Modeste, S., Ouvrier-Bufferet, C., & Gravier, S. (2010). Algorithmique et apprentissage de la preuve. *Repères IREM*, n°79, 51-72.
- National Council of Teachers of Mathematics. (1998). *The Teaching and Learning of Algorithms in School Mathematics* (1998 Yearbook of the NCTM).
- Schuster, A. (2004). About traveling salesmen and telephone networks combinatorial optimization problems at high school. *ZDM*, 36(2), 77-81.
- Discrete mathematics and Proof in the High School (2004). *ZDM*, 36(2), 44-84 and 36(3), 82-116.

# PROOF: A GAME FOR PEDANTS?

JOANNA MAMONA-DOWNS\* & MARTIN DOWNS

\*University of Patras, Greece

*In this paper, we propose to examine the types of argument that are deemed acceptable at tertiary level mathematics and under which circumstances, and why the expectancy that a proof is required is sometimes relaxed. We specialise on the status of proof in cases where mathematical modelling takes place, and on tasks whose informal resolution rests on two or more mathematical milieu. On occasion, can the insistence on a proof be regarded as pedantry?*

Key words: Proof, Credited Argument, Definitional Tautness, Mathematization, Modelling.

## INTRODUCTION

Students attending Mathematics courses at the tertiary level often have difficulty to understand the status of proof (Jones, 2000). Further there is evidence that students find the standards applied to proof production somehow arbitrary (Gondek et al, 2009). What is the line between a mere argument and a proof? For this question, some would consider a spectrum might be more appropriate rather than a demarcation line. However, such a viewpoint leaves students in the dark as to when their work should end, or whether their final product qualifies as a proof. It is useful for the researcher to distinguish a proof itself from an argument where basically all the essential ideas behind the proof have been collated but are not articulated in terms of ratified mathematical systems. We shall call such an argument a credited argument; we will discuss our choice of wording in the next section, contrasting it with similar notions employed by other educators. Also we shall give our position of what a proof is in the context of this paper.

If we acknowledge the notion of a credited argument as well as proof, fairly natural educational issues arise, including the following:

- In what circumstances is credited argument acceptable as a final output? The answer to this question might explain why lecturers can be perceived inconsistent in the level of strictness in argumentation they use.
- We do not say that all proof productions necessarily go through a preliminary stage corresponding to a credited argument. However, for those that do, the backing of the credited argument can give support for the student to formulate the proof, and to appreciate what the proof gives beyond the credited argument.
- One reason that argumentation may end as a credited argument is that translating it into a proof can be deemed not worth the effort. Reasons for this

could be that the undertaking would be messy and/or lacking in giving additional insight. Insisting on the proof in such circumstances might be considered as pedantry. However the lecturer might have good reasons to persist; these reasons, however, may not be immediately appreciated by the students.

- A task may invite a mathematical treatment whereas the context is not strictly ‘mathematical’ itself. Sometimes, the situation can be resolved at the level of a credited argument, where the reasoning retains references to the extra-mathematical context. Because of this, it is not a proof. However, if the task is modelled into another format within a recognized mathematical framework, a proof of the model may be available. Is such a proof, though, to be considered as a proof of the original version of the task?
- The solution of a task may ‘import’ tools from another mathematical perspective from the one that is most natural to assume. The fitting together of different mathematical perspectives sometimes can be made at a perceptual level and as a result aspects of argumentation can be glossed over. In such a case, we have a credited argument but not a proof.

The aims of the paper are rather wide and illustrative in character. The main part consists of descriptions of three tasks that demonstrate the difference between a credited argument and a proof, and the particular issues raised above. In particular, we discuss whether a ‘translation’ from a credited argument to a proof is always merited. Before doing this, we include a more theoretical section where our terminology is put on a firmer basis, with reference to other related research.

## BACKGROUND DISCUSSION

This section will define and situate the notion of credited argument that we introduce in this paper. To do this we first consider the notion of ‘truth status’ vis-à-vis proof.

There are many perspectives put forward in the literature concerning how different types of (mathematical) argumentation could be contrasted to proof, or (on the other side) identified with proof. We do not have the space or the expertise to be expansive here. We specialize straight away to consider models of argumentation that are not considered as proofs but in a sense embrace all the central ideas on which a proof would be based. There is a sense of certainty but a lack of usage of the requisite mathematical tools in order to present a proof. Hence, the argumentation attains a ‘truth status’ but is not completely verified as a proof. Intuition can play a large part in the truth status; Fischbein (1987) says that ‘the concept of intuition ...expresses a fundamental, very consistent tendency in the human mind: the quest for certitude’. In the past, an argument that carried conviction both personally and for others was taken as a qualification for proof (Hersh, 1993), or at least conviction is often a prerequisite for seeking a proof (De Villiers, 1990); today, a consensus on conviction could be accepted as an indication

of truth but not necessarily of proof (c.f., Segal, 2000). Another notion that educators employ is a warrant. According to Rodd (2000), a warrant is a justification / rationale *for the belief* in mathematical propositions; a warrant acts ‘as the lever for (mathematical) knowledge’, where ‘knowledge generally entails “truth”’. Durand-Guerrier (2008) considers the difference of truth and validity via logical aspects in mathematical proof. In particular, she quotes Tarski: “the truth of a proposition lies in its agreement (or correspondence) with reality; or a proposition is true if it designates an existent state of things”.

All the models or positions mentioned above are attributive (rather than operative); they characterize argumentation in terms of a completed output of thought. In this paper, we stress the possibility that argumentation that promotes confidence in the truth status also can be a basis to form a proof. In this case, care is required both in specifying what argumentation conveys ‘truth’ and in one’s stance of what proof is. We consider proof first. We take a line similar to Thurston (1995) purporting that proof production takes place in the milieu of a certain mathematical language; its’ vocabulary is technical but also allows a generous allowance of informal expression in using technical terms. It is difficult for students to attain this language, and ‘the language is not alive except to those who use it’. The technical vocabulary is based on explicit mathematical definitions (Section 2.1, Mamona-Downs & Downs, 2005). If an argument in the mathematical language is challenged, then one always has the recourse to argue more rigorously in terms of first - principle definitions and their known logical consequences. We characterize an argument as a proof if it has this recourse; we then say that the argument has ‘definitional tautness’.

Now we consider the situation of having a line of reasoning lacking definitional tautness but satisfactorily conveys the truth status; we shall call this a credited argument. This begs the question how we regard the truth status in this context. We find difficulties in confronting this issue directly. The direction we take is to imagine an expert to examine the argument and recognise channels that would allow it to be translated into the mathematical language. This stance has unsatisfying aspects; in particular, one would like to say that an individual can feel certainty in his or her argument without exterior authority. However, the reference to a proof clearly is the most reliable source to merit the truth status of an argument. A student, then, might believe that his/her reasoning determines the truth; but this belief must be justified itself, perhaps requiring external ratification. The word ‘credited’ carries allusions to both ‘confidence felt in the veracity in a body’ and endorsement, explaining our choice in wording of the term credited argument.

The forming of a credited argument always indicates a lack in the usage of relevant mathematical tools. Sometimes the tools are not available simply because the student has not been introduced to a requisite mathematical theory. This has led some researchers to make a distinction between what is called ‘disciplinary proof’ (i.e., proof as viewed by professional mathematicians) and ‘developmental proof’ referring

to the stages of learning where different types of tools pertinent to proof making are realized. (See ICMI Study 19, discussion document.) The interest of this paper, though, is when the students are familiar with the tools needed; this means that potentially the student is equipped to ‘convert’ a credited argument into a proof. This suggests a two-stage process in producing a proof, leading to the issues mentioned in the introduction. (There are other models for which two stages are identified in forming a proof; for example, ‘pre-conjecture’ and ‘post-conjecture’ stages are discussed in Pedemonte, 2007, but conjectures are not stressed in this paper. Other papers, such as Mamona-Downs & Downs, 2009, and Zazkis, 2000, consider students’ ability to convert mental argumentation into a proof presentation.)

One of the issues brought up in the introduction was modelling; this might need some clarification. A credited argument might be available within the context (i.e., prior to modelling), whilst the proof only verifies the model (and arguably not the actual act of modelling). Another term that is often used by educators is ‘mathematization’, that seems to convey the same kind of purpose of rendering a non-mathematical situation into a controlled mathematical environment. However, we regard that an act of mathematization retains a closer reference to the intuitive reasoning available in the context (compared to an act of modelling), so it is easier to accept that a credited argument can act as a ‘template’ for the proof. This point will be illustrated later in the paper.

## EXPOSITIONAL EXAMPLES

Here we illustrate some issues concerning the difference between credited argument and proof, as raised in the introduction.

The first task is widely known, but we go further in its solution than is usual. The task is:

*For a unit square  $8 \times 8$  array, the bottom-left square and the top-right square are removed. Show that the resultant figure cannot be covered by rectangular tiles all of dimension  $1 \times 2$ .*

The usual way to demonstrate this is as follows. Suppose that for the  $8 \times 8$  array, the squares are painted white or black like a chessboard. Then there are 32 squares painted white, the other 32 squares black. Now the opposite corner squares removed will have the same colour, white say. Hence the resultant configuration has 30 squares painted white, 32 black. But each tile must cover one white square and one black. So for a tiling, a necessary condition is that there are an equal number of white tiles as they are black. Hence a tiling cannot exist.

The argument is highly contextual, however carries a strong sense of conviction. What is remarkable is the assertive character it has. Every claim made reads like a fiat; ‘it is so’. There could be an objection in that its declarative manner rests much

on perception and in this way seems to be at odds to deductive reasoning. At the same time, it seems difficult to get around this. How can you introduce 'definitional tautness' in such a physical situation? This problem seems to be compounded when in the argument itself the notion of colours is brought in; even in the process of solving the problem, another undefined notion is introduced.

Despite this, we believe that many mathematicians would accept the solution as being a valid answer. However, if one were to ask whether it constitutes a proof, the response likely would not be so unanimous. If you gave them time, some probably would say that the problem lies in the task rather than the method: "the task, if you want a fully grounded argumentation, should have been...". Below, we suggest a plausible candidate for such an alternative form of the task:

Let  $S := \{1, 2, \dots, 8\}$ .

Let  $T := S^2 \setminus \{(1, 1), (8, 8)\}$ .

Let  $W$  be the set of subsets of  $T$  of two elements of the form:

$\{(u, v), (u + 1, v)\}$  or  $\{(u, v), (u, v + 1)\}$ .

Prove that there cannot be a subset  $R$  of  $W$  satisfying both

$r_1, r_2$  are different elements of  $R \Rightarrow r_1 \cap r_2 = \emptyset$

and

$\{t \in T : t \in r \text{ for some element } r \text{ of } R\} = T$ .

At first sight, the second task might seem radically different to the original. In a way, indeed it is; where are the objects and actions understood on a physical level that motivated the exercise in the first place? Moreover, if you were presented the second task independently, it would seem highly contrived; which eccentric would dream up such a convoluted seeming creature? No, there is no motivation unless the tasks are regarded as a pair, and it is not so difficult to see how the two are (isomorphically) connected. The sets and the conditions that appear abstractly in the second task are readily 'lifted' to the environment of the first with contextual meaning. Hence, the set  $T$  represents the depleted array,  $W$  the potential positions in the array that one tile can take,  $R$  a tiling of  $T$ : the two conditions explain what we mean by a tiling in the set theoretical setting. To prove the proposition, one constructs the sets:

$T_1 := \{(u, v) \in T : u + v \text{ is odd}\}$

$T_2 := \{(u, v) \in T : u + v \text{ is even}\}$

that accounts for the colouring. By showing that  $|T_1| \neq |T_2|$ , one proves that  $R$  cannot exist. The details here are left to the reader.

## Working Group 1

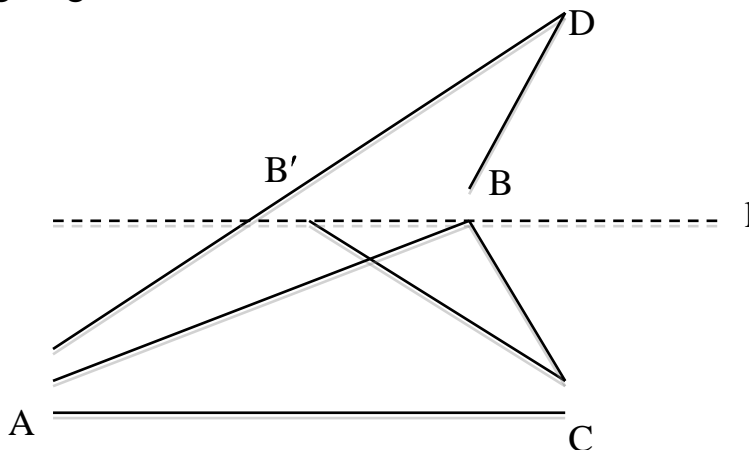
Psychologically, one can either 'identify' the two tasks regarding them essentially the same, or separate the two whilst acknowledging they have completely consonant structure. In the first case, one might say that an act of 'mathematization' has taken place, whilst for the second an act of modelling. By the word mathematization the second version of the task would be regarded as a channel to give the appropriate tools to prove the first task, whereas the modelling viewpoint would suggest that the proof status holds only for the second task. As a central construction transfers in the *solution* (i.e., the notion of colouring to the sets  $T_1$  and  $T_2$ ) we regard that here mathematization is the more appropriate term to use of the two, as insight is given why the original contextual argument works.

Suppose that a teacher presented the original task; after expounding the credited argument, what would be the pedagogical advantages in converting it into a form that permits a proof? Well, the decision whether to stop at the more intuitive and contextual level or to go further depends on the teachers' own aims. By continuing, though, students have an opportunity to judge whether an argument is a proof or not, and to see that problem solving is not only for obtaining results but also has a role in the forming of mathematization or mathematical modelling. (Obtaining the phrasing of the second task is the result of considerable reflection.) Further, a sense of structural commonality is conveyed, as is the recourse to fundamental mathematical ideas such as a set. Finally, aspects of the modelling or mathematization used in one task could be emulated in other tasks (for example, other problems involving 'arrays' could be susceptible to treatments involving Cartesian products).

Let us now proceed to another example:

*Let  $A, C$  be points of the plane. Let  $l$  be any line parallel to  $AC$ . Let  $S$  be the set of triangles  $ABC$  where  $B$  is a point on  $l$ . Show that the triangle of  $S$  that has the least perimeter is isosceles.*

This task is conducive to conventional calculus tools for optimization of functions. However we discuss a solution that retains its basis in geometry. We consider the following diagram:





What prompted us to make this diagram? Well,  $B'$  is a point on  $l$  such that the triangle  $AB'C$  is isosceles. The triangle  $ABC$  is drawn as a generic object satisfying the conditions. As the two triangles share the same basis, the problem reduces to show that:

$$2|AB'| < |AB| + |BC| \quad \text{for all } B \neq B' \text{ on the line } l.$$

To show this, we construct the triangle  $B'DB$  that represents the reflection of triangle  $B'BC$  in the line  $l$ . As a reflection preserves lengths, we have from the resultant triangle  $ADB$ :

$$|AD| = 2|AB'| < |AB| + |BD| = |AB| + |BC|. \quad (1)$$

This argument is certainly persuasive, however there are points of awkwardness in it. It rests mostly on Euclidean geometry, for which the recourse to reflection is made mostly on a perceptive level; for instance the fact that the points  $A$ ,  $B'$  and  $D$  are co-linear is assumed or regarded obvious in the argument. The reflection is not treated according to definitional tautness.

A little shift in how to read the constructive elements of the diagram removes these difficulties. Now we suppose that  $D$  is the point for which  $B'$  is the mid-point of  $AD$ . Consider the triangles  $B'DB$  and  $B'CB$ . The lengths of two sides are shared as the angle between, so they are congruent. This means that  $|BC| = |BD|$  and we are in the position to state the inequality (1) on a firmer ground.

Hence we have demonstrated the proposition via procedures that are in common currency in Euclidean Geometry, and thus we have crafted a proof. We succeeded to circumvent the problem of introducing a reflection. But we have paid a price in doing this. The first argument involving a reflection is more influential in forming the diagram; the second just exploits it.

This example, we contend, illustrates an almost universal phenomenon in formulating a proof; there are 'opposite forces' in basing an argument conceptually against concerns about means of full explication. When elements are introduced that are 'foreign' to the mathematical field in which the task environment is set, either these elements have to be assimilated or have to be sidestepped as in the example above. Obviously it is preferable to have a proof that seems transparent once it has been exposed, and this sentiment has been passed on by many mathematicians as well as educators. The phrase "A good proof is one that makes you wiser", accredited to the eminent mathematician Yu. I. Manin, has become almost a maxim. However, the situation in reality is more complicated. The advantage in distinguishing a credited argument from a proof is that a teacher or student can gauge which of the two 'makes you wiser'.

## AN EXAMPLE THAT SHOWS ASPECTS OF STUDENTS' BEHAVIOUR

The example recounted here concerns a student activity. The eight students are, at the time of writing, participants of a master's program in Didactics of Mathematics; all the students had recently graduated from a Mathematics department. For the activity, a task with a 'realistic' context, an informal argument and a mathematical model is the material given to the students; what was left for them to do was to produce an argument in terms of the model. (The example serves as an indicator of issues rather than an empirical analysis, so details of methodology are omitted.)

The givens:

Task: Let  $g(n,k)$  denotes the number of ways of placing  $k$  indistinguishable lions in  $n$  cages (in a row) such that no cage contains more than one lion and no two lions are put in consecutive cages. Show that  $g(n,k) = \binom{n-k+1}{k}$  (i.e., the number of ways we can choose  $k$  things out  $n-k+1$ ).

An informal argument: Suppose that we have a legal positioning of the lions. Then, except possibly for the  $k^{\text{th}}$  lion, there is an empty cage to the immediate right to each cage containing a lion. Imagine removing these  $k-1$  cages. Then we have  $n-(k-1)$  cages remaining and  $k$  lions, but now lions can be put into adjacent cages. Hence we are free to place the  $k$  lions wherever we want into the  $n-k+1$ .

Mathematical modeling of the task: Let  $S = \{0, 1\}$ . Let  $S_{n,k}$  be the subset of the Cartesian product  $S^n$  where an element of  $S_{n,k}$  has exactly  $k$  of its components taking the value 1. Suppose that  $T_{n,k} \subset S_{n,k}$  is defined by:

$$(x_1, x_2, \dots, x_n) \in T_{n,k} \Leftrightarrow \forall i \in \{2, 3, \dots, n-1\}, \text{ if } a_i=1 \text{ then } a_{i-1}=0 \text{ and } a_{i+1}=0; \text{ if } a_1=1, \text{ then } a_2=0; \text{ if } a_n=1, \text{ then } a_{n-1}=0.$$

What the students were asked to do was to prove that  $|T_{n,k}|$  equals  $|S_{n-k+1, k}|$ .

There are two options for a student to approach this assignment. The first is to translate the sets of the model back to the contextual families of objects and properties (e. g. lions, cages, in a row...) found in the original task environment. Then the set theoretical task is treated through the less precise but more semantic setting of its 'isomorphic' realistic-like task. The disadvantage is not only that you shift from a situation that avails the tools allowing proof to one that does not, but you also have the problem to explicitly express the grounds of the transfer itself. The second option is to keep your work within the set theoretical setting, using the contextual version rather like a 'template' to guide the argument but keeping its influence implicit in the exposition. In this case, the direction would be to construct a bijection between that  $|T_{n,k}|$  and  $|S_{n-k+1, k}|$ . The advantage here is that you are in a

position to give a direct proof; a disadvantage is that what seems immediate in the other task can transfer to messy constructions mathematically.

Of the eight students that participated, seven students clearly took the first option with various degrees of success in expounding the consonant structure between the two tasks. Their reasoning was in terms of the physically understood objects that appear in the description of the original task. Just one student adopted the second option; in his work, clearly the sets were the central actors, with just a few aside references to the context of the first task.

What is the significance of this in educational terms? The authors designed the assignment wishing to test the students' ability to argue in the environment of a model or a system arising from an act of mathematization. The aim was to push forward students' working from a credited argument to a proof via a structurally isomorphic setting. This aim backfired in the way described above. In fact this was anticipated by the teacher (one of the authors); the assignment was set as homework, and the next class was devoted to open debate about the differences between credited argument and proof, and in what circumstances is it useful to try to render a credited argument into a proof. (Such a class debate was possible because it took place within a course, on problem solving and proof, which was part of a master's program in Didactics of Mathematics.) We do not have the space to describe the dialogue that took place, but the overall opinion was that the process of bringing up the mathematical model, or mathematization, was over pedantic. But in forming this opinion, we would have to accept that, in some cases, argumentation without the definitional tautness to qualify it as a proof can give an acceptable mathematical result, even at the tertiary level.

## CONCLUSION

Mathematics students at university are often confused about the nature of proof, and worry whether what they write for a solution is in an acceptable form from the point of view of their teacher. Such confusion is natural because teachers can be inconsistent in what they accept. (For example, the notion of diagrammatic proof is accepted by some mathematicians, not by others.) Subtle hints of the level of deductive reasoning expected can be conveyed; for example, if the directive of a task is in the form 'show that', rather than 'prove that', there is an expectancy that a more relaxed argumentation is allowed. There is a lot of 'etiquette of standards' that students are supposed to pick up by themselves: this clearly pertains to the notion of didactical contract, due to Brousseau (1984). In reality, perhaps mathematicians on their own are not equipped to convey to their students what standards are demanded for different circumstances; educators should take the initiative to assist.

In this paper, we aimed to explain why for certain types of tasks an argument naturally ends even when its form falls below the level expected for a proof. Here we did not insist on formal proof, which we regard as an ideal that in practice is

rarely respected. However, we did insist that a proof is put on a firm mathematical basis where all objects and actions concerned can be explicitly defined. We pointed out and illustrated some situations where the need for a proof is debatable. In particular, we considered problems that are far from being trivial but are conducive to mental argumentation embracing perceptual elements; also we considered cases where the most natural approach involves more than one mathematical tradition or theory. Are we obliged to model in the first case, and alter the argument such that it fits within a consistent mathematical setting in the latter? We leave this question as an open issue, but we stress that, for students, there is a danger of seeming to be engaged in a game that only a pedant would be interested in. This problem is aggravated in the case where we model one task by another; are we proving the model or the original? In this respect, we raised the issue of the relative meanings of the terms ‘mathematical modelling’ and ‘mathematization’ that deserves more research inquiry.

## REFERENCES

- Brousseau, G. (1984). The crucial role of the didactical contract in the analysis and construction of situations in teaching and learning mathematics. In Steiner, H. (ed.), *Theory of Mathematics Education, Paper 54*. Institut für Didaktik der Mathematik der Universität Bielefeld, pp. 110-19.
- de Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras*, 24, 17-24.
- Durand-Guerrier, V. (2008). Truth versus validity in mathematical proof, *ZDM: The International Journal on Mathematics Education* 40 (3), 373-384.
- Fischbein, E. (1987). *Intuition in Science and Mathematics*. Dordrecht, Holland: Kluwer Academic Publishers, p.13.
- Gondek, V., Rösken, B. & Törner, G. (2009). Algebra vs. Analysis: University Students’ Views on Role of Proofs. *Proceedings of the ICMI Study 19: Proof and Proving in Mathematics Education*. Volume I, pp. 178-183.
- Hanna, G. et al. (2009). ICMI Study 19: Proof and Proving in Mathematics Education: Discussion Document.
- Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics*, 24 (4), 389-399.
- Jones, K. (2000). The student experience of mathematical proof at university level. *IJMEST*, 31(1), 53-60.
- Mamona-Downs, J. and Downs, M. (2005). The identity of problem solving. *Journal of Mathematical Behavior* 24, 385-401.

### Working Group 1

- Mamona-Downs, J. & Downs, M. (2009). "Necessary Realalignments from Mental Argumentation to Proof presentation". *Proceedings of CERME 6, electronic form*.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23-41.
- Rodd, M. M. (2000). On Mathematical Warrants: Proof does not always warrant, and a warrant may be other than a proof. *Mathematical Thinking and Learning*, 2(3), 221-244.
- Rota, G. C. (1997). *Indiscrete Thoughts*. Birkhauser, Boston, p. 142.
- Segal, J. (2000). Learning about Mathematical Proof: Conviction and Validity. *Journal of Mathematical Behavior*, 18(2), pp. 191-210.
- Thurston, W.P. (1995). On Proof and Progress in Mathematics. *For the Learning of Mathematics*, 15 (1), 29-37.
- Zazkis, R. (2000): Using code-switching as a tool for learning mathematical language. *For the Learning of Mathematics*, 20 (3), 38-43.

# TWO BEAUTIFUL PROOFS OF PICK'S THEOREM

Manya Raman, Lars-Daniel Öhman

Umeå University

*We present two different proofs of Pick's theorem commonly held to be beautiful and attempt to identify features that give rise to the sensation of beauty. In particular, we discuss two concepts, generality and specificity, that appear to contribute to beauty in different ways. We also discuss possible implications of this work for the teaching of mathematics, especially in countries in which discussions of beauty and aesthetics are notably absent from curricular documents.*

**Keywords:** Beauty, Aesthetics, Proof, Pick's theorem, Motivation

## INTRODUCTION

The claim that mathematics contains elements of deep beauty seems uncontroversial. The literature abounds with references to this beauty and characterizations of it. For instance Chandrasekhar, a Nobel prize winning physicist, once wrote that “a discovery motivated by a search after the beautiful in mathematics should find its exact replica in Nature persuades me to say that beauty is that to which the human mind responds at its deepest and most profound” (Chandrasekhar, 1987). And Hardy, in his so-called apology for mathematics, asserted, “The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way” (Hardy, 1940). We even see references to the beauty of mathematics in poetry, such as Edna St. Millay's famous line, “Euclid alone has looked at beauty bare” (Millay, 1941).

What it is less clear is why mathematics appears to us as beautiful. Hardy claimed that the sense of beauty comes, at least in part, from a sense of surprise (Hardy, 1940). Rota refutes this view. He gives an example of Morley's theorem which states that adjacent angle trisectors of an arbitrary triangle meet in an equilateral triangle (see Figure 1), claiming that this theorem while surprising is not beautiful (Rota, 1997). Rota suggests instead that what characterizes beauty is enlightenment, an admittedly fuzzy concept which he claims mathematicians do their best to avoid (Rota, 1997).

Another answer to the question of why mathematics appears as beautiful comes from Scarry, a professor of English at Harvard, whose work has recently gathered attention even in mathematical circles. In her book “On Beauty and Being Just,” she claims that beauty is, in essence, compelling. It draws us to itself. This claim resonates with Poincaré who said, “The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it; and he takes pleasure in it because it is beautiful” (Poincaré, 1908). The strong claim is that beauty has some

## Working Group 1

sort of allure, similar to that of pollen to the bee, which draws people's interest, allows it to replicate, and secures its future.

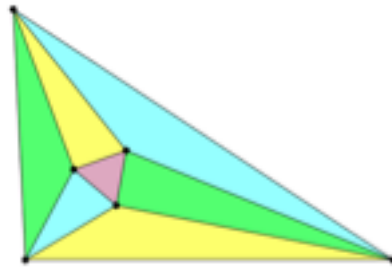


Figure 1: Morley's theorem

## PREVIOUS RESEARCH AND MOTIVATION

If beauty is compelling, then it seems natural to ask whether it can play a motivational role in the teaching of mathematics. However, there has been surprisingly little research about this question, and related questions, in the area of mathematics education (ESM, 2002). Most of the work done on this area falls under the broader, related notion of aesthetics, which involves a host of affective components in addition to the experience of beauty, such as the experience of pleasure. One recent example comes from Sinclair (2002) who created a model to describe three important functions that aesthetics play in the working lives of mathematicians: generative, motivational, and evaluative<sup>1</sup>. Burton (2004) used this framework to study the aesthetic judgements of mathematicians, looking at the connections between affective experiences of mathematics and intuitions and/or insight that the pursuit of mathematics often provides. Other examples, both from philosophy and mathematics education, include Mack (2006), Tymoczko (1993), and Wang (2001).

This study differs from the previously mentioned in that it specifically investigates the notion of beauty; we are not concerned in this paper with the affective responses, or in fact any other psychological processes related to the aesthetic experience. Another feature of the current study that differs from those previously mentioned is that the main source of data is the mathematics itself, though we draw on pilot data from an interview with one mathematician to help support our analysis, and in a future study will look more closely at interview data. We do not depart from a particular theoretical position, but rather hope to build such a position through a systematic examination of the mathematics, as demonstrated in the analysis below.

A brief look at how beauty is treated in curricular documents in different countries provides some motivation for the eventual outcomes of this project<sup>2</sup>. In some countries, such as China, the aesthetic nature of mathematics is actively researched and explicitly mentioned in the curriculum (e.g. Fu, 2004; Li, 2003; Ministry of China, 2008)<sup>3</sup>. However, in other countries, such as the United States, Sweden, and

Finland, there is little or no mention of beauty. In Sweden, there is some explicit mention of beauty at the compulsory level, but not at the high school level (Skolverket, 2000). Moreover, little or no information is given about how beauty should appear in details such as task selection or teaching practices. In the United States, beauty is not listed among the five content and process strands that affect all K-12 levels (NCTM, 2000). Similarly, we found no mention of beauty in the Finnish curriculum (which does stress many affective qualities, such as "courage" in solving problems)<sup>4</sup>. The kind of work done here could eventually serve a purpose both in raising awareness of beauty in countries that do not currently emphasize it and in articulating some of the features of beauty that could be operationalized in curricular documents.

### CURRENT PROJECT

The research described here is in its initial stages. The ultimate goal, similar to that described in Burton (2004), is to develop a theoretical model for beauty against which we can compare the views held by working mathematicians. To begin creating this model, we have proceeded fairly simple-mindedly. We looked through the literature to find proofs that are commonly held to be beautiful (e.g. Aigner et al., 2010; Wells, 1990). We wanted to choose theorems that would be fairly uncontroversial but not overly discussed (such as a proof of  $e^{\pi i} = -1$ ). We chose to focus on the beauty found in the proofs, not in the theorems themselves (though the two are often linked). We also asked mathematicians to suggest proofs that they consider beautiful, and now have a small collection of these.

One theorem that appeared both in our literature search and as a suggestion from a mathematician was Pick's theorem, which provides a simple formula for finding the area of a lattice polygon. This theorem is simple enough to be understood and verified by middle schools students, while the statement and proofs of the theorem have relevance for research mathematicians<sup>5</sup>. This feature makes the theorem particularly useful for our study, since on the one hand it is challenging enough to present to mathematicians to obtain data, but on the other hand it is accessible enough to allow us to investigate whether school age children appreciate beauty (or have capacity to appreciate beauty) in similar ways.

Below we will examine two proofs of the theorem, the first suggested by a mathematician in our pilot study, and the second found in Aigner et al., 2010<sup>6</sup>. These two particular proofs were chosen because they are similar in many respects, except for one which we would like to highlight, namely a respect that potentially gives rise to a sense of beauty. The two features of beauty that we will highlight (among many more that could be chosen, with other proofs, or even considering different parts of these proofs, such as the lemmas that support them) are those of *generality* and *specificity*. We suspect that these are two features that appear in many instances of



mathematical beauty and that they will show up as important features when we interview mathematicians about their judgements of these two particular proofs<sup>7</sup>.

## PICK'S THEOREM

Pick's theorem gives a simple formula for calculating the area of a lattice polygon, which is a polygon constructed on a grid of evenly spaced points.

**Theorem:** Let  $A$  be the area of a lattice polygon, let  $I$  be the number of interior lattice points, and let  $B$  be the number of boundary lattice points, including vertices. Pick's theorem says that  $A = I + B/2 - 1$ .

For example, in the lattice polygon given in Figure 2a there are 16 boundary points and 1 interior point, so the area is  $1 + 16/2 - 1 = 8$ . (One can confirm this is correct by counting the number of squares and triangles inside the polygon.)

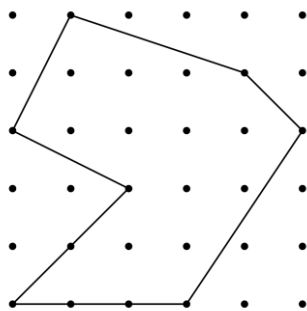


Figure 2a: Lattice Polygon

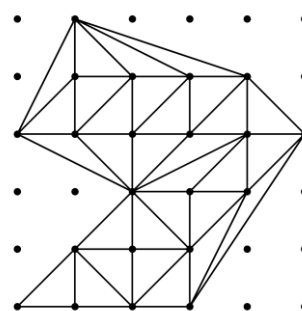


Figure 2b: Triangulated Lattice Polygon

Below we give two different proofs of Pick's theorem, which we claim are beautiful in different ways. Both proofs use a method of double counting based on a triangulation of the polygon (see Figure 2b). In the first proof, we count the angles inside the triangles in two different ways. In the second proof we interpret the figure as a graph and count the number of edges, using Euler's formula to relate the numbers of edges, faces, and vertices of the figure. We will draw on the following three lemmas, which we state here without proof.

*Lemma 1:* Any lattice polygon can be triangulated by elementary triangles.

*Lemma 2:* The area of any elementary triangle in the lattice  $\mathbb{Z}^2$  is  $1/2$ .

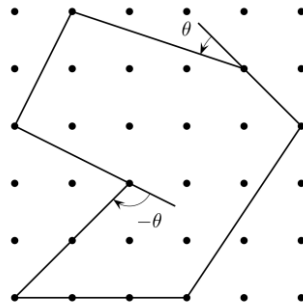
*Lemma 3:* Let  $f$  be the number of faces,  $e$  be the number of edges and  $v$  the number of vertices in a connected plane graph. Then  $v - e + f = 2$ .

**Proof 1** (using angles). We begin by partitioning the polygon into elementary triangles, which is possible by Lemma 1. (See Figure 2b.) We now sum up the internal angles of all these triangles in two different ways. On one hand, the angle sum of any triangle is  $\pi$  so the sum of all the angles is  $S = N \pi$ .

On the other hand, at each interior point  $i$ , the angles do not add up to  $\pi$ , but if we add the interior angles of the vertices, we get  $k \cdot \pi - 2\pi$ , where  $k$  is the number of

vertices, since the sum of the exterior angles is  $2\pi$  (see Figure 3). Let  $I$  be the number of interior points and  $B$  be the number of boundary points. In all, the sum of the angles at the boundary points is  $B \cdot \pi - 2\pi$ , and the sum of the angles at interior points is  $I \cdot 2\pi$ . Therefore,  $S = I \cdot 2\pi + B \cdot \pi - 2\pi$ .

We conclude that  $N \cdot \pi = I \cdot 2\pi + B \cdot \pi - 2\pi$ , so cancelling  $\pi$  we get  $N = 2 \cdot I + B - 2$ . Since by Lemma 2 the area of any elementary triangle is  $\frac{1}{2}$ , we have  $A = \frac{1}{2} \cdot N$  and thus  $A = I + \frac{1}{2}B - 1$ .  $\square$



**Figure 3: Polygon with one exterior angle marked**

**Proof 2** (using Euler's Formula). We begin by partitioning the polygon into elementary triangles, which is possible by Lemma 1. (See Figure 2b.) We then interpret the triangulation as a graph (network), where vertices in the graph are vertices of the triangulation, and edges in the graph are edges of the triangles in the triangulation. This graph subdivides the plane into  $f$  faces, one of which is the unbounded face (the area outside the polygon), and the remaining  $f - 1$  of these are the triangles inside the polygon. By Lemma 2 the area of each triangle is  $\frac{1}{2}$ , and thus  $A = \frac{1}{2} (f - 1)$ . This of course proves nothing; it is a simple consequence of how we defined  $f$ .

An *interior* edge borders on two triangles (the blue edges marked in Figure 2b), and a *boundary* edge borders on a single triangle and forms part of the boundary of the polygon itself.

Let  $e_{int}$  be the number of interior edges, and  $e_{bd}$  be the number of boundary edges. Counting the number of edges in two different ways, we get

$$3(f - 1) = e_{int} + e_{bd} (*)$$

(Here we are overcounting to get the total number of edges of the collection of triangles. The left hand side counts these edges using the fact that each triangle has 3 edges. The right hand side counts them using the fact that each interior edge contributes to two triangles while each exterior edge contributes to one.)

We can also observe that the number of boundary edges is the same as the number of boundary vertices,  $B = e_{bd}$  and that the number of vertices in the network is the sum of all the interior and boundary points,  $v = I + B$ .

## Working Group 1

Using Euler's formula and substituting for  $v$ , we get that  $(I + B) - e + f = 2$ , or by rearranging,  $e - f = (I + B) - 2$ , where  $e = e_{int} + e_{bd}$  is the total number of edges. With some clever algebraic rearrangements, starting with (\*), we get

$$\begin{aligned} f &= -2f + 3 + 2e_{int} + e_{bd} \\ &= -2f + 3 + 2e - e_{bd} \\ &= 2(e - f) - e_{bd} + 3 \\ &= 2(I + B - 2) - B + 3 \\ &= 2I - B - 1. \end{aligned}$$

Thus we get  $A = \frac{1}{2}(f - 1) = \frac{1}{2}((2I - B - 1) - 1) = I - \frac{1}{2}B - 1$ .  $\square$

## ANALYSIS AND DISCUSSION

To what extent are each of these proofs beautiful? We begin with some data from a mathematician who thought the first proof was beautiful. One way in which the proof is beautiful to him is that it gives meaning to the terms  $I$ ,  $B/2$ , and  $-1$ . He explains, "In particular, I like that you can see that each boundary lattice point contributes half as much total angle as each interior lattice point." He also said that he likes proofs that get information by counting things in different ways. The particular choice of counting angle measures, though, both contributed and detracted from the sense of beauty in this proof. He says, "The fact that the proof involves angles is beautiful in the sense that it is unexpected, but also ugly in that it breaks some symmetry." Pick's theorem, as stated, holds for any lattice polygons, regardless of whether the lattice itself is transformed in a way that preserves area. However the argument involving angle measures does not. If you shear the triangle, the angle measures change. Thus the introduction of the new quantity does not have the same property as the figure itself, which this mathematician referred to as "unnatural."

In contrast, the second proof, using Euler's formula, uses only quantities that are invariant under transformation. What seems beautiful about it is that it turns out to be an application of Euler's formula. One gets a sense of "even here, this method can apply!" But whereas the second proof is more general than the first (we introduce no auxiliary concepts) it is much less intuitive. The first proof, besides the sophisticated application of double counting, is fairly elementary. Even a grade school child can count the angle measures in both ways described above. However, the second proof requires a bit more machinery to understand. First one must conceptualize the plane in such a way that Euler's formula applies (which includes the somewhat strange step of considering the complement of the polygon in the plane as a face in itself.) Also, in applying Euler's formula, one is resting on a result which by itself is not obvious. Even if one really believes Euler's formula and feels comfortable using it to get the result, one doesn't get a full understanding of the proof if one doesn't in turn

understand why Euler's formula is true. This reliance on heavy theory seems to be an aspect which detracts from the beauty of the second proof.

We see then that in each of the proofs there is some feature that contributes to the beauty and some feature that detracts. It turns out in this case that the features are complementary. The feature that contributes to the beauty of the first proof is missing in the second and vice versa. For instance, in the second proof, what makes it beautiful is some sort of generality. This particular proof fits into a family of proofs all of which are instances of Euler's formula. In the first proof, what makes it beautiful is some sort of specificity. The surprising use of angle measures in the double counting introduces some unexpected element, which on the one hand breaks the harmony of the proof, but on the other hand— perhaps because of that breaking— becomes a compelling feature of the argument.

Both proofs appear to contain an element of surprise, but the nature of that surprise is almost opposite. In the first case the surprise arises from the specificity. We contend that the pleasure one gets from reading the proof is similar to the feeling of finding a specific tool, like the correct size hexagonal screwdriver for one particular screw. In the second case the surprise comes from the feeling of generality. The sense of fitting in arises from there being a set of objects that have a similar property. It is a wonderful, unexpected finding that this second proof is one of those kinds of proofs. To continue the tool analogy, Euler's formula is the monkey wrench, that is suitable for a great number of different situations.

## CONCLUDING COMMENTS AND NEXT STEPS

To claim that generality and specificity contribute to the beauty of these proofs through some element of surprise, we must return to Rota's criticism that beauty arises out a feeling of enlightenment rather than surprise. His critique was grounded on the fact that there are proofs that are surprising, but nonetheless not beautiful. For now we leave this as an open question, with the possibility that surprise might be a necessary (or at least contributing) but not sufficient condition for beauty. We note, however, that what seems similar about surprise and enlightenment is some sort of allure, something that grabs the mind's interest. And it might be this allure, or ``compelling"-ness referring back to Scarry again, that is the defining characteristic of beauty.

It seems fairly obvious to say that for a proof to be compelling, it must on the one hand be not too simple, and on the other hand not too complex for the mind to grasp. It might be that the features of generality and specificity are what keeps these two particular proofs appropriately compelling. The specificity of the first one makes the proof technically accessible. The generality of the second one imparts a certain status. Another striking characteristic of these two features is the fact that they play mirrored roles with each other— they are in a sense duals— and the way in which they

mirror each other is that one contributes to beauty in exactly the way that the other detracts.

At first the fact that two seemingly opposite characteristics could both give rise to beauty might seem contradictory, but we offer another interpretation: that beauty arises from the interplay of a sort of access and restraint. A potentially beautiful object which was completely accessible might not appear beautiful, just as a potentially beautiful object that is completely hidden would never be able to be experienced as beautiful. The proof based on Euler's theorem brings out some sort of hidden structure; the proof based on angle measures provides a specific instantiation of an otherwise seemingly common sort of mathematical tool.

Generality and specificity might not be just two features of beauty— they might turn out to be *exactly* the aspects of mathematical expression which provide the needed tension to give rise to the sense of beauty. We do not rule out that there might be other features that give rise to beauty, but from our preliminary analysis, we are willing to commit that the fact that we found these two particular features here is not surprising, nor idiosyncratic.

This study was meant as a first step into a rather large inquiry domain. The goal was to make the pursuit of a study of beauty in mathematics tractable, both in terms of methods and potential results. This study gives rise to a few hypotheses that we would like to investigate (and invite others to investigate!) in future studies. These include:

- (i) The features of generality and specificity are not idiosyncratic. They appear in a wide number of proofs commonly held to be beautiful.
- (ii) There is consensus among mathematicians, not just about which proofs and/or theorems are beautiful, but also about what gives rise to the sense of beauty.
- (iii) The fact that generality and specificity are related, as duals, is also not a coincidence. If there are other features that give rise to beauty, they will also be related in a way that creates some sort of tension, and the sense of beauty that arises will be related to this tension.

## NOTES

1. A description of these functions is given in Sinclair (2002): "The most recognized and public of the three roles of the aesthetic is the evaluative; it concerns the aesthetic nature of mathematical entities and is involved in judgments about the beauty, elegance, and significance of entities such as proofs and theorems. The generative role of the aesthetic is a guiding one and involves non-propositional modes of reasoning used in the process of inquiry. I use the term generative because it is described as being responsible for generating new ideas and insights that could not be derived by logical steps alone. Lastly, the motivational role refers to the aesthetic responses that attract mathematicians to certain problems and even to certain fields of mathematics."

## Working Group 1

2. Thanks to the following people who provided information on statements about beauty and aesthetics in curricular documents: Antti Viholainen (Finland), Kirsti Hemmi (Sweden), Aihui Peng (China). We welcome examples from other countries, especially those that incorporate beauty in a meaningful way.
3. Some examples of curricular statements include "appreciate the aesthetic value of mathematics theorems and mathematics methods", "experience the flexibility, the elegance (similar to the beauty, but higher than beauty, and ingenuity of mathematical proof", "experience the beauty of figure".
4. See [http://www.oph.fi/english/publications/2009/national\\_core\\_curricula\\_for\\_basic\\_education](http://www.oph.fi/english/publications/2009/national_core_curricula_for_basic_education).
5. See [http://www.cut-the-knot.org/ctk/Pick\\_proof.shtml](http://www.cut-the-knot.org/ctk/Pick_proof.shtml) for a web application of a classic way of introducing the task to middle school students. And see Sally & Sally (2007) for a lovely exposition of how this task can be made relevant to people of all ages, from school children to research mathematicians, not just in terms of verifying the theorem, but in terms of really understanding the underlying ideas. Yet another proof of the theorem, using a heat model, can be found in Hanna & Jahnke (2007).
6. These two proofs correspond closely to the two proofs given in Sally & Sally (2007).
7. To be clear, our goal here is not to establish that these two features capture all aspects of mathematical beauty, but rather to suggest that they are two features that could give rise to the sensation. The argument presented in the paper is a sketch what a mature argument about beauty in mathematical proof might look like, using the mathematics itself (rather than psychological or sociological data) as our primary data source.

## REFERENCES

- Aigner, M., Ziegler, G., & Hofmann, K. (2010). *Proofs from THE BOOK*, Dordrecht: Springer-Verlag.
- Chandrashekar, S. (1987) *Truth and Beauty. Aesthetics and Motivations in Science*. Chicago: University of Chicago Press.
- ESM Editorial. (2002). Reflections on Educational Studies in Mathematics. *Educational Studies in Mathematics*, 50(2), 251-257.
- Hanna, G. & Jahnke H. N. (2007). *Proving and Modelling* in Blum, W., Galbraith, P.L., Henn, H., & Niss, M. (Eds.), *Modelling and applications in mathematics education. The 14th ICMI Study*. pp. 145-152. New York: Springer.
- Hardy, G.H. (1940). *A Mathematician's Apology*. Cambridge: University Press.
- Poincare, H. (1908). *Science and Method*. Reprinted in 1952. New York: Dover.
- Rota, G. (1997). *Indiscrete Thoughts*. Boston: Birkhauser.
- Scarry, E. (1998). *On Beauty and Being Just*. Princeton University Press.
- Sinclair, N. (2004). The Roles of the Aesthetic in Mathematical Inquiry. *Mathematical Thinking and Learning*, 6(3), 261-284.

# MULTIMODAL DERIVATION AND PROOF IN ALGEBRA

Reinert A. Rinvold

Hedmark University College, Norway

Andreas Lorange

NLA University College, Norway

*This paper is a combination of a theoretical paper and a case study. The theoretical purpose is to introduce the concept of multimodal proof into the context of algebra in mathematics education. The case study follows two teacher training students conjecturing and proving results in an environment of visual and physical number patterns. The case study is used both to discuss what multimodal proof can be in this environment and to show some of the potential of such reasoning in the learning of proof.*

Key words: multimodal proof, proof environment, algebra, physical number patterns.

## INTRODUCTION

The interest in visual proof and visual reasoning has been increasing over the last decades. This is the case both inside mathematics and in mathematics education. Several references are given in Bardelle (2009). Barwise and Etchemendy (1994) has developed a computer based proof environment called Hyperproof. Logic students carry out proofs with this program that combines sentential formal logic rules with visual reasoning related to positions and sizes of boxes and pyramids. Barwise and Etchemendy (1996) use the phrase heterogeneous proof for this kind of proofs. This name points to the two distinct kinds of reasoning involved. Oberlander, Monaghan, Cox, Stenning and Tobin (1999) characterize hyperproofs as multimodal, because of the use of both graphical and sentential methods. We think that also other modalities are relevant for mathematical proof. In the context of mathematics education it is possible to involve students in proof like activities which not only uses the sentential and visual modality, but also the tactile, speech and motor action. We introduce a general concept of multimodal proof that intends to cover both formal proofs, visual proofs and proof like student activities. We are not claiming that multimodal proof should be seen as legitimate rigorous mathematical proof. The phrase “visual proof” is widely used even if many mathematicians do not accept such reasoning as part of rigorous proof. In the same way multimodal proof may be seen as a kind of intuitive proof.

This paper analyzes one empirical case study with teacher training students to show what multimodal argumentation and proof can mean. The learning of proof is difficult for students, especially if their background in formal mathematics is weak. Proving in the context of visual and physical number patterns gives meaning and allows for the students’ intuitive thinking. Seeing the conjecturing and arguing

process with number patterns as multimodal proof, makes clearer the potential this kind of activity has for proof learning. This uncovers structure in the students' thinking similar to traditional proof but in other modalities. Those similarities explain why we use the term “proof”, and not just multimodal argumentation.

## THE CONCEPT OF MULTIMODAL PROOF

In traditional formal proof, the emphasis is on the modality of written symbols and sentential reasoning. A multimodal proof is a generalized proof which beside written symbols and sentential reasoning can also include the visual modality, speech, the tactile and motor action. Gestures are included in the latter. Multimodality means that more than one modality is involved. Barwise and Etchemendy (1996) make a point that visual and non-visual proofs are not separate worlds. Visual proof is, of course, highly dependent on the visual modality, but the mathematician looking at such proofs invokes both language and mathematical concepts in his thoughts. Moreover, proofs not mentioning any picture or diagram have visual aspects. The first argument is that written mathematical symbols by themselves are visual (Rinvold, 2007). Secondly, mathematicians almost always relate intuition to their proofs. This intuition is very often connected with visual ideas, for instance see Sfard (1994).

It is not the intention of this paper to give a precise definition of what a multimodal proof is. This has to be done in forthcoming papers. At the moment we make a first step. In order to increase precision, attention can be restricted to subclasses of the general concept. The concept of proof environment may be useful. The proof environment is the available and allowed resources, objects, operations and modalities. In a sense we have one concept of multimodal proof for each possible proof environment.

A formal proof can be seen as symbols on paper. Multimodal proofs can also be mediated by books, as for instance in Nelsen (1993, 2000). The inclusion of gestures, speech and motor action, means however, that multimodal proof has to be linked to human cognition. More specifically, we are talking about sensuous cognition, cf. Radford (2009). We adapt the semiotic-cultural perspective. “Thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts.” (p. XXXVI). In the environment of our case study, gestures, tactility and motor action is considered both to be genuine parts of cognition and multimodal derivation.

A difference between text based proof and multimodal proof is the non-linearity of the latter. In the “proofs without words” of Nelsen (1993, 2000), some proofs are given holistically by diagrams with limited guidance to the reader. Traditional proofs are given step by step. Numbering of formulas and names of applied theorems are ways to guide the reader. When the carriers of information are physical, gestures are an important way of directing the attention. More specifically, pointing gestures are



used for this. The research literature uses the name *deictic gestures*. According to Sabena (2008) “*deictic gestures*: indicate objects, events, or locations in the concrete world” (p. 21). Some words in written language or oral speech have a similar function. Radford (2009) gives the examples “top” and “bottom” (p. XLI). These kinds of words are called “spatial deictics”, (p. XLI). Other pointing phrases are “this one” and “that one”. Both linguistic and non-linguistic deictics are central to multimodal deduction. In classical proof we use mathematical deictics as “equation 25” or “theorem 4”, which means that neither this kind of deduction is completely linear.

## **ALGEBRAIC THINKING IN MATHEMATICS EDUCATION**

In mathematics education the multimodal perspective has been used to analyze student thinking, especially in algebra and functions. Radford, Edwards and Arzarello (2009) introduce what they called “construction of mathematical meaning from the perspective of multimodality”:

... taking into account the range of cognitive, physical, and perceptual resources that people utilize when working with mathematical ideas. These resources of modalities include both oral and written symbolic communication as well as drawing, gesture, the manipulation of physical and electronic artefacts, and various kinds of bodily motion. (Radford, Edwards and Arzarello 2009, p. 91)

Our use of the word multimodal is inspired by and compatible with the perspective of those researchers. Radford (2009) have used a multimodal approach to the introduction of students to the elementary parts of algebra. We share the context of visual number patterns in algebra with Radford, but our focus on proof is new.

Radford classifies algebraic thinking into three forms, factual, contextual and symbolic. Following Radford, contextual thinking acknowledges a general figure through the concept of figure number, but “still supposes a spatially situated relationship between the individual and the object of knowledge...” (p. 9). The students’ reasoning depends on the particular context and perspective. When the students start to use symbols for variables, but still think contextually, Radford observed a phenomenon which he called iconic formulas or formulas as narratives. Formulas are not simplified, but divided into parts. Each part tells a story about the corresponding part of the pattern described by the formula. We identified both the use of contextual thinking and iconic formulas by the students in our case study. Contextual algebraic thinking is not necessarily a low level of thinking, but the students also need to master the symbolic form in order to use multimodal thinking in a mature way.

## **LEGITIMACY OF MULTIMODAL PROOF**

In pure mathematics, visual proofs are still not widely accepted as legitimate part of rigorous proof. However, arguments have been put forward that may change this in

the future. Barwise and Etchemendy (1996) have done work in mathematical logic which argue for the legitimacy of some kinds of visual reasoning. In philosophy of mathematics Brown (2008) argues for the possible validity of pictures in proofs. Jamnik, Bundy and Green (1997) and other studies in the field of automated reasoning have also indirectly provided good arguments for the case of visual proof. They have shown that typical diagrammatic proofs can be formalized with recursive  $\omega$ -logic and automated. This is an argument for the legitimacy of a wide class of visual proofs. These proofs are based on generic reasoning. In such proof the general is proved through special cases. One or two instances are used to describe how to carry out the proof for any instance. Recursive  $\omega$ -logic is an alternative to the classical induction approach, but equally valid. In  $\omega$ -logic a formula is proved if you have a proof for each instance of the formula. A recursive  $\omega$ -proof also has an algorithm to find the proof of each given instance. The use of physical figures and number patterns in our case study are confined to generic reasoning. The validity of multimodal reasoning in these kinds of environments thus rests on the correctness of generic arguments.

Conditions for legitimate formal reasoning are well developed. Similar analyses of visual reasoning are still in an early phase. The work of Barwise and Etchemendy (1996) is relevant also for multimodal proof, but the possible legitimacy of multimodal proof is not yet analysed in depth. An idea of Barwise and Etchemendy (1996) that can be useful for such analysis is the concept of information. “Valid deductive inference is often described as the extraction of making explicit of information that is only implicit in information already obtained.” (p. 4). Mathematical formulas and written text contain information. Classical proofs show how the information given by a theorem is already implicit in axioms and established mathematical results. Also pictures, diagrams and physical patterns contain information. A map, for instance, contain lots of information. If you know the use of a map and compass, you can deduce how to get from A to B. As in classical proof, some deductions are valid and others not. You do not get from A to B if you do not use the map and compass in legitimate ways.

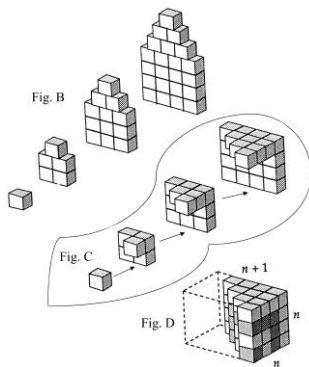
## **METHODOLOGY**

The paper uses one single successful case study with teacher training students to give examples and to discuss what multimodal argumentation and proving can mean. The students are in a problem solving process leading to derivation of formulas. We do not see a straight line of reasoning, but also some mistakes and dead ends. As such, the students are proving, not presenting a proof. The data gives examples of multimodal proving, but much of the argumentation could with some refinement be part of a multimodal proof. We thus see continuity between proving and multimodal proof. Within a traditional proof paradigm, more radical changes are needed to go from the students' argumentation to proof. However, we do not claim that this kind

of continuity between argumentation and proof is always the case within the multimodal paradigm.

In the spring of 2009 five pairs of teacher training students took part in a study conducted at Hedmark University College and NLA University College. All of the students were following a course in number theory at one of the two colleges. The pairs were asked to investigate number patterns represented by visual and physical figures. Their work is immediately followed by an interview by one or both researchers. Both the student investigations and the interviews were videotaped. One episode from one of the pairs is chosen for analysis. The selected pair clearly was the most successful. We observed a complex interplay between the students and between visual diagrams, physical figures, speech, gestures and formulas.

## THE START OF THE CASE STUDY



The students Erik and Jon are given a problem sheet with two equivalent visual number patterns B and C and asked to find out as much as possible about the sequence. Also included is a figure D. The latter figure results when the four figures in C are joined. In front of the students are physical versions of C and D built from plastic cubes. More cubes are available so that they may build their own figures. Each student also has a personal note sheet. Before we come into the story, the students have spent about 20 minutes. Among other things, they have derived an explicit formula for the sequence in B by decomposing each figure into a square and a triangle on top of it. They also observed that B and C give the same sequence.

## THE FORMULA TO BE PROVED

We will follow the derivation of a recursive formula for the sequence in B. The students were not asked to prove such a formula. They discovered and proved the formula themselves. In the problem solving process the given recursive formula was the end product. They first wrote the square part of the formula and then added the triangle part. As can be seen,  $F_{n-1}$  was written at a later stage.

$$F_n = \underbrace{F_{n-1}^2}_{\square} + \underbrace{n-1}_{\triangle}$$

$F_n$  refers to the  $n$ 'th figural number from the left in B or C. The formula is a combination of standard mathematical symbols, curly brackets and drawn figures. The latter signify the square and triangle part of the formula. The formula to be proved is not part of standard notation. The drawn square and triangle link the formula to the multimodal context of the activity. Following Radford (2009), the formula could be seen as iconic or as telling a narrative. This means that the formula mirrors the division of each figure into a square part and a triangle part.

In the following analysis, we use the students' argumentation to show what a multimodal proof can be. Deictics has a central role. We also look for generality in the students' thinking. One example is the use of indices. It is not mature, but nonetheless they think generally through them.

## THE BEGINNING OF THE DERIVATION

The start of the derivation is an oral description of the task:

105 Jon: We are going to find a formula to find the next one when we know this one.

Descriptions, definitions and reformulations are often the start of direct proof. No deictic gestures are used in 105. This may be interpreted as generality in the students' reasoning. Then they enter a journey outside the main path, but after a while Jon reformulates 105 with gestures and symbols on his note sheet. He has already written " $F_n = F_{n-1} +$ ".

139 Jon: To find  $F_n$  [pointing at the sign  $F_n$  on his note sheet] it's the last one [circular pointing gesture around  $F_{n-1}$ ] plus something more [points to the right of +].

Both deictic gestures and deictic speech turn up already at this juncture when standard mathematical symbols appear. The last deictic gesture is abstract pointing. According to Sabena (2008), abstract pointing is "when there is no actual physical pointed object, rather the pointed empty space houses an introduced reference, ..." (p. 22). Then Jon introduces the visual B-pattern on the problem sheet. The second and third figure in this pattern are denoted  $F_2(B)$  and  $F_3(B)$ .

141 Jon: It becomes this [circular pointing gesture around  $F_2(B)$ ] plus something more [points in direction of  $F_3(B)$ ].

The deictics shows the connection when 141 is derived from 139. Note the striking similarity between gestures linked to corresponding terms, for instance the circular pointing gestures used both for the sign  $F_{n-1}$  and the visual figure  $F_2(B)$ .

## THE LEMMA

At this juncture both students start to manipulate the physical figures. Jon puts  $F_2$  and  $F_3$  together and then takes one of these figures in each of his hands. After a short period of silence Jon expands what he wrote in 139 to this formula:

146 Jon:  $F_n = F_{n-1} + (K_n - K_{n-1}) + (T_n - T_{n-1})$

$K_n$  and  $T_n$  refer to the  $n$ 'th square numbers ('Kvadrattall' in Norwegian) and triangular numbers.

147 Jon: This one is nice [smiling]! But, is it correct?

The question about correctness means that 146 at this stage is a conjecture or a lemma to be proved. The lemma splits the problem of 105 into two parts. Jon's smile and description of formula 146 as "nice" may be an indication that this is the kind of answer which is expected in mathematics. But, we will see that the formula is used more as a vehicle of thought than in the standard way. Careful readers may have noticed that the indices for the triangular number part are wrong. The students, however, do not comment on this "mistake", but get it right when  $T_n - T_{n-1}$  is replaced by  $n - 1$ . The indices  $n$  and  $n - 1$  at this stage seem to have the operational meaning "the next one" and "this one". One argument for this is the complete similarity between Jon's derivations of the triangle part and the square part of 146. Because of this similarity we only show Jon's derivation of the square number part:



149 Jon: From this one [pointing to physical  $F_2$ ] to this [pointing to  $F_3$ ], this square number is added [lay his fingers down on  $F_3$ ] minus the square number we had earlier [makes a circular pointing movement around the bottom of  $F_2$ ].  $K_n$  minus  $K_{n-1}$ .

In fact, Jon uses 141, but he has replaced the visual B-patterns with the physical figures. 149 is the first time physical figures are explicitly used in the derivation. Deictic gestures now have a vital place in the derivation to show which part of the figures which is in focus. Note how the physical figures give possibilities for clearer and more precise gestures compared with the visual B- and C-figures.

## THE FINAL PART OF THE DERIVATION

Now we come to a striking example of multimodal proving involving physical objects and gestures. The lemma (146) is proved, and the students return to the derivation of the recursive formula. They are going to replace each part of the lemma with a simplification.

152 Jon: [Laughs] I doubt the formula should look like this.

153 Jon: It isn't very nice, but [smiles and laughs] [silence]

154 Jon: Yes, gets right, yes, but probably has to be rewritten in one way or another.

At this moment Jon realizes that the lemma can be simplified and starts to think how. We will follow the simplification of the square number part. The reasoning for the triangle part is similar. Jon has changed his mind and does not find the formula

## Working Group 1

“nice” anymore. Jon’s question is triggering Erik to make a contribution. He takes the physical  $F_3$  and puts it on top of  $F_4$ :



- 159 Erik: What is added then? It is the sides, the sides of the first quadrilateral number plus one. [He moves his finger along the side while talking.]
- 160 Erik: Like quadrilateral numbers...
- 161 Jon:  $n$  minus one then
- 162 Erik: Two times minus one plus one. [He writes a formula on his note sheet.]

$$F_n = F_{n-1} + 2(n-1) + 1$$

- 163 Erik: The corner is included. [He points to the corner between the mentioned sides.]

Erik locates the square numbers as the bottom part of the physical figures. He physically compares the bottom squares of  $F_4$  and  $F_3$ . The way the figures are relatively placed allows him to show one of the added sides with a sliding finger gesture. The plus one is explained by pointing to the corner below the triangular number part of the white physical  $F_4$ .

This explanation by Erik is an example of reasoning which can be refined. His idea is good, but the triangular number on top of the bottom square obscures the reasoning. In fact, Jon does not understand, and Erik follows up by an improvement. Erik takes off the triangular parts of  $F_2$  and  $F_3$  and places  $K_2$  on top of  $K_3$ .



Then he shows a sliding finger movement along each of the two sides and points with a finger to the corner. Now it is obvious to Jon what is going on.

## THE GENERALITY OF THE REASONING

The interview indicated that the concept of figure number was a safe ground which resolved potential dangers in general reasoning. Again, this corresponds to the contextual form of reasoning in the classification of Radford (2009).

- 341 Jon: We noticed by looking at the drawing that the square number equals the figure number.

342 Researcher: Yes.

343 Jon: And the triangle number is one less.

Except for the non-standard use of indices in the lemma, they always related figure numbers correctly to symbolic variables. This indicates that the students were aware of the generality of their reasoning. Their switching between different values of  $n$  when relating to physical figures strengthens that conclusion. In “The final part of the derivation” Jon uses  $n = 3$  and Erik  $n = 4$ , but they do not comment on the difference. Jon and Erik behave as if they are talking about the same thing. A few minutes later Erik repeats his sliding finger gesture argument with  $n = 3$ , also with no comments. Jon was convinced that the reasoning done for a particular value of  $n$  could be done for all other values as well. They have tested the formula in the theorem for  $n = 4$ , but not for  $n = 5$ . When the researcher asked for reasons to trust the formula, Jon repeated the derivation from the third and fourth physical figures. Then the following dialog ends the interview relating to the recursive formula:

428 Jon: You really see that it’s logical both from the figures and from the plastic cubes.

429 Researcher: Yes, you could have built something similar if we considered number five from four, for instance?

430 Jon: Yes! [Looking convinced]

## CONCLUSION AND QUESTIONS FOR RESEARCH

Traditional proof has been thought to consist only of sentential reasoning. Visual proof in an active way uses visual information. Multimodal proof can also include the use of physical objects, the tactile, gesture and other kinds of motor action. Especially the latter modalities transcend the traditional concept of proof.

The analysis of data has shown that quite advanced multimodal proving is possible for students even if their form of algebraic thinking is partly contextual and only to some extent symbolic. There is a form of structure in their proving even if they are not trained in formal proof methods. A possible explanation is the resources of intuitive thinking which is opened by this approach. A conclusion is that this kind of proof activities has a potential in proof learning. The data analysis has also shown the important role of deictics in multimodal reasoning.

The concepts of multimodal proof and derivation need further clarification and development in order to discern valid proof from other kinds of activity and presentation. We think that the concepts of proof environment and extraction of information will be helpful in this. Further research is needed both to find conditions for valid reasoning and to investigate the role of deictics in multimodal proof. The latter may also be a key to better understanding of visual proof. In mathematics education more empirical studies are necessary, combined with development of better design and teaching approaches. A question of research is how to support the development of students reasoning to include a symbolic form of thinking.

## REFERENCES

- Allwein, G. & Barwise, J. (1996) *Logical Reasoning with Diagrams*. Oxford University Press.
- Bardelle, C. (2009). Visual proof: An Experiment. In Durand-Guerrier, V., Soury-Lavergne, S. & Arzarello, F (Eds.), *Proceedings of CERME 6*, (p. 251-260), Lyon France.
- Barwise J. & Etchemendy J. (1996). Visual Information and Valid Reasoning. In G. Allwein and J. Barwise (Eds.), *Logical Reasoning with Diagrams* (p. 3-25). Oxford University Press.
- Barwise, J. & Etchemendy, J. (1994). *Hyperproof*. Stanford, CA: CSLI Publications.
- Jamnik, M., Bundy, A. & Green, I. (1997). Automation of Diagrammatic Reasoning. *AAA-I Technical Report FS-97-03*.
- Brown, J. (2008). *Philosophy of Mathematics. A Contemporary Introduction to the World of Proofs and Pictures*. Second edition. Routledge.
- Nelsen, R. (1993). *Proofs Without Words: Exercises in Visual Thinking*. The Mathematical Association of America.
- Nelsen, R. (2000). *Proofs Without Words II: More Exercises in Visual Thinking*. The Mathematical Association of America.
- Oberlander, J., Monaghan, P., Cox R., Stenning K. & Tobin, R. (1999). Unnatural Language Processing: An Empirical Study of Multimodal Proof Styles. *Journal of Logic, Language, and Information*, vol. 8, no. 3, Special Issue on Efficacy of Diagrammatic Reasoning 363-384.
- Radford, L. (2009). Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective. In Durand-Guerrier, V., Soury-Lavergne, S. & Arzarello, F (Eds.), *Proceedings of CERME 6*, (p. XXXI-LIII), Lyon France.
- Radford, L., Edwards & Arzarello (2009). Introduction: beyond words. *Educational Studies in Mathematics* (2009) 70: 91-95.
- Rinvold, R. A. (2007). Metaphors and image schemata in concept formation and reasoning. In Demetra Pitta-Pantazi, D. & Philippou, G. (Eds.). *Proceedings of CERME 5*, (p. 171-180), Larnaca, Cyprus.
- Sabena, C. (2008). On the semiotics of gestures. In L. Radford, G. Schubring and F. Seeger (Eds.) *Semiotics in Mathematics Education: Epistemology, History, Classroom, and Culture* (p. 19-38). Sense Publishers.
- Sfard, A. (1994). Reification as the Birth of Metaphor. *For the Learning of Mathematics* 14, 1 (February 1994).



# THE VIEW OF MATHEMATICS AND ARGUMENTATION

Antti Viholainen

Umeå Mathematics Education Research Center

Umeå University

*The view of mathematics affects how the role and function of argumentation is seen. If mathematics is considered as an axiomatic system, arguments have to be based on definitions, axioms and previously proven theorems, and the function of argumentation is mainly to verify and systematise a statement. On the other hand, if mathematics is seen as thinking and learning process, which requires conceptual and holistic understanding, then arguments based on more concrete representations are important. In this latter case, the function of argumentation is to engender understanding by explaining. These two types of argumentation provide a framework, which could also be applied in the design of teaching.*

**Key words:** argumentation, formal and informal reasoning, mathematical views, representation.

## INTRODUCTION

What kinds of arguments should be favoured in mathematics education? What kinds of arguments should students learn to understand, produce and present? These are important questions in design of teaching practices, teaching materials, assessment etc. In this paper some aspects to these questions will be presented on the basis of different views of mathematics and results of some earlier studies about mathematical reasoning.

Arguments can be seen both as elements and as products of a mathematical reasoning process. Often an aim of a reasoning process is to construct an argument. This process may include inductive, deductive or abductive reasoning, use of intuitions, making conjectures and testing of them etc. Both cognitive and affective factors influence this process (Furinghetti & Morselli, 2009). It may also include construction of sub-arguments, which are needed in other parts of reasoning. Also, different kinds of representations may be used in the reasoning process.

Toulmin's (2003) model of argumentation in mind, it can be said that the aim of argumentation is to construct an explanation (*a warrant*) for why the information concerning the initial state (*the data*) necessitates the statement which is argued (*the conclusion*). In some cases, a justification for the authority of the warrant (*backing*) is also needed. The same conclusion can often be argued by using different kind of arguments.

In order to answer to the question why *proofs* are needed in mathematics, the following functions are often presented: *verification*, *explanation*, *systematisation*, *discovery* and *communication* (Hanna, 2000; de Villiers 1999). Argumentation has a

broader meaning than the term proof, but it can be thought to have these same functions too. In the following, more specifically the functions of verification, explanation and systemisation are evaluated with respect to different views of mathematics.

## **DIFFERENT VIEWS ABOUT MATHEMATICS AND ARGUMENTATION**

Mathematics can be considered either as *a toolbox*, as *a system* or as *a process* (Törner & Grigutsch, 1994). The toolbox-view means that mathematics is seen as a set of skills, the system-view means that mathematics is a logical and rigorous system and according to the process-view mathematics is a constructive problem solving process (Törner, 1998). Ernest (1989) has presented a corresponding division by defining *the instrumentalist view* to mean that mathematics is seen as “an accumulation of facts, rules and skills to be used in the pursuance of some external end” (p. 250), *the Platonist view* to mean that mathematics is “a static but unified body of certain knowledge” (p. 250), and the *problem solving view*, for one, to mean that mathematics is “a dynamic, continually expanding field of human creation and invention” (p. 250).

### **The toolbox/instrumentalist view and argumentation**

The view of mathematics has an influence on how teaching and learning of mathematics are seen. On the basis of several studies, Beswick (2005) has connected the instrumentalist view to content-focused teaching in which emphasis is on performances and learning is seen as a passive reception of knowledge. At least, the toolbox/instrumentalist view alone is neither sufficient nor desirable, because deep understanding, knowledge construction and learners active role have been omitted from it, and mathematics as such is not seen interesting. Therefore, the role of argumentation is only to ensure the correctness of facts and rules, that is, only verification can be considered as a function of argumentation.

### **The system/Platonist view and argumentation**

The Platonist view is often connected to an objectivistic worldview. According to Ernest, the Platonist view involves understanding mathematics as a consistent, connected and objective structure. Mathematical objects are seen to be real and exist independently of human (Brown, 2005). Mathematical statements are considered to be objectively true or false and their truth-values are also seen to be independent from human. In addition, mathematical knowledge is seen to be non-empirical. This kind of objectivistic view of knowledge implies easily that, in the classroom, the teacher is seen as an explainer and the learning is seen as a reception of knowledge. According to Beswick (ibid.), the Platonist view also implies content-focused teaching. However, the emphasis is on understanding: Learning is seen as an active construction of understanding, but through assimilation of received knowledge.

On the other hand, the view about mathematics as a system is so salient that it cannot be omitted. The building of an axiomatic system can be seen as an essential goal in mathematics [1]. *Systemisation* means that various known results are ordered into a deductive system, and it has usually been considered as one important function of proof and proving (Hanna, 2000; de Villiers, 1999). According to De Villiers (ibid.), systemisation is useful, because it “helps to identify inconsistencies, circular arguments and hidden or not explicitly stated assumptions” and because it “unifies and simplifies mathematical theories by integrating unrelated statements, theorems and concepts with one another, thus leading to an economical presentation of results” (p. 277). In addition, de Villiers mentions a global perspective and easiness in applications as benefits of systematisation.

Mathematics can yet be considered as a consisted and connected structure without any global or objective meaning. It may be seen either as a personal or socially shared construction, which works as a frame of reference in mathematical reasoning. It is not seen as an objective system, but the wideness in which the system is socially shared may vary. This kind of view is well compatible with the problem solving view, which is discussed in the next section.

If mathematics is considered as an axiomatic system, an important function of argumentation is to connect a statement to the system. Therefore, it is important that the argument is based on the elements of the existing system. By applying Toulmin’s model, the concept of a *formal argument* can be defined in the following way:

An argument is *formal*, if its warrants are based on definitions, axioms and previously proven theorems, i.e. the elements of an axiomatic system.

Usually formal arguments are rigorous and detailed, and, thus, they remove all doubts and uncertainty about the truth of a statement. Therefore, in addition to systemisation, verification is their important function too.

### **The process/problem solving view and argumentation**

According to Beswick (ibid.), the problem solving view can be connected to learner-focused teaching, in which learning is seen as autonomous exploration of the learner’s own interests. Beswick sees the process/problem solving view to be in accordance with the principles of the constructivist learning views. If mathematics is looked from this point of view, it is important that the learners understand the content conceptually and holistically and that they can connect it to their earlier experiences, either inside or outside of the field of mathematics. In addition, invention of new creative ideas is important. According to de Villiers (ibid.), the aim of an explanation is to help an individual to understand the reasons, why a statement is true, in other words, to provide an insight into why the statement follows from the given data. This function of argumentation is crucial when mathematics is seen from the process/problem solving view.

Weber and Alcock (2004) have presented a categorization for proof production, which contrasts the functions of verification and explanation. According to their categorization, *a syntactic proof production* refers to reasoning in which inferences are drawn using only symbolic manipulation in a logically permissible way and *a semantic proof production* means that different kind of internally meaningful representations or mental images (Weber and Alcock use the term instantiations) are used to guide the reasoning [2]. Weber and Alcock regard the semantic proof production primarily as a support for the syntactic proof production so that it guides the process of choosing proper facts and theorems to apply. By the semantic proof production an individual can in a meaningful way make sense of the claim, get suggestions about inferences that could be drawn and become convinced at an intuitive level about the truth of the claim.

A similar contrast between the functions of verification and explanation is observed in Raman's (2002; 2003) categorisation of arguments into private and public ones. According to her *a public argument* has to be sufficiently rigorous for a particular mathematical authority, like a teacher at school, and it has to reveal step-by-step the progress of inference and justifications for each step. Instead, *a private argument* is an argument engendering understanding and having an essential role in facilitating conceptual and holistic understanding of relationships between concepts. According to her, private arguments are often based on empirical or visual data.

Construction of mental images is important for understanding explanations but also to construct explanations. According to Presmeg (2006a), a mental imagery may occur in various modalities, such as sight, hearing, smell, taste or touch, but in mathematical thinking the visual modality is the most prevalent one. Therefore, it is understandable that the role of visual representations in learning of mathematics and in mathematical reasoning is an issue that has widely aroused interest and vivid discussion among mathematics educators (Presmeg, 2006b).

The following definition of *an informal argument* emphasises the need for explanations as a function of argumentation:

An argument is *informal*, if its warrants (cf. Toulmin's model) are based on concrete interpretations of mathematical concepts, which may be based on visual or other illustrative representations.

According to this definition, the characteristic of informal arguments is that mathematical concepts are interpreted by using illustrative representations. Perhaps, visual representations are the most important ones, but, in addition to them, mathematical concepts can be illustrated, for example, by relating them to some physical context. However, the illustrative effect of representations and the explanatory effect of arguments based on them may depend on personal experiences, situational factors and the field of mathematics. In the next section, an example concerning the concept of derivative is presented.

## **FORMAL AND INFORMAL ARGUMENTS**

### **An example: Formal and informal reasoning concerning the concept of derivative**

The formal definition of derivative in the case of a real-valued function of a single variable is based on the concepts of function, limit and real numbers. If these concepts have been defined earlier in teaching, then the aforementioned formal definition connects the concept of derivative to the axiomatic system. [3]

By using visual representations, the meaning of the derivative can be described by referring to the steepness of the graph of a function. It can be explained that the sign of the derivative reveals whether the graph is going up or down, and the absolute value describes how steep the uphill or downhill of the graph is. It can also be said that the derivative at a given point is the slope of a tangent line drawn to the graph at this point. For more dynamic visualisation the derivative can be illustrated by sliding a pencil along the graph from left to right so that the pencil always lies on the tangent line, and the nib of the pencil points in the direction of the movement (Hähkiöniemi, 2006).

An instantaneous rate of change can be regarded as a physical interpretation of derivative. For example, instantaneous speed is the derivative of the total distance travelled as a function of time, instantaneous acceleration is the derivative of the speed as a function of time, and the electric current is the derivative of the flowing electric charge through a surface.

Next the theorem stating that the derivative of a constant function is everywhere zero is considered. By implementing the formal definition of derivative, this theorem can be proven through a short calculation. Visually, the same result can be reasoned by explaining that because the graph of a constant function is a horizontal straight line, it does not have any uphill or downhill and, therefore, the tangent drawn to the graph is everywhere a horizontal straight line, whose slope is zero. Physically, the same thing can be reasoned by explaining that if a quantity is constant, its value does not change, and thus the rate of change is everywhere zero.

### **Relationship between formal and informal arguments**

Previous examples illustrated how formal arguments usually serve for the functions of systematisation and verification, and how informal arguments often serve for the function of explanation. However, this categorization is not absolute: A formal argument may be explanative, but this requires that its overall central ideas are recognised. On the other hand, an informal argument may in some cases be general and rigorous enough so that it is sufficient to verify a statement. Especially, the role of visual arguments has occasionally raised vivid discussion among researchers of mathematics education (Presmeg, 2006b). Several researchers have proposed that

visual arguments should be considered as an intended and accepted form of final arguments (Arcavi, 2003; Dreyfus, 1994; Rodd, 2000).

Construction of formal arguments often requires exact and detailed analytic reasoning based on symbolic representations and procedural skills to carry out calculations and other technical procedures. However, informal arguments may reveal holistic features and wider trends, which, yet, may also be very important in the construction process of the argument, by simplifying and concretising the problem situation. Experienced mathematicians are often able to utilize informal elements, like visualization, in an effective way in their reasoning. Stylianou (2002) noticed that mathematicians use visualization in a very systematic way, so that in their reasoning the visual and analytic steps were very closely connected and they interact with each other. Also Raman (2002; 2003) found that mathematicians considered visual and formal arguments closely connected so that the visual arguments in an essential way contributed to inventing ideas in construction of the formal argument. Mathematicians were able to use and construct heuristic/informal and procedural/formal ideas simultaneously so that both ideas clarified each other. Instead, students could not recognise connections between visual and formal arguments. Stylianou's and Silver's (2004) study revealed that mathematicians also saw a wide variety of problems where visualization could be used, whereas students considered visual representations useful mostly in geometrical problems.

The importance of formal and informal arguments is dependent on personal and institutional needs. As well, they may have different roles depending on the field of mathematics. Especially, students may have different tendencies with respect to formal and informal reasoning depending on the field of area. Weber and Alcock (ibid.) found that in the case of algebra students' reasoning was too much restricted around the formal definitions of the concepts, but in the case of analysis several studies have reported students' tendencies to use informal approaches without sufficient connections to the formal theory (Juter, 2005; Pinto, 1998; Viholainen, 2006; 2007; 2008; Vinner, 1991).

### **The use of the division of arguments into formal and informal ones**

On the basis of the aforementioned definitions, it could be possible to categorize arguments into informal and formal ones. According to these definitions, the decisive difference between formal and informal arguments is in the natures of the warrants. It is not decisive, how much and what kinds of representations have been used in reasoning, when the arguments are constructed. For example, the use of visualization as an aid of thinking in reasoning does not make the argument informal. On the other hand, in construction of an informal argument, the applied visual or physical interpretations may be justified by using formal definitions, but this does not make the argument formal. In this latter case, the definitions work only as a backing (cf. Toulmin's model) for the used informal interpretations. Therefore, the categorisation of arguments based on these definitions has to be made on the basis of the final

forms of the arguments, not on the basis of the reasoning processes. In this way the categorisation presented in this paper differs in an essential way from Weber's and Alcock's and Raman's categorisations.

Weber and Alcock see the semantic proof production, as well as Raman sees the private arguments, as important element of a reasoning process, but not as a final goal. Instead, the definitions of formal and informal arguments make it possible to consider any of them as a final goal of argumentation. No mutual order between the types of arguments with respect to importance follows from the definitions. Therefore, this division provides a framework for design of teaching, especially, for design of argumentation tasks, in which both the need of systematisation and verification based on the view of mathematics as a system and the need of explanation based on the view of mathematics as a problem solving process are equally taken into account. Traditionally, only formal arguments are considered as intended and desirable in mathematics, but in order to consider the need of explanation in the design of teaching, opportunities to exercise both understanding and producing more explanative arguments should also be provided for the learners.

## CONCLUSION

The presented division of arguments into formal and informal ones can be used as a starting framework in determining what kinds of arguments students should learn to understand and produce. However, this division does not cover all features of mathematical reasoning: It concerns mainly deductive arguments, but it does not cover, for example, inductive and abductive arguments and arguments, whose warrants are based on some authority. These kinds of arguments may have an important role in mathematical reasoning, especially, in the affective level. The broad variety of different forms of reasoning comes out, for example, in Harel's and Sowder's (1998) classification of proof schemes. It is possible to extend the presented framework on the basis of Harel's and Sowder's classifications.

In addition, it should be noted that the nature of mathematics as an axiomatic system is explicit mostly in the tertiary-level. As well, aforementioned studies about mathematical reasoning concern mainly the tertiary level. Therefore, more studies about applicability of this division into the lower levels of mathematics education are needed. In addition, the nature and the purpose of informal arguments may differ depending on the field of mathematics and this difference could be investigated further.

## NOTES

1. In 1931, Gödel proved incompleteness theorems, which showed that it is impossible to construct a complete and consistent axiomatic system, in which all theorems concerning natural numbers could be proven. Due to that, an ambitious attempt to build a complete and consistent axiomatic system including all mathematics proved to be impossible.

2. Alcock and English (2008; 2009) and Weber (2009) have later modified these definitions.
3. In practice, the concept of real numbers is rarely defined properly before presenting the definition for the concept of derivative.

## REFERENCES

- Alcock, L. & English, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics*, 69, 111-129.
- Alcock, L. & English, M. (2009). Representation systems and undergraduate proof production: A comment on Weber. *The Journal of Mathematical Behavior*, 28, 209-211.
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215-241.
- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52, 215-241.
- Beswick, K. (2005). The beliefs/practice connection in broadly defined contexts. *Mathematics Education Research Journal* 17(2), 39-68.
- Brown, J. R. (2005). Naturalism, pictures, and Platonic intuitions. In P. Mancosu, K. F. Jørgensen & S. A. Pedersen (Eds.), *Visualization, explanation and reasoning styles in mathematics* (pp. 57-73). Dordrecht: Springer.
- De Villiers, M. (1999). The role and function of proof with Sketchpad. In M. de Villiers (Ed.), *Rethinking Proof with Sketchpad* (pp. 3-10). Emeryville, CA, USA: Key Curriculum Press.
- Dreyfus, T. (1994). Imagery and reasoning in mathematics and mathematics education. In D. F. Robitaille, D. H. Wheeler & C. Kieran (Eds.), *Selected lectures from the seventh international congress on mathematical education* (pp. 107-122). Quebec, Canada: Les Presses d'Université Laval.
- Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249-253). New York: Falmer.
- Furinghetti, F. & Morselli, F. (2009). Every unsuccessful problem solver is unsuccessful in his or her own way: affective and cognitive factors in proving. *Educational Studies in Mathematics*, 70, 71-90.
- Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, 44, 5 - 23.
- Harel, G. & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. H. Schoenfeld, L. Kaput & E. Dubinsky (Eds.), *Research in collegiate mathematics education, III* (pp. 234-283). Providence, Rhode Island: American Mathematical Society.



## Working Group 1

- Hähkiöniemi, M. (2006). Perceiving the derivative: the case of Susanna. *Nordic Studies in Mathematics Education*, 11, 51-73.
- Juter, K. (2005). Limits of functions: traces of students' concept images. *Nordic Studies in Mathematics Education*, 3-4, 65-82.
- Pinto, M. M. F. (1998). *Students' understanding of real analysis*, Doctoral thesis, University of Warwick, UK.
- Presmeg, N. C. (2006a). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of the 30<sup>th</sup> Conference of the international Group for the Psychology of Mathematics Education* (Vol. 1, pp. 19-34).
- Presmeg, N. C. (2006b). Research on visualization in learning and teaching mathematics: emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp.205-236). Rotterdam, Netherlands: Sense Publishers.
- Raman, M. (2002). *Proof and justification in collegiate calculus* (Doctoral thesis), Berkeley, USA: University of California.
- Raman, M. (2003). Key ideas: what are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319-325.
- Rodd, M. M. (2000). On mathematical warrants: proof does not always warrant, and a warrant may be other than a proof. *Mathematical Thinking and Learning*, 2, 221-244.
- Stylianou, D. A. (2002). On the interaction of visualization and analysis: the negotiation of a visual representation in expert problem solving. *Journal of Mathematical Behavior*, 21, 303-317.
- Stylianou, D. A. & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: an examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, 6(4), 353-387.
- Toulmin, S. E. (2003). *The uses of argument*. Cambridge, UK: Cambridge University Press.
- Törner, G. (1998). Self-estimating teachers' views on mathematics teaching – modifying Dionne's approach. In S. Berenson, K. Dawkins, M. Blanton, W. Coulombe, J. Kolb, K. Norwood & L. Stiff (Eds.), *Proceedings of the Twentieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 627-634). Columbus, OH, USA.
- Törner, G., & Grigutsch, S. (1994). "Mathematische Weltbilder" bei Studienanfängern – eine Erhebung. *Journal für Mathematik-Didaktik*, 15(3/4), 211-251.

### Working Group 1

- Viholainen, A. (2006). Why is a discontinuous function differentiable? In J. Novotná, H. Moraová, M. Krátká & N. Stehliková (Eds.), *Proceedings of The 30<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol 5, pp. 329-336), Prague, Czech Republic: Charles University, Faculty of Education.
- Viholainen, A. (2007). Students' choices between informal and formal reasoning in a task concerning differentiability. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of The Fifth Congress of the European Society for Research in Mathematics Education* (pp. 2369-2378), Nicosia, Cyprus: University of Cyprus, Department of Education.
- Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Montana Mathematics Enthusiast*, 5(2&3), 231-248.
- Vinner, S. (1991). The role of definitions in teaching and learning. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.
- Weber, K. & Alcock, L. (2004). Semantic and syntactic proof production, *Educational Studies in Mathematics*, 56, 209-234.
- Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics. *The Journal of Mathematical Behavior*, 28, 200-208.

# UPPER SECONDARY SCHOOL TEACHERS' VIEWS OF PROOF AND THE RELEVANCE OF PROOF IN TEACHING MATHEMATICS

Emelie Reuterswärd

Stockholm University, Sweden

Kirsti Hemmi

Mälardalen University,

Sweden

*This paper reports some results from a case-study on five Swedish upper secondary school teachers' views of proof. We describe the teachers' views of what constitutes proof and focus particularly on their views on the role and relevance of proof in teaching mathematics. We identified two basically different ways of relating to the teaching of proof in our data that consists of interview transcripts and protocols of observations of lessons. We discuss these views from a socio-cultural perspective on learning and Wenger's concepts of community of practice, reification and participation.*

**Keywords:** Mathematical proof, teachers' views, community of practice, reification, participation.

## INTRODUCTION

The proposed new curriculum for upper secondary school mathematics in Sweden is manifesting a more explicit focus on proof and proving (Swedish National Agency for Education, 2010). The same trend is visible in curricula also in other countries. The guidelines from NCTM already in 2000 recommend that every student from pre-kindergarten through grade 12 experience proof and proving in different ways in their school mathematics (NCTM, 2000). Proof has also recently obtained a stronger status in curricula in Italy (Furinghetti & Morselli, 2010) and Estonia (Hemmi et al., 2010). Common to many of these curriculum reforms is the emphasis on the relevance of proof for all students. However, it is not clear to what extent this emphasis is coherent with teachers' views of proofs and the role of proof in teaching mathematics.

The American secondary school teachers in Knuth's qualitative interview study (2002) express the view that formal and less formal (Knuth's classification) proofs are not something beneficial for all students. Instead the teachers mainly find this kind of proof appropriate for students studying science or other mathematically intensive educations. This view is contrary to the visions put forth in NCTM 2000. Informal proof – i.e. explanations or empirically based arguments – is not considered as valid proofs but regarded by the teachers in Knuth's study as a vital part of the education for all students. The view that proof – in the traditional sense (c.f. Reid, 2005) – is not something for all students is also present in the teachers' responses in a multinational pilot study carried out in Sweden, Estonia and Finland (Hemmi et al., 2010). Although distinguishing several functions that proof can serve (c.f. de

Villiers, 1990; Hanna, 2000; Hemmi, 2006), the teachers mainly consider proofs and proving relevant for students studying courses that prepare the students for university studies in programs that include mathematics, for example technology and natural science. Both studies actualise the question of what constitutes proof in upper secondary mathematics as well as what kind of proofs and proving activities are appropriate and relevant for different kinds of students.

Research suggests that the way that curricular guidelines are implemented in the classrooms is greatly influenced by the teachers' experience, views and conceptions (Corey & Gamoran, 2006; Remillard, 2005). This underlines the fact that school reforms are never launched into a vacuum. Rather they are introduced into a community of teachers with their own conceptions of proofs and about what role proofs can or should play in teaching mathematics. To understand how reforms are going to be implemented and, on a larger scale, what role proofs play in the teaching of mathematics, it is important to understand teachers' conceptions of proof and proving. This is not the least true in Sweden, where the national curriculum leaves substantial space for local interpretation and appliances.

In this paper we report parts of the results of a case-study about five Swedish upper secondary school teachers' conceptions of proof (Reuterswärd, 2008). We discuss the teachers' views on what constitutes proof in upper secondary school mathematics and analyse the ways in which the teachers talk about the role and relevance of proof in their teaching.

We focus in particular on the following questions:

- What constitutes proof according to upper secondary school teachers?
- What are upper secondary school teachers' views on the role and relevance of proof in teaching mathematics?

### **Theoretical stances**

The analysis is based on a socio-cultural perspective where learning is considered to take place through active participation in a community of practice. Proof is according to this perspective seen as an artifact that can mediate mathematical knowledge. Wenger's (1998) concepts negotiation of meaning, reification and participation are used as central tools of analysis to understand and discuss teachers' talk about proof (c.f. Hemmi, 2006; Reuterswärd, 2008).

By a community of practice we mean people who share experiences and have a common goal or purpose of some kind. Both a group of teachers at an upper secondary school as well as a group of students studying mathematics fall under this definition. Every community of practice has its artifacts, or reifications; tools like a calculator, a mathematical formula or a proof, mediating mathematical knowledge. They create focus points around which teaching can be organized. But the symbol for pi ( $\pi$ ) or the proof for the Pythagorean Theorem do not carry any meaning of their

own. Without people participating in negotiating their meaning, they are mute, meaningless (c.f. Wenger, 1998, p 52-54). It is when people engage in a mathematical practice and participate in negotiating the meaning of proofs that they can be meaningful and convince, explain and communicate mathematics.

The meaning of reifications must be negotiated within the community of practice. Hence, proofs taken from a mathematical practice must be renegotiated, within upper secondary school. It is therefore not self-evident that proofs will play the same role in school as they do within the mathematical science. Pushing to extremes, the role of proof needs to be renegotiated within every single classroom. The aim of this paper is to shed light on how teachers renegotiate the meaning and role of proof within an upper secondary school context.

## **Method**

The paper is based on a case-study aimed to describe what constitutes proof in upper secondary mathematics from a teachers' perspective (Reuterswärd, 2008). The case-study had a phenomenographic approach and hence sought to qualitatively describe teachers' views and conceptions, believing that a concept can only be perceived in a number of qualitatively different ways. The study was conducted at one ordinary Swedish upper secondary school that includes the Social- and Natural Science Programs. Five teachers were chosen to participate in the study. They were of different ages, both genders, with different teaching background and at the moment of the data gathering teaching different groups of students. Such a heterogeneous selection of participants can represent qualitatively different views (Marton & Booth, 1997).

The data was collected using semi-structured interviews. In order to enhance the richness of the data the teachers were given an outline of what themes were going to be discussed during the interview. Hence, they had the chance to reflect on the questions beforehand. Several days before the interviews they also received a questionnaire with statements chosen from a questionnaire that was piloted by Hemmi during the time of the data gathering. By choosing a number 1-4 the teachers stated to what degree they agreed with the statements. Thus, we had written material to come back to during the interviews and thereby inconsistencies or uncertainties could be resolved. In the beginning of every interview the teachers were asked to describe how they defined a mathematical proof. The interview focused on the following themes: how the teachers perceived student attitudes toward proofs, how the teachers worked with proof and proving in their teaching and how they viewed proof in general. Every interview was tape recorded and transcribed. Unstructured observations of lessons, at least two with every teacher, were also used to further triangulate the method and validate the results.

## RESULTS

### Teachers' views of proof

There is a broad consensus among the teachers about what constitutes mathematical proof. Proof is something specific for mathematics that from certain premises, through logical argument, step by step, deduces the truth of a statement. Some of the teachers stress that the premises should be axioms or theorems that have previously been proved. The characteristics of proof are a mathematical language, an unassailable logic and a specific structure. The teachers hereby distinguish between proof and other types of arguments. Proofs for them are more formal:

Teacher: It [a proof] is that one, from certain premises, with given presuppositions, can strictly, step by step so to say, that is totally true, arrive at something that has to be general.

Teacher: If someone says proof you feel that it is a bit heavier piece, so to say.

Teacher: It follows that, hence, implies, is equivalent with. It's a bit of those things. You use some of those words that are associated with proofs. That's when I think it's a real proof.

Emphasizing that proof through logical derivation, structure and formal language deduces the truth of a statement, these teachers can be said to embrace a *traditional concept of proof* (c.f. Reid, 2005). This view of proof was also represented in Knuth's (2002) study by American teachers, referred to as *formal* or *less formal* proof.

Although the teachers see the formal mathematical language, structure and logic as distinguishing proofs from other types of arguments, there is some uncertainty as to where to draw the line. Tasks like "show that left hand side equals the right hand side" and using general methods to solve problems are examples of areas that the teachers mean "*are in the vicinity of proof*" but for which the term "proof" is not used. This view is coherent with the view that Knuth calls *less formal* proof: a general argument that lacks a rigorous mathematical structure.

One teacher also gives a slightly more informal definition of what proof can be in upper secondary school:

Teacher: Proof in the sense that they [the students] should understand that it's not arbitrary, then you do it most every day. To make them understand that mathematics is a logically built system, and it's not a coincidence that we have to do things in a certain way.

This can be compared to what Knuth calls *informal proof* i.e. justifying by explanations or examples. There is, however, no doubt that examples and empirical investigations are not considered valid proof, neither in Knuth's nor in Reuterswärd's study, and the concept *informal proof* is therefore somewhat unfortunate.

The difficulty to draw the line to what counts as proof is not a problem reserved to school. Instead it is a forever ongoing discussion for which different answers are given within separate mathematical disciplines and historical contexts (Hanna & Jahnke, 1993). In Wenger's (1998) terms we can say that the meaning of an artifact has to be negotiated within every practice.

### **The role of proof in the classroom**

Common to all the teachers in this study is that they all think of proof as central to mathematical thinking, and that they distinguish several functions that proof can serve in teaching mathematics. They see proving as a desirable competence and as a suitable challenge for the well achieving students. They also agree that the role of proof in the classroom must be determined in relation to every single student group. Their views on how this valuation should be made, especially considering the low achieving students or students in the Social Science Program, are however different. Two separate views were distinguished in the study. This should not be understood in a way that the teachers can be said to solely support one view or the other. On the contrary, the teachers' ambivalence might be the most prominent feature of their views on the role of proof in teaching mathematics.

One view embraces the thesis that *proof is something for all students* and that it is important that the teacher takes the step from examples and informal reasoning to proof. The explanatory function of proof, in particular, motivates that proof is something for all students, even the low achieving ones.

Teacher: I don't really think that you should assert anything to students without in some way proving it. If it is sometimes easier to make a geometric proof or a purely theoretical proof... /.../ I definitely don't think you should just state it: now you do like this. Why should you do like that?

Teacher: With the students I've got I know that there are many who think that, if there is anything you should leave out, it's proof. And then I have to say that I don't quite agree because that could be what gets them back on track.

The underlying idea here is that understanding why something is the way it is in mathematics, is a vital condition for the students to experience mathematics as meaningful, and to be able to use it in the right way. As a teacher you therefore run the risk of losing students if you don't take the step to more formal arguments. The basic assumption is that students can and want to understand.

Furthermore, according to this view, proofs don't have to be difficult. There are abstract and concrete, easier and more difficult proofs, and according to this view it's up to the teacher to present them in a way that can appeal to every student.

The other view is characterized by the idea that *proof is not necessarily something for all students*. The teachers share the basic positive attitude towards proof, ("*I am pro proof, I really am.*") but mean that the students lack the necessary qualities to

## Working Group 1

realize the meaning potential of proof in the classroom. The students lack sufficient mathematical knowledge or sufficient experience of proof:

Teacher: If you can hardly calculate the area of a triangle it's really hard to explain what it [proof] is.

Teacher: It will be really difficult because they've hardly done it at all before. They'll only scream when I do it.

Teacher: If they don't accept the proof it's because they have gaps in their [mathematical] knowledge. Then it's really hard to accept proofs.

According to this view, the students' insufficient previous knowledge makes it impossible for them to participate in negotiating the meaning of proof. The teachers mention, for example, that the students need to know the quadratic rule in order to understand the proof of the quadratic formula, and master similarity to understand one of the most common proofs of the Pythagorean Theorem. As a teacher you therefore run the risk of losing students if you conduct general or algebraic reasoning.

The basic assumption here is that proof generally is something advanced that only a few students have the ability to master. According to the teachers some students are also not interested in proof; they are content with examples and only want to know 'what to do'. Proof is not necessarily something that is relevant for all students, in many cases it's enough to explain through examples:

Teacher: These Social Science students for example, I don't know if they have any use of it later, when they graduate, because it's probably not too many of them who'll choose a mathematical education.

### **Reification, participation and negotiation of meaning**

Some of the teachers' statements express that understanding mathematics makes the students more capable of using it correctly, more inclined to thinking math is fun and more apt to remembering what they have learned. In Wenger's terms: understanding mathematics enhances the students' inclination and ability to *participate* in the community of practice of mathematics as it is exercised in the classroom.

This is the leading idea in the view that we have called *proof is something for all students*. According to this view, understanding why something is the way it is in mathematics is a requirement if the students are to experience mathematics as meaningful and to be able to use it in the right way. As a teacher you therefore run the risk of losing students if you do not take the step towards such general arguments as proofs. In other words: Without *reifications* like proofs, the students may experience mathematics as meaningless or hard to understand. The reification offers a necessary structure, an abstraction to tie the knowledge to. The other view, which we have called *proof is not necessarily something for all students*, means instead that some students lack both the motivation and the knowledge to be able to participate in



negotiating the meaning of proof. According to this line of thought, one runs the risk of losing the students if one *does* take the step towards such general arguments as proofs. Thus, the reasoning is turned over, and it is claimed that without possibility to participate – without previous knowledge or motivation – the reification is meaningless anyway. In line with this idea, Hanna and Jahnke (1993, p. 434) point out that the students need well-founded previous knowledge to be able to negotiate the meaning of proofs.

We can see that both views express the same main goal; to make the students understand mathematics. The different views can be described as different answers to the question of what proportion between reification and participation that is the most meaningful to the students. This can be compared to Wenger's (1998, p. 65) remark that the proportion between reification and participation always has to be negotiated, and that different proportions lead to different possibilities to create meaning. An abundance of mathematical artifacts such as symbols, formulas or proofs do not create any meaning without the possibility of participation. But in the same way participating without reifications can seem meaningless. The sole use of examples, empirical investigations and lots of time can be experienced as useless without mathematical reifications to focus the practice on; without concepts, symbols or formulas to structure the mathematical knowledge. It is in the tension between reifications and participation that meaning can be created in the community of practice.

All the teachers in this study express that proof and general arguments are something desirable. But like we have described above, some of them state that empirical investigations or examples can replace proof in groups where the students are not susceptible to formal reasoning. Even those teachers who in principle mean that proof is something for all students, point out that using informal methods is a good first step towards proof. Informal methods can include letting the students see examples, investigating certain cases (empirical investigation) and trying to find patterns. We can interpret these methods as strategies to increase the students' motivation and ability to take active part and engage in understanding reifications like proof.

Although the teachers express understanding as the main goal of their teaching, it is not self-evident that all students identify with the community's goal to understand. Above all, not all of them are interested in proof. "*Some students are only like "what to do?"*" as one of the teachers in this study put it. We can interpret it as if the students simply are not interested in identifying with the goal of the community of practice. Maybe they want to practice law and do not see any use in specifically mathematical reifications like proof. It is also possible that they want to understand, but that they through years of failure have developed an identity of non-participation (Wenger, 1998, p. 165) – 'I don't know mathematics'. That means that they have

developed an identity where they do not see themselves as participants in the community of practice.

The concept *community of practice* can also help us to see that the teachers have several different practices to consider when they value the role of proof in their teaching. Some teachers point out, for example, that one reason not to deal with more proofs is that “the students haven’t experienced them before, in compulsory school.” This is in turn probably related to the fact that proof is not at all mentioned in the Swedish steering documents for compulsory school (Swedish National Agency for Education, 2000). But also future practices play a role in the teachers’ views. Upper secondary school is meant to prepare for higher education and future professions. Which these are likely to be, matter to some teachers.

## DISCUSSION

In this paper we have reported parts of the results of a case study of five Swedish upper secondary school teachers’ views of proof (Reuterswärd, 2008). We have particularly focused on the teachers’ conceptions of proof and the role and relevance of proofs in the context of upper secondary school mathematics. In doing so we have presented two different views on the role of proof in upper secondary mathematics that we’ve called *proof is something for all students* and *proof is not necessarily something for all students*.

In the beginning of the paper we drew attention to the fact that many curricula around the world manifest a revaluation of the role of proofs in teaching mathematics. Common to the guidelines in the NCTM (2000), Italy (2003) and the proposed new curriculum for upper secondary school in Sweden<sup>2</sup> (2010), is the emphasis on the relevance of proofs for all students. This rhyme well with the view that we have called: *proof is something for all students*. But how will these reforms turn out in a community of teachers where one of the views is that *proof is not necessarily something for all students*? This question is not the least relevant considering that this view is expressed not only by the teachers in this study but also by American teachers (Knuth, 2002) and other Swedish teachers (Hemmi et.al, 2010). These results raise the need for these new curricula to be anchored in the community of teachers supposed to realize these visions.

We can only speculate as to what has been the driving force behind more focus on proof and proving in the curricula. We find it likely that it has something to do with the extensive functions – such as explanation, verification, communication and transfer (c.f. de Villiers, 1990, Hanna, 2000, Hemmi, 2006) – that proofs can serve in teaching mathematics. It is therefore important to notice that the teachers in Knuth’s (2002) study seem to regard proofs mostly as a topic to study, rather than using them

---

<sup>2</sup> Vocational programs not included.

to teach mathematics. This view is not shared by the Swedish teachers in this study. They chiefly see proof as a tool that can mediate mathematics. This is probably partly due to the fact that proof is hardly mentioned in the present mathematics curriculum in Sweden. Yet, one can ask whether a sudden focus on proofs in the new curriculum has the effect that the teachers start viewing them first and foremost as a topic to teach, rather than a gateway to mathematics. Yet, both approaches are needed in the teaching of mathematics (c.f. Hemmi, 2008). This highlights once again the need for the curriculum reforms to be anchored in the community of teachers. This can be done by making the teachers aware of the arguments behind the new guidelines, and offering them the tools to introduce proofs also in groups of students lacking motivation or prior knowledge.

This being said one can wonder what the implications are if proofs become part of the mathematics education for all students. It is well documented that many students find it difficult to understand, conduct and value proofs (e.g. Healy & Hoyles, 1998; Selden & Selden, 2003). The teacher in this study who says “*If you can’t calculate the area of a triangle, it’s really hard to understand what a proof is.*” expresses a point not to be taken lightly. Introducing proofs to these students could possibly further their identity of nonparticipation in the mathematics classroom (c.f. Wenger, 1998). On the other hand, experiencing the derivation of the area formula for a triangle could help students to find proving as meaningful. One way to proceed might be to gradually formalize the use of justifications throughout the curriculum as is the case in the American guidelines from NCTM (2000). However, it certainly calls for research studies about how these reforms are being implemented and what effects the focus on proofs has in different student groups. It also underlines the need for research exploring different ways of teaching proofs to different kinds of students.

## REFERENCES

- Corey, D. & Gamoran, S.M. (2006). Practicing change: Curriculum adaptation and teacher narrative in the context of mathematics education reform. *Curriculum Inquiry*, 36(2), 153-187.
- De Villiers, M. (1990). The role and function of proof in mathematics. *Pythagoras, South Africa*, 23, 17-24.
- Furinghetti F., & Morselli F. (2009). Leading beliefs in the teaching of proof. In J. Maaß, J. & W. Shlöglmann (Eds.), *Beliefs and attitudes in mathematics education. New research results*, pp. 59-74. Rotterdam: Sense Publishers.
- Hanna, G., & Jahnke, N. (1993). Proof and application. *Educational Studies in Mathematics*, 24, 421-438.
- Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational Studies in Mathematics*, 44, 5-23.

## Working Group 1

- Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics*. Summary of the results from a survey of the proof conceptions of students in the UK. Research report, University of London. Available [2010-09-04] at <http://www-didactique.imag.fr/preuve/Resumes/Healy/Healy98.html>.
- Hemmi, K. (2006). *Approaching proof in a community of mathematical practice*. Doctoral thesis (Monograph). Stockholm University, Department of Mathematics, Stockholm.
- Hemmi, K. (2008). Students' encounter with proof – the condition of transparency. *ZDM – The International Journal on Mathematics Education. The Special Issue on Proof*, 40, 413-426.
- Hemmi, K., Lepik, M., & Viholainen, A. (in press). Upper secondary school teachers' views of proof and proving – an explorative cross-cultural study. *Proceedings of MAVI 16, The 16<sup>th</sup> conference on Mathematical Views, Tallinn, Estonia, June 26-29, 2010*.
- Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61-82.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. New Jersey: Lawrence Erlbaum Associates.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston.
- Reid, D. (2005). The meaning of proof in mathematics education. In M. Bosch (Ed.), *European Research in Mathematics Education IV. Proceedings of CERME4*, San Feliu de Guíxols, Spain 2005. Universitat Ramon Llull, Barcelona.
- Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75, 211-246.
- Reuterswärd, E. (2008). *Vilket skulle bevisas – En fenomenografisk studie om hur gymnasielärare ser på bevis i undervisningen. (Quad erat demonstrandum – a phenomenographic study on upper secondary school teachers' conceptions of proof.)* Unpublished project work, Stockholm University, Stockholm.
- Selden, A., & Selden, J. (2003). Validations of Proofs Considered as Texts: Can Undergraduates Tell Whether an Argument Proves a Theorem? *Journal for Research in Mathematics Education*, 34, (1), 4-36.
- The Swedish National Agency for Education. (2000). *Kursplaner (Curricula)*.
- The Swedish National Agency for Education. (2010). *Matematik. Remissversion – augusti 2010 (Mathematics syllabuses under consideration – August 2010)*.
- Wenger, E. (1998). *Communities of practice. Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.

# DEVELOPMENT OF BEGINNING SKILLS IN PROVING AND PROOF-WRITING BY ELEMENTARY SCHOOL STUDENTS

Stéphane Cyr

Université du Québec à Montréal, Canada

*This paper presents the results of a study regarding the development of deductive reasoning among elementary school students. We have experienced a sequence of 8 teaching and learning lessons in order to develop their primary skills of writing proof in a geometrical context. This sequence, over a 4 months period, was tested with two classes of 26 students aged 11-12 from a single school in Quebec (Canada). Results showed an important increase in most of the students, between the beginning and the end of the sequence, in their ability to reason deductively and validate geometric statements by using theoretical properties rather than measurement.*

Keywords: Proof, teaching, mathematics, primary school, geometry.

## RESEARCH PROBLEM

It is now a well-established fact in research circles that teaching proof in secondary school is undeniably important. The significance of it may be explained particularly through the many roles that proof-writing plays in the mathematics education of students (Arsac & *al.*, 1992; Duval, 1990; Houdebine, 1990). Among other things, writing proofs, being explanatory by nature, fosters students' comprehension (Hanna, 1995) and helps them develop deductive reasoning (Reid, 1995), critical thinking as well as an ability to support lines of argumentation (Houdebine, 1990).

Key to children's education, proof-writing is also one of the most complex activities and one which secondary students experience the most difficulty with (Houdebine, 1990 ; senk, 1985). The difficulties encountered and the reasons for them are manifold. Amongst the problems observed, two seem to preoccupy experts in current mathematics education research.

The first one relates to the difficulty for students to fully understand the fundamental structure within deductive proof reasoning (Duval, 1991 ; Tanguay, 2005). When writing proofs, students often err in sequencing the inferences constitutive of their demonstration. All they see in it is a discourse, a line of argumentation where propositions are simply added and organized according to relevance only.

The second problem, in connection with the first one, lies in the way secondary school students perceive and use the representations of geometric shapes to write proofs. At the elementary school level, geometry is deemed *practical* (Perrin-Glorian, 2003), as it is linked to spatial sense, visual perception, and shape-building activities. Validating formal geometry statements implies empirical work on figures, whereas in secondary school it relies on theory and very specific axiomatic systems. Thus redefined, and as Muller (1994) mentions it, shapes constrain secondary school

students to reason through concepts rather than drawings; and this is an obstacle for them. Indeed, this alternative justification based on a deductive approach that excludes any conclusion drawn from measurements and observations of geometric figures, seems to generate a lot of difficulties for students. Research in various countries, including Canada, the United States, and France, supports that claim (Chazan, 1993 ; Muller, 1994 ; Paul, 1997). As for Balacheff (1987), he assessed that the switch from *practical* geometry to more *theoretical* geometry caused a breach in the didactic contract agreed upon by both teachers and students, which, he adds, is the main source of learning difficulties when students are introduced to proof-writing. This problem has raised an important question among many researchers: How can students be taught that a practical validating approach based on empirical observations may no longer be reliable when the time comes to write deductive proofs?

### **THEORETICAL FRAMEWORK**

For Perrin-Glorian (2003), solving the problem could imply taking action before proof-writing is being taught in secondary school. She regards the end of the elementary curriculum and the beginning of the secondary cycle as the right time of transition from practical to theoretical geometry. According to her, carefully selected situations could facilitate the switch from one kind of geometry to the other. However, she observes that there is no consensus amongst researchers on how to include the relation between practical and theoretical geometry in the elementary curriculum. We consider that her recommendation seems worth looking into for the following two reasons:

- 1 – When students are introduced to proof-writing in secondary school, not only are they faced with a new and more rigorous validating procedure like proving, as well as with a new axiomatic system, but they must also learn deductive reasoning in a context of theoretical geometry. And yet, this context of proof-writing is already a problem in itself for students, for acquiring the previously-mentioned elements may not ideally prepare students for deductive reasoning nor to the switch to theoretical geometry. Rather, students should be learning deductive reasoning in more familiar everyday situations the way they do with geometric shapes in validation situations.
- 2 – Introducing deductive reasoning and more theoretical geometry in elementary school would reduce the breach in the didactic contract described by Balacheff (1987), which is the main source of difficulties for students when first coming to grips with proving. In addition to reducing this breach, a more gradual approach to those elements certainly allows for greater continuity between elementary and secondary school curricula in mathematics.

### **From practical to theoretical geometry: construction of abstraction**

According to Parzysz (1991), geometry activates two types of space: physical space (surrounding space and concrete objects) and abstract space (idealized object). The

first is associated with a more practical geometry while the second is developed through a more theoretical geometry. Objects in a physical space are concrete and they can be observed by the senses, while objects defined in an abstract space exist only in theory or in the form of ideas: they are a mental construct (figures whose existence is ensured by statements as definitions, properties or characteristics). So the transition from a practical geometry to a more theoretical geometry implies that students develop a degree of abstraction. Indeed, a close link exists between understanding of these properties and the use of mathematical abstraction, as pointed out by several authors.

According to Rosh (1978) abstraction may focus on the *properties* of perceived objects. Piaget, for his part, believes that abstraction can take several forms. He distinguishes in fact between construction of meaning through *empirical abstraction* (focusing on objects and their properties) and *pseudo-empirical abstraction* (focusing on actions on objects and the properties of the actions). He associated to these two kind of abstraction the idea of *reflective abstraction* which occurs through mental actions on mental concepts (Piaget, 1972, p. 70). Reflective abstraction is then seen as an activity applied to mental entities rather than physical objects. For Gray and Tall (2007), who also discuss the idea of abstraction, mathematical concepts are the result of a process of abstraction that takes place on a given situation. This abstraction may take the form of « a mental image of a perceived *object* (such as a triangle), a mental *process* becoming a concept (such as counting becoming number) and a formal system (such as a permutation group) based on its properties, with the concept constructed by logical deduction » (2007, p. 23).

If we accept that one of the main purpose of teaching geometry in primary school is to gradually bring students from a physical space to a more abstract space based on the properties of objects, then, we have to consider the importance of the development of abstraction in students and thus, the internalization of properties of mathematical objects and operations on them. This guidance also highlights the importance of taking into account the existence of different forms of geometric perspective, as called by Houdement and Kuzniak (2006): geometric paradigm.

### **Geometrical paradigms**

Houdement and Kuzniak (2006) have defined three geometrical paradigms through which thinking patterns develop differently. Each paradigm is defined by the following components: objects, methods and problems, and therefore each is strongly link to a didactical contract (Houdement, 2007). Each of these paradigms reflects a sophisticated form of geometry and can thus define a specific geometric framework Kuzniak (2006, p.170). It should be noted that the three paradigms are not hierarchical in the sense that they all allow to solve geometry problems properly and efficiently. One is not better than the other, their use will depend on the context.

## Working Group 1

- *Geometry I* or *Natural Geometry* is the first paradigm whose validation originates in the real, sensible world, according to Houdement and Kuzniak (2006), and where drawing plays a central role in validation. In this context, deduction is exercised primarily through the perception and the manipulation of objects.

- *Geometry II* or *Natural Axiomatic Geometry* is the second paradigm whose link to reality is not as strong as it is for the first one. It actually aims at understanding reality through axioms, and with the help of which tangible problems may be solved. However, axiomatization is not formal since syntax is not cut off from semantics, the latter referring to reality. In this regard, the source for validation is no longer the sensible, but indeed an hypothetico-deductive process in which intuition and experience still play a role, but to a lesser extent than deductive reasoning does.

- The *Geometry III* or *Formalist Axiomatic Geometry* paradigm is very different from the two previous ones, insofar as it is disconnected from reality. The source for validation is based on logical reasoning only, and not on the sensible world or perceptions.

### **Deductive reasoning in elementary school**

Coppe, Dorier, and Moreau (2005) hold that in order for teachers to demonstrate that proving is meaningful and useful, they have to “force on students the transition to deductive reasoning” (p. 35). Indeed, elementary students are more likely to develop a proper ability for it if they can enjoy teacher’s support, even if that reasoning mode remains an integral part of the human procedural system (English, 1997). For English, textbooks and elementary school curricula must take reasoning processes into account and include informal deduction problems. Furthermore, current research shows that deductive reasoning could be within elementary students’ grasp (Braine and O’Brien, 1991; Daniel, 2005; English, 1997).

### **RESEARCH GOALS**

With this research, we wish to encourage in students a more gradual approach to acquiring preparatory abilities when it comes to proof-writing. We believe that these abilities should be taught as early as elementary school in order to minimize the previously-mentioned breach-related problem. In this light we intend to:

1 – Encourage deductive reasoning in elementary students when dealing with mathematical situations.

2 – Building from their ability to reason deductively, encourage students to transition from practical geometry (Geometry I) to more theoretical geometry (Geometry II); also, gradually bring them to become aware of the limits of empirical validation for geometric figures and realize how effective more deductive reasoning (Geometry II)-based theoretical geometry can be.



We expect the results of our research to be manifold: For one, it should facilitate the creation of activities for the elementary mathematics curriculum in order to later foster proof-writing skills in secondary students. In fact, no other study has ever looked closely into including basic proof-writing skills acquisition in the elementary curriculum so far. Finally, our research is in line with the general views of the MELS aimed at encouraging in students the gradual and ongoing acquisition of mathematical skills and knowledge between elementary and secondary school.

### **Hypothesis**

We postulate that if we altered students' relation to figures as well as the role played by properties in the validation process, we would observe in students a spontaneous form of deductive reasoning. Indeed, the change of status for figures could induce new validation processes, no longer based on perceptions or measurements, but on theoretical properties which may only be resorted to through some sort of deductive reasoning. Note that the spontaneous use of deductive reasoning is based on the principle of accommodation in the sense of Piaget.

### **METHODOLOGY**

To meet our objectives, we have opted for a design-based research methodology (Edelson, 2002). This method is cyclic in nature and each cycle consists of 5 stages: 1) Teaching sequence writing; 2) In-class testing of the sequence; 3) Retrospective analysis of experimental data; 4) In light of this analysis, reassessment of theoretical hypotheses, didactic choices, and anticipated learning paths; 5) As a result of reassessment, adjustments are made in the design of the teaching sequence and a new cycle may begin. Also, we have created activities based on works by Coppe and al. (2005), Perrin-Glorian (2003), and Houdement and Kuzniak (2006).

### **Participants**

Two Montreal elementary classes of 25 sixth-graders (11-12 y.o.) were selected for this study prior to which none of the students had ever validated geometric propositions through deductive argumentation. When doing geometry activities, they would only validate geometric situations using measuring instruments.

### **Tasks**

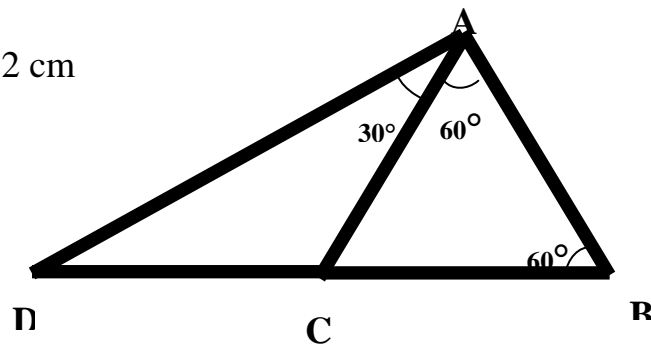
For the first stage of the design-based research methodology, we designed eight tasks that would elicit the spontaneous emergence of deductive reasoning in sixth-graders, as well as spur the transition from practical geometry (Geometry I) to theoretical geometry (Geometry II). The tasks focused on the essential knowledge as identified in the Quebec elementary education program literature, so that they would be easily merged into regular teaching planning and would not add new subject content.

## Observation and measurement-based argumentation doubting procedure

The tasks were designed to cast doubt in students on their level of certitude they might reach observing and measuring figures and show them the limits of an argumentation based on that procedure. To this end, we first proceeded to use geometric shapes that made accurate measuring difficult, that is either drawn with exact measures yet in thick and bold lines, or free-hand thus yielding very approximate results. Students could use any chosen method to do each task. Then, they had to answer questions designed to have them process their results which they also had to compare with those of other peers. As an example, here are two questions asked to students:

### Activity 1

Length of side AB = 6.2 cm

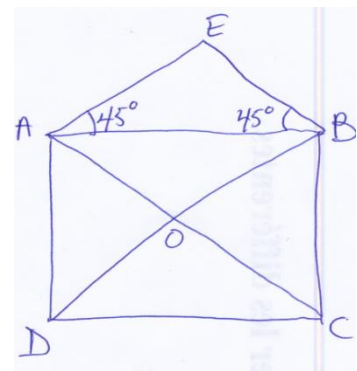


1. a) Measure angle D and segment CD. write down how you proceed and explain how you reason.
- b) Proceed differently to make up for missing measurements. Write down how you proceed and explain how you reason.
- c) Is there a discrepancy between results 1.a) and 1.b)? If yes, explain why.
2. Compare your results with those of another team.
  - a) Have you got the same results as the other team? If not, explain why.
  - b) Did you proceed as the other team did? If you proceeded differently, indicate which is the more appropriate procedure.

### Activity 2

A student draws free-hand the figure below where ABCD is a square whose diagonals AC and BD intersect at O. Then, he draws a triangle ABE on top of the square. Finally, he claims that the quadrilateral AEBO is a square.

**Note: The diagonals of a square always intersect in the middle and are perpendicular.**



Is the student right? Explain your answer.

In the second step of the methodology (In-class testing of

the sequence), we scheduled our experimentation to take place over a period of two months, one hour per week. We filmed the activities in progress so that we might later analyze how they were performed and how students reacted to them. Finally, students worked in pairs which aimed at prompting debate on validation procedure.

For the retrospective analysis of experimental data (methodological step 3), we observed the students' reactions to the tasks. We also analyzed their understanding of these tasks by using paper trail on written questionnaires, video of two teams in the tasks and the analysis of a logbook of our observations during the experiments. Our attention was also focused to the type of procedure used during the implementation of the tasks (use of measuring equipment or use of theoretical properties).

In the fourth step of our methodology, our objective was to evaluate both the type of geometric paradigm in which students could be located but also to assess how the task favored in the student the shift from paradigm 1 to paradigm 2. To support our analysis, we examined how students used the geometric figure which was provided to support their reasoning. For example, is that students based their answer and justification on intuition, observations or visual estimation, measures or geometric properties and deduction. The results of these tests have allowed us to edit questions, delete or add others to always promote the passage of natural geometry to a natural axiomatic geometry and thereby promote the use of geometric properties and use of deductive reasoning (methodological step 5). Since this experiment took place over a period of three years, we had the opportunity to repeat this sequence of activities on two other occasions with different student groups and thus make each time, adjustments to our work.

## **PARTIAL RESULTS**

At first, more than three-quarters of the students (40 students out of 50) used measurement as the only method to work out missing data for this kind of problem (pré test where they have to find missing data on a geometrical figure with the strategy of their choice). Some used mixed strategies (6 students) whereas two teams (4 students) only spontaneously resorted to a deductive approach using theoretical properties. What we mean by mixed strategies is a crossover approach that uses measurement and theoretical properties. For instance for Activity 1 above, some students used theoretical properties in simple contexts like working out the measure of the 60 degree angle inside the triangle (they used the property of the sum of angles inside any triangle), yet they turned to measurement or visual perception for the more complex areas of the situation, such as working out the length of segment CD. Most students noted that triangle ACD was isosceles, relying on their perception, then concluded that segment AC was the same length as segment AB, that is 6.2 cm.

After the first four sessions, we noticed that students had clearly improved upon their justification procedures. In simple situations where they had to work out the missing measure of an angle in a triangle or a quadrilateral, 42 students out of 50

## Working Group 1

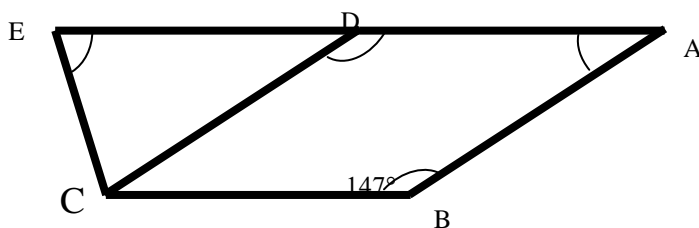
spontaneously turned to theoretical properties using a deductive approach. In more complex situations, we also observed some improvement when more than 50% of the students used a deductive approach appropriately. This improvement is partly due to the constraints imposed by the situations in which students were getting inaccurate results with measurements, thus forcing them to resort to mathematical properties and deduction to find the missing data. We also led the students to compare results between them and tried to explain the differences from one team to another. The students' comments were in line with the imprecision of the measure (which in our case was strengthened by the bold line that accentuates the vagueness on measure or the hand drawing figures).

At the end of our experimentation, all students were able to identify spontaneously the limits and lack of precision of a measurement and observation-based approach, as well as call on theoretical properties to validate simple geometry statements (problems where only one data is missing in a simple geometric figures). However, this result does not suggest that all students have acquired the ability to produce simple proofs or arguments based on properties using deductive reasoning appropriately. Besides, more complex situations remained difficult for some students leading them to use mixed strategies (measurement and deduction); and so did situations requiring validation, and where data were not provided. For example, in order to solve the problem below, students were able to use measurement as well as theoretical properties, which they knew well by then. However, the absence of numerical data significantly hampered task completion and caused students' strategies to revert to measurement and observation.

Let ABCD be a rhombus and let D be the midpoint of segment AE.

**We know that the sum of the four angles of a rhombus is 360 degrees.**

1. a) Find the measure of angles A, B, and E.



## DISCUSSION

The eight sessions that we had scheduled over a period of two months allowed us to bring most students to switch from practical (G1) to theoretical geometry (G2) spontaneously. In doing so, we had set for students an environment conducive to the use of deductive reasoning in validation situations. And yet, our experimentation validated our initial hypothesis, that is deductive reasoning shows to be a demanding process that requires time and extensive experience to be exercised properly. In fact,

it had taken four sessions before students started showing some improvement in reasoning deductively. Yet, the students who improved the least were still able to grasp the geometry concepts and skills in the tasks given in the course of our experimentation. We believe that this type of activity can be very productive when it comes to teaching and learning the geometric properties of given figures; it could also apply with most geometric concepts covered in class during the school year.

## REFERENCES

- Arsac, G., Chapiron, G., Colonna, A., Germain, G., Guichard, Y., Mante, M., (1992). *Initiation au raisonnement déductif*. Lyon : Presses Universitaires de Lyon.
- Balacheff, N. (1987). Processus de preuve et situation de validation. *Educational Studies in Mathematics*, 18, 147-176.
- Braine, M., & O'Brien, D. (1991). A theory of it : A lexical entry, reasoning program, and pragmatic principles. *Psychological Review*, 92 (2), 182-203.
- Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24, 359-387.
- Coppe, S., Dorier, J.-L., Moreau, V. (2005). Différents types de dessins dans les activités d'argumentation en classe de 5<sup>ième</sup>. *Petit x*, 68, 8-37.
- Cyr, S. (2001). Vers un enseignement signifiant de la preuve au secondaire. *Bulletin AMQ*, décembre, 19-27.
- Daniel, M.-F. (2005). *Pour l'apprentissage d'une pensée critique au primaire*. Québec, Presse de l'université du Québec.
- Duval, R. (1991). Structure du raisonnement déductif et apprentissage de la démonstration. *Educational studies in Mathematics*, Vol. 22, pp. 233-261.
- Eddie Gray & David Tall (2007). Abstraction as a natural process of mental compression. *Mathematics Education Research Journal* 19 ( 2), 23–40.
- Edelson, D. C. (2002). Design research : What we learn when we engage in design. *Journal of the Learning Sciences*, 11 (1), pp. 105-121.
- English, L.D. (1997) Intervention in children's deductive reasoning with indeterminate problems. *Contemporary educational psychology*, 22,338-362.
- Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, 15, 3, 42-49.
- Harel, G., Martin, W. G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics education*, 20, 1, 41-51.
- Houdebine, J. (1990). Démontrer ou ne pas démontrer, voilà la question. *REPERES-IREM*, 1, 5-27.

- Houdement, C., (2007) À la recherche d'une cohérence entre géométrie de l'école et géométrie du collège. *REPÈRES-IREM*, No 67, avril 2007.
- Houdement, C., Kuzniak, A., (2006) Paradigmes géométriques et enseignement de la géométrie. *Annales de didactique et de sciences cognitives*, vol 11, p.175-193, IREM Strasbourg.
- Kuzniak, A., (2006) Paradigmes et espaces de travail géométriques. Éléments d'un cadre théorique pour l'enseignement et la formation des enseignants en géométrie. *Canadian Journal of Science, Mathematics and Technology Education*, pp167-187.
- Muller. J.-P. (1994). La démonstration en géométrie en quatrième et en troisième. *REPÈRES-IREM*. 15, avril, 7-20.
- Patton, M. Q. 1990. *Qualitative Evaluation and Research Methods*. Sage Publications, Newbury Park.
- Parzysz, B. (1991). Espace, géométrie et dessin. Une ingénierie didactique pour l'apprentissage, l'enseignement et l'utilisation de la perspective parallèle au lycée. *Recherche en didactique des mathématiques*, 11, 23, p. 211-240.
- Piaget, J. (1972). *The Principles of Genetic Epistemology* (W. Mays trans.). Routledge & Kegan Paul: London.
- Paul, R. et DeBlois, L. (1998). La notion de preuve en géométrie. *Bulletin de l'AMQ*. Vol. XXXVIII (3). pp. 22-35.
- Perrin-Glorian, M.-J. (2003). Studying geometric figures at primary school from surface to point. *European Research in Mathematics Education*, p. 1-10.
- Reid, D. A. (1995). *The Need to Prove*. Thèse de doctorat (Ph. D). Faculty of Graduate Studies and Research : University of Alberta.
- Senk, S. L. (1985). How well do students write geometry proofs ? *Mathematics teacher*, 76, 6, 448-456.
- Tanguay, D. (2005). Apprentissage de la démonstration et graphes orientés. *Annales de didactique et de sciences cognitives* (IREM de Strasbourg, Université Louis Pasteur), Vol. 10, pp. 55-93.

# DESIGNING INTERCONNECTING PROBLEMS THAT SUPPORT DEVELOPMENT OF CONCEPTS AND REASONING

Margo Kondratieva

Memorial University, Canada

*In this paper I illustrate the process of designing a problem which can be repeatedly used by teachers in different mathematical courses and at various levels of complexity. The same problem can support reasoning appropriate for the context in which the problem is presented. Gradual increase of the requirements for rigor from one level to the next supports the learner's development within her natural sequence of learning modes from experimental to theoretical. The course of formalization of reasoning also affects the conceptualization process related to the object of the problem.*

Keywords: multiple-solution problem, development of mathematical thinking, necessary and sufficient conditions, dynamic geometry software, geometry of isosceles triangle.

## INTRODUCTION

Development of reasoning skills and formation of concepts is a life-long process. In particular, many mathematical concepts emerge from a child's earlier experiences in a primitive form and develop further as the child has a chance to perceive and act on physical objects, to form mental images and models, and eventually reflect, categorize, and hypothesize further properties expressed in a symbolic form. At each level of concept development, a child exhibits reasoning behaviour with the degree of rigor appropriate to the level of concept maturity. According to Bruner (1968) the sequence of learning modes, enactive-iconic-symbolic, characterizes not only grade school students but also an older learner. This idea is also consistent with van Hiele's theory of learning geometry by advancing through the stages from visualization to analysis, to informal or formal deduction and finally to rigor.

In mathematical instruction, one way to reflect this long-term continuous development of mathematical thinking is to consider the notion of an *interconnecting problem*. An interconnecting problem is characterized by the following properties (Kondratieva, 2011): (1) It allows simple formulation; (2) It allows various solutions at both elementary and advanced levels; (3) It may be solved by various mathematical tools from distinct mathematical branches, which leads to finding multiple solutions, (4) It is used in different grades and courses and can be discussed in various contexts.

It is proposed (Kondratieva, 2011) that a long-term study of a progression of mathematical ideas revolved around one interconnecting problem is useful for developing a perception of mathematics as a connected subject by all learners. Due

to the wide range of difficulty levels of its solutions, the same interconnecting problem may appear at the elementary school level, and then in progressive grades at the secondary or even tertiary level. The students, familiar with the problem from their prior hands-on experience, will use their intuition to support more elaborate techniques presented symbolically in the upper grades.

This article aims to discuss the process of creating an interconnecting problem with particular attention to the development of reasoning skills and the notion of proof. Here a Problem Designer is a mathematics teacher or mathematics curriculum developer who proceeds via the following stages: (A) choosing an initial question; (B) tailoring questions to elementary approaches; (C) upgrading to more advanced techniques; and (D) finding contexts suitable for the identified approaches and techniques within the overall curriculum.

The key point is that a problem becomes interconnecting as the Problem Designer herself experiments with the problem, identifies the types of reasoning emerging from her experiments, and starts to see different facets of its implementation in the classroom.

## **THERETICAL FRAMEWORK**

Knowing and proving are synonyms in mathematics (Rav, 1999; Balacheff, 2010). On one hand, proof, as mean for validation, reinforces precise and highly logical way of thinking based on axioms, definitions, and statements, which link and describe the properties of mathematical objects. On the other hand, proofs include mathematical methods, concepts, and strategies also applicable in problem solving situations (Hanna & Barbeau, 2010). Despite their central role in mathematics, it was observed that proofs receive insufficient appreciation and epistemological understanding from grade school students (and even their teachers), who often rely on empirical evidence rather than on formal deductions of mathematical theorems (Coe & Ruthven, 1994). This situation identifies the needs for “problems and mathematical activities that could facilitate the learning of mathematical proof” and “designing the situations so that ... the theoretical posture demonstrates all its advantages.” “The challenge is to better understand the didactical characteristics of the situation and propose a reliable model for their design”(Balacheff, 2010, p. 133). One possible approach “is centred around the idea that inventing hypotheses and testing their consequences is more productive ... than forming elaborate chain of deductions”(Jahnke, 2007, p.79). The process of making conjectures and inventing hypotheses requires mathematical intuition, which develops through students’ experiences not only in formal logical manipulations but also in experimental explorations of objects and ideas (De Villiers, 1999). Thus collecting empirical evidence (e.g. constructing and measuring) is an important part of the mathematical education of students, and it should not be rejected as such. Instead, a productive way of incorporating experimentation and proving needs to be found so that “*proofs do not replace measurements but make*



*them more intelligent*” (Jahnke, 2007, p. 83). The students should gradually move from everyday thinking in terms of “open general statements” (whose domains of validity are not completely specified) towards mathematical thinking where precision is achieved at the price of cutting ties to empirical reality. This move is possible due to several roles (besides validation statements) that proofs may play in mathematical thinking (Hanna, 2000; De Villiers, 1999). First, at the informal deduction stage, proof as explanation of empirical observations is most appropriate. Next, students “should build a small network of theorems based on empirical evidence” and become accustomed to “*hypothetico-deductive method* which is fundamental for scientific thinking” (Jahnke, 2007, p. 83). At this stage, the proof functions as a “*systematization* (the organization of various results into a deductive system of axioms, major concepts and theorems)” and “*construction* of an empirical theory”. These two stages prepare students to move towards rigorous proofs aiming at establishing truth by deduction or “*incorporation* of well-known facts into a new framework” (Hanna, 2000, p. 8).

The development of reasoning skills by the proposed scenario has an essential contribution in concept formation. As discussed in an upcoming publication (Tall et al), a child’s conceptual system evolves from the stage where several properties of an object occur simultaneously to the stage where these properties are linked by cause-effect relationship. This process results in developing *crystalline concepts* (e.g. platonic objects) with equivalent properties linked by mathematical proofs. Thus the process of maturation of reasoning skills both leads to and requires the use of more formal and structured conceptualizations of empirical objects.

From the perspective of this paper, two further ideas are of particular importance. First, learning to prove is a gradual process which requires years of mutually enhancing empirical and theoretical practices leading to concept formation as more properties, representations and relationships are being understood over time. If this structure is imposed on a learner in its final form, the effect of concept formation by the learner may not be achieved (Freudenthal, 1971). Second, teachers’ epistemological beliefs and their abilities to model the process of proving are decisive for students’ growth in this respect. Thus, teachers’ professional preparation, which facilitates them in transitioning from empirical arguments to proof, is essential (Stylianides & Stylianides, 2009). With this in mind, we now examine an example of designing an interconnecting problem.

## AN EXAMPLE OF INTERCONNECTING PROBLEM DESIGN

In Euclidean geometry an isosceles triangle is often defined as a triangle which has two equal sides. It is well known that there are many equivalent characterizations of an isosceles triangle, such as “two angles are equal”, “an angular bisector is also a median”, “two altitudes are equal”, “two bisectors are equal”, each of which reflects the axial symmetry of the triangle. Proofs that the properties are pair-wise equivalent

constitute problems of various levels of difficulty and contribute to building the conceptual understanding of the object by the learner.

Some properties of an isosceles triangle do not characterize it, however. It is an important exercise to recognize when this happens. For instance, think about the following **Problem**:

**Observation:** Consider any isosceles triangle  $ABC$ , where  $AB = AC$ . Let  $D$  be a point on  $BC$  such that  $AD$  is the angular bisector of  $BAC$ . Let  $M$  and  $N$  be midpoints of sides  $AB$  and  $AC$  respectively. Then  $DM = DN$ .

**Question:** In a triangle  $ABC$  with angular bisector  $AD$  and midpoints  $M$  and  $N$  of sides  $AB$  and  $AC$ , let the segments  $DM$  and  $DN$  have equal length. Does this property imply that  $ABC$  is isosceles?

In further subsections I discuss how to make this problem interconnecting in view of properties (1)-(4) and stages (A)-(D) of the design process outlined in the introduction.

### **A. The initial choice of problem**

The choice of the problem may be justified by several factors such as how fundamental are the objects involved in the problem, the importance of the problem in the development of strands prescribed by the curriculum, or motivational aspects (e.g. surprising result).

For instance, our Problem was chosen by the Problem Designer because it deals with isosceles triangle, the object that appears in many problem-solving situations in geometry. The Problem poses a concrete question which prepares the learner to distinguish between equivalent statements and implications, and further between necessary and sufficient conditions in more abstract theorems. This problem calls for a proof involving construction of a counter-example. When such an example is constructed, it may surprise the students and produce a cognitive shift towards understanding the concept of isosceles triangle in a wider space of its examples and non-examples.

A problem must allow a simple formulation in order to become an interconnecting one. The students should understand the question and be able to specialize and exemplify the statement of the question (Mason et al 1982). For example, in our Problem we start with an Observation which can be justified by the symmetry argument or even by simply folding the paper triangle along its axis of symmetry. In order to answer the Question students may try other examples of triangles. They will quickly realize that they have to either find an example of non-isosceles triangle with given property or proof that such a triangle does not exist. It is clear what one has to do, but not obvious how one can approach this problem. A systematic search for an example needs to be initiated by the solver.

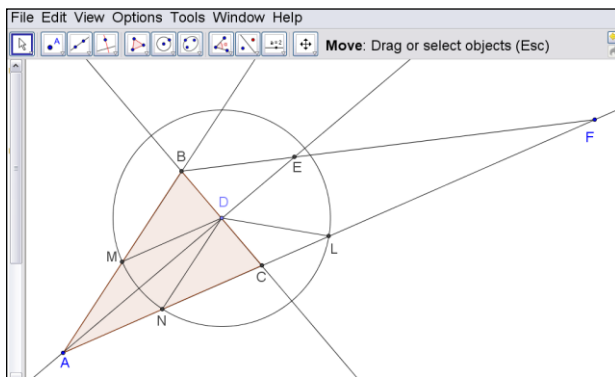
## B. Making the problem Interconnecting: elementary level

First, the Problem Designer puts herself in the position of problem solver. She starts from thinking how she can approach the problem at the most elementary level. According to Bruner's classification, this corresponds to enactive representation of the problem and involves the use of real objects and manipulatives. For a modern learner equipped with a computer, thinking at the elementary level also involves the access to virtual manipulatives, and experimentations within dynamic geometry software (DGS) environments. Through the use of dragging function, the learner receives a visualization of her problem as continuum of options. The example she is looking for may be just one static picture in this continuum.

Construction of such a continuum requires understanding of many basic mathematical ingredients of the problem as well as the properties of the software tools. Here is a protocol from a problem designer's attempt to solve the problem.

First, I examined initial configuration. I have an isosceles triangle  $ABC$ , where  $AB = AC$ . Here  $M$  is the midpoint of  $AB$ ,  $N$  is the midpoint of  $AC$ . Line  $AD$  is the angular bisector of angle  $BAC$ . We know that in this case  $DM = DN$  due to symmetry argument. I draw this triangle on the screen (Fig. 1). Points  $M$  and  $N$  lie on the circle with centre at  $D$ . But this circle also intersects the extension of side  $AC$  at point  $L$ , which means that  $DM = DL$ . Aha, I have an idea: point  $L$  could be the midpoint of the side of the required non-isosceles triangle. Now I place point  $F$  on the extension of  $AC$  such that  $AL = LF$  and look at the triangle  $ABF$ . Denote by  $E$  the intersection point of  $BF$  and the extension of angular bisector  $AD$ . If I could drag points and change the figure in such a way that  $D$  coincides with  $E$  then I will complete the task.

The Problem Designer experiments with this figure but unfortunately it does not seem to be possible to complete the task within this particular construction and she proclaims:



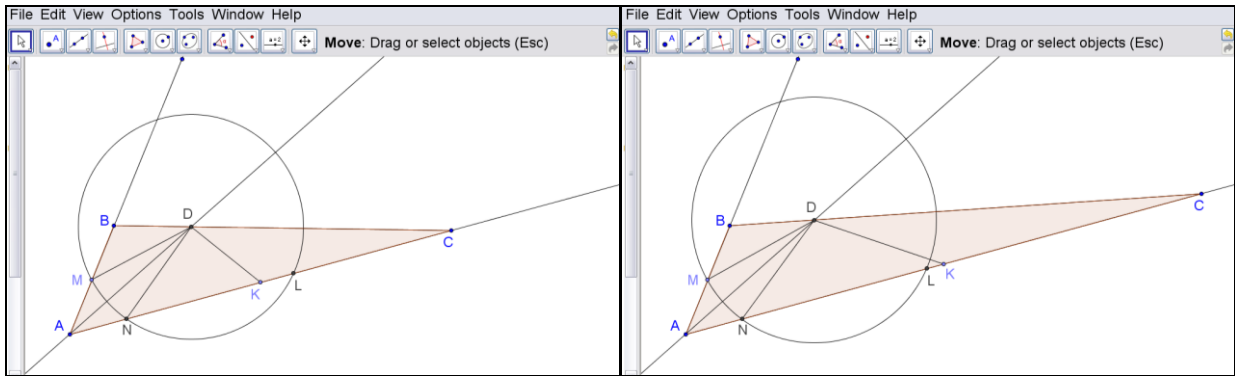
Maybe there is no such example at all. But then I have to explain why. Perhaps I should try to construct something else. Maybe I should not start with an isosceles triangle at all.

**Figure 1: Unsuccessful attempt to build an example.**

Meantime she also learned that the software has an option “reflect a point with respect to another point” which she uses to place a vertex, knowing the position of a midpoint. She continues to build her example. Finally, she succeeds in doing so, still employing the idea that the second point of intersection of the circle with the angle side is the key of the construction.

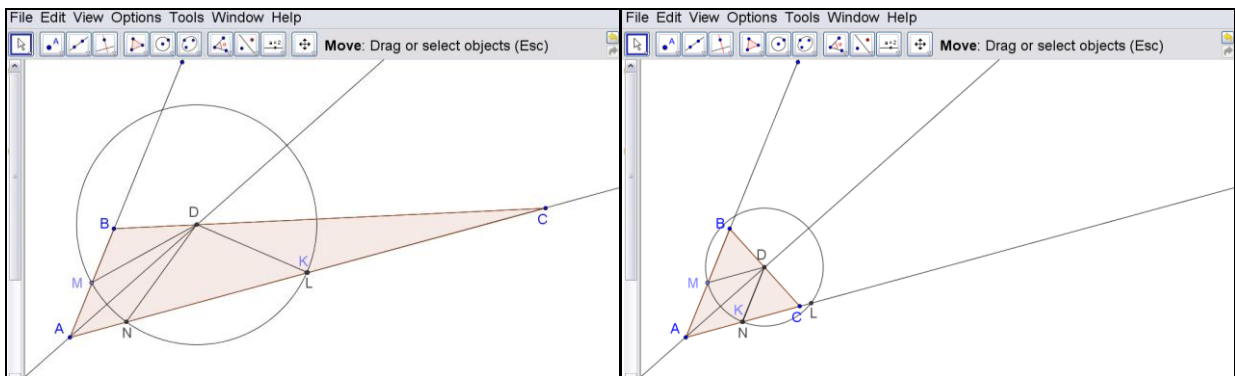
## Working Group 1

I start with an arbitrary angle with vertex at A. I place two arbitrary points M and K one on of each of the sides (see Fig 2, left). Now I place point B on one side such that  $AM = MB$  and place point C on the other side such that  $AK = KC$ . The intersection of BC with the angular bisector is called D. I draw the circle with centre at D and radius DM. This circle intersects side AC at points N and L. Now I want to make K coinciding with either N or L. I conjecture that the former case gives me an isosceles triangle and the latter, if this is possible, will produce the required example.



**Figure 2: A successful attempt to build an example: dragging K along the side AC.**

By dragging point K along the side AC I can interchange the positions of point K and L on the side of the triangle (see Figure 2, right). Thus, by dragging K along the side I achieve that points K and L coincide. (See Figure 3, left). And now I confirm that if K coincides with N then we indeed obtain an isosceles triangle (Figure 3, right).



**Figure 3: A successful attempt to build an example of a triangle where  $DM = DN = DK$ .**

Once the problem is understood at the elementary level, the Problem Designer thinks how this can be used for introduction of more advanced techniques.

### C. Connecting a problem to more advanced mathematics

Now, the Problem Designer aims to use intuition developed through the visualization of a solution for constructing a symbolic solution, which would correspond to the highest stage in Bruner's classification. In secondary school students learn equations of lines and circles. They also learn the idea that solving a system of two equations representing these curves gives the coordinates of the points of intersection of the curves. Thus to make the problem interconnecting one may try to represent

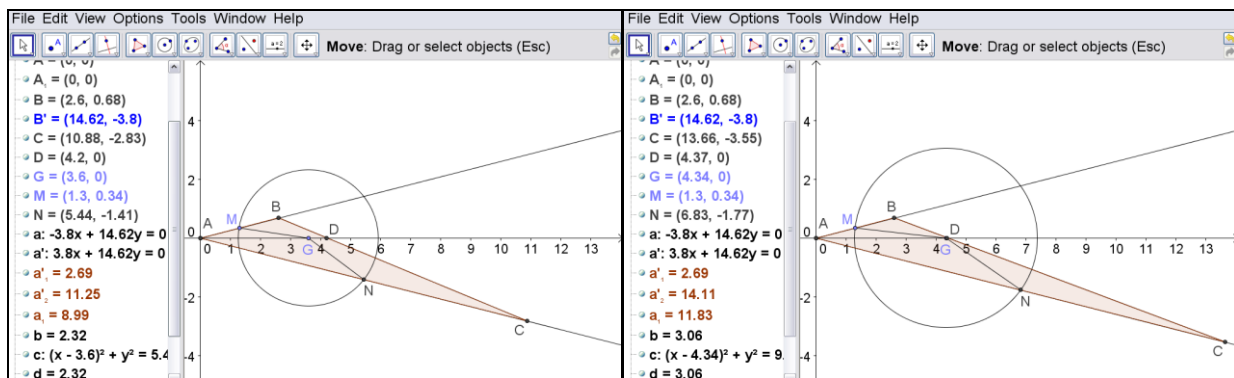
## Working Group 1

previously obtained geometric solution algebraically. Here is what the Problem Designer does next.

I was looking at the Figures 2 and 3 and understood that to make an algebraic representation I need to introduce Cartesian coordinates. Let the side AC lie along the  $x$ -axis with A at the origin. Then point K has coordinates  $(k,0)$ . I introduce a general equation of the sides and the bisector; I fix coordinates of M and B and find how change of  $k$  affects the coordinates of D, C and L. I will set coordinates of L and K equal and find  $k$  from this equation. This is a plan. ... But its implementation becomes very cumbersome! I do not think that students will benefit from it.

Then she introduces another approach aimed at easier “algebraization”.

In a coordinate system I draw two rays starting from the origin and symmetric with respect to the  $x$ -axis (see Fig 4, left). I pick an arbitrary point M on the upper ray and point G on the  $x$ -axis. The circle with centre at G and radius GM intersects the lower ray at two points. I devote by N the one which is not symmetric to M with respect to  $x$ -axis. Now I place point B on the upper ray such that  $AM=MB$ , and point C on the lower ray such that  $AN=NC$ . The segment BC intersects  $x$ -axis at point D. I can drag point G along the  $x$ -axis in the position of point D and thus I obtain the example (see Fig. 4, right).



**Figure 4: A successful attempt to build an example, which allows “algebraization”.**

Now I construct an algebraic model. The two rays have linear equations in the form  $y=mx$  and  $y=-mx$ , where the slope  $m$  can vary. Point G has coordinates  $(g,0)$  and the circle has equation  $(x-g)^2 + y^2 = r^2$ , where  $r=GM$ . In order to find points of intersections of the ray and the circle we need to solve the quadratic equation  $(x-g)^2 + (mx)^2 = r^2$ , that is to express roots  $x_1, x_2$  via  $g$  and  $r$ . Once I find the roots, I obtain coordinates of all points in terms of them:  $M(x_1, mx_1)$ ,  $B(2x_1, 2mx_1)$ ,  $N(x_2, -mx_2)$ ,  $C(2x_2, -2mx_2)$ . Now, I want point G belong to the segment BC. Equating slopes of BC and BG gives the condition:  $g = \frac{4x_1x_2}{x_1 + x_2}$ . But from the quadratic equation I find that

$x_1x_2 = \frac{g^2 - r^2}{m^2 + 1}$  and  $x_1 + x_2 = \frac{2g}{m^2 + 1}$ , thus I obtain the relation between the radius and the coordinate of the centre of the circle  $g = \sqrt{2}r$ . From this relation I find integer coordinates for the vertices of a non-isosceles triangle for which the property  $DM =$

DN can be verified. For instance, if  $A(0,0)$ ,  $M(3,1)$ ,  $B(6,2)$ ,  $N(15,-5)$ ,  $C(30,-10)$ ,  $D(10,0)$  then  $DM=DN=\sqrt{50}$ .

#### **D. Identifying levels and context suitable for problem**

Once several approaches to solve our Problem are identified, the Problem Designer needs to summarize and identify all possible places in the curriculum where this problem potentially belongs. For example, the teacher may give students complete freedom in the choice of approaches to the question. Then the teacher may introduce a DGS and let students to try to build their examples from scratch. During this activity students rethink their task in view of tools available in the computerised environment. For example, the process of constructing angular bisector or a midpoint of a segment may be an automated part of a DGS, and then the major construction pertinent to the problem is conceptualized in terms of these operations.

Alternatively, the teacher may give the students an applet such as shown in Figures 2-3, which forces the students to explain already prefabricated construction. The students may be asked to state their observations about what constraints are preserved in the applet and how important they are for building the example, or what objects are introduced and what role in the solution they play. The students shall articulate their conjectures about observed relationships, for instance “What kinds of examples are possible (e.g. acute, obtuse, right angle)?” They also may be asked to explain why the example they construct with the applet is not just a visual illusion or an approximation; how do they know that the real example with all required characteristics exists. (The role of misleading diagrams in Geometry in relation to proofs is discussed in Kondratieva, 2009).

The problem may appear in the view of the students again when they study the coordinate approach. This time an applet from Figure 4 will be useful because the drawing explicitly reveals the coordinate system and equations on the side suggest a more general algebraic approach. Here the students may be asked to use concrete equations of the lines and circle and then generalize them and analyse the situation in an algebraic form, returning to concrete examples provided by the applet for a verification of their general course of reasoning. Consideration of various cases may be supported by the applet as well.

The problem may be recalled once again when the analysis of quadratic equations is discussed, and the existence of two real roots may be related to the existence of two points of intersection of the circle with the side of the angle. Since traditionally the number of real roots is said to be defined by the discriminant of a quadratic equation, the students may be asked to investigate the connection between the parameters of the figure (radius of the circle in Fig 4) and the resulting coefficients affecting the sign of the discriminant, and draw their conclusions with justification.

Finally, the problem can serve as an illustration in the study of conditional statements (implications) and their converses, inverses and contrapositives in formal

logic, as well as a study of proofs by counter-example. Now the emphasis can be made on the logical structure of the statements because the details of building the example are familiar to the students from previous encounters with the problem.

## CONCLUSION.

In this paper, I illustrate the process of designing an interconnecting problem by using an example from Euclidean geometry. While making our Problem interconnecting, a variety of instances supporting the development of reasoning and proving skills has occurred. Being placed in the domain of mathematical activity the Problem deals with concrete objects, their properties and relationships. The experimentations with dynamic geometry software especially when the students interact with figures that are constrained to retain certain properties, forces the students to explain their actions and observations, to make and justify their conjectures. This activity accompanied by the requirement to systematise observed results pushes the learner towards the hypothetico-deductive stage of reasoning.

An algebraic approach is introduced after geometrical meaning of the model has been understood. While at this stage the focus is on setting and solving equations and development of algebraic thinking, the experience within DGS environments supports students' reasoning as they visualize the situation hidden behind variables and equations. These visualizations contribute into development of learners' intuition as well as in forming algebraic-geometric connections and perhaps a more holistic view on mathematics itself.

Behind concrete problems in mathematics often there is a more general and far-reaching agenda. For example, our Problem aims at grasping the general notion of a universally valid statement by making sense of the proclamation "For every isosceles triangle the segments DN and DM are equal." This statement has the same form as "For every isosceles triangle with  $AB=AC$  the angles B and C are equal." However, the Question is posed to identify whether this property of an isosceles triangle necessarily defines an isosceles triangle. It invites students to realise that not every property of an object in fact defines the object. This fact should be brought to the students' attention forcing them to distinguish between equivalent conditions, such as "equal sides" and "equal angles", and those for which implication works only one way and not both ways. By solving this problem students not only advance their concept of an isosceles triangle, but also build their understanding of the statement's generality, the nature of implication, the notions of necessary versus sufficient conditions, and the idea of proof by counter-example. Thus, the design of an interconnecting problem discussed above aims at fostering reasoning skills at both visual-empirical and symbolic-theoretical levels within the same mathematical question in the background. Further research on teaching practices that involve adoption or design of interconnecting problems and their affect on students' reasoning abilities will show to what extent this would indeed be possible to achieve.

## REFERENCES

- Balacheff, N. (2010). Bridging knowing and proving in mathematics: A didactical perspective. In G. Hanna, H.N. Janke, & H. Pulte (Eds.), *Explanation and proof in mathematics: philosophical and educational perspectives* (115-135). Springer.
- Bruner, J. (1968) *Toward a theory of instruction*. New York, NY: W.W. Norton & Co Inc.
- Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *British Educational Research Journal*, 2(1), 41–53.
- De Villiers, M. (1999). *Rethinking proof with Geometer's Sketchpad*. USA: Key Curriculum Press.
- Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, 3, 413-435.
- Hanna, G. (2000). Proof, explanation and exploration: an overview. *Educational studies in mathematics*, 44, 5-23.
- Hanna, G., Barbeau, E. (2000). Proofs as bearers of mathematical knowledge. In G. Hanna, H.N. Janke, & H. Pulte (Eds.), *Explanation and proof in mathematics: philosophical and educational perspectives* (85-100). Springer.
- Jahnke, H.N. (2007). Proofs and Hypotheses. *ZDM Mathematics Education*, 39, 79-86.
- Kondratieva, M. (2009). Geometrical sophisms and understanding of mathematical proofs. In F.-L. Lin, F.-J. Hsieh, G. Hanna, M. de Villiers (Eds), *Proceedings of the ICMI Study 19: Proof and proving in mathematics education* (vol. 2 pp. 3-8). Taipei: National Taiwan Normal University.
- Kondratieva, M. (2011). The promise of interconnecting problems for enriching students' experiences in mathematics. *Montana Mathematics Enthusiast*, 8(1&2), 355-381.
- Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Addison Wesley.
- Rav, Y. (1999). Why do we prove theorems? *Philosophia Mathematica* 7(3), 5-41.
- Stylianides, G.J., & Stylianides, A.J. (2009). Facilitating the transition from empirical arguments to proofs. *Journal for Research in Mathematics Education*, 40, 314-352.
- Tall, D. O., Yevdokimov, O., Koichu, B., Whiteley, W., Kondratieva, M., & Cheng, Y.-H. (to appear). The cognitive development of proof. In G. Hanna, & M. de Villiers (Eds), *ICMI 19: Proof and proving in mathematics education*. New ICMI Study Series. Springer.



# **INTRODUCTION TO THE PAPERS OFWG 2 TEACHING AND LEARNING OF NUMBER SYSTEMS AND ARITHMETIC**

Susanne Prediger<sup>1</sup>, Nada Stehlikova<sup>2</sup>,  
Joke Torbeyns<sup>3</sup>, Marja van den Heuvel-Panhuizen<sup>4</sup>

<sup>1</sup>Dortmund University (Germany), <sup>2</sup>Charles University Prague (Czech Republic),  
<sup>3</sup>University of Leuven (Belgium), <sup>4</sup>Freudenthal Institute (The Netherlands)

Working Group 2 has newly been formed for CERME 7 as a forum for presenting and discussing theoretical and empirical research on the teaching and learning of number systems and arithmetic (including models for operations in the number systems, ratio and proportion, rational numbers and number-theoretical aspects). The group intensively discussed on twelve papers and two posters addressing research for different ages, ranging from children of age 3-5 years up to pre-service teachers, with a focus on grade 2-6.

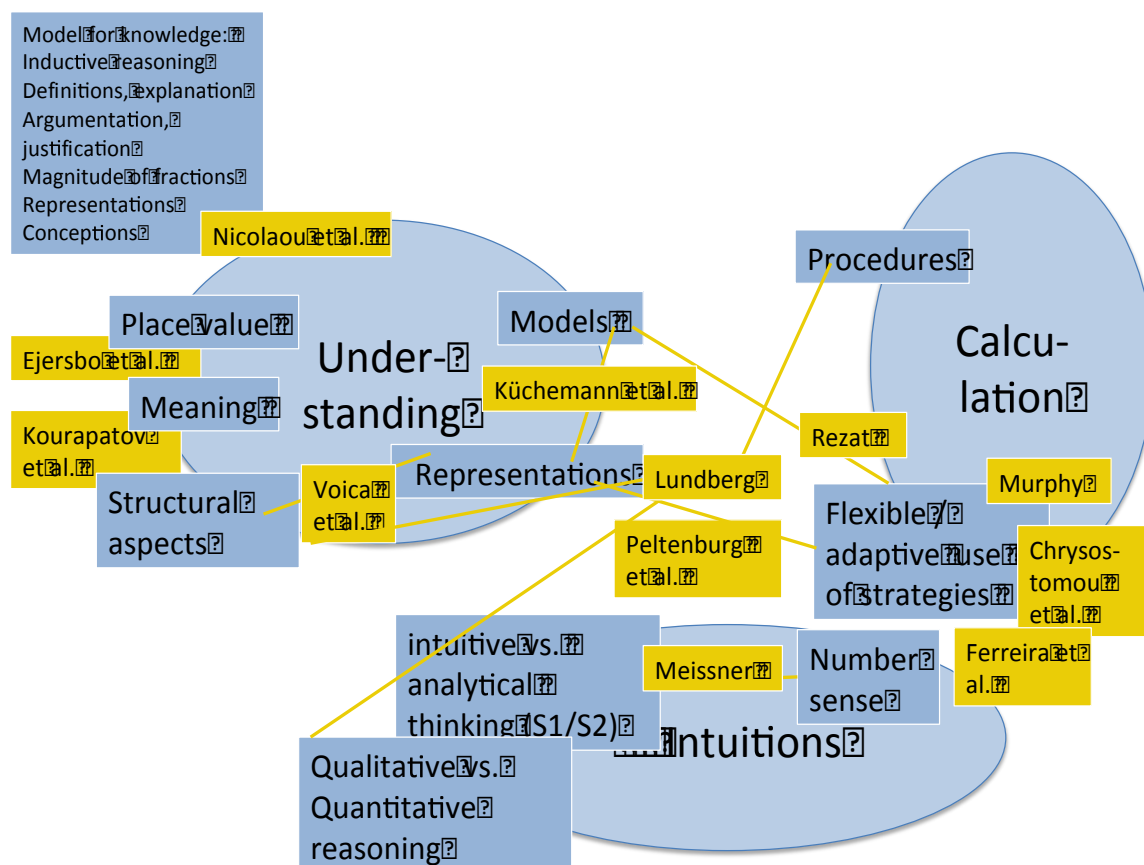
The thematic emphasis was put on

- 2 research-based specifications of domain-specific goals of what should and can be learned and in which priority;
- 3 analysis of learning processes and learning outcomes in domain-specific learning environments and classroom cultures;
- 4 new approaches to the design of meaningful and rich learning environments and assessments.

## **DIFFERENT CONSTRUCTS FOR CONCEPTUALIZING DOMAIN-SPECIFIC GOALS**

The domain-specific goals comprised knowledge and competencies on numbers and operations in basically three areas: *intuition*, *understanding* and *calculation*. Whereas the participants of the working group agreed on the theoretical constructs to conceptualize competencies in *calculations* (procedures and flexible strategies), they used different constructs for conceptualizing *understanding* and *intuition*.

The figure attempts to give an idea on the landscape of different constructs that were used in the papers for conceptualizing different aspects of understanding, intuition and calculation. One intensively discussed example is the construct *number sense* that is widely used but conceptualized in different ways.



## DIFFERENT TYPES OF RESEARCH QUESTIONS

The research questions studied in the working group give an interesting insight into priorities of some of the current research in the domain of teaching and learning arithmetic. Most papers concerned research on students' thinking, but with a different focus.

Focus on *status quo* of students' thinking:

- describing selected phenomena, e.g.
  - Which strategies for subtracting do students apply? (Peltenburg & van den Heuvel-Panhuizen, Rezat)
  - Which typical difficulties appear, e.g., while solving problems? (Voica et al.)
- searching for internal connections, e.g.
  - How is students' use of strategies connected to their number sense? (Ferreira et al)
  - To which models do students refer for explaining their strategies? (Rezat)
  - How are different components of students' knowledge on fractions connected? (Nicolaou et al.)

## Working Group 2

- searching for external connections, e.g.
  - How are students' strategies related to their preference to refer to structures in representations (spatial imagery)? (Chrysostomou et al.)

Focus on means for *supporting* students' thinking and its *development*:

- describing the development of students' thinking, e.g.
  - How does students' use of strategies develop over a certain time? (Ferreira et al., Murphy)
- specifying aspects or means that support or hinder students' thinking, e.g.
  - What mathematical / linguistic structures might hinder students in applying number concepts? (Ejersbo/Misfeldt)
  - What means (representations, models, ...) support the use of different strategies? (Peltenburg et al.)
- investigating means / conditions / contexts to support the development of students' thinking, e.g.
  - How can explorative talk in small group situations help students to develop their strategies? (Murphy)
  - How can the use of calculators contribute to developing number sense? (Meissner)
  - How do textbooks support or hinder the development of multi-faceted concepts of proportions? (Lundberg)
  - How can the double number line be used for developing multiplicative reasoning? (Kuechemann et al.)
  - Which activities helped kindergarten children to develop symmetric conceptions of equivalence? (Kourapatov et al.)

This wide spectrum of research questions reflects a large number of open points in the teaching and learning of number systems which need further research.

The discussion in the group raised some new awareness and questioned some positions that were taken for granted. Especially the following questions need further research:

- 5 What balance and what interplay between developing conceptual understanding and procedural skills for number operations can and should we aim at while designing learning environments? And in what order should we teach students understanding and procedural skills?
- 6 What does it mean to operate flexibly with numbers? What knowledge and skills are required to operate flexibly with numbers?

## Working Group 2

- 7 What roles do models and teaching strategies play in operating with numbers flexibly?
- 8 What aspects of number theory should and can be taught in the primary school grades and how can these be taught?
- 9 How can long-term learning processes from grade 1 to grade 10 be supported and analysed?
- 10 What aspects of the number curriculum in the higher grades can support the transition to tertiary mathematics study?

These questions include so-called *what-questions* that are on the one hand crucial for didactical research and development, but, on the other hand, are often neglected in research papers because of a lack of established standards how to treat them scientifically. The working group will continue to search for ways of systematically tackling them, for example by math-didactical analysis (van den Heuvel-Panhuizen & Treffers, 2009), by referring to general educational goals (e.g. Heymann, 2003) and by specifying needs of the society (Meissner in this volume).

## REFERENCES

- Heuvel-Panhuizen van den, M., Treffers, A. (2009), 'Mathe-didactical reflections on young children's understanding and application of subtraction-related principles', *Mathematical Thinking and Learning*, 11, 102–112.
- Heymann, H.W. (2003). *Why Teach Mathematics: A Focus on General Education*. Dordrecht: Kluwer.

# COGNITIVE STYLES AND THEIR RELATION TO NUMBER SENSE AND ALGEBRAIC REASONING

MarilenaChrysostomou, CharaTsingi, EleniCleanthous&Demetra Pitta-Pantazi

Department of Education, University of Cyprus

*The purpose of the present study was to examine the relationship between individuals' cognitive style and their mathematical achievement and strategies used while solving number sense and algebraic reasoning tasks. A mathematical test on number sense and algebraic reasoning and a self-report cognitive style questionnaire were administrated to 83 prospective teachers. The results indicated that spatial imagery, in contrast to the object imagery and verbal cognitive styles, is related to the achievement in algebraic reasoning and number sense. The study also revealed that as prospective teachers' spatial imagery style increases, the use of conceptual strategies in solving the tasks also increases.*

Keywords: cognitive styles, number sense, algebraic reasoning, procedural and conceptual strategies

## INTRODUCTION

The relationship between various cognitive style dimensions and mathematical achievement attracted the attention of several researchers worldwide for many years (Pitta &Christou, 2009b; Presmeg, 1986a). A number of recent studies (Anderson, Casey, Thompson, Burrage, Pezaris&Kosslyn, 2008; Kozhevnikov, Hegarty& Mayer, 2002) examined the effects of students' cognitive styles on their mathematical learning, utilizing a new approach to the visual-verbaliser cognitive style dimension. These studies, supported the view that there exist three different cognitive style dimensions, a verbal style as well as two types of visual cognitive styles, the spatial imagery and the object imagery. Despite the fact that there have been suggestions for improvements concerning the cognitive style dimensions, research so far (Anderson et al., 2008; Pitta-Pantazi&Christou, 2009b), focused mainly on the investigation of the relationship between cognitive styles and achievement in geometry and measurement concepts. What is absent in mathematics research, is the investigation of a possible relationship between *cognitive styles*, *number sense* and *algebraic reasoning*. Consequently, the purpose of the present study was to examine the relationship between cognitive styles, achievement and strategies used while solving tasks that involve *number sense* and *algebraic reasoning*.

## THEORETICAL FRAMEWORK AND RESEARCH GOALS

### Cognitive styles

Cognitive style is an individual preferred and habitual approach to organizing and representing information, which subsequently affects the way in which one perceives

and responds to events and ideas (Riding and Rayner, 1998). A number of researchers have proposed a wide variety of cognitive style dimensions, such as visualisers-verbalizers (Paivio, 1971), impulsivity-reflectiveness (Kagan, 1965), field dependency-field independency (Witkin & Asch, 1948a). In the field of mathematics education, the verbaliser/imager distinction was the one that attracted most attention (Pitta & Christou, 2009). According to this view, visualisers rely primarily on imagery when attempting to perform cognitive tasks, whereas verbalizers rely primarily on verbal-analytical strategies. However, in a recent study, Blazhenkova and Kozhevnikov (2009) suggested that there exist two distinct imagery subsystems that help individual process information in different ways (Pitta & Christou, 2009), the object imagery system and the spatial imagery system. Therefore, research provides evidence for two types of visualisers, the object visualisers and the spatial visualisers. Object visualisers have low spatial ability and use imagery to construct vivid high-resolution images of individual objects, while spatial visualisers have high spatial ability and use imagery to represent and transform spatial relations.

### **Cognitive styles and mathematics**

A number of studies have investigated the relationship between cognitive styles and mathematical achievement (e.g. Kozhevnikov et al., 2002; Presmeg, 1986a). However, their results are often conflicting. Some studies have shown that spatial imagery is an important factor of high mathematical achievement (Kozhevnikov et al. 2002) whereas other studies showed that students classified as visualisers do not tend to be among the most successful performers in mathematics (Presmeg, 1986a).

Moreover, findings from such studies revealed also certain areas of mathematics for which spatial imagery is important. For example, Kozhevnikov et al. (2002), conducted a study to compare the use of mental images by the two types of visualisers in solving problems with graphs of motion. Students with object imagery style interpreted the graphs as pictures while students with spatial imagery style constructed more schematic images and manipulated them spatially. In another study of Anderson et al. (2008), on geometry problems with geometry clues matched to cognitive styles, both spatial imagery and verbal cognitive styles were important for solving geometry problems, whereas object imagery was not.

### **Cognitive styles, number sense and algebraic reasoning**

Some studies examined mental representations or *imagery* in arithmetic and revealed differences between high and low achievers (Pitta & Gray, 1996; Gray, Pitta & Tall, 1997). Low achievers had a tendency to highlight surface details and emphasized the concrete qualities within situations (focused to the descriptive qualities of numbers) and their responses to a range of addition and subtraction combinations involved mainly *counting procedures*. On the other hand, mathematically high achievers concentrated more on the relationships and abstract qualities of numbers (Pitta & Gray, 1996) and in the addition and subtraction combinations they seemed to have a

*better sense* of the concept that enabled them to *compress* the long sequences of procedures (Gray et al., 1997). According to Gray et al. (1997), different perceptions of the objects are at the heart of different cognitive styles that lead to success and failure in elementary arithmetic.

The aforementioned strategies used by high and low achievers refer to procedural and conceptual understanding, respectively. According to Hiebert and Carpenter (1992), procedural knowledge is a sequence of actions and conceptual understanding is the knowledge that is rich in relationships. In this study, the terms conceptual and procedural strategies emerge from these definitions and are used to describe the way that prospective teachers solved the tasks. Procedural strategies involve typical and time-consuming strategies, application of formulas and generally a “sequence of rote or senseless actions”. On the other hand conceptual strategies are those that reveal insightful “understanding” regarding the concepts that are studied and the ability of a person to make mathematical judgments and to use more flexible strategies. What must be noted, is that the term “cognitive style” was treated in an “informal” way in the abovementioned studies (Gray et al., 1997; Pitta & Gray, 1996), since no use of a tool for measuring the cognitive styles was evident. However, their suggestions about a possible relation between cognitive styles and arithmetic are in accord with some older studies’ results (Navarro, Aguilar, Alcalde & Howell, 1999; Blaha, 1982). Navarro et al. (1999), found that field independence style relates to achievement in arithmetic and the study conducted by Blaha (1982), showed that reflective cognitive style, relates to achievement in arithmetic problem solving. However, these studies, did not take into consideration an appropriate cognitive style questionnaire, that is rooted in more general theory of human information processing (Kozhevnikov et al., 2002).

Despite the fact that there has been a limited number of studies examining the relationship between cognitive styles and arithmetic operations, there is a lack of studies examining the relationship between cognitive styles, number sense and algebraic reasoning. Number sense refers to a person’s general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations (McIntosh, Reys & Reys, 1992). Algebraic reasoning is a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways (Kaput, 1999). The interrelation of number sense and algebraic reasoning is reported by several researchers (Carpenter, Levi, Berman & Pligge, 2005; Steffe, 2001). Moreover, number and algebra constitute together the strand “Number and Algebra” in the mathematics curriculums of many countries e.g. New Zealand, Singapore, Hong Kong. For these reasons, number sense and algebraic reasoning were jointly considered in this article and were not separated.

The development of both, number sense and algebraic reasoning is crucial for mathematics learning. However, results from some studies (e.g. Johnson, 1998), revealed that prospective teachers' general number sense is not sufficiently developed. For these reasons it is important to investigate whether some cognitive variables (such as cognitive styles) will enable us better to understand individual differences in solving problems and tasks that involve these concepts, something that is not evident in the mathematics literature review.

As a result, the purpose of the present study was to investigate whether mathematical achievement in number sense and algebraic reasoning tasks and strategies used for solving these tasks, are related to specific cognitive style. More specifically, we sought answers to the following questions: (a) Do cognitive styles (verbal, spatial and object-imagery) predict prospective teachers' achievement in number sense and algebraic reasoning tasks? (b) Does prospective teachers' achievement in the aforementioned tasks differentiate in accordance to their cognitive style profile? (here participants were grouped into eight different cognitive style profiles) (c) Is there a relation between prospective teachers' cognitive style and the strategies they adopt in solving algebraic reasoning and number sense tasks?

## **METHODOLOGY**

### **Participants**


The participants were 83 prospective elementary school teachers. All participants have taken mathematics lessons during their lower and upper secondary education. At the university level, they attended three mathematics courses where one of them was mathematics education. A mathematical test and a self-report cognitive style questionnaire were administered to participants during two sessions.

### **The mathematical test**

The mathematical test on number sense and algebraic reasoning included 10 tasks. Examples of tasks are provided in Figure 1. A verbal and a pictorial task were employed to examine students' abilities in each content area (calculation-estimation, patterns, number divisibility, relations among numbers and problem solving with unknowns). Using two different representations (verbal and pictorial), we attempted to achieve a balanced test among the three cognitive styles that were examined in this study. Two codes were given to each answer. First, the answer was coded as correct (success=1) or incorrect (success=0). Then, a second code was given for the strategy used by the participant to complete the task. In the initial stage of the analysis many strategies were generated which were later grouped into two general categories. The first category (strategy=1) contained conceptual strategies (i.e. flexible strategies that revealed deep understanding of relationships among numbers and symbols) and the second category (strategy=2) included procedural strategies (i.e. strategies that followed step by step procedures or memorization of formulae and rules). For



example, a prospective teacher gave the following answer when asked to solve task 1 “Yes, (there is there a number divisible by 7 between 12 358 and 12 368) because the difference between the two numbers is ten and there must be at least one number that is divisible by 7” (strategy=1). On the other hand, another participant answered as follows: “I believe that the numbers 12 357 and 12 363 can be divided by 7 because the numbers 57 and 63 can be divided by 7” (strategy=2).

<p>Task 1 (number sense)</p> <p>Is there a number divisible by 7 between 12 358 and 12 368? (Zazkis&amp; Campbell, 1996)</p>	<p>Task 2 (algebraic reasoning)</p>  <p>Find the value of one star.</p>	<p>Task 3 (number sense)</p> <p>Calculate the sum of numbers between 1 and 100.</p>
--	---	---

**Figure 1: Sample Tasks of the mathematical test**

### The cognitive style questionnaire

The cognitive style questionnaire that was a translation of the Object-Spatial Imagery and Verbal Questionnaire (Blazhenkova&Kozhevnikov, 2009) contained 45 statements and examined participants’ differences in spatial imagery (15 items, e.g. *My images are more schematic and they are not colourful*), object imagery (15 item, e.g. *My images are colourful and bright*) and verbal cognitive style (15 items, e.g. *My verbal skills are excellent*). These items were statements about qualitative characteristics of the images, special favours to specific types of visual images or verbal thinking, learning and habitual preferences, preferences to certain professionsand individuals’ assessments of their skills in using spatial or object imagery or verbal processing. Participants rated the items on a 5-point Likert scale with 1 indicating total disagreement and 5 indicating total agreement. For each participant, the spatial imagery, the object imagery and the verbal scale scores were created by calculating the average score of the fifteen items of each cognitive style. The data were analysed using the statistical package SPSS and multiply methods of analysis were performed, including regression analysis, multivariate analysis of variance (MANOVA), pearson correlation and descriptive statistics. The results of this study are presented in two sections. The first section deals with the relationship between cognitive styles and achievement in number sense and algebraic reasoning, whereas the second is concerned with cognitive styles and their relationship to the strategies that prospective teachers adopt in solving the tasks.

### Cognitive styles and achievement in number sense and algebraic reasoning

In order to answer research question 1 correlation and regression analyses were conducted. Firstly, to investigate the relationship between cognitive styles and achievement in number sense and algebraic reasoning, we examined the correlations between prospective teachers’ cognitive styles and their achievement, which are presented in Table 1. As it appears from Table 1, spatial imagery cognitive style significantly correlates with prospective teachers’ total achievement score, with their

achievement in verbal tasks as well as with their achievement in pictorial tasks. However, the other cognitive styles (object and verbal) did not correlate with prospective teachers' achievement.

Cognitive styles	Total achievement	Achievement in verbal tasks	Achievement in Pictorial tasks
Spatial Imagery	.401*	.358*	.345*
Object Imagery	-.165	-.216	-.075
Verbal	-.176	-.190	-.118

\*Correlation is significant at the 0.05 level (2-tailed).

**Table 1: Correlations among achievement and spatial imagery, object imagery and verbal cognitive styles.**

Then multiple regression analyses were conducted with criterion (dependent) variables the total achievement score, the achievement in verbal tasks and the achievement in pictorial tasks, and predictors (independent) the spatial imagery, object imagery and verbal cognitive styles. The results of the multiple regressions are presented in Table 2 and provide more information about the nature of the relationships between teachers' achievement and cognitive styles. It is obvious that only the spatial imagery cognitive style is a statistically significant predictor of prospective teachers' achievement in number sense and algebraic reasoning, regardless of the mode of representation of the tasks, and it explains a respectable proportion of variance (more than 20%) in achievement in number sense and algebraic reasoning. In other words, as prospective teachers' spatial imagery cognitive style increases, their total achievement in the test, and their achievement in verbal and pictorial tasks also increase. Moreover, in order to answer research question 2 and investigate possible differences between the different profiles of cognitive styles, participants were grouped in eight different groups with respect to their spatial imagery, object imagery and verbal cognitive styles as follows: high/low Spatial, high/low Object and high/low Verbal. The mean scores of each group with regard to their total achievement score are presented in Table 3. The highest mean score corresponds to prospective teachers with high preference in spatial visualization processing and low preference in object visualization and verbal processing (group 6).

Cognitive styles	Total achievement		Achievement in verbal tasks		Achievement in pictorial tasks	
	<i>b</i>	<i>p</i>	<i>b</i>	<i>p</i>	<i>b</i>	<i>p</i>
Spatial imagery	1.339	.001*	.660	.002*	.679	.003*
Object Imagery	-.495	.222	-.390	.093	-.106	.661
Verbal	-.199	.593	-.140	.510	-.059	.790

\*Statistical significance  $p < 0.05$

**Table 2: Multiple regression analyses with dependent variables total achievement, achievement in verbal tasks and achievement in pictorial tasks, and independent variables spatial imagery, object imagery and verbal cognitive styles.**

Also, prospective teachers with high preference in spatial processing (groups 1,3,4,6) have higher scores than those with low preference in spatial processing.

To further investigate the impact spatial imagery has on achievement, prospective teachers were assigned to high and low spatial imagery groups. A multivariate analysis of variance (MANOVA) was conducted with the achievement scores in verbal and pictorial tasks as dependent variables and the preference in spatial processing as independent one. The results of the multivariate analysis showed that there were significant differences between prospective teachers achievement according to their preference in spatial processing (Pillai's  $E_{(1,81)} = 3,882$ ,  $p < 0,05$ ). More specifically, prospective teachers with high spatial imagery have significantly better achievement scores on verbal and pictorial tasks than prospective teachers with low spatial preference.

Cognitive style profiles			N	$\bar{x}$	SD	Cognitive style profiles			N	$\bar{x}$	SD
S*	O*	V*				S*	O*	V*			
1. H	H	H **	18	3.89	1.78	5. L	H	H	13	3.08	1.80
2. L	L	L **	7	3.14	1.35	6. H	L	L	9	5.56	2.65
3. H	H	L	11	4.36	2.58	7. L	L	H	9	3.22	1.39
4. H	L	H	11	4.82	2.72	8. L	H	L	5	4.00	1.23

\*S=Spatial, O=Object, V=Verbal, \*\* H= High, L=Low

**Table 3: Means of achievement score for each cognitive style profile.**

### Cognitive styles and strategies used in solving the tasks

To answer research question 3, correlation and regression analyses were conducted. The correlations among prospective teachers' strategies and cognitive styles are shown in Table 4. It appears that the spatial imagery cognitive style significantly correlated with the use of conceptual strategies ( $r=.310$ ,  $p < 0.05$ ) and not procedural strategies, while the rest of the cognitive styles did not correlate with any type of strategies.

Cognitive styles	Conceptual strategies	Procedural strategies
Spatial imagery	.310*	-.021
Object Imagery	-.124	-.092
Verbal	-.152	-.101

\*Correlation is significant at the 0.05 level (2-tailed).

**Table 4: Correlations among strategies used in number sense and algebraic reasoning tasks and spatial imagery, object imagery and verbal cognitive styles.**

To further investigate the nature of these correlations, we analysed our data using multiple regression analysis with criterion (dependent) variables the “conceptual strategies” and the “procedural strategies” and predictors (independent) variables the cognitive styles. The results are presented in Table 5. As it can be seen, the spatial imagery cognitive style is a statistically significant predictor of prospective teachers’ use of “conceptual strategies” for solving tasks with numbers concepts and algebraic reasoning and it explains the 20% of the variance in the adoption of strategies. We can conclude that as prospective teachers’ spatial imagery increases, the use of conceptual strategies in solving various tasks, also increases. On the other hand, none of the cognitive styles can predict the use of “procedural-conventional” strategies.

<b>Cognitive styles</b>	<b>Conceptual strategies</b>		<b>Procedural strategies</b>	
	<i>b</i>	<i>p</i>	<i>b</i>	<i>p</i>
<b>Spatial imagery</b>	.864	.009*	-.137	.681
<b>Object Imagery</b>	-.299	.402	-.299	.529
<b>Verbal</b>	-.184	.576	-.268	.424

\*Statistical significance  $p < 0.05$

**Table 5: Multiple regression analyses with dependent variables the strategies prospective teachers use and independent variables spatial imagery, object imagery and verbal cognitive styles.**

## DISCUSSION

Several studies revealed the important role that spatial imagery cognitive style plays in mathematical creativity, geometry and problem solving (see e.g. Pitta & Christou 2009; Anderson et al., 2008; Kozhevnikov et al., 2002). The present study moves one step further and provides evidence that spatial and object imagery may have different effects on the achievement and on the strategies that children adopt for solving tasks that involve number sense and algebraic reasoning. The results indicated that spatial imagery is the only significant predictor of prospective teachers’ achievement in number sense and algebraic reasoning tasks and that teachers with high preference on spatial processing had significantly higher scores on both verbal and pictorial tasks, than the teachers with low preference on spatial processing. This finding is in line with other studies’ results that examined cognitive styles in relation to mathematical problem solving (Kozhevnikov et al., 2002) and mathematical creativity (Pitta-Pantazi & Christou, 2009).

The present study also revealed that spatial imagery is a significant predictor of prospective teachers’ “conceptual strategies” adoption. It appears that as prospective teachers’ preference to spatial processing increases, the adoption of conceptual strategies for solving the number sense and algebraic reasoning tasks also increases. We can say that prospective teachers with high spatial imagery tend to “see” relations between numbers that others do not and consequently they are “in favour”

in using quicker and conceptual strategies that involve “understanding” or “insight”. This is in accord with previous research by Gray et al., (1997), who suggested that individuals may approach mathematics tasks in different ways, depending on their cognitive styles. On the other hand, none of the cognitive styles could predict the use of procedural-conventional strategies that involve “rote” or “senseless” actions. The latter might indicate that the adoption of “procedural-conventional” strategies is more of a result of formal instruction in schools and prospective teachers, regardless of their cognitive style, adopt the aforementioned strategies only when they cannot efficiently use the “conceptual-short strategies”. However, from these results it cannot be generalised that spatial imagery is the only cognitive style that is beneficial for number sense and algebraic reasoning, since the sample was of limited size and from a certain age group. The replication of the present study with a different and larger sample is necessary in order to inform us whether the same pattern appears in younger students.

Concluding, an interesting proposal for future research could be the investigation of in-service teachers’ cognitive styles and their relation to the strategies they teach to students for solving several mathematical problems that involve different concepts. If a certain cognitive style (e.g. spatial imagery) relates to specific strategies, then possible growth of teachers’ spatial processing could improve teachers’ strategies, which in turn could enhance students’ conceptual understanding concerning several concepts.

## REFERENCES

- Anderson, K.L., Casey, M. B., Thompson, W. L., Burrage, M. S., Perazis, E., & Kosslyn, S. M. (2008). Performance on middle school geometry problems with geometry clues matched to three different cognitive styles. *Mind, Brain, and Education*, 2(4), 188-197.
- Blaha, J. (1982). Predicting reading and arithmetic achievement with measures of reading attitudes and cognitive styles. *Perceptual and Motor Skills*, 55(1), 107-114.
- Blazhenkova, O., & Kozhevnikov, M. (2009). The New Object-Verbal Cognitive Style Model: Theory and Measurement. *Applied Cognitive Psychology*, 23(5), 638-663.
- Gray, E., Pitta, D., & Tall, D. (1997). The nature of the object as an integral component of numerical processes. In E. Pehkonen (Ed.), *Proceedings of the 21st International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 115–130). Lahti: Finland.
- Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 65-97). New York: MacMillan.

- Johnson, N.R. (1998). A Descriptive Study Of Number Sense And Related Misconceptions About Selected Rational Number Concepts Exhibited By Prospective Elementary Elementary Teachers. *Dissertation Abstracts International*, DAI-A 59/11, 4088.
- Kagan, J. (1965). Individual difference in the resolution of response uncertainty. *Journal of Personality and Social Psychology*, 2, 154–160.
- Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.
- Kozhevnikov, M., Hegarty, M., & Mayer, R.E. (2002). Revising the visualizer/verbalizer dimension: evidence for two types of visualizers. *Cognition & Instruction*, 20, 47-77.
- McIntosh, A., Reys, B., & Reys, R. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12, 2-8.
- Navarro, J. I., Aguilar, M., Alcalde, C., & Howell, R. (1999). Relationship of arithmetic problem solving and reflective-impulsive cognitive styles in third-grade students. *Psychological Reports*, 85, 179-186. doi:10.2466/PRO.85.5.179-186
- Paivio, A. (1971). *Imagery and verbal processes*. New York: Holt, Rinehart, and Winston.
- Presmeg, N. (1986a). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17, 297-311.
- Pitta-Pantazi, D., & Christou, C. (2009b). Cognitive styles, dynamic geometry and measurement performance, *Educational Studies in Mathematics*, 70, 5-26.
- Pitta-Pantazi, D., & Christou, C. (2009). Mathematical creativity and cognitive styles. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the PME 33* (Vol. 4, pp.377-384). Thessaloniki, Greece: PME
- Pitta, D., & Gray, E. M. (1996). Nouns, Adjectives and Images in Elementary Mathematics. In L. Puig and G. Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 35-42). Valencia: Spain
- Riding, R., J., & Rayner, S. (1998). *Cognitive styles and learning strategies: Understanding style differences in learning and behavior*. London: D. Fulton.
- Witkin, H. A., & Asch, S. E. (1948a). Perception of the upright in the absence of visual field. *Journal of Experimental Psychology*, 38(5), 603-614.
- Zazkis R., & Campbell, S. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, 27(5), 540-563.

# DANISH NUMBER NAMES AND NUMBER CONCEPTS

Lisser Rye Ejersbo & Morten Misfeldt

Danish School of Education, Aarhus University

*This paper raises some questions concerning the relation between Danish number names and digits in the canonical base 10 system. Our hypothesis is that in Danish, number names are more complicated than in other languages, and for this reason, Danish children have more difficulties learning and working with numbers. From this point of view, we make a theoretical investigation among different languages and how the names influence the conceptual understanding of numbers. We compare Danish, English and Japanese number names and show from semiotic and cognitive perspectives how the qualitative differences in how the first 100 numbers are named may give rise to linguistically determined differences in children's concept of numbers and in the cognitive load of arithmetic processes.*

**Keywords:** Base-10 system, two-digit number names, semiotic, cognitive perspectives

## INTRODUCTION

In this paper we investigate theoretically the Danish number names and their influence on how children may conceptualize numbers. We look at the interplay between the words we use to denote numbers and the way numbers are written in the canonical base-10 system (from here on base-10). In the Great Danish Encyclopedia, Danish number names are described as “Very old and reflecting a number concept that is primitive in relation to mathematical thinking” (Talord, 2009–2010/Our translation).

Danish number names are very complex, deriving from old number systems using base-12 and base-20. In our investigation, we are especially interested in how the Danish number names can help or hinder the development of a practical concept of numbers and arithmetic competence.

Denmark is one of the Scandinavian countries, and Danish is so similar to Swedish and Norwegian that the three languages are mutually intelligible—except for the number names. In both Swedish and Norwegian, the system for number names is similar to the English one.

Comparative investigations between different linguistic communities provide examples of how a preschool child's mother tongue influences his or her concept of numbers and understanding of place value (Miura et al., 1989; Miura et al., 1993; Miura et al., 1999). The focus of several investigations has been the connection between number names from 10–100 in different languages and the understanding of the place value in base-10. Denmark has never participated in such comparative studies.

## Working Group 2

Our hypothesis is that the Danish names for the first 100 numbers are more complicated than those in other languages, and that for this reason Danish children have more difficulties learning and working with numbers. In this paper we test this hypothesis from three different types of theoretical perspectives: comparative perspectives involving the analysis of the number names in Danish, English and Japanese, semiotic perspectives, and cognitive perspectives.

### COMPARING NUMBER NAMES

The following table contains number names in Danish, English and Japanese. The table shows how some Danish names for numbers use 20 as the base reference, while the numbers written with digits follow base-10.

	Danish	Explanation	English	Explanation	Japanese	Explanation
1	En	1	One	1	Ichi	1
2	To	2	Two	2	Ni	2
3	Tre	3	Three	3	San	3
4	Fire	4	Four	4	Shi	4
5	Fem	5	Five	5	Go	5
6	Seks	6	Six	6	Roku	6
7	Syv	7	Seven	7	Sichi	7
8	Otte	8	Eight	8	Hachi	8
9	Ni	9	Nine	9	Kyu	9
10	Ti	10	Ten	10	Juu	10
11	Elleve	11	Eleven	11	Juu-ichi	10 + 1
12	Tolv	12	Twelve	12	Juu-ni	10 + 2
13	Tretten	13	Thirteen	13	Juu-san	10 + 3
14	Fjorten	14	Fourteen	14	Juu-shi	10 + 4
15	Femten	15	Fifteen	15	Juu-go	10 + 5
16	Seksten	16	Sixteen	16	Juu-roku	10 + 6
17	Sytten	17	Seventeen	17	Juu-sichi	10 + 7
18	Atten	18	Eighteen	18	Juu-hachi	10 + 8
19	Nitten	19	Nineteen	19	Juu-kyu	10 + 9
20	Tyve	20	Twenty	20	Ni- Juu	2x10
21	Enogtyve	1 + 20	Twenty-one	20 + 1	Ni- Juu-ichi	2x10 + 1
30	Tredive	30	Thirty	3x10 or 30	San-Juu	3x10
32	Toogtredive	2 + 30	Thirty-two	30 + 2	San- juu-ni	3x10 + 2
40	Fyrre	40	Forty	4x10 or 40	Si- Juu	4x10
43	Treogfyrre	3 + 40	Forty three	4x10 + 3	Si- Juu-san	4x10 + 3
50	Halvtreds	50 (2.5x20)	Fifty	5x10	Go-Juu	5x10
54	Fireoghalvtreds	4 + 50	Fifty-four	5x10 + 4	Go- Juu-shi	5x10 + 4
60	Tres	60 (3x20)	Sixty	6x10	Roku- Juu	6x10
65	Femogtres	5 + 60	Sixty-five	6x10 + 5	Roku- Juu-go	6x10 + 5
70	Halvfjers	70 (3.5x20)	Seventy	7x10	Sichi- Juu	7x10
76	Seksoghalvfjers	6 + 70	Seventy-six	7x10 + 6	Sichi- Juu-roku	7x10 + 6
80	Firs	80 (4x20)	Eighty	8x10	Hachi- Juu	8x10
87	Syvog firs	7 + 80	Eighty-seven	8x10 + 7	Hachi- Juu-sichi	8x10 + 7
90	Halvfems	90 (4.5x20)	Ninety	9x10	Kyu- Juu	9x10
98	Otteoghalvfems	8 + 90	Ninety-eight	9x10 + 8	Kyu- Juu-hachi	9x10 + 8
100	Hundrede	100	Hundred	100	Hyaku	100

**Table 1: Number names between 1 and 100 in Danish, English and Japanese, combined with the underlying calculation ‘explained’ through these names.**



In comparing Danish and English and Japanese, we look at two main issues:

1. The number of words that must be learned by rote in each language
2. The regularity of the spoken number system; that is, the degree to which the spoken number system corresponds to the written base 10 system

We can simply count the number of rote learning instances that are required to count to one hundred. The first 100 numbers of spoken Danish can be viewed as a system containing 28 basic signs, 1–9, 10–20, 30, 40, 50, 60, 70, 80, 90, 100, whereas the written base 10 contains 10 signs and a system. In that sense it takes 28 different words to count to 100 in Danish while in Japanese it only takes 11. In English, it is debatable whether there are 13, 20 or 28 different words to be learned by rote. If we accept that the three different forms often (ten, -teen and -ty) are the same, then there are only 13 words; if we accept only that -ty is equal to ten, then there are 20; and if we do not accept that -teen and -ty are the same as ten, then there are 28 words – as in Danish. In Danish, all the numbers from 13–19 end with a -ten, but the initial syllable does not sound like the numbers (from 3–9) that they refer to and we therefore count them as different words. The Danish decade number names from 20 to 90 are not directly connected to the numbers from 2–9 and must therefore be learned by rote without a system. Compared to English, the Danish number names are harder to systematize. In English, there is a system whereby all decades start with the number (e.g. twen-, thir-, four-, ...) and end with a -ty. In Japanese there is regularity in the oral counting system exactly as in base 10 and in addition, the names for numbers between 0–9 are very short.

Danish and English have the same regularity as base-10 starting with the number twenty (in Danish *tyve*): we start counting from 1–9 between each decade. In Japanese, this regularity starts already at 10, as in base-10. The English and Danish numbers between 10 and 20 follow another system. In both languages eleven (*elleve* in Danish) and twelve (*tolv* in Danish) do not follow the same system as the numbers between 13 and 19; in fact they are reminiscences of an old base-12 system. The numbers from 13 to 19 are combinations of an ordinal number with a -ten in Danish and a -teen in English. In both languages, the numbers between 13 and 19 are named in reverse order from the digits. This irregularity continues for the Danish numbers up to 100. For instance, the Danish name for 83 is “three and ‘fours’” or in Danish “*tre-og-firs*” where ‘tre’ is Danish for ‘three’, ‘og’ is Danish for ‘and’, and ‘fire’ is Danish for ‘four’ (the inflection ‘firs’ means eighty); in numbers: 3 + 80. In English, this irregularity in the sequence of the syllables constituting the number words only exists in the numbers from 13–19.

In summary, the Danish spoken number system is in reverse order from 10–100, the English spoken number system is reversed from 10–20 and regular from 20–100, while the Japanese system is regular from 10–100.

## COMPARATIVE PERSPECTIVES

Several studies (Miura et al., 1989; Miura et al., 1993; Miura et al., 1999) comparing English-speaking American, Japanese, Chinese and Korean first graders' (aged on average 6–7 years) cognitive representation of and understanding of place value confirmed that the Asian language speakers showed a preference for using base 10 representations to construct numbers, whereas English speakers showed a preference for using a collection of units. Note that a significant difference between American and Asian number names appears between 11 and 19; exactly when the base-10 system starts to use two digits. In the investigation, children were asked to construct the numbers 11, 13, 28, 30 and 42 from sets of wooden blocks (ten blocks and unit blocks). The results showed that 91 % of the American first graders used unit blocks to represent the numbers on their first try. In contrast, about 80 % of the Asian children used ten-blocks when representing the numbers on their first try. These differences in cognitive representation were mainly ascribed to language (Miura et al., 1993). Still, the validity of this conclusion is challenged by the many other cultural and educational differences between Asian and Western children. Yet, children who share similar cultures and belong to similar school systems but have different mother tongues have also been investigated. For example, Dowker et al. (2008) compare English and Welsh students. These students have the same cultural conditions, but Welsh names for numbers are as regular as the Japanese. They conclude:

No statistically significant differences were found between schools or age groups on the scaled score on either test (arithmetic and number skills). [...] However, there were group differences in a specific area of arithmetical ability, notably, in ability to read and judge numbers pairs.

Because their investigation can more or less eliminate the cultural and educational differences we can conclude that the results indeed show an effect of linguistic differences:

Welsh-speaking children find it easier than English-speaking children to read and compare two-digit numbers, suggesting that they are better at using the principles of place value.

This raises the question of why and how different languages influence number concepts and perhaps even the ability to learn simple arithmetic.

## SEMIOTIC PERSPECTIVES ON NUMBER NAMES

The empirical studies show that some languages seem to support the development of concepts of numbers better than other languages, and our initial comparison of Danish, English and Japanese shows that the differences maybe related to the degree to which the number names and the written numbers are 'alike'.

In order to view this case from a semiotic perspective we use the concept of iconicity (Stjernfelt, 2007), which describes the likeness of a sign to what it signifies, and the

epistemological triangle (Steinbring, 2006), a model describing the relation between mathematical signs and mathematical concepts.

### **Iconicity of the base-10 number system**

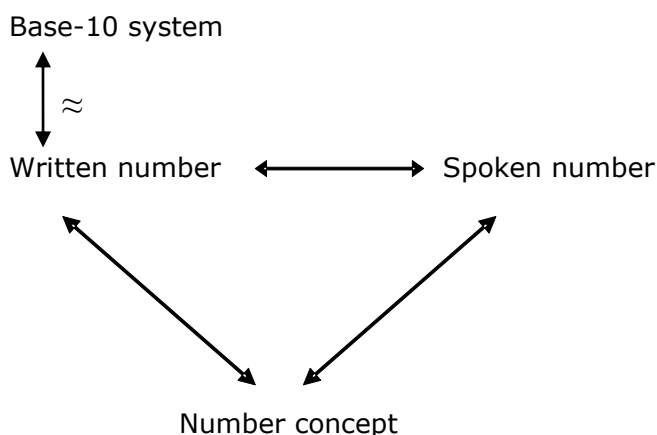
In the case of an iconic sign, the token relates to the object by similarity. Signs with strong iconicity typically “look like” their objects in some sense. For example, the roman number *iii*, which represents *three*, is typically considered iconic, because the three *i*’s correspond to the cardinality of the number three.

In order to describe how the base-10 numbers can be considered iconic we use Stjernfelt’s description of operational iconicity (Stjernfelt, 2007). The operational criterion for iconicity denotes the way a sign or a system of signs allows us to experiment and learn about what the sign signifies by extracting information from the sign that was not deliberately included by the producer of the sign. Base-10 has some qualities that make it reasonable to consider written numbers as partially iconic signs. In particular, it is possible to build any number in a position based system using only the number of different digits given by the base. It is one of the genius aspects of here the base-10, but this aspect is spoiled if the names don’t follow the numbers. A base-10 number tells us how many of each power of 10 it contains. The system allows us to create increasingly larger numbers from the ten basic numbers; furthermore, the system allows us to easily decide which one of two given natural numbers is largest. Written base ten numbers fully reflect the system. Using the operational criterion of iconicity we can say that written numbers are iconic in the sense that it is easy to determine which one of two given (natural) numbers is largest; it is easy to create larger and larger numbers; and the written base-10 numbers support a range of arithmetic algorithms (addition, subtraction, multiplications etc.). There is in principle nothing to prevent spoken number names from resembling this kind of iconicity. As simple examples of iconic spoken numbers, the additive “bum bum bum” can represent “three” in the same way as the roman number *iii*, and saying “four times ten and 3” to represent 43 is iconic in the same way as writing “43” in base-10. As we have seen, this is the situation in Japanese and for some numbers in English, but not at all in Danish.

### **Epistemology and the signification of numbers**

In order to understand how written numbers, spoken number names and concepts of numbers relate to each other we use the ‘epistemological triangle’, which connects conceptual entities to the signs that represent them and to mathematical objects in a reference context (Steinbring, 2006). Steinbring notes that there is in some cases exchangeability between the reference context and the sign/symbol, because the same sign can serve as a reference context for a mathematical concept (left side of the triangle) in some cases and as a representation of a mathematical concept (right side of the triangle) in other cases (Steinbring, 2006).

We can apply the epistemological triangle in order to understand how mother tongue influences number concepts, by viewing spoken words and written signs as reference contexts for each other. Taking point of departure in the epistemological triangle, we can represent the situation as shown below:



**Figure 1: The influence of written and spoken numbers on the number concept. The written numbers reflect the base-10 system completely.**

The epistemological triangle shows that the relations between the written number and the spoken number in terms of how the written number signifies the spoken number and how the spoken number signifies the written number, can influence the number concepts that individuals develop. Since these relations, as we have argued, differ from language to language, we should expect a difference in the number concepts arising from the different number names.

In English and especially in Danish, the signifying relations between spoken and written numbers are more complicated than in Japanese. The specific effect on the number concepts for English and Danish speaking children cannot be inferred from this analysis, but to hypothesize that this leads to more complicated concepts of numbers and possibly even to problems in learning numbers seems reasonable.

Using the concept of iconicity and the epistemological triangle for mathematical signs, we have seen that the written numbers resemble base-10, and using an operational criterion of iconicity, can be said to be iconic. We have also seen that different languages respect this iconicity in the spoken numbers to different degrees. What we have previously described as regularity between written base 10 numbers and spoken number names are cases in which the operational iconicity of the base 10 numbers are reflected in the spoken numbers as well.

Furthermore, we have applied the epistemological triangle to a situation in which written and spoken signs for numbers are considered as reference contexts for each other. We infer that these two representations of numbers affect the number concept

that pupils develop and hypothesize that this leads to more complicated number concepts for native speakers of languages in which the operational iconicity of the written numbers is less reflected in the spoken number names.

## **COGNITIVE PERSPECTIVES**

Learning to count and understand base-10 are cognitive challenges involving many small steps. We have chosen to focus on the following three aspects:

1. Oral counting
2. The cardinal principle of combining a name with a cardinal value
3. The combination of words for a number, its cardinal value and the digit sign

### **Oral counting**

Developing familiarity with the symbolic number system begins with oral counting. Children start oral counting quite early, and it is not clear if they understand what they are doing when they count. Counting appears to be learned first as a linguistic routine through which the number names are perceived as ‘sign systems’ or cultural semiotic systems that enable the symbolic representation of knowledge (Goswami, 2008).

We have seen that Japanese number names are brief and regular. English number names are somewhat longer and the regularity of the number system starts more or less at 20. In Danish, number names are generally even longer and the system has many irregular numbers. Studies of numbers and language (Sousa, 2008) show that a language with short number names loads the working memory (WM) less than a language with longer number names. There are no differences in the ability to count and numerate sets from 1 to 12 among Asian and American children, but from 13 to 100 Asian children are much better (Sousa, 2008). The development from being a novice to being an expert by gaining automaticity with numbers bigger than 12 seems to take longer if the language used has an irregular number system. Danish children have to learn many different and meaningless number words by rote, and we can now conclude that Danish number names load the children’s WM considerably when they are learning to count; in addition, there is clear evidence that syllables rated as more meaningful are easier to recall (Baddeley et al., 2009). A Danish investigation of children’s ability to count (N=140) at the age of 6 showed that more than 40 % of the children stopped counting at a number ending with 9 (Lyngsted & Knudsen, 2007). We suspect that this is due to the irregular Danish system for naming decade numbers.

### **The cardinal principle of combining a name with a cardinal value**

At 3–5 years of age, children understand more or less the five counting principles at least until the set of 10, even when they err in their counting. The five counting principles are (Gelman and Gallistel, 1978, here Siegler, 2003):

1. The *one-one principle*: Assign one and only one number word to each object.
2. The *stable order principle*: Always assign the numbers in the same order.
3. The *cardinal principle*: The last count indicates the number of objects in the set.
4. The *order irrelevance principle*: The order in which objects are counted is irrelevant.
5. The *abstraction principle*: The other principles apply to any set of objects.

Children typically learn the names of numbers as a long list of words and demonstrate knowledge of the stable order principle by almost always saying the number words in a constant order and saying the last number with emphasis (ibid). The names are developed as sounds connected to the number of objects in the sets.

The developmental shift to understanding the number name as a cardinal value requires a qualitative shift in children's representation of numbers. The cardinal principle requires an understanding of the logic behind counting (Goswami, 2008) and the ability to judge the size of a set. It relies on a representation of quantitative information in which the coding of smaller quantities is different from the coding of larger quantities (ibid). Children are born with the capacity to cognize magnitudes (Halberta et al., 2008) and distinct numerical difference in small number values, called subitization (Dehane 1997). This means that when comparing two different sets, children are for the most part capable of pointing out which set is biggest, depending on the size of the sets and the differences between them; but counting and telling the number in words seem to be harder, especially with bigger numbers.

Children's conceptual understanding of numeration depends on their being able to make a connection between a number name and its cardinal value, which they learn to do by grouping and quantifying sets of objects (Thomas et al., 2002). To group a set means to divide the set into smaller equal groups. Miura et al. (1993) showed how American, French and Swedish children used units instead of ten sticks representing two-digit numbers, and we know from our own experiences that Danish children do the same; it seems that certain languages facilitate grouping in tens whereas others do not.

### **The combination of words for a number, its cardinal value and the digit sign**

Learning how to connect the number word, its cardinal value and the digit sign is another challenge. As discussed, two different systems must be combined with different representations. Becoming an expert at combining these two systems means developing rapid access to an automatic use of written numbers and at the same time being able to multitask to solve other problems in parallel. If the two systems are iconic and support each other, the difficulties the child encounters in learning this skill will be minimized, as is the case for Japanese-speaking children. If the two systems are irregular and therefore conflict with each other, the child will have

greater problems understanding and remembering the connection between the name, the cardinal value and the sign. Duval (2006) describes this situation as a conversion between registers, and observes that the conversions that seem to be easiest for students are the ones that are congruent, meaning that the representation in the starting register is transparent to the target register.

From preliminary observations of Danish first graders, we have seen that if the task is to say the name of a written number, say 63, children often repeat the ten, twenty, thirty-rhyme and use their fingers. The children stop when they get to their sixth finger and then they know the word. This may be seen as a kind of interfering process, which in cognitive terms means that two parallel processes are in conflict with each other. The semantic treatment demands too much attention, and therefore it is not possible to multitask and complete both processes at the same time (Baddeley, 2009). This interfering effect means that learning to combine spoken and written numbers takes Danish children much longer time to automate. The logic in the base-10 disappears in the Danish language, and therefore the combination of the names and the written digits has more or less to be learned by rote.

## CONCLUSION

In this article we have theoretically investigated our hypothesis that Danish number names are more complicated than those of other languages, and therefore, that Danish children have more difficulties learning and working with numbers. We have shown through comparison of number names in different languages, combined with semiotic and cognitive arguments, how the differences in naming numbers may give rise to linguistically determined differences in how children learn number concepts as well as in the cognitive load of arithmetic processes.

We have argued that the Danish number names create comparatively great cognitive load in relation to number comparison, counting and basic arithmetic. Furthermore we have shown how number concepts are influenced by the names of numbers, and that number concepts are especially simple when the iconicity in the written base-10 numbers is also present in the words used to signify numbers. In Danish there are two main reasons that this iconicity is not present in the number names: (1) the order of digits is reversed in the numbers between 11-99, and (2) the names of the number decades do not correspond to the number of tens they represent.

We conclude from this first stage of our investigation that comparatively, Danish children find learning numbers difficult, due to the system of naming numbers in Danish. Our next step is to make empirical approaches where we intend to set up a comparative investigation of numeracy performance, comparing numbers written with digits and with words, a design research project about "School numbers" and finally investigate language reforms, with an example from the Norwegian numbers.

## REFERENCES

- Baddeley, A., Eysenck, M. W., Anderson, M. C. (2009). *Memory*. Hove and NY: Psychology Press.
- Dehaene, S. (1997). *The number sense. How the mind creates mathematics*. London: Penguin Press.
- Dowker, A., Bala, S., Lloyd, D. (2008). Linguistic influences on Mathematical Development: How Important Is the Transparency of the Counting System? In *Philosophical Psychology*, 21 (4), 523–538.
- Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in a Learning of Mathematics. *Educational Studies in Mathematics*, 61(1–2), 1–2.
- Goswami, U. (2008). *Cognitive Development. The Learning Brain*. Hove and NY: Psychology Press.
- Halberta, J., Mazzocco, M. & Feigenson, L. (2008). Individual differences in non-verbal number acuity correlate with maths achievements. *Nature* 07246, 455, 665–669.
- Lyngsted, H. og Knudtzen, G. (2007). *Matematiske begrebers udvikling hos børn i førskolestart*. Udviklingsprojekt. Ribe Seminarium.
- Miura I. T., Okamoto, Y. (1989). Comparisons of U.S. and Japanese first graders' cognitive representation of number and understanding of place value. *Journal of Educational Psychology*, 81, 109–113.
- Miura, I. T., Okamoto, Y., Vlahovic-Stetic, v., Kim, C. C., Han, J. H. (1999). Language Support for Children's Understanding of Numerical Fractions: Cross-National Comparisons. *Journal of Experimental Child Psychology*, 74, 356–365.
- Miura, I. T., Okamoto, Y., Chungsoon, C. K. Steere, M., Fayol, M. (1993). First Graders' Cognitive Representation of Number and Understanding of Place Value: Cross-National Comparison – France, Japan, Korea, Sweden, and the United States. *Journal of Educational psychology*, 85(1), 24–30.
- Siegler, R.S. (2003). Implications of Cognitive Science Research for Mathematics Education. In Kilpatrick et al, ed.: *A research Companion to Principles and standards for school mathematics*. NCTM.
- Steinbring, H. (2006). What makes a sign a mathematical sign? – an epistemological perspective on mathematical interaction. *Educational Studies of Mathematics*, 61(1), 133–162.
- Stjernfelt, F. (2007). *Diagrammatology*. Synthese Library, 336. New York: Springer.
- Thomas, N.D., Mulligan, J.T., Goldin, G.A. (2002). Children's representation and structural development of the counting sequence 1–100. *Journal of Mathematical Behavior*, 21, 117–133.
- Talord. (2009-2010). In *Den Store Danske Encyklopædi* (Open ed.) Gyldendal. Retrieved, 01.09.2010, from [http://www.denstoredanske.dk/Samfund\\_jura\\_og\\_politik/Sprog/Morfologi\\_og\\_syntaks/talord?highlight=talord](http://www.denstoredanske.dk/Samfund_jura_og_politik/Sprog/Morfologi_og_syntaks/talord?highlight=talord)



# STRATEGIES AND PROCEDURES: WHAT RELATIONSHIP WITH THE DEVELOPMENT OF NUMBER SENSE OF STUDENTS?

Elvira Ferreira<sup>1</sup> and Lurdes Serrazina<sup>2</sup>

<sup>1</sup>Higher School of Education of Torres Novas

<sup>2</sup>School of Education, Instituto Politécnico de Lisboa

*This paper analyses the procedures and strategies used by Daniel, a second grader student, when solving a set of problems, using addition and subtraction of positive whole numbers, under a classroom teaching experiment. This is a qualitative and interpretative case study, with data collection through participant observation, interviews and documents, namely, reports of classroom episodes, tasks and student's involvement in classroom activities. The results suggest that Daniel's preference for certain mathematical procedures and strategies depended on the context of the problems, namely the types of situations and the size and the structure of numbers involved.*

**Keywords:** procedures, strategies, number sense, addition and subtraction.

## INTRODUCTION

During the last decade, the goals and content of elementary mathematics education have changed internationally (Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007). The development of number sense is now an essential aspect of learning mathematics in the first school years, enabling students to solve problems involving addition and subtraction with positive whole numbers (McIntosh, Reys, & Reys, 1992). In the 21<sup>st</sup> century, “helping children develop number sense is being considered on a global scale as a key task in mathematics education” (Yang, Li, & Lin, 2008, p. 805).

This paper reports part of a study which main aim is to describe and analyse how students develop their number sense in a problem solving context using addition and subtraction of positive whole numbers, considering problems of real world addition and subtraction situations (Fuson, 1992). In particular, understanding the strategies and procedures they use in solving subtraction problems under a classroom teaching experiment. In this paper will be analysed the strategies and procedures used by Daniel, when compared with those described in the literature in the field.

## THEORETICAL FRAMEWORK

What is number sense? This is a question which the answer is not easy to obtain. Greeno (1991) states that “number sense is a term that requires theoretical analysis rather than a definition” (p. 170) and he suggests that “it may be more fruitful to view number sense as a by-product of other learning than as a goal of direct instruction” (p. 173). Dolk (2009) considers that developing number sense in the

class setting “implies giving students the opportunity to think with numbers and operations, guiding them in the way they look at numbers, and helping them to construct an active network of number relationships” (p. 5). Developing number relations also implies that students see numbers as mathematical objects.

McIntosh et al. (1992) define number sense as “a person’s understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and develop useful strategies for handling numbers and operations. It reflects an inclination to use numbers and quantitative methods as a mean of communicating, processing and interpreting information” (p. 3). They propose three strands to number sense: (i) knowledge of and facility with numbers, (ii) knowledge of and facility with operations and (iii) applying knowledge of and facility with numbers and operations to computational settings. This definition encompasses the behaviour defined by other authors as strategy use, and on the belief that promote strategy flexibility is important for all children, including younger and mathematically weaker children (Kilpatrick et al., 2001; Verschaffel et al., 2007; Verschaffel, Greer & Torbeyns, 2006).

Thus, strategies are seen as embedded within number sense. Strategies for solving particular types of problems are often presented as procedures that are followed in response to the stimulus problem. For Beishuizen (1997) strategy is the “choice out of options related to problem structure” and procedure is “the execution of computational steps related to the numbers in the problem” (p. 127).

The discrepancy between formal and informal computation procedures is currently seen as an impediment to the initial learning and understanding of mathematics (Blöte, Klein, & Beishuizen, 2000) as well as a hindrance in the development of number sense and the use of flexible number operations at the end of primary school (McIntosh et al., 1992; Treffers, 1991). The study developed by Yang (2003) demonstrates that students’ number sense can be effectively developed “through establishing a classroom environment that encourages communication, exploration, discussion, thinking and reasoning” (p. 132).

Many of the studies of children’s strategies and procedures consider mental computation methods very important in solving addition and subtraction problems (Beishuizen, 1993; 1997; Blöte et al., 2000; Buys, 2001; Klein et al., 1998; Torbeyns, Verschaffel, & Gesquière, 2006; Verschaffel et al., 2007). Such problems can be solved by three types of procedures: one type is the split method (1010); the second is the jump method (N10) and the third type is called *varying, compensation* or *short jump*. In the 1010 procedure numbers are decomposed in tens and ones which are processed separately and then put back together. The 10s (1010 stepwise) is a 1010 procedure that conceptually can be located between the 1010 and the N10 procedure. The N10 computation procedure (also the variant of N10C) starts with counting by tens up or down from the first, unsplit number. The A10 (adding-on)

procedure also starts from the first, unsplit number and goes from there to the next ten. The *varying*, *compensation* or *short jump* refers to bridging the difference in subtraction problems, like “71 - 69” in one or two steps instead of subtraction the second number from the first one (Blöte et al., 2000, p. 222) or  $86 - 25 = 85 - 25 + 1 = 60 + 1 = 61$  (Torbeyns et al., 2009, p. 80).

All these three procedures belong to “direct subtraction or indirect addition strategies” (Torbeyns et al., 2009, p. 80), when solving subtraction problems. Solving a subtraction problem by indirect addition means “that the problem is solved by adding on from the subtrahend” (Van den Heuvel-Panhuizen & Treffers, 2009, p. 108) and “direct subtraction means that the subtrahend is subtracted from the minuend (e.g.,  $71 - 29 = ?$ )” (Torbeyns et al., 2009, p. 80).

Apart from the nature of the numbers, Van den Heuvel-Panhuizen & Treffers, (2009) consider that there is more reason to calculate subtraction as an addition and so make use of the complement principle. It is important to consider subtraction as *taking away* (direct subtraction) and as determining the difference (indirect addition). The same authors say that “in the subtraction problems the context opened up the indirect addition strategy” (p.109)

The relationship between subtraction and addition is a big idea that children need to develop. Eventually, it is important that children know either strategy can be used. “Children need to understand the connection between addition and subtraction. Furthermore, they need to understand that comparison and removal contexts can both involve subtraction” (Fosnot & Dolk, 2001, p. 90). Traditionally teachers have often told learners that subtraction means “take away”. This is a superficial, trivialized notion of subtraction, if not erroneous (Fosnot & Dolk, 2001).

When students exhibit number sense, they apply efficient mental strategies and they are able to manipulate numbers mentally (Heirdsfield & Cooper, 2004). Some research (Blöte et al., 2000; Kamii & Dominick, 1998) has contended that mental computation promotes number sense if students are encouraged to formulate their own mental computation strategies.

## **METHODOLOGY**

The aim of the research is to understand how students develop their number sense in a problem solving context using addition and subtraction of positive whole numbers. In particular, to understand the strategies and procedures children use in solving addition and subtraction problems under a classroom teaching experiment. The aim of this classroom teaching experiment is to promote the development of the number sense in solving addition and subtraction problems involving different situations and different numbers. In this case, a sequence of instructional tasks was elaborated with the main goal of promoting children’s mathematics development

In the selection of tasks, we focused on four aspects: (i) context problems; (ii) selection of problems with “good” numbers to promote the use of different strategies and procedures to facilitate the discussion and permit students to ask questions. Numbers that allow multiple combinations, whose structures go further beyond the level of calculation by counting; (iii) sequence of the problems; (iv) adaptation/modification of this sequence after the analysis of each observation. In addition instructional tasks, we also considered the classroom culture and proactive role of the teacher. In the classroom culture we accounted the nature of classroom norms, social and socio-mathematical norms (Yackel & Cobb 1996)

The object of this study is a group of four children in elementary school integrated in a second grade classroom. The data collection was done during the school year 2007/2008, with observations of nine classes, where the students resolved two problems, one of addition and other of subtraction. Data collection included (i) participant observation, with reports from several lessons, one or two each month, during the school year, concentrating on the way children solve addition and subtraction problems with different structures (the sequence of instructional tasks mentioned above). All the lessons mentioned were videotaped; (ii) interviews, conducted by the first author, with the four children three months after finishing the classroom teaching experiment, which were audiotaped and transcribed; (iii) written documents, namely, reports of classroom episodes, tasks and students’ involvement in classroom activities.

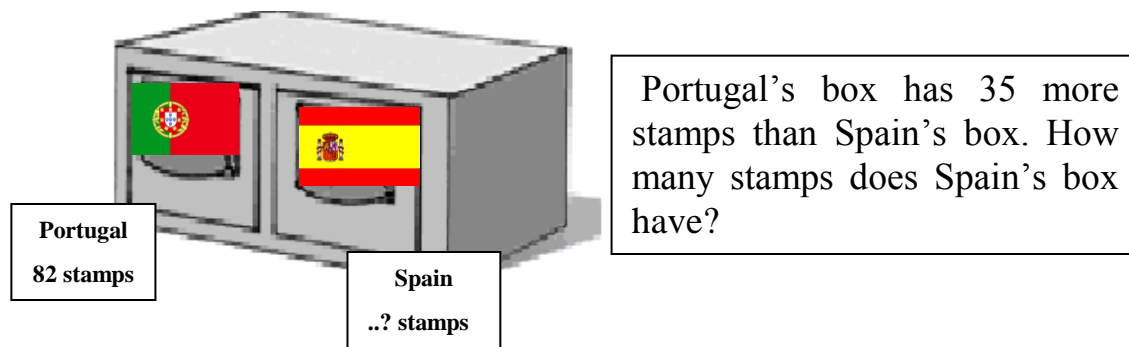
The principal source of data is participant observation with writing of researcher reports and collection of documents (Yin, 1989; Patton, 2002), by the first author of this paper, completed by the transcripts of video-taped classrooms. Each observation lasted at least two hours, and included observing children solving problems and listening to their explanations (Bogdan & Biklen, 1994; Erickson, 1986; Guba & Lincoln, 1994). According to the research plan, data analysis began simultaneously with data collection, in order to identify students’ strategies and procedures and how they were developing them. The videotaped of classroom lessons were an important source of data. The lessons were analysed according to the sequence of problems presented by the teacher, and for each one the strategies and procedures used and discussed by the students.

In this paper, we describe Daniel’s strategies and procedures for four subtraction problems during classroom teaching experiment.

## RESULTS

During classroom teaching experiment Daniel solved several real world subtraction problems (*take away, complete, compare difference unknown* and *compare referent unknown*). In this section we present the resolutions of one subtraction problem *compare difference unknown* and three subtraction problems *compare referent unknown*.

**Problem one** (*compare referent unknown*). Stamps problem (figure 1)



**Figure 1: Stamps problem**

This was the first problem *compare referent unknown* during classroom teaching experiment (3.<sup>rd</sup> observation). Daniel uses a direct subtraction strategy and he uses a mixed computation procedure, N10/A10. He starts from the first unsplit number and after he takes away 32 (figure 2)

**Figure 2: The way Daniel solved problem two**

First, he takes away 30 and he gets 52. Then  $52 - 2$  (from decomposition 5 into 2 + 3), and, finally, he takes away 3 and he gets 47, the answer of the problem.

**Problem two** (*compare difference unknown*). In the cinema, Room 1 has 215 seats. Room 2 has 98 seats. How many more seats does Room 1 have compared to Room 2? (4.<sup>th</sup> observation).

Daniel uses a direct subtraction strategy and he uses A10 computation procedure. He starts from the first unsplit number and then he takes away 98 (figure 3):

**Figure 3: The way**

**Daniel solved problem**

**one**

When he explains how he carried out the computations, he says:

**Daniel:** I did  $215 - 15$  (from 98 decomposed into  $15 + 80 + 3$ ) and I got 200. After, 200 minus 80 is 120. As I had taken away 95, there was still 3 missing so  $120 - 3$  is 117.

**Problem three** (*compare referent unknown*). 303 students from Eleanor's school went to the cinema. These are 45 more students than from John's school. How many students from John's school went to the cinema? (8.<sup>th</sup> observation)

Daniel uses an indirect addition strategy and he uses a mixed computation procedure, N10/A10. He starts from the first unsplit number and after he adds 245 and he gets 290. He says "I already knew in my head that  $45 + 45$  is 90" (a basic fact that he had already automated), and plus 200 is 290. After he adds ten and gets 300, a benchmark number (A10). Finally, he adds 3 and reaches 303 (figure 4).

Handwritten work for Problem three showing the calculation:  $158$ ,  $45 + 245 = 290$ ,  $290 + 10 = 300$ , and  $300 + 3 = 303$ . There are some corrections and additional notes in blue ink.

**Figure 4: The way Daniel solved problem three**

**Problem four** (*compare referent unknown*). For lunch, Peter ate a Big Mac which has 490 calories and Antonio ate a piece of salmon fish. The Big Mac has 295 more calories than the fish. How many calories does the fish that Antonio ate have? (9.<sup>th</sup> observation).

This is the last problem with this context during classroom teaching experiment. It is very interesting because first Daniel uses indirect addition strategy and a mixed computation procedure, N10/A10 and after he uses direct subtraction strategy and the same computation procedure (figure 5), that is, Daniel says that he can use both strategies for solving subtraction problem.

Handwritten work for Problem four showing the calculation:  $490 - 295 = 195$ ,  $195 + 295 = 490$ . There are some corrections and additional notes in blue ink.

**Figure 5: The way Daniel solved problem four**

## DISCUSSION

The results presented above show that Daniel is able to justify his computations using appropriate strategies and procedures. It seems that this is related to his mental computation ability, that is, to his flexibility with numbers and their manipulation. This seems to be related with the understanding of the meaning of numbers and operations, how he uses the reference numbers and how he recognises the reasonableness of the results. This understanding also enabled Daniel invent their own procedures. These findings are consistent with other studies (Heirsfield & Cooper, 2004; Blöte et al., 2000)

The findings of this study also indicate that Daniel used most frequently N10 e A10 procedures and this may be related to the way subtraction problems are solved. Another finding of this study indicates that there is a connection between the use of indirect addition and the context of the subtraction problems they have to solve. This seems to happen in problems of *complete*, *compare difference unknown* and *compare referent unknown*, as the context of these problems help Daniel, to understand the relationship between addition and subtraction and to use addition operation. The use of additive strategy also helped Daniel to use more efficient procedures, inventing his own procedures. Also there seems to be a relationship between the chosen strategy and effectiveness of procedures that Daniel was able to use.

Those findings also present empirical evidence, that the development of strategies and procedures on solving multi-digit addition and subtraction problems were influenced by social and socio-mathematical settings that Daniel were involved in a classroom environment that encourages communication, exploration, discussion, thinking and reasoning. These findings are consistent with other studies, namely, Kilpatrick et al. (2001), Verschaffel, et al. (2007) and Yang (2003).

## REFERENCES

- Beishuizen, M. (1993). Mental strategies and materials or models for addition and subtraction up to 100 in Dutch second grades. *Journal for Research in Mathematics Education*, 24(4), 294-323.
- Beishuizen, M. (1997). Development of mathematical strategies and procedures up to 100. In M. Beishuizen, K. P. E. Gravemeijer, & E. C. D. M. Van Lieshout (Eds.). *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 127-162). Utrecht: Utrecht University.
- Blöte, A. W., Klein, A. S., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction*, 10, 221-247.
- Bogdan, R., & Biklen, S. K. (1994). *Investigação qualitativa em educação: Uma introdução à teoria e aos métodos*. Porto: Porto Editora.

- Buys, K. (2001). Mental arithmetic. In M. van den Heuvel-Panhuizen (Ed.). *Children learn mathematics* (pp. 121-146). Utrecht: Freudenthal Institute.
- Dolk, M. (2009). Looking at numbers: Young children developing number sense. In *Actas XIX Encontro de Investigação em Educação Matemática* (pp.1-14). Vila Real: SPCE.
- Erickson, F. (1986) Qualitative methods in research on teaching. In M. C. Wittrock (Ed.). *Handbook of research on teaching* (pp. 119-161). New York, NY: MacMillan.
- Fosnot, C., & Dolk, M. (2001). *Young mathematicians at work: Constructing number sense, addition and subtraction*. Portsmouth, NH: Heinemann.
- Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. C. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 243-275). New York: Macmillan.
- Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In: J. van den Akker, K. Gravemeijer, S. McKenney, & N. Nieveen, (Eds). *Educational design research* (45-85). London: Routledge.
- Greeno, J. G. (1991). Number sense as a situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22(3), 170-218.
- Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. In Nornan Denzin & Yvonna Lincoln (Eds). *Handbook of qualitative research* (pp. 105-117). Newbury Park: Sage Publications.
- Heirdsfield, A. M., & Cooper, T. J. (2004). Factors affecting the process of proficient mental addition and subtraction: Case studies of flexible and inflexible computers. *Journal of Mathematical Behavior*, 23(4), 443-463.
- Kamii, C., & Dominick, A. (1998). The harmful effects of algorithms in grades 1-4. In L. J. Morrow & M. J. Kenney (Eds). *The teaching and learning of algorithms in school mathematics* (pp.130-140). Reston, VA: National Council of Teachers of Mathematics.
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up. Helping Children learn mathematics*. Washington, DC: National Academy Press.
- McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining basic number sense. *For the Learning of Mathematics*, 12(3), 2-8 e 44.
- Torbeyns, J., De Smedt, B., Stassens, N., Ghesquière, P., & Verschaffel, L. (2009). Solving subtraction problems by means of indirect addition. *Mathematical Thinking and Learning*, 11, 79-91.



- Treffers, A. (1991). Didactical background of a mathematics program for primary education. In L. Streefland (Ed.). *Realistic Mathematics Education in Primary School* (pp. 21-56). Utrecht: Freudenthal Institute.
- Van den Heuvel- Panhuizen, M., & Treffers, A. (2009). Mathe-didactical reflections on young children's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning*, 11, 102-112.
- Verschaffel, L., Greer, B., & Torbeyns, J. (2006). Numerical thinking. In A. Gutiérrez & P. Boero (Eds.). *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp.51-82). Rotterdam, The Netherlands: Sense Publishers.
- Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.). *Second handbook of research on mathematics teaching and learning* (pp. 557-628). Reston, VA: NCTM.
- Yang, D. C. (2003). Teaching and learning number sense - an intervention study of fifth grade students in Taiwan. *International Journal of Science and Mathematics Education*, 1(1), 115-134.
- Yang, D. C., Li, M. N., & Lin, C. (2008). A study of the performance of 5th graders in number sense and its relationship to achievement in mathematics. *International Journal of Science and Mathematics Education*, 6(4), 789-807.

# **PRESCHOOL CHILDREN'S UNDERSTANDING OF EQUALITY: OPTING FOR A NARROW OR A BROAD INTERPRETATION?**

Anatoli Kouropatov

Dina Tirosh

Tel Aviv University

*Mathematical concepts and symbols are often introduced at first in a narrow meaning. The harmful consequences of such initial presentation are widely reported in mathematics education. The present paper asks the following question: Is it essential to introduce the equal sign in a narrow meaning as an operation and not in its broader meaning as a relation? We found that preschool children are capable of addressing changes simultaneously made to two quantities (adding or subtracting the same or even a different number of elements to/from two groups of objects), a capability needed when handling relations. These findings call for looking more closely at the possibility of introducing the equal sign in its broader meaning, as a relation ("the same").*

Keywords: preschool, quantities, models, equality, equivalence

## **EQUALITY AND SCHOOL MATHEMATICS**

The meaning of a sign in mathematics is often determined by its context. This phenomenon is evident also in the mathematics studied in elementary and middle schools. For example, the minus sign has at least three uses: a negative number, a binary operator that indicates an operation of subtraction, and a unary operation, acting as a demand to replace the operand by its opposite.

The equal sign continues to attract considerable attention in elementary and middle school education (e.g., McNeil & Alibali, 2005; Vassiliki & Philippou, 2007; Mark-Zigdon & Tirosh, 2008; Molina & Ambrose, 2008; Molina, Castro & Castro, 2009). Consider the following expressions:

(a)  $2 + 3 = 5$

(b)  $2 + 3 = 4 + 1$

Various studies have reported that elementary school students tend to assign only the meaning of the expressions of type (a) to the equal sign, namely, "How many does it make together?" It has also been reported that children experience difficulty with expressions of type (b) (MacGregor & Stacey, 1999; Baroody & Benson, 2001; Keilpatrick, Swafford & Findell, 2001), which require calculating each side and verifying that the two sides are equal.

The equation of type (a) can be interpreted as a demand to perform an operation, whereas in type (b) the sign is interpreted as a relation between the two parts. The mathematical term "relation" is broader than the term "operation." Because operation is a special case of relation, viewing equality as a relation covers both situations. Nevertheless, instruction often starts with assigning a relatively narrow meaning to a

term, and the broader interpretations are introduced only later. For example, the term “number” is often introduced with reference to non-negative whole numbers, and only later to broader sets of numbers, such as rational, irrational, and real numbers.

Is it necessary to narrow the interpretation of the equal sign to that of an operation, or is it preferable to introduce the symbol in its broader meaning of a relation? This issue is of both theoretical and practical importance because the common pedagogic approach today is to initially introduce the equal sign only in the context of performing an arithmetic operation (Herscovics & Kieran, 1980; Saenz-Ludlow & Walgamuth, 1998; MacGregor & Stacey, 1999; Baroody & Benson, 2001; Keilpatrick, Swafford & Findell, 2001). Moreover, researchers have noted that the initial, narrow meaning assigned to the equal sign may have a long-lasting effect, and that children aged 5-12 tend to perceive the equal sign only as a request to calculate and not as a sign of relation (Herscovics & Kieran, 1980; Behr, Erlwanger & Nichols, 1980; Molina, Castro & Castro, 2009). The narrow interpretation of the equal sign dominates and often blocks other interpretations and references that are more general and are essential in various branches of mathematics (e.g., equations in algebra, identities in geometry and in trigonometry). There is some evidence that specific interventions can promote the interpretation of equality as a concept that expresses a relation among children aged 5-7 (McNeil & Alibali, 2005).

Two research questions arise naturally from the above description:

- 11 What is the understanding that children 3-5 years old have about equality before instruction? Specifically, do young children understand which manipulations can and cannot be performed on two quantities to maintain the equality relation?
- 12 What types of responses are evident when young children are presented with tasks that address the basic properties of an equivalence relation?

## METHODOLOGY

### Sample

Seventeen children from an upper-middle class neighbourhood in the central region of Israel participated in the study. The children were divided into two groups. Group 1 ("Older Children") included nine children aged 4-5, with an average age of 4 years and 8 months. Group 2 ("Younger Children") included eight children aged 3-4, with an average age of 3 years and 10 months. Ten boys and seven girls participated in the study.

### Research tools

#### Interviews

We used task-based interviews (Goldin, 2000) as the primary research tool because they are well suited for the age of the participants. The interviews involved the

interviewee, the interviewer, and the tasks. Interviews were pre-planned, but there was room for *ad hoc* changes initiated by the interviewer based on the interviewee's responses. This procedure allowed us to focus both on the correctness of the responses and on the thought processes involved (Steffe, Nesher, Cobb, Goldin & Greer, 1996). Each interview, conducted separately with each child by the authors of the paper, lasted about 50 minutes, in two sessions of approximately 25 minutes each. The duration matches the concentration span of most children of this age, although some children required additional sessions. The interviews took place in the children's preschool, in a separate room, in a calm and pleasant environment. All interviews were audio recorded and transcribed.

### Tasks

Two sets of tasks were developed, each addressing one of the two research questions: manipulation of two quantities (research question 1) and properties of equality as an equivalence relation (research question 2). The majority of the tasks were developed based on the ideas of Morris (2003), with some necessary adaptations to the young age of the participants. Note that no written symbols were used in the design of the tasks and in the communication with the participants.

#### 1. Manipulation of Two Quantities

We focused on two types of manipulations: reciprocal (manipulations that must be applied to both sides of the equation simultaneously), and non-reciprocal (manipulations that can be applied to one side of the equation).

##### *Reciprocal manipulations*

1a. *Equal Addition*. The researcher places two boxes on the table. In each box he inserts 5 marbles in a way that the child can see the marbles in the boxes. The child is asked to compare the quantities in the boxes. The researcher says, "Let's check what we have in our boxes. Do we have the same number of marbles, or does one of us have more?" After the child and the researcher agree that the quantities in the boxes are equal, the researcher takes the same number of marbles (3) into each hand from a pile of marbles. He shows the marbles to the child and inserts the marbles from one hand into one box and from the other hand into the other box. The researcher says, "I put these marbles in your box and these marbles in my box. What do you think? Do we have the same number of marbles in our boxes or does one of us have more?"

1b. *Different Addition*. The researcher places two boxes on the table. In each box he inserts 5 marbles in a way that the child can see the marbles in the boxes. The child is asked to compare the quantities in the boxes. The researcher says, "Let's check what we have in our boxes. Do we have the same number of marbles, or does one of us have more?" After the child and the researcher agree that the quantities in the boxes are equal, the researcher takes a different number of marbles from the pile of

marbles in each hand (3 marbles in one hand and 2 marbles in the other). He shows the marbles to the child and inserts 3 marbles in the child's box and 2 marbles in his box. The researcher asks, "I put these marbles in your box and these marbles in my box. What do you think? Do we have the same number of marbles in our boxes or does one of us have more?"

1c. *Reverse of Different Addition*. Following the Different addition (1b) task, the researcher removes the amount of marbles that were added to each box, leaving equal quantities in each box. The child is asked to compare the quantities in the boxes. The researcher says: "Do we have the same number of marbles in our boxes, or does one of us have more?"

1d. *Equal Subtraction*. This task is identical to Equal addition (1a), but instead of adding marbles the researcher removes the same amount of marbles from the two boxes.

1e. *Different Subtraction*. This task is identical to Different addition (1b), but instead of adding a different number of marbles to the boxes the researcher removes a different number of marbles from the boxes (3 from the child's box, 2 from his box).

1f. *Reverse of Different Subtraction*. Following the Different subtraction (1e) task, the researcher returns the marbles that were removed and asks, "What is in the boxes now? Do we have the same number of marbles in our boxes, or does one of us have more?"

### *Non-reciprocal manipulations*

1g. *Violation of Equality*. The researcher gives an equal number of marbles to the child and to himself and validates with the child that the amounts are equal. Additional marbles remain in a pile on the table. The researcher asks, "Can we do something so that we won't have the same number of marbles?"

1h. *Creation of Equality*. The researcher distributes a different number of marbles to the child and to himself (5 marbles to child, 3 to himself). Additional marbles remain in a pile on the table. The researcher asks, "What do you think, do we have the same number of marbles or does one of us have more? Is it possible to do something so that we will have the same the number of marbles? "

## 2. Properties of Equality as an Equivalence Relation

The equality relation is an equivalence relation. As such, it is reflexive, symmetrical, and transitive. Designing tasks to diagnose young children's comprehension of these properties (especially the reflexive property) is challenging. We developed the following three tasks to attempt to address these properties:

2a. *Reflexivity*. The researcher points to the marbles in his box and says, "I don't want these marbles in my box, but I don't want to have less or to have more. Please help me." The child is expected to substitute one or more marbles with the same

amount of marbles, taken from the pile on the table. This task examines the substitution aspect of reflexivity. It appears that this aspect can be formulated in a meaningful way to young children. The task examines whether the child can ignore such factors as colour, shape, etc., and at the same time consider only the numerosity aspect and keep the same number of objects for both parties.

2b. *Symmetry*. The researcher places two boxes on the table and inserts 5 marbles in each box. He validates with the child that the boxes contain the same number of marbles and closes the boxes. The researcher then swaps the places of the boxes, without opening them. The researcher asks, "I showed this to two children: Guy and Shay. Shay said that the boxes still have the same number of marbles. Guy said that there are more marbles in one of the boxes. Who is right?"

2c. *Transitivity*. The researcher places two boxes on the table and inserts 4 marbles in each box. He validates with the child that the boxes contain the same amount of marbles, closes the boxes, takes one box, and gives the other box to the child. He then places a third box on the table and says, "Your box and my box contain the same number of marbles. Let's prepare a box for Sigal (a preschool teacher) with the same number of marbles that is in your box. What do you think, will Sigal and I have the same number of marbles or will one of us have more?"

The order of the presentation of the tasks was carefully planned to avoid creating a fixed pattern of responses that would accidentally lead to correct answers. For example, after several tasks for which the correct answers were "equal amounts," the child was presented with a task or a group of tasks for which the correct answer was "different amounts," and *vice versa*.

## MAJOR FINDINGS

### 1. Manipulation of Two Quantities

#### *Reciprocal Manipulations of Two Quantities*

In two of the six tasks grouped under this category (Equal Addition (1a) and Equal Subtraction (1d)) the same changes (addition/subtraction) were carried out simultaneously on two equal quantities. Eight Older Children and 5 Younger Children solved the addition task correctly, and 9 Older Children and 7 Younger Children solved the subtraction task correctly. The explanations of all of the older children were based on the idea that "when you add the same thing to the same thing you get the same thing."

At first, 3 of the 5 Younger Children who provided correct responses to the Equal Addition (1a) task were mistaken in their judgement, but after reducing the initial amounts of marbles in both groups from 5 to 3, two more Younger Children provided correct answers and used the concept "when you add the same thing to the same thing you get the same thing" in their explanations. It is likely that the reduction of

quantities enabled the children to recognize the quantities. This behaviour may be related to subitizing (Dehaene, 1997).

In the Different Addition (1b) and Different Subtraction (1e) tasks, different numbers of marbles are added or subtracted simultaneously to/from equal quantities. Eight Older Children solved the Different Addition (1b) and 7 the Different Subtraction (1e) task correctly. The Younger Children, however, encountered difficulties especially with the Different Subtraction (1e) task: 6 solved the Different Addition (1b) and 2 the Different subtraction (1e) task correctly. Those who solved these tasks correctly used the concept: "If we add more to one side, this side gets more." Analysis of the interviews shows that 3 Younger Children mistakenly argued that "you took away more from me so now I have more." It is possible that for these children addition is a dominant prototype of an operation, and therefore the term "more," whether adding or taking away more, is conceived as resulting in a higher number.

In the four tasks discussed above, single changes were made on each of the quantities. The success rate of the children on these tasks was relatively high. But this was not the case for the Reverse of Different Addition (1c) and for the Reverse of Different Subtraction (1f) tasks. Only 8 children (4 in each group) succeeded in the Reverse of Different Addition (1c) task, and 6 children (5 Older Children and 1 Younger Child) succeeded in the Reverse of Different Subtraction (1f) task. The most common, incorrect response was: "If we add more to one side, this side gets more," referring only to the latter change. A possible explanation of the difficulties that children faced with these tasks is related to the sequence of actions being carried out. The children were asked to follow two-stage manipulations on both sides and/or to understand that the process of adding/subtracting the same amount to a set does not change its numerosity ( $A=B, C<D \Rightarrow A \cup C < B \cup D \Rightarrow (A \cup C) \setminus C = (B \cup D) \setminus D$ ). It is likely that such tasks require a relatively advanced computational capacity, an ability to perceive quantities as partial sets and as wholes, and a capability to memorize the different steps that were carried out. Our findings suggest that children at this age have not yet developed these abilities, and consequently, in most cases they consider only one of the changes (the last one).

#### *Non-reciprocal Manipulations of Two Quantities*

Violation and Creation of Equality (1g/1h) are two facets of the same task. Nine of the older children and 7 of the younger ones succeeded in these tasks. The most common method used by the children to solve this task was to add marbles from the pile either to themselves or to the researcher. All but three children used the pile of marbles to create or violate the equality. Three children from the older group used another method: they transferred marbles from the researcher's box to their own or *vice versa*. Except for these three older children, the participants insisted on using the pile of marbles to create or violate the equality, and objected to attempts to form

or violate the equality without extra marbles. This resistance may be related to young children's tendency to conceive the quantities in each box as a “whole” from which parts are not to be taken away (e.g., Piaget & Szemirska, 1952).

## 2. Properties of Equality as an Equivalence Relation

Two of the three tasks included in this category were Symmetry and Transitivity. The Symmetry task was correctly solved by 8 Older Children and 6 Younger Children, and the Transitivity task by 8 Older Children and 5 Younger Children. The participants who succeeded in solving the Transitivity task used a strategy that may be represented schematically as follows:  $A = 4$ ,  $B = 4$ ,  $C = 4$  and therefore  $C = A$  (“The same number in all the boxes”), and not the classic transitive strategy of  $A = B$ ,  $B = C \Rightarrow A = C$ .

To examine reflexivity we chose a “substitution” activity (the number of elements does not change if one or more elements are substituted by the same number of elements). Such a substitution may be considered reflexive. The distribution of responses to this task (3 Older Children and none of the younger ones solved the task correctly) and the nature of the accompanying explanations suggest that the majority of the children failed to understand the task. Common responses were: “If you don’t want these marbles, you can add others,” “Take more,” and “You can throw them away.” The children’s difficulty may have been the result of the phrasing of the task, but it seems that an inherent difficulty with reflexivity is the need to relate to equality within only one set. In their concrete world, the children find it challenging to relate to equality without the presence of at least two objects (how can one perceive that something is equal to itself if there is only one “thing” there).

This complexity manifests later in the difficulties that students often experience when asked to relate to equalities such as  $5=5$  and  $x=x$  (e.g., Herscovics & Kieran, 1980; Saenz-Ludlow & Walgamuth, 1998; Morris, 2003). It appears, however, that in the case of equality, the construction of meaning of symmetry, transitivity, and even reflexivity do not seem to indicate an inherent, epistemological obstacle.

## DISCUSSION

The present study addressed two mathematical concepts that are unequivocally defined: equality and operation. Equality is defined as a relation (equivalence) and as an operation (in the context of this paper, a binary relation applied to a pair of natural numbers). Formally, the concept “operation” is a special case of the concept “relation.”

The formal terms can be translated into common language familiar to children using two well-known models. In the context of equality, the term “relation” may be interpreted as: “What can be said about two quantities (more than, less than, equal to, the same)?” (Model 1). The term “operation” may be introduced by asking: “How many are there together?” (Model 2). The study focused primarily on examining



young children's knowledge of Model 1. Two major considerations for focusing on Model 1 were that (a) Model 1 is more general (Model 2 is a special case of Model 1), and (b) in reports of difficulties that students encounter when facing situations in which Model 1 must be applied, the barriers are often attributed to the primacy effect of introducing Model 2 first and using it excessively in elementary schools, while almost neglecting the use of Model 1 (e.g., Fischbein, Deri, Nello & Marino, 1985).

In the present study we attempted to examine preschool children's capability to address various manipulations of two quantities ("two sides of the equal sign") and the properties of equivalence relation. Knowledge of young children's conceptions can assist educators in decisions about introducing Model 1 at an earlier age, possibly as the model to be presented first to young children. As noted above, many mathematical symbols and concepts carry several meanings, both narrow and broad. Often, a narrow interpretation of a concept is introduced first, assuming that it is easier to grasp (for example, multiplication is introduced only as "number of times"). Such narrow presentation often results in conceptual blockages that are widely reported in the research literature (e.g., Fischbein, 1993). A related question is: What information is needed to determine whether a concept should be presented in a narrow or in a broad meaning? In the present context, the question is: Are preschool children ready for the introduction of the concept of equality (and the equal sign) as an equivalence relation?

The findings of the current study suggest that preschool children are able to correctly solve tasks involving reciprocal and non-reciprocal manipulations of two quantities, and some tasks that draw upon the properties of equality as an equivalence relation. It seems that young children are capable of addressing some changes that are made simultaneously on two quantities (adding or subtracting the same or even a different number of elements to two groups of objects). This capability is necessary for handling relations. Our findings call for a closer examination of the effectiveness of introducing the concept of equality and the equal sign in its broader form, as a relation ("the same"). The findings of our study suggest that in the case of equality, many of the substantial properties of this concept are acquired at a very early age without any intentional intervention.

To summarize, the present study suggests that young children are capable of performing tasks involving comparison of quantities. The findings support the implementation of instructional sequences that attempt to present equality and the equal sign, from the earliest stages of instruction, in its broadest sense, as a relation and not only as an operator. In other words, the study calls for a re-examination of the common practice of introducing equality and the equal sign only as an operation ("How many are there together") and not as a relation ("What can be said about two quantities"). More research, on a larger sample of children, is needed to validate these findings. Additionally, some of the tasks that are included in the interview protocol should be modified (especially those that attempted to address reflexivity),

to critically examine the findings reported in this study and the feasibility of the approach that we propose.

## REFERENCES

- Baroody, A. J., & Benson, A. (2001). Early number instruction. *Teaching Children Mathematics*, 8(3), 154-158.
- Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, 92(1), 13-15.
- Dehaene, S. (1997). *The number sense: How the mind creates Mathematics*. New York: Oxford University Press.
- Fischbein, E., Deri, M., Nello, M.S., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16(1), 3-17.
- Fischbein, E. (1993). The interactions between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. W. Scholz, R. Straser & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231-245). Dordrecht, Netherlands: Kluwer.
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematical education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517-545). Mahwah, New Jersey and London: Lawrence Erlbaum Associates.
- Herscovics, N., & Kieran, C. (1980). Constructing meaning for the concept of equation. *Mathematics Teacher*, 80(1), 572-580.
- Keilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding It Up: Helping Children Learn Mathematics*, (Vol. 2(8), pp. 7-9). Center for Education. National Research Council. National Academy Press: Washington, DC.
- MacGregor, M., & Stacey, K. (1999). A flying start to algebra. *Teaching Children Mathematics*, 6(2), 78-85.
- Mark-Zigdon, N., & Tirosh, D. (2008). Kindergarten and first graders' knowledge of the number symbols recognition and production. *Focus on Learning Problems in Mathematics*, 30(1), 1-11.
- McNeil, N., Alibali, M. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development*, 4(76), 883-899.
- Molina, M., & Ambrose, R. (2008). From an operational to a relational conception of the equal sign. Thirds graders' developing algebraic thinking. *Focus on Learning Problems in Mathematics*, 30(1), 61-80.

## Working Group 2

- Molina, M., Castro, E., & Castro, E. (2009). Elementary Students' Understanding of the Equal Sign in Number Sentences. *Electronic Journal of Research in Educational Psychology*, 17(1), 341-368.
- Morris, K. A. (2003). The development of children's understanding of equality and inequality relationships in numerical symbolic context. *Focus on Learning Problems in Mathematics*, 25(2), 8-51.
- Piaget, J., & Szeminska, A. (1952). *The Child's conception of number*. London: Routledge & Kegan Paul.
- Saenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics*, 35, 153-187.
- Steffe, L. P., Nesher, P., Cobb, P., Goldin, G. A., & Greer, B. (1996). *Theories of mathematical learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Vassiliki, A-L., & Philippou, G. (2007). Elementary school student's understanding and use of equal sign. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the CERME 5*, (pp. 825-834). Larnaca, Cyprus.

# USING THE DOUBLE NUMBER LINE TO MODEL MULTIPLICATION

Dietmar Küchemann, Jeremy Hodgen and Margaret Brown

King's College London, UK

*As part of the research of the ICCAMS project, we have been exploring Year 8 students' (age 12–13 years) understanding of multiplicative structures and developing materials to enhance this understanding. In this paper we discuss some of our work related to the double number line and its use as a model for multiplication.*

Keywords: models, multiplication, double number line

## INTRODUCTION

Vergnaud (2009) argues that a concept's meaning arises from a variety of situations and puts forward the notion of a conceptual field, as a way of binding situations and concepts together. In the case of multiplicative structures, this conceptual field is extremely complex, as can be seen, for example, from the myriad of models put forward by the Rational Number Project (eg Behr et al, 1991), or from the intricate 'learning trajectories maps' woven by Confrey et al (2009). Anghileri and Johnson (1992) identify 6 key aspects of multiplication (and division), which they list as: equal grouping, allocation/rate, number line, array, scale factor, Cartesian product. They argue that "children will need to become familiar with the different situations that embody these aspects" (ibid, p170).

Davis (2010) describes how 'concept study' can be used to draw out and develop teachers' mathematical knowledge for teaching. He discusses how a group of teachers worked on the concept of multiplication. One of their activities was to produce a list of 'realisations of multiplication', which included these items:

grouping process; repeated addition; times-ing; expanding; scaling; repeated measures; making area; making arrays; proportional increase; splitting; skip counting; transformations; stretching/compressing a number line.

Of course, this list of realisations is in no sense definite and Davis makes the important point that the teachers' conceptions of multiplication (and how it might be engaged with in the classroom) was continually shifting over time. We would argue that the same will apply to school students, though depending on the extent to which they are allowed to engage with these ideas. In this regard, it is interesting to note the guidance on multiplication offered by the National Numeracy Strategy, which was set up by the UK government to offer advice on the teaching of mathematics. In the Strategy's Framework document for primary schools (DfEE, 1999), it is suggested that students as early as Year 2 (6–7 year olds) should understand multiplication in

terms of repeated addition and rectangular arrays. Similar advice is offered for Year 3 students, and it is suggested that the idea of scaling should also be introduced. Some advice on understanding multiplication is also given for older students, but this is subordinated to advice on performing calculations. Thus the use of models such as the array is not mentioned again in this document, nor in the Strategy's subsequent Framework document (DfEE, 2001) for early secondary school (Years 7, 8 and 9). One gets the impression that by the time students reach secondary school, multiplication is somehow meant to be 'understood' and no longer needs to be supported by models. So for example, it is stated that Year 7 students should understand that "multiplication is equivalent to and is more efficient than repeated addition" (ibid, Section 4, p82), but there is little indication of what 'multiplication' means here, ie in what situations it might be modeled or realised. Unfortunately, the paucity of models in the Framework documents is mirrored in most current secondary school mathematics textbooks in the UK.

Extensive work on the didactical use of models has been undertaken in Holland, from the perspective of RME (see eg Van den Heuvel-Panhuizen, 2002). This work makes an interesting distinction between 'models of' and 'models for', whereby the development of a more formal mathematical understanding is seen as a shift from the construction and use of the former to the latter.

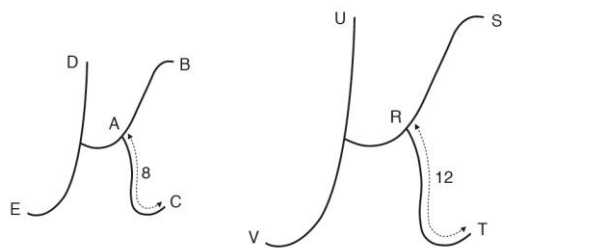
## THE DOUBLE NUMBER LINE

As part of the work of the ESRC-funded ICCAMS project<sup>1</sup> we have been exploring Year 8 students' (12–13 year olds) understanding of multiplicative structures, with the aim of developing teaching materials to enhance this understanding. From our work so far, including group interviews and short teaching sequences, but also the large-scale use of written tests, it is clear that many secondary school students have a very shaky understanding of multiplication, based in only a very limited way on models that could support and develop their understanding.

We think that one interesting and important *model for* multiplication is provided by the double number line, and it is this model that we focus on in this paper. The double number line also serves as a *model of* a variety of contexts that should be reasonably accessible to students (eg scales on a map; a one-way stretch; conversions, such as £ to €). An attraction of this model is that it offers a fairly gentle way of departing from an additive approach to multiplication. Repeated addition (perhaps modelled by skips along a number line) provides a salient and reliable (and quite efficient) model for the multiplication of (small) whole numbers (eg  $3 \times 7$  can be thought of as 3 skips, each of 7 units, along the number line, starting at 0). However, even in situations that involve multiplication by simple rational numbers, eg  $\times 1.5$ , we have found that many students stick to an additive strategy (in this case, 'rated addition').

Consider the item in Fig 1 (below), taken from the CSMS Ratio test (Hart, 1981). This was answered correctly by 14 % of a representative sample (N=309) of Year 8 students in 1976, and by a similar proportion (12%) of a representative sample (N=754) of Year 8 students in 2008. When we have interviewed students on this item, we have found that many students use an inappropriate ‘addition strategy’ (Karplus & Peterson, 1970) arguing along the lines of ‘ $8+4 = 12$ , so  $RS = 9+4 = 13$ ’. Those who do solve the task successfully tend to use an argument of this sort: ‘ $8 + \text{half of } 8 = 12$ , so  $RS = 9 + \text{half of } 9 = 9 + 4.5 = 13.5$ ’. Students tend not to go for the more direct approach of scaling by  $\times 1.5$ , along the lines of ‘ $12 = 8 \times 1.5$ , so  $RS = 9 \times 1.5 = 13.5$ ’. It is worth pointing out that this scaling approach is actually more *appropriate* here, since rated addition does not fit the geometric situation: one can’t really add a curved line of length 8 units to a smaller version of the line of length 4 units, and thereby make a larger version of the line of length 12 units! It is perhaps for this reason that the task is so difficult.

These 2 letters are the same shape. One is larger than the other.  
AC is 8 units. RT is 12 units.

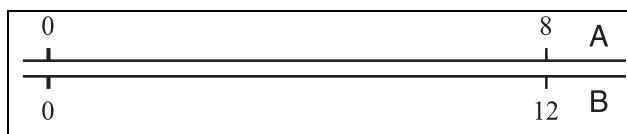


The curve AB is 9 units. How long is the curve RS ? .....

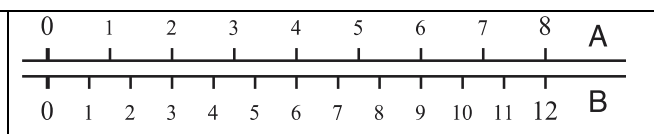
**Fig 1: Curly Ks item (Hart, 1981)**

Thus, in a study using items derived from Hart (1981) but with the numbers more closely matched, Küchemann (1989) found that while a Ks item was answered correctly by only 25% of students (N=153), a recipe item, where rated addition makes perfect sense, was answered correctly by 64% of students from a comparable sample (N=154).

The double number line provides a neat way of representing (or indeed embodying) multiplicative relations, such as  $\times 1.5$ . Consider the pair of lines A and B in Fig 2a, where 0 and 8 on line A are lined-up with 0 and 12 on line B.



**Fig 2a**

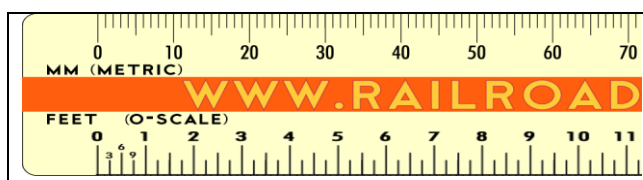


**Fig 2b**

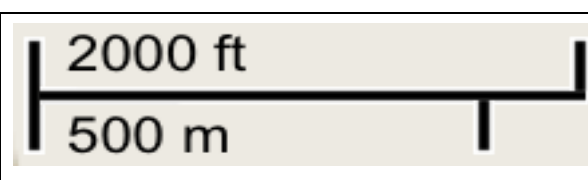
We know that  $12 = 8 \times 1.5$ , and we can thus make the number lines represent the mapping  $\times 1.5$  (or  $x \rightarrow 1.5x$ , or  $y = 1.5x$ , etc.) by drawing linear scales on each line (Fig 2b): any number on line A is now mapped onto a number positioned directly below it on line B that is 1.5 times its value. The great strength of this representation is that it is not just mapping 8 onto 12, but showing the mapping  $\times 1.5$  regardless of any particular pair of numbers that we may be considering. Thus the operation  $\times 1.5$  is brought to the fore. Of course, this may not be perceived in this way by all students. The diagram can, for example, be read as showing a move along a number line from 0 to 12, in 8 skips of 1.5 units (with line A representing the number of

skips and line B the distance skipped); in other words the diagram can also be interpreted as showing repeated addition.

Scales on a map are well known examples of double number lines. One kind shows distances on the map (measured in cm, say) and corresponding distances on the object depicted by the map (measured in km, say). The ruler in Fig 3a is of this sort. Another kind shows how distances represented on a map can be read in different units (eg feet and metres). Fig 3b shows a scale of this sort used by Google Maps.

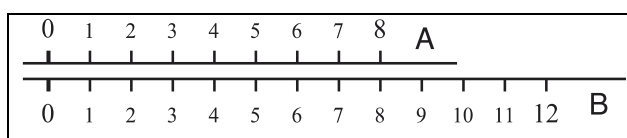


**Fig 3a: A ruler for scaling feet to mm**

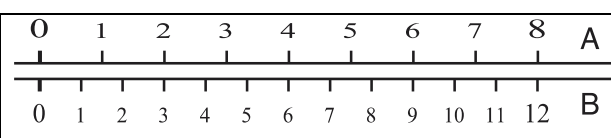


**Fig 3b: A scale used by Google Maps**

A double number line can also be used to represent an enlargement, or more specifically, a one-way-stretch. An example is shown in Fig 4b, which shows the result of stretching scale A in Fig 4a by a factor  $\times 1.5$ . (Note the similarity to Fig 2b.)



**Fig 4a: Two identical scales, A and B**



**Fig 4b: A  $\times 1.5$  stretch applied to scale A**

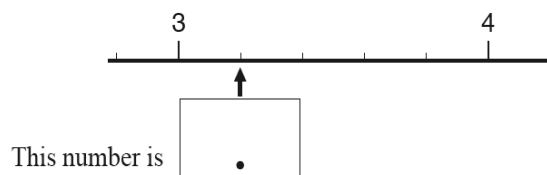
## RELATED WORK

Our approach to the didactical use of models is perhaps similar to the Dutch approach, as embodied by RME, though we would not be as strict about not handing ready-made models to students. Van den Heuvel-Panhuizen (2002), in describing RME, gives a nice illustration (ibid, Fig 11, p12) of how the number line can be used to support students' learning. She adds that

The number line begins in first grade as (A) a beaded necklace on which the students can practice (sic) all kind of counting activities. In higher grades, this chain of beads successively becomes (B) an empty number line for supporting additions and subtractions, (C) a double number line for supporting problems on ratios, and finally (D) a fraction/percentage bar for supporting working with fractions and percentages. (p12)

The fraction/percentage bar and, to a lesser extent, the double number line, feature strongly in materials developed by RME, as can be seen in the *TAL-project* materials in the Netherlands (eg, Van Galen et al, 2008) and in the *Mathematics in Context* materials in the USA (eg, Keijzer et al, 2006). The use of the double number line is not yet widespread in the UK, although the 'fraction wall', which has similarities to the fraction bar, has been around for a long time (eg, Watt et al, 1967, p103). However, since the introduction of the National Numeracy Strategy (DfEE, 1999) the

*single* number line has been used extensively in UK primary schools, which may partly explain the dramatic rise in facility of the CSMS Decimals item (Brown, 1981) shown in Fig 5. When we gave this to a representative sample ( $N=294$ ) of Year 8 students in English schools in 1977, it had a facility of 37%, but with a similar sample ( $N=767$ ) in 2008/2009 this had risen to 78%<sup>2</sup>.



**Fig 5: CSMS Decimals item 6e**

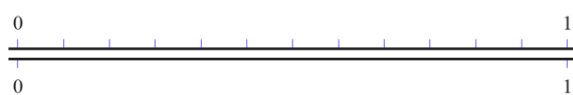
## TASKS AND FINDINGS

Our data on Year 8 students' use of the double number line are still exploratory. However, we have sufficient data to give a sense of some of the affordances of the double number line and of some of the difficulties that students encounter. Put another way, our current data suggest that the development of classroom activities involving the double number line is worth pursuing for us, but that learning to construct and use the model may be far from trivial for students. We report informal data from three sets of tasks involving students from two classes. The first task asks students to use a double number line to represent equivalent fractions or to evaluate a percentage. The second task involves a one-way-stretch, while the third asks students to convert information on a map from metres to feet.

### Fractions and percentage tasks

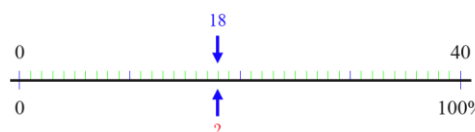
The task in Fig 6a gives some indication of whether students can, in certain circumstances, appreciate the *need* for a linear scale (although it clearly tests much more than this). The task in Fig 6b can be solved in a variety of ways. One approach is to see the double number line as representing a mapping, whose value (leaving aside the units, or in this case the %) is given by  $\times 100 \div 40$ , ie  $\times 2.5$ . We gave these tasks to an above-average attaining Year 8 class as a homework. The tasks were given cold, i.e. without any kind of introductory work.

The top number-line is divided into 12ths.  
Divide the bottom number-line into 4ths.  
("4ths" means "quarters")



**Fig. 6a**

What is 18 out of 40 as a percentage?



**Fig.6b**

For the task in Fig 6a, most students made effective use of the given scale (of 12ths) to mark off the required 4ths (as in Fig 7a). Some partitioned the line into 4 parts but seemed to ignore the given scale (Fig 7b), while a few



students tried to use the given scale but did so erroneously (Fig 7c).

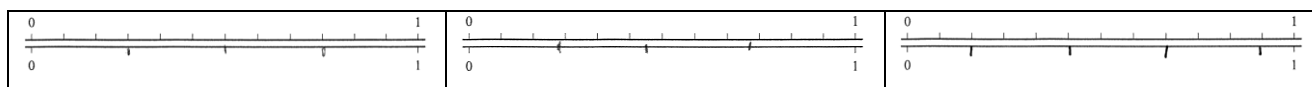


Fig 7a

Fig 7b

Fig 7c

Fig 8 shows responses to the percentage task shown in Fig 6b. Fig 8a shows a successful response to the item, but it seems likely that the student achieved this by making use of his (impressive) knowledge of fractions, decimals and percentages, rather than making use of the structure embodied by the double number line. On the other hand, the students giving the responses in Figs 8b and 8c do seem to have made use of the double number line and interpreted it successfully as representing the mapping  $\times 2.5$  or, in the case of Fig 8c, seeing this in terms of the more grounded, rated addition approach of ‘double and add a half’.

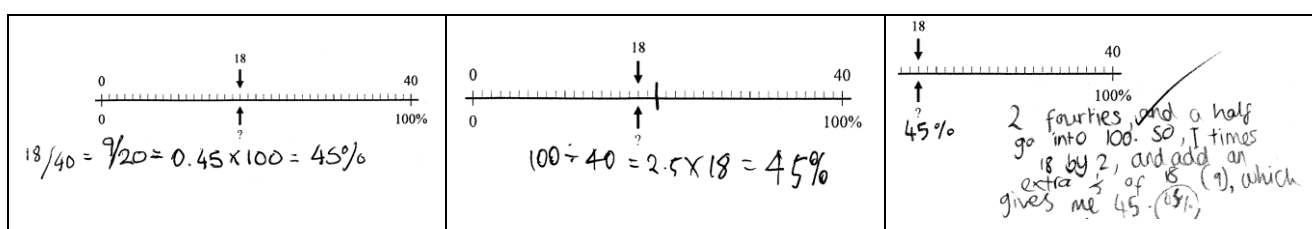


Fig 8a

Fig 8b

Fig 8c

### Elastic strip task

Part of the elastic strip task is shown in Fig 9. We tried the task as a starter activity with several Year 8 classes and also interviewed small groups of students on the task. In one Year 8 class of roughly average attainment we displayed the task on the whiteboard and also acted it out using a long elastic strip. The demonstration was quite dramatic and intriguing but the task still proved very demanding for this group. After some small-group discussion, three possible solutions emerged for the new distance of the red mark from the left hand edge, namely 16cm, 12cm and 9cm. The class teacher wrote these on the board and asked for a vote. The three responses received 12, 4 and 9 votes respectively. The 16cm response comes from using the addition strategy (either ‘the end has moved 30cm–20cm=10cm, so the 6cm mark will move 10cm’, or, less often, ‘the 6cm mark is, and will remain, 14cm from the right hand side’). We were aware of only 3 students who had themselves

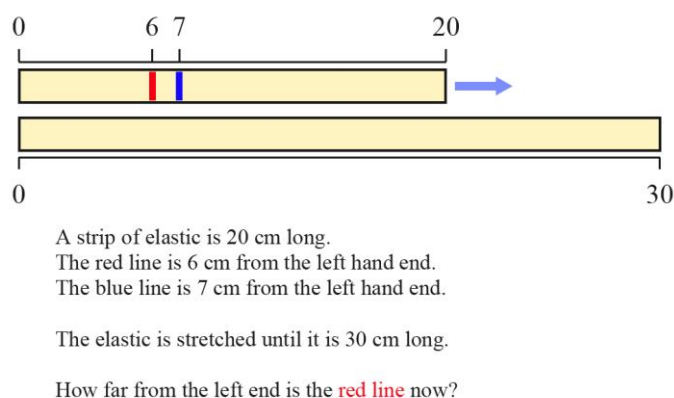
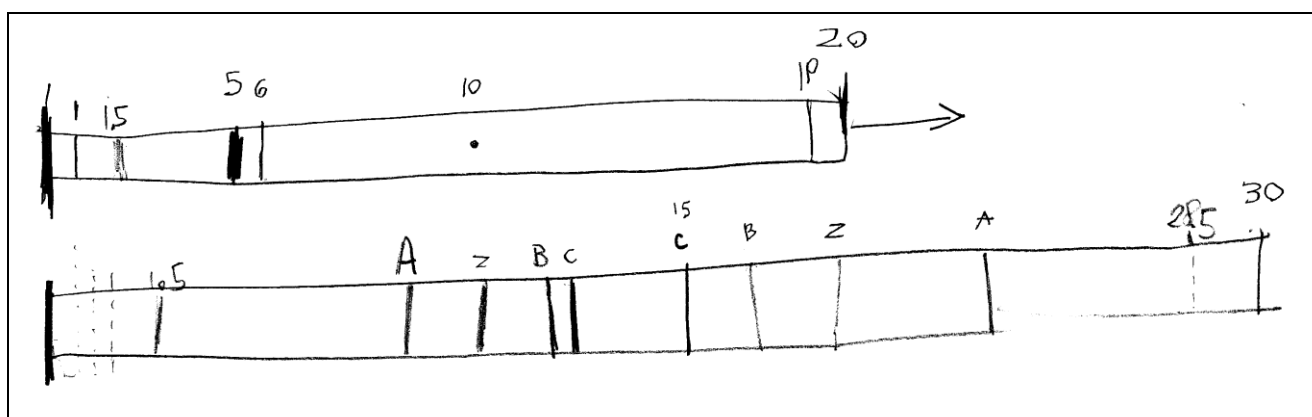


Fig 9: The elastic strip task

come up with the correct value of 9cm, though there might of course have been others. Two had used rated addition ('The stretched strip is half as long again;  $6\text{cm} + \text{half of } 6\text{cm} = 9\text{cm}$ ') while one student had used an argument based on the idea that if the strip was 10cm long the red mark would be 3cm from the end (this is quite a sophisticated argument since it would not be possible to enact this in practice).

Fig 10 shows a drawing that arose during an interview with 4 students (A, B, C and Z) from the previously mentioned above-average Year 8 class. The drawing of the 20cm strip and the 30cm stretched version was done by the interviewer. The vertical marks labelled A, Z, B, C, on the 30cm strip were drawn by the students. We started by considering the image of a line 5cm from the left hand end of the 20cm strip after the strip had been stretched to 30cm. Reading from the left, the first set of vertical lines (labelled A, Z, B and C) on the 30cm strip were students' estimates of the position of the image of the 5cm line. [The second set of lines labelled C, B, Z and A



**Fig 10: Interview responses to the elastic strip task**

were their initial estimates for the image of the central dot drawn 10cm along the 20cm strip.] Student C justified his mark in terms of the addition strategy (ie the right hand end of the strip had moved 10cm, so the 5cm mark would move 10cm). Other students thought it would move a bit less than this (and hence drew their images to the left of C's image). However, initially their estimates were purely qualitative, though eventually, after considering the image of the midpoint, they adopted a successful rated-addition approach (the 5cm mark moves half this distance, ie 2.5cm, so it ends up 7.5cm from the left hand end).

### **A map task: Westgate Close**

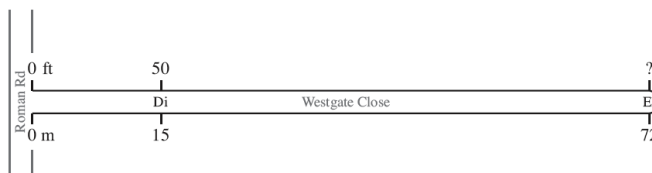
Our third task was based on a map of a short private road, Westgate Close. One version of the task is shown in Fig 11. As with the elastic strip task, we used this as a starter with some Year 8 classes. We would argue that it is more obvious in this task, than with the elastic strip, that the scales are linear (it would be very odd if for part of the road, a given number of feet matched a certain number of metres but that somewhere else on the road the same given number of feet matched a quite different number of metres!). Nonetheless, the addition strategy was still common here, with students calculating that the distance of El's house along Westgate Close was 107 ft

## Working Group 2

from Roman Road ( $50-15=35$ ,  $35+72=107$ , or  $72-15=57$ ,  $50+57=107$ ). However, the task also provoked an interesting variety of correct, or partially correct strategies. For example, several students in a low attaining Year 8 class, estimated the distance by marking-off 50 ft lengths. Often this lead to quite good

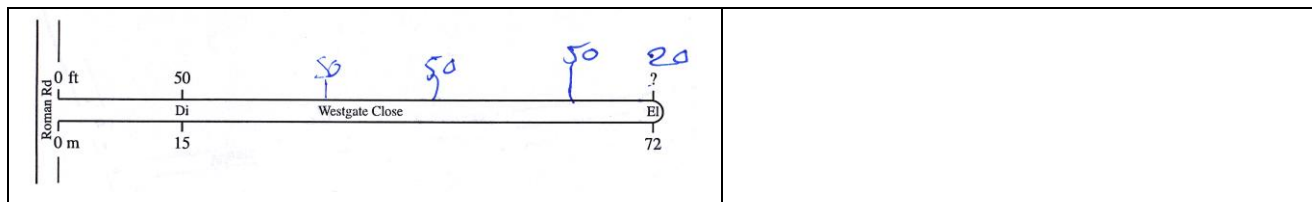
Here is another map of Westgate Close and Roman Rd.  
Di's house is 15 m, or 50 ft, along Westgate Close.  
El's house is 72 m along Westgate Close, at the very end of the road.

- Estimate El's distance in feet.
- Calculate El's distance in feet (try to find *several* ways of calculating the answer).



**Fig 11: Westgate Close**

estimates (Fig 12a), of between 200ft and 250ft (the actual answer is 240ft). Though these students simply ignored the information about the distances in metres, their estimates might stand them in good stead once they do try to calculate, especially if they use an inappropriate strategy.



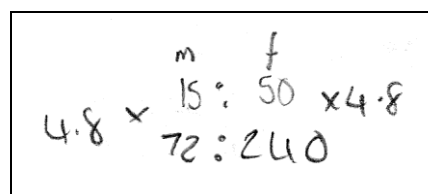
**Fig 12a**

**Fig 12b**

As with Elastic Strip, the task proved difficult for the average-attaining Year 8 class. One student (Fig 12b) used what might be called a 'function' approach to arrive at the multiplier 3.3 ( $50 \approx 15 \times 3.3$ ), and used this to calculate  $72 \times 3.3$  (but as can be seen, he made arithmetic and transcription errors to arrive at a widely-off answer of 91.87ft). Another student used an effective rated-addition strategy to move along the road in steps of 15m/50ft, to arrive at  $75\text{m}=250\text{ft}$ ; however, having thus overshoot the 72m distance by 3m, she then subtracted 3 from 250 to arrive at 247ft.

In the above-average attaining Year 8 class, one pair of students used a 'scalar' approach; they determined that 72 is 4.8 times 15, and used this multiplier to calculate the distance of El's house in feet:  $50\text{ft} \times 4.8 = 240\text{ft}$ .

This pair also made use of a ratio table (Fig 13) which may well have helped them structure their work.



**Fig 13: Ratio table**

However, not all students in this class were immune from using the addition strategy, and one pair used a hybrid scalar/addition strategy to come up with an answer of 212ft, based on the observation that 72 can be expressed as  $15 \times 4 + 12$  (leading to  $50 \times 4 + 12 = 212$ ).

## DISCUSSION

The findings reported above suggest that some students can make productive use of the double number line as a model of contexts that involve multiplication-as-scaling, even when they may have had little or no prior experience of using it in this way. But equally, there are students who do not readily see the multiplicative structure of the contexts (or of the model), which suggests that developing this insight is not a trivial matter. Nonetheless, we would argue that it is worth giving students experience of the model (both as a model *of* multiplication-as-scaling contexts, and as a model *for* multiplication-as-scaling concepts), since scaling is an important aspect of multiplication, and since it is important to be able to discern whether a context involves scaling. Also, the model is well suited to showing the contradictions inherent in the addition strategy when applied to scaling contexts (for example, in the case of a  $\times 1.5$  stretch applied to a 20cm elastic strip fixed at one end, if students use the addition strategy to argue that points on the strip will move 10cm further from the fixed end, it is fairly easy to provoke a contradiction by asking what happens to the mid-point of the strip, or to a point very near the fixed end).

The double number line also links nicely to other powerful representations and we should help students develop these, in particular links with ratio tables, and with mapping diagrams and Cartesian graphs (the latter are also based on number lines, but in the case of Cartesian graphs the lines are orthogonal rather than parallel). At the same time the double number line may not always be the most appropriate model for representing and/or solving multiplication tasks, and we would argue that students, including those at secondary school, need extensive and ongoing experience of other models of multiplication, in particular arrays and the area model.

## NOTES

1. ICCAMS (Increasing Student Competence and Confidence in Algebra and Multiplicative Structures) is a 4-year research project funded by the Economic and Social Research Council as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics (STEM) disciplines. Phase 1 of the project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England. This is followed in Phase 2 by a collaborative research study with two teacher-researchers in each of four secondary schools. The aim is to examine how formative assessment can be used to improve attainment and attitudes, and finally how the work can be disseminated on a larger scale.

2. In 2008/2009, we re-administered three of the CSMS tests that had been developed in the 1970s - Algebra, Decimals, and Ratio. In general performance on the tests was very similar to the 1970s; the most notable exceptions were some Decimals items that related to measurement, where performance improved in 2008/2009.

## REFERENCES

Anghileri, J., & Johnson, D. C. (1992). Arithmetic operations on whole numbers: multiplication and division. In T. R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods* (Second ed., pp. 157-200). Boston, MA: Allyn and Bacon.

## Working Group 2

- Behr, M., Harel, G., Post, T. & Lesh, R. (1991). The Operator Construct of Rational Number. In F. Furinghetti (Ed.) *Proceedings of PME XV Conference*, pp. 120-127. Assisi, Italy: PME.
- Brown, M. (1981). Place value and decimals. In Kath Hart (Ed.) *Children's understanding of mathematics: 11-16*. London: John Murray.
- Confrey, J., Maloney, A.P., Nguyen, K.H., Mojica, G. & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In M. Tzekaki, M. Kaldrimidou and H. Sakonidis (Eds.) *Proceedings of PME XXXIII Conference*, Vol 2, pp. 345-352. Thessaloniki, Greece: PME.
- Davis, B. (2010). Concept studies: designing settings for teachers' disciplinary knowledge. In Pinto, M.M.F. & Kawaski, T.F. (Eds.). *Proceedings of PME XXXIV Conference*, Vol 1, pp. 63-78. Belo Horizonte, Brazil: PME.
- DfEE. (1999). *Framework for teaching mathematics from Reception to Year 8*. London: Department for Education and Employment.
- DfEE. (2001). *Framework for teaching mathematics Years 7, 8 and 9*. London: Department for Education and Employment.
- Hart, K. (1981). Ratio and proportion. In Kath Hart (Ed.) *Children's understanding of mathematics: 11-16*. London: John Murray.
- Karplus, R. & Peterson, R. (1970). Intellectual development beyond elementary school II: ratio, a survey. *School Science and Mathematics*, LXX, 9, Dec 1970.
- Keijzer, R., Abels, M., Wijers, M., Brinker, L.J., Shew, J. A., Cole, B.R., & Pligge, M.A. (2006). *Ratios and rates*. In Wisconsin Center for Education Research & Freudenthal Institute (Eds.), *Mathematics in Context*. Chicago: Encyclopædia Britannica, Inc.
- Küchemann, D. (1989). The effect of setting and numerical content on the difficulty of ratio tasks. In G. Vergnaud et al (Eds.) *Proceedings of PME conference* (pp. 180-186). Paris, France.
- Van Galen, F., Feijs, E., Figueiredo, N., Gavemeijer, K., Van Herpen, E. & Keijzer, R. (2008). *Fractions, percentages, decimals and proportions*. Rotterdam: Sense Publishers.
- Van den Heuvel-Panhuizen, M. (2002). Realistic Mathematics Education as work in progress. In F.-L. Lin (Ed.), *Common Sense in Mathematics Education*. Proceedings of 2001 The Netherlands and Taiwan Conference on Mathematics Education, Taipei, Taiwan (pp. 1-42). Taipei, Taiwan: National Taiwan Normal University.
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52:83-94.
- Watt, I. D., Ford, M.J., Furst, S., Prentice, S., Renucci, A., Watt, R.M. & Yuill, G.S. (1967). *Mathematics for schools. A modern approach. General course, Book 1*. London: Foulsham.

# PROPORTION IN MATHEMATICS TEXTBOOKS IN UPPER SECONDARY SCHOOL

Anna L. V. Lundberg

Linköping University, Sweden

*Proportional reasoning and knowledge of proportion are prerequisites for success in higher studies in mathematics. The aim of this paper is to investigate what possibilities Swedish upper secondary school textbook tasks offer students to develop knowledge about proportion during the first course in mathematics. The three textbooks investigated in this paper show a variation in “know – how” of proportional reasoning but less variation regarding knowledge about proportion.*

Keywords: Proportion, Textbooks, Upper secondary school, Anthropological Theory of Didactics.

## BACKGROUND

To understand and use calculations with proportionality is one of the learning goals for grade nine in Swedish compulsory school (Skolverket, 2001). However, results from TIMSS 2007 show that 50% of the students in grade eight have difficulties solving tasks about proportionality (Mullis, 2008). But what about the students at upper secondary school? How are they managing proportions? International research also shows a predominance to use the linear model in solving proportion tasks in upper secondary school (De Bock, Verschaffel, & Janssens, 1998). My ongoing study attempts to shed light on various aspects of how proportion and proportional reasoning are exposed in textbooks at *upper secondary school* level in Sweden. According to several studies (e.g. Johansson, 2006), Swedish mathematics teachers rely on textbooks mainly in terms of exercises, making the textbook a critical factor in the classroom to study.

The aim of this paper is to present the first results from an investigation into what possibilities Swedish upper secondary school textbook tasks offer students to develop knowledge about proportion during the first course in mathematics.

## THEORETICAL FRAMEWORK

This section will present the theoretical background from which the analytical tool used for this study was developed by first discussing how the notions of proportion and proportional reasoning were interpreted, then shortly outlining the theory linked to the tool, and finally presenting the tool as constituted by its four main parts.

### Proportion and proportional reasoning

The term proportion is used when two quantities  $x$  and  $y$  are related by an equation  $y=kx$ , where  $k$  is a constant. Then  $y$  is said to be (directly) proportional to  $x$ , which may be written  $y \propto x$  (The Concise Oxford Dictionary of Mathematics, 2009). It is

also common to use the term proportion for some specific relations such as *direct proportion*  $y = k \cdot x$ , *square proportion*  $y = k \cdot x^2$ , *inverse proportion*  $y = \frac{k}{x}$ , *inverse square proportion*  $y = \frac{k}{x^2}$  and *inverse square root proportion*  $y = \frac{k}{\sqrt{x}}$ .

I am also investigating proportional reasoning tasks though it is difficult to set up a general definition of proportional reasoning, maybe because proportion is such a complex concept. I will here use the description of proportional reasoning found in Lamon (2007, p. 638).

According to Cramer and Post (1993) and several other studies there are three central types of problem situations in proportional reasoning: *numerical comparison*, *missing value*, and *qualitative prediction & comparison*. In numerical comparison problems, the answer does not call for a numerical value. The student compares two known complete rates, as in Noelling's (1980) well known orange juice problem. Lybeck (1986) among others, found that there exist two different main solution strategies: the A-form or the so-called Within Comparison, where quantities of the same unit are compared, and B-form, a Between Comparison across different units. In missing value problems three objects of numerical information in a proportion setting are specified with a fourth number to be discovered. A popular such task is the tall-man short-man problem (Karplus, Karplus, & Wollman, 1974). The third problem situation, qualitative prediction & comparison, does not demand memorized skill. These types of problems force the students to gain knowledge about the meaning of proportion with qualitative thinking (Cramer & Post, (1993).

### **Knowledge and know-how related to proportion tasks**

As this study is focused on how a specific mathematical notion is treated in the school institution in terms of types of tasks and strategies, The Anthropological Theory of Didactics, ATD; (see e.g. Bosch & Gascon, 2006) offers a useful approach. The ATD postulates an institutional conception of mathematical activity, starting from the assumption that mathematics, like any other human activity, is produced, taught, learned and diffused in social institutions. Mathematical work can be described in terms of mathematical organisation. A mathematical organisation (MO) is constituted by two levels, the *know-how* (task & techniques) and the (discursive) *knowledge* (technology & theory) related to a given task (Chevallard, 2006). Task – different kinds of tasks to be studied, Techniques – how to solve tasks, Technology – justification and explanation of the techniques, Theory – founding technology and justification of technology. In this study, there are influences from two MO's, one where proportion is defined as a 'dynamic' notion MO<sub>1</sub> and one where proportion is defined as a 'static' notion MO<sub>2</sub> (see below).

In order to study a phenomenon a Reference Epistemological Model (REM) should be created by the researcher (Bosch & Gascón, 2006). Otherwise it is difficult to be

independent in relation to the educational institutions understudy and the result may be a model that is implicitly imposed by the educational institution. The REM is a corresponding body of mathematical knowledge that is continuously developed by the research community and connected to the different steps of the didactic transposition. The transposition process describes how the mathematical knowledge is transformed from the institution of knowledge production through the educational system to the classroom (Bosch & Gascón, 2006).

### The knowledge of proportion

As a REM and theory category, the two MO's of describing proportion in textbooks was used. MO<sub>1</sub> was observed in a pilot textbook study (Lundberg & Hemmi, 2009), where it was found that a frequent way to present proportion is by the relationship  $y = k \cdot x$ , where  $y$  is dependent of  $x$  and  $k$  is a fixed constant. This has been named *adynamic notion of proportion* (Miyakawa & Winsløw, 2009), as we have different values of  $x$  as input producing specific outputs as  $y$  depending on the value of  $k$ .

Another way to describe proportion is *static* (Miyakawa & Winsløw, 2009). It is possible to identify this phenomenon in Euclid's definition of proportion (Euklides & Heath, 1956), where it is regarded as static in nature because it deals with pairs of "magnitudes" rather than numbers. A magnitude could be a length, like the diagonal of a square. For the Greeks it could not be measured in centimetres, but nevertheless multiplied in a geometric sense (e.g. enlargement). The static way of defining proportion is more general in comparison with the dynamic notion because it can be defined in  $n$ -tuples of real numbers and does not constrain proportion to pairs.

An example from a Swedish textbook (Gennow, Gustafsson, Johansson, & Silborn, 2003, p. 314) will serve as an illustration of static and dynamic definition:

"An electric radiator influences by power  $P$  (the thermal energy emitted per second) of voltage the  $U$  that the radiator has been connected to. The table shows some values of  $U$  and  $P$  that belong together. Check if there is a relation between  $P$  and  $U$  represented by  $P = k \cdot U^2$  and if so calculate  $k$ . The power has the unit Watt (W) and the voltage Volt (V).

$U$ (V)	120	160	200	240
$P$ (W)	144	256	400	576

Solution: To investigate  $k = \frac{P}{U^2}$  we put a new row in the table.

$P/U^2$ (W/V <sup>2</sup> )	0,010	0,010	0,010	0,010
-----------------------------	-------	-------	-------	-------

We obtain the same result for all pairs. This implies that the relation can be written as  $P = k \cdot U^2$  and  $k = 1,0 \cdot 10^{-2} \text{ W/V}^2$ . In electricity the unit  $\text{W/V}^2$  is denoted  $S$  (Siemens). The relation can be also be written  $P = \frac{U^2}{R}$  where  $k = \frac{1}{R}$  and  $R$  has the unit  $\Omega$  (Ohm)."

*Auth transl.*



In the beginning of the example the static definition is found in the table where the given data are n-tuples of  $U$  and  $P$ . In the solution it is necessary to switch to a dynamic definition because it is eligible to calculate  $k$  in order to check if there is a proportionality that can be expressed by the general formula  $P = k \cdot U^2$ .

### The know-how in calculating proportional tasks

To investigate solution techniques for proportion tasks a study by Hersant (2005) was used because it had a lot of similarities with this study. She found six different types of techniques in her analysis of solved proportion examples in French textbooks. To show the differences between these categories of solution techniques (1-6 below, *Auth. transl.*) I will use the following task provided by Hersant:

*If 18 meters of fabric costs 189 francs, how much will 13 meters cost?*

#### 1 Reduction by unit

If 18 meters cost 189 francs, 1 meter will cost 18 times less or  $\frac{189}{18}$ , and 13 meters will cost 13 times more than one meter or  $\frac{189}{18} \cdot 13$  where  $x = \frac{189 \cdot 13}{18}$ . The answer will be 136.50 francs. In the context of the theory of proportionalities, it will be justified by the characteristic of the property proportion expressed here as part of quantities. Two quantities  $U$  and  $V$  are proportional so when  $U$  is multiplied by 2, 3, 4... $\lambda$  (and  $\lambda$  real),  $V$  is multiplied by 2, 3, 4,... $\lambda$ .

#### 2 Multiplication by a relationship

Now consider the following solution: If 18 meters cost 189 francs, then 13 meters will cost  $\frac{13}{18} \cdot 189$ . Pay attention to that neither ratio nor proportion are used here. The technique of within measures proportion is used.

#### 3 Use of proportion

If we let the price for 13 meters of fabric be  $x$  francs then the price should be in proportion to the length of the fabric  $\frac{189}{18} = \frac{x}{13}$  so,  $189 \cdot 13 = 18 \cdot x$  and  $x = \frac{189 \cdot 13}{18}$ . This technique differs from the earlier two when the proportion involves two different measures, length and price (between measures proportion).

#### 4 Cross multiplication

Now consider the resolution that follows: Let  $x$  be the price of 13 meters of fabric. (1)  $\frac{189}{18} = \frac{x}{13}$  then (2)  $189 \cdot 13 = 18 \cdot x$  and  $x = \frac{189 \cdot 13}{18}$  that can be summarized as follows in a table:

So  $189 \cdot 13 = 18 \cdot x$  and  $x = \frac{189 \cdot 13}{18}$ . In the spirit of this technique, the equality (1) does not match proportion but more rather a technique with a formal setting of the magnitudes and detached from the theories of proportions.

**Table 1: Cross product table**

189	$x$
18	13

### **5 Use of coefficient**

Another way of arguing is as follows: The fabric costs  $\frac{189}{18}$  francs/meters, so 13 meters of fabric cost  $13 \cdot \frac{189}{18}$  francs. Here again a reduction technique to unit is used but with no connection to proportion.

### **6 Other possible techniques**

It is also possible to solve the task with a graphical solution method. However, if only one value is calculated it seems to be a waste of energy to use graphical technique unless you don't have access to a graphic calculator then it is very easy to sketch a graph.

## **RESEARCH METHODOLOGY**

To investigate what possibilities textbooks offer upper secondary students in Sweden to develop knowledge about proportion during the first course in mathematics, the following research questions were set up (in terms of the ATD): What types of textbook tasks involve proportion? What techniques are used in the given solutions of proportion tasks? What explanations and justifications (technologies/theory) are presented in the proportion tasks? To answer these questions, an analytic tool was developed to investigate a selection of textbooks.

In a study about proportion in textbooks, da Ponte and Marques (2007) used the Pisa Assessment Framework as an analysing tool. In the pilot study (Lundberg & Hemmi, 2009) this tool was evaluated but for my purpose the categorisation at a cognitive level was problematic, so there was a need to develop an analytic tool better suited for text analysis.

The textbooks selected for this paper are from the A-course at the Swedish upper secondary school. The A-course is a special case because it is mandatory for all students at upper secondary school (Skolverket, 2001), selected here because it is the beginners' course for all further studies at both the theoretical and the vocational programs of upper secondary school. In Sweden there is an open market for textbooks without regulations from the authorities. There are several textbooks on the market for this course, among which I have selected the three most commonly used in my region (three municipals). The book chapters analysed were those where proportion was expected to be one of the key notions: arithmetic, geometry and functions. There was also a limitation in the geometry chapter. Only tasks about

similarity, scale and trigonometry in the geometry chapter were analysed. The textbooks were investigated concerning both the knowledge and the “know-how” of proportion. The textbooks were analysed to determine what type of tasks were given (missing value, numerical comparison and qualitative prediction & comparison) and what kinds of proportion were used (direct proportion, inverse proportion, square proportion, and square root proportion). Finally, solution techniques presented in the textbooks and the knowledge of proportion (theories and technologies) related to the tasks found were investigated. Thus, in terms of the ATD, the analytic tool used for this study was comprised by the following categories: *Task* – missing value, numerical comparison, qualitative prediction and comparison, static or dynamic proportion, direct proportion, square proportion, inverse proportion, inverse square proportion, and inverse square root proportion. *Technique* – how to solve tasks, the six categories by Hersant are used here. *Technology* – justification and explanation of the techniques. *Theory* – the two definitions of proportion static and dynamic are used as categories here.

## FIRST RESULTS

The study is still ongoing but this paper will report some first results from the textbooks that have been analysed. In this section, the first most significant observations are presented, quoting selectively from the textbooks to illustrate the main findings. In all the textbooks, the definitions of the notions are introduced by solved examples and the examples presented in this section will therefore be structured by taking the technique used as the overarching categorisation principle, before type of task and knowledge (justification) are identified.

The number of examined tasks in total for all three textbooks were 3073 (1157, 1093 and 823). 24% (757) tasks were classified as proportion tasks. The preliminary data indicate that missing value tasks (43%) occur three times as much as numerical comparison (14%) and qualitative prediction & comparison tasks (12%). The static definition was used most often in the geometry chapter (32%) and the dynamic definition of proportion was predominant in the chapters about arithmetic (81%) and functions (72%). There were only a few justifications found in the textbooks in the geometry section. The most prevalent type of proportion was direct proportion but some examples about other kind were mentioned in the textbooks. However, all types of techniques described above were found, as shown by the following illustrative examples.

### *1 Reduction by unit*

In Swedish textbooks this solution strategy is easy to find in the chapter about arithmetic. The following example is taken from Alfredsson et al, (2007, p. 45):

In a municipality the number of citizens is increasing by 8% over one year to 70 200. How many citizens were there in the municipality before the increase?

108% is equivalent to 70 200.

1% is equivalent to  $\frac{70200}{108}$ .

100 % is equivalent to  $\frac{100 \cdot 70200}{108} = 65000$  *Auth. Transl.*

This type of example is found in two of the textbooks(5%) and missing in the third book. The task is categorized as a proportional reasoning task called missing value and is also in favour for this solution technique. The MO represented is dynamic (MO<sub>1</sub>) and direct proportion. This technique is missing in the function chapter in all three textbooks.

## 2 Multiplication by a relationship

This solution technique is found in the chapter about arithmetic in several Swedish textbooks, here Liber Pyramid (Wallin, Lithner, Wiklund, & Jacobsson, 2000, p. 43):

Anna has a salary of 17 250 SEK. She got a rise in salary with 4 %. How much is her new salary? The new salary is 100% of the old salary and the salary rise of 4 % of the same salary. The new salary will be: 104% of 17 250 SEK and that will be  $1,04 \cdot 17250 \text{ SEK} = 17940 \text{ SEK}$  *Auth Transl.*

This is a very typical kind of example in all three textbooks (18%). I interpret this solution technique to use the same technique as in Hersant's example but here different data is used. The tasks is categorized as a missing value task and the notion of proportion is dynamic (MO<sub>1</sub>) and direct proportion.

## 3 Use of proportion

This special solution technique is to be found in general in the geometry chapter. An example from a Swedish textbook (Wallin et al., 2000, p. 122):

The pentagon  $ABCDE$  is similar to the pentagon  $FGHJK$ .

Calculate the length of the sides  $a$ ,  $b$ ,  $c$ , and  $d$ .

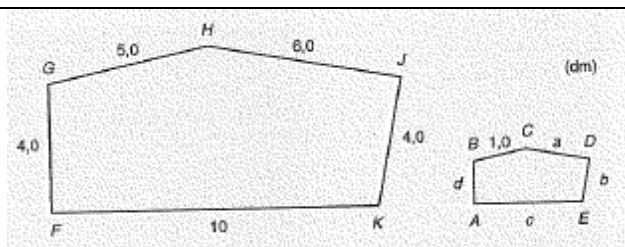
From the similarity it follows,

$$\frac{a}{6} = \frac{b}{4} = \frac{c}{10} = \frac{d}{4} = \frac{1}{5}$$

From the first and last

equality we get  $\frac{a}{6} = \frac{1}{5}$ ,  $a = \frac{6}{5} = 1,2$ . In the same way we get  $b = 0,8$ ,  $c = 2,0$  and  $d = 0,8$ .

*Auth. Transl.*



This is the most frequently used technique overall (34%) and found in all three books. The example is analysed as a missing value task and the notion of proportion is static (MO<sub>2</sub>) and direct proportion.

#### 4 Cross multiplication

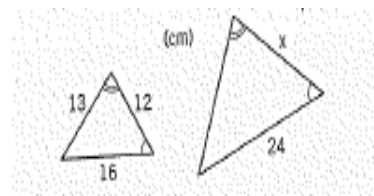
This particular solution technique is found in the chapter about geometry (Alfredsson, et al., 2008, p. 153):

In the figure the angles are marked with the same sign if they are in the same size. Calculate the length of  $x$ . The triangles are equal in two angles then they are similar and the ratio between two sides is equal.

*Auth. Transl*

$$\frac{x}{12} = \frac{24}{16}$$

$$x = 18$$



This is a very unusual solution strategy (2%) and it is only found in one textbook in the geometry chapter and static proportion. The task is analysed as a missing value task and the notion is represented as a static notion (MO<sub>2</sub>) and direct proportion.

#### 5 Use of coefficient

This category can be found in the function chapter (Gennow et al., 2003, p. 301):

In an experiment in physics, the students were measuring mass and volume for different amounts of aluminium tacks. First, the students weighed the tacks and then they poured them into a graduated measuring glass with water. The findings from one group were:

Volume (cm<sup>3</sup>) (V) 12 17 22 29 38

Mass (g) (m) 32 46 59 78 103

Determine the density (i.e. mass/volume) of aluminium if it is in proportion.

For this proportion to be valid  $k$  have to be  $k = \frac{m}{V}$ . We chose a pair of numbers far away from the origin of coordinates to increase the accuracy, draw lines to  $x$  and  $y$  axis.

Reading gives

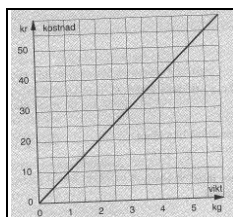
$$V = 40 \text{ cm}^3, m = 108 \text{ g}, k = \frac{108 \text{ g}}{40 \text{ cm}^3} = 2,7 \text{ g/cm}^3 \text{ Auth. Transl.}$$

This category is found in all the textbooks and is the next most frequent technique (26%) used especially used in the arithmetic chapter. The task is categorized as a missing value task and the notion is static (MO<sub>2</sub>) and direct proportion.

#### 6 Other possible techniques

An example of this solution technique comes from the chapter about functions (Wallin et al., 2000, p. 164):

## Working Group 2



The graph is showing how the cost of apples depends on the weight.

The graph is a ray with its origin in zero.

This means that we have a proportion. How much for: a) 1kg? b) 2kg?

Answer: a)10kr b)20kr.*Auth. Transl.*

This type of solution is found in all analysed textbooks (15%) and usually used in the function chapter with dynamic proportion. The example is categorized as a missing value task and the notion is dynamic ( $MO_1$ ) and direct proportion.

## DISCUSSION

The three textbooks investigated offer variations in types of tasks and techniques but the predominant task is missing value and some tasks could be exchanged for more qualitative tasks. Technique 1 and 4 are also missing completely in some of the textbooks so the techniques also have some limitations. The two notions of proportion (dynamic and static) are both represented but justifications are rare. Thus two  $MO$ 's are presented in different chapters (arithmetic, functions and geometry) with no link pointed out between them, which can be misleading for both teachers and students in their practice and might result in a predominance of the dynamic notion. It appears that the static notion is represented to a higher extent in the chapters about geometry and the dynamic notion more used in the chapter about arithmetic and functions. The theoretical description of proportion appears to be similar in all the textbooks and not presented in different approaches in parallel which is preferable. Proportion is also represented mainly as direct proportion with a few exceptions, which may be problematic as for example also inverse proportion is important for the further mathematics studies. Justifying technologies are very often missing. The explanation might be that justification is not a learning goal in the curriculum for this first basic course (Mathematics A). This study has also illustrated how the particular analytical tool developed for investigating tasks can be used as an instrument for what types of "knowledge" and "know-how" are represented in mathematics textbooks. This might be a benefit also for teachers in their practice by providing principles for the selection of tasks. However, if the students really use the techniques presented in the textbook is another research issue which will be investigated in a follow-up paper about students' solutions of proportion tasks.

## ACKNOWLEDGEMENT:

This study is funded by The Swedish National Graduate School in Mathematics, Science and Technology Education.

## REFERENCES

Alfredsson, L., Brolin, H., Erixon, P., Heikne, H., & Ristamäki, A. (2007). *Matematik 4000 kurs A blå lärobok* (1<sup>st</sup>.ed.). Stockholm: Natur & Kultur.

## Working Group 2

- Bosch, M., & Gascón, J. (2006). Twenty five years of didactic transposition. *ICMI Bulletin*, 58, 51-65.
- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *CERME 4* (pp. 21-30). Barcelona: Universitat Ramon Llull.
- Cramer, K. A., & Post, T. R. (1993). Connecting research to teaching: Proportional reasoning. *Mathematics Teacher*, 86(5), 404-407.
- da Ponte, J. P., & Marques, S. (2007). Proportion in school mathematics textbooks: A comparative study. *CERME 5* (pp. 2443-2452) Larnaca, Cyprus..
- Euklides, & Heath, T. L. (1956). *The thirteen books of Euclid's elements*. New York: Dover.
- Gennow, S., Gustafsson, I., Johansson, B., & Silborn, B. (2003). *Exponent A röd: Matematik för gymnasieskolan* (1<sup>st</sup>. ed.). Malmö: Gleerup.
- Hersant, M. (2005). La proportionnalité dans l'enseignement obligatoire en France, d'hier à aujourd'hui. *Repères, IREM*, 59(Avril), 5-41.
- Karplus, E. F., Karplus, R., & Wollman, W. (1974). Intellectual development beyond elementary school IV: Ratio, the influence of cognitive style. *School Science and Mathematics*, 74, 476-482.
- Lundberg, A. L. V., & Hemmi, K. (2009). Proportion in Swedish upper secondary school textbook tasks. In M. Lepik (Ed.), *Teaching mathematics: Retrospective and Perspectives proceedings of the 10<sup>th</sup> international conference* (pp. 252-260). Tallinn, Estonia: Tallinn University.
- Lybeck, L. (1986). Om didaktisk kunskapsbildning i matematik och naturvetenskapliga ämnen. In F. Marton (Ed.), *Fackdidaktik volym III* (pp. 149-189). Lund: Studentlitteratur.
- Miyakawa, T., & Winsløw, C. (2009). Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics*, 72(2), 199-218.
- Mullis, I. V. S. (2008). *TIMSS 2007 international mathematics report: Findings from IEA's trends in international mathematics and science study at the fourth and eighth grades*. Boston: TIMSS & PIRLS International Study Center.
- "proportion". (2009). *The Concise Oxford Dictionary of mathematics*. Retrieved 01/05, 2011, from <http://www.oxfordreference.com/views/ENTRY.html?subview=Main&entry=t82.e2298>
- Skolverket. (2001). *Compulsory school: Syllabuses* (1<sup>st</sup>. ed.). Stockholm: National Agency for Education: Fritzes.
- Wallin, H., Lithner, J., Wiklund, S., & Jacobsson, S. (2000). *NT a+b: Gymnasiematematik för naturvetenskaps- och teknikprogrammen: Kurs A och B* (1<sup>st</sup>.ed.). Stockholm: Liber.

# TEACHING ARITHMETIC FOR THE NEEDS OF THE SOCIETY

Hartwig Meissner

Westf. Wilhelms-Univ. Muenster, Germany

*In the newly formed Working Group 2 it may be allowed not only to discuss the research already done or the research in progress, but also to develop perspectives on the future needs of the society and how to approach these needs. This programmatic view into the future will not only summarize experiences from different research fields in learning arithmetic. It will also sketch necessary changes and suggest establishing a project within the European Union to face the common needs of arithmetic education in our technology based countries.*

Keywords: Number sense, estimation, calculators, paper and pencil techniques.

## INTRODUCTION

The author of this paper worked with calculators and computers in schools since the seventies. He was also a panelist at the ICME-3 Panel Discussion in 1976 on “What May Computers and Calculators Mean in Mathematical Education in the Future?”

Now, several decades later, it is interesting to see what really has happened in mathematics education in schools since the seventies. A state-of-the-art view is given in the 17th ICMI Study on *Mathematics Education and Technology – Rethinking the Terrain* (Hoyles & Lagrange 2010). But the experts at this ICMI Study discussed only briefly if the use of simple calculators had changed or if there were mental changes in the students’ computation abilities and skills. This paper will discuss some more aspects.

In 1984, after ICME-5 in Adelaide, 15 specialists from 10 countries from the Theme Group *The Role of Technology* summarized in their CALCULATOR REPORT:

- “Research investigations since 1980 have not produced dramatic new insights or results, they mainly confirm the findings of previous investigations.
- The overwhelming majority (greater than 95%) of investigations indicate that the use of calculators does not harm mathematics achievement in terms of the traditional curriculum and traditional tests.
- There are many countries with the ability rate of calculators to children of more than 80% where the school curricula ignore the existence of calculators or where the calculator use in mathematics education is not allowed for many grades” (Mohyla 1984, p. vii).

Since ICME-5 “simple *calculators* are no longer a central theme at congresses or conferences on mathematics education. *Computers* ... are the hits at the moment” (Mohyla 1984, p. 3). In the eighties investigations on the use of technology in mathematics education changed their focus. A shift can be observed from simple



## Working Group 2

calculators via scientific calculators and programmable calculators to computers. This shift also includes a shift in the age groups, from investigations related to all grades to investigations related to upper secondary grades.

But what has happened in primary schools since then? We cite some of the questions from Karlsruhe (see Athen & Kunle 1977, p. 291 ff) and sketch the answers:

ICME-3 Questions (Karlsruhe 1975)	Answers in 2010 for Primary Schools (not only in Germany)
What to do with calculators today?	In most of the European countries there are no systematic curricula for using calculators.
How will calculators ... influence teaching mathematics?	Regarding the reality in schools till now there is almost no influence calculators have on teaching mathematics in primary grades.
Is there a danger that numeracy will suffer if such calculators are introduced, ...	<b>YES</b> – in the eyes of parents, teachers, and school administrations <b>NO</b> – according to almost every research report
... or may numeracy be improved by using simple calculators?	<b>YES</b> – according to many project groups, but <b><u>a large scale investigation is necessary</u></b> to convince parents, teachers, and school administrations for implementing appropriate curricula.

## PAPER AND PENCIL TECHNIQUES

Which will be the future of paper & pencil algorithms for the four basic operations in the primary school curriculum? Already in the seventies Hans Freudenthal was warning: "We will run into a catastrophe when we today teach topics which one or two decades later will be done by calculators<sup>3</sup>." Nevertheless still in 1998 Schipper summarized that "sometimes up to 50% of the time from mathematics lessons in grades 3 and 4 is used to introduce and to train paper & pencil algorithms<sup>4</sup>." And Kaput (2002) continued warning: „The importance of the ability to serve as a poor imitation of a \$4.95 calculator is rapidly declining<sup>5</sup>." In daily life situations and for business purposes everybody uses a calculator. Simple calculators for the four basic operations are cheap and exist everywhere, they dominate our calculations in daily

---

<sup>3</sup> The text originally in German: "Wenn unser Unterricht heute darin besteht, dass wir Kindern Dinge eintrichtern, die in einem oder zwei Jahrzehnten besser von Rechenmaschinen erledigt werden, beschwoeren wir Katastrophen herauf" (Freudenthal 1973, p. 61).

<sup>4</sup> The text originally in German: "... in den Klassen 3 und 4 ... [werden] manchmal bis zu 50% der gesamten Unterrichtszeit dem Einueben der Algorithmen gewidmet." (Schipper 1998, p. 10).

<sup>5</sup> conference note

life. But in mathematics education in primary schools calculators are forbidden in almost every country, worldwide. Here we spend up to more than 100 hours in the classroom to teach and to train computing techniques as if there were no calculators. Why do we waste so much time of our children to teach them things which after primary school they will not need any longer?

We see an emotional dichotomy. Despite the research results from many research projects in many countries there is still the fear that the use of calculators in primary grades will harm mental arithmetic and estimation skills. To explain and to overcome that fear we will reflect the nature of number sense and of paper & pencil skills more carefully. We realize that the development of number sense is an intuitive and unconscious mental process while the ability to get an exact calculation result must be trained consciously. To overcome the above dichotomy we must solve the hidden dichotomy *number sense* versus *precise calculation result*. We need a new balance. Different types of examples will be given how we can further the development of number sense in a technology dominated curriculum.

## DICHOTOMY NUMBER SENSE VERSUS ALGORITHMIC SKILLS

Specialists from many research projects know that the use of calculators in primary grades does not necessarily harm mental arithmetic and estimation skills. But these “logical” arguments do not count. There still remains an emotional component against the calculator use which cannot be eliminated logically. Thus it is too simple just to claim to replace paper & pencil skills through the calculator. We need more than the ability to get a quick and exact calculation result. To remain mentally independent from the calculator we also must concentrate on automatic mental arithmetic and estimation skills. How do these skills develop? And which changes will we get when we change from paper & pencil skills to calculators?

A more profound view of mathematics learning is necessary to identify the nature of number sense and of paper & pencil skills. Learning and understanding mathematics is based on two different types of mental processes, on logical and conscious arguments (cf. *precise calculation results*) as well as on intuitive and unconscious mental processes<sup>6</sup> (cf. *number sense*). These two systems interfere.

### 1. Precise Calculation Results

There are three techniques to get precise calculation results: Paper & pencil techniques, using a calculator or computer, and mental arithmetic. Teaching and training of paper & pencil skills is time consuming and the results are less safe than

---

<sup>6</sup> For example, Vygotsky (1978) talks about spontaneous and scientific concepts, Ginsburg (1977) compares informal work and written work, or Strauss (1982) discusses common sense knowledge vs. cultural knowledge. He especially has pointed out that these two types of knowledge are quite different by nature, that they develop quite differently, and that sometimes they interfere and conflict (“U-shaped” behavior).

via pressing the calculator keys. It is obvious why the calculator technique dominates outside from school.

## 2. Mental Arithmetic

Mental arithmetic is a challenge for teachers to “teach” and for students to “learn” because of the two mental modes which are involved. On the one hand the students should be able to explain *logically and analytically* how they get the result. But on the other hand we also expect that for specific problems they can *react immediately* in a stimulus response style (stimulus response knowledge for e.g.  $1+1$  table and  $1\times 1$  table). Furthermore, we also expect such an *unconscious and intuitive* stimulus response reaction when the student gets confronted with computation mistakes. Either he/she spontaneously notices a conflict with his/her intuitive individual stimulus response knowledge or there is a spontaneous reaction like “this is too big” or “this is too small”. The latter describes a conflict between the computation result and the individual personal experiences.

## 3. Estimation Skills

Estimation is a challenging activity. Before starting computing we ask for the approximate result of a possible solution. Either the computation task is already given in the classical mathematical symbolic notation or we have to solve a word problem. For the first type of problems the estimation result can be found more easily. Here we must round the numbers and compute with rounded numbers. Estimation in this case is a special analytical and logical approximation technique (in German *Ueberschlagen*).

For word problems we usually first analyze the situation described. We then need a modeling process to get a “translation” of the word problem situation into a mathematical notation of a computation problem where we can get an estimation result via approximation. But there is an alternative strategy to estimate the result for a word problem.

Analyzing a word problem can and should stimulate also subjective domains of individual experiences related to the situation given (*Subjektive Erfahrungsbereiche*, cf. Bauersfeld 1983). Intuitively and spontaneously non-mathematical knowledge and personal experiences get stimulated, too. Estimation may then become a spontaneous and intuitive reaction like “Oh, this must be about ....”.

## 4. Estimation and Sachrechnen

To estimate spontaneously and intuitively an approximate result for a given word problem we need special experiences, environmental and daily life experiences and experiences in comparing and measuring objects. To develop these experiences the German arithmetic curricula include a special topic called *Sachrechnen* (aspects of environmental and domestic sciences). In *Sachrechnen* we *compare* objects according to their length, time, weight, etc. (*direkter / indirekter Vergleich* in

German) and we *measure* objects: Select a unit und try how often that unit fits into the object. Estimation in Sachrechnen is then quite a different mental activity, it is the internalized process of *comparing* or *measuring* (*Schaetzen* in German)<sup>7</sup>.

## 5. Concept of Numbers

In traditional German curricula for primary schools we introduce step by step the “number spaces” [0 - 20], [0 - 100], [0 - 1.000], and [0 - 1.000.000]. Thus also step by step, the “object number” gets reduced into a sequence of digits and the computation with big numbers gets reduced into manipulations with sequences of digits. Outside from school, numbers have a different meaning. Here a number is mainly a measurement number (*Groesse* in German) which describes the size (value, magnitude, ...) of an object. It consists of two parts, a *quantity number* and the appropriate *unit* like 345 km or 2 830 hours or 562 048 cents. The quantity number (*Masszahl* in German) tells us how many units we need to represent the size of that object.

## 6. Number Sense

We have summarized important aspects which are touched when we talk about number sense: “Number sense refers to an intuitive feeling for numbers and their various uses and interpretations; an appreciation for various levels of accuracy when figuring; the ability to detect arithmetical errors, and a common sense approach to using numbers. ... Above all, number sense is characterized by a desire to make sense of numerical situations” (Reys 1991).

Number sense not only refers to numbers but also to both, to conscious and to unconscious techniques to manipulate numbers, and it also includes a feeling about possible outcomes of these techniques. With a good number sense we can roughly predict the result of calculations, sometimes spontaneously (intuitively) and sometimes consciously (by approximating). Number sense also includes an intuitive feeling for additive and multiplicative structures. A central question for future curricula must be if we can develop a more effective number sense by the use of calculators than we momentarily do in our traditional curricula.

## CALCULATORS AND ARITHMETIC LEARNING

One of the first major projects to integrate the calculator use into a primary school curriculum was the Calculator-Aware-Number Project (CAN) in 1986 – 1989 in England and Wales (Ruthven 1999). But “during the 1990's, [the] curriculum and assessment system [became] much more '*calculator beware*' as a result of criticism of calculator use by mathematicians and politicians” (Ruthven 2007 in a personal

---

<sup>7</sup> Sachrechnen also includes the topic money and problem solving activities (problems from real life situations like shopping, planning an excursion, constructing a bird-cage, etc.).

communication). Thus we will start our analysis here with a warning. An unreflecting use of calculators in primary schools may damage some of the traditional goals of arithmetic education. The uncontrolled use might provoke two problems:

- Pressing keys is so easy. Why shall I still learn mental arithmetic?
- Pressing keys is so safe. Why shall I still check my calculator result?

A calculator curriculum must face these problems. But in this paper we will not discuss the PROs and CONs of the use of calculators in primary schools and how paper & pencil techniques could be replaced by a calculator use. Here we will reflect how arithmetic teaching in primary schools may benefit from the use of calculators and how the use of calculators may help to further the traditional mathematical goals. Of course, the calculator is an excellent tool to get quick and safe calculation results. But besides this property it also may serve as a didactical tool to stimulate intuitive and spontaneous ideas and activities in the teaching and learning processes. The possibility to handle a big bunch of quick calculations without any efforts allows a new working style in the class room which was not possible without calculators or computers. We will summarize and analyze some activities (for detail see Meissner 2006).

### 7. Stimulus Response Learning

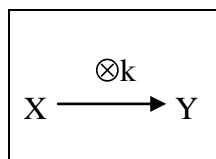
Calculators allow and facilitate stimulus response learning. This can be used in competitions to train mental arithmetic. The basic idea is to compute very quickly a given calculation problem to get an immediate feedback: *correct* or *wrong*. We developed several tasks starting with problems from the  $1+1$  and  $1\times 1$  table:

- Individual worksheets, individual training: Type the problem into the calculator and calculate the result in your head. Then press the “=” and see if you were right. If YES write down the result, if NO do the next problem. Later on work on the still open problems.
- Competitions mental computation versus calculator, who is the first? At the beginning each student wanted to be in the calculator group, later on almost nobody wanted to be there because "I am quicker in my head".
- Each student gets a worksheet, the use of calculators is allowed. Who has finished the worksheet first? There may be different worksheets according to the students' abilities.

### 8. Operators

Simple calculators with a constant facility are very important for primary schools. They can be "programmed" to work as an operator " $\otimes k$ " where  $\otimes$  stands for the four basic operations. Calculators with a constant facility allow for developing a feeling

for additive and multiplicative structures. We hide an operator  $\otimes k$  and others must find out which operator we hid:



Select a value for X, press the calculator keys, and interpret Y. Guess what  $\otimes k$  might be. If necessary select another value for X, etc. Finally select additional values for X and predict the results.

## 9. Calculator Games

There are several calculator games which use the constant facility to detect numbers and operations and to develop a feeling for additive or multiplicative structures. We will present an example with the calculator game *Hit the Target*.

Example:	
<i>input</i>	<i>display</i>

***Hit the Target:*** Find via guess and test a number  $z$  that  $z \times 17$  is in the interval  $[800,801]$ . Write a protocol of your guesses.

More general: An interval  $[a,b]$  is given and a factor  $k$ . Find a second number  $z$  via guess and test that the product " $z \times k$ " is in the interval  $[a,b]$ .

For primary schools we suggest to concentrate on integers  $k, z < 100$ .

More than 1000 guess-and-test protocols show that the students after a certain training develop excellent estimation skills. They guess a very good starting number and they develop an excellent proportional feeling. For more details on calculator games see Lange & Meissner (1980), Lange (1984), and Meissner (1987).

## 10. One-Way-Principle

Guess and test or trial and error are not considered to be a valuable mathematical behavior in mathematics education classrooms. But these components are necessary to develop spontaneous and intuitive ideas. Our experiences show that a systematic use of guess and test activities enriches creative and flexible thinking. So we developed a specific teaching method called One-Way-Principle (Meissner 2003). The One-Way-Principle is a method to use calculators or computers to explore intuitively and/or consciously many functional relationships of the type

$$X \xrightarrow{\sigma} Y \quad \text{or in case of the four basic operations } \otimes \quad X \xrightarrow{\otimes k} Y.$$

The basic idea of the One-Way-Principle is to use not reverse functions or algebraic transformations but to experience the set of variables as a "unit", as a global entity, which gets explored via guess and test.

Concentrating on the four basic operations in primary schools we can explore with simple calculators additive or multiplicative structures of the type " $a \otimes b = c$ ". Here the One-Way-Principle implies not to switch from addition to subtraction (or vice

versa) or from multiplication to division (or vice versa). Instead we have to guess " $a$ " (or " $\otimes$ " or  $b$  or " $\otimes b$ ") to use again the originally given key stroke sequence. Regardless of which variables are given and which are wanted, there is only the ONE WAY to solve all problems: Always use the same simple key stroke sequence of your calculator. The goal for the learner in the guess and test work is to discover intuitively the hidden relations between the variables and to develop a feeling how to get a good first guess (estimation) and how to reach a given target with only a few more guesses (additive resp. proportional feeling). Thus applying the One-Way-Principle furthers some of the intuitive and unconscious skills described above in no. 2 and 3.

## REDUCING PAPER & PENCIL TECHNIQUES

Again, in this paper we will not discuss how paper & pencil techniques could be replaced by calculators. We will reflect how the traditional teaching and training of paper & pencil skills could be reduced. We think the main question is not how to calculate all possible sequences of digits but to ask first for the importance of each technique.

### 11. Expanding Mental Arithmetic

Mental computations are usually done with small numbers. We suggest to expand the meaning of "small" and to concentrate the four basic operations " $a \otimes b =$ " on all  $a$  and  $b$  where  $a$  and  $b$  are one-digit- or two-digit-numbers. Adding and subtracting two-digit-numbers is already part of traditional curricula. For the multiplication of two-digit-numbers let the students themselves invent appropriate techniques. Paper and pencil should be allowed to write down results from intermediate steps.

### 12. Proportional Feeling

In parallel to the conscious techniques from no. 11 the students should also get an opportunity to develop an intuitive feeling for possible results. Playing Hit the Target would be an excellent addendum. The students can select themselves appropriate numbers for Hit the Target (small or big intervals  $[a,b]$ , no integer solution for  $z$ , ...).

### 13. "Large" Numbers

"Large" numbers in this paper are integers with at least 3 digits. Most of these multi digit numbers are unimportant in daily life because we prefer rounded numbers: Size of a swimming pool or a garbage container, distance between two cities or between the earth and the moon, weight of a lion, etc. Putting important rounded numbers on the number line we do not get an equidistant pattern. The larger the number space is the more unimportant numbers it will have. Do we still need for all these unimportant numbers the traditional paper & pencil techniques? We suggest concentrating only on calculating with "important" numbers.

### 14. Calculating with Rounded Numbers

Rounded numbers are similar to measurement numbers (*Groessen*, see no. 4 and 5). They consist of two parts, a one or two digit quantity number (*Masszahl*) and a unit (“thousands”, “millions”, etc.). To calculate with rounded numbers we can separate the two parts. We can then calculate with one or two digit numbers and apply techniques about what to do with the units. This approach also furthers Sachrechnen goals:

- Changing the unit implies converting the related quantity number. Getting experiences in changing units also furthers the development of spontaneous and intuitive reactions as described in no. 3.
- For addition and subtraction both numbers must have the same “unit”.
- For multiplication and division there are easy rules how to compute with the units. The students themselves might discover these rules.

### 15. Number Spaces

Reflecting the topics above, we also should rethink the concept of introducing numbers. It is fine to start in the first grade with [0 - 20] and then [0 - 100]. But when we start using calculators the number space gets suddenly unlimited. We need a spiral approach in which the students themselves can discover numbers and number properties in individual own subjective domains of experiences and where they can discuss their experiences. A spiral approach would also help to develop a much broader number sense.

### 16. Decimal Numbers

When we introduce calculators in primary schools, we must be aware that the students will discover very soon decimal numbers in the display. But they already have a basic knowledge of writing decimals. According to our experiences they are just happy to learn that 23.5 can be interpreted as 23 cm and 5 mm, or 12.69 as 12 € and 69 ct or 3.125 as 3 km and 125 m. And when there are more digits behind the “point”? Usually the children accept the simple answer “just ignore those digits” which corresponds to the view from above to distinguish between important and unimportant numbers. Adding or subtracting decimal numbers also becomes easy with this view. With the decimal grid problem it is even possible to develop intuitive experience about the multiplication of decimals (Barbeau & Taylor 2009, p. 210f).

## SUMMARY

There is a worldwide resistance in using calculators in primary schools, emotionally dominated by parents, teachers, and school administrations. And the experts in mathematics education are split. Teaching “only” arithmetic is in the domain of “educators”, while the use of technology is in the domain of “mathematicians” (cf. 17th ICMI Study). There is no real lobby for the needs of our primary school kids. To bridge this gap we need a new concept how to teach arithmetic in primary schools in a technology based country in the 21st century.



## REFERENCES

- Athen, H. & Kunle, H. (Eds.) (1977). *Proceedings of the Third International Congress on Mathematical Education*. Universitaet Karlsruhe.
- Barbeau, E.J. & Taylor, P.J. (Eds.) (2009). *Challenging Mathematics In and Beyond the Classroom. The 16th ICMI Study*. New York: Springer.
- Bauersfeld, H. (1983). Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In: *Lernen und Lehren von Mathematik*, Bd. 6, pp. 1-56. Aulis Koeln.
- Freudenthal, H. (1973). *Mathematik als Paedagogische Aufgabe*, Vol. 1. Stuttgart.
- Ginsburg, H. (1977). *Children's Arithmetic*. New York: Van Nostrand.
- Hoyles, C. & Lagrange, J-B, (Eds.) (2010). *Mathematics Education and Technology – Rethinking the Terrain. The 17th ICMI Study*. New York: Springer.
- Lange, B. (1984). *Zahlbegriff und Zahlgefuehl* (Dissertation). Muenster: Lit-Verlag.
- Lange, B., & Meissner, H. (1980). Taschenrechnerspiele. In: *Praxis der Mathematik*, Vol. 22, p. 174-176 (Zielwerfen), p. 245-248 (Die grosse Null), p. 308-311 (Die grosse Eins) and p. 373-375 (Faktorfinden). Aulis, Koeln.
- Meissner, H. (1987). Schuelerstrategien bei einem Taschenrechnerspiel. *Journal fuer Mathematik-Didaktik* Jg. 8, H. 1-2/1987, p. 105-128, Schoeningh Paderborn.
- Meissner, H. (2003). Constructing Mathematical Concepts with Calculators or Computers. In: *Proceedings of CERME 3*. Bellaria Italy.
- Meissner, H. (2006). *Taschenrechner in der Grundschule. mathematica didactica*, p. 5-25. Hildesheim: Franzbecker Verlag.
- Mohyla, J. (Ed.) (1984). *The Role of Technology: Theme Group 3 Report*. Adelaide.
- Reys, B. J. (1991). *Developing Number Sense in the Middle Grades: Curriculum and Evaluation Standards for School Mathematics*. USA: NCTM, Reston VA.
- Ruthven, K. (1999). Constructing a calculator-aware number curriculum. The challenges of systematic design and systematic reform. In: *Proceedings of PME-23*, Vol. 1, pp. 56-74. Haifa.
- Schipper, W. (1998). Schriftliches Rechnen-ein Fossil mit Zukunft. *Grundschulzeitschrift*, H. 119, p. 10-16.
- Strauss, S. (Ed.) (1982). *U-shaped Behavioral Growth*. New York: Academic Press.
- Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

# ANALYSING CHILDREN'S LEARNING IN ARITHMETIC THROUGH COLLABORATIVE GROUP WORK

Carol Murphy

University of Exeter

*The paper presents an example of collaborative group work with three children aged 6-7 years old, as part of a project to promote cooperative learning situations in young children's arithmetic. The premise of the project is that participation in collaborative group work will support children's cognitive development in number. Process-object duality models and the notions of reification, encapsulation or 'procept' have provided a way of examining the use of processes and objects in learning mathematics. These notions are used in relation to participation in a mathematical activity. This entails the use of two different theoretical perspectives, neo-Piagetian and neo-Vygotskian, in examining the children's mathematical thinking within a social context.*

Keywords: arithmetic, collaboration, participation, process-object duality

## INTRODUCTION

Much research has been dedicated to the development of early arithmetic and children's progression in the use of calculation strategies (for example Gray, 1991). Such studies have indicated that there are 'milestones' that show progression from simple counting strategies ('count-all' and 'count-on' strategies) and the use of commutativity ('counting-on from the larger number') to the use of number facts (additive components) and place value. It has been suggested that lower attaining children rely on counting strategies in addition and subtraction (Gray, 1991) and that a reliance on such strategies could hinder children's progression to more sophisticated, flexible strategies. The use of flexible strategies requires that children have a conceptual understanding of number and their relations.

Several studies have looked at developing children's generic thinking skills through cooperative group learning. Key to these are studies that have taught children to talk together effectively (Mercer, Wegerif & Dawes, 1999). These have shown that explicit teaching of talk strategies or *exploratory talk* can increase performance in non-verbal IQ tests and in tests for academic subjects including mathematics (Mercer & Sams, 2006). This paper presents work from a project funded by the Esmée Fairbairn Foundation and carried out with colleagues at the University of Exeter. The project introduced strategies for effective talk to young lower attaining children (ages 6-7). Our premise was that such an intervention would support children's learning in arithmetic and our aim was to examine the mechanisms of talk in relation to children's conceptual understanding of number.

## CONSTRUCTION OF MATHEMATICAL OBJECTS

Research has seen conceptual understanding of number as the cognitive construction of objects (such as number, relations and functions) and many researchers, including Dubinsky (1991); Sfard (1991); Gray & Tall (1994), have modelled this construction in terms of a process-object duality. Tall, Thomas, Davis, Gray & Simpson (2000) provided a thorough examination of the differences between these duality models but key to these is the notion of encapsulation (Dubinsky, 1991) or reification (Sfard, 1991).

In particular Gray and Tall's (1994) and Sfard's (1991) work points towards an initial focus on counting processes in children's learning in arithmetic. Sfard (1991) described a reliance on 'count all' where children count out each set. For example  $3 + 4$  becomes 1,2,3 add 1,2,3,4. That is each number is seen as a process. This is distinct from a 'count-on' strategy, for example  $3 + 4$  becomes 3 'count-on' 4,5,6,7. In this strategy three is seen as a cardinal number or object that can be 'counted-on' from. From Sfard's perspective, three is reified as an object. From Gray and Tall's perspective children see the symbol for an operation as both a process and a concept, or, in other words, as a *procept*.

It is proposed that, in order for children to progress in their use of calculations, they need to see this *dual nature*. This becomes key to children's development of flexible strategies beyond counting, and children's use of known facts to derive solutions to other calculations. This requires the knowledge of a small set of memorised facts that can be used flexibly. Gray and Tall distinguished between a *proceptual* known fact and a rote learned fact and how this knowledge is based on the notion of encapsulation or reification. In this way children will see the example  $3 + 4$  not only as a process, but also as a known fact that can be used as a reified object to derive new facts.

Much of the work on process-object duality has been based on neo-Piagetian constructivist theories. These theories are concerned with "the building (of the notion) of a mathematical object as a cognitive process that involves the learner's construction of adequate cognitive structures" (Dorfler, 2002, p.340). The project that this paper refers to examined the introduction of a didactical tool based on exploratory talk and collaboration in mathematics tasks. Such an intervention is based on neo-Vygotskian socio-cultural theories and a participatory perspective of learning in mathematics (Sfard, 2001).

Within a constructivist perspective discourse is seen to describe mental images of objects. It is possible to "ascertain whether an individual has constructed a mental object" and how the use of language indicates if an individual is conceiving the object (Tall et al., 2000, p.230). Within a socio-cultural perspective the focus is on cognition as discourse rather than on the use of discourse to understand specific examples of cognition. Speech is not only seen as a window to the inner mind to

look at the representations stored there, but speech and thought are seen as inseparable (Sfard, 2001).

The examination of the use of collaborative group work as an appropriate didactical approach in mathematics would require the investigation of how an individual develops thinking within a social context. This suggests working between socio-cultural theories that relate to participation and neo-Piagetian theories that relate to the individual. Cobb proposed the notion of *bricolage*, adapted from Gravemeijer (Cobb, 2007), where ideas are used from different theoretical perspectives to study diversity in children's mathematical thinking within the social context of a collaborative activity. Such an approach would seem appropriate in this study.

### THE STUDY

The project was based on a design experiment (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). It aimed to set up an intervention based on exploratory talk (Mercer et al, 1999) as a didactical tool with young children (ages 6-7) who were seen as low attainers in mathematics. Exploratory talk is typified by “a way of using language effectively for joint, explicit, collaborative reasoning” (Mercer et al., 1999, p. 97). Participants negotiate their understanding through constructive challenging discourse. It was anticipated that the use of exploratory talk would support children's participation in mathematics through collaborative reasoning and negotiated understanding.

Twelve teachers worked with the research team over two school terms to develop strategies to introduce exploratory talk and to trial mathematical tasks. Our aim was to analyse the group interactions and the learning of arithmetic that took place through the introduction of exploratory talk. The teachers were asked to select six focus children in each class and to engage the six children in talk and mathematics activities at least twice a week. This was managed in small groups of three children.

This paper presents excerpts from a group activity of one of the triads: Eric, Lydia and Amy. None of the children were recognised officially as bilingual learners but the teacher had identified them as lower attaining, with low confidence in their mathematics and poorer communication skills than other children in the class. Diagnostic tasks, based on those developed by the Shropshire Mathematics Centre (1996), were carried out with the focus children pre- and post-intervention. The pre- and post-tasks for addition carried out with the three children are shown in Table 1.

In the pre-tests Eric relied on ‘count-all’ strategies. Amy used a combination of ‘count-all’ and ‘count-on’ but also used an incorrect strategy. Lydia used ‘count-on’ strategies but did not always use these accurately. The post-tests indicated that the children still had a reliance on counting strategies but that there was some progression in accuracy and in the use of count on strategies and known facts. Lydia was more accurate in her use of strategies, she used known facts and made less

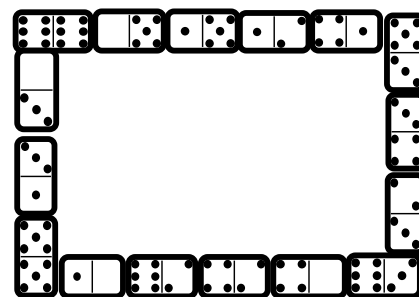
errors. Eric used more efficient count-on strategies. Amy used a more efficient strategy of counting on from the larger number. There is no attempt to present these results as evidence that the use of talk has supported children in their calculation strategies in a general sense. We have to acknowledge that the teaching may have benefited the children even if the talk strategies had not been introduced. However it does indicate that the children made some progression and in this respect it is worth looking at the mechanisms involved in a cooperative group activity.

	Lydia pre-task	Lydia post-task	Eric pre-task	Eric post-task	Amy pre-task	Amy post-task
5+7	Count-on	Count-on	Known fact	Count-on	Count-all	Known fact
7+8	Count-on with error	Count-on	Count-all	Count-on	Count-all	Count-on from larger number
6+9	Count-on	Use of known fact	Count-all	Count-on	Count-on	Count-on from larger number
10+8	Count-on with error	Known fact	Count-all	Count-on	Count-on	Count-on
8+13	Count-on with error	Count-on with error	Count-all	Count-on	Incorrect strategy	Count-all
15+6	Count-on with error	Count-on with error	Not given	Count-on	Not given	Count-all

**Table 1: Calculation strategies in pre- and post-intervention diagnostic tasks.**

This paper presents an example of one group activity with the three children. The activity took place towards the end of one term's intervention. The task was to construct a rectangle of sixteen dominoes where each join gave a total of six. An example of a possible solution is shown in figure 1.

It had been found that tasks with little or no recording supported collaborative work better. The dominoes provide a set of manipulatives that could be used by the group to arrive at a solution together. The dominoes themselves present arrays of dots that can help children in using counting and patterning ability to develop conceptual subitising as a basis for addition (Clements, 1999). Through the familiar patterns numbers can be seen as both composite parts and as a whole. The arrangement of the dominoes in a rectangle also gave a geometric problem to solve. The children were given a worksheet with the outline



**Figure 1: Domino activity.**

## Working Group 2

of blank dominoes and they used this as a template to help them organise the structure of the rectangle.

The transcript from the activity is presented in excerpts. Excerpt 1 shows the teacher modelling how to complete the task. The children are asked to agree on which domino to use next. Amy and Eric both predict a two, apparently without counting the dots and Amy selects a domino with a two. Lydia changes Amy's (correct) domino with her own. Amy then sits back from the task whilst Lydia places the next dominoes. Amy continues to observe the activity but does engage again later (see excerpt 4). Eric predicts the dominoes that are needed next. He states 'And four, there's four' and also 'Then we need a six'. Lydia expresses disagreement with a defiant 'No' or 'No, no, no...' It would seem she is checking the domino is correct by counting the dots. However, she counts all the dots on the dominoes and not just the joins. In the last line she realises the need to count the joining dots and agrees that the join makes six.

### Transcript excerpt 1:

Teacher:	The next one you've got to put sideways like that. So the four needs a what to go with it? It's got to go down there but what will it need?	Teacher points to show position on the table
Amy:	Two.	
Eric:	A two.	
Teacher:	See if you agree then what one to put.	Amy places domino with a two.
Lydia:	No (Amy inaudible), two. Five and one more.	Lydia removes the domino and replaces it with another domino (two and five). Lydia places the next domino (one and two).
Eric:	And four, there's four.	Eric points to domino (four and zero). Lydia places the domino.
Eric:	Then we need a six	Eric refers to the next domino needed.
Lydia:	No, no, no, no ... I think 1,2,3,4,5,6,7. That's seven. Ah... 5,6...1,2,3,4,5,6. Yeah that's right.	Lydia counts the previous domino (one and two) with the four of the current domino. She then changes to count the two and the four.

In excerpt 2 Eric, again, predicts the next domino but Lydia ignores this and selects an incorrect domino. Eric challenges this by saying 'You think that is going to make a six with a zero?' and offers a correct domino. Eric does not justify his choice and Lydia does not appear to question the use of the six or count the dots to verify.

## Working Group 2

### Transcript excerpt 2

Eric:	We need a six, we need a six.	Eric refers to the next domino to go with the zero.
Lydia:	Another zero.	Lydia handles a domino but rejects it.
Eric:	That one I think.	Eric points to two different dominoes that have a six.
Lydia:	We need a one.	Lydia places a domino with a one next to the zero.
Eric:	You think that's going to make a six with a zero? (Lydia shakes head). Well get a six then, get a six like that.	Eric holds up a domino (six and five). Lydia places the domino.

In excerpt 3 this routine is almost repeated as Eric predicts the next domino, Lydia ignores this and selects an incorrect domino. Eric challenges this and presents Lydia with a correct domino. Again the error is when one addend is zero and we can only speculate that this causes confusion for Lydia. Lydia does not appear to verify that the six is correct by counting and accepts Eric's offer of a solution.

### Transcript excerpt 3:

Eric:	One.	Lydia has placed domino (one and zero).
Eric:	Six.	Eric picks up domino with six but
Lydia:	Now we need a three.	Lydia places a domino with three.
Eric:	You mean that's going to make a six with a zero?	Eric holds up domino (six and four). Lydia removes domino with three and Eric places domino (six and four).
Lydia:	No. (Shakes head).	
Eric:	Two.	

In excerpt 4 Amy has rejoined the activity. She predicts the correct number of dots and checks by counting them. Eric also confirms the 'two and a four' but does not count these out. When Amy places the next domino this is challenged by Lydia using a similar question to Eric's. In this case her challenge is not supported by Eric or Amy.

### Transcript excerpt 4:

Amy:	Ahhh, I know... two, four. 1,2,3,4,5,6	Amy picks up domino with four and counts the two dots and the four.
Eric:	Two, four. Two and a four.	Amy places domino (four and three).
Amy:	Ahhh, three	Amy places domino (three and zero).
Lydia:	You think that's going to make a six?	
Eric:	Yeah.	

## Working Group 2

In excerpt 5 the children complete the rectangle. Lydia counts all the dots on both dominoes in trying to decide if the correct domino has been used and arrives at the total of 12. Eric then counts the dots that are joining and Lydia accepts this. As the children place the last domino in the rectangle Amy states that ‘the two is on there’. It may be that she is referring to the final join (a two has been placed next to a one). Eric points to the use of a two elsewhere to suggest that this is possible.

### Transcript excerpt 5:

Eric:	We need a six again	Refers to the zero on the current domino. Eric points to domino (six and three). Lydia places domino.
Eric:	Now we need a three	Eric picks up domino with three and two.
Lydia:	That’s twelve.	Lydia counts all the dots on both dominoes.
Eric:	1,2,3...4,5,6.	Eric counts the two lots of three dots. Lydia places domino (three and two).
Amy:	But the two is on there.	
Eric:	Yeah but look.	Eric points to another domino in the rectangle that has a two.

## ANALYSIS AND DISCUSSION

The use of discourse in analysing children’s mathematics presents problems in how to ‘read off’ or interpret what learners are thinking (Barwell, 2009). In the transcript from this study it is not entirely clear how the children have understood the task and if they are involved as the teacher intended. The task presented a geometric problem as well as arithmetic and at times it seems the focus of the discourse is on making the rectangle rather than finding the complements to six. The children miss the final join (although it is possible that Amy noticed this) so do not find a complete solution. However an assumption is made that the children follow the rule of making joins that add to six, at least up to the final join.

In analysing the transcript in relation to collaboration there are occasions that suggest lack of cooperation, for example when Lydia rejects Amy’s selection of domino in excerpt 1, but the children do work together to complete a rectangle following the rule. Analysis from a linguistic perspective does not indicate the children’s engagement with the characteristics of exploratory talk (cooperating, challenging and justifying) explicitly. For example, only the teacher uses the word ‘agree’ and there is little evidence of words indicating justification, for example the word ‘because’ is not used. However the children do appear to justify the use of dominoes by predicting or pointing to an example. At one point the phrase ‘Yeah but look’ is used alongside pointing. The children also verify the use of a domino by



counting the dots. The children do offer solutions and Eric is particularly proactive in this. He uses the term ‘We need a six’ and also ‘That one I think’. There is evidence of children challenging ideas and Eric uses the terms ‘You think’ and ‘You mean’ when challenging Lydia’s choice of domino.

Participation in a social context suggests the establishment of routines (Sfard, 2008). In the transcript we can see development of routines. For example, a problem is initiated ‘So the four needs a what to go with it?’ There is the offer of a possible solution ‘A two’, ‘We need a six’. The children then select the appropriate domino. In some cases the selection is verified through counting, on other occasions the selection is challenged. In effect the challenging also becomes a routine and Lydia attempts to imitate this (albeit at an inappropriate point). It is noted that participation may not always involve active engagement in the discourse. Amy sits back from the task in excerpt 1 and does not rejoin until excerpt 4. However from the video data she can be observed watching the other children until she comes in with her prediction ‘Ahhh, I know, two, four...’

How much can we see children working with numbers as processes and objects? In selecting the appropriate domino the children are finding an unknown number, the number that complements to make six. In this way the number is held as an object. This is modelled by the teacher as she asks ‘So the four needs *a what* to go with it?’ Here the teacher uses the indefinite article ‘a’ and refers to the unknown number as a noun. The term ‘We need *a...*’ (for example ‘we need *a six*’) is used frequently by the children, and maybe they are following the model of the teacher. Dorfler (2002) has suggested that the use of language provides means to express something as an object, such as the use of nouns. Dorfler also notes the use of actions on objects, such as the use of verbs. The children are often engaged in the action of counting and also use the phrase ‘You think that is *going to make...*’ suggesting that there is a use of actions on objects.

In determining the correct domino the children appear to use two strategies, either prediction, commonly used by Eric, or checking by counting, as used by Lydia. The dominoes allow the use of conceptual subitising as an early form of addition. It is possible that Eric, who is able to predict the next domino without counting, is using this strategy. Amy also shows some ability to predict but then counts to check. Lydia does not appear to use conceptual subitising as she does not predict any domino and relies on counting to verify any domino’s correct use.

## SUMMARY

It would seem that the use of exploratory talk as a didactical tool has enabled the three children to work together with a level of cooperation and that this cooperation leads to participation in mathematical routines. Collaboration and participation has engaged the children in solving a problem related to finding complements as an unknown number. This number is often referred to as a noun or object and the children use processes in determining and verifying the object. The familiar arrays represented on the dominoes can support the use of conceptual subitising and this is particularly evident with one child.

The premise of the study was that collaboration and participation in mathematical activities would support cognitive structures and in particular a proceptual knowledge. By reviewing the children's mathematics within the social context of the collaborative group work we have been able to consider the diversity of their approaches (Cobb, 2007). However this is not to say that each child's acquisition of the strategies can be determined from their participation. When a child used a process of counting it could be as a means of justification to others, not that they needed to count. The diagnostic pre- and post- tests aimed to acknowledge the children's diverse use of strategies. From these we can see how the children differ in their reliance on procedures in relation to concepts and this would seem to be reflected in the children's individual approaches. It may be too big a leap to suggest that such collaborative strategies have supported these changes but there is some initial evidence to show that the children were participating in arithmetic that involved both processes and objects and asked them to work with number in a proceptual way.

## REFERENCES

- Barwell, R. (2009). Researchers' descriptions and the construction of mathematical thinking. *Educational Studies in Mathematics*, 72, 255–269.
- Clement, D. H. (1999). Subitizing: What is it? Why teach it? *Teaching Children Mathematics*, March, 1999, 400–405.
- Cobb, P. (2007). Putting philosophy to work. In F. K. Lester, Jr. (Ed.) *Second handbook of research on mathematics teaching and learning* (3–38). Charlotte, NC: Information Age Publishing.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R. & Schauble, L. (2003). Design Experiments in Educational Research, *Educational Researcher*, 32(10), 9–13.
- Dorfler, W. (2002). Formation of mathematical objects as decision making. *Mathematical Thinking and Learning*, 4(4), 337–350.

- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D.Tall (Ed.) *Advanced mathematical thinking* (95–123). Dordrecht: Kluwer Academic Publishing.
- Gray, E. (1991). An Analysis of diverging approaches to simple arithmetic: Preference and its consequences. *Educational Studies in Mathematics*, 22, 551–574.
- Gray, E. and Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *The Journal for Research in Mathematics Education*, 26, 115–141.
- Mercer, N. & Sams, C. (2006). Teaching children how to use language to solve maths problems. *Language and Education*, 20(6), 507–528.
- Mercer, N., Wegerif, R. & Dawes, L. (1999). Children's talk and the development of reasoning in the classroom. *British Educational Research Journal*, 25(1), 95–111.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge: Cambridge University Press.
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46(1–3), 13–57.
- Sfard, A. (1991). On the dual nature of mathematical concepts: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1–36.
- Shropshire Mathematics Centre (1996). *Teaching Numeracy in a Shropshire Primary School*. Shrewsbury: Shropshire Education Services.
- Tall, D., Thomas, M., Davis, G., Gray, E. & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behaviour*, 18(2), 223–241.

# A THEORETICAL MODEL FOR UNDERSTANDING FRACTIONS AT ELEMENTARY SCHOOL

Aristoklis A. Nicolaou and Demetra Pitta-Pantazi

University of Cyprus

*In the present study we propose a theoretical model for understanding a concept of elementary school mathematics. We selected fractions for examining our model. Our theoretical model consists of six factors: inductive reasoning, explanations, justifications, conception for the magnitude of fractions, representations and connections with other concepts. The fit of the model was tested in a sample of 344 fifth and sixth grade elementary school students using confirmatory factor analysis and it was found to have very good fit with the empirical data. The results suggest that the proposed model is much comprehensive of the factors suggested for understanding fractions at the elementary school.*

Keywords: theoretical model, understanding fractions.

## INTRODUCTION

Various theoretical models have been proposed for the learning and understanding of fractions (Kieren, 1976; Behr, Lesh, Post, & Silver, 1983; Lamon, 1999; Pirie, 1994; Mack, 1990). Kieren (1976) proposed a model with five interrelated subconstructs of fractions: part-whole, ratio, operator, quotient and measure. Later on Behr et al. (1983) further developed Kieren's ideas, proposing a theoretical model linking the different interpretations of fractions to operations on fractions, fraction equivalence and problem solving. Lamon (1999) also studied fraction understanding from the perspective of fraction subconstructs. Pirie and Kieren (1994) proposed a dynamic model with levels of understanding and described key features of the model they proposed. They then used the concept of fractions to justify and explain the levels they proposed. Mack (1990) in a similar manner proposed a "more general" model which stresses the importance of students' informal knowledge towards understanding mathematical concepts, and she then examined how students' informal knowledge can be used to give meaning to fractions symbols and procedures. What we attempt in this study resembles more to the work of Pirie and Kieren (1994) and Mack (1990), i.e. we first developed a model of what constitutes mathematical understanding at the elementary school and we then examined the proposed model specifically for the concept of fractions. For the purpose of this study, we consider the term "understanding" to include conceptual understanding as well as procedural understanding.

What this study *adds* to the existing research on fraction understanding is that it seeks of those *crucial factors* that constitute understanding of fractions. By the term crucial factors we refer to those competencies that students must possess in order to understand fractions. At the same time, we encompass in the proposed model a sufficient number of factors so that they explain *as much proportion* of mathematical

understanding as possible. It must be noted that the proposed model lays in the cognitive, epistemological and semiotics domains, therefore we seek of crucial factors in the context of those domains. Of course we admit that it is impossible to include all the factors that are required for mathematical understanding. The added value of the proposed model is also situated in that it encompasses factors of mathematical understanding that were considered important by the mathematics education research and were included in national curricula (e.g. NCTM, 2000). In the section that follows the rationale for formulating the proposed model and the selection of the specific factors is explained in more detail.

## **THEORETICAL BACKGROUND AND AIMS**

### **The proposed theoretical model**

We consider that the factors that constitute understanding of fractions are: inductive reasoning, definitions and mathematical explanations, argumentation and justification, students' conception about the magnitude of fractions, representations and connections of fractions with other concepts. There are two reasons for selecting those factors. The first reason is that researchers in the field of mathematics education stress the importance of factors such as inductive reasoning (de Koning, Hamers, Sijtsma, & Vermeer, 2002), mathematical explanations (Levenson, Tsamir, & Tirosh, 2007; Niemi, 1996), argumentation and justification (Duval, 1992/1993), conception about the magnitude of fractions (Clarke & Roche, 2009), representations (Newstead & Murray, 1998; Niemi, 1996; Gagatsis, Michaelidou, & Shiakalli, 2001; Lesh, Post, & Behr, 1987) and connections of fractions with other concepts (Sweeny & Quinn, 2000; Oppenheimer & Hunting, 1999) for fraction understanding. The second reason is that national curricula (e.g. NCTM, 2000) refer to processes such as reasoning, communication, representations, and connections with other concepts that students should acquire in order to understand mathematical concepts. It should also be noted that all the aforementioned factors refer to fraction understanding both conceptually and procedurally. For example, inductive reasoning refers to processes (finding similarity, dissimilarity and integration attributes) that students need to apply (procedural knowledge), but at the same time, students should know why they apply those processes (conceptual knowledge), and these processes should be meaningful to students (Hiebert & Lefevre, 1986).

In the space below, we provide description of each factor with special reference to its importance for fraction understanding.

#### *1. Inductive reasoning*

Inductive reasoning is defined as the process that permits the extraction of general conclusions or rules from specific cases (Demetriou, Doise, & van Lieshout, 1988). Consequently, via inductive reasoning generalization can take place, the role of which is essential for understanding mathematics and the world. Especially for

understanding fractions at the elementary school level, where students' thought is at the concrete level and they use a plethora of manipulatives and materials, inductive reasoning is essential, since it permits the extraction of general rules and conclusions from specific examples. For example, the equal partitioning of a piece of chocolate, of a surface or of a group of objects can lead to the identification of the concept of fraction (de Koning, et al., 2002).

## *2. Definitions and mathematical explanations*

The ability of communication and use of language is very important according to NCTM Standards (NCTM, 2000) for understanding fractions. A lot of researchers have stressed the importance of definitions and mathematical explanations for learning and understanding mathematical concepts (Levenson, et al., 2007; Niemi, 1996). Literature review did not reveal any information as regards definitions of fractions by elementary school students. We will consider for the purpose of the present study that elementary school students can "define", but by defining we do not mean the formal definition that is required by elder students. We will consider that students can define fractions if they can express in their own words what is the meaning of the fraction, e.g. what a fraction is. Students can define either verbally or with the use of drawings, symbols and diagrams. Also, apart from those tasks that require definitions (e.g. what is a fraction), students might use more than one ways to explain other issues regarding fractions, e.g. when two fractions are equivalent.

## *3. Argumentation and justification*

Reasoning and proof are considered important factors of what constitutes understanding mathematical concepts (NCTM, 2000). According to NCTM (2000), elementary school students should be able to develop and evaluate mathematical arguments and proofs and select various types of reasoning and methods of proof. Since formal proof cannot be the case for elementary school mathematics, argumentation and justification could "substitute" what we call formal proof in the upper level of education.

Argumentation can be defined as students' ability to recognize the truth or the falsehood of a mathematical statement (Duval, 1992/1993). At the same time, students have to justify their answer. Argumentation and justification are very important because they can reveal students' conceptions about fractions, their knowledge of fractions and their errors. For example, an argument referring to what happens to the size of a fraction when increasing or decreasing the numerator and the denominator could serve as an indicator of students' understanding of fractions. Students' answers in this case will show if they understand that a fraction is a relation and not see the nominator or the denominator as two different numbers having no connection between them. We will consider that for the purpose of the present study students can justify their answer by providing numerical examples or by means of a "more general rule". More sophisticated levels of justification would provide an insight into students' understanding of fractions.

#### *4. Conception for the magnitude of fractions*

Students' sense for the magnitude of fractions is crucial for understanding, since in the case a student cannot perceive that  $\frac{1}{4}$  is smaller than  $\frac{1}{3}$ , then he/she probably does not understand the meaning of these fractional numbers and fractions in general. It is very common for some students to consider the nominator and the denominator of a fraction as two different numbers that do not constitute a unique entity, i.e. the fraction. Conception about the magnitude of fractions is essential for comparing and ordering. According to Clarke and Roche (2009), a number of researchers have highlighted the importance of students being able to give meaning to the size of a fraction and the many difficulties associated with doing so.

#### *5. Representations*

Representations are very important for understanding the concept of fraction (Lesh et al., 1987; Newstead & Murray, 1998; Gagatsis, et al., 2001). In the context of teaching fractions, children come across a great variety of representations. Further to the recognition and flexible use of various representational systems, a basic goal of teaching and learning fractions should be the development of ability to translate from one form of representation to another (Lesh, et al., 1987). Gagatsis et al. (2001) claimed that the ability to shift from one kind of representation to another is especially important for fraction understanding.

For the purpose of developing the proposed model, we consider that a student understands the concept of fraction if he/she is able to translate to iconic, symbolic and verbal representation and if he/she is able to construct drawings for fractions.

#### *6. Connections of fractions with other concepts*

Students face serious difficulties in connecting the various forms of rational numbers (Sweeny & Quinn, 2000). It is argued that students' ability to convert from one kind of rational number to the other is an indicator of understanding rational numbers (Oppenheimer & Hunting, 1999). Moreover, it seems that students' ability to see fractions as division of the numerator by the denominator is an indicator of understanding fractions (Newstead & Murray, 1998). Newstead and Murray (1998) have reported students' difficulties in doing so, thus their difficulties in understanding fractions.

For the purpose of this study, we consider a student to be adequate in connections, if he/she is able to link the concept of fraction with the concepts of decimal numbers, percentage and division of integers (division of the numerator by the denominator).

#### **Aim**

The aim of this study was to develop and empirically test the theoretical model for understanding a concept of elementary school mathematics described in the previous section. The concept of fractions was selected for testing the model in a sample of students at the upper level of elementary education (fifth and sixth grade students).

## METHODOLOGY

A test was developed for measuring the six factors considered to constitute understanding of fractions. The test was broken into two parts, since 65 tasks were needed for measuring all the factors and students would need a large amount of time to solve all the tasks at one administration. All students solved the two parts of the test. In Table 1 below, we give an example of an item for each of the six factors.

Tasks 1-7 (task 4 had two sub-tasks 4a, 4b) were used to measure inductive reasoning and were similar to the tasks developed by Christou and Papageorgiou (2007) for finding similarity, dissimilarity and integration attributes and relations. Tasks 8-13 were used to measure definitions/mathematical explanations and some of them were used by Niemi (1996) for measuring explanations as regards the concept of fraction. In tasks 14-20 statements about fractions were presented to students and they had to judge them as right or wrong and explain their way of thinking. We considered that in this way students would provide an argument about their choice and justify their choice to judge the statement as right or wrong. Some of the tasks for argumentation/justification were similar to tasks proposed by Lamon (1999) while discussing reasoning with fractions and some other tasks were used by Niemi (1996). Tasks 21a-21f and task 22 were used to measure conception about the magnitude of fractions. Tasks 21a-21f referred to fraction comparison, whereas in task 22 students had to put four fractions in the right order starting from the smallest one. Similar tasks were used by Clarke and Roche (2009) for comparing and ordering fractions. Tasks 23, 25 and 26 were previously used by Niemi (1996) and involved recognizing fractions in iconic form. Tasks 24, 30, 32 and 33 referred to writing problems that have a fraction as an answer, from an equation or on the basis of a drawing (translating to verbal representation). Tasks 27, 31, 35 and 37 asked students to construct their own drawings to show a fraction, for an equation of adding fractions and for two problems involving fractions. Tasks 28a-28f asked students to select the right fraction that could be represented by pictures (translation to symbolic form, similar tasks were used by Niemi, 1996). In tasks 29a-29c number lines were presented to students and they had to select the right fraction for each. In tasks 34 and 36 students had to solve two problems of addition and multiplication of fractions and they had to write the equation for each. Finally, in tasks 38a-38f students were asked to convert fractions to decimals, in tasks 39a-39f they had to convert fractions to percentages, while tasks 40a-40c and 41 were about the relation of fractions with the division of integers.

The test was administered to 344 fifth and sixth grade students (119 fifth grade and 225 sixth grade) from 11 different schools in Cyprus (both urban and rural areas were represented). The time period for administration was the end of the school year, so that both fifth and sixth grade students had covered the notions included in the model. The test was administered by the classroom teachers who were requested to provide no further clarification and ask students to work on their own.



**Table 1: Examples of items for each of the six factors**

<b>Inductive reasoning</b>	<p>One of the following fractions differs from the others. Find that fraction and circle it.</p> $\frac{2}{7} \quad \frac{3}{2} \quad \frac{14}{49} \quad \frac{10}{35} \quad \frac{4}{14}$		
<b>Definitions/mathematical explanations</b>	<p>Imagine that your teacher asked you to explain to one of your classmates what a fraction is. Use as many different ways you can.</p>		
<b>Argumentation/Justification</b>	<p>If I double both the numerator and the denominator of a fraction, then the formed fraction has twice value compared to the initial one.</p> <p>Justify your answer: .</p> <table border="1" style="float: right;"> <tr> <td style="padding: 2px 10px;">T</td> <td style="padding: 2px 10px;">F</td> </tr> </table> <p>.....</p>	T	F
T	F		
<b>Conception about the magnitude of fractions</b>	<p>Put the fractions <math>\frac{1}{2}</math>, <math>\frac{4}{3}</math>, <math>\frac{2}{3}</math>, <math>\frac{1}{4}</math> in order starting from the smallest one.</p>		
<b>Representations</b>	<p>Write a problem that could be solved by the equation <math>+\frac{1}{2} n \frac{1}{4}</math></p>		
<b>Connections of fractions with other concepts</b>	<p>Convert the following fractions to decimals.</p> <p>a) <math>\frac{1}{4} =</math>      b) <math>\frac{2}{5} =</math>      c) <math>\frac{3}{10} =</math>      d) <math>\frac{1}{20} =</math></p>		

The answers were coded on the basis of a coding scheme. Some tasks were marked either with 0 or 1 (categorical variables), whereas in other tasks the score could vary from 0 to 1.

For testing the fit of the proposed model, confirmatory factor analysis using MPLUS software was used with WLSMV estimator, since this kind of estimator is the most appropriate for categorical variables (Muthén & Muthén, 2004). More than one fit indices were used to evaluate the extent to which the data fit the theoretical model under investigation. More specifically, the fit indices and their optimal values were: (a) the ratio of chi-square to its degrees of freedom, which should be less than 1.96, since a significant chi-square indicates lack of satisfactory model fit, (b) the Comparative Fit Index (CFI), the values of which should be equal to or larger than 0.90, and (c) the Root Mean Square Error of Approximation (RMSEA), with acceptable values less than or equal to 0.06 (Muthén & Muthén, 2004). Moreover, confirmatory factor analysis was used to examine construct validation and “evaluate

the extent to which particular instruments actually measure one or more latent variables they are supposed to assess” (Marcoulides&Kyriakides, 2010, p. 279).

## RESULTS

After subsequent model tests, the model shown in Figure 1 proved to have very good fit to the data ( $\chi^2=338.478$ ,  $df=198$ ,  $\chi^2/df=1.71$ , CFI = 0.971, and RMSEA = 0.045)

From Figure 1, we verify that the factors we consider to constitute understanding of fractions do so very well (the fit indices are very good). Three of the factors and more specifically inductive reasoning, definitions/mathematical explanations and argumentation/justification seem to constitute a second order factor which contributes to understanding fractions. We will call this second order factor as “reasoning and informal proof” at the elementary level. Figure 1 also confirms that verbal, symbolic, iconic representations and students’ ability to construct their own drawings for fractions constitute the factor “representations” which in turn constitutes understanding fractions. In the same manner, connections of fractions with decimals, percentages and division constitute the factor “connections of fractions with other concepts” which also constitutes understanding of fractions.

**Figure 1: The proposed model for understanding fractions**

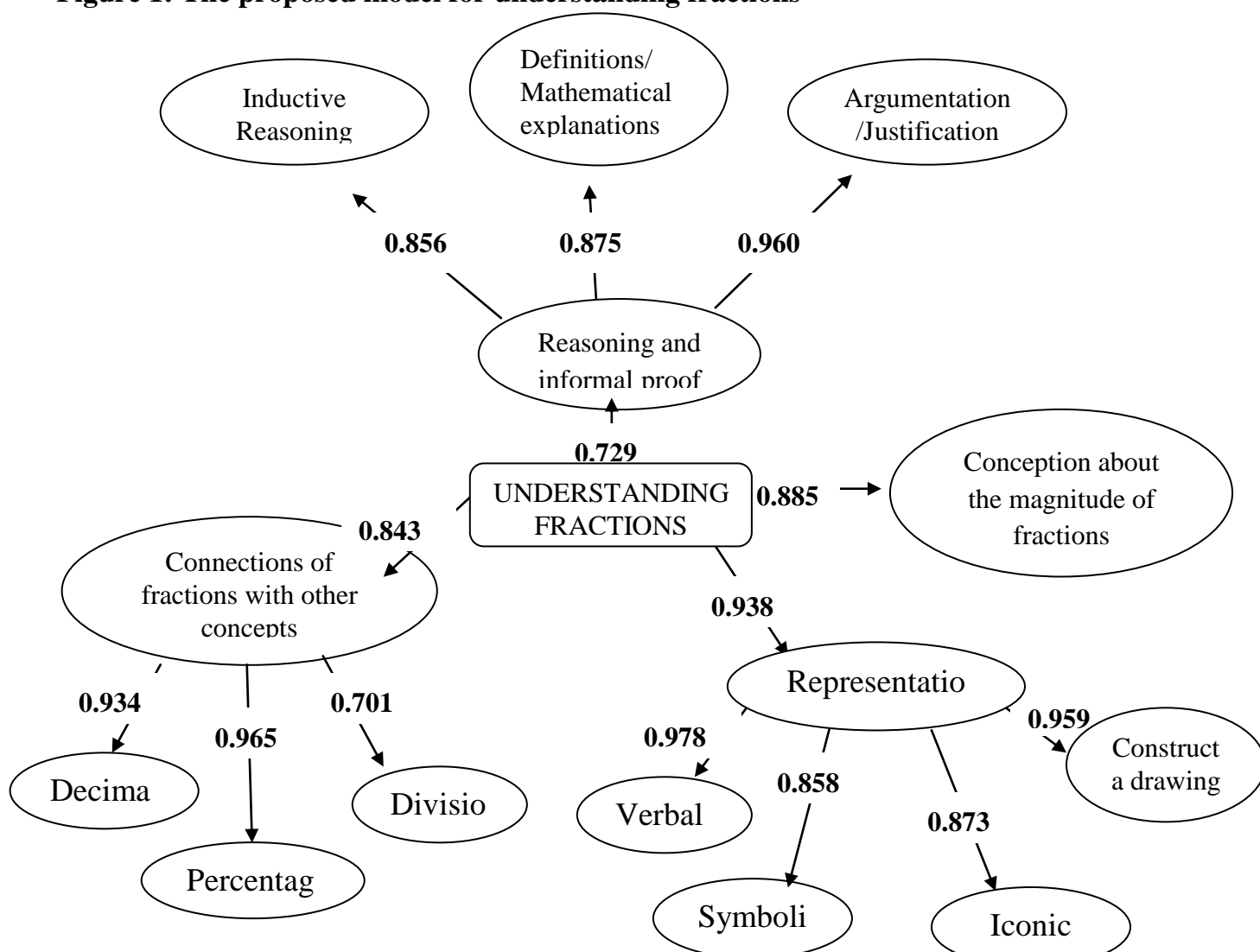


Figure 1 shows that representations have the highest contribution towards understanding fractions, followed by the conception about the magnitude of fractions, connections of fractions with other concepts and “reasoning and informal proof”. The coefficients could serve as an indicator of the importance of each factor for understanding fractions. Moreover, the square of the coefficient shows the percentage of variance of fraction understanding that is explained by the corresponding factor. For example, representations account for about 88% of the variance of fraction understanding. As regards the validity of the instrument used in the present study, this is ensured by the results of the confirmatory factor analysis, since all fit indices had optimal values ( $\chi^2/df=1.71<1.96$ , CFI = 0.971>0.9, and RMSEA = 0.045<0.06). The optimal values of the fit indices provide evidence that the instrument measures the latent variables it is supposed to measure and overall provides a means for assessing fraction understanding (Marcoulides&Kyriakides, 2010).

## DISCUSSION

The results of the statistical analysis confirmed our theoretical model and suggested that the six factors constitute understanding of fractions. Moreover, three of these factors constitute a second order factor which we call “reasoning and informal proof” at the elementary school. Therefore, we claim that for a student to understand fractions he/she should be able to engage to “reasoning and informal proof” as regards fractions, he/she should possess a conception about the magnitude of fractions, he/she must be fluent in representations and the translation from one kind of representation to the other and he/she must be able to connect fractions with other concepts.

The contribution of the factors was high (all loadings were greater than 0.7) showing that all factors had a considerable contribution towards understanding fractions. However, the contribution of representations was the highest (0.938), showing the great importance of this factor for understanding fractions, as other researchers have also claimed (Lamon, 2001; Lesh et al., 1987). The second factor in importance was conception about the magnitude of fractions with also very high loading, followed by connections with other concepts with a little bit less contribution. Therefore, both factors are considered important for understanding fractions. “Reasoning and informal proof” had the lowest contribution among the factors but it was high enough to stress the importance of this factor as well.

The findings of the present study are also important for assessment purposes as we have developed a validated test for assessing fraction understanding. Moreover, the present study provides a way for assessing representations and connections of fractions with other concepts. As shown in Figure 1, translation to verbal, iconic, and symbolic representations and the ability to construct drawings to show fractions comprise representations. In a similar manner, Panaoura, Gagatsis, Deliyianni, and Elia (2009) developed a model for understanding fraction addition. As regards

connections of fractions with other concepts, it can be claimed that for the elementary level, this can be assessed by measuring students' ability to convert fractions to decimals and percentages and by the relation of fractions with the division of integers (division of the numerator by the denominator).

The results indicated that inductive reasoning, definitions/mathematical explanations and argumentation/justification are highly correlated forming a second order factor. The relation of definitions/mathematical explanations and argumentation/justification was expected since terms such as explanations and justification are used interchangeably for reasoning at the elementary school and a relation of the two has been reported from other studies (Niemi, 1996). Explanations and justification can also provide a kind of "informal proof" at the elementary school. The relation of inductive reasoning to the other two factors can be explained by the fact that inductive reasoning is also necessary for reasoning at the elementary school.

The importance of the present study is situated in that it proposes a theoretical framework with factors that constitute understanding a concept of elementary school mathematics. In this study, fractions were selected as the concept under study. The results of the present study support our claim that we have "decomposed" understanding in factors that can be directly measured. Additionally, we believe that we have provided a sufficient number of factors to describe understanding.

Although the sample is high enough, in future the proposed theoretical model could be tested again, enhancing the validity and the reliability of the results. Moreover, from a teaching perspective, an intervention could take place for improving students' ability in the factors found to constitute understanding of fractions. Finally, the proposed theoretical model can be tested for *other concepts* beyond fractions.

## REFERENCES

- Christou, C., & Papageorgiou, E. (2007). A framework of mathematics inductive reasoning. *Learning and Instruction*, 17, 55-66.
- Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, 72 (1), 127-138.
- deKoning, E., Hamers, J. H. M., Sijtsma, K., & Vermeer, A. (2002). Teaching Inductive Reasoning in Primary Education. *Developmental Review*, 22, 211-241.
- Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*, (pp. 91-126). New York: Academic Press.
- Demetriou, A., Doise, W., & van Lieshout, C. F. M. (1988). *Life-Span Developmental Psychology*. England: John Wiley & Sons Ltd.
- Duval, R. (1992/1993). Argumenter, démontrer, expliquer: Continuité ou rupture cognitive? *Petit x*, 31, 37-61.

## Working Group 2

- Gagatsis, A., Michaelidou, E., & Shiakalli, M. (2001). *Representational theories and the learning of mathematics*. Nicosia: ERASMUS IP 1.
- Hiebert, J., & Lefevre, P. (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ, England: Lawrence Erlbaum Associates.
- Kieren, T.E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers, in R. Lesh (Ed.), *Number and Measurement: Papers from a Research Workshop ERIC/SMEAC*, Columbus, OH, pp. 101–144.
- Lamon, S. J. (1999). *Teaching fractions and ratios for understanding*. Lawrence Erlbaum Associates, Mahwah, New Jersey.
- Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Levenson, E., Tsamir, P., & Tirosh, D. (2007). Neither even nor odd: Sixth grade students' dilemmas regarding the parity of zero. *Journal of Mathematical Behavior*, 26, 83–95.
- Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21 (1), 16–32.
- Marcoulides, G. A., & Kyriakides, L. (2010). Structural Equation Modelling techniques. In B. Creemers, L. Kyriakides and P. Sammons (Eds.), *Methodological Advances in Educational Effectiveness Research* (pp. 277–303). Routledge, London and New York.
- Muthén, L., & Muthén, B. (2004). *Mplus User's Guide. Third Edition*. Los Angeles, CA: Muthén & Muthén.
- Niemi, D. (1996). Assessing conceptual understanding in Mathematics: Representations, Problem Solutions, Justifications and Explanations. *Journal of Educational Research*, 89 (6), 351–363.
- Newstead, K., & Murray, H. (1998). Young students' constructions of fraction. *Proceedings of the 22nd Conference of the International Group for Psychology of Mathematics Education*. Stellenbosch, South Africa: University of Stellenbosch.
- Oppenheimer, L., & Hunting, R. P. (1999). Relating fractions & decimals: listening to students talk. *Mathematics Teaching in the Middle School*, 4 (5), 318–321.
- Panaoura, A., Gagatsis, A., Deliyianni, E., & Elia, I. (2009). The structure of students' beliefs about the use of representations and their performance on the learning of fractions. *Educational Psychology*, 29 (6), 713–728.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: how can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26, 165–190.
- Sweeny, E., & Quinn, R. (2000). Concentration: Connecting fractions, decimals and percents. *Mathematics Teaching in the Middle School*, 5 (5), 324–328.

# BETWEEN ALGEBRA AND GEOMETRY: THE DUAL NATURE OF THE NUMBER LINE

**Ildikó Pelczer**

École Polytechnique  
Canada

**Florence Mihaela Singer**

University of Ploiești  
Romania

**Cristian Voica**

University of Bucharest  
Romania

*We analyze the statistical distribution of the answers given by 2<sup>nd</sup> to 10<sup>th</sup> graders to a set of number line problems. To structure our analysis of students' misconceptions, we identified three clusters of problems related to the number line. Our analysis shows that neglecting one of the main features of the number line can be a potential cause for misconceptions. By further exploring the students' mistakes, we found that children ignore either the geometric or the algebraic nature of the number line, making inappropriate decisions within the problem context. The errors in the problems treated within this paper seem to originate from students' lack of understanding of the dual nature of the number line, and are persistent over time.*

**Keywords:** number line, cardinal, distance, origin, direction.

## INTRODUCTION

There is a large body of literature that discusses children's early capacity of learning the numbers (e.g. Griffin & Case, 1997; Karmiloff-Smith, 1992; Singer, 2007). Children have strong primary perceptions related to the number line (Singer & Voica, 2008). Another category of neuroscience studies seems to conclude that the human mind possesses analogical representation of quantities, more precisely: numbers are automatically associated with positions in space, thus supporting the metaphor of a spatially organized internal "number line" (Dehaene, 1992, 1997; Dehaene & Cohen, 1995; Gobel, Walsh & Rushworth, 2001). However, beyond some native predispositions, students of various ages show deep conceptual difficulties in understanding the number line properties.

Formally, the number line concept includes the "empty number line" (see e.g. Beishuizen, 1999), which is filled in by taking into account three essential elements: direction, origin and unit measure. These three elements emerge from the combined nature of the number line: on the one hand, the geometric nature of its representation; on the other hand, its algebraic properties connected with the order relation on  $\mathbf{R}$ .

Ordering and direction have intuitive support from everyday experience. We acquire a basic level of understanding them by polarized descriptions of the type: bigger – smaller; taller – shorter; up – down, etc. In contrast, the origin is a theoretical construct: it is the „nodal point” that allows the construction of a reference system, endowed with a measuring unit conventionally chosen. This allows associating a number („algebraic object”) to a segment („geometrical object”). Thus, by

introducing the distance function, the number line (a geometric representation) is algebrized.

Because the number line is a basic concept, some of its features are neglected in teaching, being considered obvious. However, Booth & Siegler (2006) concluded that individual difficulties in learning the number line are correlated with lower achievement in mathematics. Their result suggests that there might be a long-term impact of these difficulties.

This paper emerged from the following questions: To what extent does the neglect of one of the three elements of the number line determine children's misconceptions in problem solving? Do these misconceptions appear in different grades?

## **METHODOLOGY**

The data reported in this study come from the statistical results of a large contest that takes place every year in Romania. At this contest, of multiple-choice type, there is a yearly participation rate of around 250,000 students from grades 2 to 12, in total.

The tests in this competition consist of a range of 24 to 40 problems, depending on the school grade. The tests cover the whole school curriculum and present students with a considerable variety of problems. In order to limit guessing, wrong answers are penalized. In this way, children who do not know the correct answer to a question are in advantage if they do not answer that question, since the no-answer choice is not scored at all. For this reason, we consider that the statistical analysis of the answers can be highly informative about children's reasoning during the test.

For our study, we have chosen problems related to the number line that have (relatively) similar text or are based on similar solving mechanisms, and grouped them in clusters. Given the focus of our first question, the clusters have been built from mathematical considerations. We thus identified three clusters of number line problems and we looked at the distribution of the students' answers to the distracters of these problems.

In the present paper, we analyze the statistical distribution of the answers given by 2<sup>nd</sup> to 10<sup>th</sup> graders to a set of number line problems. We have chosen this vertical extension of the analysis in order to see changes in misconceptions as function of age. This vertical analysis helps us to see if the errors are influenced by a certain instructional focus. Although the curriculum recommends the frequent use of the number line, it is not a common practice in the Romanian classrooms in primary grades. The classical use of the empty number line happens in grade 4, when a diagram segment is used for solving some word problems (that usually leads to the use of equations). Frequently, this is learned as an algorithm. The formal number line is used by the math teachers starting with the 5<sup>th</sup> grade.

## RESULTS

Our primary interest was to understand the mathematical sources of the students' mistakes. We looked especially at the problems in which the errors related to the properties of the number line (eventually cumulated) were higher than the percentage of correct answers. In the next sections of the paper, we focus on these problems to structure our analysis of students' misconceptions.

### The “cardinal cluster”

The first cluster contains problems that request the finding of the cardinal of an ordered set; we call it the cardinal cluster.

*Problem 1.1.* (Grade 2, 38,917 respondents): From my story book some pages are missing. Where pages are missing, I see number 12 on the left page and number 15 on the right page. How many pages are missing?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

*Problem 1.2.* (Grades 3-4, 124,641 respondents): Ana found an old book from which some pages are missing. At the place where the pages were missing, there was the number 24 on the left page; while, on the right was 45. How many pages were missing?

(A) 9; (B) 10; (C) 19; (D) 20; (E) 21.

*Problem 1.3.* (Grades 5-6, 42,439 respondents): How many integers are there between 19.03 and 2.009?

(A) 0; (B) 17; (C) -17; (D) 19; (E) 17,021.

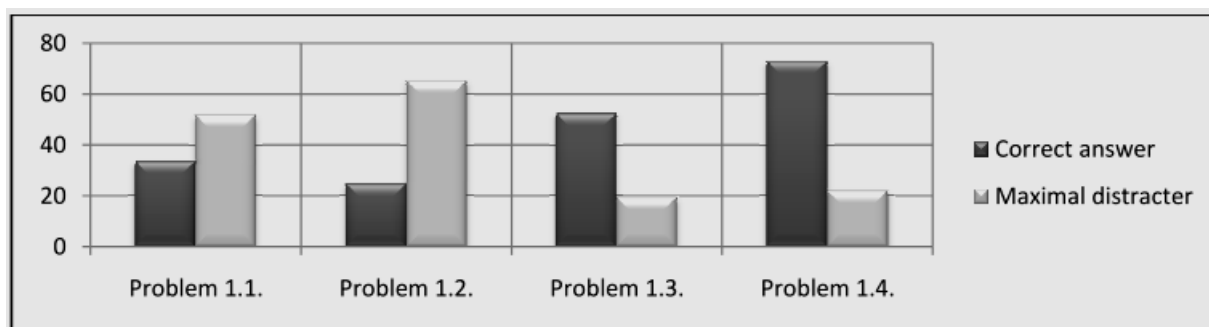
*Problem 1.4.* (Grades 7-8, 26,827 respondents): Harry brings the mail on Long Street. He has to distribute a letter to each house having an odd number. The first house has number 15 and the last 53. To how many houses did Harry give the mail?

(A) 19; (B) 20; (C) 38; (D) 39; (E) 53.

The problems from this cluster are alike since they ask to find out the length of a finite arithmetic progression of ratio 1 (problems 1.1, 1.2, 1.3), or of ratio 2 (problem 1.4). We noticed that, for each of these problems, the most frequent chosen answer refers to the computation of a length. In problem 1.1, answer (C) comes up as the difference between 15 and 12; similarly in problem 1.2, the wrong answer with the highest percentage is the one obtained from  $45 - 24$ . In the same way, in problem 1.3, the distracter with the highest percentage has been generated by performing a subtraction. A special discussion is worth for problem 1.4. While answer C is obtained by computing  $53 - 15$ , some of the students observed that the ratio is 2 and, therefore, they divided the result, thus obtaining answer A. Consequently, the answer A is also based on a reasoning that relies on the difference between two numbers. For this reason, we refer to the total of A and C choices as belonging to the same type of mistake. The statistical results are presented in Diagram 1.



## Working Group 2



**Diagram 1. Correct answers and maximal distracter for the cardinal cluster**

Because students showed an obvious preference for the distracters based on subtraction, we wondered in what situations these subtractions are yet relevant to the problem solving process. In general, such operation leads to the determination of the distance between two points situated on an axis (that is, when the points are identified by their coordinates on the axis). Thus, for Cluster 1 problems, the unit measure becomes important: most of the students do not realize that they should work with a „discrete” unit (the number of integers) instead of a „continuous” measure (distance).

Consequently, we interpret these errors as due to the confusion children make between *cardinal* and *distance*. By further exploring the nature of this error, we concluded that the children’s mistake consists in adopting a *geometrical* solution (emphasized by the distance between the points) for a situation in which the answer is *algebraic* (based on “discrete” counting).

### The “distance cluster”

To go deeply into this hypothesis, we focused on the second cluster, which we called the distance cluster. This second cluster contains problems in which one needs to find the distance between two points.

*Problem 2.1.* (Grades 3-4, 77,294 respondents): On the right side of an alley there are 9 street lamps. The distance between two neighbour street lamps is 8 m. John ran on the alley, from the first until the last street lamp. How many meters did John run?

(A) 48 m; (B) 56 m; (C) 64 m; (D) 72 m; (E) 80 m.

*Problem 2.2.* (Grades 5-6, 50,024 respondents): There are nine bus-stations, at equal distances, on a bus line. The distance between the first and the third bus-station is 600 m. What is the distance between the first and the last station?

(A) 1200 m; (B) 1500 m; (C) 1800 m; (D) 2400 m; (E) 2700 m.

*Problem 2.3.* (Grades 9-10, 19,327 respondents): The supermarket chariots are arranged in two lines. On the first line, 2.9 m long, there are 10 chariots, while on the second line, which is 4.9 m long, there are 20 chariots. What is the length of one chariot?

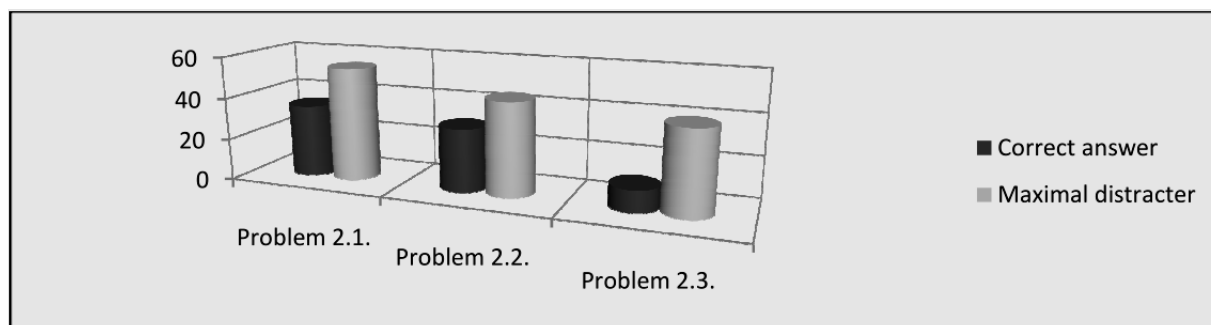


## Working Group 2

(A) 0,26 m; (B) 0,9 m; (C) 1 m; (D) 1,1 m; (E) 1,4 m

(At the contest, the problem 2.3. had an illustration of the type seen above.)

The statistical results to the three above problems are presented in Diagram 2.

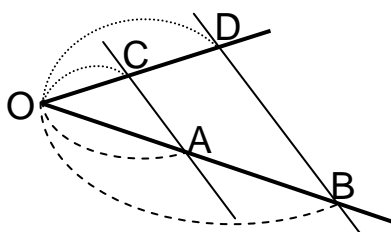


**Diagram 2. Correct answers and maximal distracters for the distance cluster**

The problems of this cluster are similar, each of them requiring the computation of a distance between two points of the number line. We can observe that in each of these cases, the most frequent error refers to the extrapolation of a proportionality relation between the discrete information (the ordinal of the „marking points” – street lamp/ bus stop) and data that are of a continuous nature (distance).

Because the proportionality-based reasoning was frequent in the sample population, we wondered in what situations this reasoning is certainly correct. The typical example for such situation would be the Thales' theorem, where one can transfer the ratio of segments situated on a line to segments situated on another line in conditions of homogeneity, meaning: the same unit of measure and, especially, the same origin (see figure 1).

$$\frac{OA}{OB} = \frac{OC}{OD}$$



**Fig. 1: A representation for the Thales' theorem**

Obviously, in the cases of the analyzed problems, we work with different origins. Why the origin is so important? When computing the length of a segment situated on the number line, we choose an origin (usually one of the end of the segment) corresponding to 0. Therefore, when we position one of the segment's endpoint over 0, we perform a translation, which can be expressed through a single algebraic operation (i.e. subtraction). On the other hand, in order to find the cardinal of a (finite) set, we count its elements starting from 1; in other words, we chose the origin in 1. This positioning has no algebraic correspondent: if the reference points (initial and final) of the interval are  $a$  and  $b$ , then to position  $a$  in 1 means to change the coordinates of  $b$  into  $b - a + 1$ . Consequently, here we have two translations (first we

take  $a$  to 0, then to 1), but children have difficulties in understanding how this succession of translations could be performed. The analysis of the second cluster shows that neglecting an essential element of the number line – namely the adequate positioning of the origin – can be a potential cause for misconceptions in algebra.

### The “directions cluster”

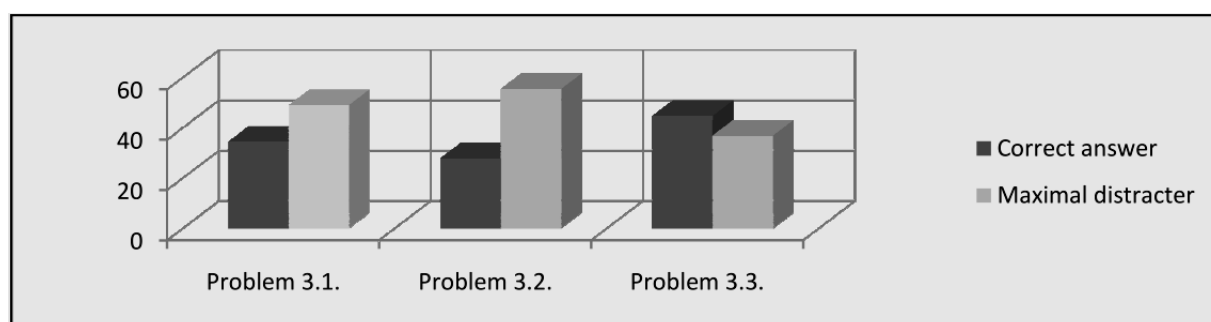
What happens if children ignore another essential element of the axis? We look at the next cluster in order to find an answer. The third cluster was called the directions cluster given that the selected problems in this category require to handle two origins and, consequently, different directions for counting.

*Problem 3.1.* (Grades 3-4, 63,059 respondents): Radu and Maria went on a trip by train. Radu got his place in the 17th car from the beginning of the train; while, Maria got hers in the 14th car from the end of the train. They were surprised to see that they travel in the same car. How many cars did the train have?  
(A) 28; (B) 29; (C) 30; (D) 31; (E) 32.

*Problem 3.2.* (Grades 5-6, 36,293 respondents): A staircase has 21 stairs. Mike and Nick count the stairs, the first from the bottom and the second from the top. They meet on the step that is the 10th for Mike. What is the number of this stair in Nick’s counting?  
(A) 21; (B) 31; (C) 11; (D) 12; (E) 10.

*Problem 3.3.* (Grades 9-10, 16,692 respondents): Sally was in the 50th position on the list of the Kangaroo contest results when counting both in increasing and decreasing order. How many children participated in the contest?  
(A) 50; (B) 75; (C) 99; (D) 100; (E) 101.

The statistical results of the three above problems are presented in Diagram 3.



**Diagram 3. Correct answers and maximal distracters for the directions cluster**

These problems involve the relationship between the cardinality of a set and one of its subsets. From this perspective, the second problem is slightly different since we have the cardinality specified here and the task requests finding the number of elements when counting from two different origins. The typical error is given by performing a simple addition (or subtraction for the second problem) and it is linked to the idea that cardinality of a set can be obtained by summing up cardinalities of subsets. This type of reasoning is correct only for disjoint sets and children seem to

forget that. Nevertheless, at a deeper level, their error is linked, once again, to the confusion between distance and the cardinal of the set of the marking points, thus to the geometrical and algebraic nature of the number line. When children sum up or subtract numbers, they work with distances on the number line, since the number is represented by its distance from the origin. But, in the above problems the questions focus on the marking points (in this sense, these problems are the opposite of the ones from Cluster 2) and play on the fact that one of the marking points is common to the two “countings”, therefore it does not define a new distance.

Most of the above tasks are of a special kind, which for many students can be understood as a trap. However, the pattern of wrong answers shows that the “trap” is not enough to explain the error. It is possible to consider that the trap invites the mistake, but actually the mistake is made by a misconception.

We conclude that, in general, the above problems cause difficulties to children because they request to see the dual nature of the number line, which is not explicitly highlighted by the information given in the texts. For example, in Cluster 2, the information is given by using “marking points”, which induces a mental image based on the geometrical interpretation of the line, but the question focuses on distances, which add an algebraic dimension. For the correct solution, one should be able to make a transfer, but for most of the children this is not evident. Therefore, they “continue” to reason in the frame that is induced by the problem text. We can see the same pattern of answers in Cluster 3: briefly, the source of errors consists in operating with distances and not with cardinals, therefore, in not being able to switch from algebraic to geometric properties.

## NEW RELEVANT CASES

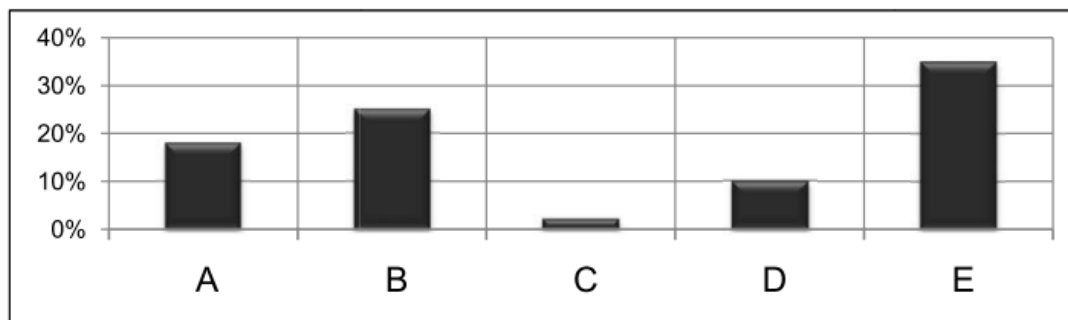
In order to get confirmation of the above assertions, we looked at two more problems where this switch between algebra and geometry is not necessary for the successful solution. The most common case for this is when an early algebraic transcription is possible or the problem needs to be treated algebraically. If the cause for the errors in these problems is geometry-algebra analogical transfer, then in problems that are similar to the ones discussed above, but, for some particular reason, do not require the interplay between algebra and geometry, we should have a different pattern of answers. We will analyze two such problems. The first discussed problem involves both distance and the cardinality of a set.

*Problem 4.* (Grades 7-8; 29,288 respondents): On both sides of a 20 m long alley, I planted roses. The distance between two neighbouring roses is 2 m long. How many roses did I plant?

(A) 22; (B) 20; (C) 12; (D) 11; (E) 10.

This problem is a „geometrical version” of problem 2.1. In both cases, we have a movement of length 2 and the computation of a cardinal. In opposition to the

problem 2.1, in which 34.80 % of students answered correctly, the statistical results to this problem are the ones presented in Diagram 4.



**Diagram 4. Statistical results to Problem 4. (The correct answer is (A). The remaining percents correspond to non-answers.)**

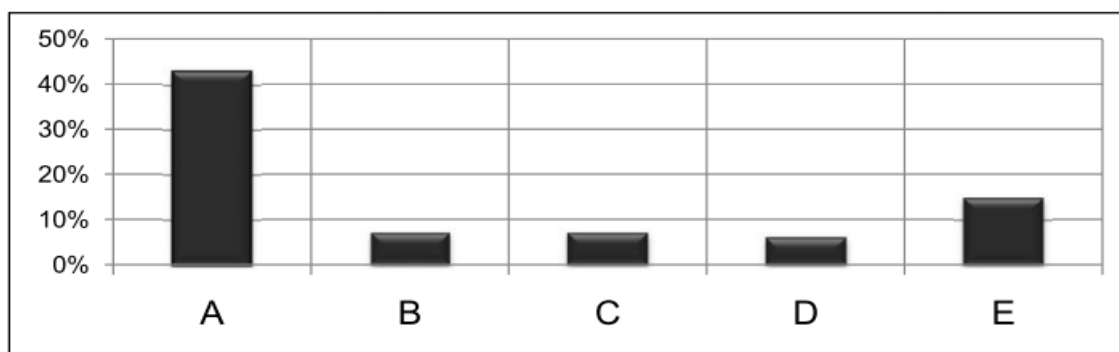
We can interpret answer D as „half correct”, with the mistake coming from neglecting a constraint of the problem’s text (that the roses are on the both sides of the alley). This is why we consider that the students who had chosen the answers (A) or (D) made a correct judgment. Similarly to the problems discussed above, we group answers B and E as belonging to the same category: here the errors come from the duality marking points – segment (distance). We can observe that we have a strong polarization here: almost 28% correct answers versus 60% typical error. We consider that the range of children’s answers to this problem can be explained by their different level of algebraic reasoning. More precisely, once the student manages to transfer from geometry to algebra, he/she can make use of both the number sets properties and an algorithmic thinking to solve the task. In order to check this hypothesis, we looked at the answers to the next problem.

*Problem 5.* (Grades 9-10; 17,620 respondents): 2 009 participants in a cross country race have arrived at the finish line. The number of persons behind John is three times bigger than the number of persons in front of John. In what position did John finish the race?  
(A) 503; (B) 2009; (C) 1005; (D) 4; (E) 1507.

The statistical results to this problem are presented in Diagram 5.

The problem 5 is similar to problem 3.3. However, while for problem 3.3 the main distracter was chosen by 36.81 % of the respondents, here the maximal distracter (E) was chosen by only 14.98 % of the students. The percent of correct answers is relatively similar. What is the source of these differences? In the case of problem 5, one needs to transcribe the information into equations and the solution does not require switching between algebra and geometry. Once the equation is written, we can proceed in a purely, algebraic way. This can explain the differences in results.

## Working Group 2



**Diagram 5. Statistical results to Problem 5. (The correct answer is (A). The remaining percents correspond to non-answers.)**

## CONCLUSIONS

As a support-representation for the real numbers, the number line has a dual conceptual structure that consists of a geometrical component that allows visualization and an algebraic component, given by the introduction of distance. Consequently, in order to understand the number line with all its properties, we need a bi-directional transfer between geometric and algebraic viewpoints. Children seem tempted, even from the very beginning, to make this shift in an involuntary manner for problems that ask for length/ continuous measure (so, in which the origin “is situated” at 0) and cardinal/ discrete measure (in which the origin “is situated” at 1). The change of the origin seems to be more natural in the first case, given that there is a corresponding algebraic operation (subtraction). In the second case we have the composition of two translations; therefore, the algebraic correspondent is more complex. The problems in the third cluster belong to the first case mentioned here, but with an additional complexity related to the fact that the final results need to be „reinterpreted” through a switch between algebra and geometry. In conclusion, the answer to the first question of our study: “To what extent does the neglect of one of the three elements of the number line determine children’s misconceptions in problem solving?” becomes more complex. It seems that the problem is not only in neglecting one of the elements of the number line, but also in the quality of the connections made between various representations (algebraic or geometrical) of some abstract concepts.

We have seen that the errors in the problems treated above originate from the duality of the number line. Similar errors appear in different grades. One reason for this persistence over time can be the fact that school curriculum does not explicitly focus on the number line duality. In younger grades, children work, mostly, with the geometric interpretation of the number line, just using an intuitive algebraic structure. Once the algebraic structure is formally introduced, students are not

reminded of the geometric aspects, and the structure remains, most often, in the realm of abstraction, lacking an intuitive interpretation.

This study is based on a statistical analysis. It allowed us to identify students' difficulties in solving number line problems and to launch some research hypotheses. Although these hypotheses were statistically validated, a qualitative study focused on these aspects could reveal more detail of students' thinking.

### **Acknowledgments.**

The Support received by the third author from the European Social Fund grant POSDRU 17/1.1/G/37412 is gratefully acknowledged.

### **REFERENCES**

- Beishuizen, M. (1999). The empty number line as a new model. In Thompson I. (ed.), *Issues in Teaching Numeracy in Primary Schools*. Buckingham: Open University Press.
- Booth, J.L. & Siegler, R.S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology*, 41, 189–201.
- Dehaene, S. (1992). Varieties of numerical abilities. *Cognition*, 44, 1–42.
- Dehaene, S. (1997). *The Number Sense: How the Mind Creates Mathematics*. Oxford Univ. Press.
- Dehaene, S. & Cohen, L. (1995). Towards an anatomical and functional model of number processing. *Mathematical Cognition*, 1, 83–120.
- Gobel, S., Walsh, V. & Rushworth, M.F.S. (2001). The mental number line and the human angular gyrus. *Neuroimage*, 14 (6), 1278–1289.
- Griffin, S. & Case, R. (1997). Rethinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education*, 3(1), 1–49.
- Karmiloff-Smith, A. (1992). *Beyond modularity: A developmental perspective on cognitive science*. Cambridge, Ma: MIT Press.
- Singer, F.M. (2007). Beyond Conceptual Change: Using Representations to Integrate Domain-Specific Structural Models in Learning Mathematics. *Mind, Brain, and Education*, 1(2), 84–97.
- Singer, F.M. & Voica, C. (2008). Between perception and intuition: thinking about infinity. *Journal of Mathematical Behavior*, 27(3), 188–205.

# **SPECIAL EDUCATION STUDENTS' ABILITY IN SOLVING SUBTRACTION PROBLEMS UP TO 100 BY ADDITION**

Marjolijn Peltenburg & Marja van den Heuvel-Panhuizen

Freudenthal Institute for Science and Mathematics Education,

Utrecht University, the Netherlands

*In this study we examined special education students' use of indirect addition for solving two-digit subtraction problems. Fifty-six students (8- to 12-year-olds), with a mathematical level of end Grade 2, did an ICT-based test on subtraction. Although most students had not been taught indirect addition they frequently applied this procedure spontaneously. For about two-thirds of the problems that have an adding-on context and for about half of all the problems with a small difference between the minuend and subtrahend indirect addition was used. The main prompt for using indirect addition were the item characteristics. Indirect addition was identified as a highly successful procedure for special education students and the best predictor of a correct answer was found in combination with a stringing strategy.*

Keywords: special education, indirect addition, information and communication technology (ICT), assessment, empty number line

## **INTRODUCTION**

At the end of primary school many special education (SE) students are considerably behind on the topic of subtraction with numbers up to 100 compared to their peers in regular education (Kraemer, Van der Schoot, & Van Rijn, 2009). To improve the achievements of SE students in solving subtraction problems, it is suggested to teach them one particular way of solving calculations (see e.g., Milo & Ruijsenaars, 2002; National Mathematics Advisory Panel, 2008).

There are several reasons for challenging this advice. Firstly, the idea of teaching only one method goes against the goal of developing numeracy in students. This goal implies that students should be able to choose a suitable strategy when solving number problems (see e.g., Van den Heuvel-Panhuizen, 2001; Warry, Galbraith, Carss, Grice, & Endean, 1992). Secondly, teaching one method implies that for solving particular problems students may have to follow an unnecessary long way to come to an answer (see e.g., Torbeyns, Ghesquière, & Verschaffel, 2009). Thirdly, using prescribed methods can lead to 'didactical ballast' (Van den Heuvel-Panhuizen, 1986) for students. This means that students have to become skilled at following the given recipes, which may not always be easy for them, because the ownership is completely on the side of the teacher or textbook author.

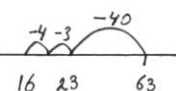
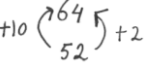
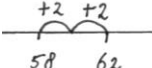
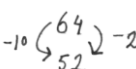
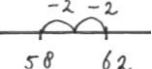
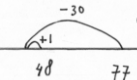
Despite these disadvantages, the idea of teaching students with mathematical difficulties one solution method is still often advocated nowadays. This plea results from the assumption that weak learners do not have the necessary



insights to choose an approach that suits a particular task (see e.g., Milo & Ruijsenaars, 2002; Timmermans & Van Lieshout, 2003). In the study reported in this paper the tenability of this claim is investigated by means of an Information and Communication Technology (ICT)-based assessment. The focus of the study is on using an *addition* procedure for solving *subtraction* problems up to 100.

### Strategies and procedures for solving subtraction problems

For solving addition and subtraction problems with numbers up to 100 generally three different types of strategies can be distinguished: splitting, stringing, and varying (Van den Heuvel-Panhuizen, 2001). These idealized *strategies* of which examples are given in Figure 1 have in common that they describe *how we deal with the numbers involved* (in splitting both numbers are decomposed in tens and ones, in stringing one number is kept as a whole number, and in varying one or both numbers are changed in order to get an easier problem).

		Strategies Number perspective		
		Splitting	Stringing	Varying
Procedures Operation perspective	DS Direct Subtraction	$63-31 =$ $60-30 = 30$ $3-1 = 2$ $30+2 = 32$	$63-47 =$ 	
	IA Indirect addition	$64-52 =$ * 	$62-58 =$ 	
	IS Indirect Subtraction	$64-52 =$ * 	$62-58 =$ 	
	MO Multiple operations			$77-29 =$  or $(77+1)-(29+1) = 78-30 = 48$

\* This way of solving this problem is not very common

**Figure 1. Relation between procedures and strategies illustrated with problems**

A different way of describing a calculation is by focusing on *how the operation is carried out*. From this perspective, two main *procedures* for solving subtraction problems can be distinguished: (1) direct subtraction (DS), which means taking away the subtrahend from the minuend (e.g., solving  $62-58=_$  by  $62-50=12$ ;  $12-2=10$  and

finally  $10-6=4$ ), and (2) indirect addition (IA), which means adding on from the subtrahend until the minuend is reached (e.g., solving  $62-58=_$  by  $58+2=60$  and  $60+2=62$ ). Together the strategies and the procedures offer a complete framework for describing how students solve subtractions up to 100. See Figure 1. DS is likely to go together with splitting or stringing. For IA and IS, stringing is the most obvious strategy, although splitting can be applied as well. Finally, when a varying strategy is applied multiple operations are required.

### **Solving subtraction problems by indirect addition**

In connection to the earlier mentioned chasm between opposing ideas about whether or not to teach one solution method to students who are weak in mathematics, there is also debate on whether SE students are able to flexibly solve subtraction problems up to 100 by applying an IA procedure. For example, a few recent intervention studies have revealed that even students in regular primary education have great difficulty to incorporate IA for solving subtraction problems up to 100 (De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009).

However, these studies are contradicted by other intervention studies that do support the claim that already in the first grades of primary mathematics education, students of different ability levels in mathematics can learn to flexibly solve subtraction problems by applying IA (Blöte, Van der Burg, & Klein, 2001; Menne, 2001).

### **Factors influencing students' procedure use**

Important factors that may influence students' procedure use when solving a subtraction problem are: (1) student characteristics, such as their general mathematics level, (see e.g., Torbeyns, De Smedt et al., 2009), (2) teaching characteristics, for example, whether or not students' have been taught a particular procedure (see e.g., Menne, 2001), and (3) problem characteristics. With respect to the latter, the influences of the following three features of a subtraction problem are discussed: (a) the numbers involved, (b) the problem format (context problems or bare number problems), and (c) the available auxiliary tools.

#### *Influence of the numbers involved*

Several studies (e.g., De Smedt et al., 2010; Torbeyns, De Smedt et al., 2009) have indicated that subtraction problems that require crossing the ten and have a small difference between the minuend and subtrahend may evoke the use of IA. However, IA may also be an efficient procedure in solving large-difference problems with a small difference around the tens and requiring crossing the ten. For example,  $82-29=_$  may be easily solved by IA (i.e.,  $29+1=30$ ;  $30+50=80$  and  $80+2=82$ , so  $1+50+2=53$ ). Finally, research suggested that small-difference problems that do not require crossing the ten (e.g.,  $47-43=_$ ) may also evoke the use of IA (Gravemeijer et al., 1993).

### *Influence of the problem format*

Two didactical phenomenological interpretations of subtraction are: (1) subtraction as taking away, and (2) as determining the difference. In the first interpretation, the matching operation is that of taking away the subtrahend from the minuend, whereas in the second interpretation bridging the difference between the subtrahend and minuend by adding on is also an option. Both interpretations need to be addressed if we want students to learn subtraction in a more complete way (Freudenthal, 1983; Van den Heuvel-Panhuizen & Treffers, 2009).

To contribute to this broad understanding of subtraction students should not only be presented bare number problems. Different studies (e.g., De Smedt et al., 2010; Torbeyns, De Smedt et al., 2009) revealed that bare number problems hardly evoke the use of IA. Context problems, on the contrary, have the possibility to open up both interpretations of subtraction (Van den Heuvel-Panhuizen, 2005).

### *Influence of the auxiliary tools*

To support students in carrying out calculation problems up to 100, different models can be used. Basically, two main models can be distinguished: group models and line models (Van den Heuvel-Panhuizen, 2001). Group models, such as rods of ten and blocks of one, are particularly appropriate to represent a splitting strategy together with DS. Line models, such as the empty number line, are mostly suitable to support a stringing strategy in combination with either DS or IA. The empty number line thus has the possibility to represent both interpretations of subtraction: taking away by jumping backwards and adding on by jumping forwards.

### **The present study**

The present study was set up to investigate whether and under which conditions SE students are able to use IA for solving subtraction problems up to 100, and whether they can solve subtraction problems correctly when applying this procedure. The purpose of the study was to clarify the role of the numbers involved, the format of the problem (context or bare number problems), the presence of a digital empty number line as an optional auxiliary tool, and the occurrence of prior instruction in IA. The study has two foci (I) students' spontaneous use of IA, i.e., applying IA without being asked to use this procedure, and (II) students' success rate when applying IA.

## **METHOD**

### **Participants**

In total, 56 students from fourteen second-grade classes in three Dutch SE schools participated in the study. The participating students (39 boys, 17 girls) were 8 to 12 years old, with a mean age of 10 years and 6 months (SD=10.4 months). All students

had a mathematical ability of level C or lower at the CITO Monitoring Test for Mathematics End Grade 2 (Janssen, Scheltens, & Kraemer, 2005).

## Materials

### *ICT-based test on subtraction problems*

An ICT-based test<sup>1</sup> was developed that contains a collection of items in which item characteristics are varied systematically. These characteristics include number characteristics and format characteristics.

The number characteristics refer to the size of the difference between the minuend and subtrahend (small means  $<7$  and large means  $>11$ ), whether or not the tens have to be crossed (e.g.,  $61-59=_$ ), and whether or not the minuend and the subtrahend are close to the ten ( $<3$ ). The format characteristics refer to whether or not the items are presented as a bare number problem (BN) or as a context problem. The latter can describe a taking-away situation (ConTA) or an adding-on situation (ConAO). Figure 2 shows a screen shot of one of the context problems that reflects an adding on situation.).



41 euro

already 29 euro paid

1234567890

answer:





**Figure 2. Train ticket item; the read aloud instruction is: “A train ticket costs 41 euro. Father has already paid 29 euro. How many more Euros does he need to pay?”**

The ICT-based test is divided into two parts. The first fifteen items do not feature the number line tool, whereas the last fifteen do. This digital empty number line operates by touch-screen technology. After a short introduction, the students worked individually on a touch-screen notebook. Students were told that they were completely free in choosing a particular solution method. As well as giving an

answer they had to report verbally how they solved the items. The students' on-screen work was recorded by means of Camtasia Studio software.

### *Online teacher questionnaire*

To collect data about the students' prior instruction on subtraction problems an online teacher questionnaire was developed. The link for the questionnaire was sent by email to the fourteen teachers responsible for teaching mathematics to the students that participated in the study. All fourteen teachers filled in and submitted the questionnaire. The questionnaire contains two questions on the topic of 'subtraction up to 100' to collect data about (1) the models and materials the teachers have used for teaching subtraction up to 100, and (2) the procedures (DS and/or IA) they have taught their students for solving subtraction problems up to 100.

## **RESULTS**

In the analysis of the data we included all the cases in which the students gave an answer to an item. Of the 1680 possible cases (56 students each doing all thirty items) 147 cases were missing. This resulted in 1533 cases to be analyzed. DS and IA were clearly the most frequently applied procedures. DS was applied in 64% of the total cases and went together almost equally often with a stringing and splitting strategy. IA was applied in 32% of the total cases; in almost 90% IA was applied in combination with a stringing strategy.

### **Different conditions and SE students' spontaneous IA use**

#### *Numbers involved*

IA was most frequently applied in small-difference problems, i.e., in 50% of the 322 cases involving items with crossing the ten and in 43% of the 324 cases involving items without crossing the ten. DS was most frequently applied in large-difference problems, i.e., in 90% of the 282 cases involving items with crossing the ten, in 78% of the 306 cases involving items without crossing the ten, and in 66% of the 299 cases involving items with a small difference around the tens and requiring crossing the ten.

#### *Problem format*

We found that an adding-on context generally goes together with IA, whereas a taking-away context mostly resulted in a DS procedure. That is, IA was applied in 68% of the 509 cases involving an adding-on context and DS was applied in 75% of the 510 cases involving a taking-away context. When solving bare number problems the students also had a strong preference for DS. That is, in 91% of the 514 cases involving bare number problems, DS was

applied.

### *Prior instruction*

The teachers' responses to the online questionnaire revealed that two different textbook series were used in the fourteen classes. Although these textbook series each contain some missing addend problems, they do not explicitly address the inverse relation between addition and subtraction. Because teachers might have given attention to IA without it being addressed in their textbooks we also asked them which procedures for solving subtraction problems they taught their students. Their answers made it clear that all teachers taught DS. Only three teachers responded that they have taught both DS and IA. This means that in total fifteen students were taught both procedures. These students applied IA in 32% of the total 419 cases they had solved; the students who were not taught IA applied this procedure in 32% of the total 1114 solved cases.

### *Use of the empty number line*

According to the data that were retrieved from the online teacher questionnaire all students were familiar with the empty number line for doing subtraction. For fifteen out of the thirty items, the students had the optional digital empty number line for solving the subtraction problems available. The fifteen items resulted in 778 cases of processed items. In 131 of these cases the empty number line was actually used for finding an answer. The students used IA in 15% of these 131 cases. In the 647 cases in which the students saw the empty number line but did not use it, IA was applied in 33% of the cases.

### **Multilevel analysis with IA use as dependent variable**

A cross-classified multilevel model was carried out with IA use as dependent variable. This analysis revealed, among other things, that the random item effect ( $SD=2.41$ ) was quite large compared to the random student effect ( $SD=.85$ ), indicating that IA use is mainly an item characteristic. This means that the application of IA is more strongly elicited by the nature of an item than by the specific preference of a student, which implies that students applied IA in a flexible, item specific way.

### **SE students' success rate in IA and DS use**

In 70% of the 489 cases in which the students applied IA their answers were correct. In the 976 cases in which they applied DS their answers were correct in only 48% of the cases.

### **Different conditions and success rate in IA and DS use**

#### *Numbers involved*

Small-difference subtraction problems were solved with the highest success rate when IA was applied. Of the 162 cases involving items that required crossing the



tens, the students solved 86% correctly with IA and of the 139 cases that did not require crossing the tens, the students solved 88% correctly with IA. When students applied DS, the highest percentage of correct answers was found in small-difference subtraction problems without crossing the ten. Of the 176 cases involving such items, the students solved 67% correctly by DS.

### *Problem format*

In all three problem formats (ConAO, ConTA and BN) the students solved more problems correctly than incorrectly when they applied IA. The highest percentage of correct answers was found in solving items that reflect taking away, i.e., 82% of the 108 cases involving taking-away items were solved correctly. When using DS, we found that in all three problems format the students solved about half of the items involved correctly. When not taking into account the procedure used and comparing the three problem formats, students appeared to be most successful in solving context problems that reflect adding on (ConAO) and least successful in solving bare number problems (BN).

### *Other conditions*

The students who had received IA instruction correctly solved 76% of the 134 total cases they had solved by IA. The students who did not receive IA instruction correctly solved 68% of the 355 total cases they had solved by IA. Moreover, the IA-instructed students correctly solved 54% of the 270 total cases they had solved with DS. The students who did not receive IA instruction correctly solved 45% of the 706 total cases they had solved with DS.

Of the 778 cases in which the empty number line was available in the items IA was applied in 235 cases. In twenty cases the students actually used the number line and in 215 cases they did not. It appeared that the percentage of correct answers in both groups was about the same, namely 75% and 73% respectively.

### **Multilevel analysis with success rate as dependent variable**

A cross-classified multilevel model was carried out with success rate use as dependent variable. This analysis was focused on revealing the influence of strategy use and procedure use on students' success rate in applying IA. It was found that only the use of a stringing strategy increased success rate significantly ( $b=.52$ ,  $SE=.19$ ,  $p<.05$ ). The best predictor of a correct answer appeared to be the combination of a stringing strategy together with the IA procedure ( $b=.96$ ,  $SE=.37$ ,  $p<.05$ ).

## **CONCLUSIONS**

Our study was limited in scope, and therefore further research with more students covering more schools is needed. Moreover, we did not carry out a detailed inventory of the students' prior instruction in IA. Therefore, information on the

quality of the instruction was missing. This might explain why we did not find any influence of prior instruction on the students' success rate in applying IA.

In general, more student characteristics and more details about their prior instruction should be taken into account to acquire a deeper understanding of SE students' potential in solving subtraction problems. Nevertheless, the present study has revealed three striking outcomes:

1. SE students are able to use IA spontaneously, i.e., without being asked to do so.
2. SE students are rather flexible in applying IA to solve subtraction problems.
3. SE students are quite successful when solving subtraction problems by IA.

We think these findings argue in favour of a reconsideration of the approach to mathematics education in SE which advocates only teaching the straightforward taking-away procedure. Such an approach clearly underestimates SE students' mathematical ability. Finally, this study has shown that solely focusing on strategies (splitting, stringing, and varying) or solely on procedures (DS and IA) is a too restricted way of investigating students' ability to solve number problems. Both should be taken into account, as our study showed that the best predictor of a correct answer is the combination of IA and stringing.

### NOTE

<sup>1</sup> The ICT-based test was developed by the authors of this paper and programmed by Barrie Kersbergen, a software developer at the Freudenthal Institute.

### REFERENCES

- Blöte, A. W., Van der Burg, E., & Klein, A. S. (2001). Students' flexibility in solving two-digit addition and subtraction problems: instruction effects. *Journal of Educational Psychology*, 93, 627–638.
- De Smedt, B., Torbeyns, J., Stassens, N., Ghesquière, P., & Verschaffel, L. (2010). Frequency, efficiency and flexibility of indirect addition in two learning environments. *Learning and Instruction*, 20, 205–215.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht / Boston / Lancaster: D. Reidel Publishing Company.
- Gravemeijer, K., Van den Heuvel-Panhuizen, M., Van Donselaar, G., Ruesink, N., Streefland, L., Vermeulen, W., Te Woerd, E., & Van der Ploeg, D. (1993). *Methoden in het reken-wiskundeonderwijs, een rijke context voor vergelijkend onderzoek [Textbook series in mathematics education, a rich context for comparative research]*. Utrecht, the Netherlands: CD-beta Press.



- Janssen, J., Scheltens, F., & Kraemer, J. (2005). *Leerling- en onderwijsvolgsysteem. Rekenen-wiskunde groep 4. Handleiding [Student and education monitoring system. Mathematics grade 2. Teacher's guide]*. Arnhem: CITO.
- Kraemer, J., Van der Schoot, F., & Van Rijn, P. (2009). *Balans van het reken- en wiskundeonderwijs in het speciaal basisonderwijs 3 [Third assessment of mathematics education at special education schools]*. Arnhem: CITO.
- Menne, J. J. M. (2001). *Met sprongen vooruit. Een productief oefenprogramma voor zwakke rekenaars in het getallengebied tot 100 - Een onderwijsexperiment*. Utrecht, the Netherlands: CD-beta Press.
- Milo, B., & Ruijsenaars, W. (2002). Strategiegebruik van leerlingen in het speciaal basisonderwijs: begeleiden of sturen? [Strategy instruction in special primary education: guided or direct instruction?] *Pedagogische Studiën*, 79, 117–129.
- National Mathematics Advisory Panel (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.
- Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Solving subtractions adaptively by means of indirect addition: Influence of task, subject, and instructional factors. *Mediterranean Journal for Research in Mathematics Education*, 8(2), 1–30.
- Torbeyns, J., Ghesquière, P., & Verschaffel, L. (2009). Efficiency and flexibility of indirect addition in the domain of multi-digit subtraction. *Learning and Instruction*, 19, 1–12.
- Timmermans, R. E., & Van Lieshout, E. C. D. M. (2003). Influence of instruction in mathematics for low performing students on strategy use. *European Journal of Special Needs Education*, 18(1), 5–16.
- Van den Heuvel-Panhuizen, M. (1986). Reactie: het rekenonderwijs op de lomschool opnieuw ter discussie. *Tijdschrift voor Orthopedagogiek*, 25(3), 137–145.
- Van den Heuvel-Panhuizen, M., (Ed.) (2001). *Children learn mathematics*. Utrecht: Freudenthal Institute, Utrecht University.
- Van den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics*, 25(2), 2–23.
- Van den Heuvel-Panhuizen, M., & Treffers, A. (2009). Mathe-didactical reflections on young children's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning*, 11(1), 102–112.
- Warry, M. C., Galbraith, P. L., Carss, M. C., Grice, R. D., & Endean, L. (1992). Towards numeracy for the third millennium: A study of the future of mathematics and mathematics education. *Educational Studies in Mathematics*, 23, 569–593.

# MENTAL CALCULATION STRATEGIES FOR ADDITION AND SUBTRACTION IN THE SET OF RATIONAL NUMBERS

Sebastian Rezat

Justus-Liebig-University, Giessen

*Studies on mental calculation strategies usually focus on elementary school students and accordingly on problems in the set of natural numbers. But, mental calculation is also an issue at secondary school in the context of other number sets. In the paper an exploratory study of eight eighth grade students' mental calculation strategies for addition and subtraction problems in the set of rational numbers is presented. The study focuses on the analysis of the used strategies and on students' adaptive expertise in the choice of the strategies.*

Keywords: mental calculation strategies, adaptive expertise, rational numbers, didactical models of rational numbers

## RATIONALE

A considerable body of research investigated issues related to mental calculation of elementary school students in the set of natural numbers. This body of research mainly focuses on three aspects: Firstly, strategies students are using are identified (Reys, Reys, Nohda, & Emori, 1995; Selter, 2001). Secondly, the issue of adaptive expertise, i.e. flexible and adaptive strategy use, is addressed (Blöte, Klein, & Beishuizen, 2000; Threlfall, 2002; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). Finally, factors affecting different aspects of mental calculation, e.g. performance or adaptive expertise are analysed (Heinze, Marschick, & Lipowsky, 2009; Heirdsfield & Cooper, 2004).

The development of mental calculation strategies and their flexible and adaptive use by students in secondary school related to other number sets than the natural numbers has been rarely investigated so far. This is surprising, because on the one hand students in secondary school have had the opportunity to gain more experience in the number system. In Germany the set of natural numbers is extended to the integers, the rationals, and the real numbers in secondary school. Getting familiar with new numbers and their computation might possibly have positive effects on the development of number sense. Besides, metacognitive and affective factors number sense is regarded as one of the main influential factors on mental calculation performance and adaptive expertise (Heirdsfield & Cooper, 2004). On the other hand, mental calculation is a subject that is addressed in secondary school standards and national curricula. The NCTM-Standards expect that “In grades 6–8 all students should [...] select appropriate methods and tools for computing with fractions and decimals from among mental computation”. The call for mental calculation is continued in the upper grades: “In grades 9–12 all students should [...] develop fluency in operations with real numbers, vectors, and matrices, using mental

computation or paper-and-pencil calculations” (National Council of Teachers of Mathematics, 2000). Similar expectations can be found in the English National Curriculum (Qualifications and Curriculum Development Agency, 1999) and the German ‘Bildungsstandards’ (Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2005).

In this paper an exploratory study of eight eighth grade students’ mental calculation strategies in the set of rational numbers is presented.

## **THEORETICAL FRAMEWORK**

### **Mental calculation strategies**

Research on mental calculation for addition and subtraction problems in the set of natural numbers has identified numerous strategies that can be divided into different groups. Threlfall (2009) firstly distinguishes “approach strategies” and “number-transformation strategies”. He defines an approach strategy in mental calculation as “the general form of mathematical cognition used for the problem—for example counting, or recall, or application of a learned method, or visualisation of a procedure, or exploiting known number relations” (Threlfall, 2009, p. 541). A number-transformation strategy in mental calculation is “the detailed way in which the numbers have been transformed to arrive at a solution” (Threlfall, 2009, p. 542). In the literature varying conceptualizations of these strategies can be found (see e.g. Threlfall, 2002 for an overview). In this paper Threlfall’s terminology is adopted. ‘Mental calculation strategy’ is used as an overarching term whenever it is referred to both, an approach and a number-transformation strategy.

In this section an a priori analysis of possible approach- and number-transformation strategies in the set of rational numbers is presented.

From a mathematical point of view an analysis of mental calculation strategies for addition and subtraction in the set of rational numbers might focus on different aspects:

13 Addition and subtraction with integers

14 Addition and subtraction with fractions

15 Addition and subtraction with decimals

Addition and subtraction with fractions (2) is carried out by treating the nominator and the denominator separately. Therefore, it is likely that the number-transformation strategies students use for solving addition and subtractions problems with fractions are the same as with natural numbers and integers. Furthermore, in everyday life mental calculation with fractions is hardly needed in contrast to mental calculation with integers and decimals (Profke, 1991). Therefore, the study reported in this paper focuses on mental calculation strategies related to integers (1) and decimals (3).

Besides the occurrence of negative numbers a major novelty in calculating with integers is a zero-transition, i.e. crossing the zero-point in either direction – from positive to negative or vice versa – during the calculation process. In order to avoid the zero-transition, addition and subtraction of integers can be reduced to addition and subtraction of natural numbers either by definition or by proving the following rules: For any integers  $n, m$

$$\begin{array}{ll}
 (-n) + n = 0 & 0 - n = -n \\
 (-n) + m = m - n, \text{ if } m > n & (-n) - 0 = -n \\
 (-n) + m = -(n - m), \text{ if } n > m & 0 - (-n) = n \\
 (-n) + 0 = -n & (-n) - (-m) = (-n) + m \\
 (-n) + (-m) = -(n + m) & (-n) - m = -(n+m) \\
 & m - (-n) = n + m
 \end{array}$$

Therefore, mental addition and subtraction of integers can be approached by applying these definitions / rules combined with number-transformation strategies for natural numbers.

Another way of approaching mental addition and subtraction problems with integers is by referring to the mental image of the number line. In Germany a common way of introducing negative numbers is the extension of the number line. This geometrical model is sometimes derived from temperature, altitudinal, or monetary (bank-account-balance-model) contexts (Vollrath & Weigand, 2007). Table 1 summarizes the 5 different approach strategies that were identified in the a priori analysis:

<i>transformation</i>	<i>number line</i>	<i>bank-account-balance model (b-a-b-m)</i>	<i>temperature scale model</i>	<i>altitude model</i>
refers to an approach strategy where the original problem is transformed into an equivalent problem in the set of natural numbers according to the above stated rules	refers to an approach strategy where students solve the problem with reference to an internal image of the number line.	refers to an approach strategy where students solve the problem with reference to monetary contexts of deposit, withdrawal and depts.	refers to an approach strategy where students solve the problem with reference to a temperature context.	refers to an approach strategy where students solve the problem with reference to an altitude context.

**Table 1: Idealized approach strategies for addition and subtraction problems in the set of rational numbers.**

Decimals are introduced in Germany according to two major approaches: They are either regarded as a special kind of common fractions, e.g.  $23,45 = 2345/100$ , or they are introduced via an extension of the place-value-system (Padberg, 2009). In the former case, addition and subtraction of decimals is reduced to addition and subtraction of fractions, which is – as we have seen earlier – just a special case of addition and subtraction with integers. In the latter case, the number-transformation rules for decimals are traced back to the number-transformation rules of natural numbers.

In summary, all addition and subtraction problems with rational numbers can be approached by reducing them to equivalent problems with natural numbers. Therefore, it is likely that the actual transformation of numbers in order to arrive at a solution is carried out with number-transformation strategies for natural numbers.

In this paper I will refer to the conceptualization of idealized number-transformation strategies for addition and subtraction in the set of natural numbers that is put forward by Heinze et al. (2009), because it is appropriate for the German situation in the way that it comprises the strategies that are well known in German arithmetic literature. An overview of the different strategies is given in table 2.

<i>Stepwise strategy</i>	<i>Split strategy</i>	<i>Compensation strategy</i>	<i>Simplifying strategy</i>	<i>Indirect addition (ind. add.)</i>
<u><math>45 + 12 = 57</math></u>	<u><math>45 + 12 = 57</math></u>	<u><math>16 + 38 = 54</math></u>	<u><math>59 + 11 = 70</math></u>	<u><math>53 - 49 = 4</math></u>
$45 + 10 = 55$	$40 + 10 = 50$	$16 + 40 = 56$	$60 + 10 = 70$	$49 + 4 = 53$
$55 + 2 = 57$	$5 + 2 = 7$	$56 - 2 = 54$		
	$50 + 7 = 57$			

**Table 2: Idealized number-transformation strategies for addition and subtraction according to Heinze et al. (2009).**

Whereas the stepwise strategy, the split strategy, the compensation strategy, and indirect addition can be applied to integers and to decimals without any modification a variation of the simplifying strategy is specific to the set of integers: Problems like  $-47+23$  can be solved by simplifying into  $-(50-3) + (20+3) = -50+20+6$ .

### **Flexible and adaptive strategy use**

The use of the terms flexibility and adaptivity is not consistent in the literature (Selter, 2009). Based on a literature review Verschaffel et al. suggest that flexibility refers to “switching (smoothly) between different strategies” and adaptivity puts more emphasis on “selecting the most appropriate strategy” (Verschaffel, et al., 2009, p. 337). As a working definition they suggest that an adaptive choice of strategy is characterized by “the conscious or unconscious selection and use of the most appropriate solution strategy on a given mathematical item or problem, for a

given individual, in a given sociocultural context” (Verschaffel, et al., 2009, p. 343). This definition implies that strategies can be adaptive related to different aspects: they can adapt to task characteristics as well as to individual or sociocultural conditions. Furthermore, adaptivity seems to imply flexibility. This is even more evident in Selter’s slight variation of Verschaffel et al.’s definition: “Adaptivity is the ability to creatively develop or to flexibly select and use an appropriate solution strategy in a (un)conscious way on a given mathematical item or problem, for a given individual, in a given sociocultural context” (Selter, 2009, p. 624). Whereas flexibility might be operationalised by showing at least a certain number of different strategies, the question when a strategy is considered to be appropriate and which criteria are relevant to this are “critical and challenging [...] fundamental theoretical questions in this context (Heinze, Star, & Verschaffel, 2009, p. 536). Therefore, a normative perspective is usually taken in order to decide whether a strategy adapts to a given problem or not.

The findings of studies on students’ mental number-transformation strategies for addition and subtraction problems in the set of natural numbers indicate that elementary school students hardly choose among different strategies with respect to task characteristics, but favor one or two strategies that they apply to every problem (Selter, 2001; Torbeyns, et al., 2009; Torbeyns, Verschaffel, & Ghesquière, 2006).

Based on the discussion of mental calculation strategies and adaptive strategy use the following research questions are addressed in the study presented in this paper:

- Which approach and number-transformation strategies do students use for mental calculation in the set of rational numbers? Do students use strategies that are specific to the rational numbers or do they refer to strategies that they are familiar with from the set of natural numbers?
- Do students in secondary grade use mental calculation strategies adaptively according to tasks characteristics?

## **STUDY DESIGN AND METHODOLOGY**

Data on students’ mental calculation strategies were collected in video recorded interviews. The tasks were read out loud to the students and the students were asked to solve the tasks mentally without using any notes. Afterwards the students were asked to explain the way they solved the task [1].

Eight eighth grade students from a German comprehensive school took part in the study. The school is located in a rural area in Germany. According to their grades in mathematics the students are considered to be medium-achieving students.

The problems posed in this study can be grouped into three different categories:

- Addition and subtraction problems with natural numbers (tab. 3: P1 – P4)

- Addition and subtraction problems with integers (tab. 3: P5 – P8)
- Addition and subtraction problems with positive and negative decimals (tab.3: P9 – P14)

Whereas addition and subtraction problems with natural numbers (category 1) were included in order to get an idea which mental computation strategies students use in the set of natural numbers, problems of categories 2 and 3 relate directly to the main aims of the study. Since there are hardly any findings about students' mental calculation proficiencies with decimal numbers only numbers with one decimal were included in order to keep the numbers simple. Since a zero-transition is a major novelty when students are introduced to calculating with rational numbers two problems containing a zero-transition (P7, P14) were included.

In this study adaptivity is investigated related to task characteristics (research question 2). A normative perspective is taken in order to decide if a strategy adapts to a task or not. Therefore, problems suggesting the application of different number-transformation strategies were included. From a normative perspective the stepwise or the split strategy are most appropriate for problems P1, P2, P3, P9, P10, P11, P12. The compensation strategy is appropriate for problems P3, P7, P8, P13, P14. P13 is also suitable for applying the indirect addition strategy. P4 and P5 might be solved by a variation of the simplifying strategy, since e.g.  $53-27=(50+3)-(30-3)=50-30+6$ .

## FINDINGS

Table 3 provides an overview of used approach and number-transformation strategies related to problems P1-P14.

The analysis of the eight students' approach strategies reveals that the main strategy applied to problems containing negative numbers was the transformation into equivalent problems with natural numbers using laws for addition and subtraction of integers, e.g.  $-11 + 28 = 28 - 11$ . Only the problems containing a zero-transition (P7 and P14) were approached by referring to a mental image of the number line or to the bank-account-balance-model. Student 1 approaches problem P4 through the number line model:

Interviewer: Minus 11 plus 28?<sup>8</sup>

Student 1: Plus 17!

Interviewer: Correct!

Student 1: I subtracted 11 from 28 so that I will be at 0 and then I added the rest.

Interviewer: And how exactly did you subtract?

Student 1: I took 28 minus 11 so that I am at 0, is 17, then 0 plus 17 is 17.

---

<sup>8</sup> The language of the original interview data is German. All translations were carried out by the author.

problem	S1	S2	S3	S4	S5	S6	S7	S8
(P1) 46+23	stepwise	stepwise	split	split	stepwise	split	stepwise	split
(P2) 57-24	stepwise	stepwise	stepwise	stepwise	stepwise	split	stepwise	split
(P3) 38+16	stepwise	stepwise	stepwise	split	stepwise	split	stepwise	split
(P4) 53-27	stepwise	stepwise	stepwise	stepwise	stepwise	ind. add.	stepwise	split
(P5) -47+23	$-(47-23)^9$ , stepwise	$-(47-23)$ , stepwise	$-(47-23)$ , ?	$-(47-23)$ ; stepwise	$-(47-23)$ ; stepwise	?	$-(47-23)$ ; stepwise	$-(47-23)$ ; stepwise
(P6) -12-25	$-(12+25)$ , stepwise	$-(12+25)$ , stepwise	$-(12+25)$ ; stepwise	$-(12+25)$	$-(12+25)$	$-(12+25)$ ; split	$-(12+25)$ ; stepwise	$-(12+25)$ ; split
(P7) -11+28	number line 28-11, ?	-11+1=-10; $-(10-7)=-3$ ; $-3+20=17$	number line -11 up to 0 is 11; 28-11, ?	number line 28-11; ?	28-11; ?	number line ind. add.	28-11; stepwise	28-11
(P8) -33-19	$-(33+19)$ , stepwise	$-(33+19)$ , stepwise	$-(33+19)$ ; split	$-(33+19)$ ; stepwise	$-(33+19)$	$-(33-19)$ ; split	$-(33+19)$	$-(33+19)$ ; split
(P9) 3.5+1.2	stepwise	stepwise	split	split	split	split	35+12	split
(P10) 7.2-2.1	stepwise	stepwise	split	split	stepwise	?	stepwise	split
(P11) -7.6+2.3	$-(7.6-2.3)$ , stepwise	$-(7.6-2.3)$ , stepwise	$-(7.6-2.3)$ ; split	$-(7.6-2.3)$ ; split	$-(7.6-2.3)$ ; stepwise	$-(7.6-2.3)$ ; split	76-23; split	$-(7.6-2.3)$ ; split
(P12) 3.8+2.6	stepwise	stepwise	split	split	split	split	stepwise	split
(P13) 6.4-3.8	stepwise	stepwise	split	split	stepwise	ind. add.	64-38; split	split
(P14) 0.8-2.9	$-(2.9-0.8)$ , stepwise	$0.8-0.9=-0.1$ ; $-(0.1+2)=-2.1$ ; stepwise	$-(2.9-0.8)$ ; ?	b-a-b-m $-(2.9-0.8)$ $2-0.8=1.2$ ; $1.2+0.9=2.1$	$-(2.9-0.8)$	number line $-(2.9-0.8)$	$-(29-0.8)$	b-a-b-m $2.9-0.8$ ; split

Table 3: Approach and number-transformation strategies used by students S1-S8.

<sup>9</sup> Whenever students use the approach strategy 'transformation' the actual transformation will be specified in the table.



## Working Group 2

Student 4 refers to the bank-account-balance-model in order to calculate the algebraic sign of the result of problem P14:

Interviewer: 0.8 minus 2.9?

Student 4: makes minus 2.1, right?

Interviewer: Yes!

[...]

Student 4: I subtracted 0.8 from 2, makes 1.2, then I add 9, 8 are missing up to 2, and I have got 9, makes 2.1, it's minus, because I am plus 0.8, when I have got 8 Cent deposit on my account and I make a withdrawal of 2.9 Euro, it's negative.

Regarding number-transformation strategies, the analysis reveals that all the problems were solved using familiar number-transformation strategies from the set of natural numbers. No strategies specifically related to rational numbers were observed. Six of the eight students solved problems with decimal numbers by applying either the *stepwise strategy* or the *split strategy*. Two of them (S1, S2) used the *stepwise strategy* for every problem, no matter if it was a problem with natural numbers, integers or decimals. S8 almost exclusively used the *split strategy*. Three students (S3, S4, S5) used the *stepwise* and the *split strategy*. Only two students (S7, S8) apply other strategies than the *stepwise* or the *split strategy*: S7 transformed some of the problems to problems with natural numbers by leaving the decimal point out and setting it correctly after the calculation. S6 also applied the indirect addition strategy once to a subtraction problem with decimals.

In terms of adaptive strategy use, the analysis of the used number-transformation strategies reveals that students do not apply a variety of strategies to the posed problems: Three of the eight students (S1, S2, and S8) use the same number-transformation strategy for almost all problems. The rest of the students use mainly two strategies. Although the compensation strategy would have been appropriate for problems P3, P7, P8, P12, P13, P14 none of the students used this strategy.

However, the three students using the *stepwise* and the *split strategy* (S3, S4, and S5) show the tendency to solve problems with integers using the *stepwise strategy* and problems with decimals according to the *split strategy*. This might be an indication for adaptive strategy use. But, the reasons for the strategy choice cannot be derived from the data.

## DISCUSSION AND CONCLUSION

In the present study all problems from the set of rational numbers were solved according to number-transformation strategies that students are familiar with from the set of natural numbers. No strategies specific to the set of rational numbers were observed. Furthermore, no problems calculating with decimals were observed. This

might be due to the fact that no problems containing numbers with a different number of decimal places were included, e.g. 1.23-2.5. This is a shortcoming of the present study and should be approached in further research.

The students in the present study hardly choose adaptively among different computation strategies with respect to task characteristics. In none of the cases where a compensation strategy would have been appropriate the students applied this strategy. Furthermore, students in the study favour the *stepwise* and the *split strategy*. Regarding the small scope of the study these results cannot be generalized. Further research is needed to support the findings from this study.

In terms of approach strategies, it is remarkable that students only refer to didactic models of negative numbers (number line, bank-account-balance-model) associated with problems that contain a zero-transition. This might indicate that problems with zero-transition present particular difficulties to students. This hypothesis could be approached in further research. Furthermore, the role of didactical models in mental calculation is an issue which has not been investigated very much so far.

The findings from the present study reflect results from previous studies on mental calculation in the set of natural numbers: students apply only one or two strategies to almost all problems without choosing strategies adaptively to task characteristics. To investigate factors affecting adaptive expertise in mental calculation is therefore not only an issue for further studies related to the set of natural numbers, but also related to the set of rational numbers.

## NOTES

1. Data was collected by Jens Hubert in the context of his final thesis for earning a teaching degree in secondary school.

## REFERENCES

- Blöte, A. W., Klein, A. S., & Beishuizen, M. (2000). Mental computation and conceptual understanding. *Learning and Instruction*, 10(3), 221-247.
- Heinze, A., Marschick, F., & Lipowsky, F. (2009). Addition and subtraction of three-digit numbers: adaptive strategy use and the influence of instruction in German third grade. *ZDM*, 41(5), 591-604.
- Heinze, A., Star, J., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM*, 41(5), 535-540.
- Heirdsfield, A. M., & Cooper, T. J. (2004). Factors affecting the process of proficient mental addition and subtraction: case studies of flexible and inflexible computers. *The Journal of Mathematical Behavior*, 23(4), 443-463.
- National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics*. Reston, Va.: National Council of Teachers of Mathematics.

- Padberg, F. (2009). *Didaktik der Bruchrechnung*. Heidelberg: Spektrum.
- Profke, L. (1991). Bruchrechnung im Mathematikunterricht. In H. Postel & u. a. (Eds.), *Mathematik lehren und lernen. Festschrift für Heinz Giesel* (pp. 143-155). Hannover: Schroedel.
- Qualifications and Curriculum Development Agency. (1999). *Mathematics: the National Curriculum for England. Key stages 1-4: Qualifications and Curriculum Authority*.
- Reys, R. E., Reys, B. J., Nohda, N., & Emori, H. (1995). Mental Computation Performance and Strategy Use of Japanese Students in Grades 2, 4, 6, and 8. *Journal for Research in Mathematics Education*, 26(4), 304-326.
- Selter, C. (2001). Addition and Subtraction of Three-digit Numbers: German Elementary Children's Success, Methods and Strategies. *Educational Studies in Mathematics*, 47(2), 145-173.
- Selter, C. (2009). Creativity, flexibility, adaptivity, and strategy use in mathematics. *ZDM*, 41(5), 619-625.
- Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland. (2005). *Bildungsstandards im Fach Mathematik für den Hauptschulabschluss (Jahrgangsstufe 9). Beschluss vom 15.10.2004*. München: Luchterhand.
- Threlfall, J. (2002). Flexible Mental Calculation. *Educational Studies in Mathematics*, 50(1), 29-47.
- Threlfall, J. (2009). Strategies and flexibility in mental calculation. *ZDM*, 41(5), 541-555.
- Torbeyns, J., De Smedt, B., Ghesquière, P., & Verschaffel, L. (2009). Jump or compensate? Strategy flexibility in the number domain up to 100. *ZDM*, 41(5), 581-590.
- Torbeyns, J., Verschaffel, L., & Ghesquière, P. (2006). The development of children's adaptive expertise in the number domain 20 to 100. *Cognition and Instruction*, 24, 439-465.
- Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education*, 24(3), 335-359.
- Vollrath, H.-J., & Weigand, H.-G. (2007). *Algebra in der Sekundarstufe*. München: Elsevier, Spektrum.

# **INTRODUCTION TO THE PAPERS OF WG 3:**

## **ALGEBRAIC THINKING**

María C. Cañadas<sup>a</sup>, Thérèse Dooley<sup>b</sup>, Jeremy Hodgen<sup>c</sup>, Reinhard Oldenburg<sup>d</sup>

<sup>a</sup>University of Granada, <sup>b</sup>Dublin City University,

<sup>c</sup>King's College London, <sup>d</sup>Frankfurt University

In CERME7, WG 3 “Algebraic thinking” continued the work carried out in previous CERME conferences (Ainley, Bagni, Hefendehl-Hebeker, & Lagrange, 2009).

The 13 papers were considered in four themes:

### **The transition to algebraic symbolisation**

Caspi and Sfard investigate the discourse of 7th grade Israeli students as they move from informal meta-arithmetic toward formal algebra. By examining a historical example, they show how students’ discourse, whilst informal and ambiguous, contains some algebra-like features, not normally found in everyday discourse. Dooley examines a group of primary pupils in Ireland aged 9-11 years. She uses the epistemic actions of recognising, building-with and constructing to analyse and describe the development of algebraic reasoning amongst the pupils. She argues that in some case the use of “vague” language facilitated this development. Drawing on a design science approach, Gerhard uses interviews with secondary students in Germany to exemplify the use of an analytic tool examining the transition from arithmetic to algebra. She argues that it is important to distinguish the transition from arithmetical to algebraic thinking and that from numbers to variables. Pytlak analyses a child’s solution to a matchstick sequence task drawing on a wider study of primary children in Poland. She demonstrates how relatively sophisticated algebraic thinking can be achieved with geometric and numeric approaches but without the use of symbols.

### **Equations and symbolisation**

In an intervention study of 135 primary children in Cyprus, Alexandrou-Leonidou and Philippou found that the children were capable of developing the dual meaning of the equal sign. This understanding, in turn, enabled the children to solve equations in multiple representation formats. By conducting a survey of 113 students in Turkey, Didiş, Baş and Erbaş examine students understandings and errors in relation to the solving quadratic equations. Their findings add further weight to the literature highlighting the ubiquity and problems of a purely instrumental, or procedural, understanding.

### **Technology**

Drawing both on the historical development of mathematics and on examples of Italian students, Chiappini demonstrates how AlNuSet software can enable students to overcome crucial epistemological obstacles in the move from arithmetic to

### Working Group 3

algebra, specifically negative numbers and the equivalence of different algebraic forms.

Hewitt discusses the work of a group of 9-10 year olds in England as they engaged with formal algebra for the first time using the software Grid Algebra. He outlines six perspectives from the literature on algebraic activity and uses these to analyse the students' activity in order to examine what constitutes algebraic activity.

Working in Italy, Maffei and Mariotti use Aplusix CAS to examine the interplay between different representations of algebra: standard (symbolic) representation, tree representation and natural language. They demonstrate that natural language has a dual role as a representation in itself and in describing the other representations. Nobre, Amado, Carreira and da Ponte show how a generic spreadsheet, Excel, can enable students to engage with algebraic structure without the need for algebraic symbolisation. Indeed, the three Grade 8 Portuguese students, were able to model and solve a complex problem involving simultaneous equalities and inequalities.

### **Generalisation**

A. Barbosa reports on her analysis of the strategies used by 54 Portuguese students in 6th Grade working on generalization tasks as they participated in an intervention study. Students achieved better results with near generalisation than with far generalisation problems. Reporting on a survey of 359 Spanish Secondary students, Cañadas, Castro and Castro outline the different approaches to generalisation adopted. They find that students use graphical approaches infrequently and generally only when the problem was presented graphically. Chua and Hoyles discuss differences in the generalisation strategies used by 13 year old students in Singapore from the Express (higher attaining) course and from the Normal course. Express students were more flexible, adopting a numerical approach for a linear problem, but using a constructive approach for a quadratic problem.

### **GENERAL REFLECTIONS**

Algebraic thinking is a “mature” domain within mathematics education research (Kieran, 2006). Indeed, alongside multiplicative reasoning, algebra is perhaps the most extensively researched area in mathematics education. The papers and posters reflect this and all the papers and posters drew on this body of research. Unsurprisingly given this research history, there were many aspects of consensus across the group, but there were also significant differences.

### **Points of consensus**

In relation to the practice of teaching and learning algebraic thinking, there was general agreement that:

### Working Group 3

- Doing algebraic thinking provides considerable insight into school mathematics, but translating these insights into general classroom practice is not straightforward.
- Classrooms around Europe and elsewhere tend to be dominated by procedures and manipulation. Skemp's (1976) seminal work is still of considerable relevance.
- The promise of technology has largely yet to be realised in most classrooms.
- There are many approaches to algebra and learners should acquire many ways to look at and work with algebra.

All participants agreed on the importance of multiple perspectives, of talk and discourse, of rich tasks and of children's existing and naïve (mis-)understandings [1].

Reflecting the three plenary lectures at CERME-7, key overarching issues in the group discussions included a recognition of the importance of the teacher (*Sierpinska*), the importance of pupils experiencing "surprise" (*Hannula*) and the relationship between arithmetic and algebra (*Mariotti*).

#### Points of difference

The issue of "early algebra" and the relationship to / transition from arithmetic continues to be a thorny one, which generated much debate. The question as to whether there is a clear cognitive gap between (generalised) arithmetic and algebra remains an open one. Similarly, there was disagreement on whether there exists one *best* or *ideal* learning trajectory or whether there are several *good-enough* learning trajectories or whether learning is inevitably somewhat idiosyncratic. An international conference inevitably (and usefully) highlights issues of language and meaning. Working Group 3 was no exception. For example, whilst all agreed on the importance of talk and discourse, some participants preferred the more general term of "talk" and others preferred the more specific and theory-laden "discourse". Related to this, theory was used differently by different participants. Some opted for a pragmatic use of theory to solve and illuminate research problems as and when they occurred. Others attempted to draw synergies between different theoretical approaches in order to inform research.

#### ISSUES FOR FUTURE RESEARCH

We have already noted the concern with early algebra. Whilst this concern in part reflects a current theme in the literature (Kaput, Carraher & Blanton, 2007), it also responds to the policy context in which some countries (such as Portugal) are introducing algebra earlier. This policy imperative highlights several important issues for Working Group 3 and CERME more generally. Re-contextualisation – the translation of "existing" knowledge into new settings and contexts - is a valid and important field of study and we note that the replication of existing research has been

### Working Group 3

somewhat undervalued in mathematics education as a field generally. However, in re-contextualising or replicating existing work, researchers need to demonstrate the contribution they make to the field as a whole through stronger literature reviews.

The issue of translating research knowledge into practice in general was a concern for almost all participants. The mismatch between what can be achieved in experimental settings and the general practice in the majority of classrooms is a serious concern. So, for example, in considering how to realise the potential of technology, the group discussed how technology can help children do something that they would not otherwise do and then how teaching can enable children to understand “independent of” technology. Similarly, the group identified a need for further research into understanding group dynamics specific to algebraic thinking.

### LOOKING FORWARD TO CERME-8

Finally, in looking forward to CERME-8, the group discussed ways of continuing and extending existing studies by:

- Identifying research collaborations with a view to replicating studies in different national / cultural contexts.
- Reporting follow-on studies to CERME-7 papers and posters.
- Examining the same research problem / dataset using different theoretical lenses and methodologies.

We hope that the majority of the participants will return the CERME-8.

### NOTES

1. However, the issue of children’s understandings was conceptualised differently with some using the notion of misconceptions and others rejecting this as too cognitive.

### REFERENCES

- Ainley, J., Bagni, G. T., Hefendehl-Hebeker, & Lagrange, J.-B. (2009). Algebraic thinking and mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the CERME 6* (pp. 415-419). Lyon, France: Institut National De Recherche Pédagogique (INRP).
- Kaput, J. J., Carraher, D., & Blanton, M. L. (2007). *Algebra in the early grades*. Mahwah, NJ: Lawrence Erlbaum.
- Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the Psychology of Mathematics Education: Past, present and future* (pp. 11-49). Rotterdam: Sense.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.

# CAN THEY “SEE” THE EQUALITY?

Vassiliki Alexandrou-Leonidou\* & George N. Philippou\*\*

\*University of Cyprus \*\*University of Nicosia

*The concept of equality is basic and common in mathematics, but primary school students have limited understanding (operational meaning only) of the symbol used to represent it. The present study focuses on primary school students' abilities to recognize equality in different representation formats (iconic, verbal and symbolic) and reports on a teaching experiment that aimed at helping primary school students develop the dual meaning of the equal sign (operational and relational). Analysing data from 135 3rd to 6th grade Cypriot students who participated in the teaching experiment, we found that they are capable of developing the dual meaning of the equal sign and that this understanding has a significant effect on their ability to solve equations in multiple representation formats.*

## THEORETICAL FRAMEWORK AND RESEARCH GOALS

Deep understanding of the notion of equality is a necessary prerequisite for primary school students in order to proceed to upper school mathematics. Likewise, the equal sign ( $=$ ), although is one of the most basic ones along with the numbers and the operation signs, it becomes a serious obstacle, if students do not understand it's meaning. However, research in developmental psychology and mathematics education over the last 20 years has indicated that many primary school students (ages 7 to 11) have an inadequate understanding of the equal sign (Behr, Erlwanger&Nichols, 1980; Carpenter, Franke&Levi, 2003; Jones, 2008; Kieran, 1981; McNeil &Alibali, 2005; Rittle-Johnson &Alibali, 1999). As has been documented, instead of interpreting it as a relational symbol of mathematical equivalence, most students interpret it as an operational symbol, meaning “find the total” or “put the answer”.

Linking multiple representations of the same concept may help students enhance their learning, as it helps them express generality, an asset that, according to Mason, Drury and Bills (2007), gives students powerful understanding of a concept. Radford (2003) argues that generalisation develops through three levels: factual, where the generalisation focus remains at the level of the material to be generalised; contextual, which is more abstract and descriptive and where explanations of generalisation are language driven; and symbolic, where algebraic notation (including letters) is used to describe the generalisation. Generalisation is a necessary objective of every mathematics lesson (Mason et al., 2007). Hence, the opportunities and the frequency that learners are given to express generalities through actions and words may facilitate their appreciation of what algebra can do for them.

As it has been stated a long time ago, every learner who arrives at school, walking and talking, has displayed the power to perceive and express generality (Whitehead,



1932;Gattegno, 1970). The critical issue is that the extent to which learners manage to use these powers depends on methods and the approaches adopted by the teacher in the mathematics classroom. Thus, the need for experimental intervention studies appears in order to practically improve the situation and document certain methodology and appropriate teaching approaches.

In this respect, the main goal of the present study was to describe primary school students' abilities to recognize equality in different representation formats (iconic, verbal and symbolic) and to develop and test a teaching experiment (TE) that aimed at helping learners understand the dual meaning of the equal sign (operational and relational), as a mean to foster their ability to recognize equality in different representation formats.

## **METHODOLOGY**

The TE aimed at helping students understand the concept of equality and the structure of different representation formats of equality. It also aimed at developing students' understanding of the dual meaning of the equal sign, that is, its operational and its relational meaning. Practically, the TE was planned to give students the opportunity to work in activities involving multiple representations of the concept of equality. The effective involvement in such activities was expected to help them link the representations of this concept and proceed to generalize on the idea of equality.

Specifically, the planned activities involved working with objects(such as the number scale), with icons (such as with a picture of a scale with different weights), with words (such as with equivalent word expressions), and with symbols (such as with equivalent number sentences).Students were asked to complete missing weights or numbers on a scale to make it balance, to match and produce equivalent expressions in words and symbols and to identify and make equivalent vectors. They were also asked to match and produce verbal expressions of a symbolic expression and vice-versa, in order to link different representation formats. In addition, they were asked to use the equal sign to show equivalency of expressions in multiple representation formats, such as words, symbols and vectors. Students were given the opportunity to express general statements about the use of the equal sign, i.e. "it is used when we want to show that two things are equal" or "whatever exists on the left hand side of an equal sign should be equal to what there is on the right hand side".

Data were collected through two tests. Test 1 (T1) was designed to measure students' understanding of the equal sign and Test 2 (T2) was designed to measure ability to solve equations of similar structure, which were represented in different formats. T1 comprised of three parts and aimed to grasp the type of the students' understanding of the equal sign. The first part of T1 required students to write an informal definition of the equal sign in three contexts (the sign on its own, the sign at the end in a mathematical sentence and the sign between two equivalent mathematical sentences). These tasks were used in Knuth et al. (2006) study in a similar way. The

second part of T1 required students to complete equalities of different structure, that is, different number of operations, with the equal sign and the unknown at different positions, i.e.  $a + b = \_ + d$ ,  $a + b + c = a + \_$ . Only single digit numbers were used in the first and the second part of T1, to avoid students' difficulties with the operations' algorithms. The first four tasks in the third part required that students used the four operations and their own numbers to create a given result (These tasks were originally used by Saenz-Ludlow & Walgamuth (1998)). The remaining tasks in this part asked students to construct equalities with four numbers of their own, using all four operations, i.e.  $\_ + \_ = \_ + \_$ . (This task was originally used by Witherspoon (1999), while the rest were developed for the purposes of this research work (i.e.  $\_ \div \_ = \_ \times \_$ )).

T2 included equations in two different syntax, three different structures and six different representation formats. The two types of syntax used were the "start unknown" type (the unknown quantity was prior to the equal sign, i.e.  $7 + x + 6 = 20$ ) and the "result unknown" type (the unknown quantity was after the equal sign, i.e.  $26 - 9 - 7 = p$ ). The three structure types were  $a + b + c =$ ,  $a - b - c =$  and  $a \times b + c =$ . The six representation formats used were word descriptions (i.e. "When I add 5 to 8 and subtract 3, what is the answer?"), word problems (i.e. "Chris gathered 9 shells from the beach, he put them in his collection and he now has 17 altogether. How many shells did he have at the beginning?"), pictures (see Fig. 3 in Appendix), diagrams (see Fig. 4 in Appendix), symbolic equations with an unknown quantity shown with a shape (i.e.  $21 - 6 - \diamond = 10$ ) and symbolic equations with a letter (i.e.  $8 \times m - 6 = 18$ ).

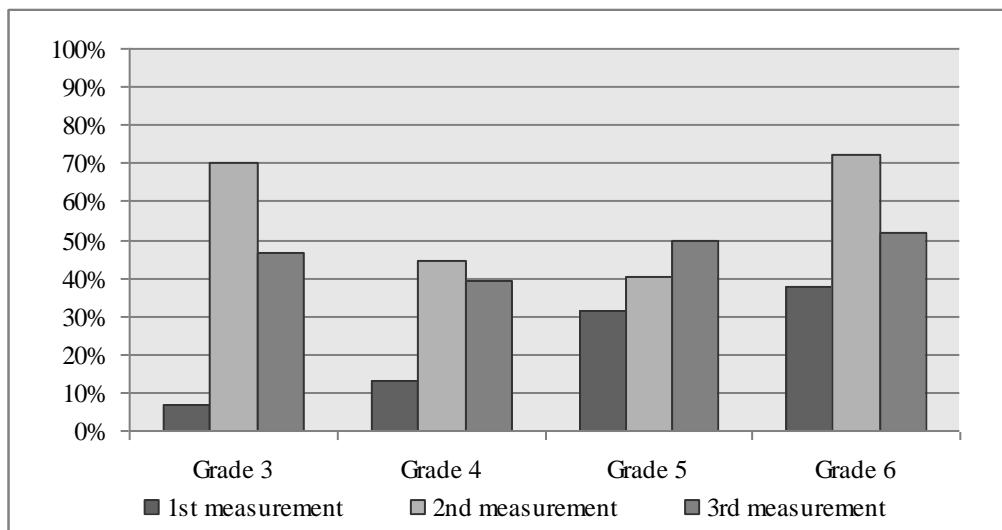
Participants in the study were 135 students (66 male and 69 female) from two primary schools in Nicosia district in Cyprus, 30 were 3rd graders, 38 were 4th graders, 38 were 5th graders, and 29 were 6th graders. The TE lasted for four months. Students participated in a 40 minute lesson every week. All teaching was undertaken by the first author. The students completed T1 three times, at the beginning, in the middle and after the end of the TE. They also completed T2 twice, before and after the end of the TE.

Data were analyzed qualitatively and quantitatively. The definitions, provided by the students in the first part of T1, were analyzed qualitatively, as they were categorized according to the content of the definition. The definitions were coded as "relational", if students indicated that the equal sign represents a relationship of equivalence, and as "operational", if they indicated that it announces the result or it gives the direction to do the operations. No response or unclear responses were coded as "other". Data from the remaining tasks from T1 and from T2 were analyzed quantitatively. Correct responses to the tasks of T1 and T2 were coded as 1 and wrong responses were coded as 0. Each student's score for each representation format of the equations in T2 was estimated by the sum of correct responses to the four equations of each format.

SPSS was used for descriptive and inferential statistical analysis of the data. Descriptive analysis was used to describe students' abilities to understand the double meaning of the equal sign and to handle algebraic tasks in different syntax, structure and representation format. Inferential analysis was used in order to evaluate the TE outcome, that is, whether it succeeded to help students develop relational understanding of the equal sign and whether it enhanced their ability to solve equations.

## RESULTS

The number of students with relational understanding of the equal sign (when presented on its own/T1-Task 1) increased at the end of the TE (Fig. 1). A paired t-test for the whole student population showed a statistically significant difference  $\{t(134)=-4,91, p=0,001\}$  between the first ( $M=1,85, SD=0,69$ ) and the second ( $M=2,26, SD=0,77$ ) measurement.



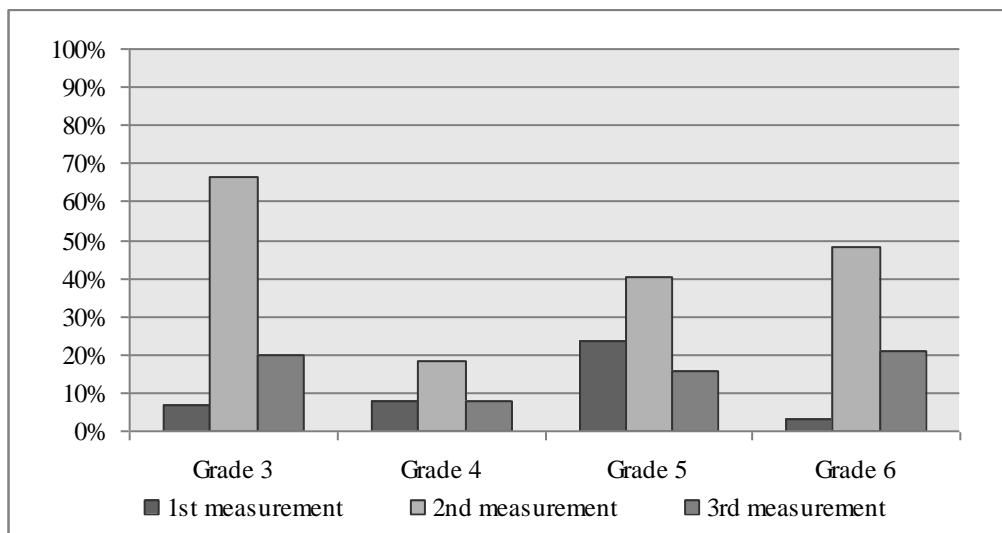
**Figure 1. Percentage of students with relational understanding of the equal sign (T1-Task 1) by grade level**

This indicates that on the total more students gave definitions indicating relational understanding at the second measurement (Fig. 1). Significant differences  $\{t(134)=5,32, p=0,001\}$  were also observed between the second and third ( $M=1,79, SD=0,81$ ) measurement. This outcome indicated that the number of students giving relational understanding decreased. As shown in Figure 1, more third and sixth graders kept the relational understanding at the end of the TE than fourth and fifth graders whose definitions with relational understanding were even more limited.

Kruskal-Wallis criterion revealed statistically significant differences in the students' Mean Rank by grade in the first ( $K-W=19,78, df=3, p=0,001$ ) and the second ( $K-W=18,04, df=3, p=0,001$ ) measurement of the level of understanding of the equal sign when it was presented on its own. Mean Rank increased from lower grades to

higher ones, suggesting that the percentage of students with relational understanding increased the higher their grade was. There were no statistically significant differences between the Mean Rank in the third measurement. This can be explained by the fact that students from all four grades participated in the same TE and had developed analogous understanding for the equal sign.

The third task of T1 asked students to give a definition for the equal sign when it was presented in an equation. The context of this task, as opposed to the first one, was expected to lead students to the relational meaning of the symbol and to the structural characteristics of an equation. At the end of the TE, more students gave definitions of the equal sign as a relational symbol in this context (Fig. 2). A paired t-test between the first measurement ( $M=1,85$ ,  $SD=0,88$ ) and the second ( $M=2,29$ ,  $SD=0,93$ ) showed statistically significant difference [ $t(134)=-4,72$ ,  $p=0,001$ ]. The difference between the second and third measurement ( $M=2,07$ ,  $SD=1,07$ ) was also significant [ $t(134)=2,05$ ,  $p=0,042$ ]. It was also found that more students showed relational understanding of the equal sign in the middle of the TE than at the end of it.



**Figure 2. Percentage of students with relational understanding of the equal sign (T1-Task 3) by grade level**

The Kruskal-Wallis criterion revealed statistically significant differences of the level of understanding of the equal sign when presented in an equation in the students' Mean Rank by grade in the first measurement ( $K-W=8,65$ ,  $df=3$ ,  $p=0,034$ ). The Mean rank increased by grade, suggesting that the percentage of students understanding the equal sign relationally increases as according to their grade level (age). No statistically significant differences were found between understanding of the equal sign by grade level in the second and third measurement. This can be explained by the fact that all students, irrespective of grade, participated in the same TE and therefore had developed similar understanding of the equal sign.

### Working Group 3

Multivariate analysis of variance has shown that there was statistically significant effect ( $F_{3, 126} = 4,04$ ,  $p=0,009$ ) of “time” and “grade level” on the students’ level of understanding of the equal sign among the three repetitive measurements [before TE ( $M=46,81$ ,  $SD=18,88$ ), in the middle ( $M=56,73$ ,  $SD=16,54$ ) and at the end ( $M=57,87$ ,  $SD=16,73$ ) of TE]. This can be due to the students’ exposure to the content of TE, although maturation may have played a role. Students’ ability to solve “result unknown” equations in different representation formats, different syntax and different structure was found to improve significantly [ $t(134)=-5,98$ ,  $p=0,001$ ] between the measurement before ( $M=12,79$ ,  $SD=4,56$ ) and after the TE ( $M=14,94$ ,  $SD=3,11$ ). Equations presented in pictorial and symbolic format were easier than word descriptions, word problems and diagrams, even after the end of the TE. Statistically significant differences were found in each one of the six different representation formats of the equations before and after TE, as shown in Table 1. Overall, the analysis has shown that the students’ performance in solving “result unknown” equations improved in all representation formats. The largest mean difference was observed in diagrams.

Representation format	M at the start	M at the end	t	df	p
Picture	2,46	2,75	-3,74	134	0,01
Diagram	1,37	2,08	-5,91	134	0,01
Symbol (Shape)	2,46	2,71	-3,05	134	0,01
Symbol (Letter)	2,40	2,67	-3,17	134	0,01
Word description	2,06	2,33	-3,08	134	0,01
Word problem	2,04	2,39	-3,87	134	0,01
Total	12,79	14,94	-5,98	134	0,01

**Table 1: Comparison of mean performances at the “result unknown” equations (beginning and end of the TE) by representation format and overall**

Representation format	M at the start	M at the end	t	df	p
Picture	2,26	2,59	-3.59	134	0,01
Diagram	1,30	2,08	-6,77	134	0,01
Symbol (Shape)	2,00	2,24	-2.52	134	0,01
Symbol (Letter)	1,87	2,22	-3.95	134	0,01
Word descriptions	1,67	2,12	-4,28	134	0,01
Word problem	1,73	1,98	-2.46	134	0,02
Total	10,83	13,23	-5,98	134	0,01

**Table 2: Comparison of mean performances at the “start unknown” equations (beginning and end of the TE) by representation format and overall**

### Working Group 3

Comparison of students' mean performance in solving "start unknown" equations before and after TE has shown statistically significant differences (before TE  $M=10,83$ ,  $SD=5,09$ , after TE  $M=13,23$ ,  $SD=4,31$ ,  $t(134)=-5,98$ ,  $p=0,001$ ). Variation in the level of difficulty of the six different types of representation formats before and after TE was found. In particular, at the end of TE diagrams were found to be the second easiest types of equations after pictures, whereas at the beginning they were the most difficult.

Table 2 shows comparisons of students' mean performance before and after TE in solving "start unknown" equations in different representation formats. Results show that in each one of the six representation formats of the equations students' performance improved after TE, especially at diagrams and symbolic equations (with letter).

Independent Variable	$\beta$	p	F	df	R <sup>2</sup>
Gender	0,06	0,52	0,40	1,134	0,00
Grade level	0,32	0,00	15,50	1,134	0,10
Equal sign level of understanding – 1 <sup>st</sup> meas.	0,06	0,51	0,40	1,134	0,00
Equal sign level of understanding – 2 <sup>nd</sup> meas.	0,20	0,02	5,60	1,134	0,04
Equal sign level of understanding – 3 <sup>rd</sup> meas.	0,39	0,00	23,60	1,134	0,15
Ability to construct equalities – 1 <sup>st</sup> meas.	0,38	0,00	22,70	1,134	0,15
Ability to construct equalities – 2 <sup>nd</sup> meas.	0,55	0,00	57,30	1,134	0,30
Ability to construct equalities – 3 <sup>rd</sup> meas.	0,64	0,00	90,40	1,134	0,41
Performance in T1 – 1 <sup>st</sup> meas.	0,51	0,00	47,70	1,134	0,25
Performance in T1 – 2 <sup>nd</sup> meas.	0,56	0,00	59,80	1,134	0,31
Performance in T1 – 3 <sup>rd</sup> meas.	0,65	0,00	99,60	1,134	0,43
Performance in T2 – 1 <sup>st</sup> meas.	0,50	0,00	45,20	1,134	0,25

**Table 3: Regression coefficients of independent variables predicting students' ability to solve "start unknown" equations after the TE**

In order to examine which variables could predict students' ability to solve "start unknown" equations at the end of TE, linear regression analysis was performed. Table 3 shows the values of each independent variable when it was examined separately. In the next step, in order to control multi collinearity, a linear regression analysis was performed stepwise with all independent variables together. In the final regression model ( $F_{3, 130} = 45,4$ ,  $p=0,001$ , 51%) only three variables were included: Performance in T1 – 3<sup>rd</sup> measurement ( $\beta = 0,50$ ,  $p=0,001$ ), Grade level ( $\beta = 0,22$ ,  $p=0,001$ ) and Ability to construct equalities – 2<sup>nd</sup> measurement ( $\beta = 0,23$ ,  $p=0,001$ ).

It is shown that improvement of students' ability to use the equal sign relationally to construct equalities helps their ability to solve "start unknown" equations. Also, it is suggested that their ability to solve such equations is improved by studying relevant subjects through their school years, since grade level has been found to play an important role. Finally, the values of the regression coefficients demonstrate that, when students develop relational understanding as opposed to operational understanding only, are more capable of solving "start unknown" equations.

Generally, results support the idea that equation solving ability can be improved through the development of the dual meaning of the equal sign. More specifically, it is suggested that a teaching program can help students understand that the equal sign represents a relation and subsequently improve their ability to solve equations in different structure, syntax and representation format.

## DISCUSSION

Understanding the concept of equality is an important prerequisite for students' abilities to solve equations. As was documented by the outcome of the TE, when students are able to construct equalities with their own numbers using all four number operations, they are more successful at solving equations in multiple representations. Although primary school students' (grade 3-6) understanding of the equal sign is generally limited to its operational meaning, they are capable to develop its dual meaning when they are exposed to teaching that aims at developing the relational meaning of the symbol. Thus the concept of equality needs to be emphasized in primary school math curriculum, in order to help them build robust knowledge for further algebra study in later years.

The teaching of the relational meaning of the equal sign does not happen during the years of primary schooling and this omission of the mathematics curriculum leads students to a limited understanding of it. Besides, its pervasive use as an operational sign helps them build strong misunderstandings of its meaning. Consequently, the sooner in their school life students become aware that the equal symbol is both an operational and a relational symbol, the fewer misconceptions they will have. Limited understanding of the equal sign (that is, only operational meaning of the sign) may hamper the development of important algebraic ideas, such as the concept of equality and their ability to solve equations.

As it has been suggested by the outcome of the implementation of the teaching experiment discussed above, it is helpful for primary school students to both understand the dual meaning of the equal sign and use multiple representation formats for the concept of equality. The use of multiple representations of equivalent expressions gives students an insight of the concept of equality and helps them generalize. The order in which different representation formats of equality can be used seems to be an important issue for the teaching of the concept of equality. Pictures and symbolic expressions have been found to be easier for primary school

students than verbal expressions and diagrams. Thus, a teaching program aiming at developing the concept of equality to students of these ages needs to be designed on the basis of students' abilities to handle certain representation formats.

Primary school students need to understand the concept of equality and to be able to use correctly the symbol that represents this relation, in order to be able to recognize equality during the process of equation solving. The process of generalization is an important element of a teaching program that may help them "see" the equality in multiple representation formats and use it to solve equations when they begin the study of formal algebra in high school.

## REFERENCES

- Behr, M., Erlwanger, S., Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, 92, 13-15.
- Cai, J. (2004). Why do U.S. and Chinese students think differently in mathematical problem solving? Exploring the impact of early algebra learning and teachers' beliefs. *Journal of Mathematical Behavior*, 23, 135-167.
- Carpenter, T. P., Franke, M. L., Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in elementary school. Portsmouth, NH: Heinemann.
- Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics, *Educational Studies in Mathematics*, 61(1-2), 103-131.
- Gattegno, C. (1970). *What We Owe Children: the subordination of teaching to learning*. London: Routledge & Kegan Paul.
- Jones, I. (2008). A diagrammatic view of the equals sign: Arithmetical equivalence as a means not an end. *Research in Mathematics Education*, 10, 119-133.
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12(3), 317-326.
- Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does an understanding of the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, 36(4), 297-312.
- Mason, J., Drury, H., Bills, E. (2007). Explorations in the Zone of Proximal Awareness. In J. Watson, K. Beswick (Eds.) *Mathematics: Essential Research, Essential Practice* (Vol 1, pp. 42-58). Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia. Adelaide: MERGA.
- McNeil, N. M., Alibali, M. W. (2005). Knowledge Change as a Function of Mathematics Experience: All Contexts are Not Created Equal. *Journal of Cognition and Development*, 6(2), 285-308.



### Working Group 3

- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, 5(1), 37-70.
- Rittle-Johnson, B., Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology*, 91, 1-16.
- Saenz-Ludlow, A., Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol, *Educational Studies in Mathematics*, 35(2), 153-187.
- Whitehead, A. (1932). *The Aims of Education and Other Essays*, London: Williams and Norgate.
- Witherspoon, M. L. (1999). And the answer is symbolic literacy (accurate interpretation of mathematical or numerical symbols!). *Teaching Children Mathematics*, 5(7), 396-399.

## APPENDIX

Examples of tasks included in T2

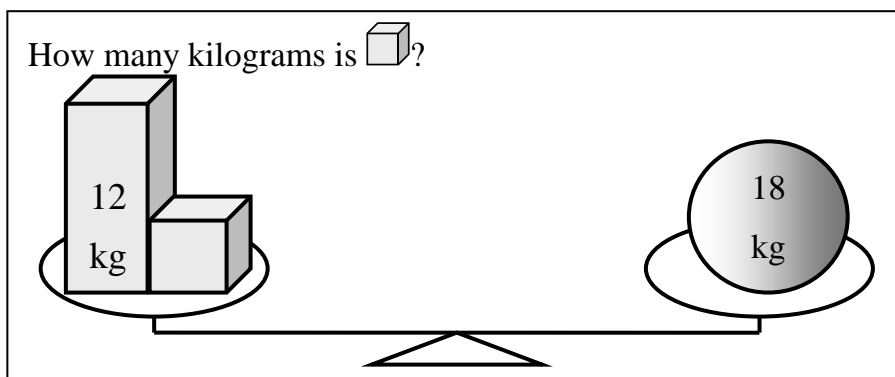


Figure 3. Picture (Start unknown)

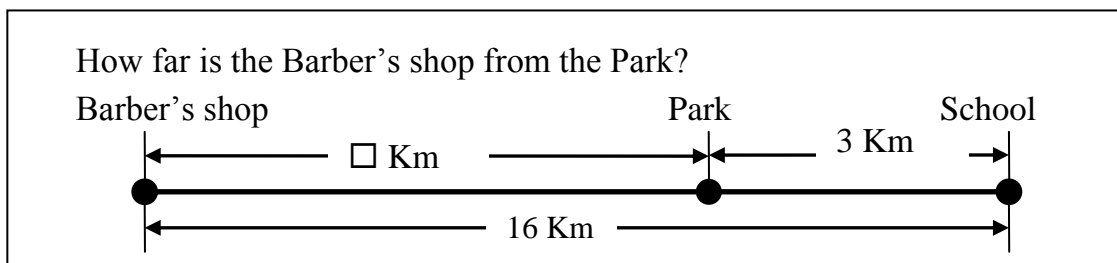


Figure 4. Diagram (Start unknown)

# **PATTERNING PROBLEMS: SIXTH GRADERS' ABILITY TO GENERALIZE**

**Ana Barbosa, School of Education of Viana do Castelo, Portugal**

*This paper analyses the performance of fifty-four 6th grade students when solving visual patterning tasks. The main goal is to understand the following features: type of generalization strategies used; difficulties that emerged from students' work; and the role played by visualization on their reasoning. In this paper I will focus on the results related to the implementation of two particular tasks.*

**Keywords:** Mathematics, problem solving, patterns, generalization.

## **INTRODUCTION**

Since the 1980s problem solving has been recognised as a fundamental part of the teaching and learning process in mathematics (NCTM, 2000). This prerogative is still current in the recent curricular guidelines of several countries. However, international studies (SIAEP, TIMSS, PISA) show that Portuguese students perform badly problem solving (Amaro, Cardoso & Reis, 1994; OECD, 2004; Ramalho, 1994). These results along with similar difficulties observed in classroom experiences, are a matter of serious concern to the researchers and educators' community in Portugal. This study approaches problem solving through the exploration of visual patterning tasks: (1) pattern generalization may contribute to the development of abilities related to problem solving, through emphasising the analysis of particular cases, organizing data in a systematic way, conjecturing and generalizing. Working with numeric, geometric and pictorial patterns may be helpful in building a positive and meaningful image of mathematics and contribute to the development of several skills related to problem solving and algebraic thinking (NCTM, 2000; Vale, Barbosa, Barbosa, Borralho, Cabrita, Fonseca, et. al., 2009); (2) Geometry is considered a source that can help students to develop abilities such as visualization, reasoning and argumentation. Visualization is essential but its role has not always been emphasized in students' mathematical experiences (Healy & Hoyles, 1996; Presmeg, 2006). Portuguese teachers privilege numeric aspects over geometric ones in classrooms. This study aims to understand how 6<sup>th</sup> grade students (11-12 years old) solve problems involving visual patterns, addressing the following research questions: (a) how can we characterize students' generalization strategies? Which difficulties do 6<sup>th</sup> grade students have when solving pattern exploration tasks? (b) Which difficulties do 6<sup>th</sup> grade students have when solving pattern exploration tasks? (c) What is the role of visualization on students' reasoning?

## **THEORETICAL FRAMEWORK**

The mathematics curricula of many countries include significant components related to patterns, including: searching for patterns in different contexts; using and understanding symbols and variables that represent patterns; and generalizing. Curricular guidelines reflect an enthusiastic view about the role of patterns in

mathematics. Some mathematicians go even further to define mathematics as the science of patterns (Devlin, 2002; Steen, 1990), highlighting the centrality of this theme. The Portuguese curriculum considers the importance of developing abilities like searching and exploring numeric and geometric patterns, as well as solving problems, looking for regularities, conjecturing and generalizing (ME-DGIDC, 2007). Pattern generalization can be achieved through a given strategy, but different students may use diverse approaches to accomplish generalization. There are significant works concerning students' generalization strategies, from pre-kindergarten to secondary school. The revision of several researchers' frameworks (Lannin, 2005; Lannin, Barker & Townsend, 2006; Orton & Orton, 1999; Rivera & Becker, 2005; Stacey, 1989) led me to develop the categorization (Barbosa, 2010) shown in Table 1.

Strategy		Description
<i>Counting (C)</i>		Drawing a figure and counting the desired elements.
<i>Whole-object</i>	No adjustment ( $W_1$ )	Considering a term of the sequence as unit and using multiples of that unit.
	Numeric adjustment ( $W_2$ )	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on numeric properties.
	Visual adjustment ( $W_3$ )	Considering a term of the sequence as unit and using multiples of that unit. A final adjustment is made based on the context of the problem.
<i>Difference</i>	Recursive ( $D_1$ )	Extending the sequence using the common difference, building on previous terms.
	Rate - no adjustment ( $D_2$ )	Using the common difference as a multiplying factor without proceeding to a final adjustment.
	Rate - adjustment ( $D_3$ )	Using the common difference as a multiplying factor and proceeding to an adjustment of the result.
<i>Explicit (E)</i>		Discovering a rule, based on the context of the problem, that allows the immediate calculation of any output value given the correspondent input value.
<i>Guess and check (GC)</i>		Guessing a rule by trying multiple input values to check its' validity.

**Table 1: Generalization Strategies Framework**

These categories will be clarified in this paper, through some examples.

Patterning activities can be developed in a variety of contexts (numeric, geometric, concrete, visual) and may promote the use of different approaches. Gardner (1993) claims that some individuals recognize regularities spatially or visually, while others notice them logically or analytically. This duality has caused much controversy. Many investigators stress the importance of visualization in problem solving (Presmeg, 2006; Shama & Dreyfus, 1994), while others claim that visualization should only be used as a complement to analytic reasoning (Goldenberg, 1996; Tall, 1991). Those perspectives reflect the importance of using and developing visual abilities, enhancing students' mathematical experiences. Nonetheless, teachers must consider that *seeing* an image can lead to different interpretations depending on the

individual. A figure can be apprehended perceptually when the image is interpreted as a whole, or discursively, if the individual identifies the spatial disposition of the elements that compose the figure (Duval, 1998). Considering patterning tasks and the generalization process, the discursive apprehension of a pattern can be of different nature. Either by seeing sets of disjoint visual cues that form the initial figure (constructive generalization) or by identifying overlaps, whose elements are counted more than once involving a subsequent subtraction (deconstructive generalization) (Rivera & Becker, 2008).

Depending on the type of task, some strategies may be more adequate than others and, on the other hand, can even lead students to difficulties or incorrect answers. It is fundamental that students understand the potential and limitations of each approach.

## METHOD

I performed a qualitative approach (Erickson, 1986) with a case study design (Yin, 1989) with fifty four sixth-grade students (11-12 years old), from three different schools in the North of Portugal, over the course of a school year. These students solved seven tasks during six months, working in 27 pairs. Two pairs from each school were selected for clinical interviews. The tasks used in the study required near generalization (the order of the term allows the use of strategies like making a drawing or using a recursive method) and far generalization (the use of recursive methods is not adequate, implies the finding of a rule) and featured increasing and decreasing linear patterns as well as non linear ones. This paper reports results related to the application of two tasks.

## RESULTS

### Generalization strategies

The first task was *Pins and Cards*. It involves an increasing linear pattern, illustrated with a visual representation of the third element of the sequence. Table 2 shows the strategies used to solve the problem and its connection with the level of generalization. Following Table 1, the strategies were abbreviated. No answer or imperceptible strategy was categorized as NC. The first column of Table 2 summarizes the number of pairs of students that used a given strategy in each of the three questions of this task, based on the categories described in Table 1.

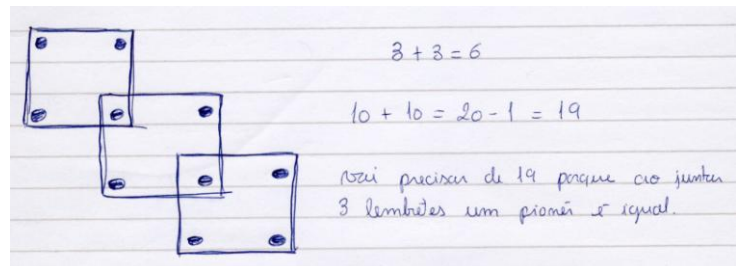
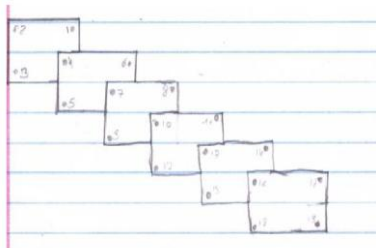
	C	W <sub>1</sub>	W <sub>2</sub>	W <sub>3</sub>	W	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D	E	GC	NC	Total
1.	16	8	-	1	9	1	1	-	2	-	-	-	27
2.	-	3	2	1	6	3	1	-	4	12	-	5	
3.	-	2	1	-	3	-	4	3	7	9	-	8	

**Table 2: Summary of the strategies used by the 27 pairs of students**

The first question of this task requires near generalization (students must find the 6<sup>th</sup> element of the sequence). It can be solved using recursive reasoning, either by making a drawing of the requested term of the sequence and counting its elements,

### Working Group 3

using the *counting* strategy, or by extending the sequence through the identification of the common difference (recursive strategy). A counting strategy was the predominant approach in near generalization tasks and it always led to a correct answer. Figure 1a reflects one example of counting strategy.



**Figure 1a, 1b. Counting strategy and  $W_3$  strategy to solve question 1**

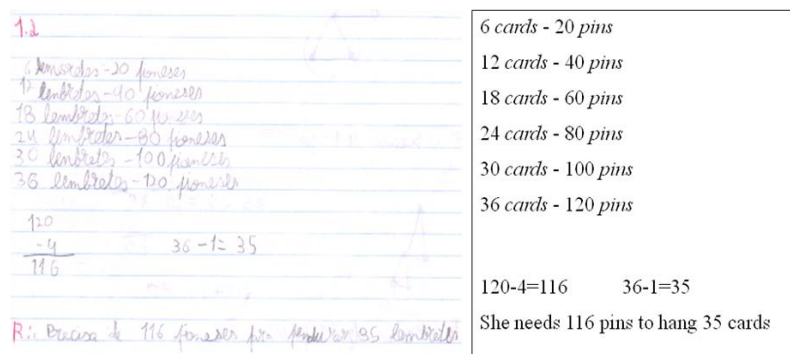
The *whole-object* strategy also emerged in some pairs of student. This approach is associated to direct proportion situations and the problem does not fit this model. For this strategy to be adequate, students had to make a final adjustment based on the context. Eight pairs of students used proportional reasoning, doubling the number of pins associated to the three cards. Only one pair adjusted the result obtained duplicating the number of pins of the three cards (see Figure 1b).

Only one pair of students extended the sequence using the common difference and achieves a correct answer. This is astonishing because this kind of task tends to promote *recursive* thinking (e. g. Orton & Orton, 1999; Stacey 1989), especially when near generalization is involved. We found another case in which the *difference* strategy was employed but in an incorrect way. These students used a multiple of the common difference, without adjusting the result. The *explicit* and *guess and check* strategies were not applied to solve this question.

Although both questions 2 and 3 require far generalization, the third question of task 1 involves reverse thinking. As expected, when approaching far generalization students revealed more difficulties and can be seen by the increasing number of NC answers in Table 2. We can also notice that students dropped the *counting* strategy when solving these two questions. Some pairs of students started using this strategy but gave it up, claiming that “there were too many cards”. As an alternative, *explicit* strategies prevailed. Those who relied on this approach, identifying an immediate relationship between the two variables, presented a high level of efficiency, making a discursive apprehension of the image. Some students *saw* that each card needed three pins and the last one four, deducing that the rule was  $3(n-1)+4$  ( $n$ , number of cards). Other pairs of students *saw* the pattern differently, considering that each card had three pins adding one more at the end. The rule was  $3n+1$  ( $n$ , number of cards). This fact reinforces that individuals might see the same pattern differently (Rivera & Becker, 2008), originating equivalent expressions. In spite of finding varied forms of representing the same pattern, these generalizations were all constructive. Two pairs of students performed this strategy inadequately in the last question, mixing pins and cards. The *whole-object* strategy remained in this question. Some students

### Working Group 3

considered multiples of known terms of the sequence (Figure 2). Students used proportional reasoning to determine the number of pins and when adjusting the result, they neglected the problem context., and using only numeric properties, they obtained an incorrect answer.



**Figure 2: W<sub>2</sub> strategy used to solve question 2**

Comparing the first question with the other questions the use of the *difference* strategy increases. Some students gave up *counting*, as the order of the term increased, and started by the common difference between terms. In the third question of the task, we noticed that three pairs of students applied a strategy that had not been used before (D<sub>3</sub>). The difference between consecutive terms is three pins, so students used this fact to approach the number of pins. Knowing the structure of the pattern, they were able to criticize the result, adjusting it correctly.

It is important to note that nearly 25% of the responses to questions 2 and 3 of the first task were not categorized. This is because these two questions imply far generalization, involving a more abstract reasoning.

The *Sole Mio Pizzeria* task was solved four months later. This problem is very similar to the first one, exhibiting an increasing linear pattern and involving near generalization (question 1) and far generalization (questions 2 and 3), with a visual representation of the third and fourth terms of the sequence. The strategies used by the students are shown in Table 2, using the same structure of Table 1:

	C	W <sub>1</sub>	W <sub>2</sub>	W <sub>3</sub>	W	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D	E	GC	NC	Total
1.	21	-	-	-	-	4	-	-	4	2	-	-	27
2.	1	-	-	-	-	3	-	1	4	22	-	-	
3.	-	-	-	-	-	2	3	-	5	14	5	3	

**Table 3: Summary of the strategies used by the 27 pairs of students**

There was a lack of preference for the *whole-object* strategy in this problem. A linear pattern is involved but the use of proportional reasoning is not adequate, unless an adjustment is made based on the context. This adjustment is more complex in this problem than in the previous one. This might justify the absence of this approach. *Counting* was once again the predominant strategy in near generalization (question 1), leading to the students to a correct answer. It was applied by most of the students and this preference increased comparing to the previous task. Other

strategies emerged in this first question but were used only by a minority of the students. Four pairs choose recursive reasoning to extend successfully the sequence to the 10<sup>th</sup> term and two pairs applied an explicit reasoning. We highlight that in the first task explicit strategies only appeared when students were dealing with far generalization so it is surprising that they used it at this stage, showing that they immediately discovered the structure of the pattern. As in the first task, when dealing with far generalization, students did not recognize the usefulness of *counting* and that is why it has no expression in Table 3, as they progress to far generalization. On the other hand, *explicit* reasoning prevailed being implemented by even more students in a successful way. All of them described the pattern as  $2n+2$ ,  $n$  being the number of pizzas. They frequently referred that “in front of each pizza there are two people and one more at each end of the table”, describing the pattern in a constructive way. As an alternative, some students adopted a recursive approach, through the extension of the sequence using the common difference. Similarly to what happened in the first task, there were three pairs that considered multiples of the common difference but neglected to adjust the result, basing their work just on numeric relations. Table 3 also shows the use of the new strategy *guess and check* was only applied in far generalization, when reverse thinking was involved. Students identified the relation between the two variables and then tried some numbers until they achieved the intended result (Figure 3).

**Figure 3: Guess and check strategy used to solve question 3**

### Difficulties emerging from students' work

When solving the first task some students struggled with cognitive difficulties that led to incorrect answers. In pairs of students who made false assumptions about the use of direct proportion, attention tended to focus only on numeric attributes with no appreciation of the sequence structure. The use of strategies based on recursive reasoning was not always adequate, particularly when far generalization questions were involved. The recursive approach through the use of  $D_2$  lacked a final adjustment based on the context of the problem, because students only considered a multiple of the common difference. Also, when students used explicit strategies, the model was not always correctly applied. These errors might be linked to the extensive students' experience manipulating numbers without meaning, making no sense of what the coefficients in the linear pattern represent. Results from the second task show that the level of students' efficiency increased. They displayed a greater level of awareness in the selection of the proper strategies. For example, the



inadequate use of direct proportion was no longer observed. In spite of these differences, questions that involved reverse thinking provoked a shift on the type of approaches used by the students.

### **The role of visualization in students' reasoning**

Presmeg (2006) states that a strategy is of visual nature if the image/drawing plays a central role in obtaining the answer, either directly or as a starting point for finding the rule. In this sense the following strategies are included in this group: *counting* (C), *whole-object with visual adjustment* ( $W_3$ ), *difference with rate-adjustment* ( $D_3$ ) and *explicit* (E). *Counting* was always a successful strategy but only useful in solving near generalization questions. Drawing a picture of a certain object and counting its elements was an action used in near generalization questions and does not lead to a generalized strategy. Strategy  $W_3$  was only used by one pair of students, when solving the first task. They had only applied it correctly in near generalization. This type of reasoning involves a higher level of abstraction in visualization. In spite of not being one of the most frequent strategies, students who used  $D_3$  always reached a correct answer. This fact enhances the relevance of understanding the context surrounding the problem. Finally, the application of an *explicit* strategy leads to a high level of efficacy. Students based their work on sequence structure, referring to the relation between the variables. Only a few pairs of students “disconnected” the sequence from the context and used a mix of different variables.

### **DISCUSSION AND CONCLUSION**

In this study, patterning tasks were selected to set an environment to analyse students' generalization strategies, difficulties, as well as the impact of the use of visual strategies when solving this type of problems.

Concerning the research questions posed, some significant findings are: (a) a variety of strategies were identified, although the frequencies observed were different; (b) students achieved better results in near generalization questions than in far generalization questions and, even with some experience with patterning activities acquired along the study, reverse thinking was still complex for many of them; (c) some of the pairs worked exclusively on number contexts using inadequate strategies like the application of direct proportion, using multiples of the difference between consecutive terms without a final adjustment and mixing variables. As a result of the study, this tendency was gradually inverted as most students understood the limitations of some of those strategies; (d) in some cases, students revealed difficulties in finding a functional relation, frequently generalizing rules that were verified for particular cases or showing a fixation for a recursive strategy; (e) visualization proved to be a useful ability in different situations like making a drawing and counting its elements, to solve near generalization tasks, and “seeing” the structure of the pattern, finding an *explicit* strategy to solve far generalization tasks; (f) the application of visual strategies allowed students to find different



expressions to represent the same pattern; (g) it was also evident that they privileged constructive generalizations, *seeing* the structure of the pattern as a set of disjoint elements.

To conclude, it is important to provide tasks which allow the application of a diversity of strategies and to encourage the students to use and understand the potential of visual strategies, establishing a relationship between the number context and the visual context to better understand the meaning of numbers and variables. The connection between parallel approaches and the exploration of the potentialities and limitations of each strategy can contribute to the development of a more flexible reasoning, essential to problem solving.

## REFERENCES

- Amaro, G., Cardoso, F., & Reis, P. (1994). *Terceiro Estudo Internacional de Matemática e Ciências, Relatório Internacional, Desempenho de alunos em Matemática e Ciências: 7.º e 8.º anos*. Lisboa: IIE.
- Barbosa, A. (2010). *A resolução de problemas que envolvem a generalização de padrões em contextos visuais: um estudo longitudinal com alunos do 2.º ciclo do ensino básico*. Tese de Doutoramento em Estudos da Criança: Universidade do Minho.
- Devlin, K. (2002). *Matemática: A ciência dos padrões*. Porto: Porto Editora.
- Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana & V. Villani (Eds.), *Perspectives in the teaching of geometry for the 21<sup>st</sup> Century* (pp. 29-83). Boston: Kluwer.
- Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York, NY: Macmillan.
- Gardner, H. (1993). *Multiple intelligences: The theory in practice*. New York: Basic Books.
- Goldenberg, E. P. (1996). "Habits of Mind" as an organizer for the curriculum. *Journal of Education*, 178(1), 13-34.
- Healy, L., & Hoyles, C. (1996). Seeing, doing and expressing: An evaluation of task sequences for supporting algebraic thinking. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the PME20* (Vol. 3, pp. 67-74). Valencia, Spain.
- Lannin, J. (2005). Developing algebraic reasoning through generalization. *Mathematics Teaching in the Middle School*, 8(7), 342-348.
- Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic generalization strategies: factors influencing student strategy selection. *Mathematics Education Research Journal*, 18(3), pp. 3-28.
- ME-DGIDC (2007). *Programa de matemática do ensino básico*. Lisboa: Ministério da Educação, Departamento de Educação Básica.

### Working Group 3

- NCTM (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- OECD (2004). *PISA 2003: Relatório internacional, o rendimento dos alunos em Matemática*. Lisboa: Santillana-Constância.
- Orton, A., & Orton, J. (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 104-120). London: Cassel.
- Presmeg, N. (2006). Research on visualization in learning and teaching mathematics: Emergence from psychology. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205-235). Dordrecht: Sense Publishers.
- Ramalho, G. (1994). *As nossas crianças e a Matemática. Caracterização da participação dos alunos portugueses no "Second International Assessment of Educational Progress"*. Lisboa: DEPGF.
- Rivera, F., & Becker, J. (2005). Figural and numerical modes of generalizing in Algebra. In *Mathematics Teaching in the Middle School*, 11(4), pp. 198-203.
- Rivera, F., & Becker, J. (2008). Abduction in pattern generalization. In Woo, J. H., Lew, H. C., Park, K. S., & Seo, D. Y. (Eds.). *Proceedings of the PME31* (Vol. 4, pp. 97-104). Seoul: PME.
- Shama, G., & Dreyfus, T. (1994). Visual, algebraic and mixed strategies in visually presented linear programming problems. *Educational Studies in Mathematics*, 26, 45-70.
- Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20(2), 147-164.
- Steen, L. (1990). *On the shoulders of giants: New approaches to numeracy*. Washington, DC: National Academy Press.
- Tall, D. (1991). *Advanced mathematical thinking*. The Netherlands: Kluwer Academic Publishers.
- Vale, I., Barbosa, A., Barbosa, E., Borralho, A., Cabrita, I., Fonseca, L., & et. al. (2009). *Padrões no Ensino e Aprendizagem da Matemática: Propostas curriculares para o ensino básico*. Viana do Castelo: ESEVC.
- Yin, R. K. (1989). *Case study research: Design and methods*. Newbury Park, CA: Sage.

# THE ROLE OF TECHNOLOGY IN DEVELOPING PRINCIPLES OF SYMBOLICAL ALGEBRA

Giampaolo Chiappini

Istituto per le Tecnologie Didattiche, Consiglio Nazionale delle Ricerche (CNR),  
Genova, Italy

*For a significant percentage of students, the current teaching of algebra is unable to develop skills, knowledge and forms of control that are necessary to fully master this domain of knowledge. This paper builds on Peacock argument that the difficulties of teaching and learning are due to epistemological obstacles rooted in the development of the Symbolical Algebra during the 19th century. The paper shows the role of AlNuSet system, particularly its two environments named Algebraic Line (AL) and Algebraic Manipulator (AM), in the conceptual development of Symbolical Algebra.*

## INTRODUCTION

Moving from arithmetic to algebra, students must overcome many obstacles that can be correlated with those that mathematicians have faced during the 18th century and the first half of the 19th century, with the development of Symbolical Algebra. In fact, the definitive overcoming of a vision of algebra as Universal Arithmetic, where the meaning of symbols is legitimated by the semantic references of numerical nature, dates back to that time. In 1830 Peacock distinguished between Symbolical Algebra and Arithmetical Algebra (Peacock, 2004a, 2004b). According to Peacock, Arithmetical Algebra differs from Arithmetic for the use of letters that allow operation on indeterminate quantities, namely on quantities whose value is not specified. In this algebra, however, the operations are those of Arithmetic, with the same natural limitations that they have in this knowledge domain, so that an expression like  $a-b$  has a sense only for  $b < a$  (Peacock, 2004a). With Symbolical Algebra, the meaning of symbols becomes operational, namely defined according to the operation (and its properties). In Symbolical Algebra, symbols can represent any kind of quantity that is incorporated into them through specific operations. The extension of the operational field, however, is not only the result of a process of generalization, but it marks a real change in terms of principles and culture of the discipline. Peacock established two principles from which the algebra becomes operational and deductive (Peacock, 2004b). The first principle states that the result of an operation that is impossible for arithmetic, as  $a-b$  for  $b > a$ , can be regarded as a mathematical reality. The second principle is the principle of permanence of equivalent forms. It attests that the equivalence between two expressions can be deduced from the properties of operations that are combined in their form, and this guarantees the equality of the results of calculations of the two expressions, for each

value assigned to the variable. Peacock's work can be very useful for understanding and framing epistemological obstacles encountered in teaching algebra.

### **NATURE OF THE OBSTACLES IN LEARNING ALGEBRA**

A reflection on the work of Peacock for educational purposes was performed by Menghini (1994). In this work the author stresses the importance of presenting algebra as a tool to quantitatively interpret real-world situations. She also highlights the need for an axiomatic approach to allow students to internalize the principles developed by Peacock, to understand the jump that takes place in the passage from arithmetic to algebra and to develop knowledge and meta-control skills that are specific of the Symbolical Algebra after Peacock. For example, moving from arithmetic to algebra, it is important to understand that algebraic symbols can represent not only the numbers used in arithmetic, but also other numbers, such as negative numbers. That is, it is important to understand that the operations of arithmetic are not only transferred into the new system of signs of algebra, but in the context of algebra their meaning and their operational field may be extended. It is also important to understand that if two expressions are equal to each other for any value of their variable, this can be demonstrated through the use of transformation rules that guarantee the preservation of equivalence in transformation. Note that, in building the described knowledge and skills, it is necessary that students learn to perform various forms of control, and more precisely: the control of an algebraic expression on the numeric level, i.e. the ability to control what the expression denotes when the value of the variable is modified (Arzarello et al, 2001); the control of the expression at a syntax level, i.e. the ability to transform it by applying rules which guarantee the equivalence in the transformation (see Kieran, 2006); the control of transformation of the expression at a theoretical level, i.e. the ability to justify the algebraic transformation of an expression as logical-deductive activity within an axiomatic framework (Cerulli & Mariotti, 2002; Pedemonte, 2010). Using the terminology of Peacock, the first control level is typical of the Arithmetical Algebra, while the other control levels are specific of the Symbolical Algebra.

### **NEED FOR A NEW TEACHING OF ALGEBRA**

The current teaching of algebra is not able to promote the conceptual development described above with a significant percentage of students. The causes are certainly different. I believe that this development cannot be achieved through the mechanical and repetitive exercises of the traditional didactics of algebra. It is necessary to allow the students to make meaningful experiences of contents and algebraic concepts in a sort of Didactical Laboratory of Mathematics that only the use of appropriate tools can help to structure (Chiappini & Reggiani, 2003). In the following I shall refer to the system AlNuSet to highlight some practices centred on its use which can facilitate the development of the two principles developed by Peacock concerning Symbolical Algebra. AlNuSet is a system that was developed at ITD-CNR in the ReMath EU project to improve the teaching and learning of algebra (Chiappini & al,

2008; Pedemonte & Chiappini, 2008; Chiappini & Pedemonte, 2010). AlNuSet consists of three environments: the Algebraic Line (AL), the Algebraic Manipulator (AM), the Function Environment (FE). ). In a previous paper presented at CERME 6 (Chiappini & Pedemonte, 2010) the main ideas underlying the development and design of these three components of AlNuSet have been presented and its design has been illustrated and justified theoretically within the anthropological framework. In that paper it has been shown that the instrumented techniques available in the three environments of AlNuSet structure a new phenomenological space where algebraic objects, relations and phenomena are reified by means of representative events that fall under the visual, spatial and motor perception of students and teachers. Moreover, it has been evidenced that in the phenomenological space determined by the use of the instrumented technique of AlNuSet, algebra can become a matter of investigation.

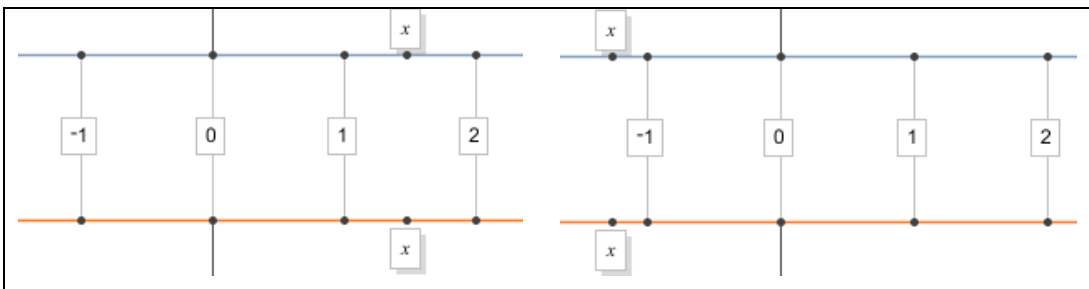
In this article I will present some teaching practices supported by the use of the AL and of the AM in the construction of knowledge, skills and abilities concerning the extension of the operational field of the subtraction (with the construction of different meanings that the sign "-" assumes in algebra) and the principle of equivalent forms (within the development of a deductive and axiomatic vision of algebra).

## **THE HISTORICAL ROOTS OF THE DIFFICULTIES REGARDING THE USE OF THE SIGN "-"**

In the early development of algebra and the solution of equations with negative roots, a broad debate took place among mathematicians about the nature of negative numbers. The transformation of the negative number into a mathematical entity, well-founded at conceptual level, overcome considerable disagreements in the community of mathematicians. Many mathematicians of the 17th and 18th century referred to these numbers as "false" or "absurd" because they could not conceive a quantity that was "less than nothing" and therefore had great difficulty conceptualizing it as a number, namely as an ideal entity with specific properties. Indeed, for a long time, numbers as +4 and -4 were defined as a quantity respectively to add and to subtract, as an operational value and no predicative value was assigned to the signs "+" and "-". It took a long time to achieve consensus about validity of the different meanings to the sign "-" in writing as -4 or  $a-b$  for  $b > a$ . The work of Peacock has played a crucial role in this development. The difficulties that students face extending subtraction to negative numbers reflect the difficulties that characterized the conceptualization of these numbers on the historical level. In fact, many students have difficulties interpreting the meaning of the sign "-" placed before a number, as in -3. It is not easy for the teacher to help them overcome these difficulties. Below I will highlight the role of mediation provided by the AlNuSet in developing these specific meanings.

## THE EXPERIENCE OF THE EXTENSION OF THE OPERATIONAL FIELD OF SUBTRACTION

In AlNuSet, the Algebraic Line (AL) is a normal number line empowered with new operative and representative opportunities of algebraic nature through the exploitation of digital technology. The starting point of the transformation of the number line into the AL is the possibility to associate a letter to a point on the line. The figure below shows two states of the AL of AlNuSet after the editing of the letter  $x$ . In this environment the letter  $x$  becomes something very concrete and tangible - a mobile point on the line that can be dragged with the mouse. If the point is not moved,  $x$  assumes a definite value on the line. Dragging the point with the mouse, the value that  $x$  assumes on the line changes.



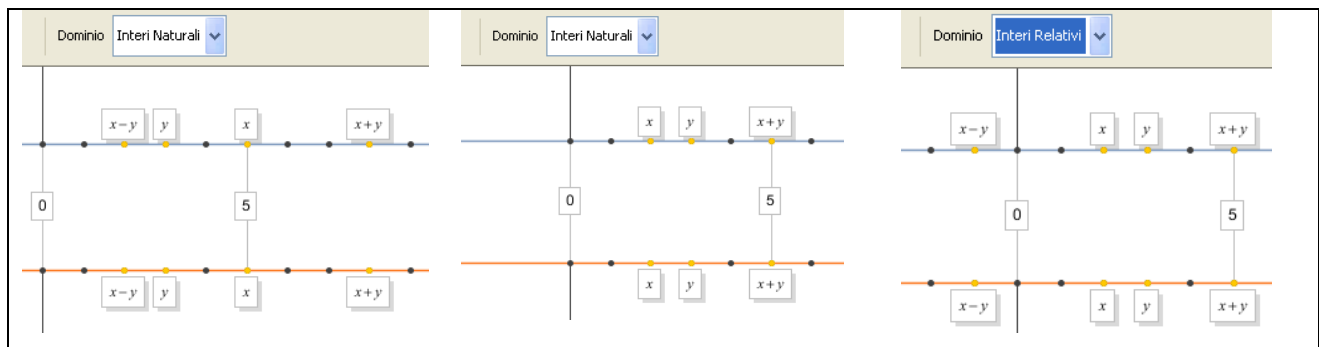
**Fig 1: Two states of the Algebraic Line after the editing of the letter  $x$**

This environment offers different operative and representative possibilities of quantitative nature to operate with algebraic expressions, among these are the possibilities of :

- editing algebraic expressions that are automatically represented on the line associated with points that indicate the value of the expression according to the value assumed by their variables on the same line,
- dragging the mobile point corresponding to algebraic variables, whilst maintaining the relationship between points, expressions and values.

In addition, the AL can be instantiated in various numerical domains, including natural numbers, integers, rational numbers and real numbers. On the line only numbers and expressions that are compatible with the choice made are visible. These features have great importance at didactical level. Consider this didactical situation:

Working with natural numbers, associate the variables  $x$  and  $y$  with mobile points on the line, and edit the expressions  $x+y$  and  $x-y$  (first image of the Fig. 2). Drag the points  $x$  and  $y$ . Can you explain what happens? Repeat working with the integers.



**Fig 2: Three images highlighting crucial representative events of the didactical situation**

Working with natural numbers, mobile points  $x$  and  $y$  can be dragged only to 0 or positive integers. By dragging points  $x$  and  $y$ , it is easy to verify that the expression  $x+y$  is always represented on the line, whatever the value of  $x$  and  $y$  is, while when  $y > x$ , an important representative phenomenon occurs: the expression  $x-y$  and the point associated to it disappear from the line (second image of Fig 2). This experience can be used to reflect on the domain in which an expression is defined, to investigate and clarify the conditions under which the subtraction is closed in  $N$ . When the domain is extended to include the integers, however, the expression is defined for any value of  $x$  and  $y$  (third image of Fig 2). In other words, it is possible to make a concrete experience of the principle of Peacock according to which the expression  $x-y$  assumes a status of mathematical reality when  $y > x$ , through the extension of the operational domain of the subtraction to the relative integer. These features make the AL an important educational tool for learning to control the domain of existence of expressions through new types of activities such as:

Under what conditions will the following expressions will be represented on the AL if you select the domain of natural numbers:  $6 \cdot x - y$ ;  $2 \cdot x - 4 \cdot y$ ;  $2 - x - 8 \cdot y$ .

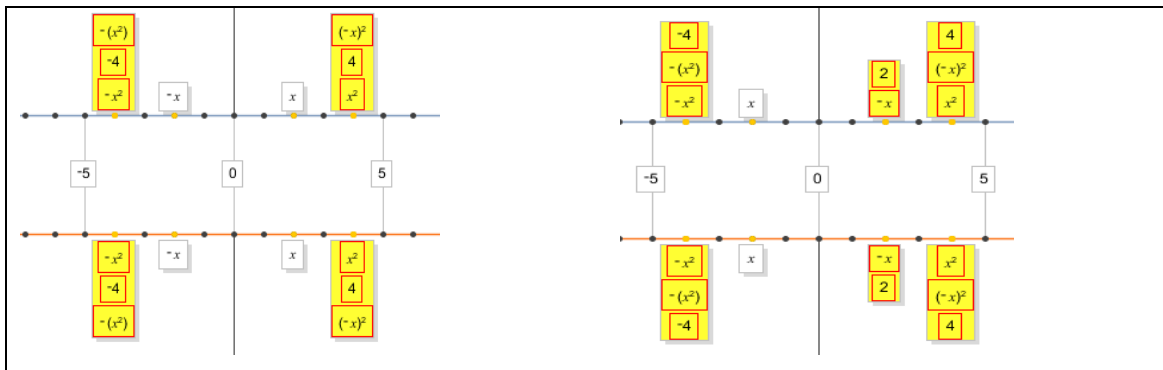
It is important to note that through tasks such as these, the AL becomes a kind of laboratory where it is possible to explore the domain of existence of an expression, to make conjectures on the conditions of its existence and to validate these *hypotheses*.

## **OPPOSITE AND EQUIVALENT EXPRESSIONS: THE EXPERIENCE WITH THE ALGEBRAIC LINE OF ALNUSET**

Consider the following assertion: “The two expressions  $-x$  and  $-x^2$  considered in the domain of relative integers always represent a negative number”. What do you think about this statement? Justify your answer. Construct the two expressions on the AL and verify your answer using what is displayed on the AL during the interaction. Is there any difference among the following expressions:  $-x^2$  and  $(-x)^2$  and  $-(-x)^2$ ?

Solving this task in an experimentation with AlNuSet, many students aged 15 answered that “ $-x$  is a negative number and  $-x^2$  is always a positive number because the even power of a negative number is positive”. This answer shows that these students are not able to control either the use of the “-” sign with the meaning of opposite and the connection of the algebraic notation with their referential objects.

Then these students represented the expression  $-x$  and  $-x^2$  and the other algebraic expressions indicated in the task on the algebraic line of AlNuSet (see Fig 3).



**Fig. 3: Two images of the exploration of the didactical situation on the AL**

They dragged the variable  $x$  and observed that the point corresponding to  $-x^2$  on the algebraic line is always located on negative numbers while the point corresponding to  $-x$  is positive when  $x$  is negative and vice-versa “We have verified with AlNuSet that what we have written is false, so the assertion reported in the text that  $-x^2$  is always negative is true”. “With AlNuSet we have verified that  $-x^2$  is a negative number,  $(-x)^2$  is a positive number and  $-(-x)^2$  is a negative number coincident with  $-x^2$ ” Some students were quite amazed by these result. A pair of students wrote: “ $-x^2$  and  $(-x)^2$  are the same thing because making the square you always obtain a positive number...” and after the verification with AlNuSet “...Ah, hence they are not the same thing, because in one expression the minus sign is inside the parenthesis while in the other it is not”.

The features of AlNuSet have been exploited both to destabilize students’ wrong conceptions regarding the connection of the algebraic rules used in a sign and its referential object and to develop new appropriate conceptions of this connection. “Through the observation of the line it emerges that, except at 0,  $x$  and  $-x$  are opposite on the line. Moreover  $-x^2$  and  $(-x)^2$  are not the same thing, they are opposite while  $-x^2$  is always equal to  $-(-x)^2$ ”. The example shows the mediating role of two important features of the algebraic line, namely that:

- two expressions are opposite when their respective points on the line are *always* symmetric about the point 0,
- two expression are equivalent when they are *always* associated with the same point on the line.

These features have been exploited to mediate both the comprehension of the symbolic function of the sign “-” and the development of the notions of equivalent and opposite expressions. Concerning the notion of equivalent expression, let me consider another task:

Explain what the expression  $3 \cdot x + 1$  represents, considering  $x$  as natural number. Write an equivalent expression and use the Algebraic Line to verify their equivalence.



A pair of students wrote, “The expression  $3 \cdot x + 1$  represents the triple of  $x + 1$ , and stated  $3 \cdot (x + 1)$  to be an equivalent expression to  $3 \cdot x + 1$ . However, when they represented the two expressions on the AL, they observed that they are not equivalent because they do not refer to the same point on the line while dragging the variable  $x$  along the line. The emergence of a contradiction between the performed hypothesis and the results visualized on the algebraic line helped them reflect on the structure of the two expressions through the connection of the two expressions to their referential objects on the AL. “Using AlNuSet we have seen that the triple of  $x + 1$  is  $(x + 1) \cdot 3$  while  $3x + 1$  is 3 times a natural number plus 1”. Successively they conjecture  $2x \cdot x + 1$  to be equivalent to the expression  $3 \cdot x + 1$ . They verify with AlNuSet that also this hypothesis is wrong, they produce  $2x + x + 1$  as equivalent expression, that successively transforms into  $(2 + 1) \cdot x + 1$  and verify that this is equivalent to the given expression (see next paragraph). The two reported examples highlight the mediating role of AlNuSet in the appropriation of the algebraic symbolism through the comprehension of how algebraic operations characterize the expressions and determine what they denote. In particular the last example shows the construction of the notion of equivalent expressions through a quantitative approach. In the successive sections I will analyze the role of AlNuSet in the development of equivalent expressions through an operational approach coherent with the Peacock’s principle of permanence of equivalent forms.

## **DEVELOPMENT OF KNOWLEDGE, SKILLS AND ABILITIES TO CONTROL THE ALGEBRAIC MANIPULATION**

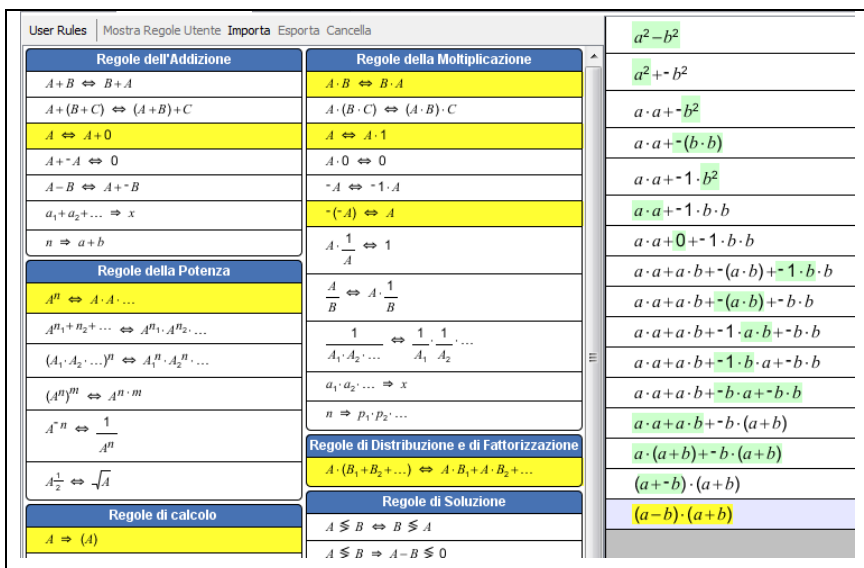
Much of current algebraic educational practice is unable to promote the comprehension of the notion of equivalent expressions through an axiomatic and deductive vision of the algebraic transformation coherent with Peacock’s principle of permanence of equivalent forms. This is evidenced by the large number of students who, despite years of work with literal expressions, are not aware that the algebraic manipulation preserves the equivalence in the transformation, that equivalence between two expressions can be inferred from the properties of operations combined in their form, and that through the use of such properties more complex transformation rules can be proved. In this regard we note that in ordinary educational practice, the properties of operations are presented to students, but these properties are not used to perform algebraic transformations. The steps of transformation based explicitly on the properties of operations do need to be “condensed” into more powerful transformation rules. In ordinary educational practice, teachers tend to this too early in the students’ learning. This occurs because teachers consider the basic steps of transformation as “obvious”, and because the costs of their explicit use (in terms of cognitive effort, time, and even consumption of paper) are not considered compatible with the constraints of schooling. For these reasons, the approach to algebraic transformation, centred on a long-term use of the basic properties of operations, is not commonplace in ordinary teaching practice. To promote the development of appropriate skills in algebra, both at the operative level

(ability to follow the rules) and at the conceptual level, a transformation in the educational practice is necessary. I believe that a new educational approach should be based on the use of the properties of operations and should not be separated from a deductive, properly mediated axiomatic approach.

## THE ALGEBRAIC MANIPULATOR OF ALNUSET

The AM of AlNuSet was designed to approach algebraic transformation according to the perspective outlined above. Figure 4 below shows the interface of this manipulator. It is divided into two distinct spaces: the space where symbolic manipulation commands available for the transformation activities are reported (in the figure only a part of the commands is visible); the space where the algebraic expression or proposition is inserted to be manipulated and where the transformation is realised (in the figure, an example of algebraic transformation).

In the interface Algebraic Manipulator (AM) (see Fig. 4) makes available commands for the algebraic transformation that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. When a sub-expression is selected, only the commands of the interface that can be applied to it are automatically activated and highlighted to the user. This feature is very important from an educational standpoint because it allows students to explore the connection among the rules of transformation available in the interface, how they may be applied and the result that the application of a rule produces.



**Fig 4: The interface of the Algebraic Manipulator of AlNuSet**

Another feature of this AM allows students to create new transformation rules, once these have been demonstrated using the available commands. These new rules can be saved and included in the interface to be used in subsequent transformations. This feature is essential to mediate a didactic of algebraic transformation based on axiomatic and deductive approach, centred on the use of some basic axioms

(properties of operations) and the demonstration of progressively more complex rules for the algebraic transformation (theorems).

### EXAMPLES OF USE OF THE AM OF ALNUSET IN AN AXIOMATIC AND DEDUCTIVE FRAME OF ALGEBRAIC TRANSFORMATION

These are the commands concerning the operation of subtraction that are available in the interface of this AM.

$A + ^{-}A \Leftrightarrow 0$	$A - B \Leftrightarrow A + ^{-}B$	$^{-}A \Leftrightarrow ^{-}1 \cdot A$	$^{-} (^{-}A) \Leftrightarrow A$
-------------------------------	-----------------------------------	---------------------------------------	----------------------------------

These commands correspond to properties of the subtraction that students previously explored on the AL through the quantitative approach of this environment. In the Algebraic Manipulator these properties become operational tools to manipulate numerical and/or literal expressions, preserving the equivalence in the transformation.

In the following example, the commands used to perform the transformative steps are reported in the second column. These transformative steps are unlikely to emerge in the standard educational practice. On the contrary, this approach is important to learn to work consciously with the transformation rules of algebra. Often these rules are not entirely clear even to those who seem able to solve correctly. For example, several students who in the ordinary practice of algebra successfully transform  $5 \cdot x - x$  into  $4 \cdot x$ , very often are not aware of the fact that behind their result there is the application of the neutral element of the product and the distributive property of product with respect to the sum. Not to mention those for which  $5 \cdot x - x = 5$ .

$5 \cdot x - x$	$A \Leftrightarrow A \cdot 1$	$(5 + ^{-}1) \cdot x$	$n \Rightarrow a + b$
$5 \cdot x - x \cdot 1$	$A \cdot B \Leftrightarrow B \cdot A$	$(4 + 1 + ^{-}1) \cdot x$	$A + ^{-}A \Leftrightarrow 0$
$5 \cdot x - 1 \cdot x$	$A - B \Leftrightarrow A + ^{-}B$	$(4 + 0) \cdot x$	$A \Leftrightarrow A + 0$
$5 \cdot x + ^{-}(1 \cdot x)$	<i>Semplifica()</i>	$(4) \cdot x$	<i>Semplifica ()</i>
$5 \cdot x + ^{-}1 \cdot x$	$A \cdot (B_1 + B_2 + \dots) \Leftrightarrow A \cdot B_1 + A \cdot B_2 + \dots$	$4 \cdot x$	

The following examples concern the proof of the equivalence respectively between  $-x^2$  and  $^{-}(-x)^2$  and between  $3x+1$  and  $2x+x+1$  after the activity performed on the Algebraic Line.

## Working Group 3

$-(-x)^2$	$-A \Leftrightarrow -1 \cdot A$	$3 \cdot x + 1$	$n \Rightarrow a + b$
$-(-1 \cdot x)^2$	$(A_1 \cdot A_2 \cdot \dots)^n \Leftrightarrow A_1^n \cdot A_2^n \cdot \dots$	$(2+1) \cdot x + 1$	$A \cdot (B_1 + B_2 + \dots) \Leftrightarrow A \cdot B_1 + A \cdot B_2 + \dots$
$-((-1)^2 \cdot x^2)$	$A^n \Leftrightarrow A \cdot A \cdot \dots$	$2 \cdot x + 1 \cdot x + 1$	$A \Leftrightarrow A \cdot 1$
$-((-1) \cdot (-1) \cdot x^2)$	$1 \Leftrightarrow -1 \cdot -1$	$2 \cdot x + x + 1$	
$-(1 \cdot x^2)$	$A \Leftrightarrow A \cdot 1$		
$-(x^2)$	$-A \Leftrightarrow -1 \cdot A$		
$-1 \cdot (x^2)$	Semplifica ()		
$-1 \cdot x^2$	$-A \Leftrightarrow -1 \cdot A$		
$-x^2$			

The activity with the AM allows students to experience that the transformational algebraic activity is the demonstration of equivalence of different algebraic forms. The characteristics of the manipulator allow the teacher to introduce abstract notions in the algebraic activity such as those of proof, axiom, theorem, equivalent expressions exploiting what the system exhibits in the interaction to construct an idea of these notions.

## CONCLUSIONS

In this article we have shown that when moving from arithmetic to algebra, technology can help to overcome some crucial epistemological obstacles. It has been shown that AlNuSet can be easily exploited to allow students to experiment with the two principles developed by Peacock and to grasp the axiomatic and deductive nature of symbolic Algebra. It has been highlighted that the two environments of AlNuSet allow students to interact with algebraic objects and concepts, of abstract and formal nature, in a concrete and tangible way. This is possible because digital technology allows to create phenomena related to algebraic concepts that can be controlled through a visual, spatial and motor experience. In this way, it is possible to have a direct and concrete experience with mathematical objects and concepts. This favours the construction of sense for these objects and concepts and the development of forms of control needed to master the language of algebra at the symbolic level.

## REFERENCES

- Arzarello F., Bazzini L., Chiappini G., (2001), A Model for analysing algebraic processes of thinking. In R. Sutherland et al. (Eds.), *Perspectives on school algebra* (pp. 61-81). Dordrecht: Kluwer.
- Cerulli, M., & Mariotti, M. A. (2002). L'Algebrista: un micromonde pour l'enseignement et l'apprentissage de l'algèbre. *Sciences et techniques éducatives*, 9, 149-170.
- Chiappini, G. & Reggiani, M. (2003). Toward a didactical practice based on mathematics laboratory activities. Proceedings of the Third Conference of the

European Society for Research in Mathematics Education (CERME 3), Bellaria, Italy.

Chiappini G., Pedemonte B (2010). The design of new digital artefacts as key factor to innovate the teaching and learning of algebra: the case of AlNuSet. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of Sixth Conference of European Research in Mathematics Education (CERME 6)*. Lyon, France: Institut National De Recherche Pédagogique (INRP).

Chiappini G., Pedemonte B., Robotti E., (2008). Using AlNuSet to construct the notions of equivalence and equality in algebra. In Kendall M. and Samways B (Eds.), *Learning to Live in the Knowledge Society* (pp. 345-348). New York: Springer.

Kieran, C. (2006). Research on the learning and teaching of algebra. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the Psychology of Mathematics Education: Past, present and future* (pp. 11-49). Rotterdam: Sense.

Menghini M. (1994). Form in Algebra: Reflecting, with Peacock, on Upper Secondary School Teaching, *For the learning of Mathematics*, 14, 9-14.

Peacock G. (2004a). *Treatise on Algebra. Volumes I: Arithmetical Algebra*. New York: Dover Publications.

Peacock G. (2004b). *Treatise on Algebra. Volumes II: Arithmetical Algebra*. New York: Dover Publications.

Pedemonte, B. & Chiappini, G. (2008). Algebra On Numerical SETs: a system for teaching and learning algebra. *International Journal of Continuing Engineering Education and Life-Long Learning*, 18(5/6), 627-639.

Pedemonte B., 2010, The algebraic manipulator of ALNUSET: a tool to prove. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of Sixth Conference of European Research in Mathematics Education (CERME 6)*. Lyon, France: Institut National De Recherche Pédagogique (INRP).

# SECONDARY SCHOOL STUDENTS' PERCEPTION OF BEST HELP GENERALISING STRATEGIES

Boon Liang Chua<sup>a</sup> and Celia Hoyles<sup>b</sup>

<sup>a</sup>National Institute of Education, Nanyang Technological University

<sup>b</sup>London Knowledge Lab, University of London

*This paper reports a study whose aim was to examine secondary school student choices of generalising strategy to determine what they would judge as the most helpful strategy for expressing generality. Data were collected from 45 Secondary One students through administering a questionnaire which contained one linear and one quadratic generalising tasks. The students had to select and justify the strategy that they believed would best help them to establish the rules. The data were analysed and revealed differences in the judgements of the more able and the less able students regarding the best-help strategy. The more able students' choices of best-help generalising strategy seemed to vary across the two different types of tasks whereas those of the less able students appeared to remain unchanged.*

**Keywords:** Pattern generalisation, student beliefs, generalising strategy

## BACKGROUND

Modelling mathematical problems as examples to demonstrate how the problems in question can be worked out is an important teaching activity that features strongly in most mathematics lessons. When illustrating examples, teachers tend to introduce their methods to show students the way to deal with these problems. They could have picked these methods based on their beliefs about what students are capable of understanding and how they will learn best (Chua & Hoyles, 2010b). All this is done with the good intention of wanting to help students experience some success. And students generally accept and follow the teachers' methods. But some teachers may go as far as to enforce their methods and get students to comply. So the big issue here is what students think of their teachers' methods. Do the teachers' methods help them understand the examples and learn mathematics better?

For generalisation of number pattern, the literature identified several kinds of generalising strategies used by students to establish a rule between the term and its position in the pattern (Drury, 2007; Lannin, 2005; Lee & Freiman, 2006; Rivera & Becker, 2008; Steele, 2008). In some of these recent studies, the students were even presented with different-looking rules that could be used to describe the same underpinning pattern and asked to justify how these rules could all be equivalent to one another (Drury, 2007; Lee & Freiman, 2006; Rivera & Becker, 2008). Such an activity challenges them to use different strategies to come up with multiple ways of seeing the same pattern. However, none of these studies went further to ask students for the kind of strategy that they believe will best help them to construct those rules. Thus the present study sought to fill in this gap by examining secondary school

student choices of generalising strategy to ascertain what they would judge as the most helpful strategy to establish the functional rule for deriving any term in the pattern. It is hoped that the findings of the present study could provide valuable insights for teachers, teacher educators and curriculum developers.

## THEORETICAL FRAMEWORK

Pattern generalising tasks are a common feature of school mathematics in many countries. Generally, researchers concur that such tasks are a powerful vehicle not only for introducing the notion of variables (Mason, 1996) but also for developing two core aspects of algebraic thinking: the emphasis on relationships among quantities like the inputs and outputs (Radford, 2008) and the idea of expressing an explicit rule using letters to represent numerical values of the outputs (Kaput, 2008). Apart from these merits, pattern generalising tasks are also useful for developing the notion of equivalence of algebraic expressions as well. A typical generalising task involves facilities like identifying a numerical pattern, extending the pattern to make a near and far generalisation, and articulating the functional relationship underpinning the pattern using symbols.

There is a wealth of research that examines students' generalising strategies and reasoning when they deal with pattern generalising tasks. Students had been found to use a variety of strategies for constructing the functional rule underpinning the pattern depicted in the tasks. For instance, Rivera and Becker (2008) established three types of strategy that students employed: (1) *numerical*, which uses only cues established from any pattern that is listed as a sequence of numbers or tabulated in a table to derive the rule, (2) *figural*, which only applies in generalising tasks that depict the pattern using diagrams, and relies totally on visual cues established directly from the structure of the figures to derive the rule, and (3) a combination of both the numerical and figural approaches.

Different types of strategies do exist even within the *numerical* solutions. Bezuska and Kenney (2008) identified three such strategies that involve recursion: (1) *comparison*, where the terms in a given number sequence are compared with corresponding terms of another sequence whose rule is already known, (2) *repeated substitution*, where each subsequent term in a number sequence is expressed in terms of the immediate term preceding it, and (3) *the method of differences*, also known as finite differences in Mathematics, which is an algorithm for finding explicit formulae that are polynomial equations.

The *figural* solutions were further distinguished into two different categories by Rivera and Becker (2008): (1) *constructive generalisation*, which occurs when the diagram given in a generalising task is viewed as a composite diagram made up of non-overlapping components and the rule is directly expressed as a sum of the various sub-components, and (2) *deconstructive generalisation*, which happens when the diagram is visualised as being made up of components that overlap, and the rule

is expressed by separately counting each component of the diagram and then subtracting any parts that overlap.

Apart from these two kinds of *figural* strategy, Chua and Hoyles (2010a) introduced two other strategies into the existing classification scheme developed by Rivera and Becker (2008). One of them occurs when one or more components of the original diagram are rearranged into something more familiar. This newly reconfigured figure then unveiled the pattern structure and facilitated the construction of the functional rule. The other happens when the original diagram is viewed as part of a larger composite figure, from which the functional rule is generated by subtracting the sub-components from this composite figure.

To sum up, the literature review leads us to recognise the diverse ways of constructing the functional rule that represents the pattern in a generalising task. Hence this present study aims to add to the body of work on pattern generalisation by seeking to answer some of these questions: Which strategies would students believe would best help them to work out the rule? How would more able students' choices of best-help strategies compare with those of the less able students? If the rule underpinning the pattern were to change from a linear to a quadratic relationship, would the best-help strategies that students considered for the former case change to suit the latter?

## METHODS

Student data were collected through a questionnaire administered to 45 Secondary One students (aged 13 years) from a secondary school. 29 of the students came from the Express course and 16 from the Normal (Academic) course. The students were placed in these courses based on their performance at a national examination taken at the end of their primary education when they were 12 years old. These students, 22 boys and 23 girls, were selected by the school according to their Mathematics grade in the national examination. Amongst the Express students who were considered academically more able than the Normal students, 15 scored an A or A\* (high distinction) for Mathematics while the remaining 14 scored a B or C. All the 16 Normal (Academic) students scored a B or C because no one obtained A or A\*.

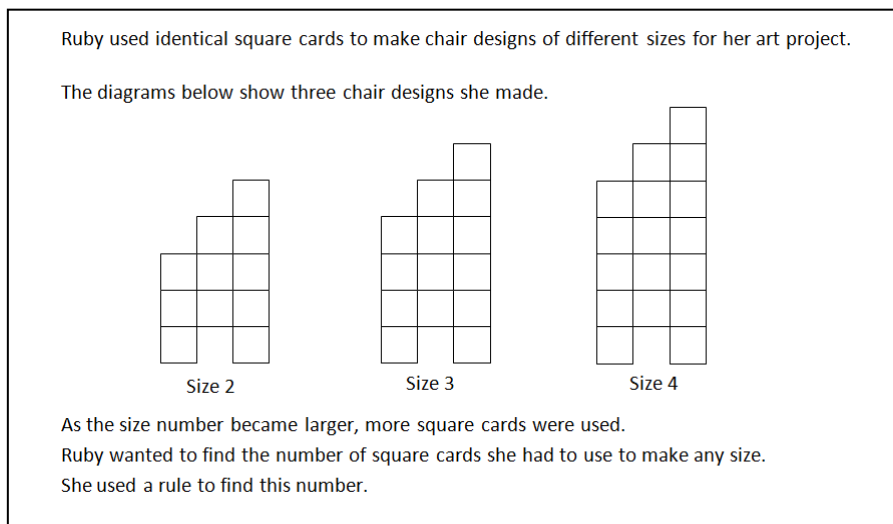
These students had already learnt the topic of number patterns, which is part of the Singapore mathematics curriculum, before participating in this study. So they should be able to continue any pattern, whether presented as a sequence of either numbers or figures, for a few more terms, make a near and far generalisation and derive the functional rule in the form of an algebraic expression for predicting any term. Further, they should also be far more familiar in dealing with linear patterns than with non-linear ones, which are less common in their mathematics textbook.

Before administering the questionnaire, a worksheet comprising the two generalising tasks that were used in the questionnaire was distributed to every student. The two tasks, *High Chair* and *Christmas Party Decoration*, are presented in Figures 1 and 2

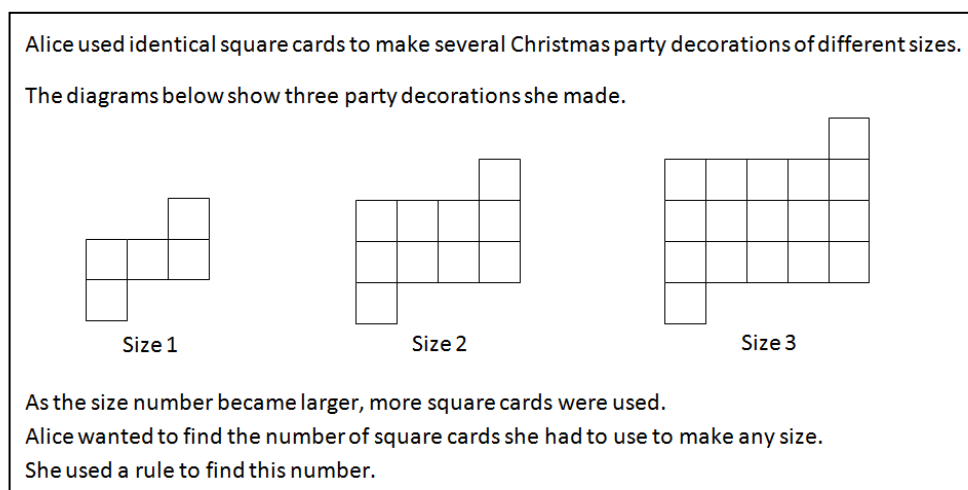


### Working Group 3

respectively below. The first task involves a linear rule whereas the latter involves a quadratic rule. These two tasks differ from the typical textbook tasks in that they are less structured, thus allowing a greater scope for exploring the pattern structure. The students were asked to individually work out the functional rules in terms of the size number using any strategy that they were familiar with. The purpose was to prepare and familiarise them with these tasks so that they could better understand the questionnaire tasks that they had to do later.



**Figure 1. *High Chair***

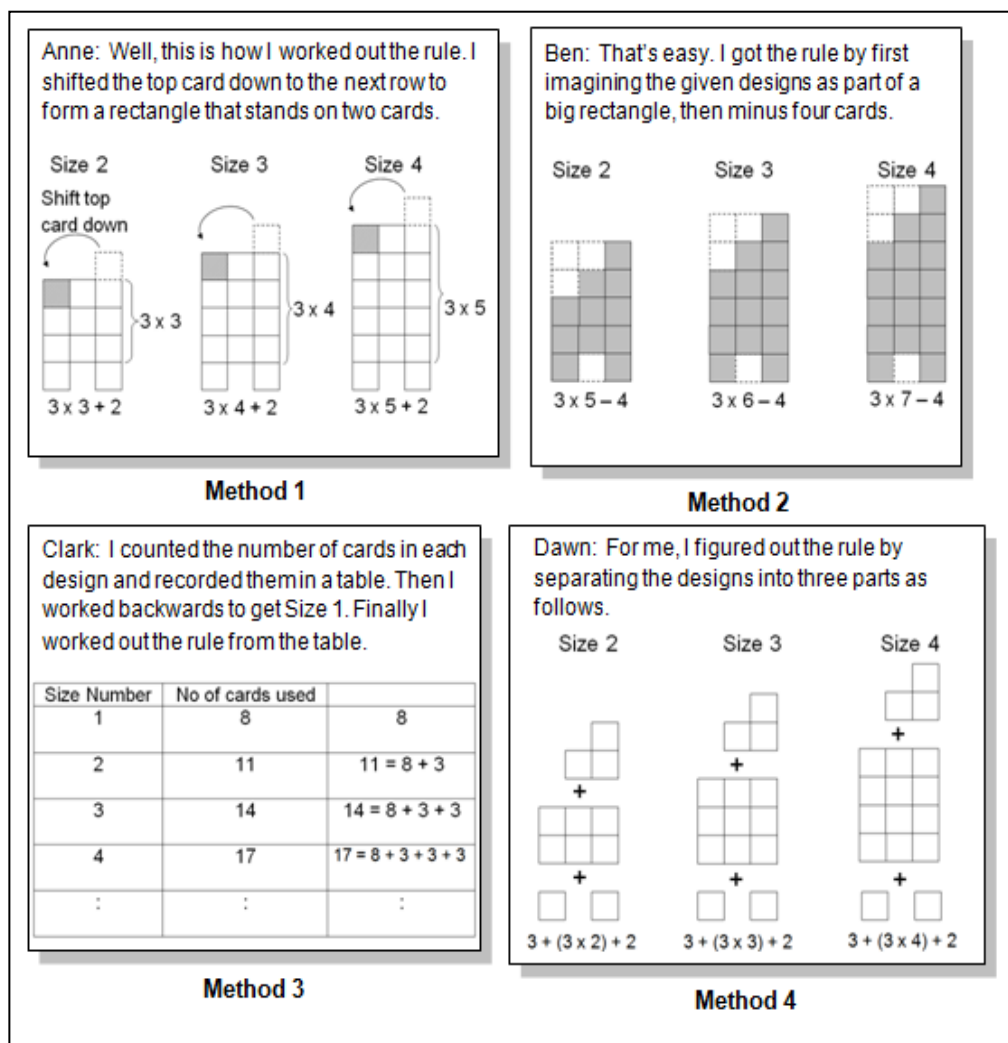


**Figure 2. Christmas Party Decoration**

Subsequently, the questionnaire containing those two generalising tasks, each accompanied by four possible student solutions, was distributed to each student. Figures 3 and 4 below show the four distinct student solutions for the two respective tasks. Set in a context of a discussion amongst four students, each student solution represented a different way of constructing the rule based on the classification scheme described above. Take, for instance, the solutions in *High Chair*. Method 1

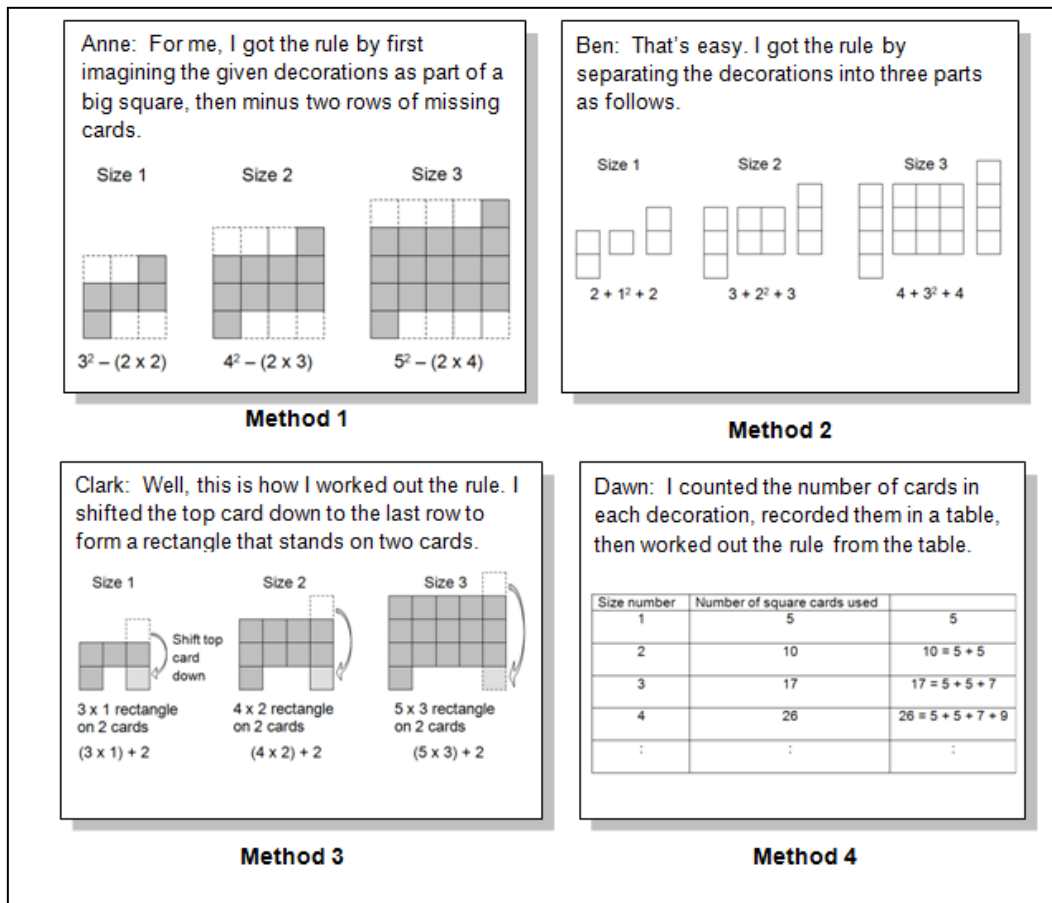
### Working Group 3

involves rearranging the original figures into something more familiar (S3). In Method 2, the original figures are viewed as part of a larger rectangle with four missing cards (S4). Method 3 uses a *numerical* strategy (S1) known as the repeated substitution strategy (Bezuszka & Kenney, 2008) while Method 4 employs a *constructive* strategy (S2). For *Christmas Party Decoration*, Methods 1, 2, 3 and 4 correspond to S4, S2, S3 and S1 respectively. The students were asked to choose the method that they believed would best help them to construct the functional rule. In addition, they had to provide justifications for their choices of the best-help method.



**Figure 3. Student solutions to High Chair**

All 45 questionnaires were collected and analysed to determine the student choices of method that they thought would best help them to work out the rule. The frequencies of the four student methods for each generalising task were then counted. The student justifications were looked into to gain a better understanding of the reasons behind their choices of best-help strategies.



**Figure 4. Student solutions to Christmas Party Decoration**

## RESULTS

This section presents the findings to the following two questions that guided this study.

1. Which strategies would students believe would best help them to work out the rule for High Chair?

Table 1 shows that the numerical solution S1 was the top choice of best-help strategies amongst the Express students in this study, with 13 of them selecting it. Following it, in descending order, are S2, S4 and S3. There were nearly an equal number of students choosing S2 and S4, with another three preferring S3. Taking these numbers of students collectively, 16 of the 29 Express students believed that a figural method would best help them to derive the rule. Similarly, a significant number of the Normal (Academic) students (69%) also found the numerical method S1 most helpful. As for the rest, three chose S2, two selected S3 and none opted for S4.

### Working Group 3

<i>High Chair</i>	Best-help Method to students				
	S1	S2	S3	S4	Total
Express Students (n = 29)	13	7	3	6	29
Normal (Academic) students (n=16)	11	3	2	0	16

S1: Numerical; S2: Constructive; S3: rearranging the original figures; S4: viewing the original figures as part of a larger rectangle

**Table 1: Student Choices of Best-help Method for High Chair**

#### 2. Which strategies would students believe would best help them to work out the rule for Christmas Party Decoration?

As Table 2 clearly shows, the Express students' choice that topped the list of best-help strategies for this quadratic generalising task was S2, with as many as 13 students choosing it. This was then followed in descending order by S1, S4 and S3. Of the remaining number of students, seven picked the numerical solution S1 and nearly the same number of them selected S3 and S4. Collectively, over 75% of the Express students believed that a figural method would best help them to derive the rule. On the other hand, 75% of the Normal (Academic) students found the numerical method S1 most helpful. For the rest of them, two each chose S2 and S3, and none opted for S4.

<i>Christmas Party Decoration</i>	Best-help Method to students				
	S1	S2	S3	S4	Total
Express Students (n = 29)	7	13	4	5	29
Normal (Academic) students (n=16)	12	2	2	0	16

S1: Numerical; S2: Constructive; S3: rearranging the original figures; S4: viewing the original figures as part of a larger rectangle

**Table 2: Student Choices of Best-help Method for Christmas Party Decoration**

## DISCUSSION

For the less able Normal (Academic) students, their choices of best-help strategy did not seem to vary very much between the linear generalising task and the quadratic task. Their top choice was the numerical strategy S1, followed in descending order by S2 and S3. S4 was not picked by them at all. The high frequencies of these students choosing the numerical method in both generalising tasks clearly suggest that a substantial majority of them prefer to work out the functional rule using this method compared to any of the other three given figural methods. An examination of their justifications revealed that its popularity lies in its simplicity for them to represent the changes across the different cases without having to draw any diagrams, thus making the workings easier to understand. In addition, some students

found that using the table of values is a well-organised and systematic way for them to detect the pattern and derive the rule.

Unlike the Normal (Academic) students, the numerical method emerged the top choice for the Express students only for the linear generalising task but slipped to the second position for the quadratic task. What is interesting to note about this finding is that some of these students seemed to be more aware of the applicability of this strategy to the quadratic task than their Normal (Academic) peers. They might have realised that while the numerical method shows how the pattern grows clearly in a table, the derivation of the quadratic rule is not as straightforward and easy as it appears. In fact, it is anticipated that such a method would pose a real challenge to all the Secondary One participating students if they were asked to use it to establish the rule. Therefore, it is not at all surprising to find some of these Express students abandoning the numerical strategy for a figural one in *Christmas Party Decoration*, thus causing a dip in its frequency by nearly one-half as compared to that for *High Chair*. As for the Normal (Academic) students who are regarded academically weaker, it is rather expected of them to not recognise the real difficulty of employing the numerical strategy to obtain the quadratic rule.

The popularity of the numerical strategy could also be traced to another plausible reason as suggested in a few students' justifications. The students explained that the numerical method was picked as the best-help strategy because it was the only method demonstrated by their mathematics teachers. This student revelation is consistent with evidence from another recent study of ours, which showed that the majority of the participating secondary school mathematics teachers would use the numerical strategy in class to show students how to work out the rule underpinning a pattern (Chua & Hoyles, 2010b). The student revelation also highlights a precarious situation students could be facing when they are only taught, in particular, what Bezuska and Kenney (2008) called the *repeated substitution* strategy and lack exposure to other types of generalising strategies. They could be misled to think that such a strategy is an effective method that can work easily for all types of generalising tasks.

Some valuable insights have also emerged from the students' justifications of their choice of strategy. There were students who preferred the numerical method due to its clarity and simplicity, which made pattern detection and understanding easy. Subsequently, this led to the ease of obtaining a rule, a view which Bezuska and Kenney (2008) had also pointed out. Those who eschewed this method generally found it time consuming, confusing and tedious to set up a table of values. To those who opted for figural methods, the pattern structure was easier to visualise because the explicit link between the size number and the number of cards used was more noticeable. That was why figural methods were found to be more helpful in deriving the rule quickly. Despite the evidence that figural methods can offer insight to pattern structure, some students still shunned such methods in favour of the

numerical method because they found them tiresome to draw and difficult to visualise the diagrams.

## CONCLUSION

The present study provides a window for teachers, teacher educators as well as curriculum developers to understand which generalising strategies would facilitate student visualisation of the structure underpinning the pattern. The findings showed that the Normal (Academic) students seemed to prefer the numerical method to the figural method for working out the functional rule whereas the Express students tended to favour the figural method. Such research-based knowledge is useful to the teaching and learning of number patterns, teacher training as well as curriculum design. For instance, teachers seeking an idea of what might be an appropriate generalising strategy to employ in class when demonstrating examples can use the findings to help them make informed decisions. Aligning their choices of generalising strategies with that preferred by students can support the efficacy of teaching and learning outcomes.

Looking from another perspective, the findings of this study also draw attention to a few implications for teachers. First, teachers will need to look into the assumptions that they are making when deciding on the kind of strategies to use in class. For pattern generalisation in particular, teachers will need to be keen observers of how their students express generality to find out how they process the strategies. Second, teachers will also need to be familiar with the different generalising strategies so that they can lead students to work out the functional rule. Finally, while the findings may be preliminary since the present study is still on-going, they appear to hold promise of creating a greater awareness amongst teachers, teacher educators and curriculum developers of what students are actually capable of doing and learning. By making an attempt to understand how students visualise patterns can help teachers and teacher educators in planning more effective teaching and learning experiences, and curriculum developers in curriculum design to improve students' ability to make generalisations.

## REFERENCES

- Bezuszkas, S. J., & Kenney, M. J. (2008). The three R's: Recursive thinking, recursion, and recursive formulas. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics (Seventieth Yearbook)* (pp. 81 - 97). Reston, VA: NCTM.
- Chua, B. L., & Hoyles, C. (2010a). Generalisation and perceptual agility: How did teachers fare in a quadratic generalising problem? *Research in Mathematics Education*, 12(1), 71 - 72.
- Chua, B. L., & Hoyles, C. (2010b). Teacher and student choices of generalising strategies: A tale of two views? In Y. Shimizu, Y. Sekiguchi & K. Hino (Eds.),

### Working Group 3

*Proceedings of the 5th East Asia Regional Conference on Mathematics Education* (Vol. 2, pp. 24 - 31). Tokyo, Japan: EARCOME.

- Drury, H. (2007). Do you see what I see? Issues arising from a lesson on expressing generality. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 862 - 871). Larnaca, Cyprus: University of Cyprus.
- Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5 - 17). New York: Lawrence Erlbaum Associates.
- Lannin, J. K. (2005). Generalising and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematics Thinking and Learning*, 7(3), 231 - 258.
- Lee, L., & Freiman, V. (2006). Developing algebraic thinking through pattern exploration. *Mathematical Teaching in the Middle School*, 11(9), 428 - 433.
- Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65 - 86). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalisations of patterns in different contexts. *ZDM: International Journal in Mathematics Education*, 40, 83 - 96.
- Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalisations involving linear figural patterns. *ZDM: International Journal in Mathematics Education*, 40, 65 - 82.
- Steele, D. (2008). Seventh-grade students' representations for pictorial growth and change problems. *ZDM: International Journal in Mathematics Education*, 40, 97 - 110.

# USING EPISTEMIC ACTIONS TO TRACE THE DEVELOPMENT OF ALGEBRAIC REASONING IN A PRIMARY CLASSROOM

Thérèse Dooley

Centre for the Advancement of Science Teaching and Learning,  
St. Patrick's College, Dublin City University

*In this paper, the epistemic actions of recognising, building-with and constructing (RBC) are used to analyse and describe the development of algebraic reasoning by primary pupils in a whole-class setting. The lesson concerned finding the sum of positive integers, 1-100. On the basis of mathematical principles developed for the lesson, transcripts of whole-class discussion were coded using the RBC framework. Some of these epistemic actions were inferred by the language used by pupils – for example, they tended to use linguistic hedges when conjecturing ('building-with') but used language of greater certitude when 'constructing'. It also emerged that the use of 'vague language' facilitated collaborative construction of mathematical ideas.*

## INTRODUCTION

Blanton and Kaput (2008) argue that algebraic thinking, which they describe as “purposeful generalization of mathematical ideas and the expression of generalities with increasingly sophisticated symbol systems” (pp. 362 – 3) should lie at the heart of classroom mathematical practice. They go on to say that

[a]lgebraic thinking thrives in an instructional context that both elicits students' thinking and uses it to build a climate of conjecture and argumentation so that conjectures can be established or rejected as valid mathematical claims, especially conjectures regarding the generality of claims. (p.363)

In this paper, I describe a framework that can be used to analyse and describe the growth of algebraic reasoning in the context of whole-class discussion. In particular, it shows that a 'conjecturing atmosphere' (Mason, 2008) is central to such growth.

## THEORETICAL FRAMEWORK

### RBC Epistemic Actions

A theoretical framework that has been used to describe the construction of new mathematical ideas is 'Abstraction in Context' (AiC) (Schwartz, Dreyfus, and Hershkowitz, 2009). Related epistemic or observable actions are 'recognising' (R) (using a familiar structure), 'building-with' (B) (using available structural knowledge to deal with the problem at hand) and 'constructing' (C) (building more complex structures from simpler structures). These three epistemic actions are not linear but nested. In other words, 'recognising' (R) and 'building with' (B) do not precede the process of 'constructing' (C) but, rather, are nested within it. While the RBC model has been used for the analysis of a variety of mathematical topics, C-actions often



concern the generality of claims (e.g., Schwartz *et al.*, 2009). As such it is pertinent to algebraic thinking as described above. The RBC model of abstraction described by Hershkowitz, Schwartz, and Dreyfus (2001) was based on data derived from a teaching interview with one student who had a computerised tool at her disposal. However, they suggested that epistemic actions might be distributed among participants. In a paper that I presented at CERME 5, I suggested that, in the context of whole-class discussion, “[o]ne pupil’s ‘recognising’ led to ‘building with’ by another and to ‘construction’ of new ideas and strategies by others” (Dooley, 2007: 1658). Hershkowitz (2009) has since described the process where different individuals contribute different building blocks to the construction of new mathematical knowledge as one of ‘collective abstraction’.

A partial construct (PaCC) obtains if a C-action only partially matches elements of underlying mathematical principles, that is, intended constructs such as concepts, methods, or strategies (Ron, Hershkowitz, and Dreyfus, 2008). PaCCs are apparent in cases where a student’s incorrect answer overshadows meaningful knowledge s/he has constructed or, conversely, where a correct answer masks a gap in knowledge. The RBC model has been found to be an effective tool for tracing these partial constructs (Schwartz, *et al.*, 2009).

In order to investigate the construction of new mathematical ideas by pupils I conducted a ‘classroom design experiment’ (Cobb, Gresalfi, and Hodge, 2009) in three different primary schools in Ireland. I taught 32 lessons, some of which extended over a few class periods, in all. Data collected included field notes, audiotapes of whole-class and group interactions, pupils’ written artefacts, digital photographs, interviews with teachers and, in two of the schools, pupil diaries and post-lesson interviews with small groups of pupils. Data collection and data analysis were interwoven. Retrospective analysis was conducted on micro- (between lessons) and macro- (between and after cycles of research) levels. For each lesson I identified ‘mathematical principles’, that is, the constructs that pupils might be expected to develop over the course of a lesson, and these informed a hypothetical learning trajectory. Other principles arose *a posteriori* and were included in the analytic framework. Using the computer aided qualitative data analysis software package, Nvivo, I first coded all pupils’ turns as ‘R’, ‘B’ or ‘C’ with reference to these principles. Thus I was able to trace the mathematical constructions of (some) individual pupils. A difficulty I encountered was that I had to infer the epistemic action from pupils’ verbal protocol<sup>1</sup>. Their use of ‘hedges’ and ‘pronouns’ facilitated this process.

### **Hedges and Pronouns**

Rowland (2000) developed a taxonomy of hedges with reference to the discourse of mathematical conjecture. The first major type of hedge, a ‘shield’ indicates some uncertainty in the mind of the speaker in relation to a proposition. In the statement, “I think that the last digit of an even number is 0, 2, 4, 6, or 8”, the speaker injects a

level of vagueness into his/her mathematical assertions and thus implicitly invites feedback on his/her conjecture about a method of identifying an even (or non-even) number. There are two types of shield: (a) a ‘plausibility shield’ (e.g. ‘I think’, ‘probably’, ‘maybe’) which can suggest some doubt on the part of the contributor that the statement will withstand scrutiny and (b) an ‘attribution shield’ (e.g. ‘According to’) in which some degree or quality of knowledge is implicated to a third party. The second major category of hedges are termed ‘approximators’. The effect of the approximator is to modify the proposition rather than to invite comment on it. One subcategory of the approximator is the ‘rounder’ which comprises adverbs of estimation such as ‘about’, ‘around’ and ‘approximately’. The second type of approximator is the ‘adaptor’ – it indicates vagueness concerning class membership such as ‘somewhat’, ‘sort of’, e.g., “Zero is sort of an even number”.

In the analysis of lesson transcripts, it emerged pupils tended to use vague language (e.g., ‘probably’ ‘might’ ‘I think’) when conjecturing, an action coded as ‘building-with’. In turn, the language of a constructing action was marked by certitude – in particular, pupils often used pronouns such as ‘it’ or ‘you’ to signify generalisation (Rowland, 1999, 2000). There follows an account of a lesson on the Story of Gauss that took place with a group of pupils aged 9 – 10 years. An overview of this lesson was given at CERME 6 where my role in the constructing processes of a number of pupils was described using an improvisational metaphor (see Dooley, 2009). In this paper there is a more fine-grained analysis of the development of new ideas by one pupil, Anne. However, her construction had embedded within it the contributions of others in the class and thus their input is also described and analyzed.

## THE STORY OF GAUSS

The young Gauss astounded his teacher when, as a school pupil, he rapidly calculated the sum of numbers from 1 to 100. He did so by adding 1 to 100, 2 to 99 and so on and thus found fifty 101s. The power of this story, according to Rowland (2001), is that it provides a means of finding the sum of the first  $2k$  integers for any integer  $2k$  (that is,  $k(2k + 1)$ ). A ‘generic example’ is an example that is representative of a class of objects (Balacheff, 1987). What makes Gauss’ story special in the view of Rowland (2001) is that it is “generic among generic examples” (p.43), that is, it elucidates the nature of a generic example.

## Mathematical Principles

The Gauss lesson that is the focus of this paper followed a lesson on the ‘Handshakes’ problem. The pupils had developed an explicit formula for this problem, that is, that the number of handshakes could be determined by application of the formula,  $\frac{1}{2}n(n-1)$ , where  $n$  is the number of people. Some pupils were able to verify this formula structurally. Since they had calculated the number of handshakes in the case of 100 people, I anticipated that they might be able solve Gauss’ problem with relative ease (that is,  $4950 + 100$ ). I also thought that they might make use of

the formula although I expected that this would pose a challenge since the number of handshakes for  $n$  people is  $\frac{1}{2} n(n-1)$ , the formula for  $1 + 2 + 3 + \dots + (n - 1)$ . Pupils of this age could also be expected to use ‘flexible computation methods’ (Greeno, 1991). One of the ways they might do this is to add numbers that pair to 100 (a method that is referred to in future as ‘compatible pairs’)<sup>2</sup>, i.e.,  $100 + 0, 99 + 1, 98 + 2 \dots$

In order to use this method to solve the sum of the first  $n$  positive integers, pupils have to observe that if  $n$  is even, (a) the number of pairs is  $\frac{1}{2} n$  and (b) that  $\frac{1}{2} n$  (‘50’ in the example above) does not have a ‘compatible partner’. An interesting pattern emerges when sums of decades are examined. Using the notation  $s(m, n) = m + (m + 1) + \dots + n$ ,  $s(1, 10) = 55$ ;  $s(11, 20) = 155$  (that is,  $s(1, 10) + 100$ );  $s(21, 30) = 255$  (that is,  $s(1, 10) + 200$ ) etc. Similarly,  $s(1, 100) = 5050$ ;  $s(101, 200) = 15050$  (that is,  $s(1, 100) + 10000$ ) etc. One of the conflicts that pupils have to overcome in order to use either of these patterns effectively is the ‘illusion of linearity’ (De Bock, Van Dooren, Janssens, and Verschaffel, 2002), that is,  $s(1, kn) = ks(1, n)$ . For example, pupils might assume that  $s(1, 100) = 10s(1, 10)$ , that is,  $s(1, 100) = 10(55)$ . On a more global level, another principle that pupils might be expected to attain is the extension of this example to other classes, such as the sum of the first  $n$ -hundred integers.

A summary of a possible learning trajectory is as follows:

- (i) Formation of an association with ‘Handshakes’.
- (ii) Finding a solution by adding compatible pairs/decades or by transforming the formula for ‘Handshakes’; noticing that linearity does not apply.
- (iii) Expanding the solution method into a more general structure.

## ANALYSIS OF ANNE’S CONSTRUCTION OF INSIGHT

Early in the lesson, Anne suggested that the sum of the first 100 integers might be found by doubling the sum of one to five (that is, fifteen) and then multiplying 30 by ten, that is,

- |    |       |  |
|----|-------|--|
| 56 | Anne: | If you add one to five, that’s fifteen...  |
| 57 | TD:   | Hm, hm.  |
| 58 | Anne: | ... and then fifteen and fifteen is thirty, so then if you multiply that by ten. |

Her insight about the inefficacy of this strategy occurred a short while later and was prompted by an observation made by Alan. The turns chosen for analysis are therefore turns 66 – 86 and 91 – 95 since they provide the trace of her constructions<sup>3</sup>.

### Working Group 3

	Pupil Action	Epistemic Action (RBC)
66 Alan: Em, well, I don't think Anne's one is right.	Alan evaluates idea proposed by Anne in Phase 2.	Building-with
67 TD: Why not?		
68 Alan: Cos ninety-nine plus ninety-eight plus ninety-seven plus ninety-six to ninety would be around over five hundred and when ...	Estimation of sum of '90s'	Recognising
69 //Ch: Oh!	Exclamation	
70 TD: Ok, so you are thinking that, you think, you disagree with Anne because you are thinking, what Alan is doing now ... Alan is thinking ninety - I haven't forgotten you now, Enda, alright, I will be with you in a moment - you are thinking ninety plus ninety-one plus ninety-two plus ninety-three would give you approximately how much?		
71 Alan: Em, I don't know.		
72 TD: But it's...		
73 Alan: But it would probably be over five hundred.	Estimation of sum of 90s	Recognising
74 TD: It would be over five hundred, so in that section, if you are thinking about all those numbers there that would give you about, even just adding ninety to a hundred so you are thinking that would give you about five hundred - I will be with you in about one minute alright. Barry?		
75 Barry: Eh, well, I disagree with Anne as well because eh I counted, I counted up all the numbers up to ten and I got fifty-five.	Further evaluation of Anne's idea	Building-with
76 TD: You counted, ok so actually in your head, so you went, so ok so you added, Barry, you went one plus two plus three and so on up to ten ( <i>writes on blackboard</i> ) and you just counted that section there and you said that's about fifty-five. And then Alan was saying that ninety plus ninety-one ( <i>writes on blackboard</i> ) up as far as a hundred would give you. About how much what would that give you if you just added those numbers there, ninety, about how much, you don't have to give the exact answer, about how much would that be if you went ninety, ninety-one, ninety-two, ninety-three added all those numbers there, what would that be about? Barry?		
77 Barry: A hundred and eighty one.		
78 TD: It would be about a hundred and eighty one, that's if you went ninety plus ninety-one plus ninety-two plus ninety-three ...		
79 Barry: No eh...		
80 TD: ... plus ninety-four plus ninety-five plus ninety-six plus ninety-seven plus ninety-eight plus ninety-nine plus a hundred? About how much would that be?		
81 Barry: Eh ... a thousand.	Estimation of sum of 90s	Recognising
82 TD: It could be yeah, it could probably be a thousand.		
83 Ch: Ah!		

### Working Group 3

84	TD: Ok, so that would give you about ... Do you agree with that, that that would give you about a thousand?		
85	Barry: No around nine hundred because there's not really a hundred in the nineties.	Estimation of sum of 90s	Building-with
86	TD: In the nineties, so you're thinking that that would be around nine hundred. Ok. So this is now what we are thinking now, alright that eh this number here is about nine hundred and Enda what you might do for me if you just take out ... is just write down your idea in your diary for me please. Alright?		
[ ] <sup>4</sup>			
90	TD: Yes, so it's not just, and as Barry said that if you were going ninety, or Alan made this point, that ninety, ninety-one all the way up to a hundred gives you about nine hundred, who else said about, oh yes and Barry was saying that one up as far as ten gives you fifty-five. And we are thinking all the numbers one plus two plus three plus four plus five plus six plus seven plus eight plus nine plus ten plus eleven plus twelve (speaking quickly) ... counting up, adding up all the numbers from one up to a hundred. Anne?		
91	Anne: I don't ... my answer wouldn't work.	Anne evaluates re-her early input.	Building-with
92	TD: What were you thinking your answer was?		
93	Anne: I thought it would be thirty multiplied by a hundred but it wouldn't work.	Anne repeats earlier idea.	
94	TD: Why would it not work?		
95	Anne: Em, because you would have to em ... cos I did eh one plus two plus three plus four plus five and then em I got fifteen and then I added fifteen and fifteen equals thirty but then it would be more em ... because you would have to add six, seven and that ...	Anne rationalises why her earlier idea would not work.	Constructing

### Epistemic Actions (RBC)

I classified Alan's input in turn 66 as 'building-with' since he was reflecting on a process. The use of the words 'Well, I think...' in this instance can be explained by 'politeness theory' described by Rowland (2000: 86) as follows:

[P]oliteness theory is constructed to account for some indirect features of conversation: it claims that speakers avoid threats to the 'face' of those they address by various of vagueness, and thereby implicate their meanings rather than assert them directly.

It seems that Alan was reasonably certain that Anne was incorrect but he may not have wanted to threaten her 'positive' face (that is, her desire for approval). Use of the discourse marker, 'Well' and of the plausibility shield, 'I think' represent a form of redress (Bills, 2000; Rowland, 2000). In other words, he may have wanted to inject some vagueness into his criticism of her conjecture. His justification for this criticism was quite fuzzy as he was unsure about the sum of the numbers in the 90s. In turn 68 he used the rounder, 'around' followed immediately by a second rounder, 'over', to suggest that 'five hundred' was an estimate. When I endeavoured to get a

### Working Group 3

more precise solution from him he used a plausibility shield to signify his uncertainty: “It would probably be over five hundred”.

Barry has also built-with in turn 75. It seems that he too has reflected on Anne’s input and has rejected it on the basis that the sum of one to ten is 55. He spoke with some conviction about this sum and, like Alan, his use of the word, ‘Well’ at the beginning of this turn suggests that he was being polite, in a technical sense, to Anne. In turn 76, I reviewed both of their ideas and wrote them on the blackboard. In turn 91, Anne changed her mind about her earlier idea and stated emphatically that “it wouldn’t work”. Interestingly, her rationale for this was neither of the justifications provided by Alan and Barry. It has now occurred to her that the sum of six to ten is greater than the sum of one to five. It is possible that she has reflected on the fact that the sum of one to ten (55) is less than the sum of 90 to 100 (which has been estimated by Barry in turn 85 to be “around nine hundred”). She used the pronoun ‘it’ twice in turn 93, that is,

- 93 Anne: I thought it would be thirty multiplied by a hundred but it wouldn’t work.

The first ‘it’ is a referent to the solution (although in error she described it as ‘thirty multiplied by a hundred’ rather than ‘thirty multiplied by ten’) but the second ‘it’ seems to be a referent to the solution method. This is reinforced by her input in turn 95 in which she described her initial strategy:

- 95 Anne: Em, because you would have to em ... cos I did eh one plus two plus three plus four plus five and then em I got fifteen and then I added fifteen and fifteen equals thirty but then it would be more em ... because you would have to add six, seven and that ...

The ‘it’ in the latter part of this sentence is most likely a referent to the sum of six to ten. She built-with her reflection (turn 93) and constructed the idea of the ‘non-linearity’ of partial sums of consecutive integers within ten. Although turn 95 is coded as ‘constructing’, the construction action occurred over turns 66 – 86 and 91 – 95 and had nested within it recognising and building-with actions.

During group-work Anne worked with Fiona. Although the task assigned was to find the sum of 1 - 100, they extended this investigation to the sums of the first 200 and 300 positive integers. In the final plenary session both she and Fiona discussed their (shared) strategy:

- 237 Anne: I got five hundred and fifty, for two hundred I got ten...for one hundred I got five thousand and fifty. For two hundred I got ten thousand one hundred, em and for three hundred I got fifteen thousand one hundred and fifty. (...)
- 240 Fiona: Every time you are just adding five thousand and fifty to get another hundred.

In subsequent turns, attention was given to different storylines but I returned to that expressed by Anne and Fiona towards the conclusion of the lesson:

### Working Group 3

- 290 TD: ... I just wonder if you know, Fiona says that to add up to 200, if the answer is 5050 to add up to 200 would you just add another 5050 on to that? Would that give you the right answer ... Anne?
- 291 Anne: I think it ... em actually no ... eh no because ... em it would probably ... should there be hundreds em in the five thousand and ... and fifty so you would have to ... so probably there would be five ... there would probably be five thousand nine hundred and fifty.
- 292 TD: It could be. Yeah

‘Self-repairs’ (false-starts and self-corrections) proliferate her reasoning – this is common, according to Rowland (2000), when people are asked to make a justification or to give some kind of explanation. In this instance it seems that Anne has constructed a new idea. Her construction relates to the fact that the sum of 101 to 200 contains more ‘hundreds’ than the sum of 1 to 100. Her explanation abounds with plausibility shields, in particular ‘probably’, which is a means of presenting this idea without fully committing to it (Rowland, 2000). Although her estimate of the sum (101 to 200) was incorrect, this line of reasoning indicates that she has built-with her earlier ‘local’ construction (the non-linearity of partial sums of consecutive integers within ten) and has extended her thinking to other consecutive sets. Evidence for this building-with can be found in the language she used. Concerning the sum of six to ten, she said, in turn 95, that “it would be more” (than the sum of one to five). Concerning the sum of 101 to 200, she suggested, in turn 291, that there might be “hundreds ... in the five thousand”. The connection she seems to have made is that, just as there is ‘more’ in the sum of six to ten than in the sum of one to five, there is also ‘more’ in the sum of 101 to 200 than in the sum of one to 100.

Her construction is coded as ‘partial’ because her solution was incorrect (Schwartz, et al., 2009). However, this incorrect solution masks some meaningful knowledge she appears to have constructed: in her endeavour to find, unsolicited, the sum to 200 and to 300, there is an indication that she has seen the power of Gauss’ story as a generic example.

## DISCUSSION

This paper provides further evidence for the notion of collective abstraction, that is R-, B- and C-actions were distributed among a few pupils. Furthermore, Anne’s C-action had nested within it R- and B-actions both of her own and of others. This paper builds on that which I presented at CERME 5 by focusing on the role played by vague language. It appears that the use of such language is at the core of construction because it is intrinsic to conjecturing and the ‘trying out’ of ideas by pupils. Mason (2008: 65) describes this as follows:

Mathematicians work best in a conjecturing atmosphere in which conjectures are articulated in order to try them out, see how they sound and feel, to test them and so to see how to modify them as and when necessary. This is the sort of atmosphere in which mathematics thrives, and it can be established in any classroom at any age.

Linguistic hedges allow learners to test ideas without fully committing to them. The verbalization of these ideas enables both the contributor and other 'listeners' to scrutinize them. It is in this way that the use of vague language facilitates construction of new mathematical knowledge.

Another aspect of language that proliferated the conversation described in this paper was that of politeness. Many pupils prefaced their input with 'Well' as a means of 'face-saving'. Holton and Thomas (2001) suggest that one of the reasons that a child's peers act as an effective source of cognitive conflict is that feedback from other children is often less emotionally threatening than that from an adult. The fact that Anne remained engaged in the lesson after critical evaluation of her input by her classmates suggests that this may indeed have been the case for her. The climate of 'conjecture and argumentation' that was evident in this lesson facilitated the generalization of a mathematical idea, that is, the non-linearity of sums of consecutive integers. What this indicates is that pupils are capable of developing algebraic thinking if conjecturing, argumentation and the production of counter-examples are accepted classroom norms.

## NOTES

1. My stance as researcher derives from an interpretive perspective and therefore the conclusions that I draw are partial and tentative (Usher, 1996).
2. The usual summation given is  $100 + 1, 99 + 2, \dots 50 + 51$ . Such a pairing does not involve a 'lone'  $\frac{1}{2}n$  (50 in this case). However, the pairing method given in this paper ( $100 + 0, 99 + 1, \dots 50 + 0$ ) was that proposed by the pupils in the lesson described.
3. There were 298 turns in total the whole-class discussion.
4. Turns 87 - 90 concerned interaction that is not relevant to the analysis of this paper.

## REFERENCES

- Balacheff, N. (1987). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 89-110). Dordrecht: Kluwer.
- Bills, L. (2000). Politeness in teacher-student dialogue in mathematics: A socio-linguistic analysis. *For the Learning of Mathematics*, 20(2), 40 - 47.
- Blanton, M.L., Kaput, J.J. (2008). Building district capacity for teacher development in algebraic reasoning. In J.J. Kaput, D.W. Carraher, M.L. Blanton (Eds.), *Algebra in the early years* (pp. 361-388). NY and London: Lawrence Erlbaum Associates.
- Cobb, P., Gresalfi, M., Hodge, L. L. (2009). A design research perspective on the identity that students are developing in mathematics classrooms. In B. Schwartz, T. Dreyfus, R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 223 - 243). London and New York: Routledge.



- De Bock, D., Van Dooren, W., Janssens, D., Verschaffel, L. (2002). Improper use of linear reasoning: An indepth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, 50, 311 - 334.
- Dooley, T. (2007). Construction of knowledge by primary pupils: The role of whole-class interaction. In D. Pitta-Pantazi and G. Philippou (Eds.), *CERME 5*, (pp. 1658 - 1667). Larnaca, Cyprus: University of Cyprus.
- Dooley, T. (2009). A teacher's role in whole-class mathematical discussion: Facilitator of performance etiquette? In V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello (Eds.), *CERME 6*, (pp. 894 - 903). Lyon, France: INRP.
- Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education*, 22(3), 170 - 218.
- Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. Schwartz, T. Dreyfus and R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273 - 280). London and New York: Routledge.
- Hershkowitz, R., Schwartz, B., Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195 - 222.
- Holton, D., Thomas, G. (2001). Mathematical interactions and their influence on learning. In D. Clarke (Ed.), *Perspectives on practice and meaning in mathematics and science classrooms* (pp. 75 - 104). Dordrecht and London: Kluwer.
- Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher, M. L. Blanton (Eds.), *Algebra in the early years* (pp. 57 - 94). New York and London: Lawrence Erlbaum Associates.
- Ron, G., Hershkowitz, R., Dreyfus, T. (2008). The identification of partially correct constructs. In M. Goos, R. Brown, K. Makar (Eds.), *Proceedings of the MERGA 31* (pp. 421 - 427). Brisbane: MERGA.
- Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.
- Rowland, T. (2001). Generic proofs: Setting a good example. *Mathematics Teaching* 177, 40-43.
- Schwartz, B., Dreyfus, T., Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. Schwartz, T. Dreyfus, R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11-41). London and New York: Routledge.
- Usher, R. (1996). A critique of the neglected assumptions of educational research. In R. Usher and D. Scott (Eds.), *Understanding educational research* (pp. 9 - 32). London: Routledge.

# GRAPHICAL REPRESENTATION AND GENERALIZATION IN SEQUENCES PROBLEMS

María C. Cañadas, Encarnación Castro, and Enrique Castro

University of Granada, Spain

*In this paper we present different ways used by Secondary students to generalize when they try to solve problems involving sequences. 359 Spanish students solved generalization problems in a written test. These problems were posed through particular terms expressed in different representations. We present examples that illustrate different ways of achieving various types of generalization and how students express generalization. We identify graphical representation of generalization as a useful tool of getting other ways of expressing generalization, and we analyze its connection with other ways of expressing it.*

## INTRODUCTION

The number of works focused on the relation between the algebra and the expression of generalization has increased since the work of Mason, Graham, Pimm, and Gowar (1985). Some of this work deals with the idea that this connection does not seem to be direct for secondary students (e.g. Lee, 1996; Lee & Wheeler, 1987). Algebraic language is not the only way of expressing a generalization. For example, Mason and Pimm (1984) consider that natural language has a fundamental role in the generalization process, and Radford (2002) shows how some students used verbal and gestural means to express generalization. Currently we are engaged in a project in which some of the aims are related to the relation among generalization, and ways of achieving and expressing it.

In this paper, we focus attention on the generalization developed by students in a problem solving context and on the different representations used by students in generalization problems with different characteristics. In particular, this paper extends previous work by reporting on ways of achieving generalization through an inductive process, its relation with representations used, and how students in our investigation express generalization when they work on a written questionnaire constituted of problems involving sequences.

We first present the main ideas concerning our approach to generalization and different representations, focusing on different types of generalization. Secondly, we present our research questions. Then we present a general description of the methodology used. After this, we outline our findings and interpretations. We finally present the conclusions.

## ALGEBRAIC THINKING AND GENERALIZATION

Taking a semiotic approach, we consider that students are thinking algebraically when they “act in order to carry out the actions required by the generalizing task”

(Radford, 2002, p. 258). From this perspective, generalization is achieved when students are able to identify a common pattern that arises from some particular cases and to apply this commonality to other particular cases.

We refer to *empirical generalization* following Dörfler (1991). This type of generalization starts from work on particular cases and is very close to pattern identification. Cañadas and Castro (2007) developed a model to describe Secondary students' inductive reasoning. This model is comprised of seven states, of which generalization is one. From our viewpoint, inductive reasoning is equivalent to what Pólya (1967) called *induction*. Different authors, including Pólya, assert that generalization is a key state in the process of acquiring mathematical knowledge (Neubert & Binko, 1992; Mill, 1858).

Thus generalization can be seen as “pattern generalization”, which is considered one of the prominent routes for introducing students to algebra (Mason, Graham, Pimm & Gowar, 1985; Radford, 2010, p. 37). However, it is assumed that algebra is not the only way of expressing a pattern nor is algebraic thinking the only way of forming a generalization.

## REPRESENTATIONS

There is a general agreement amongst researchers of the need to distinguish between external and internal representations of students' knowledge. In our research, we focus on external representations. These representations allow the students to express concepts and ideas, since ideas must be represented externally in order to communicate them (Duval, 1999; Hiebert & Carpenter, 1992). In this paper, we pay attention to the external representations produced by students that have a trace or tangible support even when this support has a high level of abstraction.

We also consider *multiple representations* (e.g. van Someren, 1998). Multiple representations have benefits on schema construction processes; but it is not always beneficial for learning (Kolloffel, Eysink, Jong, & Wilhelm, 2009). Figueiras and Cañadas (2010) distinguish two different kinds of multiple representations: (a) combined representation, which concerns the use of different representations (as mentioned by previous authors); and (b) synthetic representations, which are multiple representations but under the additional condition we must consider them as a whole to give sense to the student's response (p. 3).

## WAYS OF EXPRESSING THE GENERALIZATION

Algebra is one way of expressing a pattern, but it is not the only one. We concur with Radford's idea of algebraic generalization. “It rests on the noticing of a local commonality that is then generalized to all the terms of the sequence and that serves as a warrant to build expressions of elements of the sequence that remain beyond the perceptual field” (Radford, 2010, p. 42). This author distinguishes this kind of generalization from *arithmetic generalization*, which is characterized by staying in

the realm of arithmetic (p. 47). Students who generalize arithmetically have identified the pattern and are usually conscious that this pattern is unpractical for other terms of the sequence.

In the context of analyzing the generalization process in problems involving sequences in a written problem solving test, Cañadas and Castro (2007) distinguish between algebraic and verbal representations as two ways of expressing the general term of a sequence. The first way concerns the use of symbols and numbers, in which each term of the sequence can be obtained by substituting the symbols with concrete numbers; and the second one refers to the use of natural language to express the generalization. These authors left an open question related to the role of graphical representation in the generalization procedure and the expression of such generalization. We tackle this question in this paper.

## RESEARCH QUESTIONS

We break down our research interests into three research questions for this paper, which concerns two central aspects of the generalization: (a) generalization process, and (b) generalization expression. These questions are:

- What is the role of graphical representation in the generalization process?
- How do the students express the generalizations achieved?
- What are the features of graphical expression of the generalization?

## METHODOLOGY

### Students

We took 359 students in years 9 and 10 of four State Spanish Schools whose teachers were close to us.

We obtained information about students' educational experiences related to generalization, problem solving, sequences, and algebra from four sources: (a) Spanish curriculum, (b) informal interviews with students' teachers, (c) mathematics textbooks used by students, and (d) students' notebooks.

Spanish Secondary curriculum does not include the generalization process explicitly. It includes reasoning as one of its main objectives. However, it contains just some actions related to inductive reasoning, such as: (a) to recognize numerical regularities, (b) to find strategies to support students' own argumentations, and (c) to formulate and to prove conjectures (Boletín Oficial del Estado, 2003).

Students had previously studied sequences. They had worked on problems using inductive reasoning, usually involving sequences, on occasion. These kinds of problems are usually presented with particular cases expressed numerically and are most of them de-contextualized. Students had begun the study of algebra between one or two years before the research commenced (depending on the year they studied

by the time of this research). These lessons included work related to interpretation of formula and algebraic expressions, and first grade equations. We consider that these students had the experience required to focus on the research questions posed. Specifically, the students had sufficient content knowledge of sequences. On the other hand, our analysis of their previous educational experiences demonstrates that they were not used to solving the kind of problems posed.

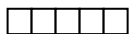
### Problems Posed

We prepared a written questionnaire with six problems involving linear and quadratic sequences. We asked students to work individually on this questionnaire for an hour. The problems of the questionnaire were selected according to our research objective and using the characteristics that arose through *subject matter analysis* (Gómez, 2007) of natural number sequences: (a) the order of the sequence, (b) the representation used in the statements, and (c) the task proposed.


In this paper we will focus on three problems which involve linear and quadratic sequences [1], and with different representations used in the statements. Each problem was focused on a “far generalization” task (Stacey, 1989). So, particular cases were presented in the problems statements to lead the students to generalize at some point. Each problem had a complementary task consisting of justifying their responses [2]. Since sequences are a particular kind of function, we took into account the four representation systems traditionally considered for functions: (a) graphical, (b) numerical, (c) verbal and (d) algebraic (Janvier, 1987). In accordance with our research objectives, problems lead the students to work on information given through particular cases expressed in a graphical, numerical or verbal context.

In what follows, we focus on three of the six problems: problems 3, 4, and 5. The first problem, presented in a graphical context through a generic example, is a familiar generalization problem that has been presented in many different versions since Küchemann’s study (1981). Problem 4 is presented in a verbal context, and problem 5 in a numerical one.

Imagine some white square tiles and some grey square tiles. They are all the same size. We make a row of white tiles:



We surround the white tiles by a single layer of grey tiles.



- How many grey tiles do you need to surround a row of 1320 white tiles?
- Justify your answer.

**Figure 1: Problem 3**

We are organizing the first round of a competition. Each team has to play two matches against the rest of the participating teams (first and second leg). Depending on whether the competition is local or national; we will have 22 or 230 teams.

- Calculate the number of matches depending if there are 22 teams and if there are 230 teams.
- Justify your answer.

**Figure 2: Problem 4**

We have the following numerical sequence:

1, 4, 7, 10, ...

- Write down the number that should be in position 234 of this sequence.
- Justify your answer.

**Figure 3: Problem 5**

## FINDINGS AND INTERPRETATION

We first used a quantitative data analysis to identify the stages of inductive reasoning model performed by each student in his/her response to each problem. Table 1 shows the number of students who expressed generalization using different representations.

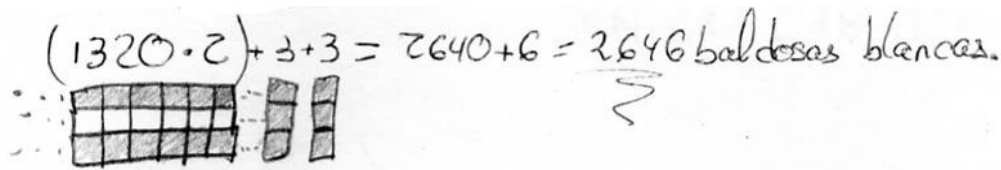
Generalization			
Arithmetic	Algebraic	Verbal	Graphic
Problem 3			
125	3	57	11
Problem 4			
174	1	69	0
Problem 5			
222	57	26	0

**Table 1. Representations used by students when generalizing**

The generalization most frequently used by the students was the arithmetical one. They generalized the pattern, got the commonality of the particular cases, and used the generalization to calculate the number of grey tiles for the required case. However, they were not able to provide an expression of *any* term of the sequence. In what follows, we mainly focus on graphical representation of generalization.

### Working Group 3

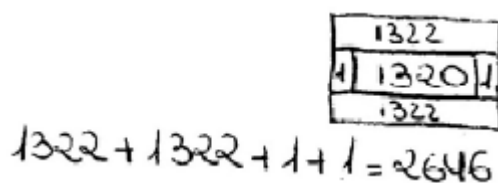
The students did not use graphical representation in the solving process except in problem 3, where the particular case was graphically presented in the problem itself. The use of graphical representation in this strategy is illustrated in one of the student's productions in Figure 4.



The image shows a handwritten equation:  $(1320 \cdot 2) + 3 + 3 = 2640 + 6 = 2646$  baldosas blancas. Below the equation is a diagram of a square frame. The frame is composed of grey tiles forming the outer border and white tiles forming the inner area. The top and bottom sides are each 13 tiles long, and the left and right sides are each 3 tiles long. The equation represents the calculation of the total number of white tiles (1320) multiplied by 2 to account for the top and bottom sides, plus 3 tiles for the left side and 3 tiles for the right side, resulting in 2646 white tiles.

**Figure 4. Generalization arithmetically and graphically expressed [Note: *Baldosas blancas* means white tiles in English]**

We can differentiate two parts in the student's drawing. First, the student considered the two vertical sides made by grey tiles and identified that these sides remain with the same number of grey tiles independent of the number of white tiles. The second part of the drawing is the part of the figure that has white and grey tiles, and the student interpreted that he needed to double the number of grey tiles to surround the white tiles. The student expressed graphically the pattern of the sequence, showing it in a different particular case from the one shown in the statement (for six white tiles instead of five). The suspension points show this student's awareness that the commonality applies to other terms of the sequence. In this context, the suspension points could be interpreted as "I should repeat *this* as many times as necessary". The student applied the commonality to the 1320 tiles in the arithmetic expression. Therefore, on one hand, the graphical representation helped the student to get the generalization; and on the other hand, the student identified the commonality in the graphical expression beyond the generic example shown in the statement. This is why we consider that the generalization is expressed graphically.



The image shows a diagram of a square frame with numbers indicating the number of tiles on each side. The top side is labeled 1322, the bottom side is labeled 1322, the left side is labeled 4, and the right side is labeled 4. Below the diagram is the equation:  $1322 + 1322 + 4 + 4 = 2646$ .

"I have to add two to the number of white tiles to the top, and two more to the bottom; plus two more to each side" (Authors' student's response translation)

**Figure 5. Generalization graphical and verbally expressed**

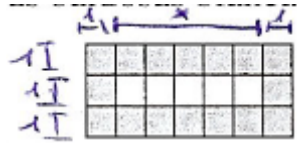
Other students went beyond graphical generalization to verbal or algebraic generalization. We illustrate this with the example shown in Figure 5.

As we observe in Figure 5, this student drew a graphical representation which represents the common features from the generic example shown in the statement, without tracing the lines that separate different tiles. S/he used this generalization to calculate the number of tiles that the problem required. Therefore, this is another

### Working Group 3

example of graphical generalization as well as arithmetical. S/he also provided a general verbal expression for any given number of white tiles, using natural language.

Moreover, some of the students used graphical and algebraic representation together to give sense to the generalization. We present an example typical of these students in Figure 6.



**Figure 6. Generalization graphical and algebraically expressed**

This student's diagram shows how s/he identified the data that remain constant and the data that change depending on the number of white tiles considered.

The three students' representations shown in Figures 4, 5, and 6 are typical examples of three groups that allow us to classify the eleven students that generalize graphically in problem 3. In the first group are the students who used suspension points to notice that the drawing would continue in the way indicated. These students used combined representation (graphical and arithmetical). In the second group are the students who used specific numbers in tiles of different sizes (the size of the tile is bigger when the number is bigger). In the third group each tile is represented by the number 1 and the number of tiles that depends on the number of white tiles is represented by  $x$ . Students in groups 2 and 3 used synthetic representations because at least two representations are considered as a whole to give sense to the generalization.

Most of the students, who expressed the generalization in problem 3, as well as in problem 4, did it verbally. For example, one student's response to problem 4 was, "The result is the number of matches that play each team against the rest of them, multiplied by two".

Most of the students generalized algebraically in problem 5. Some students even tried to use a formula which was familiar to them. The students tended to present a correct formula and used it to calculate the number in the positions requested ( $a_n = a_1 + (n-1)d$ ). This indicated that these students expressed the generalization algebraically. Some of these students worked on particular cases in the generalization process.

### DISCUSSION

We have identified four ways of expressing the generalization: (a) arithmetical, (b) algebraic, (c) graphical, and (d) verbal. Most of the student who got the generalization, expressed it arithmetically. This result is consistent with findings



from the earlier study of Becker and Rivera (2005) and with what we could expect due to students' previous knowledge.

This paper contributes to understanding the use of graphical representation in generalization (Mason & Pimm, 1984; Radford, 2002; Cañadas, 2007). This kind of representation illustrates what is common to all terms using particular cases but does not provide a general expression of any term of the sequence. Students seem particularly disposed to using it when the problem was posed using graphical representation, but not in other cases. Graphical representation of the generalization appeared to help students to generalize algebraically or verbally and thus, using these expressions, obtain particular case of the sequences. Generally students who utilized this kind of representation used it in combination with other sorts of representation. In this sense, we can consider graphical representation as a way of developing algebraic thinking and of expressing the generalization verbally or algebraically. This idea complements previous work which is mainly focused on other ways of generalization.

Graphical representation is sometimes enough for students to answer the question posed because they see the general pattern in the drawing. However, some of them revert to verbal generalization when they try to justify the answer, as Cañadas (2007) noticed. This is the main reason why verbal generalization is frequent in these problems (see Table 1).

Students generalized algebraically more frequently in the problem where particular cases were expressed numerically. This seems to be a consequence of what students were accustomed to in class. Only a low number of students generalized algebraically in problems where the statements were presented in unfamiliar representation. In particular, the lowest frequency of generalization was found in problem 4, which was presented in the least familiar way. This suggests that it is more difficult for the students to establish a relationship between algebra and generalization problems in non-numerical contexts and that, at some point, the idea of generalization has not moved from one context to others.

Unless we and other researchers have made an effort to identify and describe different kind of generalizations, one conclusion of this paper is that sometimes it is quite difficult to distinguish among them. In most cases where students used graphical generalization, they used a combined-multiple representation or synthetic representation. Multiple representations in generalization seem to be useful for students to express the generalization. The distinction between the two kinds of multiple representations of Figueiras and Cañadas (2010) is a powerful way to describe how students reach generalization. This paper shows that synthetic representation appeared in tasks with graphical representation in the statement. Graphical representation can be considered the primary in the sense that it is the one that promotes the appearance of the other(s).

As a practical consequence, it would be desirable to use tasks in different contexts to guide students to algebra as a way of generalization. Work on generalization tasks starting from particular cases expressed in different representations would be enriching to students and would promote algebraic thinking capabilities because they would relate algebra with representations different from the numerical one.

### Acknowledgement

This work has been supported by the Spanish National Plan I+D+i grants EDU2009-11337, funded by the Ministry of Education and Science and co-financed by FEDER.

### NOTES

1. The questionnaire is reproduced in Cañadas (2007, Appendix B).
2. This second task allowed us to develop other part of our objectives, which is beyond the scope of this paper.

### REFERENCES

- Becker, J. R., & Rivera, F. (2005). Generalization strategies of beginning High School algebra students. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 121-128). Melbourne: PME.
- Boletín Oficial del Estado (2003). *Real Decreto 832/2003 de 27 de junio, de Ordenación General y las Enseñanzas Comunes de la Educación Secundaria Obligatoria* (vol. BOE nº 158, pp. 25683-25743). Madrid: Ministerio de Educación y Ciencia.
- Cañadas, M. C., & Castro, E. (2007). A proposal of categorisation for analyzing inductive reasoning. *PNA*, 1(2), 67-78.
- Cañadas, M. C. (2007). *Descripción y caracterización del razonamiento inductivo utilizado por estudiantes de educación secundaria al resolver tareas relacionadas con sucesiones lineales y cuadráticas*. Granada: Departamento de Didáctica de la Matemáticas de la Universidad de Granada.
- Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. J. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63-85). Dordrecht: Kluwer Academic.
- Duval, R. (1999). *Semiosis y pensamiento humano. Registros semióticos y aprendizajes intelectuales*. México DC: Universidad del Valle.
- Figueiras, L., & Cañadas, M. C. (2010). Reasoning on transition from manipulative strategies to general procedures in solving counting problems. *Paper presented at the 7<sup>th</sup> British Congress on Mathematics Education*, Manchester, UK.

- Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: MacMillan.
- Janvier, C. (1987). Translation processes in Mathematics Education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27-32). Hillsdale, New Jersey: LEA.
- Kolloffel, B., T. H. S. Eysink, A. De Jong, & Wilhelm, P. (2009). The effects of representational format on learning combinatorics from an interactive computer simulation. *Instructional Science*, 37, 6, 503-517.
- Küchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics*: 11-16 (pp. 102-119). London: Murray.
- Lee, L. (1996). An initiation into algebraic culture through generalization activities. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 87-106). London: Kluwer.
- Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification*. Montreal: Concordia University.
- Mason, J., & Pimm, D. (1984). Generic examples: seeing the general in the particular. *Educational Studies in Mathematics*, 15(3), 277-290.
- Mason, J., Graham, A., Pimm, D., & Gowar, N. (1985). *Routes to roots of algebra*. Milton Keynes: Open University Press.
- Mill, J. S. (1858). *System of logic, raciocinative and inductive*. London: Harper & Brothers.
- Neubert, G. A., & Binko, J. B. (1992). *Inductive reasoning in the secondary classroom*. National Education Association: Washington DC.
- Pólya, G. (1967). *Le découverte des mathématiques*. Paris: DUNOD.
- Radford, L. (2002). The seen, the spoken and the written: a semiotic approach to the problem of objetification of mathematical knowledge. *For the Learning of Mathematics*, 22(2), 14-23.
- Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, 5(1), 37-70.
- Radford, L. (2010). *PNA*, 4(2), 37-62.
- Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics*, 20(2), 147-164.
- van Someren, M. W., Reimann, P., Boshuizen, H. P. A., & de Jong, T. (1998). *Learning with multiple representations*. Oxford: Elsevier.

# **THE ENTRANCE TO ALGEBRAIC DISCOURSE: INFORMAL META-ARITHMETIC AS THE FIRST STEP TOWARD FORMAL SCHOOL ALGEBRA**

Shai Caspi & Anna Sfard

The University of Haifa

*Taking as a point of departure the vision of school algebra as a formalized meta-discourse of arithmetic, we have been following six pairs of 7<sup>th</sup> grade students as they gradually modify their informal meta-arithmetic toward the "official" algebraic form of talk. In this paper we take a look at the very beginning of this process. Preliminary analyses of data have shown, unsurprisingly, that while reflecting on arithmetic processes and relations, the uninitiated 7<sup>th</sup> graders were employing colloquial means, which could not protect them against occasional ambiguities. More unexpectedly, this informal meta-arithmetic, although not supported by any previous algebraic schooling, displayed some algebra-like features, not to be normally found in everyday discourses.*

## **INTRODUCTION: STUDYING DEVELOPMENT OF ALGEBRA**

The idea that algebra is a *language*, the language in which science and other branches of mathematics are practiced, has been with us for centuries, and so was the controversy over this description (Lee, 1996). In our attempts to follow the development of school children's algebraic thinking we take as a point of departure a definition that responds to some of the concerns voiced by the objectors of the algebra-as-language approach: We define algebra as a *discourse*, that is, a form of communication. While preserving the centrality of the motif of language, this approach transfers algebra from the category of passive tools to that of human activities. This ontological change has important ramifications for how we view the development of algebraic thinking and how we investigate it. This paper is a report on the initial phase of our ongoing study of this topic. In this project, we have been following algebraic discourse of six pairs of 7<sup>th</sup> graders from its beginnings in the form of informal talk on numerical processes and relations, and through its subsequent process of its gradual formalization in school.

## **SCHOOL ALGEBRA AS FORMALIZED META-ARITHMETIC**

The definition of algebra as a discourse is a derivative of our foundational assumption that thinking is an individualized form of interpersonal communication (Sfard, 2008). To communicate either with others or with oneself, one has to act according to certain rules, implicitly shared by all the interlocutors. Different types of tasks and situations may evoke different sets of communicational regulations, that is, different discourses. Algebra can be defined as a sub-category of mathematical discourse that people employ while reflecting on arithmetical relations and processes.

Let us take a closer look at the two basic types of meta-arithmetical tasks that give rise to algebra. First, there is a question of numerical patterns, which we describe formally with the help of equalities, such as, say,  $a(b+c)=ab+ac$ . Although nothing in this latter proposition says so explicitly, this is, in fact, a piece of meta-arithmetic. Indeed, the symbolic proposition  $a(b+c)=ab+ac$  is a shortcut for the sentence *To multiply a number by a sum of other two numbers, you may first multiply each of the other two numbers by the first one and then add the results*. This type of meta-arithmetic narrative can be called *generalization*. The other algebra-generating tasks are questions about unknown quantities involved in completed numerical processes. This type of task is described in the modern algebraic language as *solving equations*. Indeed, equations, say  $2x+1=13$ , are meta-questions on numerical processes; in the present case the question is *What number, if doubled and increased by 1, would yield 13?*

According to this definition, algebraic thinking occurs whenever one scrutinizes numerical relation and processes in the search for generalization or in an attempt to find an unknown. The narratives (propositions about mathematical objects) that result from these two types of activities do not have to employ any symbolic means. Here is a rather striking historical example of pre-symbolic algebra taken from the Indian text known as *Aryabhatiya* (499 AD):

Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms. (Boyer & Mertzbach, 1989, p. 211)

Although this fact is hard to recognize, this lengthy piece presents the solution of an equation: it is a prescription for finding a number of elements in an arithmetic progression, whose first term, the difference, and the sum are given. While considering the communicational shortcomings of this intricate rendering it is easy to understand why symbolization of the discourse was one of the major trends in the further development of algebra. The symbolization was but a part of the more general historical process of *formalization* of algebraic discourse, the overall transformation that aimed at increasing the effectiveness of meta-arithmetic communication. This overall goal involved three types of action: *disambiguation*, that is, prevention of the possibility of differing interpretations of the same expressions by different interlocutors; *standardization*, supposed to ensure that all the interlocutors follow the same communicational rules; and *compression*, which turns lengthy statement such as the one quoted above into concise, easily manipulable expressions. The response to the need for disambiguation was *regulation*, that is, the constitution of strict, explicitly introduced rules of discursive conduct. Compression is attained through reification and symbolization. *Reification* means turning narratives about processes into ones about objects; Reifying usually involves introduction of nouns (e.g. *sum*, *product*) with which

to replace lengthy verb clauses. The above quote from *Aryabhatiya*, although formulated as a description of a process (a sequence of numerical operations; note the verbs *multiply*, *add*, etc.), includes compound noun clauses, such as "the square of the difference between twice the first term, and the common difference," which reify sub-sequences of computational steps. *Symbolization* means replacement of nouns, predicates, and verbs with ideograms, that is, symbols referring to objects the way words do, but without being uniquely tied to specific sounds. To make the replacement possible, a change in the grammar of the propositions may sometimes be necessary. For example, when a purely processual verbal description is translated into standard symbolic expression, the order of appearance of arithmetic operations may no longer correspond to the order of their implementation. When presented in the canonic symbolic manner, *Aryabhatiya*'s rule reincarnates into the concise expression  $(\sqrt{8Sd + (2a - d)^2} - 2a + d) / 2d$ , the special property of which is that it can be used both as a prescription for a calculation and as a *result* of this calculation.

The main difference between the two forms of algebraic discourse, *informal* and *formal*, is in the explicitness and rigorousness of regulation: of the two types of talk, only the formal is accompanied by a regulatory meta-discourse that explicitly states its meta-rules; and only in the this latter case, the meta-rules are the product of deliberate legislation, aiming at the prevention of ambiguity. The *informal/formal* dichotomy parallels the *rhetoric/symbolic* distinction, introduced by the historians of mathematics (Boyer, 1985). If we have chosen the word "parallel" rather than "identical" while describing these two distinctions, it is because the aspect of symbolization stressed in the historical dichotomy, although certainly most visible, is only one of the series of change that occurs in the transition from informal to formal algebraic discourse.

## METHOD OF STUDY

*Goal.* The overall goal of our study is to contribute to the project of mapping the development of algebraic thinking in school. If algebra is a formalized meta-arithmetic, child's algebraic discourse may be expected to emerge from discourses that the child has already mastered and which she can now try to adjust to the meta-arithmetical tasks of finding numerical patterns and investigating computational processes. In our study, therefore, the learning of algebra has been conceptualized as a gradual closing of the gap between students' informal meta-arithmetic and the formal algebraic discourse to which they are exposed in school. The aim of our investigation is to describe this process in as detailed a way as is feasible and useful. Our attention to informal algebraic discourse is motivated by its being much less researched than its formal counterpart. Although not altogether absent as an object of study, it has not been investigated in a sufficiently systematic way, and the cases of observations conducted with the eye to the formal-informal co-constitutive processes have been particularly infrequent.

*Participants and procedure.* In our longitudinal study we have been following middle school students as they progress in their informal meta-arithmetic while also making their first steps in the "official" algebraic form of talk. Six pairs of Hebrew-speaking 12-13 year old 7<sup>th</sup> grade Israeli students have been interviewed at intervals of circa four months, with the first author of this paper serving as the interviewer. So far, each round of interviewing consists of 5 to 6 meetings lasting for 60 to 90 minutes. The first round began just before the students were introduced to algebra in school. At the time this paper is being written, two rounds of interviewing have been completed, the interviews transcribed and partially analyzed, and the third round of interviewing is about to begin. We intend to conduct four rounds altogether, with the last one commencing about 18 months after the first.

*Tools.* In each round, the interviewees are asked to complete a battery of tasks that can be organized in the three-dimensional matrix, with the following binary distinctions constituting the three dimensions:

**InF** vs. **For**: the task is stated *informally* (InF), thus encouraging informal meta-arithmetical talk; or *formally* (For), e.g. by using canonic algebraic symbolism, thus inviting formal algebraic solution

**Gen** vs. **Equ**: the task invites a *generalization* (Gen) or solving an *equation* (Equ)

**ReL** vs. **Abs**: the task is set in *real-life* (ReL) or *abstract* (Abs) context

Each of the resulting 8 categories can be subdivided even further. For example, in the case of equations, we included tasks with *numerical* data (**Num**) and also tasks that ask for *parametric* (**Par**) solution (this locates this latter type of task in the mixed genre of equation-solving *and* generalization). Figure 1 presents two samples of the tasks prepared for the first, pre-algebraic round of interviews.

Task 1, Type: <InF, Gen, Abs>	Task 2, Type: <For, Equ, RealL>
<p>Given the sequence: 4, 7, 10, 13, 16....</p> <ol style="list-style-type: none"> <li>Write the next three elements of the sequence</li> <li>What number appears in the 20<sup>th</sup> place in the sequence?</li> <li>What number appears in the 50<sup>th</sup> place in the sequence?</li> <li>Write a rule for calculating the number that appears in any place in the sequence</li> </ol>	<p>On the shelf, there are books in English and in Hebrew. The number of English books exceeds the number of Hebrew ones by 8. If there are <math>n</math> books altogether, how would you calculate the number of those in English? (<b>Par</b>)</p>

Figure 1: Sample tasks from the first round of interviewing

*Analysis.* To map the development of the discourse, we describe and then compare samples of students' meta-arithmetic discourse collected in the successive rounds of interviewing. The descriptions focus on four defining characteristics of the discourse: (a) its *keywords* (e.g. those that denote variables or unknowns) and their use; (b) its *visual mediators* (icons, algebraic ideograms, graphs), and their use; (c) its *routines*, that is, patterned, recurrent forms of discursive actions; and (d)

*narratives* that the interviewees endorse and label as true. The specific questions that guide our examination of some of these discursive features will be presented in the next section, along with our sample responses.

## SOME FINDINGS FROM THE FIRST ROUND OF INTERVIEWING

Our study has only begun, but even so, in this brief paper we can only present a small fraction of our findings so far. We restrict this report to the description of the students' informal activity of generalization, as observed when they tried to solve Task 1 in Fig. 1. The activity of generalization has been investigated by many researchers (Zazkis & Liljedahl, 2002; Lannin et al., 2006; Radford et al., 2007). In most of these studies the principal focus was on strategies (routines) students used to detect and describe patterns. In the present study, while employing techniques of discourse analysis, we distribute our attention evenly between all four characteristics of the discourse produced in the activity of generalizing: the use of words, visual mediators, routines and endorsed narratives. Due to the scarcity of space, we restrict our present account to the first two of them. While reporting, we list only the most salient of the observed phenomena (the salience is not formally assessed; the evaluation of relative frequencies of these phenomena is yet to be completed.)

### Words and their use

*The choice of verbal tools for generalizing.* The first question that guides our analysis of meta-arithmetic regards the verbal means students use to generalize, that is, to perform the necessary *saming*. This last term, *saming*, regards the linguistic change that is the very essence of the process of generalizing: replacing specific numbers (e.g. 3, 5, 7, ...) with a single signifier (*odd number*) so as to turn infinitely many similarly structured arithmetic expressions (e.g. *the square of 3*, *the square of 5*, etc.) into a single meta-arithmetic expression (*the square of an odd number*). The *saming* signifier is called *variable*, which in formalized algebra usually comes in the form of a Latin letter, such as  $x$  or  $y$ . In the beginning of our study, the participants have not yet been introduced to algebraic symbolism, and it is thus not surprising that when asked in Task 1, part d, to write "a rule for calculating the number that appears in any place in the sequence", they used familiar words as their *saming* devices. This way of dealing with *saming* is instantiated several times in the following rule, written by one of the participants, H (highlighted are noun clauses, each of which does the job of *saming* over a specific set of numbers):

To find a certain place in the sequence I need the place that I found (it better be round) and then 3 (or any other number that is the regularity) times what must be added to the number you have now and then to add the number you have now and the product of the regularity and what you still need, and that's it. (T&H, Task 1(d), [209]; translated from Hebrew by the authors.)

This rule was written after the girls calculated the 50<sup>th</sup> element of the sequence (part c of task 1) by adding  $30 \cdot 3$  to the 20<sup>th</sup> element, found previously, in response to part



### Working Group 3

b of Task1. Translated into formal symbolic language, it would yield the following formula:  $a_n = a_m + (n-m) \cdot 3$ , or even  $a_n = a_m + (n-m) \cdot d$ , where  $a_n$ ,  $n$ ,  $n-m$ ,  $a_{n-m}$  and  $d$  are the traditional symbols for elements of arithmetic progression. In particular, here are the formal translations of all noun clauses from the quote:

a certain place in the sequence is equivalent to  $a_n$

the place that I found seems to appear here in the double role of

- $a_m$  (note that H speaks about adding a multiple of 3 to this number) and of
- $m$  (note the parenthetical remark "it better be round")

regularity is equivalent to  $d$  (the common difference of arithmetic progression)

what must be added to the number you have now – contrary to what the words seem to be saying, the use (it is said to be multiplied by 3) indicates that it is  $n-m$  that is meant here

the number you have now is equivalent to  $a_m$

This single example alerts us to a number of important phenomena, which we observed on many occasions, in the talk of this participant as well as of many other students. The phenomena are deemed important because they probably need to be considered in planning the further process of formalization of the students' informal meta-arithmetic.

As might be expected, the student opted for generalizing words that hinted at their prospective roles in the problem. In the single proposition quoted above, the hinting is done in several ways. Some of the words function as *metaphors*, as is the case when the index of an element is called a *place*. Some other names are *metonymies*, that is, represent the whole by its part. This is the case when the  $n^{\text{th}}$  element,  $a_n$  is called *the place I found*. Interestingly, there is also the "reverse" of metonymy: In the expression *what must be added to the number you have now*, which is meant to signify  $n-m$ , the whole (*what must be added*) appears in the role of its own part: of a multiplier with the help of which the addend is to be produced. Finally, the student made use of the *genus*, that is, of a broader category to which the given object belongs. This is the case when the common difference of the given progression, 3, is called *regularity* and when the girl refers to an element of the sequence as *number* (admittedly, this latter word was suggested by the designers of the task). To overcome overgeneralizations, H uses specifying descriptions, such as *the number you have now*. The term *regularity* has not been restricted by an additional description and this fact has two ramifications: First, the student has achieved a higher level of generalization than required by the authors of the task (the authors asked for a computational rule for the sequence in which the specific number, 3, must be added in the transition from any element to its successor). In a sense, therefore, she did even better than expected. Second, however, there is no hint in the generalizing word *regularity* that the regularities considered in the problem are those that produce arithmetic progressions. The resulting rule, therefore, is not self-

explanatory and may even be dismissed by some interpreters as offering only a special case of what it promises to present.

*The use of words (syntax).* The main question asked with respect to the syntax of the participants' informally composed generalizing propositions regards the degree of reification: Do the propositions speak about doing (calculations) or about properties of objects? Indeed, reifying is the key move toward disambiguation and condensation of meta-arithmetic narratives and may thus be seen as a "signature" feature of formal algebraic sentences.

The formerly discussed lengthy proposition from *Aryabhata*, although processual in its general tone, contained noun clauses that reified several of its sub-processes. It is striking that a similar partial reification appeared in our young participants' informal meta-arithmetic sentences. Note, for example, the H's clause *the product of the regularity and what you still need* that speaks about a result (product) of an operation (multiplication) rather than about the operation as such. This property is even more salient in another version of the rule for calculating any element of the given arithmetic sequence, which the same student, H, produced toward the end of the session:

the place times the regularity of the sequence plus one (T&H, Task 1(d), [228]).

This time, the "rule" does not even sound as a prescription for action: It does not contain any verbs (*times* and *plus* are not verbs!) and does not constitute a full sentence. Unlike in the case of the previous version, no structural change would be necessary to translate it into the canonic symbolic formula  $n \cdot d + 1$ .

### **Visual mediators and their use**

The salient property of our participants' meta-arithmetic was the scarcity of visual mediation other than arithmetical (numerical) expressions. Those of the students, who did try to express their rules with the help of ideograms, used either letters or markers such as boxes or lines. Thus, for example, the two students whose work was discussed above presented the simplified version of their rule as  $\square \cdot 3 + 1$  (H&T, 210). It should be stressed that in most cases, the students' interpretation of boxes was different from that of letters: Whereas letters functioned mainly as *names* of objects, the box was usually understood as a marker of a *physical space* for numbers. Indeed, unlike in the case of letters, which were supposed to signify the same number in all their appearances, identically looking boxes (squares) were often used indiscriminately for all the variables in the problem. Thus, in our study, some of the students presented rules such as this one in the form  $\square \cdot 3 + 1 = \square$ . Interestingly, one of the participants wrote  $x \cdot 3 + 1 = x$  (A&S, 49), the use clearly inspired by his former experience with squares functioning as delineators of a physical space for numbers.

**DISCUSSION: WHERE THE STUDENTS ARE AND WHAT COMES NEXT**

With an eye to the ultimate goal of informing instruction, we focused our efforts on identifying dissimilarities between students' informal meta-arithmetic and the formal algebra taught in schools. Let us stress that the discussion that follows and the tentative answers given in the end are grounded in a body of data much richer and more extensive than could be presented in this brief paper.

Colloquial, informally developed discourses are known for their occasional blurriness and vagueness. Therefore, it did not come to us as surprise that upon close examination, the texts produced by our participants, although quite impressive in their resourcefulness, proved also full of ambiguities. Consider H's complex prescription for calculating any element of the arithmetic progression. Here, H used a single noun for a number of purposes (see her metaphoric use of the word *place* for the index of an element and the metonymic use of the same term for the element itself) and, on another occasion, referred to a single object in a number of ways (e.g., note the difference between the expressions *the place I found* and *the number you have now*, both of which were used with reference to the previously calculated element  $a_m$ .) She also used generic names which were all too general and, as such, could be easily misinterpreted by her interlocutors. To overcome overgeneralizations, H employed specifying descriptions, such as *the number you have now*. However, this type of specification, being context-dependent (note the use of the deictic words *you* and *now*) could not possibly bar multiple interpretations.

All this said, our study, so far, has resulted also in some less predictable findings. On the basis of our own previous research (Sfard & Linchevski, 1994), we conjectured that the students' informal meta-arithmetic would be about processes rather than objects. It is because of this prediction that we were careful to formulate the first tasks in processual language. For example, in Task 1, part d we asked for the *rule for calculating* any element of the sequence rather than inquiring about what such generic element *is*. We were thus quite surprised to find out remarkable structural similarities between the students' verbal meta-arithmetic and the formal reified algebra. Two possible explanations come to mind when we try to account for this finding. First, structures of algebraic formulas are not unlike those of arithmetic expressions, and thus our students might just be building on their knowledge of the latter type of structure. Second, it is possible that these days algebra is simply "in the air": elements of algebraic discourse may be present in other school discourses well before its formal introduction in the 7<sup>th</sup> grade. With the help of media, algebraic forms of expression may even be infiltrating colloquial discourses. To check these conjectures, we decided to broaden our study and to conduct similar interviews with 6<sup>th</sup> and 5<sup>th</sup> grade students.

Whatever the results of these latter investigations, we believe that one of the present tentative conclusions from our study is unlikely to change: While much work must be invested in formalization of students' informal meta-arithmetic, the resources with

which children are coming to their algebra classrooms may be a much better foundation for the development of formal algebraic discourse than could be expected on the basis of what is known about their mathematical education so far. The more knowledgeable we are about these resources, the better our chances for helping the students in closing the gap between their informal meta-arithmetic and the formal algebra taught in school. Above all, we need this knowledge to be able to teach in such a way as to preserve the all-important link between the two discourses.

#### References

- Boyer, C. B. (1985). *A history of mathematics*. Princeton, NJ: Princeton University Press.
- Boyer, C. B., & Merzbach, U. C. (1989). *A history of mathematics* (2nd ed.). New York: Wiley.
- Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic Generalisation Strategies: Factors Influencing Student Strategy Selection. *Mathematics Education Research Journal* 8 (3), 3-28.
- Lee, L. (1996). An initiation to algebraic culture through generalization activities. In N. Bednarz, C. Kieran & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching*. Dordrecht: Kluwer Academic Publishers.
- Radford, L., Bardini, C., & Sabena, C. (2007). Perceiving the General: The Multisemiotic Dimension of Students' Algebraic Activity. *Journal for Research in Mathematics Education*, 38 (5 ), 507-530.
- Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge, UK: Cambridge University Press.
- Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics*, 26, 191-228.
- Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49, 379-402.

# STUDENTS' REASONING IN QUADRATIC EQUATIONS WITH ONE UNKNOWN

**M. Gözde Didiş, Sinem Baş, A. Kürşat Erbaş**

*Middle East Technical University*

*This study examined 10<sup>th</sup> grade students' procedures for solving quadratic equations with one unknown. An open-ended test was designed and administered to 113 students in a high school in Antalya, Turkey. The data were analyzed in terms of the students' foci while they were answering the questions. The results revealed that factoring the quadratic equations was challenging to them, particularly when students experienced them in a different structure from what they were used to. Furthermore, although students knew some rules related to solving quadratics, they applied these rules without thinking about why they did so, nor whether what they were doing was mathematically correct. We concluded that the students' understanding in solving quadratic equations is instrumental (or procedural), rather than relational (or conceptual).*

**Keywords:** Quadratic equations, instrumental understanding, relational understanding

## INTRODUCTION

For many secondary school students, solving quadratic equations is one of the most conceptually challenging subjects in the curriculum (Vaiyavutjamai, Ellerton, & Clements, 2005). In Turkey, where a national mathematics curriculum for elementary and secondary levels has been implemented, the teaching and learning of quadratic equations are introduced through factorization, the quadratic formula, and completing the square by using symbolic algorithms. Of these techniques, students typically prefer factorization when the quadratic is obviously factorable. With this technique, students can solve the quadratic equations quickly without paying attention to their structure and conceptual meaning (Sönnerhed, 2009). However, as Taylor and Mittag (2001) suggest, the factorization technique is only symbolic in its nature. Since students simply memorize the procedures and formulas to solve quadratic equations, they have little understanding of the meaning of quadratic equations, and do not understand what to do and why. This can be described using Skemp's (1976) categorization of mathematical understanding as either instrumental or relational. He simply described instrumental understanding as "rules without reasons" and relational understanding as "knowing both what to do and why" (p. 20). Using the language of Skemp, it can be said that students can perform instrumentally to solve the quadratic equations by applying the factorization technique; however, few develop relational understanding.

Although quadratic equations play an important role in secondary school algebra curricula around the world, it appears that studies concerning teaching and learning

quadratic equations are quite scarce in algebra education research (Kieran, 2007; Vaiyavutjamai & Clements, 2006). Therefore, this study was designed to widen the research considering students' reasoning when engaging in different types of quadratic equations in one unknown. In particular, this study investigated students' processes for solving quadratic equations with one unknown by using the factorization technique.

The findings of this study may provide teachers with insight into the reasoning that leads to the common mistakes that students make while solving quadratic equations, and hence guide them in creating a more efficient pedagogical design for teaching how to solve quadratics.

### **Challenges faced by Students in Solving Quadratic Equations**

According to Kotsopoulos (2007), for many secondary school students, solving quadratic equations is one of the most conceptually challenging aspects in the high school curriculum. She indicated that many students encounter difficulties recalling main multiplication facts, which directly influences their ability to engage in quadratics. And, since the factorization technique of solving quadratic equations requires students to be able to rapidly find factors, factoring simple quadratics (i.e.,  $x^2+bx+c=0$  where  $b, c \in R$ ) becomes a quite challenge, while non-simple quadratics (i.e.,  $ax^2+bx+c=0$   $a, b, c \in R$  and  $a \neq 1$ ) become nearly impossible. Moreover, students encounter crucial difficulties in factoring quadratic equations if they are presented in non-standard forms. For example, factoring  $x^2+3x+1=x+4$  is challenging for students, since the equation is not presented in standard form (Kotsopoulos, 2007). Similarly, Bossè and Nandakumar (2005) stated that the factoring techniques for solving quadratic equations are problematic for students. They indicated that students can find factoring the quadratics considerably more complicated when the leading coefficient or constant in the quadratic has many pairs of possible factors.

Skemp's (1976) description of instrumental and relational understanding can be used as a framework to discuss the difficulties students have with factoring quadratic equations. While an instrumental understanding of factorizing quadratic equations with one unknown requires memorizing rules for equations presented in particular structures, relational understanding enables students to apply these rules to different structures easily (Reason, 2003). That is, when students have relational understanding, they can transfer knowledge of both what rules (and formulas) can be used and why these rules work from one situation to another (Skemp, 2002).

Lima (2008) found that students may perceive quadratic equations just like they do calculations. Since they focus mostly on the symbols used to perform operations, they may not be aware of the concepts that are involved. Vaiyavutjamai and Clements (2006) explain that students' difficulties with quadratic equations arise from the lack of both instrumental and relational understanding of the associated

mathematics. They found several misconceptions regarding variables which were obstacles to understanding quadratic equations. For example, students thought that the two  $x$  symbols in the equation  $(x-3) \cdot (x-5) = 0$  stood for different variables, even though most of them obtained the correct solutions  $x=3$  and  $x=5$ . Hence, they concluded that students' performance in that context reflect rote learning and a lack of relational understanding.

## **METHODOLOGY**

### **Participants and the Instrument**

The sample of this study consisted of 113 students in four 10<sup>th</sup> grade classes, and this study was performed in a high school in Antalya, Turkey during the spring term 2009-2010.

For the purpose of the study, a questionnaire was developed by the authors since no test to specifically explore students' errors and understanding was available. The test questions were carefully selected from secondary mathematics textbooks and from research regarding quadratic equations (e.g., Crouse & Sloyer, 1977). All questions used in this questionnaire were selected to measure the study objective of "determine the roots and solution set of [a] quadratic equation in one unknown". During the selection process, two mathematics educators and a mathematics teacher were consulted about whether the content of the selected questions were consistent with the objective of the test. In light of their suggestions, seven open ended question were developed. Although the format of the all of the questions was open-ended, they varied in type so as to be consistent with the objective of the study. Questions 1 to 4 were in the standard format in which students were expected to "find the solution set of the given quadratic equation". These questions were based on procedural skills, and they were mostly used to detect students' procedural abilities in solving quadratic equations in different structures. On the other hand, questions 5 to 7 introduced a mathematical scenario that included both a quadratic equation and a solution belonging to it. In these type of questions, students were expected to determine "whether the solutions [belonging to] the equations were correct or not, and to make judgment about their decision". Therefore, in addition to procedural skills, these questions were used to detect students' understanding of and reasoning level when dealing with quadratic equations.

The mathematics teacher administered the questionnaire during the regular class period and the students were given 30 minutes to complete it.

### **Analysis of Data**

Initially, the responses given to each question were given scores of either 1 or 0. A score of 1 was given for answers that were mathematically correct in terms of both solution process and final answer. A score of 0 was given for answers that were either omitted or incorrect in terms of either solution process or final answer. Then,

in order to obtain a general view of the students' performance, the percentage of correct, incorrect and omitted questions were calculated. The aim of this process was descriptive analysis. Afterwards, qualitative data analysis was conducted. The subjects' responses were studied in order to provide substantial information about their type of understanding. The aim of this analysis was to identify the common mistakes that students made while solving the quadratic equations. Therefore, the incorrect answers for all questions have been analyzed item by item with respect to the students' focus when they solved the questions in the test situation. Students' types of mistakes were coded by two researchers of this study who worked initially separately. Next, the mistakes were both combined and renamed based on their common features, and then they were classified by two researchers together. Lastly, these mistakes were interpreted in terms of students' instrumental understanding and relational understanding.

## RESULT

The first item in the instrument was related to finding the roots of a quadratic equation given in standard form (e.g.,  $ax^2+bx+c=0$  where  $a, b, c \in R$ ). Almost all students correctly solved this equation by factorization. In the following questions, quadratic equations were given in different structures (e.g.,  $ax^2-bx=0, c=0$ ). In these types of questions, just 64% of them solved the equation  $ax^2-bx=0$ , correctly. When the solution processes of students who made mistakes (36%) were analyzed, it was recognized that their mistakes were based on two different types.

$$\begin{array}{l}
 x^2 = 2x \\
 x - x = 2 - x \\
 \underline{x = 2} \\
 C = \{2, 3\}
 \end{array}$$

**Figure 1: Find the solution set of  $x^2-2x=0$ : An example of students' first type of mistake**

$$\begin{array}{l}
 x^2 - 2x = 0 \\
 x^2 - 2x + 1 = 0 \\
 (x-2) \cdot (x+1) = 0 \\
 C = \{+1, -2\}
 \end{array}$$

**Figure 2: Find the solution set of  $x^2-2x=0$ : An example of students' second type of mistake**



“Find the

$$\begin{array}{l}
 x \cdot (x-1) = 12 \\
 4 \cdot (4-1) = 12 \\
 4 \cdot 3 = 12
 \end{array}
 \quad
 \underline{x = 4}$$

**Figure 3: “ $x^2-2x=12$ ”: An example of students’ mistakes when just the form of equation changed.**

In the first type of incorrect solution (see Figure 1), students carried the term  $-2x$  from left side to the right, and then “simplified” by “dividing” both sides of the equation by  $x$ . Consequently, they ignored one of the roots of the equation, which is 0. In the second type of incorrect solution (see Figure 2), students tried to factorize the equation. Here, students perceived the form  $ax^2-bx=0$  just like  $ax^2+b=0$  and treated  $2x$  as a constant of 2 to be factorised. When the form of the equation was changed instead of the structure (e.g.,  $ax^2+bx=c$  where  $a, b, c \neq 0$ ), 12% of the students incorrectly solved the quadratic as in Figure 3. Because the constant term was in the right side, they did not perceive that the equation was in standard form. In this type of solution, they were able to find only one of the roots, 4.

Statements <u>Question 5</u>	Students’ types of responses with their reasoning			
	I. IV.	II.	III.	
<p>To solve the equation “<math>(x-3) \cdot (x-2) = 0</math>” for real numbers, Ali answered in a single line that: “<math>x=3</math> or <math>x=2</math>” <b>Is this answer correct? If it is correct, how can you show it correctness?</b></p>	<p><b>“The answer is Right”</b> Since I wrote <math>(x-3) \cdot (x-2) = 0</math> as <math>x^2-5x+6=0</math> then I factorize to find roots of it. from <math>(x-3)=0</math> and from <math>(x-2)=0</math> “<math>x=3</math> and <math>x=2</math>”</p>	<p><b>“The answer is Right”</b> Because <math>(x-3)=0</math> <math>(x-2)=0</math> <math>x=3</math> <math>x=2</math></p>	<p><b>“The answer is Right”</b>. Since we substitute “3 and 2” into <math>x</math>, the equation is provided. (explanation made only with words)</p>	<p><b>“The answer is Right”</b>. If the <math>x=2</math> and <math>x=3</math> are substituted into the equation <math>(3-3) \cdot (2-2)=0</math> <math>0 \cdot 0 = 0</math></p>

**Table 1: Common examples of students’ types of responses with their reasoning for question 5.**

### Working Group 3

Although all of the students stated that Ali's answer was correct by choosing either one of the statements I, II, III, and IV, their justifications were different. For instance, in statement I, students first transformed the factorized expression into the standard form, and then factorized the expression again in the same way and found the roots by rote. In statement II, students unconsciously applied the null factor law. In statement III, the way of justification for solution was based on substitution method. In all of these three statements, they could not clearly justify why the solution is correct. In statement IV, students substituted  $x=3$  into  $(x-3)$  and  $x=2$  into  $(x-2)$  simultaneously, and concluded that the solution was correct since  $0 \cdot 0 = 0$ . Namely, they thought that the two  $x$  symbols stood for different numbers and did not appear to appreciate  $x$  as a variable.

<u>Statements</u>	<u>Students' types of responses with their reasoning</u>			
	<u>I.</u>	<u>II.</u>	<u>III.</u>	<u>IV.</u>
<u>Question 6</u>				
<p>A student hands in the following work for the following problem.</p> <p>Solve ;</p> $x^2 - 14x + 24 = 3$ $(x-12) \cdot (x-2) = 3$ $(x-12) \cdot (x-2) = 3 \cdot 1$ $x-12=3 \quad x-2=1$ $x=15 \quad x=3$ <p>Solution set= {3, 15}</p> <p><b><i>Is the student correct? Explain your answer with its reasons?</i></b></p>	<p><b><i>"The answer is Wrong"</i></b></p> <p>Because, firstly, 3 must carry the left side of the equation and equalize the 0. Then, the other operations must be done. In this way, the equation should be</p> $x^2 - 14x + 21 = 0$	<p><b><i>"The answer is Wrong".</i></b></p> <p>Because when we substitute 3 and 15 for <math>x</math>, the equation is not correct.</p>	<p><b><i>"The answer is Right"</i></b></p> <p>Since the result is equal to 3, we equate 3 rather than 0 while factoring it. Therefore, the result is true.</p> <p>Students again solve as:</p> $x^2 - 14x + 24 = 3$ $(x-12) \cdot (x-2) = 3$ $(x-12) \cdot (x-2) = 3 \cdot 1$ $x-12=3 \quad x-2=1$ $x=15 \quad x=3$ <p>{3, 15}</p>	<p><b><i>"The answer is Wrong"</i></b></p> <p>Since the equations are separated as <math>(3,1)</math> there is no error when <math>(x-12)=3</math> however, there is error when <math>(x-2)=1</math>.</p> <p>It must be <math>(x-2)=3</math> then, <math>x=5</math>.</p> <p>Therefore, the solution will be {5, 15} rather than {3, 15}.</p>

**Table 2: Common examples of students' types of responses with their reasoning for question 6.**

### Working Group 3

In statements I and II (see Table 2), students were aware of the error in the solution of the given question. However, to explain the reasons for the mistake, they mainly referred to procedures in explanations similar to the responses in statements I, II, III for question 5 (see Table 1). In statement III, students incorrectly stated that the answer was right. Looking at the statement “since the result is equal to 3, we equate to 3 rather than 0 while factoring it”, it can be said that they wrongly tried to transfer the null factor law to this context. That is, they equated the factors of equation  $x^2 - 14x + 24$  with the integer factors of 3. In statement IV, students correctly claimed “the answer of the question wrong”; however, their explanations were fully erroneous. Similar to statement III, these students tried to apply the null factor law to the equation. Nonetheless, in this case, they only equated the factors to 3 rather than to the factors of 3. In both statements III and IV, students did not check whether the roots they found were appropriate or not.

<u>Statements</u>  <u>Question 7</u>	<u>Students' types of responses with their reasoning</u>			
	<u>I.</u>  <u>IV.</u>	<u>II.</u>	<u>III.</u>	
<p>The solution of the quadratic equation <math>2x^2 = 3x</math> is given in the following;</p> <p><b>According to you, is this solution correct or not? Explain your answer with its reasons?</b></p> <p><b>Solution:</b></p> <p>I. step <math>2x^2 = 3x</math></p> <p>II. step <math>2 \cdot x \cdot x = 3 \cdot x</math></p> <p>III. step <math>2 \cdot x = 3</math></p> <p>IV. step <math>x = 3/2</math></p> <p>Solution set = <math>\{3/2\}</math></p>	<p><b><i>“The answer is Wrong”</i></b></p> <p>Because <math>3x</math> must be carried the left side of the equation and equalized the 0. Then,</p> <p><math>2x^2 = 3x</math></p> <p><math>2x^2 - 3x = 0</math></p> <p><math>x \cdot (2x - 3) = 0</math></p> <p><math>x = 0, \quad x = 3/2.</math></p>	<p><b><i>“The answer is Right”</i></b></p> <p>The solution is right; however, it must be added 0 to the solution set.</p>	<p><b><i>“The answer is Right”</i></b></p> <p>Because when we substitute the value for <math>x</math>, the equation is satisfied.</p>	<p><b><i>“The answer is Right”</i></b></p> <p><math>2x^2 = 3x</math> and <math>x^2</math> is opened.</p> <p><math>2 \cdot x \cdot x = 3 \cdot x</math></p> <p>Yes the <math>x</math> is simplified.</p> <p><math>2x = 3</math> so <math>x = 3/2.</math></p>

**Table 3: Common examples of students' types of responses with their reasoning for question 7.**

In statement I (see Table 3), students stated that the answer was correct. They explained an appropriate procedure required for solving the equation. Since they memorized the rule without its reasons, they could only explain how the procedure should be carried out. In statement II, on the other hand, students were aware that the roots of the equation were 0 and  $3/2$ . However, they did not recognize that when  $x$  was simultaneously canceled from both sides, the root 0 disappeared. Furthermore, in statement III, the explanation for solution was just based on the substitution method. In statement IV, students incorrectly stated that the answer was right. Like in statement II, students were not aware of the missing root 0 when canceling  $x$  in the equation

## DISCUSSION

The results indicate that most of the students used the factorization technique to solve quadratic equations. This result supports Bosse and Nandakumar (2005), who claimed that a large percentage of the students preferred to apply the factorization techniques to find the solutions of quadratic equations. Also, in parallel with the results of these authors and Kotsopoulos (2007), the result of this study revealed that factoring the quadratic equations was challenging when they were presented to students in non-standard forms and structures. After looking at the examples of students' solutions (see Figures 1, 2, and 3), it can be said that the students knew some rules (or procedures) related to solving quadratics. However, they tried to apply these rules thinking about neither why they did so, nor whether if what they were doing was mathematically correct. These results give some clues about students' instrumental understanding of solving quadratic equations with one unknown. However, to make an exact judgment about students' relational or instrumental understanding as Skemp (2002) defined, in-depth interviews with individual students are required. Furthermore, results also indicate that students incorrectly tried to transfer some rules from one form of equation to another (e.g., in Figure 2). This can be considered another clue to students' instrumental understanding (Reason, 2003).

When students were asked to examine a solution process of a quadratic equation and judge whether it was correct (i.e., in questions 5, 6, and 7), the results give additional clues about their reasoning in solving quadratics. In question 5, for example, although most of the students were aware of the correctness of the result, they did not explain the underlying null factor law used to solve the quadratics by factorization. The responses also reveal their misunderstanding of the unknown concept in a quadratic equation (see the statement IV, in question 5), which is consistent with the results of Vaiyavutjamai and Clements (2006). Students were not aware that the two's in the equation represent a specific unknown when dealing with equations in the form  $(x-a) \cdot (x-b) = 0$ . All of these can be regarded as clues to students' instrumental understanding. As stated by Lima (2008), and Vaiyavutjamai and Clements (2006), students knew how to get correct answers but were not aware of what their answers represented.

Similar interpretations can be made for the responses of students to question 6. There are two salient points related to their reasoning in explaining the given solution. First, although students were expected to explain the reason(s) why the given solution process was wrong, they could not detect the conceptual errors in the solution. They just presented some rules or procedures to solve the quadratic. Second, as was clear from statements III and IV (see Table 2), due to their lack of conceptual understanding of the null factor law in solving quadratics given in standard form, they wrongly transferred this principle to a quadratic in a non-suitable form. This can also be a clue for students' instrumental understanding. Because when students relationally understand a rule, they can use it in a different context (Reason, 2003). Similar inferences can be made for the students' responses to questions 7 where they did not offer any explanation for why canceling  $s$  was wrong. In other words, they did not recognize that when  $x$  was simultaneously canceled from both sides, the root 0 disappeared. Also, consistent with the results reported by Bossé and Nandakumar's (2005) and Kotsopoulos' (2007), although students knew the null factor law, they could not apply it appropriately when the structure of equation was changed.

Collectively, all these results suggest that in general students attempt to solve the quadratic equations as quickly as possible without paying much attention to their structures and conceptual meaning (Sönnerhed, 2009). Although we cannot be sure if their reasoning was based on instrumental or relational understanding without in-depth interviews with students, their written answers provide clues to their reasoning, and it can be said that their reasoning underlying solving quadratic equations was based on instrumental understanding.

Having instrumental understanding does not generally cause trouble for students. It is much easier to obtain and use than relational understanding, simply because it requires less sophisticated knowledge, and with instrumental understanding, students can generally learn how to obtain the right answers more quickly. However, it necessitates memorizing, and without relational understanding the learning cannot be adapted to new tasks, and students cannot give real reasons for their answers (Skemp, 2002). For that reason, greater attention should be given to how the concept is introduced to reduce the possibility of students learning by rote. Any solution mechanism must allow students to understand the meaning of the process that they apply in order to arrive at the correct answer; otherwise, the mechanism they learn is likely to be a source of error (Blanco & Garrote, 2007).

### **Recommendation**

As a result of this study, several suggestions can be made to contribute to improving the teaching of quadratic equations. Since factorizing the quadratic equations was challenging when they are presented in non-standard forms and structures, it would be better if teachers introduce various kinds of quadratic equations using a variety of different structures rather than only in one or two standard forms. On the other hand,

it would be also helpful for students to understand the factorization techniques as relational when teachers clearly emphasize meaning of the null factor rather than presenting it just as rule. In addition, because the students can attribute different meanings to the symbols (Küchemann, 1981), their understanding of the meanings of the algebraic symbols needs to be taken into account. Therefore, if teachers emphasize the meaning of the algebraic symbols, it would also be useful for students to understand what the symbols represent in quadratic equations. Moreover, when teachers encourage students to use different techniques while solving quadratic equations, students' learning may improve, and they may also gain a conceptually understanding. Similar recommendations can also be found in the related literature (e.g., Bossé & Nandakumar, 2005; Sönnnerhed, 2009).

Undoubtedly, teachers play an important role in encouraging students to learn relationally. We believe that this is the most important part of teachers' pedagogical content knowledge. However, research studies demonstrate a lack of secondary school mathematics teachers' pedagogical content knowledge in this respect (Vaiyavutjamai, Ellerton, & Clements, 2005). Indeed, there is a need to research teachers' knowledge about students' difficulties concerning quadratic equations.

## REFERENCES

- Blanco, L. J., & Garrote, M. (2007). Difficulties in learning inequalities in students of the first year of pre-university education in Spain. *Eurasia Journal of Mathematics, Science & Technology Education*, 3(3), 221-229.
- Bossé, M. J., & Nandakumar, N. R. (2005). The factorability of quadratics: Motivation for more techniques (section A). *Teaching Mathematics and its Applications*, 24(4), 143-153.
- Crouse, J. R., & Sloyer, W. C. (1977). *Mathematical questions from the classroom*. USA: Prindle, Weber & Schmidt.
- Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. Lester (ed.), *Second Handbook of Research on Mathematics Teaching and Learning: A project of the National Council of Teachers of Mathematics. Vol II* (pp. 669-705). Charlotte, NC: Information Age Publishing.
- Kotsopoulos, D. (2007). Unraveling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher*, 63(2), 19-24.
- Küchemann, D. E. (1981). Algebra. In Hart, K., Brown, M. L., Küchemann, D. E., Kerslake, D., Ruddock, G., & McCartney, M. (Eds.), *Children's understanding of Mathematics: 11-16* (pp. 102-119). London: John Murray.
- Lima, R. N. (2008). *Procedural embodiment and quadratic equations*. Retrieved April 1, 2010, from <http://tsg.icme11.org/document/get/701>.

### Working Group 3

- Reason, M. (2003). Relational, instrumental and creative understanding. *Mathematics Teaching*, 184, 5-7.
- Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.
- Skemp, R. R. (2002). *Mathematics in the primary school*. London: Routledge Falmer.
- Sönnerhed, W. W. (2009). Alternative approaches of solving quadratic equations in mathematics teaching: An empirical study of mathematics textbooks and teaching material in Swedish Upper-secondary school. Retrieved April 5, 2010, from [http://www.ipd.gu.se/digitalAssets/1272/1272539\\_plansem\\_wei.pdf](http://www.ipd.gu.se/digitalAssets/1272/1272539_plansem_wei.pdf).
- Taylor, S. E., & Mittag, K. C. (2001). Seven wonders of the ancient and modern quadratic world. *Mathematics Teacher*, 94, 349-351.
- Vaiyavutjamai, P., Ellerton, N. F., & Clements, M. A. (2005). *Students' attempts to solve two elementary quadratic equations: A study in three nations*. Retrieved April 1, 2010, from [www.merga.net.au/documents/RP852005.pdf](http://www.merga.net.au/documents/RP852005.pdf).
- Vaiyavutjamai, P., & Clements, M. A. (2006). Effects of classroom instruction on students' understanding of quadratic equations. *Mathematics Education Research Journal*, 18(1), 47-77.

# INVESTIGATING THE INFLUENCE OF STUDENTS' PREVIOUS KNOWLEDGE ON THEIR CONCEPT OF VARIABLES BY USING AN ANALYSIS TOOL CONSIDERING TEACHING REALITY

Sandra Gerhard

Goethe-University Frankfurt am Main

*Screening the literature on school algebra one finds various suggestions on how to implement algebra in school. But various articles also deal with problems concerning the transition between arithmetic and algebra. In order to improve school practice, teaching interventions that focus on the question of how algebraic knowledge interacts with arithmetic knowledge should be analysed while keeping in mind the influence of praxis conditions like mathematical socialisation and the didactic contract. The aim of this paper is not to give a final answer to this question, but to introduce an analysis tool meeting the above-mentioned requirements and to exemplify how such an analysis tool, according to mathematics education as design science, may help to identify integral parts of the teaching intervention.*

*Key words: variables, previous knowledge, approach to algebra, design research*

## INTRODUCTION

If one talks about school algebra the question is not only *how* to teach algebra but also *what* students are supposed to learn when they are taught about something called “algebra”. Since school algebra is a broad subject we first of all have to narrow this field.

## THEORETICAL FRAMEWORK

This doctoral project is based on the perspective that learning to deal with symbolic algebra, namely with variables, is important for students (Dörfler, 2008). There are two opposing approaches towards variables. One focuses on practicing algebraic thinking without letter variables and introducing letter variables later. Linchevski (1995) calls this ‘algebra without letters’. The associated counterpart is arithmetic with letters. A more suitable opposite to algebra without letters is starting algebra with letters or to symbolize algebraic reality into being (Sfard, 2000). This perception allows different approaches to algebra and is not limited to arithmetic.

A closer examination of approaching variables via generalized arithmetic reveals advantages and disadvantages. On the one hand, variables are numbers replaced by letters and the letters can be subjected to the same operations as numbers. At a first glance it seems that students simply have to put their “number template” (Sfard, 2000) onto the letter variables and everything should be self-evident to them. On the other hand, students deal with numbers in a different way to the way they are expected to deal with letters. After a long term arithmetic education of up to six



years, students usually believe that mathematics is all about computing arithmetic problems, the equal sign merely being a sign for “calculate now” (Carragher & Schliemann, 2007, p. 670). McNeil (2004) emphasises a strong connection between children’s equation learning difficulties and their existing knowledge. An alternative may be to avoid long term arithmetic education without algebraic contents. Then there would be less time to adopt undesirable patterns and more time to deal intensely with variables.

But teaching variables early is not a panacea either. On looking closely at the various problems students have with algebraic thinking and letter variables (Carragher & Schliemann, 2007) it becomes evident that one cannot blame long term arithmetic education alone. Reasons for the problems can also be found in the nature of algebra itself. In order to develop algebraic knowledge with the help of variables, students do not simply have to transform concrete-descriptive knowledge into abstract-formal knowledge (Malle, 1993), but rather they need to develop a “flexibility in thought to move between the process to carry out mathematical task and the concept to be mentally manipulated as a part of a wider mental schema, [an] ‘amalgam of process and concepts’ [called] ‘procept’ ” (Gray & Tall, 1994, p. 1).

Regardless of the chosen approach to variables, students must and will transfer their new knowledge to different contexts at some point. This is the nature of mathematics. Thus, the new knowledge will be affected by the student’s previous arithmetic knowledge anyway. So one of the questions to bear in mind while designing a learning environment for introducing variables is how the new algebraic knowledge will interact with previous arithmetic knowledge.

## **METHODOLOGY**

### **Design-based Research**

Like Wittmann (1995), the author sees mathematics education as a design science. Hence this research project is based on the paradigm of Design-Based Research which “blends empirical educational research with the theory driven design of learning environments” (The Design-Based Research Collective, 2003, p. 5). It has to be emphasised that design-based research represents an integrated approach to research allowing for all facets of everyday classroom life in contrast to research methods focusing on controlling variables. The research product consists of a design-based theory where the theory explicates when and why the design works.

The author of this paper is acting in the double role of researcher and teacher. This is a critical factor in respect of the authentic setting. This being said, it also offers opportunities. First of all as teacher-researcher, the author incorporates her implicit knowledge into the teaching process which, according to Design-Based Research, is desirable as long the implicit knowledge is the object of research. Secondly, follow-up interviews have also been conducted by the teacher-researcher. Thus the interviews may also provide information about the implicit knowledge of the teacher

and the didactic contract (*see Pichat & Ricco, 2001*) between teacher and student as will be shown later in this paper.

### **Designing a Learning Environment: the Teaching Intervention**

The aim of the doctoral project is the development of a theory-driven learning environment for introducing variables in early grades. The Measure Up–project (Dougherty, 2008) was chosen as starting point and redesigned for introducing variables in primary school.

Based on the analysis of a pre-study in lower elementary school (Gerhard, 2009) the learning environment was modified for a 5<sup>th</sup> grade of a grammar school. Focusing on the authentic setting, the teacher was included and the design was based on the federal curriculum, the curriculum of the mathematics section of the school and the textbook. The intervention “algebra” replaced the topic “calculation of length, area and volume” whereas this topic was embedded in the topic “algebra”.

The following principles underpinned the teaching intervention. The students were told to do mathematics without numbers, to find as many relations as they could, to write relations as equations and inequations to convert equations through “basic transformation rules” (Malle, 1993, p. 219) and to make auxiliary drawings.

### **Developing a Theory: the Interviews**

For the development of theory the research focuses in particular on the question of how new algebraic knowledge might interact with previous arithmetic knowledge. The findings, by giving important information about how the learning environment affects students’ understanding of variables, are an integral part of the theory-driven design. Therefore, additional problem-centred, half-standardised interviews were conducted. During the interview students were confronted with problems that were designed to challenge their previous arithmetical knowledge and their knowledge about variables. Therefore arithmetic rather than algebraic word problems were chosen. They were modified by using letters instead of numbers.

#### **Holiday in France I**

Stefan will have a holiday in France; he wants to hike from Lyon to Lourdes.

From Buchheim to Lourdes it takes **a** km. From his home he firstly goes by bus **b** kilometres to Lyon. From Lyon to Lourdes he wants to hike for **c** days. How much kilometres does he have to hike every day at an average?

**Figure 1: Sample problem**

The sample problem (see Figure 1) was designed considering the following aspects:

- The students have met variables as general undetermined numbers used for modelling. The task, an arithmetic word problem with letters instead of numbers, uses variables in the same way.

### Working Group 3

- The structure of the word problem ( $b + (c \cdot L) = a$ ) was chosen firstly. The narrative was developed to go with the structure with  $L$  as solution. So the students have to conduct inverse operations of addition and multiplication.
- The students have met variables in a geometric context. The task also includes geometric quantities.
- The structure of the task asks for a multiplication. The students have seldom met the multiplication of variables as in area but they are accustomed to multiplying quantities by time or money. By choosing the kilometre as the unit of measurement for  $a$  and  $b$  and days as the unit of measurement for  $c$ ,  $L$  becomes an intensive quantity with the unit 'kilometres per day'. This may cause difficulties but, due to everyday life experience, it is most likely that students treat  $c$  like a natural number without a unit.
- It is not expected that students will present  $L = (a-b):c$  as a solution. Taking into account students' previous experience with word problems, it is more likely that the students will use sub-steps.

During the interview the students had to solve this word problem and a similar word problem with numbers in place of the variables to find out if there were different approaches to the two tasks depending on the previous arithmetical knowledge but independent of the chosen context. The interviews were transcribed and the content of the transcripts together with the written products created by the students during the interview are objects of this analysis.

### THE ANALYSIS INSTRUMENT

To get a holistic view on the complex epistemic process of how the previous arithmetical knowledge interacts with the knowledge about variables taught and how these two aspects are connected to become the student's new knowledge, different parameters have to be taken into account. Therefore an appropriate analysis tool had to be developed.

#### General Analysis: Malle's 3-Step-Model

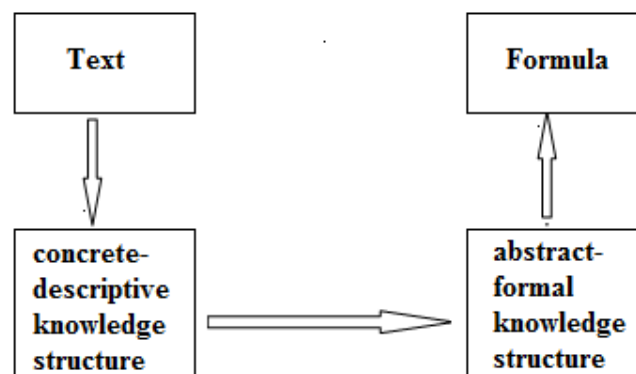


Figure 2: From text to formula, a 3-Step-Model (Malle 1993, p. 99, own translation)

If we look at students working on a task, we look at a solution-oriented process. Thus we first need a tool that helps us to describe this process in a way that allows identification of important moments of transition. The 3-Step-Model of Malle (1993, see Figure 2) fulfils these needs.

The concept “knowledge structure” is defined as follows:

“A knowledge structure contains schematic, cross-linked knowledge concerning a certain scope, which has been constructed under certain conditions and has been optionally stored in the long-term memory where it can be accessed again.” (Malle, 1993, p. 98, own translation)

Certainly the transition process from text to formula is not as direct as it is shown in Figure 2. In particular steps 2 and 3 are closely intertwined and the respective steps can be passed through iteratively. (Malle, 1993, p. 99). Hence, as already remarked above, particular attention should be paid to students’ procepts, in other words the interaction of students’ procedural and conceptual view of the task.

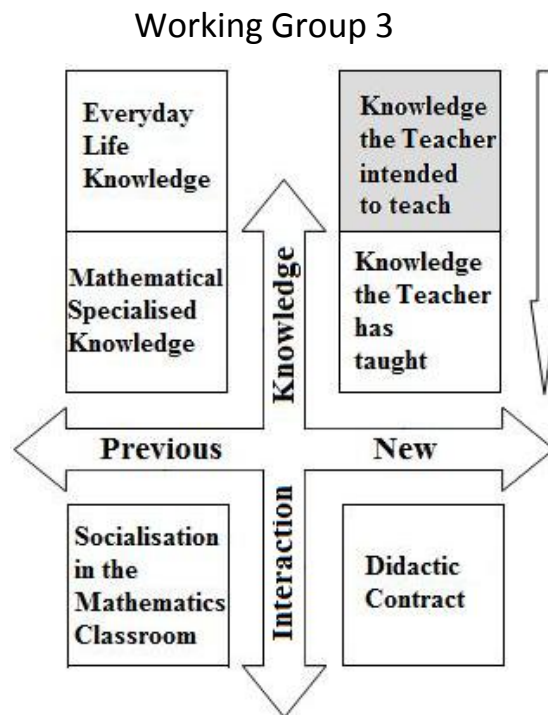
#### **Detailed analysis: searching for reasons using an interdependence model**

After analysing the interplay of procedural and conceptual knowledge, we have to address the issue as to how this interplay is affected by such factors as the actual classroom and interview setting. Therefore, we have to tease out the extent of the effect of these different influential factors.

On one hand we have to identify the **Previous Knowledge** in terms of Everyday Life Knowledge and Mathematical Specialised Knowledge. On the other we have to look at the **New Knowledge** in terms of Knowledge the Teacher has taught. This implicit knowledge of the teacher has to be identified and related to the Knowledge the Teacher intended to teach to reconstruct the students’ understanding thereof.

In order to address the authentic setting we additionally have to take into account the perspective of social interaction (*see Pichat & Ricco, 2001*). The notion of **Previous Interaction** represents the Socialisation in the Mathematics Classroom concerning the handling of mathematical situations. Previous interaction has become independent of the teacher because the student already has internalised these habits. The term **New Interaction** or actual interaction deals with the actual Didactic Contract in an interview or classroom situation. The actions carried out here are not internalised yet and depend profoundly on those persons with whom the students are interacting. Like previous and new knowledge, previous and new interaction may interfere with each other producing conflicts from which new integrated knowledge may emerge.

The considerations above lead to a Framework of an Interdependence Analysis Model (see Figure 3).



**Figure 3: Framework of an Interdependence Analysis Model**

## ANALYSIS

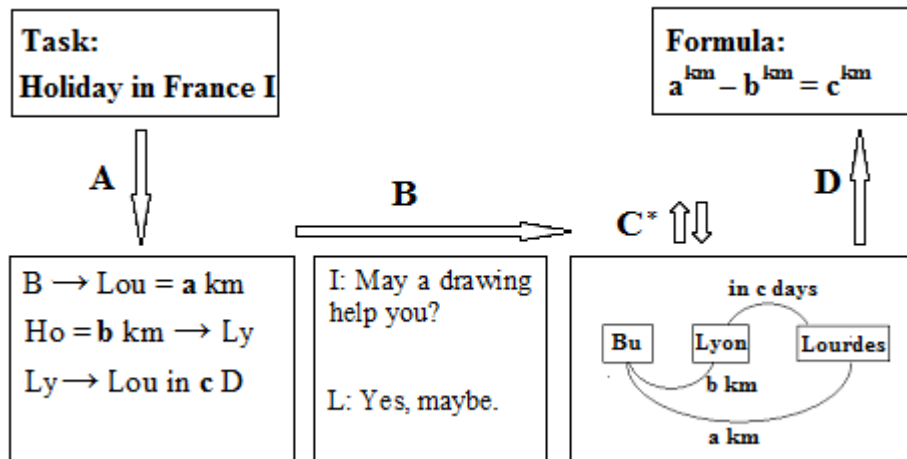
After describing the background of the analysis tool its application will be exemplified in the case of Daniela. We start with a description of Daniela's background followed by a general and detailed analysis.

### Daniela's Background

Daniela attended the algebra lessons at the end of grade 5. She is a good student with good marks in mathematics lessons. The evaluation of her arithmetical skills at end of grade 5 showed a performance below class average with difficulties with written procedures and inverse operations.

### General Analysis

If one compares Figure 4 to Figure 2, the 3-Step-Model seems to fit perfectly. At first (A) Daniela is visualising her concrete-descriptive knowledge about the task by an external representation. The three lines match exactly the first three sentences of the word problem and contain the relevant information. Then (B) she again is visualising the task by means of an auxiliary drawing which contains information about her abstract-formal knowledge structure. She tries a (incorrect) generalisation (C, see Figure 5) and finally translates her abstract-formal knowledge into a formula (D). Fortunately we are able to observe nearly the whole transition process here because she is visualising her internal concrete-descriptive and abstract-formal knowledge by external representations.



**Figure 4: First classification of Daniela working with the sample problem**

	<p>L: Then he has to walk <b>b</b> kilometres, well if, if, if in Lyon...then is exactly in the middle.</p> <p>I: Is it written there that Lyon is exactly in the middle?</p> <p>L: No (laughs)</p> <p>I: Well that would be nice, wouldn't it? But you don't know.</p> <p>L: (quietly) From Lyon that are <b>c</b> days and from Buchheim that are also...</p>
--	---

**Figure 5: Transition C**

Allowing for a process-concept-view we can describe the transitions as following:

A) The variables themselves do not activate a conceptual view on the task. The three lines reproduce exactly the (travelling) process that is given with the text.

B) The suggestion to use a drawing may initiate the transition from a procedural to a conceptual view but for the drawing process, conceptual knowledge does not need to be applied. She simply translates her concrete-formal knowledge step by step into a sketch. She may still be thinking procedurally from within the travelling process:

Daniela:      Actually this is only an intermediate place, this village.

C) The drawing itself allows Daniela to develop a conceptual view. Lyon changes its state of an “intermediate place” to “exactly in the middle”. Daniela’s statements change from “in Lyon” to “Lyon is” back to “from Lyon”. For a moment she moves out of the travelling process into a conceptual meta-level and then back into the process without actively controlling this interplay.

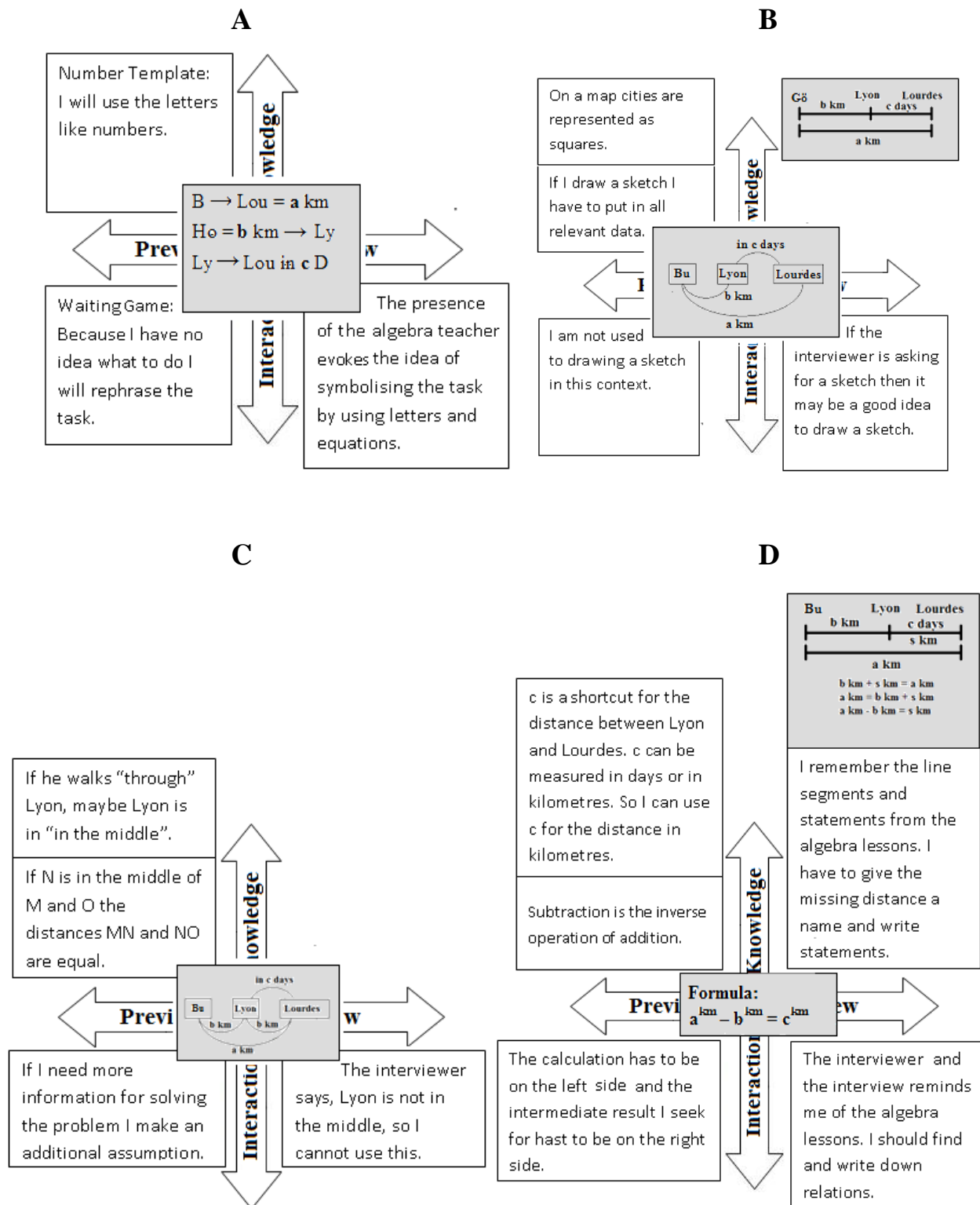
D) Now there is a time of 60 seconds where nothing observable happens, before – we are tempted to say suddenly – the following is happens:

Daniela:      (quietly) Oh, minus **b** kilometre (writes “a-b” and writes “km” in small letters above the letters **a** and **b** and then writes „= c km“)

## Working Group 3

If Daniela looks at the drawing, the arches that connect Buchheim, Lyon and Lourdes may all be seen as distances and she may see  $b+c=a$ . This interpretation is supported by the fact that Daniela added the unit “km” to all the letters. This is a conceptual view on the drawing. But she is writing  $a \text{ km} - b \text{ km} = c \text{ km}$ . This result- and process-oriented way seem to eclipse the conceptual view on the task.

### Detailed analysis



## PRELIMINARY RESULTS

To complete the demonstration of our *modus operandi* we allow ourselves to draw first exemplary conclusions about the interplay of Daniela's previous arithmetic knowledge and her new algebraic knowledge

- A) The presence of letters as part of an arithmetical task is not strong enough to activate Daniela's algebraic knowledge. Her arithmetical knowledge is dominating.
- B) Using auxiliary drawings is not a self-evident part of Daniela's mathematical socialisation. But her drawing allows her to change from a procedural to a conceptual view and to associate the task with the algebra lessons. The drawing acts as a mediator between the arithmetic task and Daniela's knowledge from her algebra lessons.
- C) The algebraic symbol system allows one generalised description of several different cases without requiring a case distinction. But Daniela wants to work with a direct relation of the two distances Buchheim-Lyon and Lyon-Lourdes. This is easy with number distances, because the order is part of the numbers. But here it is impossible, because the direct relation of the two variables is not defined.
- D) It took 60 seconds for Daniela to recall how to conduct the required subtraction. She may have known that she had to conduct a subtraction but had difficulties finding out which letter represented the minuend and which letter represented the subtrahend. Later she made the following remark:

Daniela: ...but with numbers, well, you already know from the beginning, what minus what equals what...

Again Daniela's difficulties are a result of the fact that she cannot get the order of the variables at first sight. Instead she has to take the information about the order of the variables out of the text, something she is not used to.

## PERSPECTIVE

This passage of Daniela's interview was chosen as an example of findings that were also present in interviews with other students. A further analysis of the interviews will show if students can be categorised according to the interaction of their previous knowledge with exposure to variables and if there are differences between students with low- and high-achievements in arithmetic. The analysis will also show if the analysis tool can meet the claim for which it was designed.

What we already can see is that we have to differentiate between the transition from arithmetical to algebraic thinking on the one hand and the transition from numbers to variables on the other hand. The apparent difference of algebraic thinking with numbers to algebraic thinking with letters deserves further investigations.



## REFERENCES

- Carraher, D.W. & Schliemann, A.D. (2007). Early algebra and algebraic reasoning. In F.K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics* (pp. 669-705). Charlotte, NC: IAP & NCTM.
- Dörfler, W. (2008). En route from patterns to algebra: comments and reflections. *ZDM*, 40(1), 143-160.
- Dougherty, B. (2008). Measure Up: A quantitative view of early algebra. In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 389-412). New York: Erlbaum
- Gerhard, S. (2009). Problem solving without numbers. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 499-508). Lyon, France: INRP.
- Gray, E. & Tall, D. (1994). Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140.
- Linchevski, L. (1995). Algebra with numbers and arithmetic with letters: A definition of pre-algebra. *Journal of Mathematical Behaviour*, 14, 113-120.
- Malle, G. (1993). *Didaktische Probleme der elementaren Algebra*. [Didactical problems of elementary algebra.] Braunschweig, Germany: Vieweg.
- McNeil, N. (2004). Don't teach me  $2 + 2$  equals 4: Knowledge of arithmetic operations hinders equation learning. In K. Forbus, D. Gentner & T. Regier. (Eds.), *Proceedings of the Twenty-sixth Annual Conference of the Cognitive Science Society* (pp. 938-943). Mahwah, NJ: Erlbaum.
- Pichat, M. & Ricco, G. (2001). *Mathematical problem solving in didactic institutions as a complex system: The case of elementary calculus*. *Journal of Mathematical Behaviour*, 20, 43-53.
- Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspective. *Journal of Mathematical Behaviour*, 14, 15-39.
- Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. Yackel & K. McClain (Eds.), *Symbolizing and communicating: perspectives on mathematical discourse, tools and instructional design*, (37-98). Mahwah, NJ: Erlbaum.
- The Design-Based Research Collective (2003). Design-Based Research. An emerging paradigm for educational inquiry. *Educational Researcher*, 32(1), 5-8.
- Wittmann, E. Ch. (1995). Mathematics education as a “Design Science”. *Educational Studies in Mathematics*, 29, 355-374.

# WHAT IS ALGEBRAIC ACTIVITY? CONSIDERATION OF 9-10 YEAR OLDS LEARNING TO SOLVE LINEAR EQUATIONS

Dave Hewitt

School of Education, University of Birmingham, UK

*This paper tries to raise issues about what constitutes algebraic activity through looking at number of episodes from a series of three lessons taught to 9-10 year olds using the software Grid Algebra[1]. From different viewpoints the work the students achieved could be viewed as anything from impressive algebraic activity after relatively short time of teaching, to feeling as if no algebraic activity took place at all. The aim of the paper is to raise issues rather than come to a particular position. It ends up highlighting the fact that such viewpoints are the result of us considering certain things are important and to encourage pursuit of what those are and why we give them that significance.*

## SOME DIFFICULTIES AND SUCCESSES WITH ALGEBRA

There has been much research reporting difficulties students have with algebra. Küchemann (1981) highlighted the fact that many students had considerable difficulty in developing meaning for letters. Difficulties students experience are not restricted to letters as Collis (1974; 1975) identified a tendency for 6 to 10 year olds to want to replace two numbers connected by an operation with a single number. He described this as students struggling with a *lack of closure*. Sfard and Linchevski (1994) talked about the need for students to be able to see an expression as an object as well as a process to be carried out. The equals sign has also been shown to have meanings for students where certain correct mathematical forms of statements are deemed to be unacceptable. Behr et al. (1980) showed that 6-7 year olds viewed the equals sign as a *do something signal*. Kieran (1981) pointed out that this was not just an issue with younger students but something which carried on throughout elementary school, into high school and even college as well. These issues with the equals sign still persist as shown in more recent studies (Knuth et al., 2005; Linsell and Allan, 2010).

Over the last 10 years there has been a number of reports on what students are able to do, rather than what they are not able to do. Younger children in primary schools have been shown to be able to work with algebraic ideas, use letters as unknowns and operate on letters without having to know their values (Schliemann et al., 2003). An example of this is students of 8-9 years of age being able to explain why  $N + 3 - 5 + 4$  must be equal to  $N + 2$  whatever the value of  $N$  (Carraher et al., 2001). Projects based on the ideas of Davydov have engaged 6 year old students with relational ideas using letters before formal work on numbers (Dougherty and Zilliox, 2003).

There is an interesting contrast between studies showing the difficulties that students have with algebra, and unquestionably continue to have with algebra in many secondary mathematics classrooms in particular, and the increasing evidence that

young children are able to engage with algebraic ideas and begin to work with more formal notation. Within this contrast I feel are questions about what we actually perceive algebra to be along with questions about pedagogic approaches which might be adopted as a consequence. The terms *pre-algebra* or *early algebra* have often been used, maybe as a way of being able to avoid contention as to whether something a student does might be deemed as algebraic or not. This is something I will now pursue by considering some different ways of viewing what it is to work algebraically.

## WHAT IS ALGEBRA?

In jest, algebra has been described as the study of the 24<sup>th</sup> letter of the alphabet. If algebra is not as simple as the appearance of a letter then the issue of when does algebra begin is one which has been debated over some time. Mary Boole (1931, p. 1231) described the move from arithmetic to algebra in terms of acknowledging “the fact of our own ignorance” which leads to explicitly labelling an unknown. Filloy and Rojano (1989) talked about a *didactic cut* between arithmetic and algebra, this occurring when a letter appears on both sides of an equation. However, Herscovics and Linchevski (1994) argued that it was not about the form of the equation but about when a student begins to work with the unknown. For example, the equation  $2x+4=19-3x$  could be solved by trying different numbers for the letter  $x$  whereas someone might change the equation into the form  $5x=15$  in which case they have the worked with the unknown. For Herscovics and Linchevski, the issue was more about the human activity of how someone worked on an equation rather than the form the equation took. They talked about a *cognitive gap* with many students having difficulty with working spontaneously with or on the unknown. The shift away from symbols themselves onto human activity is one which Radford has followed, looking at algebraic activity in terms of semiotics where “mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a genuine sense, by actions, gestures and other types of signs” (Radford, 2009, p. 112). He has recently argued for attention to shift from our obsession with mathematical symbolism and onto what he calls the *zone of emergence of algebraic thinking* (Radford, 2010a) where the expression of general rules can take place with the use of words, actions and gestures. Mason has for a long time considered algebraic activity in terms of expressing generality (Mason, 1996) and seeing the general in the particular and the particular in the general, where the existence of symbols is not central to the consideration of when algebraic activity takes place. He has talked about three pairs of powers which students bring with them into the classroom: imagining and expressing; specialising and generalising; and conjecturing and reasoning (Mason, 2002) and challenges us as teachers to consider whether we are stimulating these powers or trying to do the work for the students. The notion of *powers* which students bring with them has its roots with Gattegno (1971) who argued that we all possess powers of the mind, which are attributes of being human. These powers are used by very young children in their early learning before they ever enter a school

and remain in daily use throughout all our lives. Gattegno (1988) argued that it is helpful to consider algebraic activity in a wider sense than only in a mathematical context. He spoke of algebra as operations upon operations which can be manifested within the learning of language (such as noticing a rule in the way verbs change tense) as much as within mathematics. As such, algebraic activity is an attribute of the mind and so everyone has already worked algebraically and continues to do so. The issue is then more concerned with Mason's challenge and whether a student's powers of working algebraically are called upon within a mathematics classroom when working on the topic of algebra!

I have offered a brief summary of different ways in which algebra might be viewed and in particular I will analyse students' work during a series of three lessons in terms of the following perspectives:

- Algebra as appearance of letters
- Algebra as working with equations with a letter on both sides of the equation
- Algebra as working with or on the unknown
- Algebra as an expression of generality using actions, words and gestures
- Algebra as seeing the general in the particular and the particular in the general
- Algebra as an attribute of the mind: operations upon operations

## THE STUDY

I carried out a series of three lessons with a mixed ability group of 21 9-10 year olds in an inner city primary school. These students had never met the use of letters formally within their lessons and had never been introduced to formal algebraic notation. The students' attainment levels were based by their teachers on the UK National Curriculum levels where most 6-7 year olds are expected to achieve a level 2 and most 10-11 year olds are expected to achieve a level 4. The range of teacher assessed levels for these students was as follows: 2 (level 2); 13 (level 3); 3 (level 4); and 3 (level 5). The lessons were taught by myself and nearly all of the lesson time was spent either using the computer software *Grid Algebra* with occasional time spent on pen and paper activities related to the software. It is important for the reader to be aware that I wore three hats during this study; researcher, teacher and also the person who had developed the software. As such my comments and analysis have to be read with this in mind. One significant aspect about the way of working with the students was that at no time was anything explained to the students, including the particular appearance of formal notation. Instead, certain actions were carried out using the software on an interactive whiteboard, challenges were given to the students and questions were asked.

The teaching sessions were video recorded along with times when individual students worked in a computer room with computer generated tasks. Some pairs of students' work on computers was captured using *Camtasia* software, which records everything they are doing on the computer. Written work was also collected in.

### Working Group 3

These were all analysed through a coding process which was based upon themes and links which developed through the analysis process.

### GRID ALGEBRA

The software is based upon a multiplication grid with the one times table in the top row, the two times table underneath that, the three times table underneath that and so on (see Figure 1 where only the first two rows are shown).

1	1	2	3	4	5	1
2	2	4	6	8	10	2

**Figure 1: the first two rows of the grid**

**Figure 2: some movements on the grid**

A key feature of the software concerns the relationship between numbers in this multiplication grid. For example, moving from one number to the next number in the one times table would involve adding one and a number such as 2 can be picked up and dragged to the next cell and it would show in notation  $2+1$  (see Figure 2). There would now be a peeled back corner in that cell showing that there is also another expression in the cell, in this case the original number 3. On each click of the peeled back corner the expressions in that cell would be revealed one at a time in a cycle of all expressions which have been entered into that cell. Likewise other movements are possible with any movement to the right resulting in addition, to the left would result in subtraction, a movement down would produce multiplication and up division (see Figure 2). Once a movement has taken place the resultant expression becomes an object which can be moved once again. Thus in Figure 2 the number 5 in row one has been moved one cell to the left, producing  $5-1$ , and then that has then been picked up and dragged down from the one to the two times table to produce  $2(5-1)$ . Likewise the 6 in the two times table (row 2) has been moved twice to produce  $\frac{6-4}{2}$ . There are a large number of other features to the software but only those relevant to particular incidents below will be mentioned.

I will now describe a number of incidents which happened over the three lessons and later I will look at these in terms of the different views about what might constitute working algebraically. These incidents are chosen so as to get a general sense of the development of activities which took place over the three lessons, although it should be noted that there were several additional activities to these which took place.

### Episode 1

At the beginning of the first lesson, students were shown the grid with the times tables shown as in Figure 1. After two minutes of them describing what they saw and which numbers might come next in each row (the grid could be scrolled so that they could see which numbers do appear next), a pre-prepared grid was loaded which showed the same grid but with some of the numbers rubbed out. Below the grid was

### Working Group 3

a ‘number box’ which, when scrolled through, contained the numbers from 1 to 200. The students were asked to come up and drag an appropriate number from the number box into one of the empty cells in the grid. If it was correct the number would stay in the cell. If it was wrong a sign would indicate this and the number

1			14		
2	24		28		

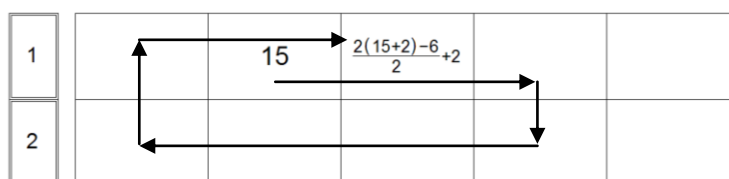
would drift off into a ‘bin’. Chris (pseudonyms are used for all the students) dragged the number 12 into the shaded cell in Figure 3 and I asked how he worked out that it was 12.

**Figure 3: which number should go into the shaded cell and why?**

He said “If it’s the one times table it’s going to be plotting one up or one down so I just counted two down from 14 which is 12.” As he did this he pointed from 14 back to 12. Abbas said that he could explain it differently and said that he halved the 24. He came up and pointed from the 24 up to the cell which now had 12 in it. Such activities continued with grids having more rows, fewer numbers given and with a greater ‘space’ between any number given and the highlighted cell.

### Episode 2

Here I will describe a series of incidents where the class were all together using the Interactive Whiteboard (IWB). Towards the end of the first lesson I had placed the number 15 into a cell in an otherwise empty grid. I made a journey with 15 as indicated by the arrows in Figure 4, and rubbed out all the expressions in the cells along the route except for the final expression. Note that the arrows did not appear on the IWB. They only appear here for clarity of description.



**Figure 4: A journey made with the number 15.**

After rubbing out the middle stages I then announced that I had forgotten what I did to make that expression and asked them to re-create the journey I had made. During this task a student came up to the board and was successfully given the directions of how to re-create it by fellow students. However, I noted these directions were in the form of *across*, *down*, etc. Mathematical operations were not mentioned. The second lesson I repeated this task with other journeys but worked on the language so that mathematical operations were being used to describe what operations were carried

out with increasingly complex expressions such as  $2\left(\frac{2\left(\frac{2(33-2)+2}{2}+3\right)-2}{2}-1\right)-4$ .

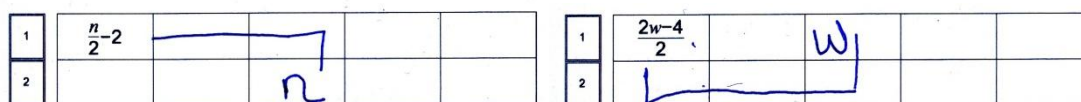
Collectively the students were successful at re-creating these expressions with very few, if any, incorrect movements on the grid. Each time I began a new journey I talked about starting with my “favourite number” which changed every time I did this activity. Then I continued with a same activity but started with a letter and took that letter on a journey rather than a number. So, for example, they were able to tell me the order of operations with the expression  $\frac{2\left(\frac{r}{2}-2\right)-2}{2}+2$ .

### Episode 3

Julie (a level 5 student) was working on a computer generated task from the software where she was told that  $x=4$  and had to drag the correct number from the number box into the cell which had within it the expression  $2(x-2+2)+8$ . She said: *Four. Ex equals four. Four take away two plus... that's just the same as saying four [pointing to  $x-2+2$ ] times two equals eight, plus eight equals sixteen.* She talked through her thinking with several of these tasks and she got many of them incorrect as her arithmetic was often faulty even though she correctly said what operations had to be carried out.

### Episode 4

Abbas (level 3) was working on a paper exercise where an expression involving a letter was written on one cell on a grid and the task was to find which cell the letter must have come from. A colleague of mine asked him a question of why he chose to undo the dividing first with  $\frac{2w-4}{2}$  but did not do so with the expression  $\frac{n}{2}-2$  from the previous question (see Figure 5, note that both these are correct).



**Figure 5: Inverse journey task to find where the letter was originally**

In his explanation he moved his pen rapidly horizontally between  $\frac{n}{2}$  and 2 in the expression  $\frac{n}{2}-2$  saying “*These two are together so it just tells me that I need to do these two first. That’s why I had to do that last because these two had to so I, so I knew I had to do that, that, that, um, first.*” He struggled to express himself in words and the action seemed to hold more meaning than the words did.

### Episode 5

At the end of the final lesson, Julie was working on a sheet of equations to solve. She was talking through solving  $\frac{f-8}{2}+6=57$  and was able to express clearly the operations she carried out to solve this (no working was written on the paper, just the



answer). Having described taking away the 6 she went on to say *Now you times by two* and at the same time she used her pen in a downwards stroking gesture from the division line to the 2 underneath it.

Other students were also able to solve equations, some with support of the grid and others without, like Julie. This included one of the level 2 students who could solve equations with the aid of the grid and write the solutions in correct notation, such as  $\frac{24}{2} - 3$  for the equation  $24 = 2(p + 3)$ .

## ANALYSIS

All students were working with confidence with formal algebraic expressions of reasonable complexity and the fact that students of this age were doing so could be viewed as impressive after only three lessons. Many of the students were able to solve linear equations, albeit some still needing the support of the grid, and express their answers in formal notation. However, it is another matter to consider whether they were doing any algebra or not, and if so, when that algebraic work started. I will now consider each of the six ways of viewing algebra mentioned earlier with reference to these episodes.

### Algebra as appearance of letters

This would mean there was a shift from arithmetic to algebra when I introduced a letter in Episode 2. The interesting thing with regard to this viewpoint is that the students really did not meet a conceptual difficulty in this transition. There was an initial reaction to the idea of a letter but as I did not react to that and moved on to the activity quickly, they found they could do the activity just as well as they had done earlier when a letter was not present.

### Algebra as working with equations with a letter on both sides of the equation

This never happened throughout the three lessons and so that would imply that the students never began working algebraically and stayed in the realm of arithmetic.

### Algebra as working with or on the unknown

The students in this study did not manipulate an unknown from one side of an equation to the other. However, they did work with the idea of an unknown and this was manifested in the particular example in Episode 3 of Julie recognising that  $x - 2 + 2$  did not change the value of  $x$ . Even though she was substituting in a particular value for  $x$  I argue that her awareness was of the generality of  $-2 + 2$  and not the particularity of  $x$  being 4 in this case. So from this viewpoint she might be working algebraically even though she struggled with the arithmetic. This is similar to Carraher et al. (2001) reporting that their 8-9 year olds were able to articulate why  $N + 3 - 5 + 4$  was equal to  $N + 2$ . They could account for this irrespective of the value of  $N$ . Other students in my study were also able to work successfully with the unknown



by working out solutions for the linear equations given at the end of the third lesson (Episode 5).

### **Algebra as an expression of generality using actions, words and gestures**

Here I would like to discuss Episode 4 where Abbas was rapidly moving his pen from  $\frac{n}{2}$  to 2 in the expression  $\frac{n}{2} - 2$  when trying to explain why he did not start by undoing division in this case. The rapid speed of the pen movement was striking and it seemed to be expressing what he was struggling to express in words. I would like to argue that he had a sense of generality of which operation he would undo first and that this was expressed with a gesture more effectively than words. However, this generality concerned a notational convention and as such might be considered qualitatively different to Radford's (2010b) example of a rule for the number of squares in a geometrically arranged sequence. I argue that such a geometric sequence is also arbitrary in its nature since there are not reasons why the squares must have been arranged how they were. It was a human construct in a similar way to mathematical notation. So did Abbas reveal algebraic thinking within that gesture? In Episode 5, Julie used a downwards stroke of her pen from the division line to the 2 below whilst saying "multiply by two". The combination of this gesture on the division line whilst saying "multiplication" revealed that she could see one operation and think of its inverse at the same time.

### **Algebra as seeing the general in the particular and the particular in the general**

In Episode 2 the nature of the activity of re-creating journeys was such that attention was placed on the mathematical operations rather than any particular starting number I used. This focus of attention allowed a letter to be introduced without causing too many issues for the students, since they often never paid attention to the start number anyway. I deliberately varied the start number to try to develop a sense of variation and also irrelevance. Fujii and Stephens (2001) talked about the idea of a *quasi-variable* where numbers were used to demonstrate a mathematical relationship which would be true irrespective of the numbers, such as  $78+49-49=78$ . The particularity of the number in both cases is irrelevant. In this way the general can be seen through the particular and indeed as the teacher I tried to judge when this was the case for most of the students so that I introduced a letter when that sense of generality was already present.

### **Algebra as an attribute of the mind**

Tahta (1981) has talked about inner and outer meanings of activities. In Episode 1 an outer meaning might be to place the correct number into the highlighted cell, whereas the inner meaning in the design of such a task for myself was for students to begin to form mathematical connections between different cells on the grid (in preparation for the later activities involving movements). The students' explicit attention might have been with the numbers, whereas the work they had to do to

achieve placing a correct number was to work out relationships between different cells on the grid. I argue that students were working with operations in order to carry out these tasks and the awareness of equivalence of different sets of operations was certainly operating upon operations. So with this view of algebra, the students were working algebraically already with the initial 'number' activity.

## CONCLUDING THOUGHTS

How we view algebraic activity changes when we feel students have started such activity. It might be argued anything from the students in this study not doing any algebra at all over the three lessons to them working algebraically from the very first activity. Naming is an act to label that which is deemed to be significant and what is important is what someone wishes to stress. So the fruits of a discussion about what constitutes algebraic activity can come from what each person reveals to be particularly significant for them in the developing process students make within their work towards algebra and within the algebra curriculum. Not only what is significant but why it is significant. That is what I feel is particularly useful in considering the question *what is algebra?*

## NOTES

1. *Grid Algebra* is available from the Association of Teachers of Mathematics at <http://www.atm.org.uk/shop/products/sof071.html>

## REFERENCES

- Behr, M., Erlwanger, S. and Nichols, E. (1980), 'How children view the equals sign', *Mathematics Teaching*, 92, pp. 13-15.
- Boole, M. (1931) *Collected works*, London: The C. W. Daniel Company.
- Carraher, D., Schliemann, A., D. and Brizuela, B., M. (2001), 'Can young students operate on unknowns?' in M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, Utrecht, The Netherlands, Freudenthal Institute, pp. 130-140.
- Collis, K. F. (1974), 'Cognitive development and mathematics learning', *Paper presented at the Psychology of Mathematics Workshop*, Centre for Science Education, , Chelsea College, London.
- Collis, K. F. (1975), 'The development of formal reasoning', *Report of a Social Science Research Council sponsored project (HR 2434/1)*, Newcastle, NSW, Australia, University of Newcastle.
- Dougherty, B. J. and Zilliox, J. (2003), 'Voyaging from theory to practice in teaching and learning: a view from Hawai'i', in N. A. Pateman, B. J. Dougherty and J. T. Zilliox (Eds), *Proceedings of the 27th Conference of the International Group for*

*the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA*, Vol. 1, Hawai'i, College of Education, University of Hawai'i, pp. 17-23.

- Filloy, E. and Rojano, T. (1989), 'Solving equations, the transition from arithmetic to algebra', *For the Learning of Mathematics*, 9(2), pp. 19-25.
- Fujii, T. and Stephens, M. (2001), 'Fostering an understanding of algebraic generalisation through numerical expressions: the role of quasi-variables', in H. Chick, K. Stacey, J. Vincent and J. Vincent (Eds), *Proceedings of the 12th Study conference of the International Commission on Mathematical Instruction: The Future of the Teaching and Learning of Algebra*, Vol. 1, Melbourne, Australia, The University of Melbourne, pp. 258-264.
- Gattegno, C. (1971) *What we owe children. The subordination of teaching to learning*, London: Routledge and Kegan Paul Ltd.
- Gattegno, C. (1988) *The science of education. Part 2b: the awareness of mathematization*, New York: Educational Solutions.
- Herscovics, N. and Linchevski, L. (1994), 'A cognitive gap between arithmetic and algebra', *Educational Studies in Mathematics*, 27, pp. 59-78.
- Kieran, C. (1981), 'Concepts associated with the equality symbol', *Educational Studies in Mathematics*, 12(3), pp. 317-326.
- Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A. and Stephens, A. C. (2005), 'Middle school students' understanding of core algebraic concepts: equivalence & variable', *ZDM*, 37(1), pp. 68-76.
- Küchemann, D. (1981), 'Algebra', in K. M. Hart (Ed.), *Children's understanding of mathematics: 11-16*, London: John Murray, pp. 102-119.
- Linsell, C. and Allan, R. (2010), 'Prerequisite skills for learning algebra', in M. Pinto, M. and T. Kawasaki, F (Eds), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, Belo Horizonte, Brazil, PME, pp. 217-224.
- Mason, J. (1996), 'Expressing generality and roots of algebra', in N. Bednarz, C. Kieran and L. Lee (Eds), *Approaches to algebra. Perspectives for research and teaching*, Dordrecht: Kluwer, pp. 65-86.
- Mason, J. (2002), 'Generalisation and algebra: exploiting children's powers', in L. Haggarty (Ed.), *Aspects of teaching secondary mathematics*, London: RoutledgeFalmer, pp. 105-120.
- Radford, L. (2009), 'Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings', *Educational Studies in Mathematics*, 70(2), pp. 111-126.

### Working Group 3

- Radford, L. (2010a), 'Algebraic thinking from a cultural semiotic perspective', *Research in Mathematics Education*, 12(1), pp. 1-19.
- Radford, L. (2010b), 'Elementary forms of algebraic thinking in young students', in M. Pinto, M and T. Kawasaki, F (Eds), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 4, Belo Horizonte, Brazil, PME, pp. 73-80.
- Schliemann, A., Carraher, D., Brizuela, B., Earnest, D., Goodrow, A., Lara-Roth, S. and Peled, I. (2003), 'Algebra in elementary school', in N. A. Pateman, B. J. Dougherty and J. T. Zilliox (Eds), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education held jointly with the 25th Conference of PME-NA*, Vol. 4, Hawai'i, College of Education, University of Hawai'i, pp. 127-134.
- Sfard, A. and Linchevski, L. (1994), 'The gains and pitfalls of reification - the case of algebra', *Educational Studies in Mathematics*, 26, pp. 191-228.
- Tahta, D. (1981), 'Some thoughts arising from the new Nicolet films', *Mathematics Teaching*, 94, pp. 25-29.

# THE ROLE OF DISCURSIVE ARTEFACTS IN MAKING THE STRUCTURE OF AN ALGEBRAIC EXPRESSION EMERGE

Laura Maffei and M.Alessandra Mariotti

Dipartimento di Scienze Matematiche ed Informatiche, Università di Siena

*The paper is based on a research study aiming at exploiting the potentialities of the natural language in the introduction to algebra calculation. By means of activities involving both the standard representation of algebraic expressions and the tree representation available in the Aplusix CAS, the natural language is used as a tool to firstly mediate the meaning of structure of algebraic expressions, and then the meaning of some specific structures, i.e. expanded and factorized expressions. After a description of the theoretical framework, the design of a teaching experiment is presented, and some results are discussed.*

**Keywords:** natural language, structure, algebraic calculation, Aplusix

## INTRODUCTION

Many studies deal with difficulties encountered by students (9-10 grade) in performing algebraic tasks and explain them as a lack of structural view of algebraic expressions (Greeno 1982; Freudenthal, 1983). But, how is it possible to make structural aspects emerge? To give a possible answer to this question we propose first of all an epistemological analysis of the resources students usually exploit to face algebraic tasks. We hypothesize that the usual representation students are used to treating algebraic expressions with, could not provide support in highlighting the ‘structure’ of algebraic expressions. In this perspective, we agree with Duval’s claim (Duval, 2006) about the need of interacting with different representation systems for developing mathematical meanings.

In the following, we are going to discuss a different type of representation, the tree-representation (TR), implemented in the Aplusix CAS (Nicaud & al., 2004). According to Morgan & al. (2009), we can classify the difference between the standard representation and the tree-representation (TR) as an ‘epistemological distance’, that is, a difference between the affordances of each system with respect to specific aspects of the mathematical concept in focus. Such a difference may be exploited for developing the meaning of ‘algebraic expression’ and specifically to make the meaning of structure emerge. As it will be discussed below, in the semiotic game between the different mathematical representation systems, a crucial role is played by natural language (NL). NL may be considered as acting at a meta-level, where we observe the emergence of specific ‘linguistic expressions’, that the teacher may use for exploiting the potentialities of TR with respect to the meaning of structure of algebraic expressions.

The research study<sup>1</sup> is based on an experiment conducted in the First Year of the Upper Secondary School in Italy, when students starts the Algebra course.

## THE THEORY OF SEMIOTIC MEDIATION

The research is framed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) which allows us to analyse the potentialities of different kinds of representations of the same mathematical object (i.e. algebraic expression) with respect to a specific meaning (i.e. structural interpretation). Three different semiotic systems of representation are used, the standard representation (SR), the tree representation (TR) and the natural language (NL), which are considered as *artefacts* according to the Theory of Semiotic Mediation (TSM). TSM emphasizes how the use of artefacts to perform specific tasks helps students develop *personal meanings* (Leont'ev, 1976) related to the actual use of them. Then, through *mathematical discussions* (Bartolini Bussi, 1999) under the guidance of an expert (typically the teacher), students' personal meanings may gradually evolve into mathematical meanings. In this perspective, the evolution from the personal sphere to the mathematical domain strictly depends on the teacher's didactical actions. An artefact, intentionally used by the teacher, who is aware of her/his crucial role in managing this process, is called a *tool of semiotic mediation* (Bartolini Bussi & Mariotti, 2008, p. 754). In this perspective, the functioning of an artefact as a tool of semiotic mediation is concerned with the double relationship that such an artefact has both with the meanings emerging from its use to accomplish a specific task, and with the mathematical meanings evoked in such a use, as they are recognized by an expert. This twofold relation is called *semiotic potential* of an artefact (Bartolini Bussi & Mariotti, 2008, p. 754).

## NATURAL LANGUAGE AS A NEW REPRESENTATION SYSTEM: THE SEMIOTIC POTENTIAL OF LINGUISTIC EXPRESSIONS

According to our theoretical frame, both SR and TR are considered artefacts. In fact, each of these has a specific semiotic potential with respect to the meaning of algebraic expression, in particular TR can be exploited as a tool of semiotic mediation with respect to the meaning of 'structure' of an algebraic expression. In the semiotic game between SR and TR, NL plays a crucial role in making mathematical meanings emerge and develop. Specifically, in the context of conversion tasks (Duval, 2006) from SR/TR to NL and vice versa, NL fosters the interaction between SR and TR.

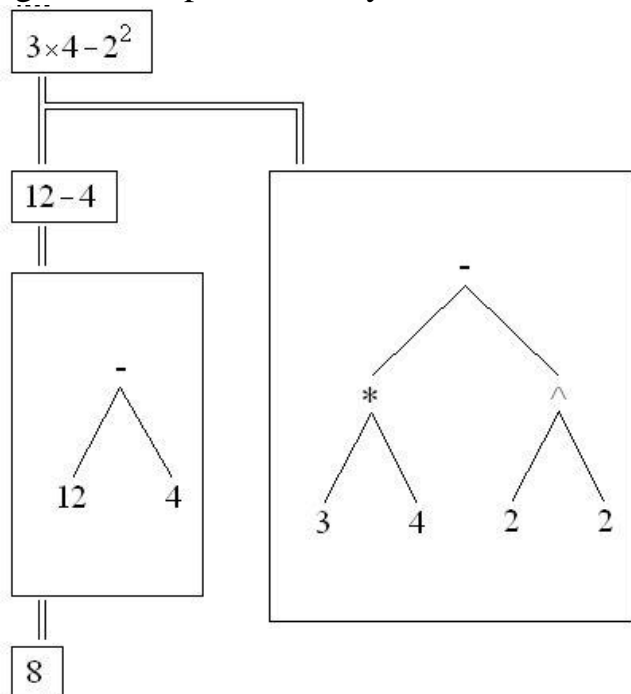
As shown in the following, the repeated use of specific linguistic expressions, develops linguistic patterns which can be viewed, according to Sfard (2001), as 'discursive templates'. The peculiarities of the 'linguistic expressions/discursive templates' and their use in the semiotic mediation process allow us to consider them as artefacts, that we refer to as *discursive artefacts*. Moreover, because of their origin

in the conversion tasks, these discursive artefacts maintain a tight relationship with SR and TR.

The teacher's didactical action consists in making the discursive artefacts serve as 'tools of semiotic mediation' with respect to the meanings of 'structure of algebraic expression', and of 'expanded/factorized expression'.

### TREE REPRESENTATION AND STRUCTURAL INTERPRETATION

We now explain our hypothesis concerning how TR can facilitate a structural interpretation of an algebraic expression, and consequently the production of linguistic expressions which make the structure of an algebraic expression emerge. Within the CAS, students can work with both SR or TR and switch between them (Fig. 1). The software automatically gives feedback on the equivalence between the algebraic expressions by means of different signs which appear between two



**Figure 1.** Switching from the standard representation and the tree representation of an algebraic expression in the Aplusix CAS.

trees and substituting them with the resulting values. Algebraically inconsistent selections are not allowed, generating an error message. Selection and substitution of (sub-)trees may induce the user to conceive the expression made of chunks (i.e. sub-expressions); in other words to conceive it structurally.

### THE TEACHING-LEARNING SCENARIO

According to the assumptions described above, and focusing on the development of the discursive artefacts, the design of teaching-learning process is organized, as suggested by the TSM in *didactical cycles* (Bartolini Bussi & Mariotti, 2008, p.754

consequent boxes enclosing the expressions. Since TR consists of a bi-dimensional structure, it can support a twofold interpretation. Actually, a tree may be interpreted ('read') both from the bottom, that is starting from the leaves, and from the top, that is starting from the root. While proceeding bottom-up implies a procedural view of the represented algebraic expression, a top-down interpretation implies a structural view.

Moreover, when calculating an expression in TR, both a procedural and a structural interpretation emerge at the same time. In fact, according to the functioning of the TR representation environment, calculation can be accomplished only by selecting (sub-)

). Each cycle starts with activities based on the artefact (in our case, NL-artefact, SR-artefact, TR-artefact), then it continues with written reports, and it ends with a classroom discussion guided by the teacher. Discursive templates emerge and develop through semiotic tasks assigned to the students; then, in the collective discussions, the teacher exploits the semiotic potential of specific discursive templates, triggering a semiotic mediation process. In this respect specific discursive expressions become *discursive artefacts* and are used by the teacher as tools of semiotic mediation.

The tasks given in the domain of arithmetic, mainly consisting in conversions between different representations, aim at making the discursive template emerge. Afterwards, in the introduction to algebra, the design of the tasks is based on conversions between SR and NL, and vice versa, with the aim of making the discursive templates re-emerge.

### **Emergence of discursive template**

The phase in arithmetic starts with an activity in paper and pencil. Students are asked to '*write at least two different ways of reading each numerical expression given*'. The objective is to collect students' NL interpretations of numerical expressions. We hypothesize three different ways of reading, which can reveal three different interpretations (Maffei & Mariotti, 2010). The *linear interpretation* consists in translating the inscriptions that the numerical expressions are made of, word by word, from left to right; the *procedural interpretation* consists in translating into words the calculation procedure to be accomplished to compute the value of the numerical expression; the *structural interpretation* consists in translating into words the structure of the numerical expression and recursively of its sub-expressions.

The subsequent activities, to be carried out, both in paper and pencil and in the Aplusix environment, aim at combining the use of both SR and TR in making the structural interpretation emerge. According to our hypothesis, concerning the crucial role played by the functioning of TR in Aplusix in making the structure emerge, the tasks given at the beginning involve the use of TR, and then gradually do not necessarily require its use, although coming back to TR is always possible. In order to move towards algebra, we hypothesize the potentialities of 'empty trees', that is expressions in TR with internal nodes (i.e. operators) and branches, but without leaves (i.e. numbers). The semiotic potential of the empty tree consists in representing the relation between terms of an algebraic expression, i.e. its structure. Empty trees could emerge either in students' solution of tasks, or, in alternative, through the teacher's intervention in a classroom discussion. In any case, empty tree may become a shared sign (Bartolini Bussi & Mariotti, 2008) referring to the structure of an expression.

The main objective of this first phase is to make the different interpretations become 'shared templates' to be used in analysing and confronting different expressions, and



specifically in detecting the structure of expressions given in different representation systems. In other words, the shared templates become ‘discursive artefacts’.

Once students are introduced to algebraic calculation they are newly confronted with conversion tasks from SR to NL (the plan in the algebra context does not include the use of TR). The discursive artefacts originating in arithmetic are exploited by the teacher and the students in order to recognize different algebraic expressions having the same structure.

#### **Mediation of discursive artefacts**

The teaching-learning sequence in algebra continues in making the discursive artefacts become tools of semiotic mediation with respect to the meaning of ‘expanded expression’ and ‘factorized expression’. More precisely, an expanded algebraic expression will be defined as an expression that can be interpreted as a sum of monomials, while a factorized algebraic expression as one that can be interpreted as a power/product of polynomials. In terms of templates, the first structure can be associated with the discursive template ‘*the sum of ...*’, the second one with ‘*the product (power) of...*’. The teacher can use these kinds of templates and/or some more elaborated ones (e.g. ‘*the product of the sum between...and the difference between...*’) in order to introduce the meaning of expanding and factorizing.

#### **SOME RESULTS**

According to the two phases described above, we are going to present some results. Namely, the analysis of some students’ protocols and excerpts from classroom discussions that we consider significant to show how discursive artefacts emerge and develop, and how they are used by the teacher as tools of semiotic mediation.

#### **The emergence and the development of the discursive artefacts**

The teaching sequence in arithmetic ends with a discussion which aims at comparing the solution given to the following task: ‘*Consider the following numerical expressions given in SR, in TR and in NL and group those having the same structure*’.

Let’s now examine an excerpt from the follow-up discussion in which the teacher leads students to make the meaning of structure emerge and develop. The teacher asks students to explain how they solved the task. The first student who intervenes, says that he preferred “*using the trees*”, that is “*converting in TR any expression given either in SR or in NL*”, as the teacher states more precisely. Then Marco takes part in the discussion.

Marco: [...] I considered only expressions given in standard and in tree and I thought how I could put them in natural language. Just some words, not only the whole expression, but just some words were enough to make a comparison with the expressions given in natural language.

Teacher: Can you please explain in a clearer way what you want to say?

### Working Group 3

Marco: Let's see. If you have a long expression that you can read starting with 'the product of' it cannot be structurally equal to an expression given as 'the sum of'. Well, to be honest I sometimes used trees to compare expressions, but even in these cases I put them into words.

Marco classifies expressions according to the structural interpretations that he obtains through converting the numerical expressions, which are given in SR/TR, into natural languages. It is evident how Marco uses specific discursive templates as artefacts to analyse and compare different expressions.

After a number of students' interventions addressing the use of different discursive artefacts; finally, the teacher leads the students to sum up the discussion, and at a certain point explicitly asks how they could define the structure of an expression.

Gabriele: For me it is a tree without leaves.

Rita: Yes, it can be considered as an empty tree.

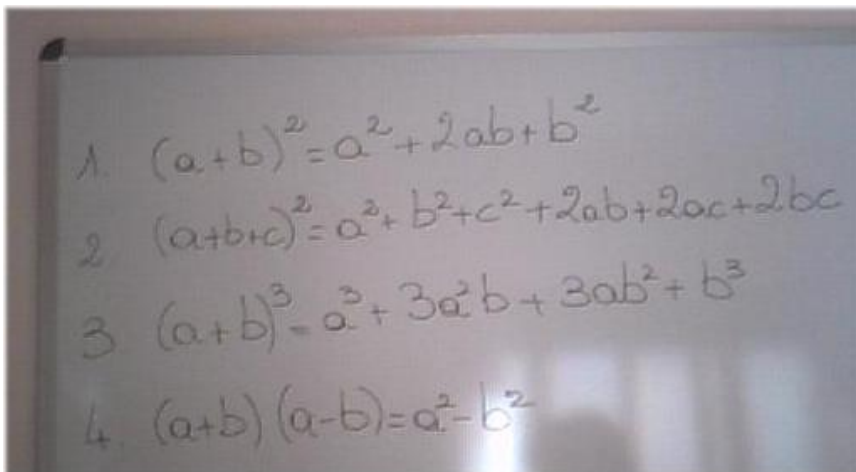
Teacher: So, coming back to the task, how can we determine two or more expressions having the same structure?

Daniele: As I said before, if they have the same tree.

Teacher: What does 'the same tree' mean?

Rita: The trees of the two expressions considered without leaves are identical, that is they have the same empty tree.

At this point, it seems that students refer the structure of a numerical expression to its



**Figure 2.** The formulas of the 'main products'.

representation as an empty tree. It means that they seem to be ready to substitute to leaves any number, that is to be ready to place unknowns in the leaves. In other words, TR has unfolded its semiotic potential in 'converting numerical expressions into potentially literal expressions'.

Later in the teaching sequence, the discursive artefacts originating within the arithmetic domain re-emerge while students are confronted with conversion tasks between different representations in algebra. Students produce new discursive artefacts related to converting into NL expressions which represent specific formulas used to speed up algebraic calculations, i.e. the 'main products' (Fig. 2).

The task given to students consists in obtaining the expanded part of the formulas in the Aplusix CAS through following instructions: ‘*write an equivalent expression not containing parentheses*’. They are also asked to ‘*write a conversion into NL of all the expressions produced in order to obtain the final expression*’.

Let’s now examine the classroom discussion after the task mentioned above. First of all the teacher writes on the blackboard all the formulas for the main products (Fig. 2) and then starts the discussion asking students to read the conversions into NL of each algebraic expression written. Most of the NL-expressions offered by students are based on structural interpretations. For instance, most of the students interpret the fourth expression (Fig. 2) as follows: “*the product between the sum of a and b and the difference of a and b is equivalent to the difference between the square of a and the square of b*”.

The discursive artefacts originated in arithmetic, now re-emerge in the algebra domain. More precisely, it seems that they mainly consist in structural discursive templates, i.e. those originated by the use of TR.

### The discursive artefacts become semiotic mediators

At this point when students are able to manipulate algebraic expressions, that is to produce chains of algebraic expressions (even very long chains) using both operations’ properties and formulas of the main products, the teacher introduces the notions of ‘expanded expression’ and ‘factorized expression’. She decides to do that by exploiting the discursive artefacts that are already available in the class. She starts by commenting on the chain that one student has produced in Aplusix, showing it to the classroom.

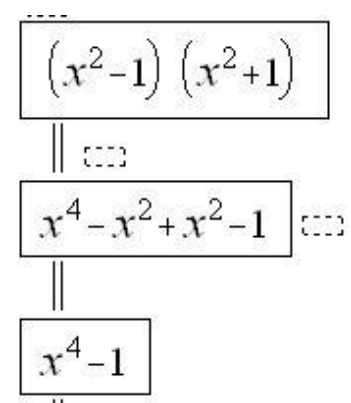
Teacher: In this chain of equivalence Matteo produced, we have two expressions without parentheses. Notice that the last expression is shorter than the second because Matteo summed together two similar monomials. Well, from now on, we are going to call ‘expanded expression’ this type of expressions without parentheses and which are expressed as ‘the sum of’ and do not contain monomials which are similar. But, we have to define another type of expression. So, how could we read the expanded expression?

Matteo: The difference between.

Teacher: The difference between x raised to the fourth and one. Well, let’s go on. How can you read the first expressions?

Chorus: The product of.

Teacher: It’s ok, the product of. Other suggestions?



**Figure 3.** Chain of equivalent expressions in the Aplusix CAS.

### Working Group 3

Alessandra: The second is a sum of monomials, and the third is easy to read, it is the difference of two squares. [...]

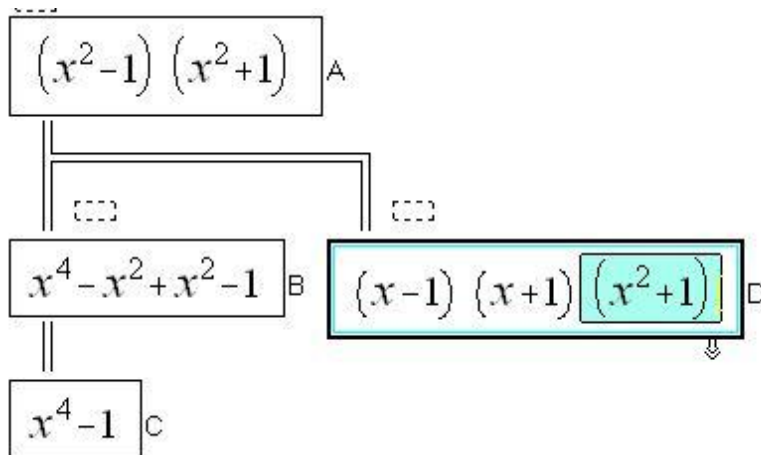
Teacher: Now, suppose I want to produce new expressions which are equivalent to those already written, and these expressions should have parentheses. Does anyone have an idea? I'll put some labels on the expressions in the chain. *(She puts A, B, C, see Fig. 4).*

Marco: I think one can start from the first expression, namely from the second parenthesis.

Teacher: Please Marco, come here and write on the screen what you are saying.

*(Marco goes and writes the expressions in a new box deriving it from the first box, the teacher names D the new expression, see Fig. 4). [...]*

Teacher: Have we finished the process or can we go on?



**Figure 4.** Chain of equivalence expressions in the Aplusix CAS where a sub-expression is

Marco: The first two parentheses have polynomials of degree one, the third has one polynomial of degree two *(the teacher highlights it, see Fig. 4)*, but we have finished since we cannot write this as a product of two parentheses of degree one.

Teacher: Ok, then we will better clarify what 'going on' and 'stopping' mean, do you have any other cases in which you can go on?

Valentina: When for instance you have the sum of a squared, two times a and b, and b squared. In this case I can write the square of the sum of a and b and then nothing else, I would stop.

Then, all the main products are converted into NL; therefore each SR-formula has its counterpart in a NL-formula.

Teacher: To conclude, in the future I can ask you to produce an expression having the form of a sum of non-similar monomials, or to produce an expression having the product-power form. So, now I will give you time to work on these types of tasks.

The teacher, on the basis of the discursive artefacts students have used repeatedly, creates new discursive artefacts. Specifically, she coins the 'product-power' expression and uses it as a tool of semiotic mediation to make the meaning of factorizing emerge. The 'product-power' expression emerges during the discussion

following the factorization task when students produced NL expressions starting with ‘the product of’, ‘the power of’. The following excerpt shows how the teacher introduces the definition of ‘factorized form’ of an expression using the product-power as a discursive artefact.

Teacher: So let’s go back to the chain we analysed last time. We said that expression B has the form of a sum of different monomials, and B and D have the form of product-power. So I can definitely call expanded form the first form. Now, remember the difference between B and D: in expression B I can go on putting it into a product-power form, while in D I have completed the process. Well, I will call expression D ‘factorized form’, while expression B does not have a name, we can continue to refer to it as a product-power form.

After that, the subsequent task given to students consists in observing a chain of equivalent expressions in Aplusix and in deciding which ones are in expanded form, which ones are in product-power form, and which ones are in factorized form, classifying the different forms of the expressions appearing on the screen. Students are also asked to explain how they solve the task’. Massimiliano gives the following comment on the solution he offers. *“As far the expanded form is concerned, I chose L because it is the form which is made of a sum of monomials which are different and for that reason I did not choose D. As far as the product-power forms are concerned I chose A, B, H because if we try to see them as trees then the formula ‘the product between...’ appears. As factorized forms I chose F and G because they could not be simplified anymore.”*

Massimiliano justifies his choices (which are correct) in terms of discursive artefacts (the ‘product between ...’ for the factorized expression). The explicit reference to TR is significant because it shows the relationship the discursive artefact maintains with its origin: the tree representation.

## CONCLUSION

Our research highlights how the role of natural language goes beyond the necessity for teacher and students to communicate in both verbal and written situations. In fact, the natural language, on the one hand, is used to focus on the specific features of both the standard representation and the tree representation, and, on the other hand, it becomes itself a representation system for algebraic expressions, in which the emergence of specific templates may be exploited to express and compare different algebraic structures. We stress how the semiotic potential of the different artefacts in use does not emerge spontaneously, rather it needs a specific didactic organization. It’s only through both the design of specific tasks and the teacher’s action in managing classroom discussions that the different discourse artefacts become tools of semiotic mediation to make the structural interpretation of algebraic expressions emerge.

## NOTES

1. The study has been developed within the European project ‘Representing mathematics with digital media’ (‘Remath’).

## REFERENCES

- Bartolini Bussi, M.G. (1999). Verbal interaction in mathematics classroom: a Vygotskian analysis. In A. Sierpinska et al. (Eds.), *Language and Communication in Mathematics Classroom*. NCTM.
- Bartolini Bussi, M. G., & Mariotti M. A. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, D. Tirosh (Eds.), *Handbook of international research in mathematics education*, 2nd revised edition. (pp. 720-749). Mahwah, NJ: Lawrence Erlbaum Associates.
- Duval, R. (2006). A Cognitive Analysis of Problems of Comprehension in the Learning of Mathematics. *Educational Studies in Mathematics* 61(1-2), 103-131.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht: Reidel Publishing Company.
- Greeno, J. G. (1982). A cognitive learning analysis of algebra, Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.
- Leont’ev, A. N. (Ed.). (1976). *Problemi dello sviluppo psichico*. Roma: Editori Riuniti-Mir.
- Maffei, L., Mariotti, M.A. (2010). Tree representation in the Aplusix CAS: a tool of semiotic mediation for the meaning of algebraic expressions. In M.F. Pinto, & T.F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (pp. 249-256). Belo Horizonte, Brazil: PME.
- Morgan, C., Mariotti, M.A., Maffei, L. (2009). Representation in computational environment: epistemological and social distance. *International Journal of Computers for Mathematical Learning*, 14(3), 241-263.
- Nicaud, J-F., Bouhineau, D., Chaachoua, H. (2004). Mixing microworld and Cas features in building computer systems that help students learn algebra. *International Journal. of Computer for Mathematical Learning*, 9(2), 169-211.
- Sfard, A. ( 2001). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, C. Walter (Eds), *Proceedings of 21st Conference of PME-NA* (pp. 23-44). Columbus, Ohio: PME-NA.

# ALGEBRAIC THINKING OF GRADE 8 STUDENTS IN SOLVING WORD PROBLEMS WITH A SPREADSHEET

Sandra Nobre, Escola E.B. 2, 3 Professor Paula Nogueira

Nélia Amado & Susana Carreira, FCT, University of Algarve & UIDEF

João Pedro da Ponte, Institute of Education, University of Lisbon

*This paper describes and discusses the activity of grade 8 students on two word problems, using a spreadsheet. We look at particular uses of the spreadsheet, namely at the students' representations, as ways of eliciting forms of algebraic thinking involved in solving the problems, which entailed dealing with inequalities. We aim to see how the spreadsheet allows the solution of formally impracticable problems at students' level of algebra knowledge, by making them treatable through the computational logic that is intrinsic to the operating modes of the spreadsheet. The protocols of the problem solving sessions provided ways to describe and interpret the relationships that students established between the variables in the problems and their representations in the spreadsheet.*

**Keywords:** algebraic thinking, inequalities, spreadsheet, representations.

## INTRODUCTION

Representations have a dual role in learning and in mathematical communication. These resources serve the purpose of communicating with others about a problem or an idea but also constitute tools that help to achieve an understanding of a property, a concept or a problem (Dufour-Janvier, Bednarz & Belanger, 1987). This is one of the reasons why we consider students' use of representations as a lens from which we can grasp the meaning involved in the mathematical processes of solving a problem.

Spreadsheets have great potential for the construction of algebraic concepts, including the establishment of functional relationships, the representation of sequences or the use of recursive procedures in solving mathematical problems. The use of spreadsheets in problem solving has been deeply investigated by several authors (e.g., Ainley et al., 2004; Rojano, 2002) and revealed interesting processes in the development of algebraic thinking, particularly with regard to the transition from arithmetic to algebra. Within a spreadsheet environment, the symbolic representation of the relations present in a problem is initiated through the nomination of columns and writing of formulas. This is considered a stimulating environment that fosters an understanding of the relations of dependence between variables and encourages students to submit solutions gradually more algebraic and moving away from arithmetical methods (Rojano, 2002). These aspects encouraged us to carry out an analysis of how grade 8 students create their representations, how they conceive and display the problem conditions on the spreadsheet and how they achieve a solution.

## **PROBLEM SOLVING AND THE LEARNING OF ALGEBRA**

Contextual problem solving is an important type of task leading to algebraic activity.

According to Kieran (2004) the work in algebra can be divided into three areas: generational, transformational and global/meta-level activities. Generational activities correspond to the construction and interpretation of algebraic objects. Transformational activities include simplifying algebraic expressions, solving equations and inequalities and manipulating expressions. Finally, global/meta-level activities involve problem solving and mathematical modelling, including pattern generalization and analysis of variation.

The nature of algebraic reasoning depends on the age and mathematical experience of the students. Students at a more advanced level may naturally use symbolic expressions and equations instead of numbers and operations. But for students who have not yet learned the algebraic notation, the more general ways of thinking about numbers, operations and notations, may be effectively considered algebraic (Kieran, 2007). Contexts that involve numbers, functional relationships, regularities, and other properties, are an essential foundation for the understanding of algebraic structures. For instance, writing symbolic numerical relations may favour the use of letters. However, the use of technological tools allows other representations for such relations, as well as new forms of exploration, which may be seen as analogous to generational and transformational activities in algebra. Thus, it seems appropriate that such new representations, and the mathematical thinking associated with them, are included in the field of algebra (Kieran, 1996). Moreover, Lins & Kaput (2004) claim that algebra can be treated from the arithmetic field, since there are many properties, structures and relationships that are common to these two areas. Therefore, arithmetic and algebra may be developed as an integrated field of knowledge. In this study we adopt this perspective, considering algebraic thinking as a broad way of thinking that is not limited to the formal procedures of algebra. This entails separating algebraic thinking from algebraic symbolism (Zazkis & Liljedhal, 2002).

## **SPREADSHEETS IN THE DEVELOPMENT OF ALGEBRAIC THINKING**

A spreadsheet supports the connection between different registers (numerical, relational, and graphical). One feature that stands out in this tool is the possibility of dragging the handle of a cell containing a formula along a column. This action generates a “variable-column”. Using this tool in problem solving emphasizes the need to identify the relevant variables and encourages the search for relations of dependence between variables. The definition of intermediate relations between variables, that is, the breakdown of complex dependency relations in successive simpler relations is a process afforded by this tool, with decisive consequences in the process of problem solving (Carreira, 1992; Haspekian, 2005). As noted by Haspekian (2005) a spreadsheet also allows an algebraic organization of apparently



arithmetical solutions and this kind of hybridism, where arithmetic and algebra naturally cohabit, becomes an educational option that may help students in moving from arithmetic to algebra (Kieran, 1996).

We want to see how this particular functioning of the spreadsheet is a valid route for solving problems where the formal algebraic approach is too heavy for the students' level. More specifically, we aim to understand how far the spreadsheet, while being a means to promote algebraic thinking, can relieve the burden of formal algebraic procedures and as such can advance the possibility of solving certain types of problems. So far, research has shown the value of the spreadsheet in the transition from arithmetic thinking to algebraic thinking, but less is known about the utility of the spreadsheet to set up an alternative to formal and symbolic algebra and yet allowing the development of students' algebraic thinking in problems that are formally expressed by inequalities (Carreira, 1992; Haspekian, 2005; Rojano, 2002)

## METHODOLOGY

This study follows a qualitative and interpretative methodology. The participants are three grade 8 students (13-14 years). They had some previous opportunities to solve word problems with a spreadsheet in the classroom, from which they acquired some basics of the spreadsheet operation. Before the two tasks here presented, students had worked with the spreadsheet in solving other problems for six lessons. All problems involved relationships among variables (usually equations) and only one included a simple linear inequality. The detailed recording of student's processes was achieved with the use of *Camtasia Studio*. This software allows the simultaneous collecting of the dialogue of the students and the sequence of the computer screens that show all the actions that were performed on the computer. We were able to analyze the students' conversations while we observed their operations on a spreadsheet. This type of computer protocol is very powerful as it allows the description of the actions in real time on the computer (Weigand & Weller, 2001).

### The two problems

King Edgar of Zirtuania decided to divide their treasure of a thousand gold bars by his four sons. The royal verdict is:

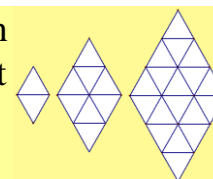
- 1- The 1<sup>st</sup> son gets twice the bars of the 2<sup>nd</sup> son.
- 2 - The 3<sup>rd</sup> son gets more bars than the first two together.
- 3 - The 4<sup>th</sup> son will receive less than the 2<sup>nd</sup> son.



What is the highest number of gold bars that the 4<sup>th</sup> son of the king may receive?

**Figure 1: The treasure of King Edgar**

From small equilateral triangles, rhombuses are formed as shown in the picture. We have 1000 triangles and we wish to make the biggest possible rhombus.



How many triangles will be used?

**Figure 2: Rhombuses with triangles**

A possible algebraic approach to the problems is presented in table 1. Solving these problems by a formal algebraic approach, namely using in equalities and systems such as these was beyond the reach of these students. Therefore, it is important to see which roads are opened by using the spreadsheet.

The treasure of King Edgar	Rhombuses with triangles
$\begin{cases} s_1 + s_2 + s_3 + s_4 = 1000 \\ s_1 = 2s_2 \\ s_3 > s_1 + s_2 \\ s_4 < s_2 \end{cases}$ $s_i - \text{number of bars of } i \quad i = 1 \dots 4 \text{ child}$	$2n^2 \leq 1000$ $n - \text{figure number}$

**Table 1: Algebraic approach to the problems**

Problem 1 contains several conditions that relate to each other and the statements “gets more” and “receives less” involve an element of ambiguity and make the problem complex, for understanding it, for translating into algebraic language and for solving it. Problem 2 entails a pictorial sequence that can be translated algebraically into a single condition. However, this condition involves a quadratic function that does not arise immediately after reading the statement of the problem. These two problems represent instances of global/meta-level activities considered by Kieran (2004), insofar as they involve functional reasoning and pattern finding strategies. They both have in common the search for a maximum value, leading to some difficulties when a purely algebraic approach is envisioned. However, a spreadsheet provides alternative approaches to both problems that may make them clearer to students, facilitating their solution process and efficiently providing a solution. We examine how students approached these problems in the classroom, the strategies they used, how they connected the variables involved and expressed that on a spreadsheet. Excerpts of Excel computer protocols are offered to further clarify the description of students’ activity.

In solving problem 1, Marcelo assigned and named a column to each of the four sons and a fifth column for the total of gold bars (table 2). Then, he started writing values in the cells corresponding to the sons in the following order: 2<sup>nd</sup>, 1<sup>st</sup>, 4<sup>th</sup> and 3<sup>rd</sup>, as follows: choosing a value for the 2<sup>nd</sup> son, then mentally doubling it for the 1<sup>st</sup> son; subtracting one unit to the 2<sup>nd</sup> son’s number of bars to get the 4<sup>th</sup> son’s; add the three values of the 2<sup>nd</sup>, 1<sup>st</sup>, and 4<sup>th</sup> sons and calculate the difference to 1000 to find the 3<sup>rd</sup> son’s number. In another column, the student entered a formula that gives the total of gold bars and served as control for the total number of bars (1000).

### Working Group 3

<table><tr><th>1º filho</th><th>2º filho</th><th>3º filho</th><th>4º filho</th><th>Total</th></tr><tr><td>200</td><td>100</td><td>601</td><td>99</td><td>1000</td></tr><tr><td>300</td><td>150</td><td>451</td><td>99</td><td>1000</td></tr><tr><td>240</td><td>120</td><td>521</td><td>119</td><td>1000</td></tr><tr><td>260</td><td>130</td><td>481</td><td>129</td><td>1000</td></tr><tr><td>290</td><td>145</td><td>436</td><td>129</td><td>1000</td></tr><tr><td>284</td><td>142</td><td>433</td><td>141</td><td>1000</td></tr><tr><td>286</td><td>143</td><td>430</td><td>141</td><td>1000</td></tr><tr><td>288</td><td>144</td><td>433</td><td>135</td><td>1000</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>	1º filho	2º filho	3º filho	4º filho	Total	200	100	601	99	1000	300	150	451	99	1000	240	120	521	119	1000	260	130	481	129	1000	290	145	436	129	1000	284	142	433	141	1000	286	143	430	141	1000	288	144	433	135	1000						<table><tr><th>1º filho</th><th>2º filho</th><th>3º filho</th><th>4º filho</th><th>Total</th></tr><tr><td>200</td><td>100</td><td>601</td><td>99</td><td>=D2+E2+F2+G2</td></tr><tr><td>300</td><td>150</td><td>451</td><td>99</td><td>=D3+E3+F3+G3</td></tr><tr><td>240</td><td>120</td><td>521</td><td>119</td><td>=D4+E4+F4+G4</td></tr><tr><td>260</td><td>130</td><td>481</td><td>129</td><td>=D5+E5+F5+G5</td></tr><tr><td>280</td><td>140</td><td>441</td><td>139</td><td>=D6+E6+F6+G6</td></tr><tr><td>290</td><td>145</td><td>436</td><td>129</td><td>=D7+E7+F7+G7</td></tr><tr><td>284</td><td>142</td><td>433</td><td>141</td><td>=D8+E8+F8+G8</td></tr><tr><td>286</td><td>143</td><td>430</td><td>141</td><td>=D9+E9+F9+G9</td></tr><tr><td>288</td><td>144</td><td>433</td><td>135</td><td>=D10+E10+F10+G10</td></tr><tr><td></td><td></td><td></td><td></td><td></td></tr></table>	1º filho	2º filho	3º filho	4º filho	Total	200	100	601	99	=D2+E2+F2+G2	300	150	451	99	=D3+E3+F3+G3	240	120	521	119	=D4+E4+F4+G4	260	130	481	129	=D5+E5+F5+G5	280	140	441	139	=D6+E6+F6+G6	290	145	436	129	=D7+E7+F7+G7	284	142	433	141	=D8+E8+F8+G8	286	143	430	141	=D9+E9+F9+G9	288	144	433	135	=D10+E10+F10+G10					
1º filho	2º filho	3º filho	4º filho	Total																																																																																																						
200	100	601	99	1000																																																																																																						
300	150	451	99	1000																																																																																																						
240	120	521	119	1000																																																																																																						
260	130	481	129	1000																																																																																																						
290	145	436	129	1000																																																																																																						
284	142	433	141	1000																																																																																																						
286	143	430	141	1000																																																																																																						
288	144	433	135	1000																																																																																																						
1º filho	2º filho	3º filho	4º filho	Total																																																																																																						
200	100	601	99	=D2+E2+F2+G2																																																																																																						
300	150	451	99	=D3+E3+F3+G3																																																																																																						
240	120	521	119	=D4+E4+F4+G4																																																																																																						
260	130	481	129	=D5+E5+F5+G5																																																																																																						
280	140	441	139	=D6+E6+F6+G6																																																																																																						
290	145	436	129	=D7+E7+F7+G7																																																																																																						
284	142	433	141	=D8+E8+F8+G8																																																																																																						
286	143	430	141	=D9+E9+F9+G9																																																																																																						
288	144	433	135	=D10+E10+F10+G10																																																																																																						
<i>xls</i> file	<i>xls</i> file (command “show formulas”)																																																																																																									

**Table 2: Print screen of Marcelo’s representation**

Although Marcelo did not display the relations between the number of bars of the four brothers -using formulas or otherwise-he kept them always present in his thinking. The task required a greater effort for the student, since in each attempt he had to recall the relations, while carrying out the calculations mentally.

Marcelo: Teacher, I found the best! [The value 139 was obtained in cell G6]. If I choose 150 [for the 2<sup>nd</sup> son]it won’t do. I’ve tried it.

Teacher: But this is not the maximum number of bars for the 4<sup>th</sup> son, is it?

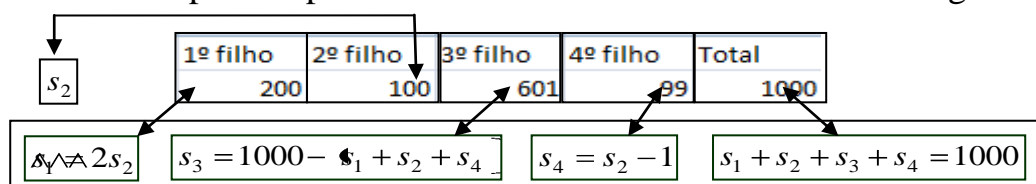
Marcelo: I went from 100 to 150, and it turns out that 150 gets worse because the other gets over 450 and the last one falls to 99.

The teacher asked Marcelo to do more experiments to which he replied that he had already made some, for example 160 and 170. So she made another suggestion:

Teacher: Here you already got an excellent value and it increased significantly from 130 to 140 [referring to column E]. So, try around these values.

The student continued to experiment, always doing the calculations mentally. He found 141, confirming that it was the best. As an answer the student wrote:“I solved this problem taking into account the conditions of the problem, making four columns, one for each child, and trying to find a higher number”.

In our view, Marcelo has developed algebraic thinking by focusing on dependence relationships between different variables to finding the optimal solution. As he stated, he took into account the five conditions of the problem and expressed them in the spreadsheet columns. From the standpoint of an algebraic approach, the student began by choosing an independent variable (the 2<sup>nd</sup>son’s number of bars) and established relationships to express the number of bars for each remaining son.



### Working Group 3

The diagram above summarizes the translation of the student's algebraic thinking in solving the problem and shows how Excel allowed dealing with simultaneous manipulation of several conditions, by means of numbers, rather than with letters and symbolic algebra. It is important to note that the condition set for the 4<sup>th</sup> son demonstrates an understanding of looking for the highest possible value, given that the difference down to the 2<sup>nd</sup> was only one bar.

Maria and Jessica (Table 3) started to solve the problem like Marcelo, with the allocation of columns to the number of gold bars for each son and another column for the total of bars. Then, they created a column of integers for the number of bars of the 2<sup>nd</sup> son; the number of bars of the 1<sup>st</sup> son was obtained by doubling the 2<sup>nd</sup> son's; the number of bars of the 3<sup>rd</sup> son was found by adding a unit to the sum of the 1<sup>st</sup> and 2<sup>nd</sup> sons' bars; the number of bars of the 4<sup>th</sup> son was obtained by subtracting one unit to 2<sup>nd</sup> son's; finally, the last column computed the sum of bars of the four sons.

<table><tr><th>2filho</th><th>1 filho</th><th>3 filho</th><th>4 filho</th><th>total de baras</th></tr><tr><td>1</td><td>2</td><td>4</td><td>0</td><td>7</td></tr><tr><td>2</td><td>4</td><td>7</td><td>1</td><td>14</td></tr></table> <table><tr><td>141</td><td>282</td><td>424</td><td>140</td><td>987</td></tr><tr><td>142</td><td>284</td><td>427</td><td>141</td><td>994</td></tr><tr><td>143</td><td>286</td><td>430</td><td>142</td><td>1001</td></tr><tr><td>144</td><td>288</td><td>433</td><td>143</td><td>1008</td></tr></table>	2filho	1 filho	3 filho	4 filho	total de baras	1	2	4	0	7	2	4	7	1	14	141	282	424	140	987	142	284	427	141	994	143	286	430	142	1001	144	288	433	143	1008	<table><tr><th>2filho</th><th>1 filho</th><th>3 filho</th><th>4 filho</th><th>total de baras</th></tr><tr><td>1</td><td>2</td><td>4</td><td>0</td><td>=D2+E2+F2+G2</td></tr><tr><td>2</td><td>4</td><td>7</td><td>1</td><td>=D3+E3+F3+G3</td></tr><tr><td>3</td><td>6</td><td>10</td><td>2</td><td>=D4+E4+F4+G4</td></tr><tr><td>4</td><td>8</td><td>13</td><td>3</td><td>=D5+E5+F5+G5</td></tr><tr><td>5</td><td>10</td><td>16</td><td>4</td><td>=D6+E6+F6+G6</td></tr><tr><td>6</td><td>12</td><td>19</td><td>5</td><td>=D7+E7+F7+G7</td></tr></table>	2filho	1 filho	3 filho	4 filho	total de baras	1	2	4	0	=D2+E2+F2+G2	2	4	7	1	=D3+E3+F3+G3	3	6	10	2	=D4+E4+F4+G4	4	8	13	3	=D5+E5+F5+G5	5	10	16	4	=D6+E6+F6+G6	6	12	19	5	=D7+E7+F7+G7
2filho	1 filho	3 filho	4 filho	total de baras																																																																			
1	2	4	0	7																																																																			
2	4	7	1	14																																																																			
141	282	424	140	987																																																																			
142	284	427	141	994																																																																			
143	286	430	142	1001																																																																			
144	288	433	143	1008																																																																			
2filho	1 filho	3 filho	4 filho	total de baras																																																																			
1	2	4	0	=D2+E2+F2+G2																																																																			
2	4	7	1	=D3+E3+F3+G3																																																																			
3	6	10	2	=D4+E4+F4+G4																																																																			
4	8	13	3	=D5+E5+F5+G5																																																																			
5	10	16	4	=D6+E6+F6+G6																																																																			
6	12	19	5	=D7+E7+F7+G7																																																																			
<i>xls</i> file	<i>xls</i> file (command “show formulas”)																																																																						

**Table 3: Print screen of Maria e Jessica's representation**

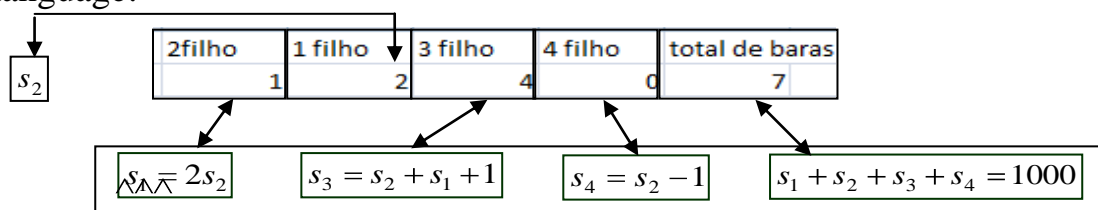
At one point the students got a value higher than 1000 in the last column and concluded that it was necessary to remove a bar from one of the sons. Yet, it was necessary to realize that one bar could only be taken from the 4<sup>th</sup> in accordance with the terms of the problem.

Maria: It shows 1001, it is wrong!

Teacher: And now?

Maria: Take one out! Take one out from the 4<sup>th</sup>!... The largest number of bars that the 4<sup>th</sup> can receive is 141.

The diagram below shows the relations as they would be expressed symbolically in algebraic language:



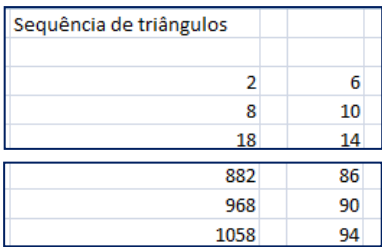
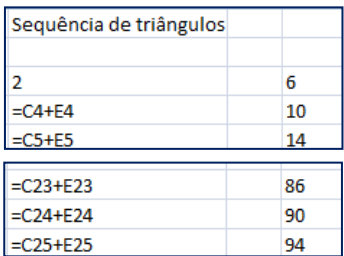
These students formulated two conditions intended to obtain the optimal solution: first, the difference between the number of bars of the 4<sup>th</sup> and the 2<sup>nd</sup> son must be one

### Working Group 3

unit and, second, the difference between the number of bars of the 3<sup>rd</sup> and the sum of the 1<sup>st</sup> and 2<sup>nd</sup> must also be one unit.

In both solutions, the data show that students use relationships between variables but they do it with numbers through the use of the spreadsheet. The fact that they are working with numbers does not deviate them from the mathematical structure of the problem. On the contrary, it helps them to better understand the problem and to deal with a set of simultaneous conditions of different nature: equations, inequalities and a free variable. We believe that the thinking involved in either approach is consistent with the perspective of Kieran (2007) and Lins & Kaput (2004) on genuine algebraic thinking development.

For the second problem, Marcelo (Table 4) started to introduce the inputs 2, 8, 18. Then, he selected these three cells as a cluster and tried to drag them (Figure 3), noticing that the numbers generated were not all integers.

	
<i>xls</i> file	<i>xls</i> file (command “show formulas”)

**Table 4: Print screen of Marcelo’s representation**

He eventually abandoned the dragging and called the teacher:

Marcelo: I don’t know if this works... How do I do this? Is there an easier way?

Teacher: To move from the 1<sup>st</sup> to the 2<sup>nd</sup> how much did you add?

The student writes in cell E4 the number 6.

Teacher: And from the 2<sup>nd</sup> to the 3<sup>rd</sup> how much do you add?

The student wrote in cell E5 the number 10, followed by 14 and 18.

Teacher: What are you going to do now?

Marcelo: If I pull it down [referring to column E] and then by adding this column plus this one [referring to column C and column E]...

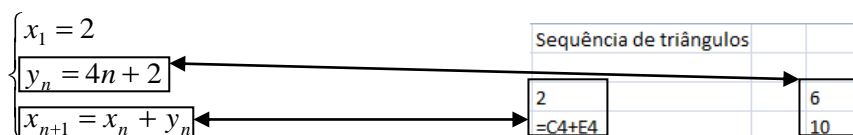
The student inserted the formula “=C4+E4” (below the first term of the sequence) and generated a variable-column:

Marcelo: 968! It’s what we will use from 1000. We have 1000, so it can’t be more than 1000 and 1058 already exceeds.

### Working Group 3

The student tried to find a pattern in the number of triangles. The construction of additional figures did not help the student to find a pattern based on the figure. One useful approach was to look at the differences between the consecutive terms.

From an algebraic point of view, this student is using a recursive method to generate the sequence of triangles with the help of the arithmetic progression which gives the difference between consecutive terms. Excel easily allows handling a recursive approach. Somehow it was no longer necessary to find  $n^{\text{th}}$  element to solve the inequality, although the mathematical structure of the problem remained visible.



Maria and Jessica (table 5) addressed the problem with a similar strategy, noticing that dragging the values 2, 8 and 18 did not produce the sequence of rhombi presented in the problem. At one point they called the teacher:

<table><tr><th>nº de figuras</th><th>nº de triangulos</th><th>(+)4</th></tr><tr><td>1</td><td>2</td><td>6</td></tr><tr><td>2</td><td>8</td><td>10</td></tr><tr><td>3</td><td>18</td><td>14</td></tr><tr><td>4</td><td>32</td><td>18</td></tr><tr><td>5</td><td>50</td><td>22</td></tr><tr><td>6</td><td>72</td><td>26</td></tr><tr><td>7</td><td>98</td><td>30</td></tr><tr><td>8</td><td>128</td><td>34</td></tr><tr><td>21</td><td>882</td><td>86</td></tr><tr><td>22</td><td>968</td><td>90</td></tr><tr><td>23</td><td>1058</td><td>94</td></tr></table>	nº de figuras	nº de triangulos	(+)4	1	2	6	2	8	10	3	18	14	4	32	18	5	50	22	6	72	26	7	98	30	8	128	34	21	882	86	22	968	90	23	1058	94	<table><tr><th>nº de figuras</th><th>nº de triangulos</th><th>(+)4</th></tr><tr><td>1</td><td>2</td><td>6</td></tr><tr><td>2</td><td>8</td><td>10</td></tr><tr><td>3</td><td>18</td><td>14</td></tr><tr><td>4</td><td>32</td><td>18</td></tr><tr><td>5</td><td>50</td><td>22</td></tr><tr><td>6</td><td>72</td><td>26</td></tr><tr><td>7</td><td>=G7+F7</td><td>30</td></tr><tr><td>8</td><td>=G8+F8</td><td>34</td></tr><tr><td>21</td><td>=G21+F21</td><td>86</td></tr><tr><td>22</td><td>=G22+F22</td><td>90</td></tr><tr><td>23</td><td>=G23+F23</td><td>94</td></tr></table>	nº de figuras	nº de triangulos	(+)4	1	2	6	2	8	10	3	18	14	4	32	18	5	50	22	6	72	26	7	=G7+F7	30	8	=G8+F8	34	21	=G21+F21	86	22	=G22+F22	90	23	=G23+F23	94
nº de figuras	nº de triangulos	(+)4																																																																							
1	2	6																																																																							
2	8	10																																																																							
3	18	14																																																																							
4	32	18																																																																							
5	50	22																																																																							
6	72	26																																																																							
7	98	30																																																																							
8	128	34																																																																							
21	882	86																																																																							
22	968	90																																																																							
23	1058	94																																																																							
nº de figuras	nº de triangulos	(+)4																																																																							
1	2	6																																																																							
2	8	10																																																																							
3	18	14																																																																							
4	32	18																																																																							
5	50	22																																																																							
6	72	26																																																																							
7	=G7+F7	30																																																																							
8	=G8+F8	34																																																																							
21	=G21+F21	86																																																																							
22	=G22+F22	90																																																																							
23	=G23+F23	94																																																																							
<i>xls</i> file	<i>xls</i> file (command “show formulas”)																																																																								

**Table 5: Print screen of Maria and Jessica's representation**

Jessica: From this one to this one it goes 6 and from this one to this one it goes 10.

Maria: From 2 to 8 it goes 6...From 8 to 18it goes 10.

Jessica: Oh teacher, we don't know how to continue.

Teacher: Have you already drawn the next figure to see if there is any relation?

They drew it on paper, but only half of the picture.

Maria: 14, 15 and 16. Then 16 plus16 is...32

Teacher: And now?... How many will the next one have?

Maria: 50.

The students were still looking for a relation between the numbers.

Maria: I know what that is... Look... The link is...

Jessica: The number plus 4.

This was the decisive moment to build the column with the differences between consecutive rhombuses, and then a formula for the number of triangles.

## CONCLUDING REMARKS

Our main aim was to understand the role of the spreadsheet in solving two word problems, which are expressed by inequalities, and examine how the solutions reflect students' algebraic thinking, regardless of the use of algebraic symbolism. It was not our intention to consider what students have done without the use of technology, since any of the problems demanded an algebraic knowledge that was beyond the level of students. In any case pencil and paper solutions could certainly come up with methods based on trial and error. We interpreted the students' processes based on the spreadsheet in light of what would be a possible use of symbolic algebra. Thus we intended to make clear students' algebraic thinking in establishing the relationships involved in the problems. In the first problem, four columns corresponded to the four sons and the column for the 2<sup>nd</sup> son was reserved for the introduction of initial values (the *input*), serving as a column for the independent variable. The remaining columns were constructed through relations of dependence. For the second problem, the students were not able to express the general term of the sequence, but by counting the number of triangles in the sequence of rhombuses they used the differences between consecutive terms to generate the former sequence recursively. As reported in some studies students when confronted with more demanding sequences tend to use the difference method (Orton & Orton, 1999). We found that the spreadsheet helped the students to establish relations between variables, expressed through numerical sequences generated by the computer, and also with the use of formulas to produce variable-columns. We claim that algebraic thinking was fostered by the affordances of the Excel in generating the rules of the problems. This result resonates with other investigations such as Ainley et al. (2004) but it also highlights the structure of students' algebraic thinking expressed in a particular representation system. It provided a clear indicator of how students interpreted the problems in light of their mathematical knowledge and their knowledge of the tool. The analysis allows us to make inferences about what is gained in using Excel to solve algebraic problems, and helps to understand the relationship between the symbolic language of Excel and the algebraic language. The use of Excel can be seen as means to fill the gap between the algebraic thinking and the ability to use algebraic notation to express such thinking. The lack of algebraic notation and formal algebra methods does not prove the absence of algebraic thinking. The kind of algebraic thinking that emerges from the use of the spreadsheet is the kind that belongs to global algebraic activities (Kieran, 2004). We highlight the following features of the spreadsheet in algebraic problem solving: (i) *It was a way to anticipate complex algebraic problems*; our study shows how the spreadsheet was a tool that allowed 8<sup>th</sup> grade students to solve two problems that were impracticable from the point view of formal

algebra. On the other hand it anticipated forms of algebraic reasoning involved in the problems that were elicited by the representation systems embedded in the spreadsheet; (ii) *It helped to understand the conditions in the problems*; students clearly understood the relations between the several variables involved and were able to express such conditions and restrictions appropriately. These were not expressed in algebraic notation but instead with the language of Excel (iii) *It led to a numerical approach of an algebraic problem*; students found ways to represent the problem through numerical variable-columns without losing the structure of the problems. Our perspective of algebraic thinking stresses the distinction between algebraic notation and algebraic structures, separated by a gap that is often underestimated. We suggest that this gap can be gainfully filled with suitable spreadsheet activities.

Rather than insisting on any particular symbolic notation, this gap should be accepted and used as a venue for students to practice their algebraic thinking. They should have the opportunity to engage in situations that promote such thinking without the constraints of formal symbolism (Zazkis & Liljedhal, 2002, p. 400).

## REFERENCES

- Ainley, J., Bills, L., & Wilson, K. (2004). Construing meanings and utilities within algebraic tasks. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28<sup>th</sup> PME Conference* (Vol. 2, pp. 1-8). Bergen, Norway.
- Carreira, S. (1992). *A aprendizagem da Trigonometria num contexto de aplicações e modelação com recurso à folha de cálculo* (Tese de Mestrado). Lisboa: APM.
- Dufour-Janvier, B., Bednarz, N., & Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 109-122). Hillsdale, NJ: Lawrence Erlbaum.
- Haspekian, M. (2005). An 'instrumental approach' to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *International Journal of Computers for Mathematical Learning*, 10(2), 109-141.
- Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. M. Alvares, B. Hodgson, C. Laborde & A. Pérez (Eds.), *International Congress on Mathematical Education 8: Selected lectures* (pp. 271-290). Seville: SAEM Thales.
- Kieran, C. (2004). The core of algebra: Reflections on its main activities. In K. Stacey et al. (Eds.), *The future of teaching and learning of algebra: The 12<sup>th</sup> ICMI Study* (pp. 21-34). Dordrecht: Kluwer.
- Kieran, C. (2007). Developing algebraic reasoning: the role of sequenced tasks and teacher questions from the primary to the early secondary school levels. *Quadrante*, 16(1), 5-26.



### Working Group 3

- Lins, R., & Kaput, J. (2004). The early development of algebraic reasoning: the current state of the field. In K. Stacey et al. (Eds). *The future of teaching and learning of algebra: The 12th ICMI Study*. (pp. 47-70). Dordrecht: Kluwer.
- Orton, A., & Orton, J. (1999). 'Pattern and the approach to algebra'. In A. Orton (ed.), *Pattern in the Teaching and Learning of Mathematics*. (pp. 104-120) London: Continuum.
- Rojano, T. (2002). Mathematics learning in the junior secondary school: Students' access to significant mathematical ideas. In L. English, M. B. Bussi, G. A. Jones, R. A. Lesh & D. Tirosh (Eds.), *Handbook of international research in mathematics education* (pp. 143-161). Mahwah, NJ: Lawrence Erlbaum.
- Weigand, H., & Weller, H. (2001). Changes in working styles in a computer algebra environment: The case of functions. *International Journal of Computers in Mathematical Learning*, 6(1), 87-111.
- Zazkis, R., & Liljedhal, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49, 379-402.

# ALGEBRAIC REASONINGS AMONG PRIMARY SCHOOL 4TH GRADE PUPILS

Marta Pytlak

University of Rzeszow

*The term “early algebraic thinking” concerns many phenomena connected with the introduction to the generalization process on a lower educational level. Algebraic thinking does not necessarily consist of formal algebraic symbolism. Young students who did not learn formal algebra in school are capable of formalizing a verbal rule in which spontaneous algebraic thinking is evident. In my work, I wish to present the way in which primary school 4<sup>th</sup> grade pupils deal with a series of arithmetical-geometrical tasks that lead to generalization of relations that exist there.*

## THEORETICAL FRAMEWORK

In the Polish mathematical curriculum, we can distinguish two branches: arithmetic and algebra. Algebra, as a branch of science, first appears in the 6<sup>th</sup> grade of primary school. So far, in school practice, there has been a conviction that actions on algebraic objects such as variables, unknowns, parameters are the central feature of algebraic thinking (Polish curriculum: [www.reformaprogramowa.men.gov.pl](http://www.reformaprogramowa.men.gov.pl)). Initially, school algebra is focused on the introduction and use of letters. A letter is treated as some kind of a code that abbreviates the recording of a verbal or visual situation such as coding the number of carriages in a drawn two-colored train. At gymnasium level pupils are very frequently asked to use recordings of algebraic notions, regardless of previously formed intuitions.

I am of the view, that the essence of algebraic thinking is not merely the use of algebraic symbolism. Using a letter has been neither a necessary nor a sufficient condition for algebraic thinking. It is Radford's opinion that “there is a conceptual sphere, where pupils can start their algebraic thinking even if they do not refer (or at least in a big extend) to symbolic language” (Radford, 2009, p.XXXV). Nevertheless, a letter is a generally accepted tool for expressing generality, on a certain stage of its understanding. In algebra, thinking is focused on relations because objects themselves can be indefinite (Radford 2005, 2009). Using algebraic symbolism for writing solutions is an external picture of “algebraic thinking”.

If a student is to use algebraic language (symbolic language) he or she has to understand its basic component – a letter. Letters in algebra are used in at least 4 types of meanings: as *general names*, *changeable values*, as *unknowns*, and as *constants* (Turnau, 1990). Each of these meanings appears in different forms that depend on the context. Moreover, the meaning of a letter in an algebraic record can change during the process of solving a task. This poses an additional difficulty in mastering and applying algebraic language.

Another conviction is that before starting algebra, students need to be competent in the sphere of arithmetic (Hejny and Littler, 2003). Then, algebra appears as a ‘superstructure’ of arithmetic, that is, - arithmetical thinking is a basis for algebraic thinking.

My understanding is different. Arithmetic is, first of all, a science about numbers, in particular natural and integral numbers. Arithmetic in this sense is focused on the object - a number. Thus, arithmetical thinking concerns mainly numbers and using them – it is strictly connected with specific mathematics actions that lead to an unequivocal result. Algebra, in my understanding, is mainly a science about relations. However, there is an approach which combines arithmetical and algebraic thinking. Pupils get a series of tasks in which they have to discover and notice certain regularities. Then, they have to formulate noticed rules and finally write them down using symbolic language. It is done by generalizing arithmetical reflections through modifying constants (which can, according to Turnau, lead to understanding a letter as a *general name*). Then, students move to actions and reasoning typical for algebra. Initially, these are ‘early-algebraic’ thoughts connected with so called ‘early algebra’, which is gaining more and more attention in literature (Mutschler, 2005).

Developing mathematical thinking is inextricably linked with the process of generalization. It is clearly stated in the TGM (Theory of Generic Model) theory (Hejny 2002, 2004, 2005). According to this theory, cognition happens on two levels: generalization (understood locally, connected with a certain type of situation) and abstraction. These levels have a common part – a generic model. For the first part it can be treated as an ending stage and for the second as a start.

The process of building new knowledge starts from gathering experiences which are kept in mind as isolated models of certain situations. If this set of experiences is large enough, connections among similar isolated models will appear. This net will become more and more dense and certain general objects that represent a broader group (of concepts, reasoning) will appear.

The moment when one general model, representing features of all models (a general model of a certain situation) replaces a couple of isolated models, students being to build mathematical abstract knowledge. Therefore if he/she is not able to create a general model for a certain situation, he/she will not be able to develop abstract knowledge.

### **THE AIM OF THE RESEARCH**

The research shown here is the part of wider research concerning the development of students’ algebraic thinking. The focus is on building their personal web of cognitive connections during solving the task connected with discovering mathematical regularity. My research question were as follows:

### Working Group 3

- In what way do 9-10 years old students „think” about regularities and what are their thinking processes while solving tasks in which they have to discover and use noticed rules?
- Do the proposed series of task help to shift attention from arithmetical relations to thinking about general relations?

## METHODOLOGY

The research was carried out in November 2009 among students from the fourth grade of a primary school. Twenty 9-10 years old students working in pairs took part. The research contained four follow-up meetings, during which pairs of students solved tasks. The researcher talked with every group of students while they were solving the tasks. All meetings were video-recorded. After the research, the report was presented.

The students had work-sheets, matches (black sticks), ball-point and a calculator. Before they started their work, they had been informed that they could solve this task in any way they would recognize as suitable; their work would not be graded; the teacher would be videotaping their work and that they could write everything which they thought was recognize as important on the work sheet. The research material consisted of work sheets filled by students, as well as the film recording their work and a protocols record from it.

The research tool consisted of four sheets and each of them consisted of two tasks. The tasks were as following: the students make a match pattern consisting of geometrical figures – one with time there are triangles and another with time there are squares with a side length of one match. In the first two sheets the figures were arranged separately, in the second two, they were connected in one row. The following sheets concerned: (1) separated triangles, (2) separated squares, (3) connected squares and (4) connected triangles. In each of the sheets the problem was presented in a frame of two following tasks. They were constructed in such a way in order to inspire students to search for and discover occurring rules.

Sheet I

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 **separated** triangles, which length of each sides equals one match?

Number	of	1	2	3	4	5	6	7
triangles								
Number	of							
matches								

2. And how many matches do you need to construct :

a) 10 triangles

b) 25 triangles

c) 161 triangles?

Sheet IV

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 **connected in one row** triangles, which length of each sides equals one match?

Number	of	1	2	3	4	5	6	7
triangles								
Number	of							
matches								

2. And how many matches do you need to construct :

a) 10 triangles

b) 25 triangles

c) 161 triangles?

**Figure 1. Research tool – sheet I and sheet IV**

### Working Group 3

In the first task the students had to give the number of matches needed to arrange from one to seven triangles or squares consecutively. The question was, - ‘How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 of such figures?’ The results were to be written in the table. In the task 2, there was a question about a number of matches that would be needed to construct 10, 25 and 161 of such figures (Littler, Benson 2005). In order to answer these questions correctly, the students had to discover the rule occurring in the first task.

The focus of this task was to perceive the relevant general rule. According to mathematical (algebraic) language the rules were as follows:  $3n$ ,  $4n$ ,  $3n+1$ ,  $2n+1$ . The two last rules were equivalent to following statements:  $4(n-1)+3$ ,  $2(n-1)+3$ . Perceiving these rules was connected with the ability to generalize regularities appearing in the initial series (that is an arithmetic sequence; i.e. for the third sheet it was 4, 7, 10, 13...) and with ability to draw the conclusion from previous experience.

The choice of the tasks and the order of the sheets were not random, they were clearly inspired by TGM theory of M. Hejny (Hejny 2005). The problem was to check if the students would benefit from their earlier experience while solving the new tasks. This task and the way of its presentation (four following sessions with two-three days break between meeting) were something new for students. So far, during maths lessons they had not solved the tasks concerned with the perception of the appeared rules and generalization of noticed regularities.

### DEPICTED RESEARCH RESULTS

In my paper I would like to take a closer look at the work of one of my student. It is representative of wider group. Similar reasonings were found in other students. The boy, initially, worked together with his friend but his last two sheets were solved individually. In my analysis, I will focus on the third and the fourth work-sheets.

#### Sheet III - Kacper

After getting the sheet, Kacper noticed that it concerned squares again. Although he arranged a couple of elements correctly, at the beginning he used the strategy of the previous sheet. Fortunately, he quickly corrected his mistake and used the rule ‘add 3 to the previous number’ for the chart. Physical manipulations (arranging figures from matches) were the basis for generalizations.

- 12 Kacper: Y... ten squares it will be ... 4 [*he is writing 4*] and three times nine [*he is writing under 4: 3x9*] twenty seven ... thirty one
- 13 Teacher: How did you count it?
- 14 Kacper: Well because the first square has four [*he is showing on the first square and moving away others sticks*] and then we always attached in this way [*he is adding three sticks to the first square*] and ten times like this. [*he is talking and in this same time he is counting and writing*] (...)

### Working Group 3

- 15 Kacper: Twenty five ... four [*he is writing 4*] three times twenty four ... [*he is writing 3x24 and he is counting*] seventy six ... and one hundred sixty one ... four plus ... one hundred sixty times three ... four hundred eighty four.

Calculations in the column for task 2 from the sheet no.3 Application of the first rule by Kacper.

- 16 Teacher: Ok., and if it will be one thousand squares to arranging?

- 17 Kacper: [*he is writing 4*] four plus nine hundred ninety nine times three [*he is writing algorithm of multiplication 999x3, counting and to the result he adding 4*]. Three thousand one.

Kacper, on the basis of the arranged pattern, worked out a rule which he had applied to calculating the number of matches in task 2. For this, he used an interesting record: he did not write down his calculations in a line but in a column. Above each column there was a number 4 which was added to the result at the end. In some cases this addition took place in the pupil's mind, without any record of that.

- 27 Teacher: Good. So if I gave you a certain number of squares, what advice would I get from you? How do I count the necessary matches?

- 28 Kacper: First, from this number I subtract one and this is one so it is four. So I'm left with this number and we multiply it by three.

When asked how to calculate the number of matches if I know the number of squares, the boy gave a general rule that shows the process of calculating the required number. The pupil could also write down the given rule in words.

### Summary of work on the worksheet III

Kacper started to work on the task from gathering physical experiences connected with arranging the current pattern. Immediately after getting the task, he noticed a similarity with the previous one, from worksheet II – the same basic figure. After analyzing the content, he noticed the difference in the layout of figures – this is why he arranged three elements before recording the data in the chart. Maybe the first association with the worksheet II was so strong that, regardless of correctly arranged figures, Kacper transferred the previous rule concerning separate squares onto the current situation. He quickly noticed his mistake and, after that, he counted correctly in the following part of the task.

Kacper did not need to arrange figures to solve previous tasks (in the first and the second worksheet) This time, the boy arranged three elements. It means that in this case, arrangement of figures was treated as a means of discovering a rule. While arranging, the boy was focused on finding relations. The knowledge he possesses is efficient to the extent that arranging even a small part of the pattern enabled him to

create a general model for the chart. The manipulation he made facilitates a geometrical goal – discovering new phenomena.

From his puzzle, Kacper isolates the first element – a complete square and the others – incomplete ones. This observation helps him to generate a new way of calculating the number of matches for a specific number of squares. It is strictly connected with the way of arranging. The record of it shows the boy's thinking structure:  $4+(n-1) \times 3$ .

Justification that the pupil gives when asked by the teacher about the way of counting the number of matches in task 2 (14) shows that he drew conclusions after arranging only 3 elements. He is also aware of the fact that these conclusions will be valid later on. Therefore, he has developed a general model for the situation concerning the whole task, and this model works within the limits of a big quantifier.

Kacper generalizes the noticed relation, and the generalization he makes has the power of a big quantifier. When describing his rule, he interchangeably operates with sizes 'one square' 'four matches' and while giving numbers from 1 to 4 he does not add determiners such as 'a square, a match'. Also in his record, he limits his description only to the 'instruction of dealing with numbers' in which 1 is equivalent to 4. For him, such a rule is very clear and legible and he does not feel the need to specify it. He deals well with operating with two arbitrary quantities (the number of matches and the number of squares). Intuitively, he uses proportions. A written record that the pupil makes is meaningful in relation to his own actions. It is not objective, it is strictly connected with a description of the situation that the boy found himself in. The record that appeared on a piece of paper is actually not the record of a general rule but only the description of that situation (Radford 2009). It shows that Kacper has a considerable level of algebraic thinking. Since, in this case, his thinking concerns a specific. Local situation, it is not at a general level.

#### Sheet IV – Kacper

After reading the task content, Kacper arranged one triangle and, initially, applied the worksheet I strategy again (rule "multiply number of triangles by 3"). When the teacher focused his attention on the way the triangles are arranged (this time they are joined together), the boy arranged the following triangles and then, filled in the chart correctly (using the rule "add 2 to the previous number of matches").

- |    |          |  |
|----|----------|--|
| 13 | Kacper:  | <i>[He quietly reads task 2 content, starts his calculations]</i> eighteen... twenty-one |
| 14 | Teacher: | How did you count it?  |
| 15 | Kacper:  | So I left three of them, because I need three for the first one, and then two times nine |

In the task 2, he applied a rule which was analogous to the previous task (as in worksheet III), but which took into account the current situation.

He proceeded with this type of work throughout the whole of task two while giving answers to additional teacher's questions. When the teacher started asking questions

### Working Group 3

about the number of matches for 1000, 10000 and one million squares, Kacper surprisingly changed his strategy, which is clearly visible in the following part of their conversation. The change of strategy resulted in discovering a new rule in the task.

- 27 Teacher: Mhm, twenty thousand and one. And what if I wanted to arrange one million?
- 28 Kacper: [*immediately, without any consideration*] two million and one
- 29 Teacher: How did you count it so fast?
- 30 Kacper: Well, cause when I did that [*points at his record*] the ending was one...so it's the same here...because it seems...
- 31 Teacher: How did it happen?
- 32 Kacper: Because these...I subtracted one from this and multiplied by two, but if I add such one but times two, because there is one more match added, because it is like a million of squares but one more match
- 33 Teacher: A million of triangles, now we have triangles.
- 34 Kacper: Yes,... [*he sighs, takes sticks and starts arranging a triangle*] So, like you see here, I arrange further [*'further' hand movement*] and then like this first one, it has three and it looks as if two and one more.
- 35 Teacher: I see, so when I told you: one million; you multiplied this million by two, right? Is there any other way of counting it?
- 36 Kacper: Right.

When justifying the correctness of the newly discovered rule, first, the pupil refers to the results of the previous calculations. He shows that all previous results ended in 1. Then, he tries to describe the procedure, in which he highlights the meaning of 'a single match' (32). Seeing that this explanation is not sufficient for the experiment, he refers to the arranged pattern. He unfolds the first element (a complete triangle) into two parts: 1(a "closing" stick) 2 ("incomplete" triangle ) The multitude of explanations indicates that the reconstruction of the previous knowledge took place in the child's mind. He associated numeral results with previous triangle arrangement strategies, which led him to a new puzzle interpretation. He is aware that this way is the correct one, so, in order to check the effectiveness of the new rule, he chooses the number of triangles from the task 2b).

- 41 Teacher: So, let us check if it works for the previous ones [*points at the chart and task 2*] Would it fit here?
- 42 Kacper: Yes [*he chooses 25 triangles and counts aloud*] twenty five times two, this is fifty plus one...fifty-one

Kacper, without any difficulty, could say it for any number of triangles. When asked about the rule, he gave the newly discovered rule: multiply the number of triangles by 2 and add 1.



### Commentary – description of Kacper’s work on worksheet IV

As the boy was beginning the task, he had knowledge and experience from the work of three previous worksheets. He did not analyze the content of the first task in detail. Seeing the word ‘triangle’, he arranged one figure, and then he started to fill in the chart using the rule which was present in the first worksheet. The information about the kind of figure that he has now was more important for him than the one about the correct position of it. Only the teacher’s remark and the arrangement of three initial elements of the current puzzle helped him to focus on the pattern structure. For the chart, Kacper uses the rule ‘plus 2’ which presents his way of arranging the following elements. The boy, while already while filling in the chart, created a general model for the situation described in the task. Even while writing down the number of matches for 4 triangles he was aware of the generality.

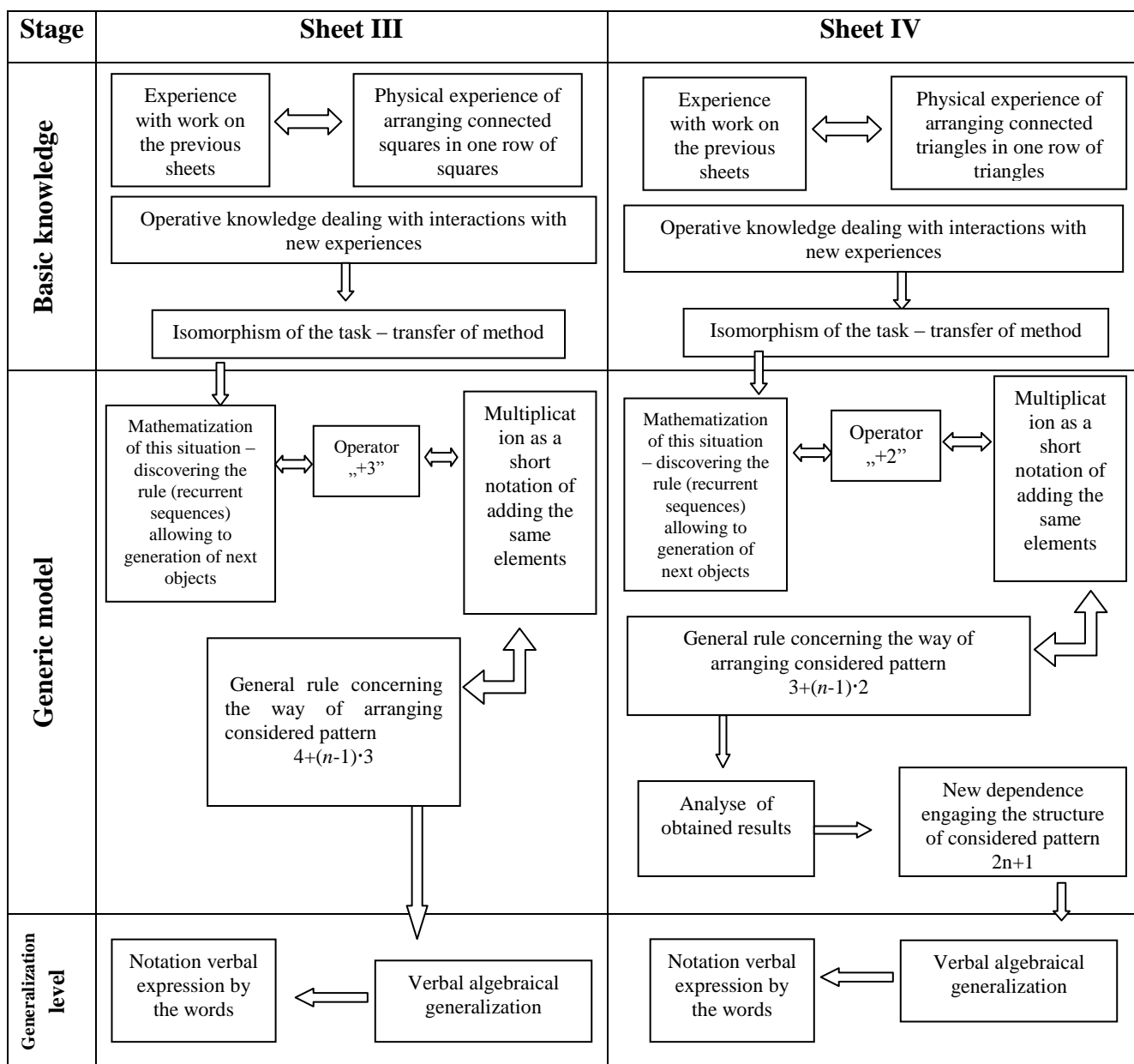


Figure 2.

### Working Group 3

While justifying the way of calculating the number of matches, he stresses the fact that the operation of adding 2 will be continued until the end (5). Therefore, we assume that he possesses a certain level of generality of the situation. The fact that he stresses it himself, proves that this knowledge is important for him. Experiences from the work on the task from worksheet III are the basis of it. Isomorphism of the task and the way of work resulted in creating a general model for the task ( specifically for the chart ), even though the boy arranged only 4 triangles.

As the boy moves to task 2, he uses an analogous method to the one used in the previous worksheet. He unfolds the puzzles into two elements: one complete triangle and the rest of incomplete ones, consisting of two matches (16) This is a clear reference to the worked out method. Because of the task's isomorphism, there is a shift from the worksheet III to the worksheet IV.

For Kacper, the whole task is one, coherent entirety. He does not treat particular stages (chart, task 2, additional teacher's questions) as separate elements. He immediately analyses the results. Because of this stance, he could see a new relation that appeared between the number of triangles and matches. This relation was discovered on a purely arithmetical basis (analysis of results for 1000, 10000 and one million triangles), and after he justified it geometrically. The boy could link this arithmetical relation to the puzzle. He noticed, that instead of building one, complete triangle (isolating the whole element) and then adding two matches at a time, he could start from building incomplete triangles and, in the end, 'close' the entirety with one single match.

When asked about one million triangles, the boy, first, looked at the previous results. The previous examples were quite suggestive: 10 triangles -21 matches, 1000 triangles – 2001 matches, 10000 triangles – 20001 matches. The pupil, noticing the analogy between the following examples, was able to see regularity, a certain arithmetical relation. This relation helped him to get the result much more quickly than the previously used rule. It was because of this that he decided to apply this new relation. This discovery was very important for him and, therefore, he used this newly discovered principle as a general rule for the whole task.

This discovery would not be possible without the boy's willingness to look for new solutions. During research, it was clearly seen that the boy uses his previous experiences, while solving new tasks and considering the follow-up examples. He was not only motivated to getting the result but he constantly analyzed his data. His reflective stance was a crucial element for enhancing the process of discovering regularities.

Creating new relations, joining an arithmetical structure with a geometrical representation was possible because of the fact that Kacper possesses general, geometrical knowledge (in the sense of a big quantifier, objects as classes of abstraction). This knowledge is operational and it is not rigid.

The pupil is absolutely aware of the fact that this new rule is correct. The examination of the rule for the previously obtained quantities is made only for external recipients (41). The chosen numbers 25 which are to be checked are also not accidental. It proves that only task 2 can be treated as the essence of the matter. The chart is only an introduction for him.

## SUMMARY

The pupil solved the task from all worksheets correctly. He could notice different rules that appeared in particular worksheets. Moreover, he could generalize them. Generalized rules were written down by him in words. There was no symbolic record of it because it is quite difficult for a primary school pupil. It is possible that while solving follow-up tasks from the presented series the pupil would try to present long descriptions in an abbreviated form. This would consequently lead to application of symbolism – initially, his own symbolism, and then, generally established algebraic one. We may successfully introduce the world of algebra to pupils, even at the initial stage of their education. However, it is very important how we do it.

## REFERENCES

- Hejný, M., and Littler, G. (2003). The beginnings of algebraic thinking. In: C. Bergssten and B. Grevholm, (eds), *Challenges in Mathematics Education*. (pp.107-118). Linköping. MADIF 3
- Hejný, M. (2004). Mechanizmus poznávací procesu. In: M. Hejný and J. Novotná and N. Stehliková (eds), *Dvacet pět kapitol z didaktiky matematiky*, (pp. 23-42). Praha. Univerzita Karlova v Praze, Pedagogická Faculta.
- Hejný, M. and Kratochvílová, J. (2005). From experience through generic model to abstract knowledge, Proc. of CERME4, Feb. 17-21, Sant Feliu de Guixols, Spain, <http://www.erne.unito.it/CERME4/> (pp. 311-321).
- Littler, G.H. and Benson, D. (2005). Patterns leading to Algebra. In: IIATM-Implementation of Innovation Approaches to the Teaching of Mathematics, (pp. 199-258). Comenius 2.1
- Mutschler, B. (2005). Early algebra – Processes and concepts of fourth graders solving algebraic problem. Proc. of CERME4, Feb. 17-21, Sant Feliu de Guixols, Spain, <http://www.erne.unito.it/CERME4/> (pp. 706-716).
- Radford, L. and Bardini, C. and Sabena, C. (2005). Perceptual semiosis and the microgenesis of algebraic generalizations, Proc. of CERME4, Feb. 17-21, Sant Feliu de Guixols, Spain, <http://www.erne.unito.it/CERME4/> (pp. 684-695)
- Radford, L. (2009). Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective. Proc. Of CERME6, Jan.28-Feb.1, Lyon, France. <http://www.inrp.fr/editions/editions-electroniques/cerme6/> . (pp. XXXIII-LIII)
- Turnau, S. (1990). O algebrze w szkole podstawowej. In: Wykłady o nauczaniu matematyki, (pp. 67-85). Warszawa. PWN.

# INTRODUCTION TO THE PAPERS OF WG 4: GEOMETRY TEACHING AND LEARNING

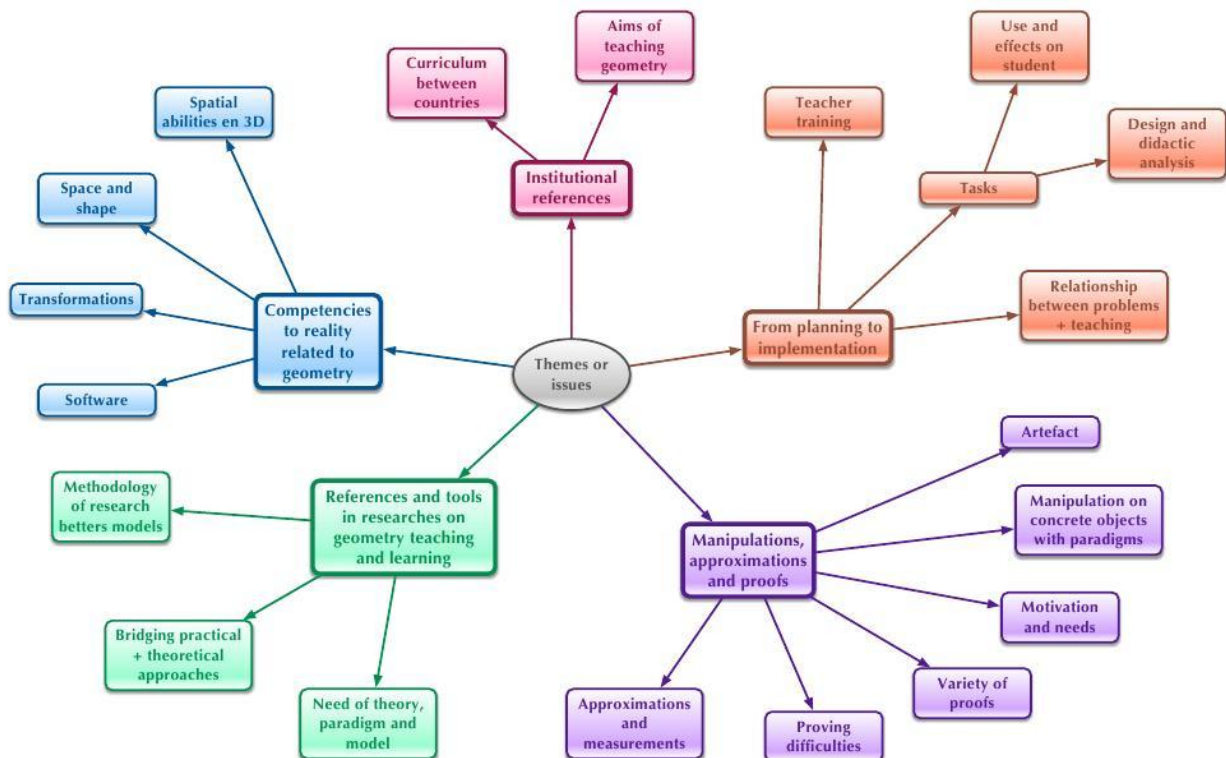
Kuzniak Alain (France),

Philippe Richard (Quebec), Athanasios Gagatsis (Cyprus), Sava Grozdev (Bulgaria)

Precious help: Kate Mackrell & Carlo Marchini

The Working Group 4 on Geometry had more than 25 participants from 12 countries all over Europe and from Quebec. During the sessions, the participants discussed 17 papers prepared for the Working Group and selected among 22 initial proposals and 15 have been retained for publication. In the continuity of the former Cerme sessions, some points can be considered as a common background and readers and future participants are invited to have a look on the former reports to know more about these points.

Some points have been developed during former sessions favoring a common approach and discussions on the following specific topics : Educational goals and curriculum in geometry, use of geometrical figures and diagrams, understanding and use of concepts and proof in geometry. More than presenting in details the work of our group, we will only give some aspects of our final discussion about the main trends involved in didactics of geometry. This can be summarized with the following colored diagram.



## **REFERENCES AND TOOLS IN RESEARCHES ON GEOMETRY TEACHING AND LEARNING**

A characteristic of the group is certainly the attention the participants give to link theoretical and empirical aspects of research in geometry education. As mentioned in Cerme6 report, two approaches of using theory in research can be distinguished: First, theory can serve as a starting point for initiating a research study. For instance, the need to empirically validate or extend specific theories may motivate an investigation. Second, theory can act as a lens to look into the data. For example, different phenomena and behaviors observed in mathematics classes may evoke ideas to the teacher or the researcher for starting research. To start from phenomena or data is a valid first approach to research. In this case, theory may enable the teacher or the researcher to better understand and interpret the collected data.

The theoretical framework is considered important when designing a research experiment. It informs the a priori analysis undertaken before the research is carried out. Undertaking an a priori analysis helps the researchers to set up a didactic situation and to predict the common mistakes and misconception students could make. After the research has been carried out the same theoretical framework is used when carrying out the a posteriori analysis. The a priori and a posteriori analyses are then compared. Did what we thought would happen actually happen?

There were a number of theories which were used by the geometry working group when analysing the teaching and learning of geometry. Different researchers often used the same theoretical frameworks in different ways. For a cognitive and semiotic approach, the Van Hiele levels, Fischbein's notion of the figural concept and Duval's registers were used. For an epistemological and didactical approach, researchers used the geometrical paradigms and geometrical work spaces described by Kuzniak and Houdement.

It was noted that researchers from different countries tend to use different approaches to research and also that teaching practices differ between countries. This can be advantageous when sharing experiences but sometimes leads to misunderstandings between researchers.

## **MANIPULATION, APPROXIMATION AND PROOF**

The reasoning is expressed by manipulating objects (material or informatics objects) or by means of the language tools (natural language, mathematical signs, figural register), and it can be performed as well during approaches of discovery (inductive reasoning, experimental proof) as of validation (deductive reasoning, mathematical proof). The mutual relations between proof and approximation were highlighted in the solving of real-life problems, from the process of modelling towards the interpretation of a geometrical solution in terms of the original problem, including identifying the limitations of the solution. Approximation raises the question of the perceptual information's limits, the reliability of the figural register and the use of

discrete models to represent continuous phenomena during some instrumented approaches with dynamic geometry software. In addition to the discretization processes, the idea of approximation appears as a fundamental tool in the conceptual construction of the geometrical objects that use measurements and the variation over time, the notion of order of magnitude, the approximate, involutorial or recursive shape in the modelling to interpretation cycle, and the dynamic equilibrium between “local” and “global” representation of the objects – for example, to perceive the infinity of the straight line with a segment, to visualise a movement with static figures, to see a 3D figure on a plane representation, etc.

Mathematical proof may be a way of facing approximations by sharing new rules for validation that are not based essentially of the material contingency or the discretization processes, so that one can leave the sensitive experience and control the results in the geometrical model.

The link between proof and manipulation appeared especially in the teaching situation and the use of the didactic materials. It is the nature and the quality of these materials that were generally approached, in particular:

- The comprehension of the space of the possibilities with the a priori or the experimented use of the material, and its feedbacks when it is interactive;
- The physical consistency of the manipulative objects, the logical coherence of the properties of these objects and the geometrical models that articulate them, of which objects defined using a dynamic geometry environment;
- The semiotic and instrumented domains of validity in the use of the manipulated objects and its relations to the geometrical models.

If the influence of the material in the proving tasks depends strongly of the teaching situation and the didactic contract, it is because the interaction of the pupil with the material is in the heart of the formation of the geometrical concepts and processes.

Finally, it seems obvious that the concepts of approximation and manipulation are intrinsically dependent of each other insofar as the approximation is used to model discrete properties on manipulation, as with measurements, and the manipulation gives meaning in the interpretation of approximation on concrete objects.

To examine manipulation, approximation and proof in the teaching of geometry led us to focus on the three interrelated aspects: using representation, the role of these representations, and the functions of proof. These three components have a strong impact on conceptualization in geometry, and on proving activity. We shall conclude that studying these three aspects closely would contribute our better understanding of complex nature of mechanisms of the learning and teaching of geometry.

## **COMPETENCIES TO REALITY RELATED TO GEOMETRY**

Under this title, we underline some aspects of the ‘geometrical eye’ which allows to identify geometry in reality and had always justified the learning of geometry. The word ‘reality’ can be discussed, and we must extend it by including the virtual reality offered by computers.

Some papers focus on geometric transformations and its relationships to movement. For more than two thousand years movement disappeared from geometry and was hidden in the concept of geometric transformation which assumed the role of movement in the exploration of a timeless space. In 1872 Felix Klein’s research was focused on the invariance by transformation and properties of geometric transformations as organizing principle of all kind of geometries. As a consequence, the concept of space depends on possible transformations. If these concepts of invariance and transformation are relevant from cultural point of view, they are also a structuring elements of geometrical knowledge at school. Various studies show that isometries are suitable also for young pupils as an appropriate way for introducing geometrical thinking. Non-isometric transformations are important too and can help students and teachers to ground their own intuition on change and movement.

The use of software in geometry (DGS) makes possible continuous and visible transformations of a drawing. However this kind of change is often not a geometrical transformation in the meaning of Klein. Nevertheless some studies show its importance for the exploration of the geometrical domain and for the learning of proof and the impact on the geometrical work space especially with impact on visualisation and use of appropriate language and representations.

### ***On geometric transformations***

1. Carlo Marchini and Paola Vighi. Innovative early teaching of isometries.
2. Edyta Jagoda and Ewa Swoboda. STATIC AND DYNAMIC APPROACH TO FORMING THE CONCEPT OF ROTATION
3. Xenia Xistouri and Demetra Pitta-Pantazi. ELEMENTARY STUDENTS’ TRANSFORMATIONAL GEOMETRY ABILITIES AND COGNITIVE STYLE

### ***From geometric transformations to teacher training***

4. Xhevdet Thaqi, Joaquin Giménez and Nuria Rosich. Geometrical transformations as viewed by prospective teachers
5. Lina Fonseca and Elisabete Cunha. Preservice teachers and the learning of geometry

### ***Spatial abilities, figure reasoning***

6. Eleni Deliyianni, Athanasios Gagatsis, Annita Monoyiou, Paraskevi Michael, Panayiota Kalogirou and Alain Kuzniak. TOWARDS A COMPREHENSIVE

#### Working group 4

##### THEORETICAL MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING and its relation with proof.

7. Athanasios Gagatsis, Paraskevi Michael, Eleni Deliyianni, Annita Monoyiou and Alain Kuzniak. Secondary Students behavior in proof tasks: understanding and the influence of the geometrical figure.
8. Annette Braconne-Michoux. RELATIONS BETWEEN GEOMETRICAL PARADIGMS AND VAN HIELE LEVELS

##### *On curriculum and general geometrical work*

9. Boris Girnat. GEOMETRY AS PROPAEDEUTIC TO MODEL BUILDING – A REFLECTION ON SECONDARY SCHOOL TEACHERS' BELIEFS.
10. Alain Kuzniak. Geometric work at the end of compulsory education
11. Caroline Bulf, Anne-Cécile Mathé and Joris Mithalal. Language in the geometry classroom.

##### *Reasoning and technology*

12. Taro Fujita, Keith Jones, Susumu Kunimune, Hiroyuki Kumakura and Shinichiro Matsumoto. PROOFS AND REFUTATIONS IN LOWER SECONDARY SCHOOL GEOMETRY.
13. Jürgen Steinwandel and Matthias Ludwig. IDENTIFYING THE STRUCTURE OF REGULAR AND SEMIREGULAR SOLIDS – A COMPARATIVE STUDY BETWEEN DIFFERENT FORMS OF REPRESENTATION.

##### *Dynamic environments*

14. Sue Forsythe. Generating shapes in a dynamic environment.
15. Kate Mackrell. Integrating number, algebra, and geometry with interactive geometry software



# INNOVATIVE EARLY TEACHING OF ISOMETRIES<sup>1</sup>

Carlo Marchini & Paola Vighi

Mathematics Department – University of Parma - ITALY

*By a teaching experiment, we introduced isometries and their practice in an experimental sample of 3<sup>rd</sup> graders (8 years old), assuming that learning and training of isometries might affect the standard geometrical understanding in primary school. Comparison with a control sample of the same grade supports our hypothesis.*

**Keywords:** Teaching experiment, Plane isometries, Primary School.

## THE ITALIAN PRIMARY SCHOOL SCENARIO ABOUT ISOMETRIES

In only one learning target of the Italian primary school new ‘curriculum’, isometries are quoted: «To recognize rotated, translated, reflected figures.» (MPI, 2007). This topic must be treated in Grade 4 and/or 5, only by observation and recognition. The official document does not pay attention to the different difficulties of diverse kinds of plane isometries (Xistouri & Pitta-Pantazi, 2011). Practice with these concepts has not been suitably developed. The official curriculum fosters a nominalistic approach to geometry, by introducing it in grade 3 (more often in grade 4 and 5) as «To recognize, to denominate and to describe geometrical shapes» and also «To measure segments by the means of both metre or arbitrary units; to connect the measure practice to the knowledge regarding numbers and operations».

## THEORETICAL FRAMEWORK

Felix Klein in 1872 reorganized geometry suggesting that the ‘content’ of its foundation is the concept of a group of transformations, therefore the learning of geometrical transformation is, at least, a cultural target. Nevertheless these subjects run into difficulties in Italian schools. Indeed, the topic is assumed to be unrelated to ‘true’ geometry (Iadrosa & Malara, 1998); it is perceived as a ‘different kind’ of geometry for which the teacher has neither traditions nor standards for its teaching and assessment, hence it is relegated to an occasional and phenomenological practice.

Research literature suggests classroom activities using isometries for middle school (e.g. Gorini (2007) and Bulf (2010)) or exploring using available software. Research about the aims, introduction, and use of isometries in primary school received little attention. Contributions of Swoboda (2005, 2006, 2007), Jagoda (2009) and Marchini & Vighi (2007, 2009) show the early presence of intuitions about isometries, from kindergarten. Marchini *et al.* (2008, 2009) analysed the presence of intuitions about isometries and continuity. We assume that standard teaching does not pay attention to the children’s attitudes, preferring geometry made by computation and formulas.

Our teaching experiment in grade 3 proposes an innovation for improving geometry learning, starting from Swoboda's outcomes. We hypothesize that by enhancing the learning of isometries the standard geometry learning will improve.

From a cognitive point of view, isometries offer a worthwhile training since they require mastery simultaneously of static and dynamic aspects (Jagoda & Swoboda, 2011). Our experiment uses artefacts which are concrete pieces of paper. They encourage a continuous development between different cognitive levels: they 'call on' geometric figures by the means of the drawings on them, taking into account its idiosyncratic features. The same tile can be 'read' differently, depending on a pupil's attention (Marchini *et al.*, 2009). We think, thence, that our approach should develop a kind of 'flexibility', i.e. the use of a variety of strategies and/or the skill of adaptive strategy choice to task specific characteristics, as a resource for mastering everyday life problems.

In the 3<sup>rd</sup> stage of the experiment we verified, by Escher's drawings, whether the practice with isometries can be extended from simple drawings to complex non-standard ones.

### **THE EXPERIMENT: Aims and planning**

Experiment aims: 1<sup>st</sup> - Are isometries a suitable topic for grade 3 pupils? 2<sup>nd</sup> - Do plane isometries learning affect the (above) 'standard' Italian school geometry?

Positive answers to these questions can support our proposal of the innovative introduction (1<sup>st</sup> aim); isometries could play a relevant role for integrating deeply the traditional teacher's practice in geometry with transformations (2<sup>nd</sup> aim) as useful tools for improving the learning of 'standard geometry' and geometrical culture.

We planned: an experimental sample (ES) (40 learners), a control sample (CS) (39 pupils), a pre-test (PT) in both samples, a treatment in ES, and the final test (FT) in both samples, one school year later [2]. The time delay between ES treatment and FT was necessary since the 'standard' geometry was introduced in grade 4 in the same way in both samples, and we need it in order to detect a possible influence of isometries on standard problems. Treatment and assessment tests were the researcher's duty; ES teachers recorded the treatment sessions, assured the discipline, and administered tests. We asked ES and CS teachers to continue her/his projected teaching, without reference to the PT. In particular, in the second school year, ES teachers avoided reference to isometries. This 'contract' aimed at the similarity of both samples.

### **THE EXPERIMENT: Pre-Test**

Table 1 is the 'portrait' of ES and CS (Table 1) offered us by the PT administered in (2008/2009) before treatment. It resumes the rate of success, and the related probability of  $\chi^2$ -Test for the statistical relevance of the score differences between

## Working group 4

samples [4]. Test consists in three sheets, here named *Shepherds*, *Pizza* and *Patterns* [3].

Sample rates	Shepherds					Pizza				Patterns					PT
	Sh <sub>1</sub>	Sh <sub>2</sub>	Sh <sub>3</sub>	Sh <sub>4</sub>	Tot	Pz <sub>1</sub>	Pz <sub>2</sub>	Pz <sub>3</sub>	Tot	Pa <sub>1</sub>	Pa <sub>2</sub>	Pa <sub>3</sub>	Pa <sub>4</sub>	Tot	Tot
ES	45	82	32	24	45	55	29	45	32	76	53	74	24	57	49
CS	12	88	9	24	33	6	0	44	13	79	73	88	48	72	43
$\chi^2$ -Test	<b>0.21</b>	43	<b>1.76</b>	99		<b>0.00</b>	<b>0.07</b>	96		80	8	13	<b>2.91</b>		

**Table 1. Results of the PT: rate of exact answers and  $\chi^2$ -test probability**

Issues employ everyday language, avoid mathematical terms, require only visual estimate regarding perimeters and area and isometries.

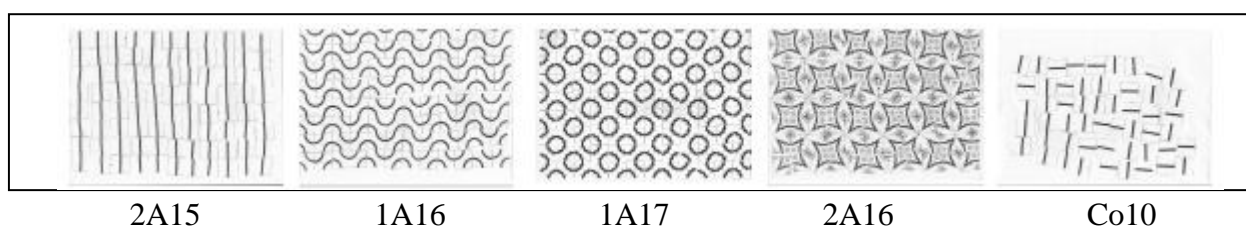
Sample	No.	Shepherds			Pizza			Patterns			PT		
		Aver. score	$\sigma$	$z$	Aver. score	$\sigma$	$z$	Aver. score	$\sigma$	$z$	Aver. score	$\sigma$	$z$
ES	38	1.82	1.05	<b>2.49</b>	1.29	1.05	<b>4.01</b>	2.26	1.14	-	5.37	2.61	1.37
CS	33	1.33	0.53		0.48	0.61		2.88	0.88	<b>2.56</b>	4.70	1.40	

**Table 2. Results of the PT: statistical test about average scores**

Table 2 aggregates results of *Shepherds*, *Pizza* and *Patterns* questions, with the statistic 'z'. Global 'portraits' of the two samples do not show relevant differences ( $z = 1.37$ ). In PT difference in patterns issues is statistically probable favourable to CS children. It can be surprising since the right solutions require a 'sensitivity' to isometries such as intuition of translation for Pa<sub>1</sub> and of rotations for tasks Pa<sub>2</sub> and Pa<sub>4</sub>. This 'feeling' could be useful for solving tasks Pz<sub>1</sub> (translation/symmetry), Sh<sub>1</sub> and Sh<sub>3</sub> (rotation), but the results of these issues show a statistically relevant superiority of ES children.

## THE EXPERIMENT: Treatment for the experimental sample

The treatment was planned for six sessions of two hours each, one session per week, in three stages, with non-quantitative approach. Its realization lasted 16 hours.



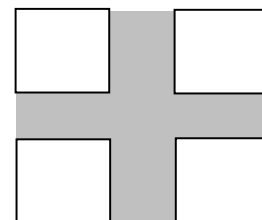
**Figure 1. Treatment protocols in the order of their presentation to ES classrooms**

*The 1<sup>st</sup> stage* (6 hours). In sessions 1 and 2, ES pupils discussed in groups the first four documents of Figure 1, realized by other pupils in another experiment (Marchini & Vighi, 2007) in which it is possible to individuate the use of isometries; for each one they recorded the worth aspects, they assessed protocol, and they presented orally their 'conclusions'. Afterwards, Carlo made an 'institutionalisation' activity, drawing on the black board four squares as in Figure 2, reproducing by a schematic way the four consecutive tiles as they appear in the upper left corner of the document. Carlo asked eventual 'tie-in' between two consecutive tiles, for recognizing rotated, translated, reflected figures (MPI, 2007). For 2A15, in a class the word *trasloco* (move)

came out immediately. For 1A16, in both ES classes, the word *specchio* (mirror) came out without difficulty, looking at horizontal tiles disposition. Hence, during the 1<sup>st</sup> session the presence of translation ( $T$ ) and (axial) symmetry ( $S$ ) in the construction of protocols were detected. Pupils realized that  $T$  works in 2A15 both from left to right and from up to down, in 1A16,  $S$  acts from left to right and  $T$  going down.

During the 2<sup>nd</sup> session, pupils analysed the 3<sup>rd</sup> and the 4<sup>th</sup> protocols. In the ‘institutionalisation’ phase, they found in 1A17 the application of  $S$  in the horizontal and in the vertical. Pupils’ analyses of 2A16 agreed that its author made a ‘mistake’. The protocol was chosen on purpose, for making evident the presence of a rule in the construction of the protocol, by the means of its violation. The different appearance of protocols 1A17 and 2A16 could hide the fact that the construction rule is the same. In the ‘institutionalisation’ phase this identity came out. In Vigatto school pupils suggested that the mistake was the fact that a tile was turned. Carlo seized the opportunity to introduce rotation ( $R$  - *rotazione*), avoiding the discussion of Co10. During 3<sup>rd</sup> session, in Vicofertile school, the discussion of Co10 allowed the introduction of rotation.

*The 2<sup>nd</sup> stage* (4 hours). It aims at the consideration of isometries as mathematical objects. We prepared a card game: three ‘playing’ cards with the letters  $T$ ,  $S$  and  $R$  in a jar, and the game board of Figure 2 as an array of two times two squares (on the blackboard). The four squares determine a cross (it is grey in Figure 2). Now the rule is to



**Figure 2. The game board**

draw, with restitution, four times one card at a time and to write the letter of that card in the empty cross arms, in this conventional order: top - left - right - bottom. Finally a tile is placed (or drawn) in the top left board square. Pupils copied the stuff in their exercise-book. We gave a homework task with the sequence of letters  $R$ ,  $S$ ,  $R$  and  $T$ , and a tile of the type used in 2A16. Children proved the task was impossible. To single out simple instances of impossible tasks is an effective way to introduce pupils to control procedures. In the homework discussion, pupils suggested possible different rotations of multiples of ‘one quarter clockwise rotation’. Therefore the cards  $R_1$ ,  $R_2$  and  $R_3$  were added in the urn instead of  $R$ . The playing with the new card game concluded the 2<sup>nd</sup> stage. In this way pupils produced protocols on the basis of simple [5] rules (Marchini & Vighi, 2011) showing a good mastery of isometries. The card game with isometries could be used in every school environment to introduce isometries, both as procedure and mathematical objects. In our experiment we avoided ‘structural’ properties of functional composition, but these topics can be useful in other grades.

*The 3<sup>rd</sup> stage* (6 hours). Paola recalled plane isometries through Escher’s paintings: Escher’s 28 (shortly  $E_{28}$ ) for translations;  $E_{79}$  for rotations and  $E_{12}$  for symmetry.  $E_{55}$  presents shapes suitable as a summary of previous plane isometries. The aim was to attach attractive and affective aspects to transformations. For each drawing Paola asked children, in sequence, “What can you see in this drawing?”, then to individuate

with letters or colours many figures obtained from a starting figure by a suitable isometry. In particular, for rotation, she required to individuate ‘rotation centres’. The shape complexity hampered only a few pupils.

### **THE FINAL TEST: Analysis of FT**

We chose, on purpose, a test which is far enough (in time and topics) from the teaching of isometries. The tasks can be considered suitable for children having a standard teaching of geometry; nevertheless the issues require geometrical thinking since straightforward applications of rules for perimeter and area are not enough.

The FT (in Enclosure) was administered in 50 minutes at the end of grade 4 in school year 2009/2010. It was inspired to some PT items. The issues of the test were problematic for children, since they learnt perimeters of rectangles and a few about area. The leading idea was to assign problems about three roughly ‘rectangular’ figures in which we gave the measure of the length for some segments (represented in proportion); for solving them the application of isometries can be useful or necessary. Some data are missing; they can be found by geometrical thinking and arithmetic computation (with an implicit didactic contract suggesting that what looks like a rectangular shape is a rectangle, or congruent-like parts of the same shape, are congruent). In our opinion, the identifying the missing data requires a sort of deduction in a ‘natural axiomatic’.

For perimeter of Shape<sub>1</sub>, six data are given and two are missing; for perimeter of Shape<sub>2</sub>, eight data are given and four are missing [6]. Therefore the computation of the perimeter involves long addition with decimals. In the first case it is necessary to solve the equation  $x+y = 5.0+1.5$ ; in the second case, the solution is found by the means of the equations  $6.5 = 3.0+x+1.5$  and  $1.0+2.0+y = 4.5$ . The drawings help to avoid algebraic computations since evidence suggests solutions with the help of a deduction in a ‘natural axiomatic’.

For Area<sub>1</sub>, it could be useful to add  $5.0+1.5$ , and then to make  $4.5 \times 6.5$ . Otherwise the shape can be divided into two rectangles. For Area<sub>2</sub> and Area<sub>3</sub>, all the necessary data are given, and searching the missing data can only corroborate the hypotheses of congruence of some pairs of pieces from the same shape. The difficulty of computing Area<sub>3</sub> can be solved correctly in two different ways.

16 The solution can be found only by insight (Divišová & Stehliková, 2010) i.e. by an intuition of congruence, similar to the one required in items of PT, since in Shape<sub>3</sub> there is a semi-circular part for which area 4<sup>th</sup> graders do not know a formula. This way requires only the computation of  $4.5 \times 6.5$ . Thence pupil can return back to Area<sub>1</sub> and Area<sub>2</sub>, recognizing a local isometry [7] which can be applied to small rectangles.

17 The simplest way of computing areas Area<sub>1</sub> and Area<sub>2</sub> is to think that all the shapes

become equal to a rectangle  $4.5 \text{ cm} \times 6.5 \text{ cm}$ , by shape suitable decomposition. Congruence of parts of each shape can be proved by appealing to local plane isometries. The direct computation of  $\text{Area}_1$  and  $\text{Area}_2$  could be considered a sort of ‘distractor’, even if it, with the sameness of results, can give the good hint for  $\text{Area}_3$ , by recognizing the role of local isometries for semicircles.

A different analysis of the results is in (Vighi & Marchini, 2011).

### THE FINAL TEST: Results of FT

We assessed FT protocols in various ways. The simplest is to assign the score 1 for the correct numerical value and 0 for the wrong or missing numerical value, as a measure of the understanding (Kilpatrick, 2009). Another kind of data is the average number of children which try to solve the problems. We consider an attempt as a positive behaviour towards the topics, so we label it as ‘confidence’. Lastly we look at solving procedures, disregarding possible mistakes, the ‘competence’ (Godino, 2003). We considered a right procedure the product  $4.5 \times 6.5$ , or for  $\text{Area}_1$  and  $\text{Area}_2$ , a suitable shape decomposition in rectangles with the corresponding products. As to perimeters we considered a right procedure when all the missing data are found and summed to the given one, or even if one of them was forgotten, for scarce attention, but the finding of the other missing data is a sufficient proof of competence.

Average	Perim <sub>1</sub>	Area <sub>1</sub>	Perim <sub>2</sub>	Area <sub>2</sub>	Area <sub>3</sub>	Perimeters	Areas	Total
ES score	0.18	0.13	0.11	0.24	0.18	0.29	0.55	0.84
CS score	0.27	0.06	0.03	0.09	0.03	0.30	0.18	0.48
Statistical test	$\chi^2=37$	$\chi^2=32$	$\chi^2=22$	$\chi^2=10$	$\chi^2=4.08$	$z = -0.10$	$z = 1.91$	$z = 1.34$
ES confidence	0.82	0.79	0.82	0.76	0.76	1.63	2.32	3.95
CS confidence	0.94	0.67	0.85	0.39	0.36	1.79	1.42	3.21
Statistical test	$\chi^2=12$	$\chi^2=24$	$\chi^2=71$	$\chi^2=0.16$	$\chi^2=0.07$	$z = -1.03$	$z = 3.39$	$z = 2.06$
ES competence	0.32	0.29	0.26	0.39	0.42	0.58	1.11	1.68
CS competence	0.39	0.12	0.24	0.12	0.06	0.65	0.30	0.91
Statistical tests	$\chi^2=49$	$\chi^2=8$	$\chi^2=84$	$\chi^2=0.94$	$\chi^2=0.05$	$z = -0.14$	$z = 3.35$	$z = 2.32$

**Table 3: Results of the FT: Average values and statistical tests**

The low average score in Table 3 results from many mistakes in computations and/or in procedures. We aggregate the scores for perimeters, areas and global. Without another table of quantitative data, we can conclude by ‘z’ statistic that differences between the scores of the two samples are not statistically significant.

Table 3 shows a big difference between confidence, competence and understanding of the topics, using these names in our meaning. CS children are more confident about perimeter, but the difference is not statistically significant. Moreover ES children show a greater confidence with area than the CS children, and this difference is statistically significant. The  $\chi^2$ -test states that this fact is due to the confidence with  $\text{Area}_2$  and  $\text{Area}_3$ . With data aggregation (perimeter+area), samples difference in confidence is statistically probable only for  $\text{Shape}_2$ ; difference in competence for

Area<sub>2</sub> and for Area<sub>3</sub> are statistically significant.

### **THE FINAL TEST: Analysis and interpretation of the FT results**

In school year 2009/2010, in both samples, measure of length, perimeters for triangles and rectangles were taught and area was introduced briefly. The CS teacher also dealt with isometries. This substantial knowledge ‘sameness’ of both samples is confirmed by the global FT results of Table 3 for Shape<sub>1</sub>, which is closer to standards.

ES confidence presents similar results for perimeter and area. For CS, instead, the differences about perimeters are statistically significant, both for confidence and competence. Moreover there are statistically probable differences for confidence with Area<sub>1</sub> and Area<sub>2</sub>, and for Area<sub>1</sub> and Area<sub>3</sub>.

The greater number of missing data could be an obstacle for determining Perim<sub>2</sub>: from Table 3 it seems that the reductions in average scores should be provoked by mistakes in the sum because of the number of addends. Only one protocol seems to determine smartly Perim<sub>2</sub> by adding 1 cm (or 0.5 cm twice) to Perim<sub>1</sub>.

Few protocols used an explicit procedure. We got 7 protocols (6 of them in ES) in which pupils obtained the same results all for the three problems with area (even if wrong). As regards scores, the number of right computations is lessened: we detect 4 cases, 1 of them in CS. Equality of Area<sub>2</sub> and Area<sub>3</sub> only is affirmed by 3 ES pupils and 1 CS child.

An ‘isometric thinking’ can be present in 10 ES pupils since at least one perimeter is computed wrongly by the sum  $4.5 + 6.5 + 4.5 + 6.5$ . In fact, 5 of them applied the correct procedure at least once for area (4 of them for both areas). They seem to be aware of isometries, but unaware that local isometry preserves area, but does not preserve perimeters.

In Area<sub>2</sub> and Area<sub>3</sub> are involved local roto-translations; for Area<sub>1</sub>, instead, we can consider local translation or local axial symmetry (as stated explicitly by one CS pupil). It could be relevant to the fact that the application of procedure identifying Area<sub>2</sub> and Area<sub>3</sub> is the most frequent (10 ES, 1 CS) and when a pupil individuates the equality of Area<sub>1</sub> (the simplest) and Area<sub>3</sub> (the most difficult), then s/he possibly comes back for obtaining the same result for all the shapes. On the basis of the previous remarks, of Table 3, and by the fact that children can determine Area<sub>3</sub> only by intuition of a local roto-translation, we can state that in CS a method based on local translation / symmetry is applied more than the one requiring roto-translation. The ‘equality’ of average confidence for Areas in ES could be justified with the previous learning of isometries.

We cannot exclude the point that even if the same school topics were presented in both samples, other factors could affect the results.

### **COMPARISON BETWEEN PT AND FT**

We can compare the results of FT and PT taking in account the ‘sameness’ of intuition/knowledge necessary for solving the tasks. The relevant differences, mainly for area questions, between the two samples in FT can follow from an evident initial difference in PT results as regards to problems from which FT issues were inspired. Notice that in the PT we did not ask quantitative results, therefore we think as unsuitable to compare PT with the FT scores. The difficulties of the passage from qualitative to quantitative can justify some results of Table 4.

A qualitative treatment can be assessed with children’s confidence and competence. Thence we compare the PT and FT looking at the average number of children who improved (equalled, made worse) their performance from PT to FT [8]. The last two columns of Table 4 are obtained by aggregation of all FT task results the PT task results.

Average no. children	Perim <sub>1</sub> & Perim <sub>2</sub> /Sh <sub>4</sub>		Area <sub>1</sub> /Pz <sub>1</sub> & Pa <sub>1</sub>		Area <sub>2</sub> & Area <sub>3</sub> /Sh <sub>1</sub> , Sh <sub>3</sub> , Pa <sub>2</sub> & Pa <sub>4</sub>		FT/PT	
	Confid.	Comp.	Confid.	Comp.	Confid.	Comp.	Confid.	Comp.
ES improve	0.66	0.29	0.42	0.11	0.76	0.39	0.82	0.29
ES equal	0.29	0.53	0.45	0.32	0.08	0.21	0	0
ES worsen	0.05	0.18	0.13	0.58	0.16	0.39	0.18	0.71
<b>ES balance</b>	<b>0.61</b>	<b>0.11</b>	<b>0.29</b>	<b>-0.47</b>	<b>0.61</b>	<b>0.00</b>	<b>0.63</b>	<b>-0.42</b>
CS improve	0.70	0.33	0.64	0.12	0.36	0.09	0.36	0.06
CS equal	0.24	0.45	0.09	0.21	0.24	0.18	0	0
CS worsen	0.06	0.21	0.27	0.67	0.39	0.73	0.64	0.94
<b>CS balance</b>	<b>0.64</b>	<b>0.12</b>	<b>0.36</b>	<b>0.06</b>	<b>-0.03</b>	<b>-0.64</b>	<b>-0.27</b>	<b>-0.88</b>

**Table 4. Comparison of FT and PT: Average number of change in performance**

Negative sign in Table 4 is a warning: the ‘intuitive’ test gave a better result than the quantitative task. This fact could be a consequence of little attention to a practice promoting the evolution of child’s idea towards a more complete knowledge.

## CONCLUSION AND DISCUSSION

The research has clear aims, but during its implementation we faced other issues:

- *Does practice with non-conventional shapes help pupils in FT tasks? b) Does the treatment improve flexibility in our meaning? c) Does the treatment improve pupils’ performance in FT tasks?*

Research 1<sup>st</sup> aim had a big number of corroborations, during the treatment (Marchini & Vighi, 2011) and also with the permanence of the taught concepts one year later e.g. by words or drawings mention of isometries in the 50% of ES pupils. Only 5 CS pupils prove their acquaintance with isometries; they testify that the same arguments were introduced in their CS class and, by results comparison, the relevance of treatment in ES. Therefore the 2<sup>nd</sup> aim of our research has been achieved.

Area problems distinguish the most (Table 3) the confidence difference between samples. The ‘distance’ of our shapes from school practice can be measured by the



average confidence, which values diminish from Shape<sub>1</sub> to Shape<sub>3</sub> for CS. The same values are nearly constant for ES pupils which show familiarity with complex shapes. Perimeter and area of FT shapes cannot be found by application of ‘one’ rule. They could block diligent pupils able to solve standard exercises. The FT tasks require insight (Divišová & Stehliková, 2010) or more flexibility and an inventory of geometrical tools going far beyond of the simple formulas for rectangles. Flexibility also has been helped by practice with isometries and complex and non-regular shapes. It is worth the improvement of competence of ES in comparison with CS for area questions (Table 3). Therefore issues a) and b) have positive answers.

Issue c) has a more complex answer. The samples present many differences which are favourable to ES versus CS, even if, often, without a statistical relevance. Table 3 affirms that ES pupils show a greater confidence with FT questions since there was an improvement of performance from PT to FT (Table 4). In this sense the treatment had a good effect. We can assume ES children were in better position for connecting new and treatment knowledge (Mayer, 2002). But the competence performances (Table 4) do not support this statement, even if diminution is favourable to ES. Therefore we could conclude that there is a wide field of research to be investigated assessing our issue c).

## NOTES

1. Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Mathematics Department, Parma University, Italy.

2. We thank teachers Ferrarini (Vigatto - PR), Tomasini (Vicofertile - PR) for their participation to ES. and Mancastropa (‘Corazza’ of Parma) for CS. From PT to post-test (here named *final test*, for a distinct acronym, FT) the samples changed for the absence of some pupils. Our samples for the statistics are reduced to the ES 38 pupils and the CS 33 children which took part to all PT and FT activities

3. The PT presented three sheets (30 minutes each) which were administered in different days. The issues are freely inspired from literature: Shepherds from Marchetti *et al.* (2006), Pizza from Vighi (2010) and Pattern from I.Q. folklore. Notice that Vigatto schoolboys treated the original Vighi (2009) issues in the school year 2007/2008 (when they are 2<sup>nd</sup> graders). We assume that difference of questions and elapsed time made this previous experience irrelevant.

4. In the tables we single out with boldface font the relevance of a statistical test.

Datum is	In $\chi^2$ -test probability	In statistic z
Statistically significant	$p \leq 1\%$	$z < -2.58$ or $2.58 < z$
Statistically probable	$1\% < p \leq 5\%$	$z < -1.96$ or $1.96 < z$

5. I.e. the rule which is explicated for the first four tiles is extended to the whole protocol (cf. Marchini & Vighi, 2011).

6. With Perim<sub>n</sub> (Area<sub>n</sub>) we refer to the task of computing perimeter (area) of Shape<sub>n</sub>.

7. With ‘local isometry’ we want to consider a bijection such as some part of the figure remains fixed and some other parts of the same figure are isometric. Thence a ‘local isometry’ could be globally an example of a non-isometric transformation. The Enclosure examples can explain this concept.

8. The comparison of FT with PT is realized as follow. For each pupil the change of performance in confidence (in competence) is given by the sign of difference between the sum of results of FT task, and the corresponding PT issues, both normalized at 1, dividing by the number of tasks.

## REFERENCES

Bulf, C. (2010). The effects of the concept of symmetry on learning geometry at French Secondary School. *Proceedings CERME 6*, 726 - 735.

- Divišová, B. & Stehliková N. (2010). Ability to see in geometry or a geometric eye. In Maj, B., Swoboda, E. & Tatsis, K. (Eds.) *Motivation via natural differentiation in mathematics*, Rzeszów, WUR, 131- 140.
- Godino, J.D. (2003). Competenza e comprensione matematica: che cosa sono e come si ottengono, *La Matematica e la sua Didattica*. 1, 4-16.
- Gorini, A. (2007). Scoprire con le isometrie. *L'insegnamento della matematica e delle scienze integrate*. vol. 30a n. 1, 57 – 70.
- Iaderosa, R. & Malara, N.A. (1998). Sulle difficoltà incontrate da allievi di 12-13 anni nell'apprendimento delle isometrie piane, *L'insegnamento della matematica e delle scienze integrate*, Vol. 21a, n.3, 235 – 255.
- Jagoda, E. (2009). Substantial learning environment “Tiles”, *Didactica Mathematicae* **32**, 121 -147.
- Kilpatrick, J. (2009). Conceptual understanding as a strand of mathematical proficiency, *Proceedings SEMT '09*, 42 – 53.
- Marchetti, P., Medici, P., Vighi, P. & Zaccomer, E. (2006). Comparing perimeters and area. Children's pre-conceptions and spontaneous procedures. *Proceedings CERME 4*, 766 – 776.
- Marchini, C., Swoboda, E. & Vighi, P. (2008). How to reveal geometrical independent thinking in the lower primary years, in Maj, B., Pytlak, M. & Swoboda, E. *Supporting independent thinking through mathematical education*, Rzeszów: 82 – 88.
- Marchini, C., Swoboda, E. & Vighi, P. (2009). Indagine sulle prime intuizioni geometriche mediante ‘mosaici’. Un esperimento nella scuola italiana. *La Matematica e la sua didattica*, **23**, n.1, 61-88.
- Marchini, C. & Vighi, P. (2007). Geometrical shapes as a revealing tools - a phenomenological/quantitative analysis of protocols, *Proceedings CERME 5*. 1032 – 1041.
- Marchini, C. & Vighi, P. (2009). Can we develop geometrical understanding by focusing on isometries? A teaching experiment by the means of geometrical artefacts. *Proceedings SEMT '09*, 169-176.
- Marchini, C. & Vighi, P. (2011). Can we develop geometrical understanding by focusing to isometries? Poster, CERME 7.
- Mayer, R.E. (2002). Rote versus meaningful learning. *Theory Into Practice*, **41**, 226 – 232.
- Swoboda, E. (2005). Structures of Van Hiele's visual level in work of 5-7 years old children', *Proceedings SEMT '05 – International Symposium Elementary Maths Teaching*, 299 – 306.

#### Working group 4

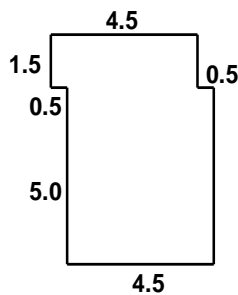
- Swoboda, E. (2006). *Przestrzeń, regularności geometryczne i kształty w uczeniu się i nauczaniu dzieci*. Rzeszów (Poland): Wydawnictwo Uniwersytetu Rzeszowskiego.
- Swoboda, E. (2007). Intuition of geometrical relations in plane in works of 4-7 year old children, *Proceedings SEMT '07. International Symposium Elementary Maths Teaching*. 249-257.
- Vighi, P. (2010). Investigating comparison between surfaces. *Proceedings Cerme 6*, 716 – 725.
- Vighi, P. & Marchini, C. (2011). The gap between Learning and Teaching *Geometry, These Proceedings*.

---

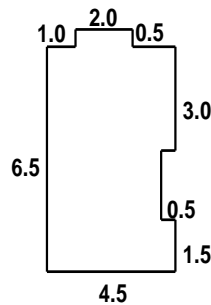
#### Enclosure

Calculate perimeter and area of Shapes 1 and 2 (measures are in centimetres). Then explain your solution.

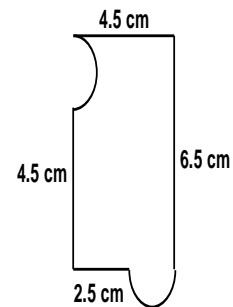
Calculate the area of Shape 3



Shape<sub>1</sub>



Shape<sub>2</sub>



Shape<sub>3</sub>

# STATIC AND DYNAMIC APPROACH TO FORMING THE CONCEPT OF ROTATION

Edyta Jagoda<sup>1</sup>, Ewa Swoboda<sup>2</sup>

<sup>1</sup>Charter School No.1 Rzeszów, <sup>2</sup>University of Rzeszów, Poland

*Our research was based on arranging many different didactical situations that could be the source of intuitions for different geometrical transformations. In this case study, we deal with the concept of rotation. On the basis of different experiences, we were looking for the answer to a research question: how should we create a coherent picture of a geometrical transformation that enables us to understand the concept both statically and dynamically. The analysis of students' behavior displayed a number of common properties related both to static and dynamic understanding of the concept. Our work also reveals that intuitions can contain elements which are inconsistent with the definition of rotation*

Key-words: isometries, geometrical intuitions, rotation.

## INTRODUCTION

According to Piaget, in the case of logical – mathematical concepts, we encounter the interplay of operations, separated not from the perceived objects, but from the actions taken on them (Piaget & Inhelder, 1999). In Piaget's view, the transformation of reality is of fundamental significance and action is the tool for that transformation. In accordance with this idea, perception (vision) influences the formation of logical – mathematical thinking to a small degree; including geometrical thinking. To support such a standpoint, Aebli quotes Piaget's views:

Investigating what activity is itself, we repeatedly verified its great importance contrary to the importance of an image. Geometrical view is, indeed, an active one as it mostly consists of potential actions, shortened schemata of effective actions or anticipatory schemata of future actions: in case of the lack of effective action, the view is inadequate (Aebli, 1982).

Uncritical implementation of Piaget's views in the field of geometry methodics raises many objections worldwide (Clements, Battista, 1992; Clements et al., 1999, van Hiele, 1986). It is generally believed that the development of geometrical concepts is different from that of arithmetical ones (Gray, Pinto, Pitta & Tall, 1999; Hejný, 1995, Vopěnka, 1989). The process of forming geometrical concepts has been the focus of a number of theories, of which van Hiele's (1986) is the most popular. He describes the first level of understanding as "visual", connected with non-verbal thinking. At this level the emphasis is placed on the ability to recognize shapes, which are perceived as a 'whole' and connected with visual prototypes. Not much is mentioned about the role of action, although a didactical conception of the theory suggests activities with objects (de Lange, 1987).

### **Static Arrangement Figure To Figure**

If we accept the fact that the visual is of significant importance in the first level of geometrical cognition, we also have to consider psychological provisions concerning cognition. Results of psychological research (Kaufman, 1979) confirm that in the process of grasping shapes pictorial designates are of great importance. In addition to that, dominance of the whole over the part is the regularity in perceiving shapes. The rules of structuring an image investigated in view of the information analysis system suggest that regular, symmetrical forms and shapes are the most easily recognized, as one element can be predicted from another (Grabowska & Budohoska, 1992). Regularities, groups creating some logical wholeness can be elements of a composition regulated through visual perception.

W. Demidow (1989) gives a broad account of the research conducted by physiologists concerning the mechanisms governing the recognition of shapes. We can also find there information about invariant transformations conducted by our eyesight. For example, pictures of different sizes are invariant (unchangeable) to the organ of sight (the eyesight identifies them), and the same happens while changing the position of an object - but only up to 15 degrees. The mirror image is not invariant even though children are born with such a property of perception; as humans develop, the eye loses the invariance of mirror images.

These remarks have an essential meaning in a geometrical environment, referred to as 'patterns'. Creating bands or mosaics was unequivocally assessed by van Hiele as operating on the visual level that did not require internalization of actions. He refers to the structures of the first level as optical, structures of appearance; they are manifested in recognizing regularities or certain wholeness. According to this theory all perceived regularities are classified as visual structures. The things that inspire children, propel them to action and which undergo control and are reflected upon are: rhythm, order and regularity. Such action seems to be in accordance with the original meaning of Greek 'symmetros' which stood for 'harmonious', 'well-proportioned'.

The aforementioned sense of order tends to be verified visually by children. During the creation of geometrical compositions, the creative process is regulated by perception. Hence, propaedeutics of geometrical figure to figure relationships may reside in the sense of certain order, harmony - specific arrangement of a surface or available fragment of space.

This leads us to the conclusion that in situations where balance is present, stemming from an appropriate arrangement of elements that constitute an image, there is no need to introduce movement. Children working in an environment of visual regularities do not resort to the idea of movement, placing one object onto the other. This interpretation resembles the assessment of the mosaics that have been created by humans from the earliest days of history. According to some historians of this

discipline, mathematical relations can already be identified in the geometrical decorations of items created by the late ice-age man. In the book by Kordos (2005, p. 23) we read that:

It is worth paying attention to the richness of geometrical forms used in decorations. In particular, it is worth seeing that the ribbon ornaments from the Neolithic period all had 7 one-dimensional crystallographic groups on the surface. (...) However, we cannot be certain that some kind of geometrical reflection was followed.

Therefore, it seems that recognition of a specific figure to figure position is only a static image of this relationship, not connected with the movement of one object onto the other. On a certain level and in a certain context, it will be a rather general depiction, adjusted by the perception of certain regularity.

### **Dynamic Understanding**

The understanding of relationships between figures as a dynamic arrangement of space is placed, so to say, at the opposite pole. Acts of perception are important but they are not a sufficient source of geometrical cognition. Szemińska (1991, p.131) states that perception gives us only static images; through these we can only catch some states, whereas by actions we can understand what causes them. It also guides us to the possibilities of creating dynamic images.

The history of mathematics as a scientific discipline shows the importance of the transition from a static to a dynamic interpretation of geometrical objects (Kvasz, 2000). This can be seen in Greek mathematics, in which the traces of general reasoning were based on dynamic object transformations. A significant part of this geometry was based on constructions, which – in an indirect way – required the use of translations, rotations and mirror reflections. The overt description of symmetry as a transformation appeared rather late in mathematics – as it can be linked to the Erlangen Programme of F. Klein – but the dynamic approach itself is crucial for geometry. Geometrical reasoning requires mental transformation of objects.

In order to understand translation, rotation or axial symmetry as a transformation it is necessary to conceive the specific movement that is transforming the initial figure into the final one. From a didactic point of view, it is important that such conception stems from mental reflection on the phenomenon of movement. This is the idea of transformations as a function. If we want to trace the origins of this concept in the realm of physics, physical movement of an object will be suitable. Nevertheless, such transformation happens in a given time and while making a movement and we can possibly trace the trajectory of an object. Everyday experience does not offer the possibility of recording consecutive stages of the object's movement. On the other hand, Piaget's widely known results (Piaget & Inhelder, 1973) show that children (on the pre-operational level) have great difficulties in movement reproduction – they are not able to foresee a movement of an object in space. The process of acquiring such skills is lengthy and gradual (Szemińska, 1991). During

manipulations, the child's attention should be focused on *action*, not on the *result of action*. It requires a different type of reflection than the one that accompanies his or her perception.

Such a problem is unnoticeable in school practice. Generally it is stated that creating their own patterns is a good starting point for children's understanding of transformations. In some handbooks for teachers, there are suggestions to do *exercises with changing the figure position such as drawing patterns, mosaics where translation, rotation and mirror symmetry are used...* Jones, Mooney, (2003), while analyzing school curricula in United Kingdom states, that *the link between symmetry and the various transformations is not always made explicit. In the "Framework", for instance, rotation appears to be considered solely as a transformation and the opportunity is missed to extend this to include rotational symmetry, even though the latter is specified in the statutory National Curriculum.*

## RESEARCH ORGANIZATION

Our research was based on arranging many different didactical situations that could be the source of intuitions for different geometrical transformations. Tiles were the basic tool for all situations. Patterns imprinted on tiles were different but they all had one general rule: on a piece of paper, one has to create an arrangement of tiles. Pupils from different age groups – from 5 year old children to gymnasium students took part in the research. This was a multistage research that started in 2002 and lasted until now. The organization of various stages was different. In some situations the work of a large group of students was analysed (more than thousand pupils), in the others – we observed lessons with only 20 students. Children worked at groups, during their regular activities. Some working sessions were videotaped. We observed and analyzed children's behavior during their work and we analyzed their worksheets. On this basis, we estimated to what level classes that we propose can be treated as the basis for creating the picture of specific isometries by children.

In this case study, we will deal with the concept of rotation. We will present examples of results and commentaries connected with the following didactical situations:

- 18 Creating a tiled floor by 4-6 year old children
- 19 Creating a free tile composition by 10 – 13 year old children
- 20 'Guided puzzle' with a suggested subject area and musical background.
- 21 'Domino' task ( 12 year old children )

The first two situations concerned static situations, and the following two were associated with movement.

Research questions were as follows:

1. What intuitions connected with rotation will appear either in static situations or dynamic ones?
2. Which of those intuitions should we enhance and which should we discourage during the further stages of mathematical education?
3. On the basis of different experiences, how should we create a coherent picture of a geometrical transformation that functions as the one that enables us to understand the concept both statically and dynamically?

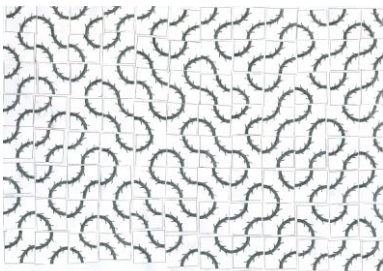
## RESEARCH AND OBSERVATION RESULTS

In children's work, intuitions of rotation appeared in different forms. Each of them stressed a different rotation property that together exist in a mathematical understanding of rotation on a plane.

### STATIC SITUATIONS

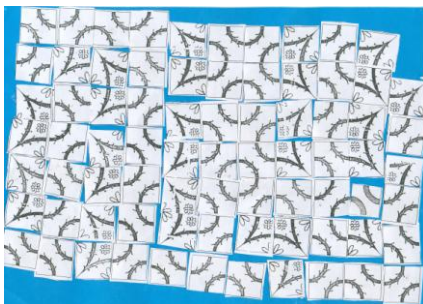
#### A. Puzzles – filling the floor

- a) Arrangement of one *figure to* another using a particular angle.

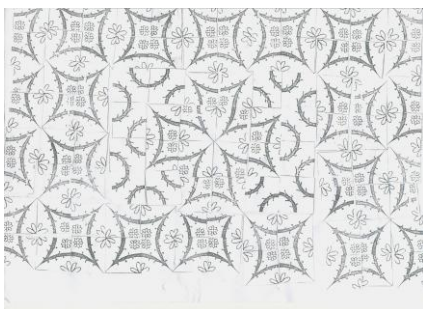


Ania, 6 years old. In this work, the child tried to fill the plane with congruent figures arranged one to another using a particular angle. The size of the angle was dependent on the shape of the tool but it is clearly seen that the child was interested in a frequent change in the basic figure's position.

- b) Arrangement of figures around a particular centre. Figures in this arrangement do not have to change their positions in relation to privileged directions. They only 'surround' one chosen element.



Julia, 6 years old. The process of gluing started from the central part of the paper sheet. Then, the girl tried to surround the circle with bells but she managed to do it only partially. In the following part, she focused on different regularities.

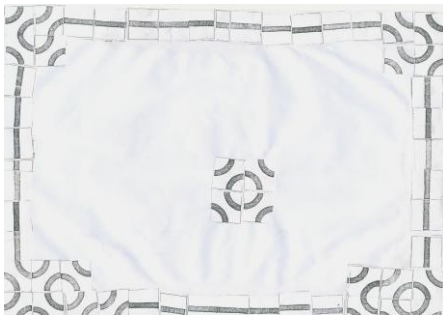


Milosz, 6 years old. A four-leaf clover was a central figure in his composition. The rest stressed the central figure, surrounding it with contrasting elements and closing it within symmetrical frames.



## Working group 4

c) ‘Along the edge’ arrangement, directed inwards. In certain places, especially at the corners of a sheet of paper, motifs are created. They are directed to a common point of the inside – diagonal intersection

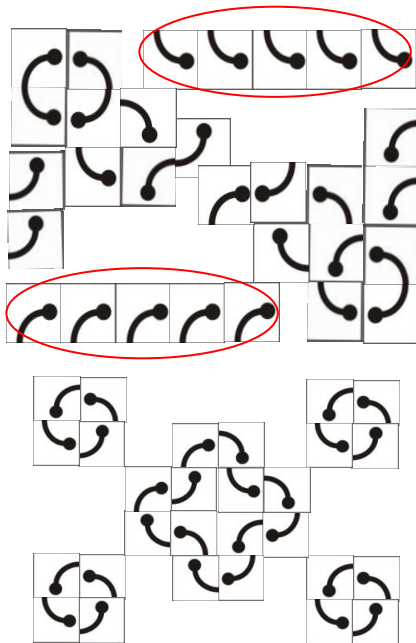


Boy, 7 years old. He started gluing from the frame. He managed to construct two elements in the upper parts such that they were symmetrical to each other. However, despite visible attempts to create corresponding elements, he could not repeat the same thing for the lower parts. In the central part, he only placed a symmetrical figure consisting of 4 tiles.

B.

### Puzzles – free activities

Older pupils, whose task in the initial stage was to arrange a free tile composition, worked in similar ways to those described above. Much of their work spontaneously realized the idea of symmetry – axial symmetry was the most frequent one. Going ‘beyond’ such arrangement did not happen often. In the observed group there were only two works where we can find traces of the idea of rotation.



Krzysiu, 13 years old. The pupil started his work from circles placed in the opposite corners of the paper (but the placement of dots is not correct). Then, he diagonally moved inwards. At the end, he glued elements at the corners. The placement of glued elements shows that point symmetry was present in his arrangement strategy.

Ala, 13 years old. Her work is an example of a perfect rotation, while the parts of the puzzle have an opposite orientation (the central part – clockwise orientation, circles in the corners – anticlockwise orientation). Rotation was present in the girl's arrangement strategy from the beginning because it is impossible to see axial symmetry in her puzzle.

Both of these works can be classified by categories distinguished in 6 year old children's work. The first of these reflects the ‘along the edges’ arrangement idea, where the corresponding elements are in the opposite corners of a rectangular sheet of paper. Some deviations from the arrangement reflecting real rotations (elements marked with an oval) indicate that the child, during her work, did not make any manipulations of a sheet of paper to check the arrangement of elements.

The second work is dynamic but compositionally close to the ‘around the centre’ arrangement. A closed central composition arranged according to a rotational movement by a closed wavy line sticks out in the foreground. Rotation also organizes four smaller sets surrounding the centre, but the direction is different. It is clearly seen that movements are local. The whole composition is enclosed within a symmetrical area determined by a sheet of paper.

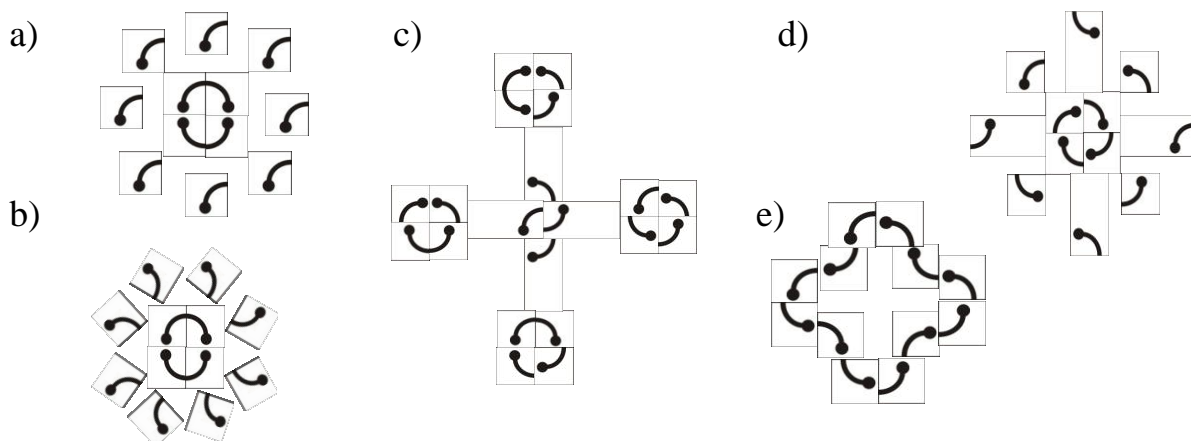
## DYNAMIC SITUATIONS

### C. „Directional puzzles”

In another stage of the project with specially prepared music, the children created puzzles with a specific theme. Music functions in a natural way, speaking of a sort of transition and changing from moment A to moment B. In the assumptions, we referred to building dynamic associations with the visual representations created by the child. The suggested theme was *merry-go-round*—the use of rotation.

The music and the topic actually inspired pupils to create compositions in which one could see relations connected with the idea of rotation. Placing several tiles required many full turns of single elements. Observation of the process proved very interesting. Some of the works started by distinguishing the central element and then other elements were arranged around it — rotational arrangement. A two-dimensional pattern with rotational symmetry was formed. Maintaining only rotational symmetry was difficult, as can be seen in the children’s work (fig. b,c).

Regardless of the external similarity of these creations to work created in the previously described stages, these puzzles tried to represent the idea of a specific movement. This results mainly from the way of organizing pupil’s work. Some children added tiles with regard to the rhythm of the music. There was also



work where axial symmetry gradually transformed into rotations. Observation of the children’s work did not pose any doubts that they try to match the constructed arrangement with rotation—the pupil would draw an oval line with their finger, trying to see if the tiles go round one after another (fig.e). While doing this, the pupil

## Working group 4

would adjust the tiles and rearrange them in such a way that the dot pattern would represent rotational and not mirror symmetry.

### D. “Dominoes”

The pupils (10-12 years old) had the following task:

How many different “domino” blocks can be created by using two squared tiles with the motif presented in the picture below?

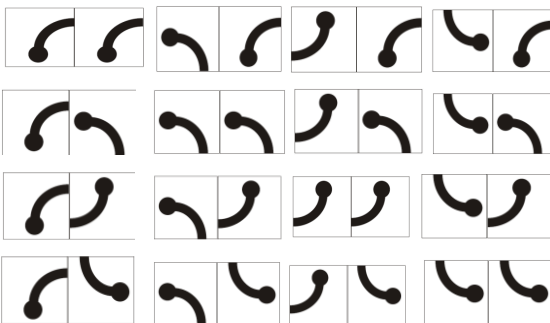


The first solutions were random. Pupils, sitting close to one another, could not recognize whether they had the same or different solutions. They also could not say whether they had all the possible solutions. Moreover, they did not know if the arrangements in front of them were actually different from one another or the same.

Such a situation was a good starting point for discussion and for a more ordered way of looking for

a solution.

An ‘unfailing’ strategy was proposed by a different pupil from the class. Below, we present his arrangement.



Analysis of this arrangement was a starting point for examining the position of one object in relation to another. This arrangement had the layout of a column: the boy started his arrangement from the first tile. A transition from one ‘domino block’ to another happened through a conscious rotation of the second tile.

Here, notions like ‘rotation by 90 degrees, 180 degrees, 270 degrees’ were appearing spontaneously. Although each domino block presented a relation of a rotation of two congruent figures, this relation was not the subject of research at this stage. For pupils, the way of constructing the whole series of dominos was very important, and this construction happened through rotating one of the tiles.

### OBSERVATION CONCLUSIONS

We conclude, that in the *static* recognition of rotation certain specific properties can be found:

- Rotation is understood locally.
  - ✓ The center of rotation is an element which sticks out in the foreground. One of the figures can serve this function.

There are elements which are inconsistent with the definition of rotation:

- During arrangement on a plane, there may be a lot of figures which are not congruent to one another.
- Figures that determine rotation themselves (around a given centre) are not rotated towards one another by a given angle. Their shift is rather parallel.

At the intuitional level of comprehending rotation, a *dynamic approach* can be characterized by the following:

- Movement representations are varied, strictly connected with a physical movement representative.
- They express only single definitions of rotation properties.
- A physical rotation of one figure results in the removal of the rotation center from the interest domain.

## SUMMARY

We believe that regular ways of filling a sheet of paper with tiles can be treated as an intuition of geometrical transformations, even though initially they are not connected with the interiorization of movement. As far as quality is concerned, this knowledge is different from the mathematician's knowledge, mainly because of the fact that it functions by totally different rules. If we want certain relations to be clear to children, we need to introduce a 'rich structure' in which not only two figures (e.g. polygons) remain in a particular relationship with each other but a certain fragment of space is organized according to this relationship. Introduction to the understanding of geometrical relationships that function in mathematics as a science is created through the feeling of regularity on a statically organized plane. Here, a child can arrange and organize tiles on a sheet of paper and ideas arise at the moment of reflection on what he or she sees.

The exterior effects of the work of pupils who create representations for a rotation both in a static and dynamic environment do not vary that much. It does not mean though, that these approaches can be identified with each other. In both approaches, the organization of pupil's work which pointed at different pictures associated with the performed activity was different. However, since these works (as a final effect) look alike, they give the possibility of building an integrated static-dynamic picture.

Presented didactical examples are not just 'clear' models of the mathematical notion of rotating by any angle. It is consistent with our understanding of the constructivist approach towards the creation of mathematical concepts. A pupil should function in such a rich learning environment that, through gaining various experiences and reflecting upon them, he would be able to create his own understanding of isometric transformations.

In spite of this, the relationship between visual recognition of geometrical objects and actions that can lead to the creation of dynamic images of such objects needs further investigation.

## REFERENCES

- Aebli, H. (1982). *Dydaktyka psychologiczna. Zastosowanie psychologii Piageta do dydaktyki*. PWN, wyd. drugie poprawione, Warszawa.
- Clements, D.H. & Battista, M.T. (1992). Geometry and spatial reasoning. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York, Mcmillan, 420-464.
- Clements, D.H., Swaminathan, S., Hannibal, M.A.Z., Sarama, J. (1999). Young Children's Concepts of Shape, *Journal for Research in Mathematics Education*, vol.30, nr.2, 192-212.
- Demidow, W. (1989). *Patrzyć i widzieć*. Wydawnictwo Czasopism i Książek Technicznych NOT-SIGMA, Warszawa.
- Grabowska, A. & Budohoska, W. (1992). Procesy percepcji, in: (ed.) T. Tomaszewski *Psychologia ogólna*, Wydawnictwo Naukowe PWN, Warszawa.
- Gray, E., Pinto, M., Pitta, D., Tall, D. (1999). Knowledge Construction and Diverging Thinking in Elementary and Advanced Mathematics, *Educational Studies in Mathematics*, 38, 111-113.
- Hejný, M. (1995). Development of Geometrical Concepts, *Proceedings of SEMT 95*, Prague, 13-18.
- Hiele van, P. (1986). *Structure and Insight. A Theory of Mathematics Education*. Academic Press Inc. London.
- Jones, K., Mooney, C. (2003), Making Space for Geometry in Primary Mathematics. In Thomson (ed.), *Enhancing Primary Mathematics Teaching*. London: Open University Press, pp. 3-15.
- Kaufman, L. (1979). *Perception. The World Transformed*. New York. Oxford University Press.
- Kordos, M. (2005). *Wykłady z historii matematyki*, SCRIPT, Warszawa.
- Kvasz, L. (2000). The Changes of Language in the Development of Mathematics. *Philosophia Mathematica* (Series III), 8 (1); p. 47 – 83.
- de Lange, J. (1987). *Mathematics, Insight and Meaning*. OW&OC Rijksuniversiteit Utrecht.
- Piaget, J. & Inhelder, B. (1999). *Psychologia dziecka*. Wydawnictwo Siedmioróg, Wrocław. (francuski oryginał: *La psychologie de l'enfant*, 1989).
- Szemińska, A. (1981). Ogólne uwagi o myśleniu matematycznym dziecka. In Z. Semadeni (Ed.), *Nauczanie początkowe matematyki*, tom 1, praca zbiorowa, WSiP, Warszawa.
- Vopěnka, P. (1989). *Rozprawy s Geometrii*. Panorama, Praha

# **ELEMENTARY STUDENTS' TRANSFORMATIONAL GEOMETRY ABILITIES AND COGNITIVE STYLE**

Xenia Xistouri and Demetra Pitta-Pantazi

Department of Education, University of Cyprus

*This study investigated 93 elementary students' abilities in solving transformational geometry tasks and how they relate to cognitive style. A test was developed to assess students' transformational geometry abilities, which included translation, reflection and rotation tasks. Students' cognitive styles were assessed using the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) (Blazhenkova & Kozhevnikov, 2009). Results suggest: 1) that the elementary students had average performance in solving the transformational geometry tasks, with rotation tasks being the most difficult and 2) that although both spatial imagery and object imagery cognitive styles relate to performance in transformational geometry, highly spatial imagery students perform better, because their cognitive style gives them an advantage in the case of solving the most difficult tasks, those of rotations.*

Key-words: transformational geometry, cognitive style

## **INTRODUCTION**

The growing emphasis on geometry teaching during the last few decades has modified its' traditionally Euclidian-based content, by introducing new types of geometry such as transformational geometry (Jones, 2002). There are several suggestions that there is limited research on transformational geometry (Boulter & Kirby, 1994; Hollebrands, 2003), which is imputed to its' underemphasis in mathematics curricula. However, it is considered important in supporting children's development of geometric and spatial thinking (Hollebrands, 2003) and it is related to a variety of activities in academic and every-day life, such as geometrical constructions, art, architecture, carpentry, electronics, mechanics, clothing design, geography, navigation and route following (Boulter & Kirby, 1994). Performance in geometric transformations has been previously connected to the holistic-analytic types of processing (Boulter & Kirby, 1994). However, despite its' rather obvious relation to visual imagery and the fact that it has often been connected to spatial abilities in literature, there doesn't seem to be a study that examines abilities in geometric transformations in relation to the visual-verbal cognitive style. This paper will study the relation between abilities in solving transformational geometry tasks and a new three-dimensional cognitive style model proposed by Blazhenkova and Kozhevnikov (2009) that distinguishes between Object imagery, Spatial Imagery and Verbal dimensions. Specifically, the aim of the paper was to investigate nine to eleven year old students' abilities in transformational geometry tasks of translations, reflections and rotations, and to investigate the relationship between these abilities and the students' cognitive style.

## **THEORETICAL FRAMEWORK**

### **Transformational Geometry in Mathematics Education.**

The inclusion of transformational geometry in mathematics curricula in the early 70's raised an emphasis around the importance of teaching and understanding geometric transformations (Jones, 2002). Early studies focus on providing evidence for suggesting that teaching geometric transformations in elementary and high school education is feasible and may have positive effects on students' learning of mathematics (Edwards, 1989; Williford, 1972). Later studies focus on more psychological aspects, such as students' ability and misconceptions (Kidder, 1976; Moyer, 1978), strategies for solving transformational geometry problems (Boulter & Kirby, 1994) and configurations influencing students' ability in transformational geometry (Schultz, 1983). During the early 90's, research started to focus on investigating a hierarchy that describes students' acquisition of transformational geometry (Molina, 1990).

It seems that research in transformational geometry decreased substantially around the late 80's, leaving unanswered questions on the cognitive development of transformations (Boulter & Kirby, 1994). For instance, Moyer (1978) raised questions on whether some geometric transformations are more difficult than others and emphasized the need to search for a successful sequence of learning activities in transformational geometry for children. There were also some issues raised concerning individual differences and different types of processing information in transformational geometry problem solving (Boulter & Kirby, 1994). It is thus important to understand the role that individual differences such as students' cognitive style may have in their abilities to solve transformational geometry tasks. Such information would guide educators in providing further assistance to the less able students in transformational geometry to overcome their difficulties.

### **The Object-Spatial-Verbal Cognitive Style Model.**

Cognitive styles refer to psychological dimensions representing consistencies in an individual's manner of cognitive functioning, particularly with respect to acquiring and processing information (Witkin, Moore, Goodenough, & Cox, 1977). One of the most commonly acknowledged cognitive styles dimension is the Visual-Verbal (e.g. Paivio, 1971), which describes consistencies and preferences in processing visual versus verbal information, and classifies individuals as either visualizers, who rely primarily on imagery when attempting to perform cognitive tasks, or verbalizers, who rely primarily on verbal-analytical strategies.

However, neuropsychological data suggest the existence of two distinct imagery subsystems that encode and process visual information in different ways: an object imagery system that processes the visual appearance of objects and scenes in terms of their shape, colour information and texture and a spatial imagery system that processes object location, movement, spatial relationships and transformations and

other spatial attributes of processing (Blazhenkova & Kozhevnikov, 2009). The distinction between object and spatial imagery has been also found in individual differences in imagery (Kozhevnikov, Hegarty, & Mayer, 2002). Recent behavioural and neuroimaging studies have identified two distinct types of individuals, object visualizers, who use imagery to construct vivid high-resolution images of individual objects, and spatial visualizers, who use imagery to represent and transform spatial relations (Kozhevnikov, Kosslyn, & Shephard, 2005). Based on these distinctions, Blazhenkova and Kozhevnikov (2009) have developed a self-report instrument assessing the individual differences in object imagery, spatial imagery and verbal cognitive styles, the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ).

Mathematics education researchers have often linked the verbalizers/visualizers distinction to mathematical performance (Presmeg, 1986). Nevertheless, the results of the relationship between visualisation and mathematical performance are unclear. Some studies found that visual-spatial memory is an important factor which explains the mathematical performance of students (Battista & Clements, 1998), while other studies showed that students classified as visualizers do not tend to be among the most successful performers in mathematics (Presmeg, 1986). In the case of transformational geometry, although it has not yet been linked to the verbalizers/visualizers distinction, it has been connected to spatial ability (Dixon, 1995; Kirby & Boulter, 1999), thus it is hypothesized in this study that the spatial imagery cognitive style will be related to abilities in transformational geometry tasks.

## **METHODOLOGY**

The purpose of the study is to investigate elementary school students' abilities in transformational geometry tasks and the relation of these abilities to the students' cognitive style. Ninety three students were selected to participate in the study (34 fourth-graders and 59 fifth-graders), based on their teachers' willingness to provide access to their classes during school-time. A transformational geometry ability test and a self-report cognitive style questionnaire were administered to all students at the same week, in groups of approximately 15 students.

The transformational geometry ability test was used to measure students' mathematical abilities in the concepts of translation, (axial) reflection and rotation. The test consisted of 33 tasks, of which seven were translations, fourteen were reflections and twelve were rotations. The test included multiple choice and drawing tasks which focused mainly in 1) performing a specific transformation, 2) finding the parameters of a given transformation, and 3) identifying the result of a given transformation (see Appendix for examples). It is noted that the students were taught 2-3 lessons on symmetry at every grade, and were informally introduced to the concept of transformations but not to the mathematical terms. They were given 40 minutes to solve the test during normal lesson time. Each correct response to an item in each of the tasks was assigned a positive point. Half point was assigned when a response was partially correct, for example when a requested transformation was



performed correctly, but there was no accuracy in the shapes' dimensions or orientation. The points were summed up separately for translations, reflections and rotations, in order to give the students' scores for each type of geometric transformation, and also in total to give an overall of each student's performance.

The students were then administered a modified version of the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) to assess the individual differences in spatial imagery, object imagery and verbal cognitive style. This is a self-report questionnaire, which includes 45 statements with a 5-point Likert scale for students to rate themselves on how much they agree with the content of the statement. Fifteen of the items measured object imagery preference and experiences, fifteen items measured spatial imagery preference and experiences and fifteen items measured verbal preference and experiences. Examples of the statements are: "If I were asked to choose among engineering professions or visual arts I would choose visual arts" (measuring Object Imagery dimension), "My images are more schematic than colourful and pictorial" (measuring Spatial Imagery dimension), and "I usually do not try to visualize or sketch diagrams when reading a textbook" (measuring Verbal dimension). The test was translated in Greek and was modified to be comprehensive to elementary students. The students were given 30 minutes to complete the questionnaire. For each student, the fifteen item ratings for each factor were averaged to create object imagery, spatial imagery and verbal scale scores.

## RESULTS

The main purpose of the study was to investigate elementary school students' abilities in transformational geometry tasks of translations, reflections and rotations, and how these are related to the students' cognitive style. Students' means of performance were calculated to describe their abilities in transformational geometry. The object-spatial-verbal cognitive style dimension was used as predictor variable for students' performance in transformational geometry tasks. Specifically, through multiple regression analyses with criterion (dependent) variable the students' performance in translation tasks, performance in reflection tasks, performance in rotation tasks and overall performance, and predictors (independent) the spatial imagery, object imagery and verbal cognitive style scores.

	$\bar{X}$	SD
Translations	0.59	0.26
Reflections	0.54	0.20
Rotations	0.38	0.21
Overall	0.49	0.18

**Table 1: Means and Standard Deviations for each type of tasks in the transformational geometry abilities test**

Table 1 presents the means and standard deviations for each type of transformational geometry task, as well as for overall performance. It appears that students performed

#### Working group 4

slightly better in the translation tasks ( $\bar{X}=0.59$ ,  $SD=0.26$ ), rather than the other categories, indicating that these might be the easiest type of tasks for elementary school students. Mean performance in reflections ( $\bar{X}=0.54$ ,  $SD=0.20$ ) is next, but very close to performance in translation tasks. In order to test these observations, students' means in translation tasks and in reflection tasks were compared in a paired sample t-test analysis, which showed that this mean difference is not significant ( $t=1.842$ ,  $p=0.069$ ). This finding is in accord with Moyer (1978), who found that translations are as easy as reflections. The most difficult tasks for the students seem to be the rotation tasks, where this group of students had a much lower mean performance ( $\bar{X}=0.38$ ,  $SD=0.21$ ). A paired sample t-test for comparing students' mean performance in reflection tasks and in rotation tasks revealed this mean difference is statistically significant ( $t=7.266$ ,  $p=0.000$ ). Students' overall performance mean in transformational geometry tasks is 0.49 ( $SD=0.18$ ), which is near average, considering zero as minimum value and one as maximum.

	Translations	Reflections	Rotations	Overall
Spatial Imagery Cognitive Style	.162	.231*	.236*	.266*
Object Imagery Cognitive Style	.174	.231*	.199	.254*
Verbal Cognitive Style	.031	.169	-.097	.050

\* Correlation is significant at the 0.05 level (2-tailed).

**Table 2: Correlations among performance scores and spatial imagery, object imagery and verbal cognitive styles.**

The correlations among students' cognitive styles and their performance in translations, reflections, rotations and overall are presented in Table 2. As can be seen from Table 2, the spatial imagery cognitive style is significantly correlated with students' transformational geometry abilities in reflections, rotations and overall performance, while object imagery cognitive style is significantly correlated only with reflections and overall performance. The verbal cognitive style dimension did not correlate with any of the students' abilities in transformational geometry. We further examined the nature of these correlations and non correlations between cognitive styles and students' abilities in translation, reflection and rotation tasks, as well as their overall performance in the test.

Table 3 presents the results of the multiple regressions, using the stepwise method. It should be noted that the regression analysis for students' abilities in the translation tasks did not enter any variables in the equation, which means that none of the cognitive style variables - object, spatial or verbal - can significantly predict performance in translation tasks. This was expected, as none of these variables were significantly correlated to the translation tasks, as seen in Table 2. Therefore, Table 3 presents the coefficients and levels of significance for predicting performance in reflections, rotations and overall.

	Reflections		Rotations		Overall	
	<i>b</i>	<i>p</i>	<i>b</i>	<i>p</i>	<i>B</i>	<i>p</i>
Spatial Imagery Cognitive Style	.154	.209	.327	.005**	.266	.013*
Object Imagery Cognitive Style	.231	.031*	.222	.083	.161	.183
Verbal Cognitive Style	.075	.536	-.228	.047*	-.068	.554

\* Indicate statistical significant  $p < 0.05$

\*\* Indicate statistical significant  $p < 0.01$

**Table 3: Multiple regression analyses with dependent variables reflection tasks, rotation tasks and overall performance, and independent variables spatial imagery, object imagery and verbal cognitive styles.**

One characteristic that rises from the analysis of the data is that object cognitive style can significantly predict performance in reflection tasks ( $b=0.231$ ,  $p=0.031$ ). It seems that this was the only significant factor for performance in reflection tasks, and it can predict more than 4% of the variance of performance in reflection tasks. This means that as the object cognitive style of the students increases, their performance in reflection tasks of transformational geometry increases also.

Another important characteristic presented in Table 3 is that spatial imagery cognitive style can significantly predict performance in rotation tasks ( $b=0.327$ ,  $p=0.005$ ) and the verbal cognitive style is negatively related to performance in rotation tasks ( $b=-0.228$ ,  $p=0.047$ ). The negative sign of the beta in the case of the verbal cognitive style means that as the verbal cognitive style increases, students' performance in rotations decreases. This suggests that students who tend to prefer verbal processing are not so successful in solving rotations tasks, while students who tend to prefer the spatial imagery processing perform better. Spatial cognitive style and verbal cognitive style can explain a proportion of variance of 7.8% in performance in rotations tasks.

In the same way we can interpret the negative relation between the verbal cognitive style dimension and overall performance, although it is not statistically significant. The last and most important observation from Table 3 is that spatial imagery cognitive style is the only significant predictor of students' overall performance in transformational geometry tasks ( $b=0.266$ ,  $p=0.13$ ). The proportion of variance explained by this factor is 6%. This means that as far as students tend to prefer the spatial visualization processing, their overall performance in transformational geometry is higher than those who seem to prefer the verbal and object visualization processing of information. This finding is in line with the results in a study by Kirby and Boulter (1999), who suggest that performance in transformational geometry can mainly be predicted by students' score in spatial ability tests.

## DISCUSSION

Although transformational geometry is considered important in supporting children's development of geometric and spatial thinking (Hollebrands, 2003), research in the field seems to have left unanswered questions concerning children's abilities in performing transformations. And despite the rather obvious relationship of transformational geometry to visual imagery, it has not yet been related to the visual-verbal types of processing. This study goes a step further, by investigating students' abilities in transformational geometry tasks of translations, reflections and rotations, and relating them to two distinct types of visual processing – object imagery and spatial imagery cognitive style - and verbal cognitive style.

The results of this study have shown that the students' overall performance was average in these transformational geometry tasks. An important finding of this study is an indication of hierarchy in students' understanding of transformations: translations and reflections are equally difficult to students, while rotations seem to be more difficult. This is in accord to Moyer's (1978) findings. However, another study by Schultz and Austin (1983) suggests that translations are easier compared to reflections and rotations, whose level of difficulty is influenced by the direction (vertical, horizontal, diagonal) of the transformation. Apparently more studies are required in order to clarify the hierarchy of difficulty in different types of geometric transformations, by considering configurations such as direction.

In regard to the relation between transformational geometry abilities and students' cognitive style, the results of this study show that the spatial imagery cognitive style is a significant predictor of performance in rotation tasks, but more important to overall performance in transformational geometry tasks. This was expected, considering the connection between transformational geometry performance and spatial abilities found in the literature (Dixon, 1995; Kirby & Boulter, 1999). On the other hand, the object imagery cognitive style seems to be related to transformational geometry abilities as well, but only by contributing significantly in predicting performance in solving reflection tasks. This was unexpected, since object imagery cognitive style is usually more related to visual arts rather than scientific fields (Kozhevnikov, Kosslyn, & Shephard, 2005). Perhaps further research with a bigger sample of students would help clarify and validate this finding.

However, a possible explanation for this finding could be that highly spatial visualizers were rather more flexible in their strategies and could manipulate both reflections and rotations tasks, whereas highly object visualizers were successful in reflections tasks, but couldn't handle the most difficult tasks of rotations. Object visualizers may have been able to see the image be reflected as a whole and then find its' position analytically over a single axis, but were inhibited in keeping track of the orientation of an image in space when rotated. This point of view may lead to the conjecture that, although students with high visualization abilities - either spatial or object imagery – can perform well in transformational geometry tasks, it is the ones

with high spatial visualization abilities that are flexible enough to deal with the most difficult tasks, those of rotations, and who eventually outperform others. However, since this is a quantitative study, further investigation with qualitative data of object imagery and spatial imagery cognitive style students' strategies for solving reflection and rotation tasks is needed to provide more evidence and deeper insight.

The fact that verbal cognitive style did not have a significant positive correlation to any type of tasks probably means that students with preference to verbal type of processing do not perform so well in transformational geometry tasks, and especially in the case of rotations, this type of processing preference may somehow raise more difficulties to students' task solving. It should also be noted that performance in translations tasks was not related to any of the three different types of cognitive styles. Although this finding was unexpected, it may mean that the concept of translation is more comprehensive to students, regardless to their cognitive style. Translation tasks can be solved either visually or verbally: one can either visualize the image move as a whole, or one can visualize the image move part by part, or one can verbally count the steps to the new position and copy the image. Some qualitative research could provide information on how students with different cognitive styles solve translation tasks and answer how and why it doesn't differentiate their performance.

In closing, we note that this study is a first investigation of students' abilities in transformational geometry and their relation to cognitive styles. The results presented in this paper are initial results and there appear to be some discrepancies which are not easily explained. There is still no clear picture for the relation between abilities in transformational geometry and cognitive styles, since different types of transformations seem to be related to different cognitive styles. The important finding is that there is some relation, but further investigation with a larger sample and more qualitative data is necessary to clarify its' nature. Overall, the results of this study suggest that it is helpful to know students' cognitive styles, especially to educators, to facilitate developing flexible methods of teaching transformational geometry to accommodate all types of learning, and also in providing appropriate assistance to each student to overcome their difficulties. Perhaps students could be guided into applying strategies of processing of their less preferred type of imagery effectively in tasks when necessary. This could be a challenging question for further research.

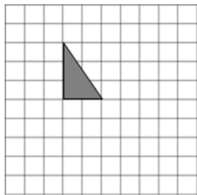
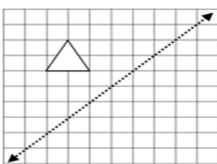
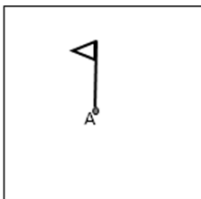
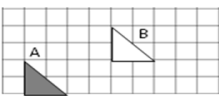
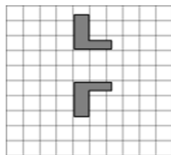
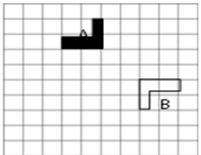
## REFERENCES

Battista, M. T., & Clements, D. H. (1998). Students' understanding of three-dimensional cube arrays: Findings from a research and curriculum development project. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 296–333). Mahwah: Lawrence Erlbaum Associates.

- Blazhenkova, O., & Kozhevnikov, M. (2009). The new object-spatial-verbal cognitive style model: Theory and measurement. *Applied Cognitive Psychology*, 23(5), 638-663. doi:10.1002/acp.1473
- Boulter, D. R., & Kirby, J. R. (1994). Identification of strategies used in solving transformational geometry problems, *Journal of Educational Research*, 87 (5), 298-303. Retrieved from <http://www.jstor.org/stable/27541933>
- Dixon, J. K. (1995). English language proficiency and spatial visualization in middle school students' construction of the concepts of reflection and rotation using the "The Geometer's Sketchpad" (Doctoral dissertation, University of Florida, 1995). *Dissertation Abstracts International*, 51(6), 9607507.
- Edwards, L. (1989). Children's learning in a computer microworld for transformation geometry (Doctoral dissertation, University of California, 1989). *Dissertation Abstracts International*, 51(6), 1943A.
- Hollebrands, K. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *Journal of Mathematical Behavior*, 22 (1), 55-72. doi:10.1016/S0732-3123(03)00004-X
- Jones, K. (2002), Issues in the teaching and learning of geometry. In Linda Haggarty (Ed), *Aspects of Teaching Secondary Mathematics* (pp. 121-139). London: RoutledgeFalmer. Retrieved from <http://eprints.soton.ac.uk/13588/>
- Kozhevnikov, M., Hegarty, M., & Mayer, R. (2002). Revising the visualizer-verbalizer dimension: Evidence for two types of visualizers, *Cognition & Instruction*, 20, 37-77. doi: 10.1207/S1532690XCI2001\_3
- Kozhevnikov, M., Kosslyn, S., & Shephard, J. (2005). Spatial versus object visualizers: A new characterization of visual cognitive style. *Memory and Cognition*, 33, 710-726. Retrieved from <http://mc.psychonomic-journals.org/>
- Kidder, R. (1976). Elementary and middle school children's comprehension of Euclidean transformations. *Journal for Research in Mathematics Education*, 7 (1), 40-52. Retrieved from <http://www.jstor.org>
- Kirby, J.R., & Boulter, D.R. ( 1999). Spatial ability and transformational geometry. *European Journal of Psychology of Education*, 14(2), 283-294. doi: 10.1007/BF03172970
- Molina, D. D. (1990). The applicability of the van Hiele theory to transformational geometry (Doctoral dissertation, The University of Texas, 1990). *Dissertation Abstracts Online*, 44, 417A.
- Moyer, J. (1978). The relationship between the mathematical structure of Euclidean transformations and the spontaneously developed cognitive structures of young children. *Journal for Research in Mathematics Education*, 9 (2), 83-92. Retrieved from <http://www.jstor.org>

- Paivio, A. (1971). *Imagery and verbal processes*. New York: Holt, Rinehart & Winston.
- Presmeg, N. C. (1986). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, 17, 297–311. doi: 10.1007/BF00305075
- Schultz, K., & Austin, J. (1983). Directional effects in transformation tasks. *Journal for Research in Mathematics Education*, 14 (2), 95-101. Retrieved from <http://www.jstor.org>
- Williford, H. J. (1972). A study of transformational geometry instruction in the primary grades. *Journal for Research in Mathematics Education*, 3, 260-271. Retrieved from <http://www.jstor.org>
- Witkin, H. A., Moore, C. A., Goodenough, D. R., & Cox, P. W. (1977). Field-dependent and field-independent cognitive styles and their educational implications. *Review of Educational Research*, 47, 1–64. Retrieved from <http://www.jstor.org>

## APPENDIX

Translations	Reflections	Rotations
<p>1. Translate the shape five boxes to the right.</p> 	<p>1. Reflect the shape over the given line.</p> 	<p>1. Rotate the shape <math>\frac{1}{4}</math> of a turn clockwise around A.</p> 
<p>2. Describe the following transformation.</p> 	<p>2. Find the line of symmetry.</p> 	<p>2. Find the centre of rotation.</p> 
<p>3. Which of the following is a translation of <b>E</b> ?</p> <p>a. <b>M</b>   b. <b>Э</b>   c. <b>E</b>   d. <b>W</b></p>	<p>3. Which of the following capital letters will <u>not</u> look the same when reflected over a vertical line?</p> <p>a. A   b. C   c. X   d. T</p>	<p>3. Which letter will look exactly the same when rotated <math>\frac{1}{2}</math> turn clockwise?</p> <p>a. D   b. E   c. M   d. Z</p>

# GEOMETRICAL TRANSFORMATIONS AS VIEWED BY PROSPECTIVE TEACHERS

Xhevdet Thaqi\*, Joaquin Giménez\*\*; Nuria Rosich\*\*

\*University of Prishtina; \*\*University of Barcelona.

*An empirical study is developed to find the meanings attributed to geometrical transformations by prospective primary teachers in Kosova and Spain. The study reveals that students' previous background had more influence than cultural differences.*

**Key words:** Geometrical transformations, prospective teachers, comparative study

## INTRODUCTION

A new challenge for teacher training in European Higher Education is to reduce international differences, considering immigration processes, exchanges, and globalization. A continuing interest of the CERME community is to understand how mathematical practices are developed by using cultural writings in different countries (Stigler et al, 2000) as culturally situated mathematical practices (Llinares & Krainer 2006). The way in which mathematics is construed by participants is a hidden variable in researching mathematic knowledge for teaching (Andrews, 2009). In particular, some authors explore differences in the use of geometry: from natural and intuitive to axiomatic perspectives (Girnát, 2008), analyzing and reconceptualising their geometric ideas, progressing to higher levels of geometrical understanding. There is a considerable amount of research about the knowledge and use of geometrical transformations in secondary school (Hoyos, 2006) but less research has been conducted for primary schools. It has also been shown that difficulties in primary students' conceptualizations depend on the weak knowledge of teachers (Law, 1991, quoted by Yanik & Flores, 2009), in particular about geometrical transformations (Pawlik, 2004).

Such studies reveal that teachers' lack of mathematical subject knowledge and confidence in mathematics are contributory factors to the low standard of pupil mathematics attainment in many countries. It has also been shown by several authors that pre-service elementary teachers have difficulties in determining: (1) the correct *attributes of transformation and motion* to move an object from one point to another; (2) the results of *transformations involving multiple combinations of figures*; (3) the use of *transformations as mathematically-general operations* which require the specification of inputs, but as particular actions, each with given 'default' or prototypic parameters. It has also been observed that the use of technological devices has strong advantages in the use of isometries, because of the possibilities of variability analysis (Harper 2003).

A recent study concerning prospective teachers' knowledge of translations and other rigid transformations (Yanik & Flores 2009) revealed that scholars (1) started by



referring to *transformations as undefined motions of a single object*, (2) followed by using *transformations as defined motions of a single object*, and (3) the understanding of *transformations as defined motions of all points* on the plane.

In Spain, studies using Van Hiele's levels found prospective primary teachers' difficulties in using symmetrical notions in an isometry task (Jaime & Gutierrez 1995), but few proposals were made to analyze teachers' ideas qualitatively before developing professional tasks. Even the term "transformation" is mentioned only at the end of secondary school and does not solve the problem of transition from the use of natural environmental geometry in primary school into secondary school axiomatic perspectives (Kuzniak & Vivier 2008). Therefore, in our research study, we focus on analyzing the influences of prospective teachers' prior cultural background before developing training activities about learning to teach geometrical transformations. We studied and compared the results in Kosova and Spain in a bridging collaborative international framework (Jaworski, 2006), where we expected to find different conceptualizations in their responses.

## METHODOLOGY

An ethnographical research case study was planned, with two groups of future teachers: 13 students from a 2nd year course at the Faculty of Teacher Training at Barcelona University (UB) in Spain, with only one prior mathematical/didactical course and 15 students from the Faculty of Education at University of Prishtina (UP) in Kosova, with two prior geometry courses based on classic Euclidean geometry, but no previous didactical training. Students were 18-22 years old. A prior curricular-cultural comparative analysis based on textbooks, official curricular proposals and teacher training materials showed deep differences between both previous preparation and cultural frameworks (Thaqi, 2009), but is not detailed in this presentation.

The results of an initial semi-structured questionnaire are considered in this paper, in order to analyze beliefs, meanings, and prototypes from transcriptions of student text. Such a questionnaire is the first step in a wider developmental study in which both groups of students have the same training about transformations in geometry (Thaqi, 2009).

An initial semi-structured questionnaire was designed by using 14 open (mainly contextualized) written questions, together with subsequent interviews considered necessary to capture students' ideas about the topic (see the main ideas in Table 1). Some other questions were added to identify reasoning and specific ethnic-cultural elements of geometrical transformations, ideas about teaching and learning, and student ideas concerning the future teaching of geometrical transformations.

Aspect of meaning of geometrical transformation	Identified Activities
Terminology. Types of transformation.	1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 14
Properties. Relations and hierarchies	1, 3, 4, 5, 6, 7, 11, 12, 14
Transformation as a process or simple change	1, 2, 3, 4, 5, 6, 7, 9, 12
Others aspects (reasoning, teaching, etc)	8, 10, 13, etc.

**Table 1: Sets of questions related to mathematical ideas about transformations.**

Further analysis about learning to teach transformations and professional activities was given from videotaped transcriptions not included in this article. Furthermore, the first sessions serve as confirmation of attributed meanings found in the questionnaire, about training practices (Stigler, Gallimore and Hiebert, 2000: 87) showing us cultural objects with specific languages or symbolic systems.

Data collected through the developmental process were analyzed using ongoing analyses. During the ongoing analysis phase, the researcher tried to understand the participant's way of thinking, not presented in this article. After each teaching episode, the research team coded and analyzed the video records of students' interactions during the given tasks. The main purpose was to find patterns and create descriptions of the development of students' mathematical knowledge over time as a hypothetical learning trajectory of participants. During ongoing analyses the researcher tested his initial hypotheses and generated new conjectures to be tested in the following teaching episodes.

## **ABOUT ATTRIBUTED MEANINGS**

Based upon student answers in both countries, we divided the results into three parts: (a) the meanings and use of geometric transformations as a mathematical object and associated examples; (b) definitions and conceptual structure, and (c) representations and non-isometric transformations.

### **Transformation as mathematical object**

To identify degrees of knowledge (table 2), we assume students develop more or less pseudo-conceptual perspectives (Vinner, 1997) by analyzing their justifications, argumentation, properties, and use of examples and counterexamples. It was not surprising that none of the students showed consolidated knowledge about the idea of transformation or the idea of transformation as a function, even in the case of projection (usually defined as a function). The majority (64%) belonged to an intermediate level in the case of the University of Prishtina (UP). The main class of mathematical objects they identified was symmetry, as we expected.

## Working group 4

Degree of knowledge about transformation as a math <b>object</b>	Barcelona	Prishtina
	N=13	N=15
22 They are able to build complete images, using terminology and justifying interpretations carefully with good statements.	•	•
23 They show some conceptual images by using prototypical examples including some relevant properties. They identify the transformation of the figure without any explicit explanation about properties.	46%	64%
24 No answer or no meaningful explanations. Poor images, based upon examples and visual prototypical examples.	54%	36%
25 Blank or without any sense	--	--

**Table 2: Results compare between Barcelona (UB) and Prishtina (UP)**

We analyzed student text by observing their answers to find their ideas about the set characteristics and to find semiotic conflicts.

We deduced that Kosovar students (UP) assume a “*transformation perspective*” by using deep mathematical expressions. When we asked how transparent paper could be used to show the rotation, Vj indicated: “...*they draw the part of the figure through the paper to be turned in order to obtain the whole figure. Thus, it will show the rotation*” (Vj, p5:3, UP). In some other cases students identified the expression “*through displacements...*” as a way for describing the transformation that generates figures from a module.

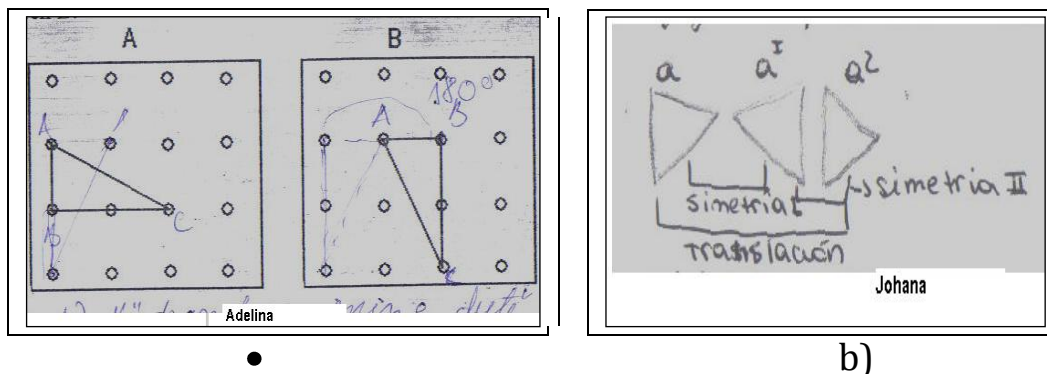
In the case of the Spanish students, transformation was mainly considered to be a simple relationship between objects and their transformed images, in which some characteristics of the object are changed (called *undefined motion* in Yanik & Flores, 2009). Change in position was not always taken into consideration: “... *the movement does not mean a change of form, but only the position, while the transformation involves change of the form*” (Al, p.9: 8, UB).

A few students told us about the invariant terms of a transformation, and gave interesting properties of transformations such as the knowledge of repetition by period  $T=2\pi$  (student Pe in UP group) when they explained the rotation of a door. When we asked “*tell us some statement to give a meaning for rotation*”, several students from UP spoke about the invariance of shape and size when talking about isometries. They used the expression “*change of a same thing*”, and others used the expression “*without changing...*” even when they talked about projectivity.

A lot of mathematical inconsistencies appeared with the Spanish students, associating rotation with the class of isometries, but then stating that a rigid movement was not a transformation. In some cases, the question seemed to promote intuitive or pseudo-conceptual knowledge, more than structured knowledge, as in the case of tiling, in which students explained rotation as the only isometric transformation: “...*I understand a movement to take some object or image and displacing it, without any change...*” (Mc, p5: 2 – 3, UB).

### About definitions, properties and structure

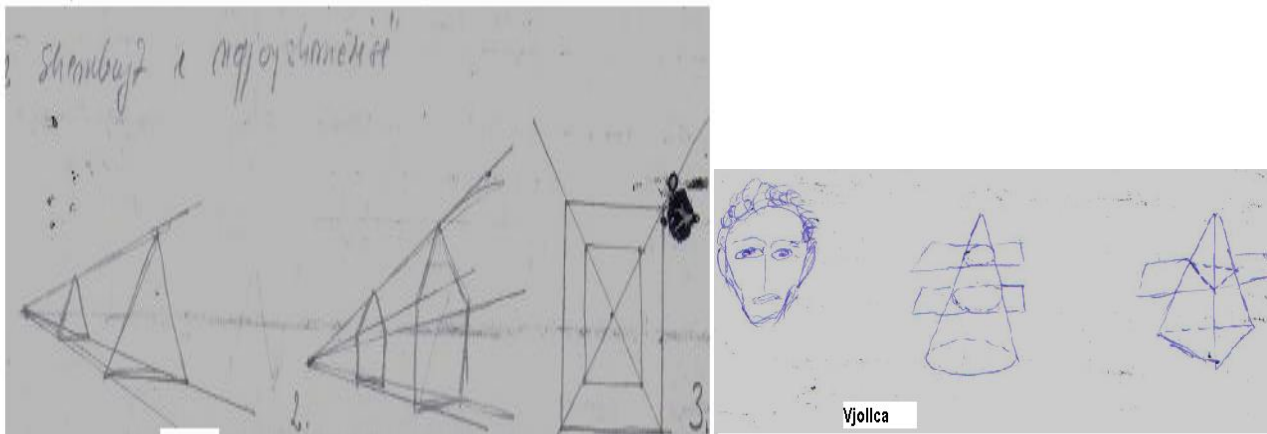
Only in two cases did we find that the (isometric) transformation was seen as defining the motion of all points on the plane. After naming the vertices of the triangle, Ad (UP) expressed the functional dependence between their positions: "... first has been the displacement of point A, and during the movement of point A, points B and C takes position presented on figure (see Fig. 1a), so the points have changed columns ... the point B has moved to the place of point A ..." (Ad, p12: 7-8, UP). Few UP students adequately affirmed that the movement was only one type of geometric transformation and established the correct relationship between the properties of conservation of size and form in isometric transformations.



**Fig.1 Transformations of triangle A onto triangle B by Ad (UP) and Jo (UB)**

In the case of UB, most of the students identified isometric transformations as repetitions, but generally reflection was not identified in that set. Another group of UB participants were limited in identification of the visual characteristics of symmetry to rotations and translations. The main image for a transformation is based upon visual understanding (transforming=deformation), and isometrics movement is interpreted as displacement. Few students visually identified the translation vector, and confirmed that the translation of a figure is equal to the product of two reflections with parallel axes (Figure 2b): *"The translation is the product of two reflections because this is the reflection of the reflection....figure a2 is the translation of initial figure a."* (Jo, p8: 4, UB).

Many UB students identified similarities as a commonsense word, without mathematical explanations. Sometimes they accepted that *"I don't know the properties of similarity"*. In general in both countries, all the necessary elements for definitions were not stated, such as change of direction in relation to some movements. When they talk about rotation, only angles were observed but not the center of rotation and the invariance of distance. Some Kosovar students (UP) established the correct relationship between conservation properties and the corresponding transformations, but only one UB student: others talked about repetitions or similar figures as commonsense. For example, student A1 stated *"It can be when you open an orange, because it can appear equal parts, but they are not"* (A1, p5:2, UB), and Jo stated *"The similarity would say that is when something is very similar to another but not the same"* (Jo, p7:1,2 UB).



**Figure 2. Drawings showing the transformational idea for similarity**

When they tried to establish a functional approach, only a figural perspective was used, arguing by means of artefacts such as paper folding, or rays of the sun. Only two students from Kosova (UP) identified a functional idea for similarity by using conic sections (Figure 2) presenting their previous mathematical knowledge.

Conservation of shape was observed in almost all UP students, but not UB, as this was viewed as a figural phenomenon. A nice description was given by student Ar: *“the shadow will be different in a human body example,... the focus of light is considered the centre of the projection, and light arrows are the right lines for the projection.”* (Ar, p10: 7, UP).

In UB, some students explained the transformation as a relationship between two different stages of an object. Other students saw the transformation as a radical change in a physical object. We were really surprised that many of the students thought that projections were not transformations as Na comment: *“There are not transformations. The shadows are the projections of an image due to the light”* (Na, p10: 4, UB).

### **About representations and non-isometric transformations.**

This is not analyzed in many studies. In fact, students are usually trained to visualize isometric transformations by using problem solving activities. We have more consolidated answers in UB than in Kosova because it is more common for students to be confronted with isoperimetric manipulation tasks. It was also expected to find that deformations were not considered globally as transformations. Some students identified projection as deformation, without considering the conservation of the shape. Ol responded to the question *“Is the shadow a transformation?”* with *“...when we work with shadows, we say it’s a work about projections because the figures have been deformed. An image is obtained from another, as we see in the overhead projector. The projected image is deformed, it stretched or enlarged”* (Ol, 10:9, UB)

Figural explanations were typical in UB, and light/shadows phenomena were only explained in terms of dependence, without any explanation about the transformed elements. Some UP students told us about dependence, indicating the main variables in projective transformation: *“the shadow depends upon the place where is observed, because the shadows grow when a light source is incident on a body and this is projected on an opaque background. If we would have a light bulb on an opaque background, we would have no shadow. It also depends on where we shine the spotlight”* (Ad, p10:3-5, UP).

One possible explanation is that Klein’s perspectives are not introduced in the curriculum and are hence unfamiliar to many teachers. Figural judgements are based upon a few prototypical examples, using incorrect comparative arguments. For example Da explained an isoperimetric transformation as follows: *“the transformation converting a rectangle 3cm x 7cm made with a 20 cm string into another rectangle using the same string, is a conservation of perimeter and area”* (Da, p14:5, UP).

## CONCLUSION

The generally low results about transformations in both countries reveal a lack of previous background, not only because of a lack of mathematical knowledge, but a lack of tasks in which transformations plays usually a restricted mathematical role. To enhance performance in geometry, students need to increase their level of geometric thinking through increased exposure to informal geometric activities throughout their training. To accomplish this goal we need to start with deepening geometric content knowledge and increasing the level of geometric thinking of prospective primary teachers. Increasing the teacher’s knowledge and level of thinking can only improve the mathematical instruction that students receive.

The acquisition of the concept of transformation is important for the development of spatial reasoning and the geometrical understanding of the immediate environment, in which phenomena are encountered that require familiarity with isometric and non-isometric transformations. Another benefit of studying the intuitive and informal aspects of transformations is the dynamic nature of transformation – transformational geometry encourages students to investigate geometric ideas through an informal and intuitive approach.

In Spain, it was found that natural images influenced the development of a functional idea of projection, but that there was a lack of deep understanding about the role of properties in definition processes. This aspect was better in the case of the Kosovar students, as expected due to the German-Russian tradition. In general, the fact that similar results appear shows that the background of mathematical content knowledge is not enough to develop these concepts. None of the students in either country had a complete concept of transformation and structure as a function and they showed different figural images in each task.

The construct of variability is needed to understand Klein's meaning of invariance for associating geometries and transformations. It also means that simple visualization is not enough to understand such concepts. More emphasis on a wider sense of contextualization is also needed to discuss the functional features of transformations. Nevertheless, the Euclidean orientation of the Kosovar curriculum for lower secondary school gives the possibility that some students can relate their prior theoretical framework with their didactical purposes as future teachers. In fact, their comments were based not only upon intuitions about transformations, but also upon relating these to mathematical knowledge.

We also found that in both countries students had not enough time to develop powerful images about types of transformations, and we suggest the need for experiences of transformations other than isometries. During developmental activities, we could reinforce such a didactical research conjecture (Thaqi, 2009) by doing professional tasks in which we insist on invariance as a phenomenon. Representing transformations with function notation requires more abstract thinking and is crucial for understanding transformations as one-to-one mappings of the points of the plane. Our results are coherent with the emergent global/punctual dialectics (Jahn, 1998) as a semiotic conflict.

Teaching isometric and non-isometric transformations permits a richer study of oft-neglected topics, as well as investigation of topics heretofore not studied in the elementary school. For example, the shadow is an important but frequently neglected topic in elementary school geometry. These visual experiences can help students develop the ability to manipulate images mentally - the essence of spatial visualization.

A problem such as "What composition of transformations will move triangle A onto triangle B?" necessitates that students formulate and test hypotheses about sequences of transformations. This transformation approach makes geometry an appealing, dynamic subject that will develop both spatial visualization and also reasoning abilities. Properly designed activities could help students bridge the gap between informal experiences and later formal study of transformations. We found the importance of using interactive environments to analyze invariance (Harper 2003) and the need to visualize when doing global transformations.

## REFERENCES

- Andrews, P. (2009). The cultural location of teachers' mathematical knowledge: another hidden variable in research on mathematical knowledge for teaching. In M. Joubert (Ed.) *Proc. of the British Society for Research into Learning Math* 29 (1).
- Bulf, C. (2008). *Etude des effets de la symétrie axiale sur la conceptualisation des isométries planes et sur la nature du travail géométrique au college* (doctoral dissertation), Université Paris Diderot, ed. Irem Paris 7.



- Girnat, (2008). *The necessity of two different types of applications in elementary geometry*. Paper presented at the CERME6, Lyon. Retrived from <http://www.inrp.fr/publications/edition-electronique/cerme6/wg5.pdf>
- Harper, J (2003). Enhancing elementary pre-service teachers' knowledge of geometric transformations through the use of dynamic geometry computer software. In C. Crawford et al. (Eds.), *Proceedings of Society for Information Technology & Teacher Education International Conference 2003* (pp. 2909-2916). Chesapeake, VA: AACE.
- Hoyos, V. (2006). Functionalities of technological tools in the learning of basic geometrical notions and properties. In C. Hoyles, J-B Lagrange, L.H. Son, and N. Sinclair (Eds.), *Proceedings of 17th ICMI*, Hanoi University of Technology.
- Jahn A.P. (1998). *Des transformations des figures aux transformations ponctuelles : étude d'une séquence d'enseignement avec Cabri-Géomètre – Relations entre aspects géométriques et fonctionnels en classe de Seconde* (doctoral dissertation), Université Joseph Fourier, Grenoble I.
- Jaime, A.; Gutiérrez, A. (1995). Guidelines for Teaching Plane Isometries in Secondary School, *The Mathematics Teacher* 88.7, pp. 591-597
- Jaworski B. (2006). Theory and Practice in Mathematics Teaching Development: critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9.2. Special Issue: Relations between Theory and Practice in Mathematics Teacher Education.
- Kuzniak, A. & Vivier, L. (2008). *A French look on the greek geometrical working space at secondary school level*. Paper presented at the CERME6, Lyon.
- Llinares, S & Krainer, K (2006). Mathematics (student) teachers and teachers educators as learners. In A. Gutierrez & P. Boero (eds) *Handbook on research in Psychology of Mathematics Education*, Rotterdam Sense Publishers. Pp 329-459.
- Pawlik, B. (2004). On false convictions concerning geometric transformations of the plane in mathematics students' reasoning, Paper presented at the ICME10, Copenhagen. Retrived from [www.icme-organisers.dk/tsg10/articulos/Pawlik\\_30\\_updated\\_paper.doc](http://www.icme-organisers.dk/tsg10/articulos/Pawlik_30_updated_paper.doc)
- Stigler, J. W., Gallimore, R., & Hiebert, J. (2000). Using video surveys to compare classrooms and teaching across cultures: examples and lessons from the TIMSS video Studies. *Educational Psychologist*, 35, 87-100.
- Thaqi, X. (2009). *Apreneder a enseñar transformaciones geométricas en Primaria desde una perspectiva cultural*. (doctoral dissertation, Barcelona University. Spain). Retrived from <http://www.tesisenxarxa.net/TDX-0521109-104606/>
- Vinner, S. (1997). The Pseudo-Conceptual and the Pseudo-Analytical Thought Processes in Mathematics Learning. *Educational Studies in Mathematics*, Volume 34, Number 2, November 1997, pp. 97-129(33)

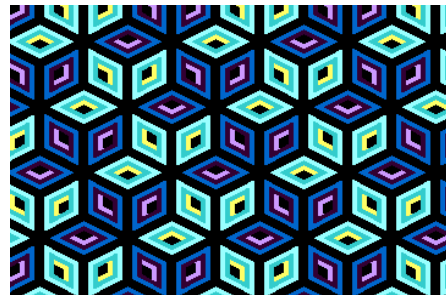


Yanik, H. B. & Flores, A. (2009). Understanding rigid geometric transformations: Jeff's learning path for translation. *The Journal of Mathematical Behavior*, [Volume 28, Issue 1](#), March 2009, pp. 41-57.

## APPENDIX

Some items of the questionnaire.

- Observe the typical Kosovar embroidery shown in the figure. There is a repeated part. Find it and draw it on the figure. Explain how to use transparent paper to show rotation as a transformation conserving size. What do you call transformations which conserve the size and shape of an object but change the position of the object?
- Observe the tiling in the following figure. There is one repeated part. Find it and draw it. Explain how to use transparent paper to show the translation as a transformation conserving size. What do you call transformations which conserve the size and shape of an object but change the position of the object?.
- Present three tasks you can use to explain symmetry and three examples to explain similarity (homothetic).
- Explain the meaning of the statement “Translation is a product of 2 symmetries”
- Transformation and movement are the same? Explain.



# **PRESERVICE TEACHERS AND THE LEARNING OF GEOMETRY**

Lina Fonseca & Elisabete Cunha

School of Education of the Polytechnic Institute of Viana do Castelo, Portugal

*The purpose of this paper is to share some ideas about an experience in teacher training. An issue that mathematics educators need to face is how to adequately work methodologies in different mathematical subjects to answer the needs of preservice teachers. Often they have little mathematical knowledge and reveal negative attitudes towards mathematics learning. In the discipline of Geometry we use a dynamic geometry application (DGA) and pattern tasks to work on geometrical concepts, to motivate preservice teachers to learn geometry and to encourage them to try to change their attitudes towards geometry.*

Keywords: Teacher training, Geometric invariant, Pattern tasks.

## **INTRODUCTION**

Nowadays in Portugal, according to Bologna, teacher training for Basic School (kindergarten to 6th grade) needs a master degree. All students who have completed secondary education and who have carried out successful tests of access to higher education can apply to these courses., although, we can have, as preservice teachers, students who have studied mathematics for twelve years and others who have only studied nine years. Besides their results, most frequently negative, they also expressed negative attitudes to learning Maths. In September 2010, a new maths programme for Basic Education (grade 1 to 9) (ME, 2007), will be generalized in all schools across our country. The goal of this document is to develop mathematical knowledge of all students in subjects like Numbers and Operations, Geometry and Measure, Algebra, Statistics and Probability, and, at the same time, to develop their capacity to solve problems, to communicate mathematically and to reason. These two issues set challenges for teacher training, such as deciding which type of methodologies to use with future teachers in ways in which they develop knowledge, capacities and attitudes which will allow them to work with children, in accordance with the new mathematics programme.

## **THEORETICAL PERSPECTIVES**

International and national documents (e.g. NCTM, 2000; DEB, 2001) argue that during the compulsory education of mathematics students should acquire knowledge and understanding of facts and basic procedures of mathematics, develop their ability to solve problems, to investigate, to formulate, to test and to evaluate conjectures, to argue mathematically, to analyze mathematical arguments presented by teachers and peers, to communicate their mathematical ideas and options, thus contributing to the organization and systematization of their mathematical thought, should develop a positive attitude towards mathematics, and also to develop their autonomy at work.

On the same line the new Maths Programme for Basic Education (ME, 2007) maintains that all students should be able to identify and understand facts and basic procedures of Mathematics, develop a positive attitude towards mathematics and must be able to work with mathematical ideas in diverse representations. They should be able to communicate their ideas and interpret other's ideas, organizing and clarifying their mathematical thinking, to elaborate maths reasoning, to solve problems, to make connections, and be autonomous. Teacher educators need to address challenges created by the principles sustained in these documents.

Mathematics is usually considered as a science of patterns and the study of patterns could be a way of motivating the students to learn mathematics (Fonseca, 2009; Rivera, 2009). When we observe aspects that repeat themselves in a given situation, if you comprehend the means that produce the repetition, if you observe relations that remain invariant when everything around them changes, if they try to understand and explain the reasons behind those relations, students develop a more profound comprehension of the matters worked on. They create a more dynamic conception regarding mathematics and can say that the patterns are in fact subjacent to mathematics, particularly to geometry.

In the scope of geometry the resource on a DGA could help students gain many experiences focusing on invariants, which allows them to see the general through the individual and to establish, analyse and evaluate conjectures and mathematical relations (Chazan & Yerushalmy, 1998; Laborde, 1993; Villiers, 2000). We consider this important for students' motivation for the comprehension and explanation of the reasons that justify the relations detected, contributing to the treatment of proof in mathematics, helping them to understand the importance of justifications that explain (Hanna & Jahnke, 1996; Villiers, 1999). We use many images in geometry connected to geometric concepts and, sometimes preservice teachers try to build reasoning having as a base mistaken drawings and only visual appreciation, as for example the fact that 'building' a triangle in which the side lengths measurement is represented by consecutive whole numbers, 1, 2 and 3 cm, by analogy with rectangular triangle sides 3, 4 and 5 cm, without having addressed the question of impossibility. This is an example that sets them into the geometric paradigm I (GI) (Houdement & Kuzniak, 2006), when you aim them to use figures representing a class of mathematical objects, they establish general relations and build justifications, even though with a resource to local axiomatic, and therefore they would have to be on the geometric paradigm II (GII). The passage from one paradigm to the other and the paradigm recognition in which they work on turns out to be a difficulty, just as referred by Parzysz (2003).

Some authors (e.g. Malone, 1996; Hefendehl-hebeker, 1998) refer to the difficulty that preservice teachers show when embracing the ideas of the proposed reforms to mathematics teaching and that those difficulties are rooted in the way their prior mathematical education took place and highlighted the beliefs and the attitudes,

many times negative, that they relatively manifest to mathematics learning and in particular to geometry (Carrillo & Contreras, 2000; Pajares, 1992; Thompson, 1992). For that reason, we share Swars, Smith, Smith & Hart's (2009) opinion that defends the focus on beliefs of preservice teachers during the initial education when new programmes are proposed, as is happening at this moment in Portugal. Therefore in their teacher training courses, preservice teachers need to be successful through the same experiences, which we hope they will develop with their future students.

Thus, the next question is unavoidable: how do we train future teachers so they can develop the principles sustained by international documents and by the mathematical national curriculum?

### **THE EXPLORATORY STUDY**

As in Portugal, according to the new maths program (ME-DGIDC, 2007), all students must develop some mathematical abilities, our problem is knowing "*How to work with preservice teachers in geometry in such a way that it is possible: (a) to develop their mathematical knowledge; (b) to develop their reasoning; (c) to develop a positive attitude toward mathematics; and (d) to gain more confidence in their capacities to do maths?*".

The exploratory study was oriented by the following general questions: (a) which geometric knowledge do preservice teachers display?; (b) which difficulties do preservice teachers display in solving the proposed tasks?; (c) which attitudes do preservice teachers display regarding geometry? We focus on an exploratory qualitative approach. To obtain more detailed information we selected two future teachers per class, from the ones that had only studied up to 9th grade in maths, that were available to participate in the study and were willing to share their difficulties. Five were between the ages of 20 and 23 and one was 33 years old. Data was collected through tasks, sorted out by all students, and even observations and semi-structured interviews with the six case students. The interviews were managed with the objective to know which geometrical knowledge was acquired, which difficulties they had in the task resolution and which attitudes they expressed regarding the study of geometry. Data analysis was in a holistic, descriptive and interpretative way.

### **Context and participants**

The exploratory study was developed in the 2009/2010 school year in a semester course (16 weeks) of geometry, with a weekly workload of four hours. Topics worked in Geometry are: plane geometry, triangles, congruent triangles and similarity, area of polygons and of the circle, space geometry, polyhedron, regular polyhedron, surfaces and volumes. Of the 79 students involved, organized into three classes, only 8% had studied mathematics from kindergarten to grade 12, with the remaining having studied mathematics until the 9th grade. The majority of the students revealed negative attitudes toward mathematics and they were "afraid" to study geometry: "to say the word [GEOMETRY] is horrifying" (S1); "I never liked

geometry” (S4); “I never understood geometry” (S2); “I can’t see anything” (S3); “I can’t see the plans” (S3).

According to the profile of the preservice teachers we decided to organize the teaching of geometry in two dimensions: work with problem solving and a DGA to look for analogies and patterns. Thus, we tried to develop the geometrical knowledge of the preservice teachers, but also giving them the possibility of experiencing a different method of learning mathematics, focusing more on their work and, we expected it, to be more challenging.

### Some tasks and discussion

To give students opportunities to develop their capacity, to notice patterns and aspects that remain invariant while everything around them is changing, we used a dynamic geometry application (Geometer'Sketchpad). We chose this application because it already existed at school, teachers were used to working with it and because in certain situations it becomes more “demanding” than other similar applications: for example, for the construction of an equilateral triangle or a square it is necessary to resort to geometrical properties and not only the use of the application.

Polygons were built with this tool and the students explored triangles and quadrilaterals properties, as the sum of the amplitudes of the internal angles, location of the notable points of the triangle and solved problems.

Task 1 - Construct a dynamic triangle. Measure the angles. What is the sum of the angles? Drag a vertex of the triangle. Does the sum remain the same? Formulate a conjecture on the sum of triangle's angles. Explain why the conjecture is true.

We began with this simple task using the DGA for exploration and students easily began to solve the task. The triangle's construction did not raise problems and the actual task indicated the following step “sum of the angles”. The conjectures arose immediately: “It's 180°”, “The sum is 180°”. Just as the actual task indicated, they dragged a vertex from the triangle and verified that the sum was unchangeable. We discussed if the invariant was characteristic of some type of triangle, having concluded that it was general. It was a property. When we asked why the conjecture presented was true they responded that the examples were there. They considered the examples sufficient, just as Kunimune, Fujita & Jones (2009) related. One of them said:

S1- We can see. Why do we need a proof?

Tutor – You know that but how do you explain the conjecture?

S1 – I see.

Tutor – Ok, you see. Tell me why this happens in triangles?

Other student goes on:

#### Working group 4

- S2 - We may make some copies of each of the angles and join them together.
- Tutor - What do you think you will get?
- S2 - ...
- Tutor - Try.
- S2 - ... [I get] an angle with  $180^\circ$ .

Students revealed difficulties in explaining why the conjecture was true. They were convinced that it was, by the experimentation carried out, but they did not know how to explain why. Even student S2 had to be helped in concluding his idea. After that, another explanation of this relation was discussed resorting to paper folding, justification that can be used with 5th and 6th grade students and a parallel was made with a proof that normally appear in maths text books in which we turn to the sketch of a straight line parallel to one of the triangle's sides passing through the opposite vertex. This proof (in GII) turned out to be more demanding to the students because they needed to resort to the various relations of congruency between angles. The preference for paper folding was unanimous because "this way I understand" (S1).

During this task the six case students revealed their knowledge regarding the types of triangles, internal angles and the straight angle. They had the capability to present the conjecture, although with some difficulties with mathematical language used, in understanding that the particular cases experienced did not explain and difficulties in the construction of an explanation for the conjecture, difficulties that confirm previous studies also carried out with preservice teachers, in spite of them being students with more mathematics education (Fonseca, 2009). They revealed expectation in the DGA exploration and showed confidence during the task and in the conjecture formulation, confidence that was conferred on them by DGA. The justification option by paper folding had an objective to captivate the student's attention and improve their attitude regarding justification constructions, that by their point of view is beyond their capacities "I can't justify... as intended ... many times I don't understand (the justifications) ... but I understood this one, I didn't know that we could do it like this" (S4).

**Task 2** - Construct a dynamic triangle. Construct an external angle of the triangle. Measure the external angle. Try to relate this angle with the triangle's internal angles. Formulate a conjecture. Drag a vertex of the triangle. Does the conjecture remain the same? Explain why your conjecture is true.

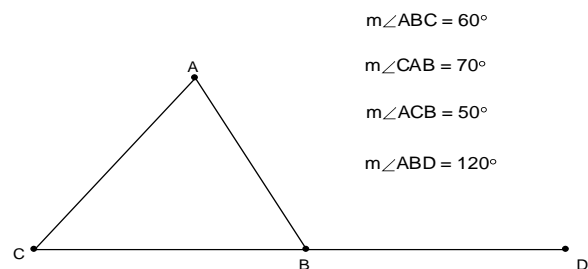
Students initiate the task by the construction of the triangle and the external angle's marking. Unlike the previous task, it was not suggested how to find the conjecture.

Various conjectures took place that not always revealed themselves to be valid, as for example, "the external (angle) is bigger (smaller) than the internal (adjacent angle)", "the external (angle) is bigger than the other (angles)" and that were abandoned with DGA's help, by the triangle's vertex dragging. We discussed the possibility of the external angle being congruent with the internal adjacent angle and the students

#### Working group 4

concluded, without difficulty, that it always happened with one of the angles, external or its internal adjacent angle was right. Other conjectures came up “both (internal and external adjacent angles) make a  $180^\circ$  angle”, “the four angles present more than  $180^\circ$ ”, but students reveal difficulties in relating the external angle with the internal ones to make the conjecture intended. After some exploration, students ran out of ideas to continue the work and teachers decided to make a suggestion – to observe the angle’s measurement values attentively, that they were obtaining, and to think of everything that they knew about triangles - which proved to be useful for the continuation of student’s work.

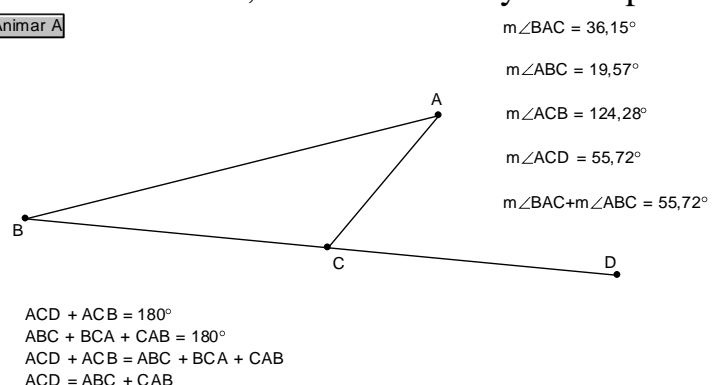
Observing the values produced by the application, some students, very few, noticed special cases (Fig. 1) they observed “120 is 70 plus 50.” Is that it?, they questioned. This aspect reveals the necessity to obtain the teacher’s agreement for the relations that they were detecting, which displays a low confidence attitude in their capacities to construct mathematical knowledge.



Using the **DGA** they tested if the relation maintained itself. When they verbalized the conjecture it already had been tested. The conjectures presented needed to be analyzed and rewritten. The following version was accepted “An external angle of a triangle is equal to the sum of the internal angles that has a different vertex”.

We want our students to understand that mathematical results always need an explanation. Why does this happen? This question was put forward. To focus the student’s attention, the teacher asks “What do you already know about angles and triangles?” The following aspects were related to, even with very little precise language: “The internal angles’ sum is  $180^\circ$ ”, “the external can be bigger, smaller or the same to the interior”, the bigger angle objects to the larger size“, “both (external and internal adjacent angles) create a  $180^\circ$  angle“. We discussed what could be useful and focused on relations implicating  $180^\circ$ .

Animar A



In each class some students, excluding case students, get an explanation accepted by their peers. It was an algebraic explanation (in GII) as shown below.

One student asked if he could do paper folding. We asked how he would like to do it and he explained “we can cut out two angles in the triangle and cover the external angle. It’s going to work” (S3). His idea was carried out with all the students. It is

understood as an explanation in GI.

During this task the six case students turned out to have knowledge regarding triangle properties, which they related to while they looked for an explanation for the relation they claimed. They had difficulties in formulating the conjecture, since it was not suggested by the task. It was necessary to guide the observation carefully by the values provided by the DGA to be able to conjecture although they showed to have difficulties with the mathematical language being used. They continued to be “trapped” into the particular cases tested and revealed difficulties in proof, to explain the conjecture, difficulties that confirm previous studies also carried out with preservice teachers (Fonseca, 2009). They continued to reveal expectations in the DGA exploration, but as it was referred to by various students “I thought that it would be easier, but it helps” (S2).

After these tasks were explored tasks relating to the sum of the internal angles of a quadrilateral, pentagon, hexagon, ... guided by teachers, students formulated a conjecture on the sum of the angles of a  $n$ -polygon. Some tasks follow each other, and students said that they had never worked like this in geometry.

In order to enhance their creativity, students were asked to prepare and submit to the class a theme associated with the areas and volumes of solids. Each class was divided into five groups and the subjects were randomly selected. To carry on with their work students needed to present exercises and problems, among other things. One of the items was tied to the posing of problems, in this case from the scope of geometry, as one of the aspects is referred to as an enhancer of creativity (e.g. Meissner, s/d).

Task 3 - For your geometric theme (cubes and quadrilateral prisms, other prisms, cylinder, pyramids, cone) present:  
(a) Examples of the solids in the world; (b) Solid Planning; (c) Formulas (areas and volume); (d) Exercises /problems.

The students had teacher support in their study and topic organization. Issues related to proposals intended to be present and would be shared among all students via a digital platform, which were discussed. During preparation or during work in the classroom, one of the most common questions was related with the need to use different formulas for calculating the surface areas of the solids. This fact frightened the students. The real need for all those formulas was discussed. Most of the students relied on their necessity, with some voices questioning this option, saying that they already knew the formula to calculate the areas of squares, rectangles, triangles, ... and knew that they had just to think about it and apply them.

They [math teachers] always asked me to do as I was taught and not otherwise. I could do it but it was wrong. Now in geometry I realized that I could do it in a different way. We just need to explain and do things right (S3).

I didn't know that I knew maths (S4).



## FINAL REMARKS

Based on data collected, we can say that the case students revealed:

(a) basic geometry content knowledge, as types of angles, triangles, quadrilaterals, polygons, solids, regular convex polyhedrons, geometrical relations in triangles and some formulas for the area and volume calculation. They revealed the ability to formulate conjectures in simple situations, in starting to communicate their reasoning and in certain situations they showed an ability to reason correctly and in an autonomous way. One of the paradigmatically cases was the student S4. In questions involving the area and volume calculations he said:

- S4 - I don't know the formulas. I don't know how to do it.
- Tutor - Which formulas don't you know?
- S4 - The lateral and total areas.
- Tutor- Find another way. Do you know any formulas?
- S4 - I do, but not these ones.
- Tutor – Can you use the ones you know?
- S4 – Can I? I don't have to use these ones?
- Tutor- What do you think?
- S4- I don't know.
- Tutor- Try.

This challenge allowed him to solve the particular questions that occurred and allowed him to understand that the formulas that “they didn't know” could be derived from the ones they knew. This was an episode that caused a change of attitude in this student regarding geometry, after that he was not so preoccupied with what he did not know, but in reason to use the knowledge he possessed. This change happened because the student shared his frustration, the teacher had the opportunity to challenge and the student accepted the challenge. Even in teacher training environments where the teacher intends that the students are comfortable to share their doubts, many times they do not and opportunities for change are lost. This is an aspect that, in our opinion, should continue to deserve the reflection of teacher educators.

(b) difficulties, as in visualizing, using mathematical language, conjecture in more complex situations, reason and making proofs, justification /explanation.

(c) commitment, striving themselves to solve proposed tasks, although showing very little confidence in their reasoning capacities and almost always hoping for the teacher's guarantee, regarding the quality of their work. In case students signs of changing of their negative attitudes to geometry were detected; changes were influenced by their responsiveness to the adopted methodologies. To develop

students' confidence in their capacities seems to be a path to the improvement of their knowledge.

After this exploratory study we conclude that most preservice teachers revealed that they understood some facts and basic geometry procedures; they worked with diverse representations (iconic, symbolic, ...); they began to communicate their reasoning; they found analogies and solved some problems. However, they revealed many difficulties in visualizing, reasoning and explaining their reasoning, in making connections between different mathematical topics, in being confident in their capacities, because most of them were always afraid of making mistakes, and being autonomous. Most of the time students need teacher support to validate the quality of their work. They said that this was a different way to learn geometry.

As mathematics teacher educators we consider that the options taken revealed themselves to be positive for the students, but we are conscious that we have a long way to go, to learn, to experiment and to reflect on our options when we decided on methodologies and tasks to present and challenge our preservice teachers, but our goal is to make them become more knowledgeable and confident in mathematics.

## REFERENCES

- Carrillo, J. & Contreras, L. (2000). El amplio campo de la resolución de problemas. In J. C. Yáñez & L. C. Contreras (Eds.), *Resolución de problemas en los albores del siglo XXI: Una visión internacional desde múltiples perspectivas y niveles educativos* (pp.13-38). Huelva: Hergué, Editora Andaluza.
- Chazan, D. & Yerushalmy, M. (1998). Charting a course for secondary geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp.67-90). London: Lawrence Erlbaum Associates Publishers.
- Conney, T. (1985). A beginning teacher's view of problem solving. In *Journal for Research in Mathematics Education*, 16 (5), 324-336.
- Conney, T. (1994). On the application of science to teaching and teacher education. In R. Biehler, R. Scholz, R. Stäßer & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp.103-116). London: Kluwer Academic Publishers.
- DEB (2001). *Currículo Nacional do Ensino Básico*. Lisboa: ME.
- Fonseca, L. (2009). Looking for patterns in geometric transformations with pre-service teachers. In I. Vale & A. Barbosa (org.) *Patterns. Multiple perspectives and Contexts in Mathematics Education*, pp.111-121. Viana do Castelo: Escola Superior de Educação, Projecto Padrões.
- Hanna, G. & Jahnke, H. N. (1996). Proof and proving. In A. J. Bishop et al. (Eds.), *International handbook of mathematics education* (pp.877-908). Netherlands: Kluwer Academic Publishers.

#### Working group 4

- Hefendehl-Hebeker, L. (1998). The practice of teaching mathematics: Experimental conditions of chance. In F. Seeger, J. Voigt & U. Waschescio (Eds.), *The culture of the mathematics classroom* (pp.104-126). Cambridge: Cambridge University Press.
- Houdement, C. & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie. *Annales de Didactique et de Sciences Cognitives*, volume 11, pp. 175-193, Irem de Strasbourg.
- Kunimune, S., Fujita, T. & Jones, K. (2009). Strengthening students' understanding of "proof" in geometry in lower secondary school. *Workinh Group 5. Cerme 6*.
- Laborde, C. (1993). The computer as part of the learning environment: the case of geometry. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp.48-67). Berlin: Springer-Verlag.
- Malone, J. (1996). Preservice secondary mathematics teachers' beliefs: two case studies of emerging and evolving perceptions. In Puig & Gutiérrez (Eds.), *Proceedings of the 20 th Conference of the International Group for the PME* (vol.3, pp.313-320). Valencia, Spain: Universitat de València.
- Meissner, H. (s/d). Creativity and Mathematics Education. <http://wwwmath.uni-muenster.de/didaktik/u/meissne/WWW/indengl.htm> (July 2010).
- ME (2007). Programa de Matemática para a Educação Básica. Lisboa: DGIDC.
- Parzysz, B. (2003). Pre-service Elementary Teachers and the Fundamental Ambiguity of Diagrams in Geometry Problem Solving. *Cerme III*.
- Rivera, F. (2009). Visuoalphanumeric Mechanisms that support Pattern Generalization. In I. Vale & A. Barbosa (org.) *Patterns. Multiple perspectives and Contexts in Mathematics Education*, pp.123-136. Viana do Castelo: Escola Superior de Educação, Projecto Padrões.
- Swars, S.L., Smith, S.Z., Smith, M.E. & Hart, L.C. (2009). A longitudinal study of effects of a developmental teacher preparation program on elementary prospective teachers' mathematical beliefs. *Journal of Mathematics Teacher Education*, 12(1), 47-66.
- Villiers, M. de (2000). Developing understanding of proof within the context of defining quadrilaterals. Outubro, 2000.
- Villiers, M. de (1999). *Rethinking proof with the Geometer's Sketchpad*. Emeryville: Key Curriculum Press.

# **TOWARDS A COMPREHENSIVE THEORETICAL MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING AND ITS RELATION WITH PROOF**

Eleni Deliyianni\*, Athanasios Gagatsis\*,

Annita Monoyiou\*, Paraskevi Michael\*, Panayiota Kalogirou\* and Alain Kuzniak\*\*

\*University of Cyprus, \*\* University Paris Diderot

*This research study examined the fit of various models regarding geometrical figure understanding and its relation with proof. Data were collected from 457 middle and high school students. Structural equation modelling affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification, construction of a figure and proof. The three second-order factors which represented the perceptual, operative and sequential apprehension were regressed to a third-order factor that corresponded to the geometrical figure understanding. Results indicated that geometrical figure understanding has a strong effect on logic apprehension. Data analysis provided support for the invariance of this structure across the two educational levels.*

## **INTRODUCTION AND THEORETICAL FRAMEWORK**

Fischbein (1993) called geometrical figures “figural concepts” since these entities are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). The double status of external representation in geometry often causes difficulties to students when dealing with geometrical problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998). Duval (1995, 1999) distinguishes four apprehensions for a “geometrical figure”: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, perceptual apprehension refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. Sequential apprehension is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. Discursive apprehension is related with the fact that mathematical

properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through operative apprehension that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one made the figure larger or narrower, or slant, while the place way refer to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations.

Recently, some researchers (Deliyianni, Elia, Gagatsis, Monoyiou, & Panaoura, 2009; Elia, Gagatsis, Deliyianni, Monoyiou, & Michael, 2009) made an effort to verify empirically some of the cognitive processes underline the geometrical figure understanding proposed by Duval (1995, 1999). Elia et al. (2009) gave emphasis on the cognitive processes involved in operative apprehension. Furthermore, Deliyianni et al. (2009) affirmed the existence of a third-order model that involved six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. Both studies also suggested the invariance of this structure across elementary and secondary school students. Keeping in mind the underlying cognitive complexity of geometrical activity (Duval, 1995) and the transition problem from one educational level to another universally (Mullins & Irvin, 2000) the main aim of this research study was to confirm a comprehensive theoretical model concerning middle and high school students' geometrical figure understanding which involves the whole spectrum of geometrical figure apprehension types, i.e. perceptual, discursive, sequential and operative apprehension and the relation between their corresponding cognitive processes. It should be mentioned that concerning discursive apprehension Harada, Gallou-Dumiel and Nohda's (2000) conceptualization is used. Harada et al. (2000) indicated that the hypothetical-deductive proof is produced by this kind of apprehension. In fact, the discursive apprehension or logic apprehension, the term which is used in the present paper, is produced by inferences based on definitions and valid procedures of proof.

## **HYPOTHESES AND METHOD**

In the present paper the following hypotheses were examined: (a) There is a relation between students' geometrical figure understanding and their performance in proof tasks, (b) Perceptual, sequential and operative apprehension influence middle (grade

9) and high (grade 10) school students' geometrical figure understanding, (c) Perceptual and recognition abilities have a differential effect on perceptual apprehension, (d) The three ways of figure modification (i.e. mereologic, optic and place way) have a differential effect on operative apprehension, (e) The abilities to construct and describe a figure's construction differentially affect sequential apprehension, (f) Inferences based on definition and procedures for proof differentially affect discursive (logic) apprehension, (g) There are similarities between middle and high school students in regard with the structure of their geometrical figure understanding and (h) Differences exist in the geometrical figure understanding performance of middle and high school students.

The study was conducted among 457 students, aged 15 to 16, of middle (Grade 9) and high (Grade 10) schools in Cyprus (252 in Grade 9, 205 Grade 10). Taking into account, Duval's (1995, 1999) apprehensions for a "geometrical figure" the a priori analysis of the test (Appendix) that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (PE1a, PE1b, PE1c, PE1d, PE1e, PE1f, PE1g), 2 (PE2a, PE2b, PE2c, PE2d, PE2e, PE2f) and 3 (PE3a, PE3b, PE3c). These tasks examine students' perceptual apprehension of a geometrical figure. The task 1 examines students' ability to name figures. The tasks 2 and 3 examine their ability to discriminate and recognize in the perceived figures several subfigures.
2. The second group of tasks includes task 4 (OP4), 5 (OP5), 6 (OP6), 7 (OP7), 8 (OP8) and 9 (OP9). These tasks examine students' operative apprehension of a geometrical figure. The tasks 4 and 5 require a reconfiguration of a given figure, the tasks 6 and 7 an optic way of modification, while the tasks 8 and 9 demand the place way of modifying two given figures in a new one in order to be solved.
3. The third group of tasks consists of the tasks 10 (SE10), 11 (SE11), 12 (SE12) and 13 (SE13) that correspond to sequential figure apprehension. The tasks 10 and 11 require students to construct a figure, while the tasks 12 and 13 investigate students' ability to describe the construction of a figure.
4. The fourth group of tasks includes the verbal problems 14 (LO14), 15 (LO15), 16 (LO16), 17 (LO17), 18 (LO18) and 19 (LO19) that correspond to logic apprehension. On the one hand, the verbal problems 14, 15, 16 and 17 demand inferences based on definitions in order to be solved. On the other hand, tasks 18 and 19 require inferences based on procedures for proof for their solution.

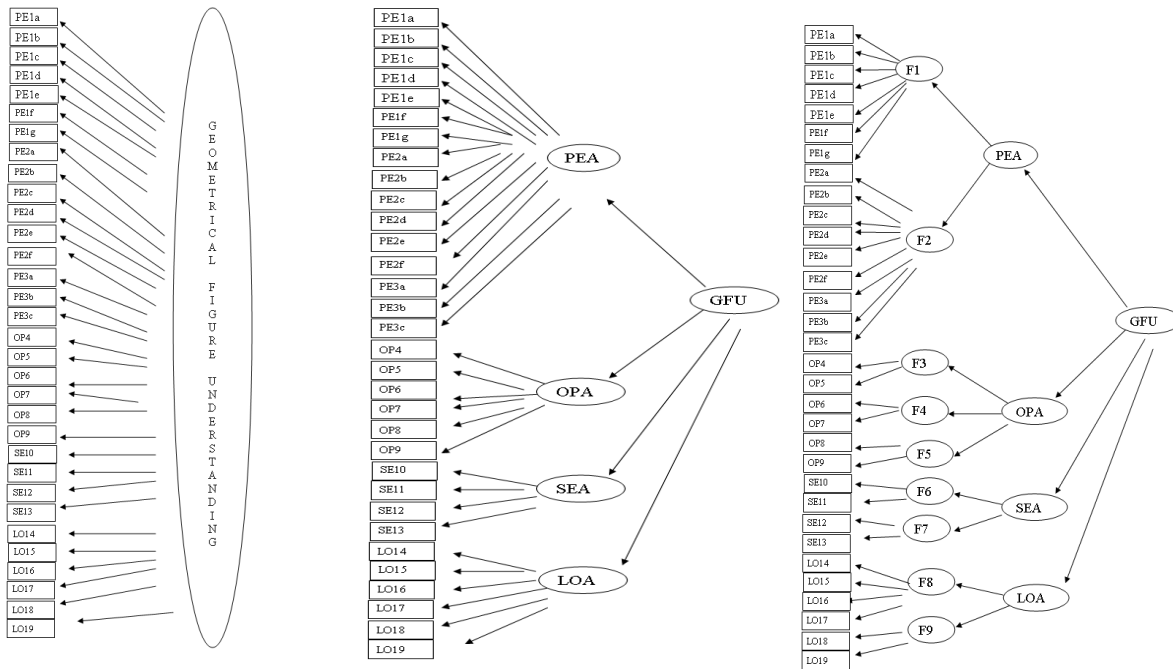
Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with PE, OP, SE and LO corresponding to perceptual, operative, sequential and logic apprehension (proof tasks), respectively, followed by the number indicating the exercise number.

Confirmatory factor analysis (CFA), by using the EQS program, was used to explore the hypotheses about the structural organization of the various dimensions investigated here (Bentler, 1995). The tenability of a model can be determined by using the following measures of goodness-of-fit:  $\chi^2$ , CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model:  $\chi^2/df < 2$ ,  $CFI > 0.9$ ,  $RMSEA < 0.06$ . A multivariate analysis of variance (MANOVA) was also performed to examine if there were statistically significant differences between middle and high school students concerning their performance in the various dimensions of the figure understanding.

## RESULTS

*Confirmatory factor analysis model.* A series of CFA models were tested and compared. Specifically, the first model involved only one first-order factor associated with all the tasks. This model was the most parsimonious, it disregarded though the related theory and past empirical work which pointed out that different cognitive processes are needed in order to solve: perceptual, operative, sequential and logic apprehension tasks. The fit of this model was poor [ $CFI = 0.52$ ,  $\chi^2(702) = 3933.98$ ,  $RMSEA = 0.10$ ]. The second model that was constructed and tested involved four first-order factors corresponding to the perceptual, operative, sequential and discursive apprehension and one second-order factor on which all the first-order factors were regressed. A chi-square difference test indicated a significant improvement in fit between the first and the second model [ $\Delta\chi^2(43) = 1505.09$ ,  $p < 0.001$ ] due to the second-order factor inclusion. However, the fit of the second model was also poor [ $CFI = 0.81$ ,  $\chi^2(459) = 1682.99$ ,  $RMSEA = 0.08$ ].

The third model took into account Deliyianni's et al. (2009) findings and moved a step forward involving sequential apprehension dimension, the three ways of figure modification in operative apprehension dimension and the deductive reasoning dimension. A chi-square difference test indicated a significant improvement in fit between the second and the third model [ $\Delta\chi^2(52) = 611.83$ ,  $p < 0.001$ ]. Besides, the fit of the third model was acceptable [ $CFI = 0.91$ ,  $\chi^2(511) = 1071.162$ ,  $RMSEA = 0.05$ ]. Even though the third model fitted the data reasonably well, the need to confirm that this was the best fitting model arose. Taking into account that visualisation is thought to be useful to some aspects of mathematical proof (Hanna & Sidoli, 2007), a fourth model was tested. Its fit was acceptable [ $CFI = 0.94$ ,  $\chi^2(444) = 815.08$ ,  $RMSEA = 0.04$ ], as well. A chi-square difference test indicated a significant improvement in fit between the third and the fourth model [ $\Delta\chi^2(67) = 256.08$ ,  $p < 0.001$ ] due to the causal relation between geometrical figure understanding and logic apprehension inclusion. The first, second and third tested models are presented in Figure 1. Factor loadings are omitted.



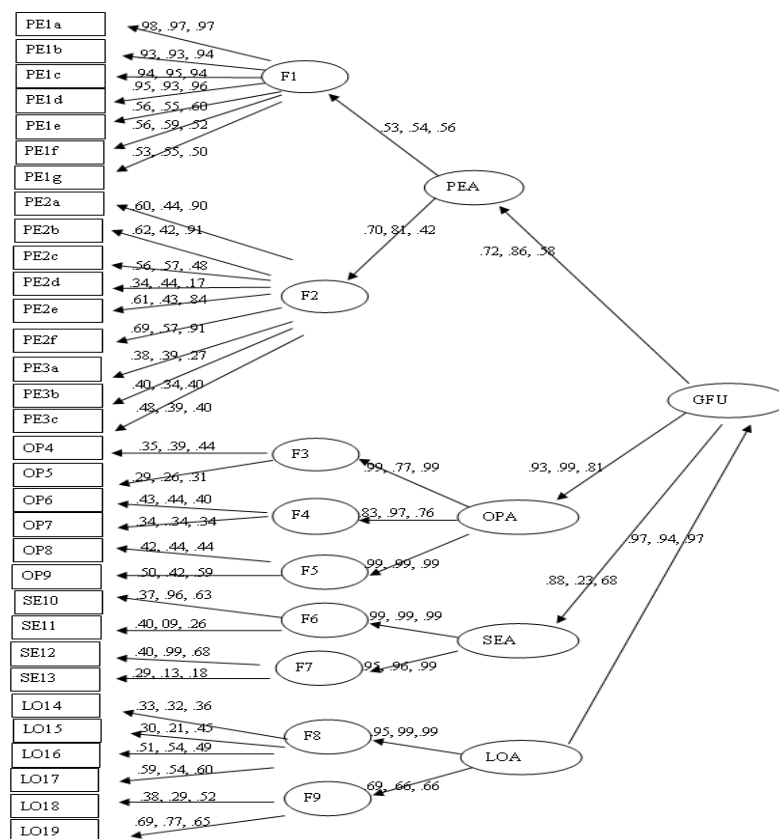
**Figure 1. The first, second and third CFA tested models**

Figure 2 shows the results of the elaborated model, which fitted the data reasonably well. The first, second and third coefficients of each factor stand for the application of the model in the whole sample (Grade 9 and 10), middle (Grade 9) and high (Grade 10) school students, respectively. Particularly, the third-order model which is considered appropriate for interpreting geometrical figure understanding, involves nine first-order factors, four second-order factors and one third-order factor. The four second-order factors correspond to the geometrical figure perceptual (PEA), operative (OPA), sequential (SEQ) and logic (LOA) apprehension, respectively. Perceptual, operative and sequential apprehensions are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our second hypothesis. On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension (hypothesis c). On the second-order factor that corresponds to operative apprehension the first-order factors F3, F4 and F5 are regressed. The first-order factor F3 consists of the tasks which require a modification of a given figure in a mereologic way. The tasks which demand an optic way of modifying a given figure compose the first-order factor F4 and the tasks demanding the place way of modifying two given figures in a new one in order to be solved constitute the first-order factor F5. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding (hypothesis d). The first-order factors F6 and F7 are regressed on the second-order factor that stands for sequential



apprehension. The first-order factor F6 refers to the tasks which demand the construction of a figure, while the first-order factor F7 consists of the tasks in which the description of a figure's construction is needed. Thus, the results indicate that the two abilities differentially affect sequential apprehension (hypothesis e). According to the factor loadings, operative apprehension is more strongly related with geometrical figure understanding than perceptual and sequential apprehension.

On the second-order factor that stands for logic apprehension the first-order factors F8 and F9 are regressed. The first-order factor F8 refers to the tasks which require inferences based on definition, while the first-order factor F9 to the tasks which inferences based on processes of proof are needed. Thus, the findings reveal that the kind of inferences has a differential effect on this kind of apprehension (hypothesis f). Loadings indicate that geometrical figure understanding have a strong effect on logic apprehension (hypothesis a).



**Figure 2. The CFA model of the geometrical figure understanding in relation with proof processes.**

To test for possible similarities between the two educational levels concerning their geometrical figure understanding the proposed three-order factor model is validated for middle and high school students separately. The fit indices of the model tested for both middle [ $\chi^2$  (445) = 658.59, CFI= 0.94, RMSEA= 0.04] and high school students are acceptable [ $\chi^2$  (438) = 659.61, CFI= 0.94, RMSEA= 0.05]. Thus, the results are in line with our hypothesis that the same geometrical figure understanding structure

holds for both the middle and the high school students. It is noteworthy that some factor loadings are higher in the group of the high school students suggesting that the specific structural organization potency increases across the ages. Besides, the factor loading in grade 10 regarding perceptual apprehension is lower than in grade 9, while the factor loading for sequential apprehension is higher than the corresponding in grade 9. This finding indicates that as students grow up are based more on mathematical properties and less on perceptual laws and cues.

*The effect of students' educational level.* Table 1 presents the means and the standard deviations for perceptual, operative, sequential and logic apprehension in the two educational levels. Overall, the effect of students' educational level is significant (Pillai's  $F_{(4, 452)}=7.03$ ,  $p<0.001$ ). In particular, the mean value of high school students in geometrical figure perceptual apprehension (PEA) is statistically significant higher ( $F_{(1,452)}=16.94$ ,  $p<0.001$ ) than the mean value of middle school students. Similarly, the mean value of high school students in operative apprehension tasks (OPA) is statistically significant higher ( $F_{(1,452)}=14.26$ ,  $p<0.001$ ) than the mean value of middle school students. In the same way, the mean value of high school students' performance in sequential apprehension tasks (SEA) is statistically significant higher in comparison with middle school students' performance ( $F_{(1,452)}=11.88$ ,  $p<0.001$ ). Even though, the performance of high school students in logic apprehension tasks (LOA) is also higher than the performance of middle school students this difference is not statistically significant ( $F_{(1,452)}=3.83$ ,  $p=0.05$ ). Therefore, the findings verify the last hypothesis stating that differences exist in the performance of middle and high school students. In particular, high school students' performance is higher in all the types of geometrical figure apprehension.

Educational Level	PEA		OPA		SEA		LOA	
	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
Middle	0.78	0.21	0.59	0.23	0.08	0.18	0.27	0.21
High	0.86	0.20	0.67	0.24	0.14	0.21	0.31	0.25

**Table 1: Means and standard deviations in the different dimensions of the geometrical figure apprehension for middle and high school students**

## CONCLUSIONS

This study examined the fit of various models regarding geometrical figure understanding and its relation with proof processes. Using structural equation modelling we constructed and verified a comprehensive model for geometrical figure understanding. Moving a step forward in relation with previous studies (e.g. Elia et al., 2009; Deliyianni et al., 2009) which verified Duval's (1995, 1999) taxonomy, the proposed model involves the whole spectrum of geometrical figure apprehension

types and the relation between their corresponding cognitive processes with cognitive processes concerning proof.

According to the results, the three second-order factors which stand for perceptual, operative and sequential apprehension are regressed to a third-order factor that corresponds to the geometrical figure understanding. Results suggest that geometrical figure understanding has a strong effect on students' performance in proof tasks. This is in line with the findings of previous research studies (e.g. Hanna & Sidoli, 2007; Giaquinto, 2007) that suggested that there is a potential contribution of visual representations to mathematical proof. Findings reveal also that operative apprehension is the one which contributes the most to geometrical figure understanding. Taking into account that visualization consists only operative apprehension (Duval, 1999) the important role of this kind of apprehension confirms empirically Duval's (1999) opinion that there is not understanding in geometry without visualization. The specific result indicates also that teaching and learning should give emphasis in this kind of apprehension since visualization is not primitive. In fact, the use of visualization requires specific training, specific to visualize each register (Duval, 1999). However, the model points out the important role of the other types of geometrical figure apprehension, as well, taking into account that even though coordination between them is needed each one is distinct from the other (Duval, 1999). Besides, findings affirmed the existence of nine first-order factors revealing the differential effect of perceptual and recognition abilities and the ways of figure modification, construction and proof. Thus, the results verified Duval's (1995, 1999) and Harada's et al. (2000) categorization, respectively.

In addition to extent our knowledge about students' geometrical figure understanding, this study may give valuable information to curriculum designers and teachers of both middle and high school education. The elaborated model offers teachers a framework of students' thinking while solving a wide range of geometrical tasks in a systematic manner within and between the two educational levels. Therefore, the proposed framework may be used as a tool in mathematics instruction and designing tasks on geometry in both middle and high school. The framework of this study appears to be useful from an assessment perspective, as well. It may provide teachers with valuable and specific information on students' thinking in geometry based on prior knowledge and enable them to enhance this thinking by giving appropriate support through the tasks focused on the competences and cognitive processes for the geometrical figure understanding and the proof.

Concerning age, it is important to stress that the structure of the processes underlying the geometrical figure understanding in relation with proof processes was invariant across the two age groups tested here. These findings enhance the validity of the proposed framework and support its potential to coherently describe and predict students' understanding in geometry irrespectively of their grade, even during the transitional phase from middle to high school. However, findings reveal differences

between middle and high school students' performance. In fact, the results provide evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, middle school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception, is enriched by the experiment and privileges self-evidence and construction. On the other hand, high school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. As a result, in the case of middle school students geometrical figure is an object of study and of validation, while in the case of high school students geometrical figure supports reasoning and "figural concept" (Fischbein, 1993). However, the knowledge produced by quantitative research studies might be too abstract and general for direct application to specific local situations, contexts, and individuals. For this reason, further research is needed to evaluate the feasibility of using this framework for developing effective instructional programs for the teaching of geometry in regular classroom situations in middle and high education.

## ACKNOWLEDGE

This paper is a part of the research project "Ability to Use Multiple Representations in Functions and Geometry: The Transition from Middle to High school" (0308(BE)/03) founded by the Research Promotion Foundation of Cyprus

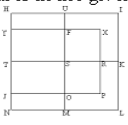
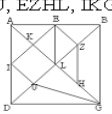

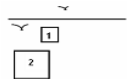
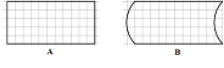
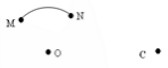
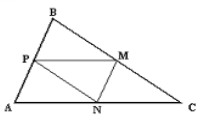
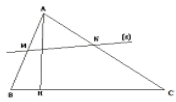
## REFERENCES

- Bentler, M. P. (1995). *EQS Structural equations program manual*. Encino, CA, Multivariate Software Inc.
- Deliyianni E., Elia I., Gagatsis A., Monoyiou A., & Panaoura A. (2009). A theoretical model of students geometrical figure understanding. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6<sup>th</sup> CERME* (pp. 696-705). Lyon, France.
- Duval, R. (1995). Geometrical Pictures: Kinds of Representation and Specific Processes. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematical education* (pp. 142- 157). Berlin, Springer.
- Duval, R. (1999). *Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for learning*. Retrieved from ERIC ED 466 379.
- Elia, I., Gagatsis, A., Deliyianni, E., Monoyiou, A., & Michael, S. (2009). A structural model of primary school students' operative apprehension of geometrical figures. In M. Tzekaki, M. Kaldrimidou, & C. Sakonidis (Eds.), *Proceedings of the PME 33* (Vol. 3, pp. 1-8). Thessaloniki, Greece: PME.
- Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, 24(2), 139-162.

## Working group 4

- Fischbein, E., & Nachlieli, T. (1998). Concepts and figures in geometrical reasoning. *International Journal of Science Education*, 20(10), 1193- 1211.
- Giaquinto, M. (2007). *Visual thinking in mathematics*. NY: Oxford University Press.
- Hanna, G., & Sidoli, N. (2007). [Visualization and proof: a brief survey of philosophical perspectives](#). *ZDM. The International Journal on Mathematics Education*, 39(1-2), 73-78.
- Harada, K., Gallou-Dumiel, E., & Nohda, N. (2000). The role of figures in geometrical proof-problem solving: Students' cognitions of geometrical figures in France and Japan. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the PME 24* (Vol. 3, pp. 25-32). Hiroshima, Japan.
- Houdement, C., & Kuzniak, A. (2003), Elementary geometry split into different geometrical paradigms. In M. Mariotti (ed.), *Proceedings of CERME 3*, Bellaria, Italy, [http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings\\_draft](http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft)
- Mesquita, A. L. (1998). On conceptual obstacles linked with external representation in geometry. *Journal of mathematical behavior*, 17(2), 183-195.
- Mullins, E. R., & Irvin, J. L. (2000). Transition into middle school. *Middle School Journal*, 31(3), 57 -60.

## APPENDIX

<p>1. Name the squares in the given figure:</p>  <p>(PE1a, PE1b, PE1c, PE1d, PE1e, PE1f, PE1g)</p>	<p>2. Recognize the figures in the parenthesis (KEZL, IEZU, EZHL, IKGU, LGU, BEL)</p>  <p>(PE2a, PE2b, PE2c, PE2d, PE2e, PE2f)</p>
<p>3. Underline the right sentence</p>  <p>(OP4)</p> <p>a. Fig. A has equal perimeter with Fig. B b. Fig. A has smaller perimeter than Fig. B c. Fig. A has bigger perimeter than Fig. B</p>	<p>4. Paris is looking the box 1 and 2 in the horizon. He says that the box 1 has exactly the same size with box 2. Is his opinion right? Explain your answer. (OP6)</p> 
<p>5. Underline the right sentence</p> <p>a. Fig. A has equal area with Fig. B b. Fig. A has smaller area than Fig. B c. Fig. A has bigger area than Fig. B</p>  <p>(OP8)</p>	<p>6. a) Draw an arc AB with centre C, equal to the arc MN with centre O. b) Describe the construction of the figure.</p>  <p>(SE10) (SE12)</p>
<p>7. The points M, N and P are the midpoints of the sides of triangle ABC. Show that the quadrilaterals APMN, BMNP, and GNPM are parallelograms. (LO14)</p> 	<p>8. In the figure below:</p> <ul style="list-style-type: none"> <li>- AB and AC are equal</li> <li>- Line (e) is parallel to BC</li> <li>- AH is perpendicular to BC</li> </ul>  <p>Underline the right sentence The length of NH</p> <p>a) is smaller than MH b) is equal to MH c) is bigger than of MH d) it cannot be determined</p> <p>(LO18)</p>

# SECONDARY STUDENTS BEHAVIOR IN PROOF TASKS: UNDERSTANDING AND THE INFLUENCE OF THE GEOMETRICAL FIGURE<sup>10</sup>

Athanasios Gagatsis\*, Paraskevi Michael\*, Eleni Deliyianni\*, Annita Monoyiou\*,  
Alain Kuzniak\*\*

\*University of Cyprus, \*\* Université Paris Diderot, France

*This research study examined and compared 109 9<sup>th</sup> graders' and 103 10<sup>th</sup> graders' behaviour in proof tasks, giving emphasis on their understanding of proof and the influence of the nature of the geometrical figure, either as an object or as an illustration. The results indicated differences, but also similarities between students of the two grades. Students were categorized into three levels according to their proof understanding, whereas the importance of the geometrical figure as a heuristic tool for proof was revealed.*

## THEORETICAL FRAMEWORK

Nowadays, the development of a sense of proof constitutes an important objective of Mathematical Education. There seems to be a general trend towards including the theme of proof in the curriculum (Mariotti, 2006). Marrades and Gutiérrez (2000) defined proof as any justification which satisfies the requirements of abstraction, rigor, language, etc. demanded by professional mathematicians to accept a mathematical statement as valid within an axiomatic system. Mathematical proof in a wide sense includes formal proofs but also any attempt made by students to convince themselves, the teacher or other students of the truth of a mathematical statement or conjecture by means of explanations, verifications or justifications (Fiallo & Gutiérrez, 2007).

Even among mathematicians there is a diversity of opinion regarding the role and functions of proof (Healy & Hoyles, 2000). According to Marrades and Gutierrez (2000) the main objectives of mathematical proof are to verify or justify the correctness of a statement, to illuminate or explain why a statement is true, to systematize results obtained in a deductive system (a system of axioms, definitions, accepted theorems, etc.), to discover new theorems, to communicate or transmit mathematical knowledge and to provide intellectual challenge to the author of a proof. For Mariotti (2006) the purpose of proof is validation. Based on de Villiers (2002) some of the functions of proofs are: explanation, discovery, communication, intellectual challenge and systematisation.

---

<sup>10</sup> This paper is in the context of the research project "Ability to use multiple representations in Functions and Geometry: The Transition from Middle to High school" of the Research Promotion Foundation of Cyprus (IΠE) [ANΘΡΩΠΙΣΤΙΚΕΣ/ΠΑΙΔΙ/0308(BE)/03].

Duval (2000) considers that the cognitive processes underlying the understanding of a mathematical proof require students to realise that a valid reasoning can produce intrinsically an evidence of necessity and become aware of the discrepancy between a valid reasoning and a non-valid reasoning. Kunimune, Fujita and Jones (2008) examined students' understanding of proof in terms of the 'Generality of proof' and the 'Construction of proof'. Considering these two aspects, they proposed three levels of understanding. In Level I students consider experimental verifications are enough to demonstrate that geometrical statements are true. In Level II students understand that proof is required to demonstrate geometrical statements are true and finally in Level III students can understand simple logical chains between theorems.

As for the understanding of the valid reasoning functions, the learner have to work in another register than language and to come back to linguistic expression, since geometry involves the mixing of two registers: the figure register in order to 'see' and the natural language register in order to 'explain' (Duval, 2000). According to Hanna (2000) diagrams and other visual aids have been used as facilitators of understanding and as heuristic accompaniments to proof, inspiring both the theorem to be proved and approaches to the proof itself.

The nature of external representations in geometry, either as an object or as an illustration, was examined by Xistouri, Nicolaou, Koukkoufis and Gagatsis (2005). According to Mesquita (1998), an external representation has the nature of an *object* when it is possible to infer geometrical relationships from the construction of the figure that may be used in geometrical reasoning and proof and when the visual perception of the figure is consistent with the verbal statements of the problem. On the contrary, when the external representation has the nature of an *illustration*, it is then impossible to directly extract a geometrical relationship from the construction of the figure, the figure seems to 'mislead' and the visual perception of the figure is in contradiction within the verbal statements.

Based on the above the purpose of this study was to examine and compare Grade 9 and Grade 10 students' geometrical proof – problem solving ability. In particular, we focussed on the following: 1) *The differences between Grade 9 and Grade 10 students' behavior in geometrical – proof tasks*, 2) *Students' understanding about the geometrical proof*, 3) *The impact of the geometrical figure on students' proving ability according to its nature: as an 'object' and as an 'illustration'*.

## METHOD

The study was conducted among 109 9<sup>th</sup> graders and 103 10<sup>th</sup> graders of mixed mathematical ability, randomly selected from secondary schools in Cyprus. The test that examined students' geometrical proof – problem solving ability consisted of 8 proof tasks: 2 proof tasks whose solution is '*based on definition*' (VEde1, VEde2) and 6 proof tasks '*based on procedures*' (VEp), according to Harada's, Gallou –

Dumiel's and Nohda's (2000) categorization of proof. The '*based on procedures*' proof tasks were distinguished into two special categories of tasks:

1. Tasks that examined the impact of the nature of the geometrical figure (Mesuita, 1998) in students' geometrical proof – problem solving ability. There were two sets of corresponding tasks in which the first task included a figure functioning as an '*object*' (VEpo1 and VEpo2) and the second a figure functioning as an '*illustration*' (VEpi1 and VEpi2)
2. The second category was comprised of a set of corresponding tasks, based on Kunimune et al. (2008), examining the 'Generality of proof'. In particular, the tasks included three different types of proof: an empirical proof (VEp1m2, VEp2m2), a semi – empirical proof (VEp1m1, VEp2m1) and a formal proof (VEp1m3, VEp2m3). Students had to declare whether they accepted each type as a proof.

Representative samples of the tasks used in the test appear in the Appendix.

For the analysis of the collected data, the hierarchical clustering of variables and Gras' implicative statistical method has been conducted using the software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). The similarity diagrams produced allow for the arrangement of students' responses to the tasks into groups according to their homogeneity. This aggregation may be indebted to the conceptual character of every group of variables. The implicative diagrams contain implicative relations, which indicate whether success to a specific task implies success to another task related to the former one.

## RESULTS

### Grade 9 students' behaviour according to the similarity diagram

The similarity diagram of the 9<sup>th</sup> graders' responses to the tasks of the test (Figure 1) is divided into two similarity clusters. In the first similarity cluster two subgroups are distinguished. In specific, the first subgroup is comprised of procedure proof tasks (VEpo1, VEpi2, VEpo2, VEp1m3 and VEpi1) and one definition proof task (VEde1). This definition task has a significant and strong relation with the second procedure task that includes a figure as an illustration (VEpi2). In the second similarity group of the similarity diagram the variables concerning the acceptance of the semi – empirical proof in the two corresponding tasks (VEp1m1 and VEp2m1) are significantly connected with the solution of the second definition proof task of the test (VEde2). These three variables are significantly related with the acceptance of the formal proof in the second procedure task of this type (VEp2m3).

From the 9<sup>th</sup> graders' similarity diagram concerning the nature of the geometrical figure we notice that the tasks in which the figure functions as an illustration are involved in significant relations. This is indicative of the fact that 9<sup>th</sup> graders behavior during the solution of proof tasks is influenced by the nature of the



geometrical figure and specifically when the geometrical figure functions as an illustration. The second subgroup consists of two variables that are related to the acceptance of an empirical proof in a set of corresponding tasks (VEp1m2 and VEp2m2).

We also notice that as regards the acceptance of the three types of proof, there is a distinct group of students. In this group belong students who accept the empirical proof (VEp1m2 and VEp2m2) ( $\bar{X}=0.11$ ,  $SD=0.22$ ). Furthermore, the significant relation between the variables VEp1m1 and VEp2m1 allows us for the consideration of a second group of students, those who accept the semi – empirical proof ( $\bar{X}=0.10$ ,  $SD=0.21$ ). The definition tasks are located into different clusters, showing that students' behavior is not differentiated in definition and procedures tasks.

However there seems to be a relation between students of this group and some students of the formal proof group. This might be indicative of some common characteristic between these two groups. The two variables representing the formal proof are separated into two different similarity clusters. This is indicative of a not stable behavior of students yet for this type of proof tasks. The acceptance of the formal proof in the first task is connected with the variables concerning the nature of the geometrical figure. Thus we can infer that the nature of the geometrical figure affects, at a degree, the 9<sup>th</sup> graders proving ability. Particularly it seems that students who accept the formal proof ( $\bar{X}=0.07$ ,  $SD=0.20$ ), are more able to overcome the negative influence of the figure functioning as an illustration.

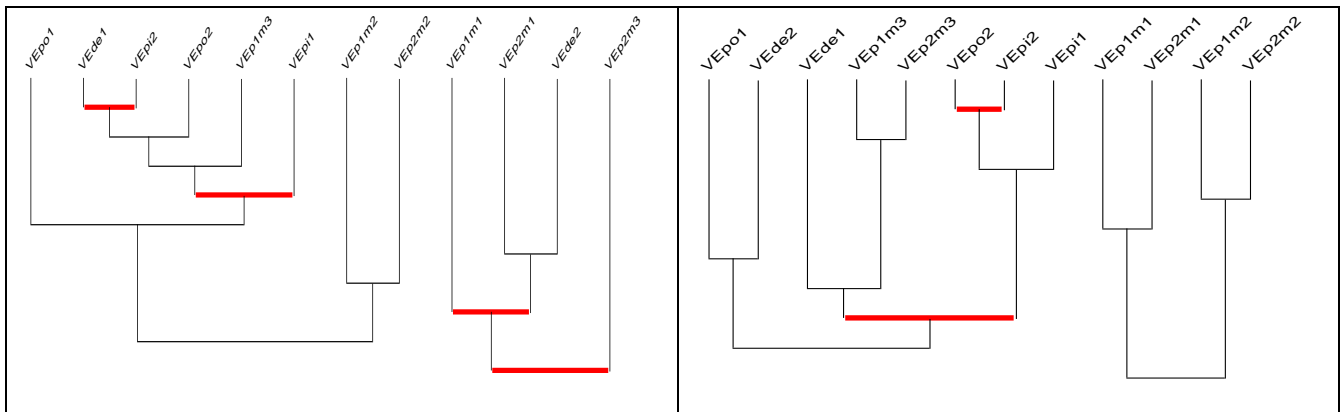
### **Grade 10 students' behaviour according to the similarity diagram**

Two similarity clusters are identified in Figure 2. Cluster 1 consists of two subgroups. The first subgroup includes a procedure proof task accompanied with a figure operating as an object (VEpo1) linked to the second definition proof task (VEde2). In the second subgroup important relations are observed between some variables. In particular, there is a connection between the variables VEp1m3 and VEp2m3, indicating students' stable behavior in accepting the formal proof in the two corresponding tasks. These two variables are also connected with a definition proof task (VEde1). Another significant relation is formed between the variables VEpo2 and VEpi2, which concern two corresponding procedure proof tasks, having a figure as an object and as an illustration respectively. This important relation indicates that the presence of the geometrical figure as an illustration does not negatively influence students' behavior in this proof task, since their behavior remains stable in the two corresponding tasks. The two variables concerning the first task that examines the nature of the geometrical figure (VEpo1 and VEpi1) are separated. The variable that concerns the figure as an illustration (VEpi) is linked with variables VEpo2 and VEpi2. In the second cluster two subgroups are also found. The first subgroup is formed by the two variables concerning the acceptance of a semi – empirical proof (VEp1m1 and VEp2m1). The second subgroup is

#### Working group 4

comprised of the two variables related to the acceptance of an empirical proof (VEp1m2 and VEp2m2).

In the similarity diagram of the 10<sup>th</sup> graders three distinct groups of students are identified. There are students who accept the empirical proof ( $\bar{X}=0.14$ ,  $SD=0.27$ ), students who accept the semi – empirical proof ( $\bar{X}=0.22$ ,  $SD=0.33$ ) and students that accept the formal proof ( $\bar{X}=0.07$ ,  $SD=0.20$ ). The first two groups are situated in the same similarity cluster, indicating that students of these two groups have many common characteristics in their proving behavior. The significant relations of this group with the variables representing the role of the geometrical figure leads us to the conclusion that the students that are mostly able to overcome the negative influence of the geometrical figure as an illustration are those who are also able to recognize and accept the formal proof. Moreover, the location of the definition tasks in the same similarity cluster shows a more stable behavior of students in these tasks, but again no differentiation related to procedure proof tasks.



**Figure 1: Similarity diagram for Grade 9 students**

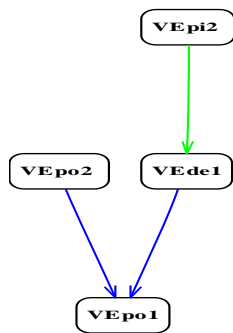
**Figure 2: Similarity diagram for Grade 10 students**

#### Grade 9 and 10 students' behaviour according to the implicative diagrams

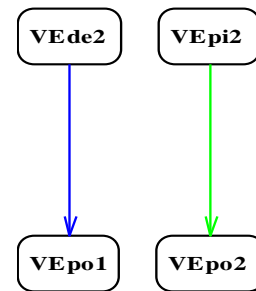
In the 9<sup>th</sup> graders' implicative diagram (Figure 3) two implicative chains are formed. The first implicative chain provides important information concerning students' behavior related to the nature of the geometrical figure. In particular, it is obvious that students who solved the second task including the figure as an object (VEpo2) were also able to solve the first task of the same type (VEpo1). Thus, there is a relation based on the common function of the figure, as an object in this case. From the second implicative chain we can infer that the solution of the second task including an illustration figure (VEpi2) leads to the solution of the definition task 1 (VEde1). This implicative chain ends with the solution of the first task including a figure as an object (VEpo1).

In the implicative diagram of the 10<sup>th</sup> graders (Figure 4) two distinct implicative chains are formed. The first implicative chain consists of two tasks. The solution of

the definition task 2 (VEde2) leads to the solution of the first procedure task which is accompanied by a geometrical figure as an object (VEpo1). The second implicative chain is comprised of two corresponding procedure tasks. Particularly, if students succeeded to the solution of the task with a figure as an illustration (VEpi2), which is proved to be more difficult for them, then they were also able to solve correctly the corresponding task including a figure as an object (VEpo2). This shows that students are not misled by the figure as an illustration in this task.



**Figure 3: Implicative diagram for Grade 9 students**



**Figure 4: Implicative diagram for Grade 10 students**

### **Grade 9 and 10 students' performance according to the nature of the geometrical figure**

Students' performance relatively to the nature of the geometrical figure is presented in table 1. We can see that students of both grades perform better in the tasks in which the figure has the nature of an object. Specifically, the paired sample t test revealed that the means in task 1 are statistically significant higher for the figure as an object for both grades [grade 9:  $t(108)=-8,1$ , grade 10:  $t(102)=8,2$ ]. This is also the case for task 2, although the difference is statistically significant only in grade 10 [ $t(102)=3,8$ ]. These results are in line with the outcomes of the similarity concerning the influences in students' behaviour and those of the implicative diagrams regarding the difficulty of the tasks.

	Task 1				Task 2			
	object		illustration		object		Illustration	
	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD	$\bar{X}$	SD
Grade 9 (N=109)	0,48	0,50	0,06	0,25	0,11	0,31	0,06	0,23
Grade 10 (N=103)	0,60	0,49	0,14	0,34	0,26	0,44	0,10	0,30

**Table 1: Means and standard deviations for grade 9 and grade 10 students' performance in tasks examining the influence of the nature of the geometrical figure**

## CONCLUSIONS

The aim of this research study was to examine 9<sup>th</sup> and 10<sup>th</sup> graders proving behavior, especially in respect to the understanding of proof and the role of the nature of the geometrical figure, either as an object or as an illustration. The results revealed that there are mainly differences, but also some common characteristics in the way students solve geometrical proof tasks.

Considering students' understanding of proof, we can classify them into three levels. Level I refers to the acceptance of the empirical proof, Level II refers to the acceptance of the semi – empirical proof, while Level III concerns the acceptance of formal proof. Referring to the 9<sup>th</sup> graders, they can be clearly situated into Level I and Level II. Although there are students that seem to have reached Level III, they cannot be clearly distinguished as a group. The relations revealed from the similarity diagram between the students of Level II and Level III indicate that there are some common characteristics between students of these two groups. Thus, we can say that 9<sup>th</sup> graders are more situated in the first two levels of proof understanding and some of them are at a transitional stage to the third level. The situation differs for Grade 10 students, since there is a quite clear classification of the students into the three levels of proof understanding. 10<sup>th</sup> graders who reach Level III constitute an independent group, since they do not have any common characteristics with the students of the other two levels. On the contrary, students of Levels I and II appear to have some common features. The three levels used in order to categorize students' proof understanding can be connected to the three levels proposed by Kunimune, Fujita and Jones (2008), despite the fact that we have only examined students' understanding regarding the 'Generality of proof' and not the 'Construction of proof'. Besides, the formation of the three levels of proof understanding can be indicative of the existence of the forms of geometry, as suggested by Houdement and Kuzniak (2003). We could claim that students of Level I are still situated in Geometry I (Natural Geometry), in which intuition is often linked to immediate perception and enriched by experiment. Students of Level III have characteristics of Geometry II (Natural Axiomatic Geometry), where a system of axioms is necessary. The axiom system can be uncompleted, but the demonstrations inside the system are necessary requested for progress and for reaching certainty. At last, Level II students seem to have characteristics of both Geometry I and Geometry II. Kuzniak (2011, in press) suggests that the constant emphasis on a transition towards Geometry II based on Geometry can let us suppose that a mixed Geometry (GI/G2) is possible.

Another difference between 9<sup>th</sup> graders and 10<sup>th</sup> graders concerns the definition proof tasks. Based on the similarity diagram, 10<sup>th</sup> graders' behaviour is more coherent compared to the 9<sup>th</sup> graders. Nevertheless, the implicative diagrams show that the definition tasks are more difficult for 9<sup>th</sup> and 10<sup>th</sup> graders, than the procedure tasks in which the geometrical figure functioning as an object. Furthermore, the solution of these tasks is achieved by students that belong to Level III of proof understanding for

10<sup>th</sup> graders, while for 9<sup>th</sup> graders a relation with the students of Level II is observed. A common feature for the students of both Grades is that the phenomenon of compartmentalization is not observed in their behaviour during the solution of definition and procedure tasks.

Concerning the impact of the nature of the geometrical figure on students' proving ability, it emerged that the geometrical figure plays an important role for the solution of proof tasks, since proof does not depend solely on sentential representation and visual and sentential reasoning are not mutually exclusive (Hanna, 2000). In some cases it does not influence students proving ability when it is functioning as an illustration. However it still has an impact on their behaviour, since it sometimes leads to different behaviour during the solution of corresponding tasks. Specifically, according to the implicative diagram, although 9<sup>th</sup> graders show consistency when the figure has the nature of an object, their proving behavior is influenced and they are misled when the geometrical figure functions as an illustration. This is not the case for 10<sup>th</sup> graders, since they show more consistency in the solution of tasks including a figure as an illustration. Although students of both grades seem to start overcoming the negative influence of the geometrical figure as an illustration, grade 10 students show a more stable proving performance in tasks of this type. Despite the fact that they use the figure, they do not exclusively base their proving procedures on it. The findings give support to Duval (2000) who distinguishes between a heuristic and a supportive function of the geometrical figure. Thus, Grade 9 students' proving behaviour is more affected by the nature of the figure, than the common characteristics of the exercise, while Grade 10 students seem to be in a transitional stage in which they start overcoming the negative influence of the geometrical figure on their proving ability. For Grade 9 students the geometrical figure is more an object of study and of validation, while for Grade 10 students the geometrical figure is supportive for reasoning (Houdement & Kuzniak, 2003) and concerned as a "figural concept" (Fischbein, 1993).

Despite the differences mentioned above, findings indicate an important common characteristic for Grade 9 and Grade 10 students as far as the relation between the nature of the geometrical figure and the levels of proof understanding is concerned. Students belonging to Level III are those who are able and make efforts to overcome the impact of the figure as an illustration. For these students, who are also at the level of Geometry III, the figure functions as a heuristic tool (Houdement & Kuzniak, 2003). Therefore, the overcoming of the negative influence of the geometrical figure leads students to the recognition and acceptance of the formal proof or vice versa. In this sense it is well accepted that a diagram is a legitimate component of a mathematical argument (Hanna, 2000).

It is vital to know students' conception of mathematical proof in order to understand their attempts to solve proof problems. The knowledge of what it is for them to prove a statement or what kind of arguments convince students that a statement is true is


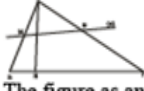
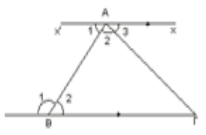
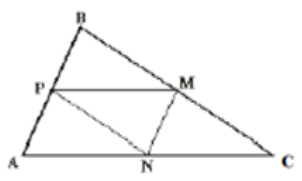
very important for teaching (Marrades & Gutiérrez, 2000). Thus the necessity for qualitative research is evident in order to gain a deeper insight in students' proving behaviour and constitutes the next step of our study. Besides, longitudinal performance investigation in geometrical figure understanding related to proof understanding for students (e.g. low achievers) as they move from Grade 9 to Grade 10 should be carried out.

## REFERENCES

- Bodin, A., Coutourier, R., & Gras, R. (2000). *CHIC : Classification Hiérarchique Implicative et Cohésive-Version sous Windows – CHIC 1.2*. Rennes, Association pour la Recherche en Didactique des Mathématiques.
- De Villiers, M. (2002) . Developing Understanding for Different Roles of Proof in Dynamic Geometry. *Proceedings of the ProfMat 2002*. Visue, Portugal, 2- 4 October 2002.
- Duval , R. (2000). Proof Understanding in Mathematics: What ways for students? *Proceedings of 2002 International Conference on Mathematics*. Retrieved from [www.math.ntnu.edu.tw](http://www.math.ntnu.edu.tw)
- Fiallo, J., & Gutiérrez, A. (2007). Analysis of conjectures and Proofs produced when learning trigonometry. Retrieved from [http://www.lettredelapreuve.it/CERME5Papers/WG4-Fiallo\\_Gutierrez.pdf](http://www.lettredelapreuve.it/CERME5Papers/WG4-Fiallo_Gutierrez.pdf)
- Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics* 24(2), 139-162.
- Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1), 5-23.
- Harada, K., Gallou-Dumiel, E., & Nohda, N. (2000). The role of figures in geometrical proof-problem solving: Students' cognitions of geometrical figures in France and Japan. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the PME24* (Vol. 3, pp. 25-32). Hiroshima, Japan.
- Healy L., & Hoyles, C. (2000). A Study of Proof Conceptions in Algebra. *Journal for Research in Mathematics Education*, 31(4), 396–428.
- Houdement, C., & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. In M. Mariotti (Ed.), *Proceedings of CERME 3*, Bellaria, Italy, [http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings\\_draft](http://www.dm.unipi.it/~didattica/CERME3/draft/proceedings_draft)
- Kunimune, S., Fujita, T., & Jones, K. (2008). “Why do we have to prove this?” Fostering students' understanding of ‘proof’ in geometry in lower secondary school. Retrieved from <http://ocs.library.utoronto.ca/index.php/icmi/8/paper/view/877/51>
- Kuzniak, A. (2011, in press). Geometric work at the end of the compulsory education. *Proceedings of the 7<sup>th</sup> Conference of the European Society for Research in Mathematics Education: Working Group 4*, Rzeszów, Poland.

- Mariotti, A. (2006). Proof and proving in mathematics education. In A. Gutiérrez, & P. Boero(Eds.). *Handbook of research on the psychology of mathematics education: Past, present and future*, (pp. 173-204). Rotterdam/Taipei: Sense Publishers.
- Marrades, R., & Gutierrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics*, 44, 87-125.
- Mesquita, A. L. (1998). On conceptual obstacles linked with external representation in geometry. *Journal of Mathematical Behavior*, 17(2), 183-195.
- Xistouri, X., Nicolaou, A., Koukkoufis, A., & Gagatsis, A. (2005). The nature of external representation in geometry and the solution of mathematical problems. In A. Gagatsis, F. Spagnolo, Gr. Makrides, & V. Farmaki (Eds.), *Proceedings of the 4<sup>th</sup> Mediterranean Conference on Mathematics Education*: (Vol. I, pp.163-174). Palermo: University of Palermo, Cyprus Mathematical Society.

## APPENDIX

<b>'Based on procedures' proof task – category 1</b>		
<p>In the figure below AB and AC are equal, line (e) is parallel to BC and AH is perpendicular to BC. Circle the right answer.</p> <p>The length of NH</p> <p>a) is smaller than MH b) is equal to MH c) is bigger than of MH d) it cannot be determined</p>	 <p>The figure as an object (VEpo1)</p>	 <p>The figure as an illustration (VEpi1)</p>
<b>'Based on procedures' proof task – category 2</b>		
<p>Read the following explanations by three students who demonstrate why the sum of inner angles of triangle is 180 degree.</p> <p><b>Student A:</b> 'I measured each angle, and they are 50, 53 and 77. 50+53+77=180. Therefore, the sum is 180 degree.' <b>Accept/Not accept as proof (VEp1m1)</b></p> <p><b>Student B:</b> 'I drew a triangle and cut each angle and put them together. They formed a straight line. Therefore, the sum is 180 degree.' <b>Accept/Not accept as proof (VEp1m2)</b></p> <p><b>Student C:</b> Demonstration by using properties of parallel line</p>		
 <p><math>xx' \parallel BC \Rightarrow \hat{A}_1 = \hat{B}_2 \text{ and } \hat{A}_3 = \hat{C} \Rightarrow \hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 180^\circ \Rightarrow \hat{B}_2 + \hat{A}_2 + \hat{C} = 180^\circ</math>.</p> <p><b>Accept/Not accept as proof (VEp1m3)</b></p>		
<b>'Based on definition' proof task</b>		
<p>The points M, N and P are the midpoints of the sides of triangle ABC. Show that the quadrilateral APMN, BMNP, and GNPM are parallelograms.</p>		
 <p>(VEde1)</p>		

# RELATIONS BETWEEN GEOMETRICAL PARADIGMS AND VAN HIELE LEVELS

Annette Braconne-Michoux

IUFM de Lyon, LEPS-LIRDHIST Université Claude Bernard Lyon1

*Following previous CERME sessions on Geometrical Thinking, this paper addresses the link between the geometrical paradigms and the van Hiele levels of thinking in geometry. It records the experimentation I conducted with pupils from grade 5 (10-11 years old, last year of Primary School) and grade 6 (11 – 12 years old, first year of Secondary School). Two main points arose: first, that a pupil tends to use a lower paradigm or van Hiele level when facing a difficult task, second, that the second van Hiele level (analysis) seems to be “the overlapping zone” between GI and GII.*

## INTRODUCTION

During CERME3 (2003), for the first time, Houdement-Kuzniak presented a paper in which they referred to the van Hiele levels theory and the geometrical paradigms. They introduced in a two dimensional table the idea that Geometry GI integrates the first two van Hiele levels of thinking in geometry (identification-visualization and analysis) while Geometry GII integrates the fourth level (formal deduction), the third level being the transitional one between GI and GII. Parzysz (2003) suggested that, from an educational point of view, the same third van Hiele level (informal deduction) could be the “overlapping zone” between GI and GII. Houdement-Kuzniak’s research had been based on pre-service teachers’ understanding of geometry, that is to say on adults working in geometry. According to the French curricula, as Parzysz (2003) put it: “roughly, GI is Primary School geometry and GII is Secondary School geometry”. If the van Hiele levels theory is to be considered as a pedagogical reference to geometry teaching, how can it relate to the geometrical paradigms when considering pupils entering Secondary School?

These suggestions had to be tested among pupils able to work in GI and approaching GII, that is to say, at grade 5 (end of Primary School, pupils aged 10 - 11) and grade 6 (first year of Secondary School, pupils aged 11 - 12). This paper reports the experimentation conducted in June 2004 among 250 primary school and 250 Secondary school pupils.

## THEORETICAL FRAMEWORK

Geometry is an important topic in the mathematics curricula in France both at Primary and Secondary School levels. Though the geometrical objects have the same name in Primary School and Secondary School, it has obviously different meanings. For instance, a square as drawn by a 8 years old pupil with ruler and set square is different from a square as sketched at free hand by a 15 years old student. The first one depends on the quality of the drawing (precision in measures and accuracy in the



use of instruments) while the second is only the graphic support for reasoning and may be used as a heuristic tool.

Since we are to use the two theories of geometrical paradigms and the van Hiele levels of thinking in geometry, we have to summarize the main aspects of both theories, particularly those relevant for the purpose of our study that is, GI and GII as far as the geometrical paradigms are concerned and the three first van Hiele levels: identification-visualization; analysis and informal deduction.

### Geometrical paradigms

As Houdement-Kuzniak (2003) put it: “our fundamental principle is that the various proposed paradigms are homogeneous: it is possible to reason inside one paradigm without knowing the nature of the other. Students and professors, and it is a source of misunderstanding, are not necessarily situated in the same one.” The following table is a summary of the description of the three different paradigms as defined by Houdement-Kuzniak. Parzysz (2003) added the lines about validations and the nature of the object as considered or studied by the person working in a specific geometrical paradigm.

	<b>GI</b>	<b>GII</b>	<b>GIII</b>
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near of the Real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms
Kind of space	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean space
Status of drawing	Object of study and of validation	Support of reasoning and “figural concept”	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-evidence	Properties and demonstration	Demonstration and links between the objects, Structure
Objects	Physical	Theoretical	
Validations	Perceptive or by use of instruments	Deductive	

**Table 1: The geometrical paradigms**

## **The Van Hiele levels<sup>11</sup>**

The van Hiele theory of levels of thinking in geometry has been less popular among French researchers than among the English speaking didactical community. We will give here the major aspects of the theory relevant to our study, that is the first three levels: identification-visualization, analysis, informal deduction.

At the first level (N1 identification-visualization)

The geometrical figures are judged according to their appearance. The pupils do not see the parts of the figure, nor do they perceive the relationship among the components of the figure. They cannot even compare figures with common properties with one another. [...] They recognize, for example, a rectangle, a square and other figures. [But] they conceive of the rectangle as completely different from the square. [To such pupils] these figures are still completely distinct. (Wirszup, 1976; p.76)

At the second level (N2 analysis)

The pupil begins to discern the components of the figures; he also establishes relationships among these components and relationships between individual figures. [...] The properties of the figures are established experimentally; they are described but not yet formally defined. At this stage, the figures are the bearers of their properties [they are recognised by these properties...] however these properties are still not connected with one another. (ibid; p.78)

At the third level (N3 informal deduction)

Students can establish relations among the properties of a figure and among the figures themselves. At this level there occurs a logical ordering of the properties of a figure and of classes of figures. The pupil is now able to discern the possibility of one property following from another, and the role of definition is clarified. The logical connections among figures and properties of figures are established by definitions. However, at this level [...] the pupil does not yet understand the role of axioms, and cannot yet see the connection of statements. [...] At the third level a square is already viewed as a rectangle and a parallelogram. (ibid; p.78)

At the fourth level (N4 formal deduction) “the students grasp the significance of deduction as a means of constructing and developing all geometric theory.” (ibid; p.78)

---

<sup>11</sup> In this text, the van Hiele levels will be numbered from 1 to 5 following Uziskin (1982), Guttiérrez & als (1991) Clements and Battista (1992) etc. ; the main reason being that some students do not reach the first level of identification-visualization.

## A new framework

If we are to consider both theories, we can assume that a pupil mastering only the first van Hiele level (N1 Identification-visualization) is not yet working in a any geometrical paradigm, that is to say even not in GI: he/she can only name a figure by global recognition and is not using any instrument or geometrical property to assess his/her answer. Though at the end of Primary School most pupils should master not only the first van Hiele level (N1) but be able to work in GI as well. At the opposite, a pupil mastering the third van Hiele level (N3 informal deduction) is definitely able to work in a more theoretical geometry that is to say GII named either “natural axiomatic” (Houdement-Kuzniak; 1998-1999) or “proto-axiomatic” (Parzysz; 2003). Such an ability is expected from the students finishing Secondary School in France.

So the question is: if a student masters the second van Hiele level (N2 analysis), is he/she working in a specific paradigm? If the answer is yes, is it GI or GII? Since for a student at level N2, the geometrical object is not necessarily a physical one but can be a meaningful free hand drawing, we cannot consider that such a pupil is working in GI. Is he/she working in GII? We cannot be sure of that either: the deductions and validations he/she makes are based on a list of properties and are not organised deductively. So we can suggest that the second van Hiele level could be an “overlapping zone” between GI and GII in terms of teaching references.

The following table summarizes these new relations between the geometrical paradigms and the van Hiele levels.

Geometrical paradigm	GI		GII	
Status of drawing	Representing itself	Object of study and of validation	Representing a class of objects	Schema of a theoretical object
Characteristic of drawing	“straight lines”	“straight lines coded”	Free hand coded drawings	
Validations	Perceptive or using instruments	Using instruments or deductive (through description and lists of properties)		Hypothetical-deductive
Van Hiele level	N1 Identification-visualization	N2 Analysis		N3 Informal deduction

**Table 2: Relations between the geometrical paradigms and the van Hiele levels**

Is such an organisation accurate? Can we detect in which geometrical paradigm a pupil at this age is working? Can we assess the van Hiele level he is mastering in a particular situation?

## METHODOLOGY

In order to assess these questions, 250 pupils at grade 5 (last year of Primary School in France; 10-11 years old) and 250 pupils at grade 6 (first year of Secondary School) were asked to answer the same tests. Each test consisted of two questions, the first one supposedly easier than the second among 17 different tasks about triangles, quadrilaterals and circles, such as illustrated in figure 1. The first tasks were numbered from 1.1 to 1.8; the second ones from 2.1 to 2.9. The tasks consisted in recognizing or identifying a specific figure among a lot of different figures, drawing a specific figure according to different conditions, identify and explain or justify some property of a figure; etc.

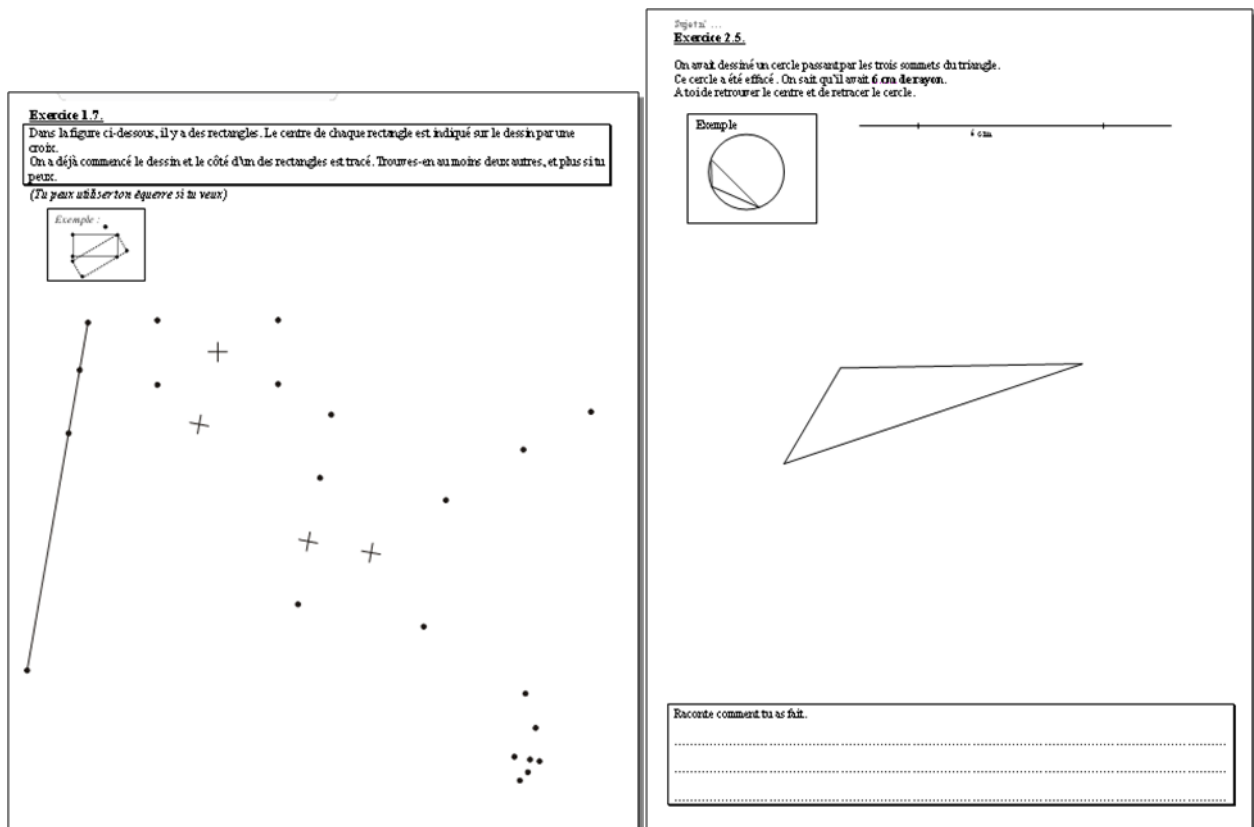


Fig 1: “The rectangles” and “the circle” questions<sup>12</sup>

<sup>12</sup> Translation:

*Exercise n°1.7.* : In the figure below, there are some rectangles. The center of each rectangle is indicated by a cross. The drawing has already been started and the side of a rectangle is drawn. Please, find at least two other rectangles and more if you can. (you may use your set square if you want to) [the biggest rectangle is 16 cm long and 11 cm wide]

*Exercise n°2.5.* : A circle passing through the three vertices of the triangle had been drawn. It was erased. We know that the radius is 6 cm. Please, find the center and draw the circle again. Tell us how you did it.

Each task could be worked out either in GI or GII. The way the pupils answered the questions was to give us some indication about the paradigms they were working in and the van Hiele level they were mastering.

As examples, the tasks illustrated in figure 1 are analysed here.

In “exercice 1.7.” a pupil mastering the first van Hiele level 1 would identify only the small rectangle on the top left part of the page and try to finish the small rectangle with one of its sides already drawn. At level 2 (analysis), the pupil could identify three out of the four rectangles. Mainly at this level, the pupil could not cope with the recognition of the square and the biggest of the four rectangles. Most of the answers at level 2 indicate that the square previously identified as a rectangle has been erased thus rejected as a particular rectangle. The pupils at level 3 (informal deduction) could identify the four rectangles.

Considering the geometrical paradigms, pupils having rejected the square or wrongly connected the vertices of the biggest rectangles were considered as working in GI at most. These pupils could not cope with the different properties of the rectangle (length of sides and right angles) at the same time, even when using the ruler and/or set square. From the traces left on the sheets, we could tell that most of these pupils had made a perceptive validation of their drawings. We considered that the pupils working in GII were able to accept the square as one of the rectangles to be identified in the figure and were very précised when searching the fourth vertex of the biggest rectangle: opposite sides equal and right angles. Some of them indicated that the lengths of the opposite sides or the half of diagonals should be equal.

In “exercice 2.5.”, the van Hiele levels were identified according to these criteria.

At level 0, the pupil did not answer or drew a circle with a radius different from 6 cm passing at most by one of the vertices.

At level 1 (identification-visualization), the circle is a 6 cm radius one but it passes by one or two of the vertices of the triangle. Others drew a circle passing by two of the vertices but the radius is not 6 cm. Not all the conditions are taken into account.

At level 2 (analysis), by trials and errors, the pupil draws a 6 cm radius circle passing by the three vertices but he cannot explain why he is successful. There are many holes in the sheet of paper. The explanation is generally: “I tried as many times as necessary.” At this van Hiele level, the pupil is able to grasp the different constraints (circle with a 6 cm radius passing by the 3 vertices) but he cannot give any theoretical explanation.

At level 3 (informal deduction) the pupil is able to use the compasses the other way round and use the vertices of the triangle as centres of 6 cm radius circles, drawing two or three arcs and using their intersection as the centre of the circumcircle of the triangle. The explanations are descriptions of the different actions and very rarely refer to the definition of a circle.

As far as the paradigms are concerned, we considered that a pupil using the compasses with a 6 cm opening and working with trials and errors was working in GI. Though he could be successful in drawing a convincing circle, he could not give any explanation of his success. The validation was a perceptive one.

As soon as a pupil was able to use the vertices of the triangle as centres of circles, he was using the definition of a circle the other way round: every point lying on a 6 cm radius circle is the centre of a 6 cm radius circle passing by the centre of the first circle. Such a pupil is working in GII, even though he cannot word it through a hypothetical-deductive reasoning.

Each answer was coded according the van Hiele theory on one side and the geometrical paradigms on the other one. Every pupil being given two tasks, we had for everyone two sets of codes: the van Hiele levels and the geometrical paradigms he could work in.

## RESULTS

First of all, as the two theories are concerned, we discovered that most of the answers given by the pupils could be identified as belonging to a specific geometrical paradigm and a van Hiele level. 12 answers out of 457 (<3%) were impossible to code in one paradigm or a van Hiele level. So we have 445 answers: 209 from grade 5 pupils and 246 from grade 6 pupils.

Grade 5				Grade 6			
Recto	GX + GI	GII	Total	Recto	GX +GI	GII	Total
Verso	(recto)	(recto)		Verso	(recto)	(recto)	
GX + GI (verso)	120	57	177	GX + GI (verso)	69	76	145
GII (verso)	18	14	32	GII (verso)	43	58	101
Total	138	71	209	Total	112	134	246

**Table 1: the geometrical paradigms (grade 5 and grade 6)<sup>13</sup>**

Considering the geometrical paradigms (see Table 1), as expected, three quarters of grade 6 pupils could work in GII when answering at least one of the two questions while only one out of 15 (<7%) from grade 5 pupils could do it. This happened mainly when the answer could be worked out without wording, like in “exercice 1.7.” where they were asked to retrieve the four rectangles, a square being among

<sup>13</sup> The code GX was used for the answers we could not clearly identify as referring to GI or GII. But, we considered too that at that level of schooling, most pupils were no longer working in G0. So we figured out that they were working in GI, even with some awkwardness.

them. As the pre-service teachers do (Jore-Lemonnier 2006), one pupil can work in both paradigms depending on the task itself or the interpretation he/she has of the task. In most cases, even if the pupil can answer the first question working in GII, when the second task seems more difficult to him/her, he/she may work in GI to give an answer. The validations are then more frequently perceptive or obtained by use of instruments rather than by reasoning.

Grade 5					Grade 6				
Recto Verso	N0 + N1 (recto)	N2 (recto)	N3 (recto)	Total	Recto Verso	N0 + N1 (recto)	N2 (recto)	N3 (recto)	Total
N0 + N1 (verso)	67	28	20	115	N0 + N1 (verso)	32	29	31	92
N2(verso)	32	13	25	70	N2(verso)	13	32	19	64
N3(verso)	8	10	6	24	N3(verso)	17	34	39	90
Total	107	51	51	209	Total	62	95	89	246

**Table 2: the van Hiele levels (grade 5 and grade 6)<sup>14</sup>**

As far as the van Hiele levels theory is concerned (see Table 2), the same report can be done. In most cases, we could identify the level mastered by a pupil when answering a specific question. As Burger and Shaughnessy (1986) and Gutiérrez (1992) report, one pupil can master different van Hiele levels at the same time depending on the subject. We observed too that, when the second question seemed more difficult than the first one, the answer given by the same pupil proved to pertain to a lower level. When considering the specific level of analysis (N2), we have this result: a quarter of grade 5 pupils and a third of grade 6 pupils were able to give at least one answer at this level. At the same time, a third of grade 5 pupils and half of grade 6 pupils proved to master level 3 (informal deduction) on one occasion at least. These answers were identified as such through worded validations relying on theoretical properties of figures.

## CONCLUSION

As expected, 57% of Primary School pupils worked only in GI and 32% of them mastered only the first van Hiele level (N1 identification-visualization) on both tasks. 28% of Secondary School pupils worked consistently in GI but 13% mastering only the first van Hiele level. Such results suggest that some pupils are working in

---

<sup>14</sup> The code N0 was used for the answers we could not clearly identify as referring to any other van Hiele level. But, we considered that at that level of schooling, most pupils knew some elements of geometry, even with some awkwardness and could be added to those who were clearly identified as mastering N1.

GI but master van Hiele level 2 (N2 analysis). At the same time, some answers were coded as GII – N2, meaning that the pupil working in GII was mastering only the van Hiele level 2.

When analysing the characteristics of the different answers identified as belonging to level 2 (analysis), it seems that this level is the “overlapping zone” between GI and GII.

We could describe as GI – N2 any situation where:

- The drawing is called “figure” and is made at scale 1 with instruments (ruler, set square, compasses, etc.).
- The pupil can read the information given in such a “coded” drawing.
- He/she can give a list of the properties of the figure but cannot work out any new piece of information or conclusion.
- He/she can identify sub-figures within a figure but cannot make any connections in terms of incidence.

We could describe as GII – N2 any situation where:

- The “figure” is a coded sketch or a free hand drawing and the pupil is able to give the list of the properties of the figure.
- The pupil can identify sub-figures in a figure but he/she cannot explain or justify the existence of such sub-figures.
- The pupil can tell what he/she did when drawing the “figure” at scale 1 but the story is not organised. It does not refer to any theoretical property or hypothetical-deductive reasoning.

These results tend to indicate that teachers should be advised that not all the pupils entering grade 6 are ready to work in GII. When proposing tasks to their pupils, the van Hiele level theory could come in hand, the focus being on level 2 (analysis).

This research was the first one on such a hypothesis: level 2 being “the overlapping zone” between GI and GII. More research has to be done particularly clinical investigation in order to understand how and when a pupil switches from one paradigm or a specific van Hiele level to another. It would be particularly interesting to investigate if a pupil entering GII is still working back and forth from GII to GI just as any professional mathematician do. Research on the tasks themselves could be fruitful and offer the teachers, activities helping the students moving forward in their learning of geometry. Investigation should be undertaken in order to connect these results with the different working geometrical spaces as identified by Kuzniak (2010) too.



## REFERENCES

- Burger, W. F. & Shaughnessy, J. M. (1986). Characterizing the van Hiele Levels of Development in Geometry. *Journal for Research in Mathematics Education*, 17(1), pp. 31-48.
- Braconne-Michoux, A. (2008). *Evolution des conceptions et de l'argumentation en géométrie chez les élèves : paradigmes et niveaux de van Hiele à l'articulation CM2-6<sup>ème</sup>*, Thèse de Doctorat Université Paris Diderot-Paris 7. [on line] <http://tel.archives-ouvertes.fr/tel-00389633/fr/>
- Clements, D.H. & Battista, M.T. (1992). Geometry and Spatial Reasoning, *Handbook of Research on Mathematics Teaching and Learning* (pp.420-464). New York, U.SA. Mac Millan Editors
- Gutiérrez, A., Jaime, A., Shaughnessy, J. M. & Burger, W. F. (1991). A Comparative Analysis of two ways of Assessing the van Hiele Levels of Thinking; *Proceedings of PME 15*. Vol. 2, pp. 109-116. Assisi, Italy
- Houdement, C., Kuzniak, A. (1998-1999). Géométrie et paradigmes géométriques. *Petit x* N°51. pp. 5-21.
- Houdement, C., Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. *Proceedings of CERME-3*. Bellaria, Italy Retrieved from [http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7\\_list.pdf](http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG7/TG7_list.pdf)
- Parzysz, B. (2003). Articulation entre perception et déduction dans une démarche géométrique en PE1. *Carnet de route de la COPIRELEM*, 2, pp. 107-125
- Kuzniak, A. (2006). Paradigmes géométriques et géométrie enseignée au Chili et en France. *Cahiers de DIDIREM Numéro spécial 6*, Université Paris 7- Denis Diderot, Paris
- Kuzniak, A. (2010). Un essai sur la nature du travail géométrique en fin de scolarité obligatoire en France. *Annales de Didactique et de Sciences cognitives*, vol 15 ;pp. 73-93, IREM de Strasbourg, France
- Jore – Lemonnier, F. (2006). *Paradigmes géométriques et formation initiale des professeurs des écoles en environnements papier-crayon et informatique*. Thèse de Doctorat de Paris 7 Diderot, Paris
- Uziskin, Z. (1982). *Van Hiele Levels and Achievement in Secondary School Geometry*. CDASSG Project, University of Chicago. Retrieved from <http://www.eric.ed.gov/>. ERIC# ED220288
- Van Hiele .P.M. (1959). La pensée de l'enfant et la géométrie. *Bulletin de l'APMEP*, 198, pp. 199-205.
- Wirszup, I. (1976). Breakthroughs in the Psychology of Learning and teaching Geometry, *Space and Geometry, Papers from a Research Workshop*, Martin, J. Larry, Ed., Bradbard, David A., Ed.. Retrieved from <http://www.eric.ed.gov/>. (ERIC# ED 132033)

# **GEOMETRY AS PROPAEDEUTIC TO MODEL BUILDING – A REFLECTION ON SECONDARY SCHOOL TEACHERS’ BELIEFS**

Boris Girnat

University of Education Freiburg, Germany

*This article presents a qualitative study of teachers’ beliefs about applying geometry, setting application-oriented beliefs in the context of the teachers’ whole geometry curricula. Surprisingly, geometry is not perceived as a field of good applications, especially not for model building. Three classes of objections are discussed and connected to a prevalent Euclidean view of geometry and a preference for proofs and problem-solving tasks. Despite these objections, applications are taught, but the analysis of the teachers’ tasks implies that the teachers’ implicit theory of applying geometry differs from didactical requirements and is not compatible with common approaches to model building. The teachers’ predominant alternative, which is called a propaedeutic use of geometry, is described in detail.*

## **CURRICULAR ASPECTS OF TEACHERS’ BELIEFS ABOUT GEOMETRY**

The research on teacher beliefs has become a prospering branch of mathematics education, revealing subtle influences on student learning. The focus of this study is secondary school teachers’ beliefs about teaching geometry in the context of applied mathematics. Beliefs are customarily understood as “psychologically held understandings, premises, or propositions about the world that are thought to be true” (Philipp 2007, p. 259). In contrast to the general notion of beliefs, this study is only interested in a subset of teacher beliefs which has a similar content, structure, and purpose to a written curriculum. This part of a teacher’s belief system is called his individual curriculum (cf. Eichler, 2007). Its aspects – content, structure, and purpose – can be explained as follows (cf. Stein, Remillard, & Smith, 2007): the purpose of an individual curriculum is equal to the function of a written curriculum, i. e. it is used to structure lessons and to guide instructional practice to the goals of education through various steps of content and method. Hence, its structure can be seen in means-end relations between mathematical content, methods, and educational goals.

Teacher beliefs about mathematical content are split into beliefs about concepts, theorems, objects, tasks, and textbooks. Beliefs about educational goals are separated into three levels of generality: content-specific abilities, general competencies, and top-level goals of education. The choice of the competencies enquired about during data collection is guided by the written curriculum the teachers have to act on: arguing, problem-solving, modelling, communicating, formalising, algorithmising, and using mathematical description and symbols (KMK 2004, p. 7).

Against this background, a qualitative study was designed to examine the individual curricula of nine teachers teaching geometry at German higher-level secondary

schools (so-called Gymnasien, in which about 35-40% of German students are taught and whose school leaving certificates are normally necessary to get access to university). To invite teachers to participate in this study, four districts of the governmental school hierarchy in different regions of Germany were contacted, each responding with a list of two or three teachers willing to participate in this study. The teachers were visited and interviewed by the author.

The focus on curricular aspects was chosen because investigations of the implementation of curricula are scarce in Germany. In addition, no study has been carried out about the ways in which the changes to the new national curriculum (KMK 2004) are reflected by teachers – especially its emphasis on the competencies mentioned above and its increased emphasis on applications and model building. This lack of information was the reason to focus this study on modelling and to examine how beliefs about this topic are integrated into the teachers' whole geometry curricula.

Individual curricula as subjective theories can provide a bidirectional contribution to mathematics education (cf. Girnat 2010): it is possible to detect disparities between prescribed goals and teacher objectives in order to discover possible errors in practice. On the other hand, individual curricula can also be analysed in comparison with didactical opinions on a cooperative level to integrate teachers as semi-professional researchers and to expedite theoretical thinking on the ground of differing views from classroom practice. Both aspects are pursued within this study.

## **THEORETICAL BACKGROUND, METHODS, SETTINGS, AND DATA**

The data were collected by semi-structured in-depth interviews, each taking about 90 minutes. They were interpreted according to the research programme of subjective theories (Groeben et al. 1988). This framework was invented by psychologists to collect and interpret complex systems of beliefs used by professionals to make their decisions when acting occupationally on the basis on a more or less commonly shared, but individually interpreted, theory, containing empirical knowledge and normative prescriptions similar to curricula or didactical theories. Due to the usual complexity of a professional's subjective theory, a qualitative approach is normally preferred and is used in this study.

To interpret the data, a so-called dialogue-hermeneutic method was invented (Scheele & Groeben, 1984) consisting of three steps: an interview to collect the main data, interpretation of the data by hermeneutic methods to define the subjective theory, and spot check observations of participant behaviour to validate whether the assumed subjective theories are in fact relevant to the teachers' practice. In this case, observations consisted of five lessons per teacher and a collection of the application-oriented tasks used in approximately the last quarter before the observation.

The teachers' geometry curricula were analysed in all the aspects mentioned above, and not limited to topics of applications, mathematisation or modelling. The reason for such a "holistic" approach is the idea that the common instructional practice is

guided by several goals of education not necessarily related to applications. Hence, application-oriented goals have to find their places within the totality of curricular aims and convictions. The central questions of this study can be only answered concerning a whole individual curriculum: what significance do the teachers attribute to application-oriented goals? Of what kind are the connections between application-oriented goals and other goals of education? Are teachers' application-oriented goals similar to or different from didactical ideas and the new written curriculum?

## THE FOCUS ON MODEL BUILDING

Before presenting some results, it is necessary to briefly sketch some didactical perspectives on applied mathematics. The most essential issue seems to be the relationship between general mathematical theories and empirical knowledge in singular situations. Kaiser-Meißner (1986, pp. 83-92) has proposed a classification whose extremities are called the pragmatic and the scientific-humanistic approach. The latter emphasises mathematical concepts and theories as the main goals of education and incorporates real-world situations mainly as subordinate tools to develop mathematical concepts and insights into manifold realistic associations. Empirical knowledge is of minor interest; the teaching process follows a mathematical taxonomy of problems, concepts, and techniques, and is not derived from empirical questions connected to real-world situations. The real-world situations are just "illustrations". The pragmatic view, in contrast, stresses empirical insights into real-world situations and includes a meta-theory about the relationship between mathematics and reality to be picked out as a central subject when teaching applied mathematics. This approach rests on three classes of educational goals (Kaiser-Meißner 1986, p. 86): 1) utilitarian aims: the situations are not selected according to mathematical taxonomies, but on the basis of the current or expected benefit to the students' lives. 2) methodological aims: the students shall obtain general competencies in and meta-knowledge of applying mathematics. 3) meta-scientific aims: applying mathematics is perceived as model building. The concept of model building can be explained by the model building cycle and has to be reflected in classroom practice as "one of the main components of the theory for teaching and learning mathematical modelling" (Kaiser, Blomhøj, & Sriraman, 2006, p. 82).

Both approaches to applying mathematics imply different standpoints on the goals of education in general; the pragmatic view sees its contribution in universal model building competencies and a preparation for life situations. The scientific-humanistic approach instead rests on the generality of mathematical theories. The new written curricula in Germany based on the national prescriptions (KMK 2004) underline the pragmatic view and introduce model building as obligatory; it was incidental or ignored in former German curricula. For this study, a simple version of the manifold modelling cycles (fig. 1) is used, which seems to be sufficient to determine if a teacher possesses a concept of model building in the sense of the contemporary academic debate and written curriculum.

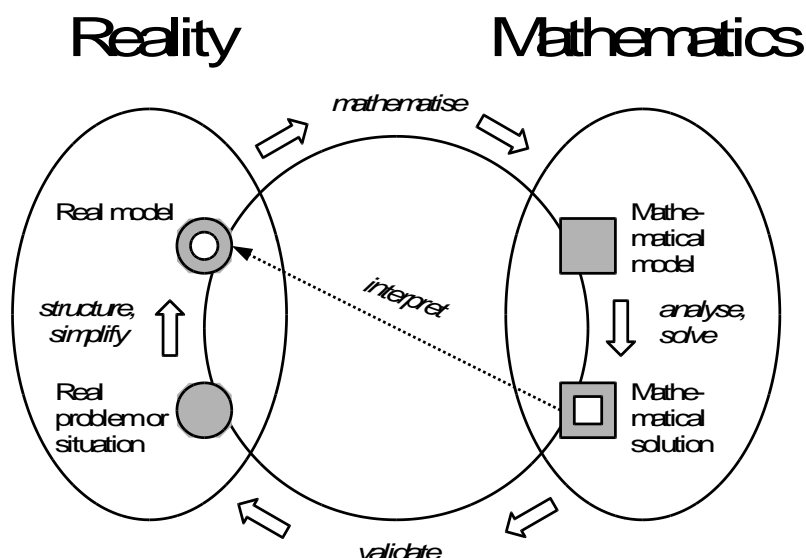


Figure 1: The modelling cycle used in this study

## FINDINGS: THREE OBJECTIONS TO GEOMETRICAL MODELLING

To summarise the results: seven of the nine teachers had objections to integrating applications into their geometry lessons, though being open-minded about modelling in other parts of school mathematics. The spectrum of objections ranges from strict exclusion to a moderate use. But even if geometrical applications are taught, the way of applying geometry differs from didactical suggestions. The interpretation of the data leads to three classes of objections which are based on different reasons.

### Ontological Aspects

The strictest opposition to an application-oriented way of teaching geometry is based on ontological beliefs about the nature of geometry and its objects. In Girnat (2009), the following classification of geometrical ontologies is proposed (fig. 2).

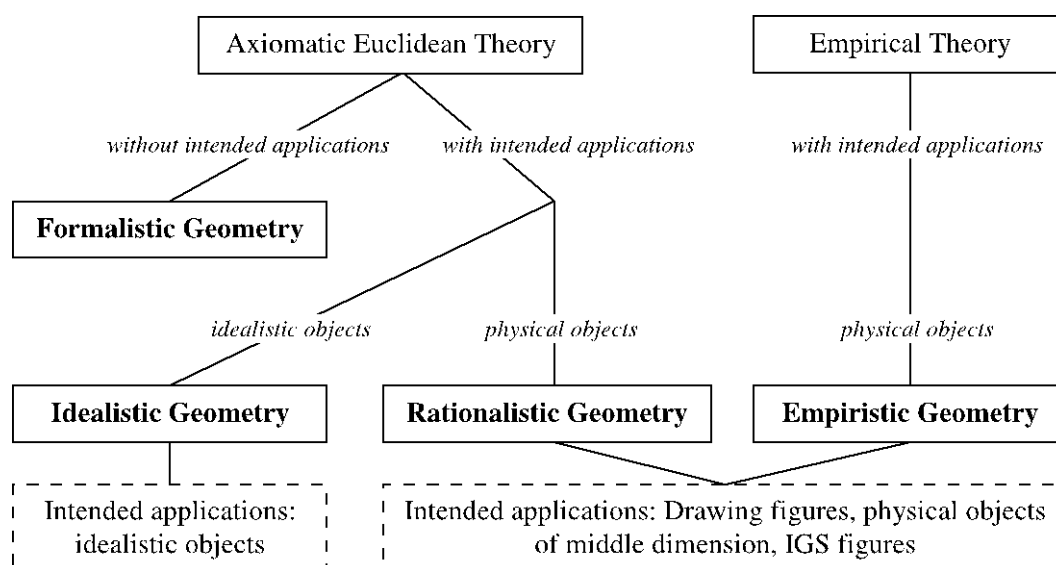


Figure 2: An ontological classification of geometry

This classification rests on two aspects: 1) the theoretical aspect: is geometry taught on the basis of a given axiomatic Euclidean theory (whose rigour may be restricted to an adequate school level) or is it derived from experience, observations, and measurement as an empirical theory? 2) intended application: does geometry refer to ideal objects in the sense of Plato or to physical objects or is it regarded as purely formalistic in the sense of Hilbert?

The two aspects of this classification follow the main ideas of the theory of geometrical working spaces (cf. Houdement & Kuzniak, 2001). According to this approach, geometry is split into three paradigms: a formalistic theory (called GIII), an idealistic theory strictly based on deductive arguments (GII), and an empirical theory based on measurement and experiments (GI). Compared to the classification used here, the theory of geometrical working spaces combines the aspects of justification (deductive or empirical) and reference (the connection of objects) to each other, leading to the consequence that an empirical reference is tacitly bounded to empirical, non-deductive methods in GI. For our purposes, it is necessary to separate these aspects and to allow a geometry which refers to empirical objects, but is mainly based on deductive arguments (just including some empirical initial conditions). This type of geometry is called the rationalistic one.

### **Idealistic Platonists: No applications intended**

Two of the teachers can be classified as exponents of an idealistic view of geometry. They do not perceive geometry as a theory of real objects, but of ideal entities which correspond to ruler-and-compass constructions and which fulfil the theorems of Euclidean geometry without any exceptions.

Mrs. D: The beauty of mathematics is the fact that everything is logical and dignified. [...] Everywhere else, there are approximations, but not in mathematics. There is everything in this status it has ideally to be in. [It is important for the students] to recognise that there are ideal things and objects in mathematics and that, in reality, they are similar, but not equal.

From this point of view, applying geometry is barred by definition. Instead, constructive descriptions are promoted to get access to ideal objects. Physical objects, typically limited to drawings, are only used as symbolic representations of the “true” ideal objects of geometry. Every empirical investigation is seen as a heuristic tool, but does not have any relevance for justifying geometrical insights.

Mrs. D: Besides proof abilities, problem solving is in fact the most important thing I want to convey in my lessons on geometry.

Mr. C: If geometry just consisted of measuring, calculations, drawing, constructing, and land surveying, then I would regard it as poor. [...] [Geometry as a] tool to get access to the real world? No, problem solving would be my favourite. Why? Problem solving is a keyword that includes everything. It is the final

goal to make students work systematically, identifying premises and drawing conclusions to solve a problem.

Classroom observations support the impression derived from the interviews: the lessons on geometry held by these two teachers are focussed on proof, construction, and problem solving tasks, using drawings only as heuristic tools.

### **Model building versus proof and problem solving tasks**

Not only by the two “idealistic” teachers are proof and problem solving tasks seen as the main aspects of teaching geometry, but also by six teachers who are not strict opponents of geometrical applications. Applying mathematics is not a top level goal, but rather is subordinated to proof and problem solving competencies. The tasks the teachers presented as good examples to convey these competencies match the typical characteristics of problem solving tasks (Holland 2007, pp. 170-195) and lead to the hypothesis that the aspects and methods demanded are contrary to the settings of a model building process. The main differences are summarised as follows (tab. 1):

<b>Aspect</b>	<b>Model building</b>	<b>Proving or problem solving task</b>
<i>Objects of interest</i>	singular situation	general theorem or configuration
<i>Access to objects</i>	by measurement and experience	by constructive descriptions
<i>Building a real model</i>	by simplifying	simplifying not allowed
<i>Mathematical treatment</i>	inventing a mathematical model	using known operators, theorems, or methods
<i>Validation</i>	empirically	by deductive arguments

**Table 1: Differences between model building and proving or problem solving tasks**

Most of the teachers want to prevent their students from getting confused by mixing the standards of modelling, proving, and problem solving. They hence typically split their courses on geometry into pure and application-oriented sections:

Mr. B: Geometry as a tool to get access to the real world is legitimately not in the first place. An application is useful to introduce a new subject, to legitimise it, and to test the competencies of this field by realistic tasks in the end. But in between, a lot has to be done without any reference to the real world, detached from these accessory parts which are not important to the mathematical theory. In between, applications are counterproductive. They seduce the students into not arguing strictly deductively.

Insofar as the deductive view of geometry is predominant and geometrical applications are mostly seen as appendices to the “serious” treatment of geometry, the

consequence is that aspects of model building are not integrated into the geometry curriculum and that geometry is “applied” to real-world problems in the majority of cases in the way that a problem-solving task is “decorated” by an empirical-sounding vocabulary, which is seen as characteristic of a rationalistic use of geometry.

### **The contrast: Geometrical applications as the main focus**

To contrast the first two types of objections, the only teacher who approves and practises application-oriented tasks extensively shall be quoted comprehensively. After arguing for teaching “geometrical modelling”, Mr. H was confronted with the question of what significance proofs and problem solving tasks had in his opinion:

Mr. H: Proofs, not in the sense of what is called a proof at university, shall demonstrate that something could be plausible, more plausible than something different. [...] My students shall be able to judge if a solution can be plausible, if the units match, if something could really have happened [...] if you throw a stone into a basin of water, then if the gauge could really rise by 3 meters. [...] They have to solve specific problems, and they have to use their geometrical tools. That needn't to be exact, that depends on the situation, and they have only to know if the methods and solutions may be plausible and realistic and how they can be used in the specific context.

Mr. H stresses the pragmatic aspects of applied mathematics, holding an empirical view of geometry at G1 level; and in doing so, he dispenses with the educational goals which the teachers mentioned above pursue by proofs and problem solving.

### **Geometrical applications as being uninteresting**

The third and last class of objections against geometrical applications is based on the assertion that they do not lead to interesting insights.

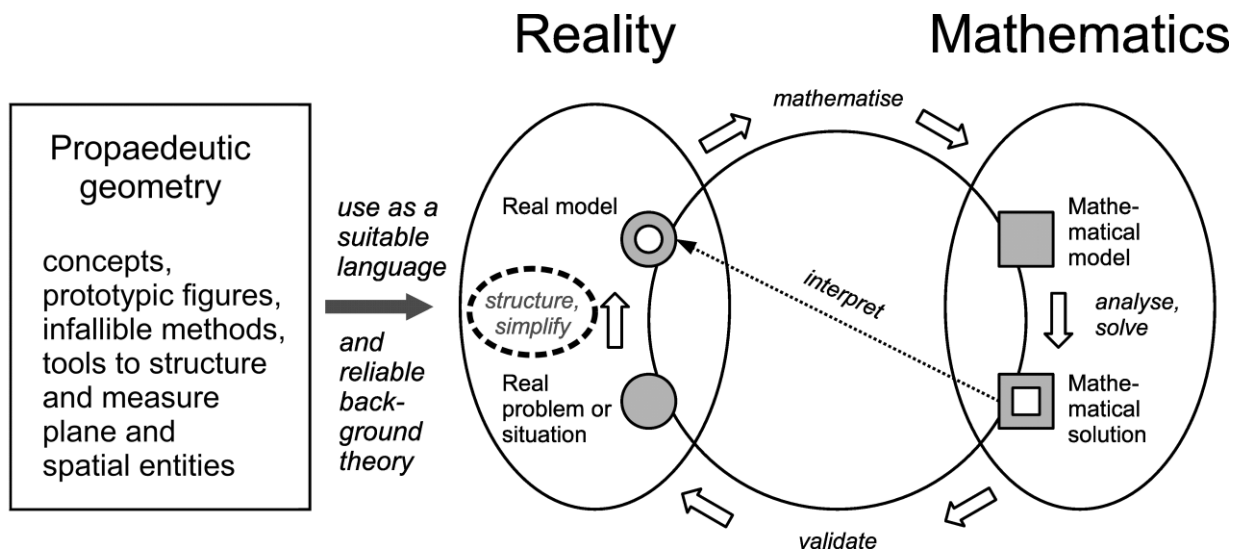
Mr. A: The better applications can be found in algebra or stochastics, per cent calculations, linear optimisation. It is important to get a deeper insight into reality by modelling. In geometry, there are such things as dividing a pizza by a compass. I saw a trainee teacher do so. That's ridiculous.

It is remarkable that this objection is focussed on purely geometrical applications. The classroom observations and the teachers' statements about “good” examples for modelling reveal that the teachers in fact use geometrical applications, but that they are small and, for themselves, uninteresting parts of more complex modelling tasks guided by non-geometrical questions. These tasks typically possess a two-step structure. In the first step, geometry is used to calculate some initial or boundary conditions, e. g. some lengths, areas, or volumes. Afterwards, these values are committed to a second, non-geometrical step which includes stochastics, algebra, optimisations or a problem derived from the natural or social sciences, e. g. some price, weight or velocity calculations. Five teachers used these two-step applications and stated that the interesting insights first and foremost arise in the second step. A typical example is mentioned and explained by Mr. B:



Mr. B: To grasp the sense of what I like to say, let us regard the following task: “A businessman wants to sell salt in small rectangular packages of 250 gram. What would be your advice to reduce the waste of material?” That’s an interesting problem providing some surprise, if you take the situation serious, and it is quite challenging, but the geometry in it is not, it’s standard, it’s only a vehicle to manage the interesting aspects, and it has to be well understood before deliberating about this problem.

If the modelling cycle is the core concept to analyse the learning and teaching of applied mathematics, then it will be difficult to reconstruct this two-step application by a cyclic structure (fig. 1). It rather seems appropriate to perceive geometry as “propaedeutic” to modelling, outside and “a priori” to the modelling cycle (fig. 3).



**Figure 3: Geometry as propaedeutic to model building**

The meaning of “propaedeutic” can be explained in three aspects: 1) A propaedeutic use of geometry is characterised by a static view on geometry: It is seen as a pre-established theory, based on rigid concepts, proved theorems, and infallible methods.

2) A propaedeutic geometry is used as a suitable language and reliable background theory to structure and simplify a situation by geometrical concepts. This way of applying geometry is different from a modelling process, since geometrical concepts are already used to structure the real situation, and not to build a mathematical model after structuring the real situation independently. Hence, the use of geometrical concepts and methods is prior to any kind of mathematisation in the sense of the modelling cycle. This aspect is best to observe in the two-step structure of the teachers’ “good” examples for modelling tasks. Geometrical concepts and theorems are already necessary to “see” geometrical objects in reality and to calculate the values of areas or volumes before the second step, the “true” modelling process, can get started. Additionally, and as a further contrast to the modelling cycle, the propaedeutic use of geometry does not include any kind of validation, since geometrical theorems and methods are treated as already proven. In the salt example,

the relevant second step is the optimisation process, based on proposals on how to shape the packages. The calculation of its shells, volumes, and cut-offs is just an algorithmic task, based on pre-established geometrical knowledge and methods.

3) The teaching method is propaedeutic, since most of the teachers follow Mr. B's suggestion to avoid connections to reality at first and to integrate realistic situations at the end of a teaching unit. The observations indicate that five of our teachers approve of modelling tasks and pose them in their lessons, but either geometrical problems are not involved and are taught separately or geometry is integrated propaedeutically into a two-step structure.

## CONCLUSIONS

The study reveals objections to an application-oriented approach to geometry based on three reasons: a traditional idealistic view of geometry, a preference for proof and problem solving competencies, and a propaedeutic treatment of geometry. In particular the pragmatic view of modelling with the model building cycle as its core concept could not be found as a part of the teachers' geometry curricula and teaching practice, though being observable in non-geometrical contexts.

It is interesting to see how several of the teachers integrate application-oriented aspects into their geometry curriculum on basis of their "non-application-oriented" view, combining a Euclidean perception of geometry with a propaedeutic use. This finding suggests some further reflections: although the classroom observations of most teachers reveal no application-oriented tasks which could be described as "good" modelling tasks in the sense of the academic debate and the new German curriculum, the teachers are not just unwilling to teach application-oriented geometry, but are focussed on educational goals connected to proving and problem solving tasks, which are also parts of the written curriculum, and which presuppose a geometrical ontology and methodology that provokes a conflict with the background theory of modelling objectively, and not only in the subjective perceptions of these teachers. The academic debate needs to propose a way to manage these conflicting demands in practice.

The observed two-step structure poses a particularly interesting question to the academic debate on modelling: is this way of teaching applied geometry just a consequence of the teachers' traditional Euclidean view of geometry or is it based on a typical way in which geometry "naturally" refers to reality? In the latter case, it would be questionable whether the model building cycle is an adequate representation of applying geometry. In contrary to common didactical debates (Kaiser et al., 2006), the findings suggests the conjecture that it may be advisable to shape the modelling debate less as a "top-down theory", establishing a single framework to be applied in every part of school mathematics identically, but more as a "bottom-up research programme", exploring the existing uses of applications in different contexts and parts of school mathematics following from the question of

whether there are ways other than the modelling cycle in which mathematics and the different parts or disciplines of school mathematics refer to reality.

## REFERENCES

- Eichler, A. (2007): Individual curricula – Teachers’ beliefs concerning stochastics instruction. *International Electronical Journal of Mathematics Education* 2(3). <http://www.iejme.com/>.
- Girnat, B. (2009). Ontological Belief and Their Impact on Teaching Elementary Geometry. In *Proceedings of the 33rd IGPME conference*, Thessaloniki (vol. 3, pp. 89-96).
- Girnat, B. (2010): Subjective Theories: A Conceptual and Methodological Specialisation of Beliefs Systems. In *Proceedings of the 15th MAVI conference*, Genoa (to appear).
- Groeben, N., Wahl, J., Schlee, D., & Scheele, B. (1988). *Das Forschungsprogramm Subjektive Theorien* (the research programme of subjective theories). Tübingen: Francke.
- Holland, G. (2007): *Geometrie in der Sekundarstufe* (geometry in lower secondary schools) Hildesheim und Berlin: Verlag Franzbecker.
- Houdement, C. & Kuzniak, A. (2001): Elementary Geometry Split into Different Geometrical Paradigms. *Proceedings of CERME 3*. Bellaria, Italy (Web).
- Kaiser-Meßmer, G. (1986). *Anwendungen im Mathematikunterricht* (applications in mathematics classrooms). Franzbecker: Bad Salzdetfurth.
- Kaiser, G., Blomhoj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *ZDM*, 38(2), 82-85.
- Kultusministerkonferenz (KMK): *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss*. (educational standards for the school leaving certificate in mathematics in secondary schools), München: Luchterhand, 2004.
- Philipp, R. A. (2007). Mathematics Teachers’ Beliefs and Affect. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 257-315). Charlotte: Information Age Publishing.
- Scheele, B., & Groeben, N. (1984). *Dialog-Konsens-Methoden zur Rekonstruktion Subjektiver Theorien* (dialogue-hermeneutic methods to reconstruct subjective theories). Tübingen: Francke Verlag.
- Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. Lester (Ed.), *Second Handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte: Information Age Publishing.

# GEOMETRIC WORK AT THE END OF COMPULSORY EDUCATION

Kuzniak Alain

Laboratoire de didactique André Revuz, University Paris-Diderot France

*Abstract. The purpose of this contribution is to define the nature of the geometric work implemented in France at the end of compulsory education. To do the study, the notions of geometric paradigms and Geometric Work Space (GWS) have been used. The reference GWS is clarified through an analysis of the curriculum written in 1996 and 2005; then the “appropriate” GWS is studied by confronting textbooks and observations in class. From this analysis, it results that the GWS are becoming more and more split and oscillate in a confusing way between the geometric paradigms. This dispersion of the GWS is largely due to the fact that the geometric work is not any more controlled by epistemological concerns but tends to adapt the mathematical level of the students.*

## INTRODUCTION

This article focuses on the nature of the geometric work implemented at the end of compulsory education in France (Grade 8 to Grade 10). These grades correspond to the end of the compulsory education and also to the end of a common education program for the quasi-totality of students. Grade 9 is the final year of the “collège unique”, called unique because it is dedicated to welcoming all the students by giving them the same learning within a common framework. Grade 10 is the first class of the Senior High School and it provides the time when students choose the school sections that are more specialized in some particular domains. Due to this, most of the students receive their last education in geometry in this grade. To examine the nature of geometric work, the notions of Geometric Working Space (GWS) and geometrical paradigms (Houdement and Kuzniak, 2003, 2006) are used. The three geometrical paradigms, named Geometry I, II and III, are not presented in detail, but some elaboration of the notion of geometric work and GWS are given to frame the questions studied in the paper.

## THE NOTION OF GEOMETRIC WORKING SPACE

### Towards a definition

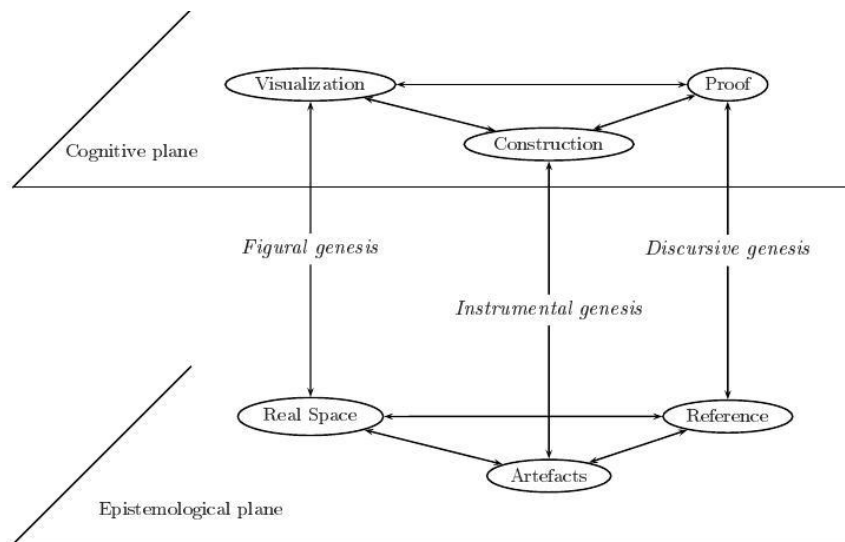
The geometric working space (GWS) is a workspace organized to ensure the work of people solving geometry problems (geometricians). It has two levels: the "components plan" and the "cognitive plan".

The "components plan" comprises three parts: the real and local space as material support with a set of concrete objects; artefacts such as drawings instruments and software available to the geometrician; a theoretical system of references made of properties organized in a way that is dependent on the geometrical paradigm.

The sole components are not sufficient to define the global meaning of a GWS which depends on the function that its designer and its users give to it. A first reorganization of these various constituents is of an epistemological nature and directed by the geometrical paradigms.

The GWS function can evolve in connection with the social and economic context which influences the educational institutions where geometry is taught. Moreover, this function depends strongly on the cognitive ability of a particular user. The

cognitive plan was introduced to clarify the cognitive processes involved in geometry. Adapting Duval (1995), the three following cognitive processes have been introduced: a visualization process with regard to space representations and the material support; a construction process depending on the instruments and on the configuration; a discursive process in relation with proving and reasoning.



As the GWS is created within the framework of school institutions, we need to introduce different levels in order to describe the diversity existing in school education.

**The reference GWS or the expected reorganization** The choice of a paradigm by the members of a community implies that problems have to be formulated and solved in a particular GWS that we name the reference or expected GWS. To describe this GWS, it is necessary to exhibit these specifics ways of working and clarify the expected style with its treatment and presentation rules. The reference GWS depends on the chosen paradigm: Geometry I, II or III.

**The “appropriate” GWS or the question of the implementation** Once the bases of the geometry taught have been determined, it remains to be worrying about its actual learning which requires the existence of GWS convenient to carry out the expected geometry with a chance of success. The experts in charge of the GWS design play a role similar to that of an architect who conceives a working space for future potential users. They develop a GWS which can be appropriate with regard to the intentions of the institution, but which can turn out to be not adequate to its expected function and not successful during its implementation in classes.

**The personal GWS** The appropriate GWS must be invested by students who use it with their knowledge and their cognitive abilities and these new GWSs are named personal GWSs. They are formed in a progressive way depending on the individual and can sometimes not be operational. Not only are students concerned by this notion but also teachers in charge of shaping it. Indeed, they have to have a clear consciousness of the nature of the GWS to avoid some misunderstandings resulting from a vague and implicit management of the interplay between paradigms.

We can finally express both questions which we aim to clarify in this paper.

Question 1: What is the current reference GWS proposed at the end of compulsory education in France?

Question 2: What are the characteristics of the corresponding appropriate GWS?

We wish to study more particularly the consequences of the division announced of the reference GWS on the appropriate GWS and also on the students' personal GWS. Our research into the characteristics of this geometric work agrees well with the systematic approach privileged by the study TIMSS (Kaiser, 1994) where the focus on various types of curricula are called intended, implemented and attained.

## **ON THE REFERENCE GWS AT THE END OF COMPULSORY EDUCATION IN FRANCE**

During periods of educational stability, access to the reference GWS is facilitated by what might be called “treaties” which organize and determine the reference corpus. For a long time, Euclid's Elements played this role and fixed the nature of the geometric work. It is no longer the case in our education system. Today only the texts of the official programs and the documents accompanying them seem fulfill this reference role. On the one hand, mathematicians are practically absent from the process of elaboration of the curriculum which is left chargeable to the school institution and the teachers. On the other hand, the absence of organization of the theoretical set of reference in a treaty explains the impression of parceled work space that we attribute to the current GWS of reference. The programmes that we study are those published in 1996 and 2005.

In school, both versions of the programme, the older and the newer one, insist on the notion of mathematical activity defined as the fact « to identify a problem, speculate a result by experimenting on examples, build an argumentation and control the obtained result. ». As for the geometry, it has assigned to it the role « to pass from a perceptive identification of figures and configurations to their characterization by properties ». In the documents accompanying the programme, it is clarified that the properties to be demonstrated can be “seen” on the drawing but that students have to understand the necessity of demonstrating this result. By using the framework of paradigms, we can assert that the curriculum is mainly concerned with the question of the transition between GI and GII. However, the passage from one geometry to the

other one is not once definitively established at a specific moment in the curriculum, and the transition seems ceaselessly put back on every new notion. The new notions are introduced and structured around geometric objects which can be seen also – and the programs insist on it – as objects of the sensitive space: triangles, circles, polygons. The geometric transformations are also used as a structuring element. In the curriculum of 1996, a new transformation (orthogonal symmetry, symmetry through a point, translation and rotation) was introduced at each level of the college. In those of 2005, translations and rotations disappeared with the effect of decreasing the global structure of the theoretical set of reference which appears more and more like a collection of objects juxtaposed.

This indecision on the final choice of the paradigm is particularly clear with the emphasis on experimental studies made to speculate on properties in every Grade. Constructions (in freehand or using drawing instruments or software) play a key role in this process. But at the same time, the notion of a minimum base (le socle) is imposed by the school institution and all the students are supposed to reach this minimal level on all the topics. In the case of geometric constructions, no formal proofs are expected and the basic knowledge is limited to the mastery of techniques useful to make constructions. At the end of the schooling, a student must know how to build and master some elementary techniques without necessarily knowing the theoretical justifications stemming from Geometry II. Through this sole expectation of techniques, we already guess the possible gliding towards an appropriate and personal GWS directed by a technological horizon in Geometry I with the accent put on the perception and the artefacts. To characterize this gliding not assumed towards Geometry I, we will speak of a surreptitious Geometry I.

By contrast and always in the syllabi of 2005, the learning of the demonstration is the object of a more steady attention. In a paragraph entitled « a progressive initiation to the demonstration » it is explained that « the question of the proof has a central place in mathematics ». The practice of the proof allows gradually the implementation of the demonstration. This distinction between proof and demonstration is new in the French education. The proof depends on social context and it can take various forms while the demonstration is fundamentally a rhetoric shape characteristic of the mathematical style. This distinction between proof and demonstration leads the authors of the curriculum to differentiate two phases in the learning process of demonstration: the reasoning and its shaping. At the same time the authors insist on the phase of discovery and it has some consequence on the work set up in the classes because it suggests introducing two different contexts in the appropriate GWS: a context of discovery and context of justification.

Dynamic geometry software is also introduced and used for changing the students' viewpoint on figures that are no longer considered only in their iconic form. By giving the possibility of moving points and of multiplying experiments, software is supposed to favour the access to the general notion of figure contrasting to the

drawing. In this optimistic view, dynamic geometry software is used to favour conjecturing and reasoning to validate the conjectures. But, the ambiguity of the appropriate GWS is again stressed by the role they have: in certain cases they can be used instead of a demonstration when students are not able to produce the expected reasoning in Geometry II. So, software is not only at the origin of conjectures but they can also guarantee the validity of a result. There is then an implicit and potential gliding towards a GWS where the geometric work is directed by experiment and artefacts. In Grade 10, in another institution, the Lycée, the goal announced by the authors of the programme is to stabilize the geometric work developed before during the Junior High School education. New notions such as isometric and similar triangles are not introduced for their own interest but for using the tools developed for proving in Junior High School. The recommended working methods have to be very close to the implemented ones at Junior High School. The starting-point in geometry must be intuitive and experimental and based on perception. Software remains a source of conjectures of properties. These must be proved and then demonstrated in a more formal way.

To conclude this part, we shall speak from a mixed Geometry (GI / GII) because if the authors of the programme insist on the difference between demonstration and experimental proofs, both forms live ceaselessly and seem also justifiable. Furthermore, the theoretical set of reference is split and does not allow to assume completely the passage to Geometry II due to the lack of an axiomatic horizon. Only, isolated and proof dedicated islands are developed and proofs have to be supported by experiments every time. This evanescence of an organized theoretical set of reference is not new and was starting at the end of “modern maths” in reaction to the “all axiomatic” of this period. What is new is, on the one hand, the ambiguous status given to instruments and to constructions and, on the other hand, the multiplicity of the demonstrative islands dependent of configurations in poor connection with each other. This multiplicity contributes to the breaking up of the theoretical set of reference.

## **TODAY APPROPRIATE GWS**

We are now going to try to appreciate the effects of this Geometry mixed and split on the appropriate GWS which we met. In other words, how is this evolution of the reference space echoed in textbooks and in practice of the teachers? To address this question, we now attempt to assess the effects of this mixed and fragmented geometry on the appropriate GWS which we met. In other words, how does this evolution of the reference GWS affect textbooks and teachers' practice?

Describing the appropriate GWS is more complex than specifying the reference GWS because it is rare to be able to have a unique source to define these GWSs. To understand their functioning, it is necessary to resort to various, sometimes contradictory, sources such as the courses of the teachers and the textbooks, very



numerous in France (where there are no textbooks accredited by the Ministry). Furthermore, it is often only possible to approach the GWS in a local way from a study of a subject or even of a type of prescribed task. For our approach of the appropriate GWS, besides our personal observations in class, we shall lean on different works realized within the Laboratory André Revuz and which give information stemming from textbooks but also from the practice in class. To describe the process of didactisation existing in class and determine the appropriate GWS, we present successively:

- A study of the notion of inscribed angle in Grade 9 which provides a first characterization of the standard appropriate GWS;
- The gliding introduced into this GWS by the large use of geometry software from the years 2000;
- The break between the standard appropriate GWS and a great part of the students engaged in another type of geometric work than the expected one by the teacher.

### **Standard appropriate GWS**

We are going to observe the treatment of the notion of the inscribed angle at Grade 9. This notion seems relevant for our study because it takes place at the end of the Junior High-school and it forces the teachers and the textbooks to integrate it into a GWS already in place and that allows us to see some stable characteristics of the GWS. Two properties appear in the Grade 9 syllabus and correspond to the properties expressed by Euclid in his book III: property 20 gives the relation between the central angle and the inscribed angle intercepting the same arc, and property 21 asserts the equality of the inscribed angles as a consequence. The implementation of the notion of the inscribed angle in class has been studied by Roditi (2004) who was able to show the close relation of the approach developed by a teacher with the proposition made in the textbook retained for our study. Roditi points out that this textbook is well known for being well-adapted to the level of the students. So in that case, the implementation of the appropriate GWS is already very influenced by the students' mathematics level. The notions of inscribed angle and central angle are introduced by an activity. Students are asked to draw these two notions from two questions on a corpus of six figures. Definitions are given a little farther by the authors of the book. They are then associated to prototypic images of the notions of acute angle and central angle. So the mode of production of the definitions is of empirical type. Based on some particular drawings, the defining process suits to an abductive way, something which is confirmed in the activity dedicated to the two fundamental properties of inscribed angles and central angles.

« Draw a circle with centre O. Draw several inscribed angles in a circle which intercept the same arc BC. Measure these angles. What conjecture can we do?

Draw a circle with centre O. Draw a central angle and an inscribed angle in this circle which intercept the same arc BC. Measure these two angles. Repeat several times these drawings. What conjecture can we do? »

This activity allows the showing of both properties, written in red in the book and presented in an order different from the Euclidean order. « If two inscribed angles in a circle intercept the same arc then they have the same measure. If, in a circle, a central angle and an inscribed angle intercept the same arc then the measure of the central angle is the double of the measure of the inscribed angle. »

Both properties are identified from very few examples. We can actually speak here of an abduction: the idea of property being present, it is sufficient to extract it from a small number of examples verifying it. The use of measuring is recommended to speculate the property even if the abductive process incites to neglect the approximation and tends to make useless the actual measurement. So, the appropriate GWS which is set up leans determinedly on Geometry I but do we really enter in Geometry II? Settling the question is not evident in this book since both properties are not demonstrated. They are admitted without knowing exactly their validation. In other words, are these properties included in a GWS directed by Geometry II or Geometry I?

Furthermore, the two properties are not presented in the order usually used in the Euclidean tradition and which allows the deducing of the property of the inscribed angles from that of the central angle. This absence of a concern with the global organization of the deductive schema takes away the appropriate GWS from Geometry II. This impression is confirmed by the study of the only use of the property in the textbook. It is asked to prove that some points lie on a straight line on a very particular configuration without any degree of generality – with specifics measures – that a more general formulation would have been able to introduce. So, the work remains joined to a particular figure without reaching the level of the generic figures which marks the entrance in Geometry II. The appropriate GWS is so characterized by the absence of generic figure and by the support on particular figures. It is also allowed to measure. The reasoning is mainly based on abduction to clear properties which are then used as techniques to give numerical values. To us, all these elements characterize a GWS, actually, rather directed by surreptitious Geometry I.

In his study, Roditi observed a class session given by a young teacher who used the earlier textbook to prepare his course. He did a certain number of changes with the goal of limiting the degrees of freedom of the students during the activity. The fact of limiting the work and the initiatives of the students allows the teacher to manage more easily the behaviour of the class. In his study, Roditi asserts that students have worked even less than the teacher was waiting, notably at the level of calculations and speculating. So, we note a phenomenon of progressive crumbling of the appropriate GWS which is more and more oriented by the teacher trying to adapt the work to the level of the students. These last ones, in a well-oiled role game, still try to simplify the task to make easier their work of student.

### **The impact of the software on the appropriate GWS**

In her Masters dissertation, Boclé (2008) described the typical situation given in the French textbooks to introduce a new notion in geometry at the end of Junior High-school. In textbooks, conceived just after 1996, the typical structure SP1 was the following:

1. Construction of some particular figures with drawing instruments.
2. Measurement on these figures by using instruments (marked ruler or protractor).
3. Conjecture of the property.
4. Institutionalization of the property accepted without proof or formally proved later.

In the textbooks printed after 2005, a new tendency appears. A new notion is introduced with digital geometric software. The typical situation SP2 is then the following one:

1. Construction of a figure with digital geometric software.
2. Measures given by the software.
3. Dragging of points to notice that the property remains true.
4. Institutionalization of the property accepted or accepted without proof or formally proved later

In both cases, to introduce the property, students have to build several figures satisfying some criteria. Thanks to the measures made on the figures, it is possible to notice an invariant then to draw a conjecture. In the textbooks written according to the programs of 2005, the activities of construction and measuring suppose the use of geometry software. Every activity starts clearly in the GWS directed by Geometry I and favouring perception and instrumentation. In both approaches, with and without software, the point 4 is the crucial point to determine the type of geometry really used and the appropriate GWS. If the property is only proved in a deductive way without any use of measuring, it is possible to enter into Geometry II. On the other hand, what happens if the property is not demonstrated? It seems that students stay in Geometry I. These typical situations well fulfil the programme instructions recommending the implementation of activities leading to conjecture of properties. The recent emphasis on the use of geometry software is taken into account in textbooks but the real contribution of this software in the transition from Geometry I to Geometry II deserves to be questioned. Indeed, the use of digital geometric software is justified in the textbooks by improving the measuring accuracy and the possibility of multiplying the examples. But a measure remains an approximation and it is thus not exact. This imprecision can create a contradiction within the class and lead some students to convince themselves on another way and then to prove without any measurement. By contrast, insisting on the precision of the software and their advantage with regard to ruler and compass constructions could risk to take away students from the necessity of proving which was one of the stakes expected within the reference GWS. In her work, Boclé tried to see if the use of software in

these typical situations favoured the transition to Geometry II or if, on the contrary, it created a blocking element. She noticed that the strength of the proof by experiment overcame the classic work on demonstration with a purely deductive proof. In that case, it seems that the use of the software in a standard situation stabilizes rather a GWS of type Geometry I and not a transition to GII.

**The break achieved in Grade 10 or when the ostension becomes demonstration.**

We are going to find again this contradiction between the work expected by the institution and the work effectively set up in the case of the teaching of similar triangles in an ordinary class at Grade 10. Similar triangles are not seen by the programs as a new notion but as an opportunity to stabilize the geometric work at the end of the compulsory education. We shall consider here only the result of a session managed by a teacher who first follows the typical way SP1 but who changes on phase 4 of institutionalization and then follows the process SP2 by using uniquely himself the software.

The activity is the first activity on similar triangles.

A sheet of paper is given to the students with a drawing: Part 1 Create a triangle DEF such that  $\angle BAC = \angle EDF$ ,  $AB = DE$

Under the figure, the following questions appear on the sheet given to the students: What can we say about angles ACB and DFE? Compare the sides of the triangles with your ruler. What can be noticed? Finish the sentence: We can speculate that if two triangles have .... then their sides are ....

For the teacher the construction does not cause a problem. He anticipated two possible configurations, what seems an interesting difficulty to him. He wants to motivate the origin of a property in Geometry I which will be completely in Geometry II when it will have been proved in the following lesson. For him, the figure is a generic example and he has not really thought about the measures given on the paper sheet. The great majority of students, but not all, undertake completely in the activity of construction which turns out long and complex. Students have difficulties with the use of their drawing instruments: the task « to make an equal angle » does not fit to a well-known technique. Furthermore, the two existing possibilities of the figures cause problems in the class since students are working on particular and not on general figures. Other students understood that the construction is not important for the teacher and they quietly wait that the course goes on. They give, by abduction, purely linguistic conjectures by trying to adapt their mathematical knowledge to the situation. At the same time, students engaged in the construction task produce very different and contradictory results but actually these results and the work of these students will be left aside by the teacher who will privilege the solution with the software Geogebra and presented on video-projector to the class. The teacher follows the structure SP2 but without making any devolution to the students. He is the unique user of the software and he proceeds to

an institutionalization which denies all the previous work of the students. On the computer, the figure is the start-point and measures are given with five digits and this even for angles. The ratio of proportionality calculated by the computer was 1.875 and was exactly the same for the three ratios. The accuracy of the measures indicated by the computer shows the students the imperfection of their work with instruments on a very violent way. Strictly speaking, the students' work is of little utility because it is left aside by the teacher. Moreover, the accuracy of the software turns it into a proof tool and a source of truth and, this, without the teacher knowing, as it can be seen in the dialogue which closes the class after the statement of the conjecture.

*Teacher : « Did we demonstrated the property? »*

*Almost of the students: « Yes! We have done a demonstration. »*

*Teacher (taken aback) « Hum.. No, it is too imprecise! »*

So after more than three years of progressive entrance in Geometry II and in spite of the curriculum which insist on the necessary awareness on the status of the statements, accepted or demonstrated, the gap between the expected work and the effective work is deep. It largely results since the appropriate GWS proposed to the students is very ambiguous itself and probably fundamentally a surreptitious Geometry I.

## CONCLUSION

From our study, some characteristic points can be drawn of the GWS implemented at the end of the compulsory education in France. The GWS of reference can be characterized as relevant to parceled Geometry II. Numerous demonstrative islets are introduced to show well the link between geometry and space intuition. Then, the emphasis is put on the necessity of developing the demonstrative work by separating it from experimental proofs and from perceptive assertions. However, this reference GWS leaves the door opened, in certain cases, to the implementation of techniques and properties only validated by students experimenting with software. Furthermore, the constant emphasis on a transition towards Geometry II based on Geometry I can let suppose that a mixed Geometry is possible. This opened door becomes a boulevard when we look on the appropriate GWS which appears particularly unstable and dependent on the students' level and the choices of the teacher. The traditional play between the Geometries I and II turns out to be particularly ambiguous because of the probing power of the software for the students. In the observed examples, the gliding towards Geometry I was favoured by the use of software which establishes a computer proof faced the axiomatic proof. This last one is weakened all the more as the theoretical set of reference given to the students does not appear, even between the lines.

Finally, the reorganization of the GWS seems more and more managed by a teacher adapting at the level of the students more than by assuming epistemological choices. The geometric work evolves by successive impoverishment what can explain the

recent attempt to abolish the discursive and figural geometry in Grade 10 for the benefit of the sole analytical geometry. Another way would be possible, in equivalence with the current social demand: to assume a Geometry I in the compulsory education. It would allow again a rich geometric work to be in place and for the GWS to be structured in a coherent way.

## REFERENCES

- Boclé, C. (2008). *Utilisation des logiciels de géométrie dynamique et espace de travail géométriques en classe de quatrième*, Master de Université Paris-Diderot.
- Duval, R. (1995). Why to teach geometry. *Icmi Studies on Geometry*, 53-58. Catania.
- Houdement, C., & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. *Proceedings of CERME 3*. Bellaria, Italy.
- Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie. *Annales de didactique et de sciences cognitives*, 11, 175-193.
- Kaiser, G., Luna, E. & Huntley, I. (1999). *International Comparisons in Mathematics*. London: Education Falmer Press.
- Roditi, E. (2004). Le théorème de l'angle inscrit au collège : analyse d'une séance d'introduction. *Petit x*, 66, 18-48.

# LANGUAGE IN THE GEOMETRY CLASSROOM

Bulf C., équipe DAESL, LACES, Université de Bordeaux

Mathé A.-C., LML, Université d'Artois

Mithalal J., LDAR, IUFM de Paris

*The purpose of this paper is to examine how it is possible to relate the way students perceive, act and talk about objects in geometry class. Based on the analysis of an "emitter-receiver" situation in a French geometry classroom at primary school, our contribution provides a theoretical tool in construction, called "frequentation mode".*

Key word: geometry, language, "frequentation mode"

## INTRODUCTION

In French primary school, manipulating objects is a usual and attractive way of teaching geometry in order for the pupils to get familiar with geometry objects and concepts. Yet, this is only a first step as there is no obvious link between material and conceptual objects (Lismont, 1999). Now, this relationship between the physical world and the geometry world necessarily induces problems of meaning due to the gap between the students' understanding and the sense that we, as teachers or researchers, give to these confrontations with objects.

Until today, French researchers (Berthelot and Salin, 1991; Gobert, 2001) in geometry education have approached this question by linking actions on selected material objects to geometrical knowledge, as Theory of didactical situations mainly focuses on material feedback from the *milieu* (Brousseau, 1998). However, these researchers' results, as well as our various experiments in teaching geometry (Bulf, 2009; Mathé, 2006), suggest that the relationship between the physical world and the geometry one is a very tough problem. Bartolini-Bussi (1994) or Sfard (2008) stress that analyzing the students and teacher's discourse in mathematics classroom is relevant to better understand learning and teaching phenomena in geometry. Our aim is to analyze how the way students act on the objects, the way they talk about them and how they act on them, are interrelated. In this paper, we give an overview of our work and introduce theoretical and methodological tools to be used for a joint analysis of physical and language facts in the geometry classroom.

## A SHORT EXAMPLE

The following example is an extract from Fregona's thesis (1990), recorded in a CM1 classroom (4<sup>th</sup> grade, 9 year-old students) at the Corem [1]. We focus on the analysis of a part of the second session of a sequence in which a "transmitter-receiver" situation is implemented. The announced aims of this lesson were to create

conditions to confront basic knowledge of figures from elementary geometry, to develop geometrical vocabulary and to use methods of figures construction.

The teacher gave the "transmitters" a cardboard-made geometrical figure and asked them to write a message, without any drawing, which should allow the "receivers" to construct a figure that can be superposed on theirs. In the session we analyzed, the teacher and students reviewed the messages referring to rectangles and organized a collective validation of the productions, by superposing the figure constructed and cut on the cardboard model figure.

For now, we focus on the following message, describing a rectangle shape: "*Take a set square and plot 19cm4mm for the larger side and 11cm7mm for the smaller side.*" There are at least two possible interpretations of this message: the "receivers" draw a triangle whereas the "emitters" thought about a rectangle. It seems there was a misunderstanding about the meaning of "set square" in this message. So, what happened, and how to deal with this? A didactical analysis might conclude that the students' understandings of "set square" were different: set square as the shape of a triangle, used to draw a triangle and on the other side, set square as the shape of a right angle, used to draw perpendicular sides (in this case, of a rectangle).

### **Naïve analysis**

The two interpretations we identified tally with Duval's ways of seeing in geometry (Duval, 2005): the "receiver" focuses on shapes with an interpretation of *iconic visualization*, whereas the "transmitter" may have described a way of representing relations between sides in a *non-iconic visualization* way.

Furthermore, we have to say that this misunderstanding was not clear to the pupils at the beginning, and three main stages are to be distinguished:

- First of all, a message with an "invisible" misunderstanding about the description of the shape;
- Next, a material feedback (the drawn shape doesn't fit the original): this revealed the misunderstanding as a fact, but it gave no explanation or solution to the problem;
- Then, a discursive interaction (arguing about what "set square" means): each student tried to *justify* how he defined "set square", used this tool, and to convince his opponent that his conception about set square was right : the meaning of set square is negotiated through language beyond a shared use and meaning. Thus, the use of set square changed and students moved from triangle to rectangle.

The first stage only doesn't make the pupils change their interpretation of "set square", whereas the negotiation about set square is about explaining and justifying and plays a significant role in the learning process. Actually, these two discursive interactions' functions are very different, and have to be clearly distinguished. Nevertheless, the role of material feedback is essential; that is why systems of representation and pupils' actions have to be taken into account simultaneously.



First of all, this preliminary analysis allows us to bring out the initial findings our development is based on. Two systems of representation are to be linked in order to understand student's actions: graphical representations and language representations. The study of each system of representation led us to use the notion of action in the sense of the following definition. *We shall define an action in a system of representation as the deliberate transformation of an object within this system* (Mithalal, 2010). The observables that we seek to identify are material actions - such a plot - or speech act - and the transformation of meaning assigned to a term.

Moreover, from the former short example, it seems that the students' geometry understanding evolved because of the combination of discursive and material activities. The various roles of these activities – description, construction, justification – as well as their interactions, have to be analyzed.

In the next section we present the theoretical model that we designed in order to get ahead with the comprehension of these two points. Our first aim is to analyze and to have an influence on the students evolving to a shared conception.

## **JOINT ANALYSIS OF MATERIAL AND DISCURSIVE ACTIONS: FREQUENTATION MODES [2]**

The first purpose of defining frequentation modes is to elaborate a tool which simultaneously takes into account students' discourse and actions. Indeed, material feedback mainly leads to local and unstable changes in the students' conception (Mithalal, 2010). At the same time, we assume that only taking into account language is not sufficient either. As a consequence, this may be the result of a dialectic relationship between manipulation and language interactions, without any subordination link.

### **Three points of view on geometrical activity**

Frequentation modes have to take into account three dimensions at the same time:

- The different *ways of seeing* in geometry (Duval, 2005) ;  
Indeed, the underlying hypothesis in Duval's work is that *ways of seeing* drawings are strongly linked to the kind of geometrical reasoning.
- The types of action undertaken by the subject and the instrument usage rules.  
We refer here to geometrical work from research directed by Marie-Jeanne Perrin-Glorian (Offre, Perrin-Glorian, Verbaere, 2006).
- The subject's discourse on objects and actions.

Therefore, defining a frequentation mode consists in describing a geometry activity by a way of seeing and considering modalities of action – drawing a plot, complying with a given instruction – consistent with a certain discourse characterized by its structure – descriptive, explanatory, etc. – and the meaning assigned to the terms by

the subject. In particular, it should be stressed that this notion is essentially local, being attached to a determined context and subject of study.

### Deeper analysis of the former example

In the example presented earlier in this paper, students explained that the group who constructed a triangle was wrong because the message received was “*Take a set square*”. Yet, they considered that a set square “*is used first of all to draw a triangle*”. These students explained to the teacher that they drew the rectangle using the graduated ruler, by measuring three consecutive sides, and then “[they checked] it was right”. The transmitters of this group said it was necessary to use a set square so that “*the angle be right*”. As we can see, the interpretation of the retroactions led students to be explicit about the meaning they gave to the set square. Here and now these two dimensions clearly appeared to be closely interrelated: the action was determined by the way students considered the set square and by its understanding of the word “set square”. It is also because they used the set square as a shape template that they interpreted the message this way. Through the longitudinal analysis of the meanings of the words “set square” and “rectangle” and of their action modes, we can start to outline two conflicting frequentation modes of the set square and the rectangle, each one being shared by many students, and which coexist in the language interactions observed:

- For those who read and interpreted the message the students considered as wrong, the word “set square” referred to a triangle template and the rectangle to a 2D surface the general looking of which was known. Consistently, they used the set square as a surface template (“indivisible” triangle) and their rectangle construction method consisted in drawing a rectangle-looking shape using the graduated ruler, and taking into account the measurements of the sides so that the rectangle drawn could be superposed on the model figure. This frequentation mode refers to an iconic visualization (called “botanist” by Duval).
- For those who transmitted the message, “rectangle” referred to a quadrilateral with two pairs of equal opposite sides and four right angles. The word “set square” applied to an instrument used to plot right angles. Their instrument usage mode was consistent with this meaning and their rectangle construction method consisted in plotting right angles and equal opposite sides using the graduated ruler and the set square. This frequentation mode rather referred to a “surveyor-geometer” or “constructor” vision of the *rectangle* figure (because the angle rightness is an invariant property of the rectangle) according to Duval’s meaning.

This analysis allows us to interpret the difficulties encountered by students as the confrontation between two contradictory frequentation modes, with outwards signs concerning at the same time the material actions made by students and their discursive activity.

### Three focuses that make frequentation modes operational

The analysis of the former example made us distinguish three consecutive parts of the pupils' co-constructing a shared frequentation mode of the set square. First, there were different interpretations of the words "set square", and various corresponding ways of using the tool; then material feedback highlighted the differences between these contradictory frequentation modes; and eventually there was a negotiation about what could be a shared suitable interpretation of "set square".

Our aim is now to better understand how the pupils' frequentation modes evolved and turned into this shared interpretation, suitable to geometry. Studying this dynamic process made us see that different stakes crystallized the opposition of these frequentation modes: physical transformations and usage of words in the students' discourses, or judgment on the validity of a drawing construction or of "set square" and "rectangle" the semantic values.

Our second concern, making a judgment of validity and semantic value, constitutes a "meta" level for the first one – drawing and description. Therefore, our analysis is based on identifying stakes, each one of them being a meta-stake for another one. By "meta", we mean that it is about judgments made on the sense or validity.

Defining meta-stakes, from the two we mentioned before, would be an infinite process. We decided to add a third one in order to take references to theory into account, and the example of the next part shows its role, but it seemed that a fourth one – epistemological – would have been irrelevant. These stakes are the following ones.

- **Stake 1**

The first stake is based on the actions students perform on drawings: plotting, constructing... We assume that these actions are a sign of their frequentation modes. We also assume that they are strongly linked to the meaning they assign to the words, which has to be analyzed from description of objects – *here, it is a rectangle* – or of actions – *take, measure, check, etc.*

Then, this stake is about discourse representatives and material actions.

- **Stake 2**

This stake is a "meta-stake 1", which means that we here focus on judgments about validity of constructions according to a description, and then judgments on semantic values of terms.

Then, we focus at the same time on making a judgment on the sense or validity of the plot drawn – meta material actions – and on changing the meaning of words – semantic action.

- **Stake 3**

The last stake refers to a theoretical framework and highlights properties on which depends the meaning and the finality of the performed actions. Therefore, concerning material actions, we focus here on a “judgment on the judgment” (invoking a theoretical reference framework), and at the same time this stake is about judgments on the validity of semantic values (which is also linked to the pupil’s theoretical point of view)

## **UNDERSTANDING THE DYNAMICS OF FREQUENTATION MODES**

### **Back to the example**

We have already described the confrontation between the two frequentation modes identified in the previous section. The physical productions were drawings, in a paper-and-pencil environment, using ruler and set square. Now, the rest of the session was about reproduction of rectangles, and mainly with the construction of four right angles; therefore our analysis focuses on the frequentation modes of the geometrical objects – rectangle, triangle, etc. – of the available artefacts – ruler, set square – and of the properties called for, in particular the notion of right angle. In the following analysis, we try to show how the three stakes we defined allow us to characterize the pupils' frequentation modes, analyze how these three parts of geometrical activity dynamically interact and which is more important. Our analysis highlights the role of this dynamic process in the evolution of the frequentation modes.

### **First move by teacher’s intervention**

The teacher intervened and went back to the expression “right angle” mentioned by the student then she established that a rectangle had four right angles. The teacher confirmed the relevancy of stressing the property of the right angles of the rectangle. By doing so, she gave students strong indications on the adequacy of the rectangle frequentation mode that paid particular attention to the right angle, and on her expectations.

### **Next: changes of some students towards a shared frequentation mode**

The teacher focused the discussion on the confrontation between the two rectangle constructions methods already mentioned, one using the set square while the other only took the graduated ruler. A student pointed out that the problem was that the message mentioned: “*take a set square and plot*”; now, she considered that “*a set square is not used to plot, but to have “right angles”*”. The teacher formulated the student’s speech: “*You mean that a set square is not used to measure ... it is used to make a right angle*”. As we can see, she did take into consideration that splitting the angle into a network of 1D elements and considering the possibility of plotting an angle is highly difficult for pupils: “to plot” meant “to draw a line” and the only instrument for this was the ruler.

Both teacher and students' interaction were about the second stake, since they tried to justify a specific way of using a set square coherent with their respective frequentation mode, and we could see the teacher changing the language form used by the student. Nevertheless, her attempts in language actions were not sufficient to make the different frequentation modes of the set square converge.

Prompted by the teacher (*"Now [...] they tell me they do not need a set square but still, you needed it"*), some students who used the set square as a surface template for drawing a triangle now considered another usage of it: the set square became an instrument used to "check right angles".

Because of language interactions, these students changed their frequentation modes of set square and rectangle to an intermediate frequentation mode : they could split rectangles and set squares into 2D sub-elements, admit the property of right angles as one of the specific properties of the rectangle and consider the set square as a right angle template. Moreover, this change occurred because of language interactions, through the simultaneous questioning of the actions performed, of their result and validity. The different terms semantic values were affected by the language interactions which, in turn, affected the usage modalities of designated objects. Thus, language activity made the notion of angle connected to "rectangle" and "set square", and the attention drawn on constituent parts rather than on shape.

### **Overcoming the contradictions and changing the frequentation mode basing on the pragmatic validation of a rectangle constructed on the blackboard**

At first, a student recalled that he only needed the graduated ruler. Spontaneously, he underlined the necessity to make sure that "the line" was "not askew". Two conflicting ways of checking the drawn figure appeared; the first one, upheld by the student, lied on the property establishing that the opposite sides of a rectangle are equal; the second one, repeatedly recalled by the teacher, called for the property laying down that the rectangle has four right angles.

If some students succeeded in making their frequentation mode change from the previous language interactions, others resisted in considering a non-iconic visualization of the rectangle in which the right angle is an invariant property of the rectangle. For these students, the set square still could not be used to check the construction of the rectangle. Here, the confrontation between the different frequentation modes appeared in the conflict between different checking means, each one calling for rectangle properties we consider as specific to each frequentation mode: for these students, the equal opposite sides that had to be measured using the ruler; for the teacher, the property of right angles. Communication couldn't work. Students were in a necessity of operating plotting whereas the teacher complied with theoretical requirements (taking into account the property of the right angle as the main criterion).

The teacher asked a student to construct a rectangle on the blackboard, using the ruler, as he said he did. He organized the validation of the rectangle by asking the student to measure the last side drawn. Then the student realized that the fourth side was too small. Immediately, students in the classroom raised “the problem of angles”. The teacher incited to explain how he could establish the figure built was not a rectangle. The student mentioned the property of equal opposite sides. Prompted by the teacher, he finally explained the property of right angles and the set square now appeared to be the instrument that had to be used to check the angle rightness. The teacher asked the student to check the rightness of angles using the set square. Once again, the set square was considered as an instrument used to check the rightness of angles but not as a plotting instrument.

During the plotting operation or when he made his first judgments on the validity of the produced figure, the student was still in a frequentation mode based on the relation to measurement and guided by an iconic visualization. At the same time, the teacher guided him towards a pragmatic validation (sides don’t have the right length) because she tried to make him change his vision mode of the figure. The mistake was found out in a mode called “surveyor-geometer” (as it lies on the reading of the measuring instrument which is the graduated ruler). Again, the teacher and the student used the same language level since they intended to judge the validity of plots (stake 2) “*yes, but he wanted to draw the line directly here. I told him he should check because ...*”. The student’s frequentation mode changed via the interpretation of the pragmatic invalidation of his action results. He eventually related the terms “askew sides” and “right angles” when he investigated the origin of the found mistake. Thus, he adapted his frequentation mode to the teacher’s one and made the set square *compulsory* to check an angle rightness. Here, the negotiation took place in a level of action and entailed a change not only in the rectangle construction method but also in the usage mode of the set square and in the meaning assigned to the terms set square and rectangle (stake 1).

## Reinvestment

The teacher organized a reinvestment phase during which she asked students to collectively write down a message allowing for a receiving student to construct a rectangle with modified side lengths. The teacher asked whether it was relevant to specify the necessity to use a set square in the message. Unanimously, students agreed that “*a rectangle can’t be plotted with a ruler*” and that “*we should know that a ruler must be used!*”. Finally, the message only retained the words rectangle, which now implicitly contained the property of right angles, and its side lengths.

The gap between the students’ point of view was reduced via a negotiation in stake 2, according to two modalities: through a questioning either of the semantic value of terms, or of the validity of the result or modalities of a physical action. We can state that here the agreement concluded by students in this stake 2 induces an agreement on actions at stake 1 (construction method, set square usage mode) and on discourse:

now the reference of the term “rectangle” is shared and includes the “4 right angles” property.

## CONCLUSION

In our work, we assume that the different students’ ways of doing geometry are related to the modalities of interaction between them and sensitive objects of the situation. We do consider that these modalities of interactions apply in two dimensions: a physical dimension which refers to the modalities of students’ physical actions and a language dimension. The students’ geometrical activities are to be seen via their frequentation modes of the various sensitive objects of the situation, in which their modalities of action and their discourse on objects form consistent wholes. Therefore we consider that the gap between various points of view results in the coexistence of conflicting frequentation modes in the students’ geometry activity. This gap is reducing as their respective frequentation modes of the various objects of the situation (rectangle, set square, right angle, etc.) progressively change until converging towards a shared one (shared discourse and action modes). Then, how can we better understand what makes it possible to negotiate a change to a shared frequentation mode, operating in the situation and that fits the framework of Euclidian geometry?

Then, we consider that the different levels of (physical as well as language) actions we identified may provide some lines of research on the possible modelling of relationships between different frequentation modes. Our first analysis suggests that these contradictions are overcome when these different types of confrontation are linked at different levels. Indeed, the analysis elements examined in this paper put forward that the conflict only appears as far as the language forms used in order to describe a sensitive object or a physical action differ for two individuals, or when on the contrary, two identical terms are used to designate different objects or actions. Therefore the contradiction comes from retroactions that emerge from a first level to be subjected to the interpretation of a second level. In return, the negotiation allowing for solving a conflict between two frequentation modes depends on the judgment made on objects or actions or on the terms that designate them: as such, the negotiation will take place in the second level, possibly calling for the third level. This negotiation makes it possible to agree not only on the modalities of action on the sensitive objects of the situation but also on the reference of the terms used (“set square” and “rectangle in our example”).

These theoretical and methodological tools are in a construction phase and should undoubtedly be improved. However, we believe that they can contribute to questions about the place of language in the Geometrical Working Space (GWS) by putting in parallel the physical and language actions of students in a geometry activity and by examining to what extent the negotiation for a shared GWS can be reached through a dialectic change in students’ ways of seeing, speaking and acting.

## NOTES

[1] Centre pour l'Observation et la Recherche sur l'Enseignement des Mathématiques (Centre for Observation and Research on Mathematics Education) based in Jules Michelet school in Talence, France, until 1998.

[2] We mean by “mode” the way a student carries out or handles a geometric concept through the consideration of three dimensions (we shall develop these three dimensions later: a way of geometrical seeing, a way of acting and a way of speaking). By “frequentation”, we refer to the degree of familiarity, the student’s understanding about the geometric concept at stake, according to his own in-school and out-of-school acquired knowledge.

## REFERENCES

- Bartolini Bussi, M. G., Mariotti, M. A. (2007). *Semiotic mediation in the mathematics classroom*. In L. English & al. (Eds), *Handbook of International Research in Mathematics Education*. LEA. 746-783.
- Bakhtine, M. (Ed. fr. 1984). *Esthétique de la création verbale*, Paris, Gallimard.
- Berthelot R., Salin M-H. (1992). L’enseignement de l’espace et de la géométrie dans l’enseignement obligatoire. Thèse de doctorat, université de Bordeaux 1.
- Brousseau, G. (1998). *Théorie des situations didactiques*. Grenoble : La Pensée sauvage.
- Bulf, C. (2009). The effects of the concept of symmetry on learning geometry at French secondary school, *Proceeding of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6)*, Conference European of Research in Mathematics Education) à Lyon, 725-735.
- Duval, R. (2005). Les conditions cognitives de l’apprentissage de la géométrie : développement de la visualisation, différenciation des raisonnements et coordination de leurs fonctionnements, *Annales de didactique et sciences cognitives*, 10, 5-53.
- Fregona, D. (1995). *Les figures planes comme « milieu » dans l’enseignement de la géométrie : interactions, contrats et transpositions didactiques*, Thèse de doctorat Université de Bordeaux I.
- Gobert, S. (2001). Questions de didactique liées aux rapports entre la géométrie et l'espace sensible, dans le cadre de l'enseignement à l'école élémentaire, Thèse de Doctorat, Université Paris 7.
- Houdement C. & Kuzniak, A. (2006) Paradigmes géométriques et enseignement de la géométrie, *Annales de didactique et de sciences cognitives*, 11, 175-193.
- Mathé, A.-C. (2006). *Jeux et enjeux de langage dans la construction d'un vocabulaire de géométrie spécifique et partagé en cycle 3 (Analyse de la portée des jeux de langage dans un Atelier de géométrie en cycle 3 et modélisation des*



#### Working group 4

*gestes de l'enseignant en situation*), Thèse de doctorat Université Claude Bernard, Lyon 1.

Mithalal, J. (2010). *Déconstruction instrumentale et déconstruction dimensionnelle dans le contexte de la géométrie dynamique tridimensionnelle*, Thèse de doctorat, Université de Grenoble.

Offre B., Perrin-Glorian, M.J., Verbaere O. (2006). Usage des instruments et des propriétés géométriques en fin de CM2, *Grand N*, 77, 7-34

Lismont, L. & Rouche, N. (Eds) (1999) *Formes et mouvements, perspectives pour l'enseignement de la géométrie*, Nivelles, Belgique.

Sfard, A. (2008). *Thinking as communicating : Human development, the growth of discourses, and mathematizing*. Cambridge, England : Cambridge university Press.

Vygotski, L. (1934). *Pensée et Langage*, Paris : La dispute (édition 1997).

# PROOFS AND REFUTATIONS IN LOWER SECONDARY SCHOOL GEOMETRY

Taro Fujita<sup>\*</sup>, Keith Jones<sup>\*\*</sup>, Susumu Kunimune<sup>\*\*\*</sup>, Hiroyuki Kumakura<sup>\*\*\*</sup> and  
Shinichiro Matsumoto<sup>\*\*\*</sup>

<sup>\*</sup>Faculty of Education, University of Plymouth, UK

<sup>\*\*</sup>School of Education, University of Southampton, UK

<sup>\*\*\*</sup>Faculty of Education, Shizuoka University, Japan

*In this paper, we report on a teaching experiment in which we focused on students tackling 3D geometry problems in which, in general, they initially tended to produce 'primitive' conjectures by relying on visual images rather than geometrical reasoning. Following the work of Larsen and Zandieh (2008), we utilise the ideas of Lakatos (1976) on managing the refutation process and how the use of counter-examples can be important in promoting the growth of students' capability with geometrical reasoning and proof. We found that students' primitive conjectures can cause an unexpected result and that this can trigger further reviewing ('Monster-barring') and modifications of the conjecture ('Exception-barring') amongst students. Whole classroom discussion followed by small group discussion allowed students to exchange various ideas and opinions and this process was important for their construction of a proof of their new conjecture ('Proof-analysis').*

Key-words: geometry, conjecturing, proof, refutation; 3D tasks

## INTRODUCTION

The teaching of geometry provides not only a key vehicle for developing learners' spatial thinking and visualisation skills in mathematics, but also a major opportunity to develop their capability with deductive reasoning and proving (Battista, 2007; Royal Society, 2001). Through classroom-based research (for example, Kunimune, Fujita & Jones, 2010; Fujita, Jones & Kunimune, 2010), we are working on several themes in the teaching of geometrical reasoning and proof at the lower secondary school level, encompassing the design of problem-solving situations in geometry for students, the integration of geometrical constructions, ways of providing students with explicit opportunities to examine the differences between experimental verifications and deductive proof, and approaches to the teaching of deductive geometry based around a set of 'already-learnt' properties which are shared and discussed within the classroom.

In this paper we extend our previous research by focusing both on the design of problem-solving situations in geometry for students *and* on the teaching of deductive geometry based around a set of 'already-learnt' properties. While designing suitable classroom tasks is very important in mathematics education (e.g. Wittmann, 1995), using such tasks with students does not necessarily lead to 'good' results: something

which Schoenfeld (1988) has illustrated in detail. Hence, additional factors need to be considered if the teaching of geometry is going to be more effective. In this paper, and following the work of Larsen and Zandieh (2008), we utilise the ideas of Lakatos (1976) to show how managing the refutation process and the use of counter-examples can be important in promoting the growth of students' capability with geometrical reasoning and proof. The tasks we use involve geometrical reasoning on simple 3D shapes - one of the topics considered by several papers from the CERME geometry working group (e.g. Mithalal, 2010; Pitallis et al, 2010).

## **REFUTATIONS IN THE PROCESS OF PROVING IN MATHEMATICS**

Given that conjecturing processes are known to be important in the teaching and learning of mathematics in general, and geometry in particular, (Cañadas et al, 2007), our focus in this paper is on the relationship between conjecture, refutation, and proof. It is known that, on the one hand, treatment and understanding of refutation and counter-examples are not straightforward for learners (e.g. Balacheff, 1991; Stylianides and Al-Murani, 2010): indeed, Potari, Zachariades and Zaslavsky (2009) show that even trainee teachers find it difficult to identify correct counter-examples to refute false statements. Yet, on the other hand, counter-examples play an important role within the process of conjecture production and proof construction. Mathematical activity, it has to be said, is not straight-forward, but rather more like a zigzag path. Mathematicians typically make a conjecture, find counter-examples, refine the conjecture, find more counter-examples and so on, during their proving process. Lakatos (1976, p. 127), in his historical and epistemological study, considered that the proof and refutation process consists of the following:

- Primitive conjecture
- Proof (a rough thought-experiment or argument, decomposing the primitive conjecture into sub-conjectures)
- 'Global' counter-examples emerge (counter to the primitive conjecture)
- Proof is re-examined as a new theorem or improved conjecture emerges

While mathematicians, historians and philosophers remain engaged in on-going discussions into the validity of this process (see, for example, Hanna 2007, p. 10), there is some evidence in the mathematics education literature that Lakatos' framework can be a useful guide to promoting students' conjecture production and proof construction process. For example, Larsen and Zandieh (2008) utilised Lakatos' framework to analyse undergraduate students' proof construction processes in abstract algebra. They categorises the types of proof and refutation activities in terms of students' responses, described in their words as follows (p. 208):

- *Monster-barring*; any response in which the counter-example is rejected on the grounds that it is not a true instance of the relevant concept

- *Exception-barring*; any response that results in a modification of the conjecture to exclude a counter-example without reference to the proof
- *Proof-analysis*; the resulting modification to the conjecture is intended to make the proof work rather than simply exclude the counter-example from the domain of the conjecture

Larsen and Zandieh showed that Lakatos' framework "can serve as heuristics for designing instruction" (p. 215). In a similar vein, Komatsu (2010) revealed how a focus on counter-examples can encourage primary school pupils to refine their conjectures and extend their reasoning to reach a correct answer in a number task. We designed the teaching experiment below with a view to giving lower secondary school students valuable opportunities based on the 'proof and refutation' framework.

## RESEARCH SETTING

The teaching experiment was undertaken in a Japanese lower secondary school where geometry has a major role in developing pupils' ideas about proof and proving. In Japan, the curriculum states that, in geometry, students must be taught to "understand the significance and methodology of proof" (JSME, 2000, p. 24. In terms of the 'paradigm of geometry' proposed by Houdement and Kuzniak (2003), Japanese geometry teaching may be characterized as within the *Geometry II* paradigm (in that axioms are not necessarily explicit and are as close as possible to natural intuition of space as experienced by students in their normal lives).

In our teaching experiment, by following the principles of the geometry curriculum, the following lessons were designed for Grades 7 and 8 students (aged 12-14);

- 21 lesson for Grade 7 (students aged 13 yrs old at the time): Introduction of 3D shapes and nets (2 lessons), Points, lines and planes (1 lesson), Positions and angles in 3D shapes (3 lessons, **our focus in this paper**), Distances of two points (2 lessons), Rotated shapes, circles and sectors (1 lessons), Surface areas volumes of 3D prisms and pyramids (2 lessons), line and rotational symmetry (1 lesson), Construction of parallel lines and tangents of circles (2 lessons), vertically opposite angles, alternate and corresponding angles in parallel lines (3 lessons), and Angles in polygons (4 lessons).
- 28 lessons for Grade 8 (students aged 13 yrs old at the time): Congruent triangles (3 lessons), Theorems and definitions in geometry (3 lessons), Constructions and properties of isosceles triangles (5 lessons), Constructions and properties of parallelogram (4 lessons), Construction of a cube (2 lessons, **our focus in this paper**), Congruent right-angled triangles (2 lessons), Relationship between triangles and quadrilaterals (2 lessons), Properties of circles (3 lessons), Parallel lines and areas (2 lessons) and Summary (2 lessons).

These lessons were implemented in one class of 40 students in a university-attached school where the teachers and researchers work together to undertake classroom-based research. The students' standard in mathematics is generally high. The regular teacher of the class, in line with Sekiguchi's (2002) account, generally considers a good lesson to be one in which the students are encouraged to share their ideas and solutions with each other.

In this paper, we focus on the lessons from *Positions and angles in 3D shapes* (taught in Grade 7) and *Construction of a cube* (taught in Grade 8). Our reason for focusing on these lessons is that, in trialling the lessons, students in general tended to produce their 'primitive' conjectures by relying on visual images rather than through geometrical reasoning. Thus, our concern is how to break this situation.

Recent studies (e.g. Christou et al, 2006; Mithalal, 2010) have shown how the use of technology and dynamic 3D geometry environments might help counter the difficulties that students have in studying the properties of 3D shapes. In this paper, we consider the method of 'proof and refutation' with practical activities and group discussions might also be effective and accessible way of teaching. In the analysis that follows, we consider this issue by using 'proof and refutation' framework of *Monster-barring*, *Exception-barring*, and *Proof-analysis*.

## ANALYSIS OF EPISODES FROM OUR CLASSROOM EXPERIMENT

### Episode 1 – what size is angle PQR in a cube?

In Grade 7 in Japanese schools, the main purpose of geometry teaching is to introduce students to geometrical reasoning through the study of 3D shapes and the angle properties of 2D shapes. In this episode (during the third of three lessons on *Positions and angles in 3D shapes*), and after learning some basic concepts of cubes and cuboids during the previous five lessons, the students were asked to investigate the size of the angle PQR in a cube ABCDEFG (see figure 1).

Of the forty students in the class, 25 of them considered that 'the angle is 90 degrees', 11 thought that 'angles will be changed' and 4 said 'I don't know'. As such, the dominant 'primitive' conjecture can be taken to be 'the angle is 90 degrees'. One student (referred to as student 1) stated his reasoning as follows:

Student 1: I think wherever P, Q, and R are, the size is 90 degree. Because angle PQR looks like 90 degrees if you look at it from the face BFGC.

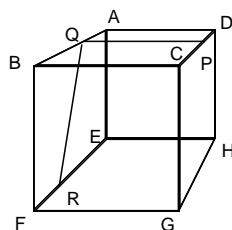


Figure 1: angle PQR in a cube

#### Working group 4

Students exchanged their ideas and opinions in groups and subsequently in whole classroom discussions which led them to modify their conjecture. The following presentations were made by students during the whole classroom discussion:

Student 2: I investigated by cutting a model of a cube. If we cut AC and AF, then we have an angle, and I think it won't be 90 degrees as the angles are formed by AC and AF.

Student 3: I also used a model, and I used protractor as well. I have got about 60 degrees, and not 90 degrees.

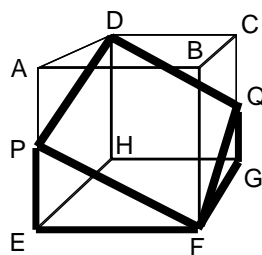
We consider these as *Exception-barring* responses, as their focus is not rejection of the 'primitive' conjecture, but the production of a new conjecture that 'angles will be changed'. After these presentations, the following idea was proposed by a student:

Student 4: I consider why Students 2's and 3's angles are 60 degrees. If we connect C and F, then there will be a triangle. It is a bit difficult to see the figure on the blackboard [as this is a 2D representation of a cube], but these lines should be the same and since all the angles are the same, this triangle should be an equilateral triangle. Therefore, angle CAF is 60 degrees.

We consider this as a *Proof-analysis* response wherein the new conjecture 'angles will be changed' is now justified by a simple proof.

#### Episode 2 – What shape is face DPFQ in a cube?

In Grade 8 in Japan, students continue to study geometry and are gradually introduced to more formal ways of geometrical reasoning. In the two lessons on *Construction of a cube* (16th and 17th lessons of their geometry work), the students undertook the following problem: 'Consider the net of a cube [see Figure 2]. Construct a net including the face DPFQ [where P and Q are the mid-points of AE and CG respectively].'



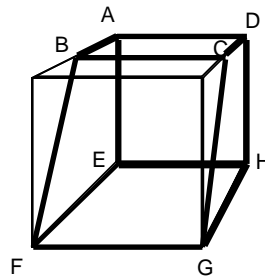
**Figure 2: a half-cube for Grade 8 students to construct**

In this task, the students were not only expected to identify the face DPFQ, but also to construct an actual net and make the model. This additional practical requirement is particularly important in the teaching experiment as we consider this is more likely to create 'unexpected situations' (such as the square DPFQ does not fit) for many students more easily than a question that solely asks students to determine the shape of the face DPFQ. In the latter case, students might say that the face DPFQ is a square, but it might be more difficult for them to recognise that it is not.

#### Working group 4

First, the teacher introduced the problem by referring to the students' experiences in Grade 7:

Teacher: Do you remember we made the solid ABCDEGH [illustrated as Figure 3]



**Figure 3: a solid ABCDEGH**

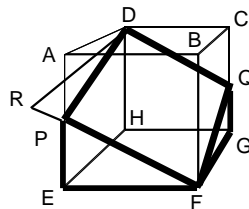
Students: Yes, I remember. I think we managed to make it.

Teacher: Yes, and today, we try the task 'Let us consider a net of this 3D shape (where P and Q are the mid-points of AE and CG respectively). Construct a net including the face DPFQ'.

In this problem, a challenging point, on the one hand, is that the quadrilateral DPFQ is not a square, but a rhombus. On the other hand, this can lead the students to making a conjecture, refuting their conjecture, modifying the conjecture and so on, until their final decisions. After investigating this task individually, the students found that their 'primitive' conjecture 'the DPFQ is a square' might not be true as a square did not fit their models. The students then started exchanging their ideas within each group. For example, students in Group A (with students referred to as A1, A2, etc) had the following discussion (relating to models represented by Figures 4 and 5):

Student A1: I think DPFQ is a square. First the original shape was a cube, and all faces are squares, and therefore  $\triangle APD \equiv \triangle EPF \equiv \triangle GQF \equiv \triangle CQD$  and all the sides are the same [note that this student's model was incomplete as the quadrilateral DPFQ did not fit perfectly].

Student A2: I thought, like you, that DPFQ is a square, but it did not fit... I drew a square first, and cut and pasted in my model.

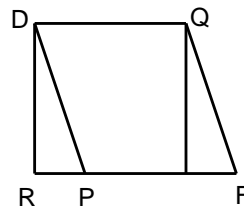


**Figure 4: the model by student A2**

Student A3: But [see Figure 5] if we follow A2's method, then I wonder if we would have a rhombus? I think, if the first shape we make is a square, then all

### Working group 4

sides should be the same,  $DQ=DR$ , and we cut  $\triangle DRP$ , and this is a right-angled triangle. Therefore,  $DP$  is longer than  $DR$ , and  $DP \neq DQ$ , and this is not a rhombus?



**Figure 5: student A3's reasoning about shape DPFQ**

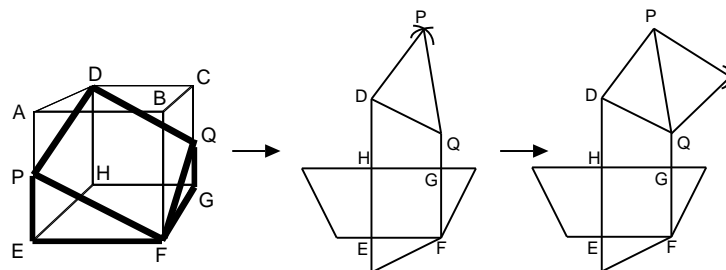
We consider the above responses as *Monster-barring* and (incorrect) *Proof-Analysis*. The students tried to reject the counter-example and keep their original conjecture by using (incorrect) reasoning. It is interesting that their *Monster-barring* led to a proof which they tried to use to justify their original conjecture.

In another group (group B), however, two students (B1 and B2) first made their models without drawing DPFQ, and then student B3 showed his answers as follows (see Figure 6):

Student B1: My method is probably cheating, but I drew a net without DPFQ, and then made a model without a lid. Then, I put my half-completed model on a piece of paper, traced DPFQ and then made the lid (DPFQ).

Student B2: My method is similar to B1, but I did it a bit differently. I also made a model without a lid, and then I measured the angle  $PDQ$ , and it was 79 degrees. I made a quadrilateral with the angle  $PDQ$  79 degrees, and then put the lid.

Student B3: I tried the method which is similar to B1 and B2, started from a net without DPFQ, and made a model. But I noticed that the length of  $PQ$ , the diagonal of DPFQ is the same as  $EG$ , the diagonal of HEFG. If we use this fact, we can construct  $\triangle DQP$  by using ruler and compass. If we can construct  $\triangle DQP$ , then we can also construct  $\triangle PQF$ , then we can complete the net [see Figure 6]



**Figure 6: the net made by student B3**

The above process can be considered as *Exception-barring*. This is because the students' original conjecture was abandoned and new ideas were searched for to make the situation consistent. Neither arguments by student B1 nor B2 were proofs.



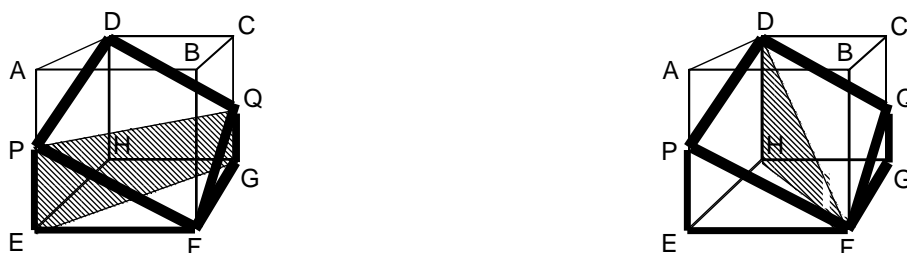
In addition, it is difficult to consider B3's argument as a proof as his method still does not explain what DPFQ is.

After the group discussion, all the group arguments were shared with the whole class. After listening to the presentation of student B3, a student G1 (from group G) added his reasoning as follows:

Student G1: I did like B3's way, but if you looked at the shape without the lid from above, we can see PQ is equal to EG, and as the four sides of DPFQ are the same, so I think it is a rhombus. I then measured PQ and then used compass to complete the face DPFQ.

Student G1's response is again *Exception-barring*, and now a new conjecture 'the face DPFQ is a rhombus' is shared in the classroom. Finally, student H1 (from group H) presented his idea and the new conjecture was proved as follows (see Figure 7):

Student H1: My idea is that I dissected the solid first. If we cut it vertically from PQ to EG, then it will be a rectangle. Therefore,  $PQ=EG$ . Also, if we cut it by connecting DH and F, then it will be a right-angled triangle, and DF is its hypotenuse and the other line is HF [and therefore, DF is longer than PQ].



**Figure 7: illustration of student H1's proof of why the face DPFQ is a rhombus**

This reasoning, triggered by group discussions and whole classroom discussion, is considered as *Proof-analysis*. It is also interesting to see that the properties of quadrilaterals and triangles are used effectively by the student to justify the reasoning. Before this lesson, in addition to the 21 lessons in Grade 7, students have already completed 15 geometry lessons in which they practiced their geometrical reasoning in using a set of already-learnt properties which are shared and discussed within the classroom. The properties of quadrilaterals and right-angled triangles were already studied, and this student (H1) used them effectively to advance his reasoning.

## DISCUSSION AND CONCLUDING COMMENTS

These episodes show that the first conjecture 'DPFQ is a square' caused an unexpected situation, and then this triggered further reviews (*Monster-barring*) and modifications of the conjecture (*Exception-barring*) amongst students. Whole classroom discussion followed by small group discussion allowed students to exchange various ideas and opinions and this process was important for their construction of a proof of their new conjecture (*Proof-analysis*).

In focusing, in this paper, on students' conjecture production and proof construction within the proof and refutation framework, we can conclude that the framework is useful not only for describing students' proving processes but also in indicating some helpful instructional approaches in geometry lessons. Through our analysis of data from our classroom-based research, we illustrate how managing students' discussions of counter-examples, both in group and whole classroom work, can act as a vehicle for promoting the development of their geometrical reasoning. We found that *Monster-barring* can sometimes lead to an incorrect proof from students (for example, students A1, A2 and A3 in the second episode). As such, *Exception-barring* and classroom discussions are important to construct legitimate proofs (*Proof-analysis*) (see student 4 in the first episode, and student H1 in the second). In future research, in addition to continuing to design suitable tasks for students, we aim to investigate other factors which could facilitate students' conjecture production and proof construction in geometry.

## NOTE

The lessons in this teaching experiment were based on the Japanese 'Course of Study' first published in 2000 (JSME, 2000).

## REFERENCES

- Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp.89-110). Dordrecht: Kluwer.
- Battista, M. T. (2007). The development of geometric and spatial thinking. In: F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*. Charlotte, NC: NCTM/Information Age Publishing.
- Cañadas, M. C., Deulofeu, J., Figueiras, L., Reid, D., & Yevdokimov, A. (2007). The conjecturing process: Perspectives in theory and implications in practice. *Journal of Teaching and Learning*, 5(1), 55-72.
- Christou, C., Jones, K., Mousoulides, N. & Pittalis, M. (2006). Developing the 3DMath dynamic geometry software: Theoretical perspectives on design, *International Journal of Technology in Mathematics Education*, 13(4), 168-174.
- Fujita, T., Jones, K. & Kunimine, S. (2010). Students' geometrical constructions and proving activities: a case of cognitive unity? *Proceedings of PME34*. Belo Horizonte, Brazil. Vol 3, pp 9-16.
- Hanna, G. (2007). The ongoing value of proof. In P. Boero (Ed.), *Theorems in School* (pp. 3-18). Rotterdam: Sense.
- Houdement, C. & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. *Proceedings of CERME 3*.

- Japanese Society of Mathematics Education (2000). *Mathematics Programme in Japan*. Tokyo: JSME.
- Komatsu, K. (2010). Counter-examples for refinement of conjectures and proofs in primary school mathematics. *Journal of Mathematical Behavior*, 29(1), 1–10.
- Kunimune, S., Fujita, T. & Jones, K. (2010). Strengthening students' understanding of 'proof' in geometry in lower secondary school. *Proceedings of CERME6* (pp756-765).
- Lakatos, I. (1976). *Proofs and Refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.
- Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics*, 67(3), 205–216.
- Mithalal, J. (2010). 3D geometry and learning of mathematical reasoning. *Proceedings of CERME6* (pp 796-805).
- Pittalis, P., Mousoulides, N. & Christou, C. (2010). Students' 3D geometry thinking profiles. *Proceedings of CERME6* (pp 816-825).
- Potari, D., Zachariades, T. and Zaslavsky, O. (2010). Mathematics teachers' reasoning for refuting students' invalid claims. *Proceedings of CERME6* (pp 281-290).
- Royal Society (2001). *Teaching and Learning Geometry 11-19*. London: Royal Society/Joint Mathematical Council.
- Schoenfeld, A. (1988). When good teaching leads to bad results: the disasters of "well taught" mathematics classes. *Educational Psychologist*, 23, 145-166.
- Sekiguchi, Y. (2002). Mathematical proof, argumentation, and classroom communication: A cultural perspective. *Tsukuba Journal of Educational Study in Mathematics*, 21, 11-20.
- Stylianides, A. J. and Al-Murani, T. (2010). Can a proof and a counterexample coexist? Students' conceptions about the relationship between proof and refutation, *Research in Mathematics Education*, 12(1), 21-36.
- Wittmann, E. C. (1995). Mathematics education as a 'design science', *Educational Studies in Mathematics*, 29, 355-374.

# **IDENTIFYING THE STRUCTURE OF REGULAR AND SEMIREGULAR SOLIDS – A COMPARATIVE STUDY BETWEEN DIFFERENT FORMS OF REPRESENTATION**

Jürgen Steinwandel, Matthias Ludwig,

University of Education Weingarten

*This paper presents the current status of a scientific study that investigates the impact of three different working environments (illustration, real model or interactive computer-animation) on the recognition and processing of spatial structures. Current literature does not give a consistent picture of this matter. This inconsistency could have several causes, e.g. the spatial-geometrical and arithmetic skills of the test persons, or the complexity of the spatial-geometrical task. We tested the working environment with  $n=242$  students from grade 5 to 9 (10 years to 15 years old students). We selected the students from three types of secondary schools in Germany, "Hauptschule (low track), Realschule (middle track) and Gymnasium (high track)". It emerges that the working environment "model" is superior to the other two forms of representation-like illustration or interactive animation. Furthermore, we are exploring whether it is possible to quantify the complexity of a regular or semi-regular solid.*

## **INTRODUCTION**

To detect spatial structures of polyhedra (e.g. number of faces, edges or vertices, shapes of their faces, symmetries etc.) students use models (e.g. from the school's collection), illustrations or pictures (e.g. textbooks), as well as partly-interactive computer animation in school. Studies in this field have mainly focused on the torsion angle between two illustrations of a solid that are to be compared and analyzed here for example the processing speed or the frequency of errors (Peters et. al, 1995). Other variables, for example the age or gender, were taken into consideration. Further studies analyzed the impact of computer environments and their training effects (Souvignier, 2000, Hartmann & Reiss, 2000; Hellmich et. al, 2002; Ahmad, 2009). The findings of these studies are inconsistent or contradictory, and in this paper we aim to contribute to clarifying the situation.

## **THEORETICAL FRAMEWORK**

Numerous authors strongly emphasize the great importance of spatial abilities both as a factor of human intelligence and a human skill with practical importance in life generally. However, it seems to be very difficult to define this complex construct. Different authors use terms such as spatial ability, spatial imagination, spatial orientation, spatial awareness and spatial thinking to name just a few. Glück (2006) suggests the high complexity of the underlying issue as a possible cause of the

inconsistent definition and application of this term. Glück herself uses the term spatial imagination performance and means "the use of visual, nonverbal information [...], which is stored in the memory, usually transformed in some way or manipulated and / or which has to be retrieved." In contrast to this, Franke uses the term "spatial ability" as a generic term and distinguishes between visual perception and spatial imagination (see Franke, 2007).

Linn and Petersen (1985, 1986) identified three basic factors as part of a meta-analysis:

- *Spatial perception* is defined as the ability to determine spatial relations despite distracting information.
- *Spatial visualisation* is the ability to manipulate complex spatial information when several stages are needed to produce the correct solution.
- *Mental rotation* is defined as the ability to rotate, in imagination, quickly and accurately two- or three-dimensional figures.

It should be noted that these three factors cannot be considered in isolation to each other because of their various dependencies and influences. It is, for example, highly unlikely that a child has the ability for mental rotation while not possessing abilities in the other two areas. In recent years the analysis of the factor "mental rotation" has become a key area. Numerous studies focused on this aspect with diverse emphasis in their observation, e.g. the ability to learn "mental rotation" (see Glück, 2006, Hellmich & Hartmann, 2002; Souvignier, 2000 etc.). Furthermore, factors or dispositions such as gender, performance groups, age, mother language etc. have been evaluated (see Hirnstein & Bayer et.al, 2009; Kruger & Krist, 2009; Peters & Battista, 2008). The related results show trends but mainly do not give definite answers that remain consistent even under detailed observations. For example, male subjects frequently called into question gender-specific advantages with regards to spatial abilities by carrying out analysis with various filterings of the independent variables (see Jordan et al., 2002).

Another field of research evaluates the interrelation between several factors, e.g. the dependence of the processing speed and the processing correctness with regards to the rotation angle of a distorted body (see Shepard & Metzler, 1971; ter Horst & van Lier et.al, 2010 etc.). To some extent the findings demonstrated an almost linear interrelationship. However, these studies remain limited to composite cubes, which play a minor role in the classroom. Investigations with regards to solids that are introduced in grade 5 to 9 (students from 10 years to 15 years) such as cubes, cuboids, pyramids etc. play a very minor role. In our investigation we try to describe and to investigate the complexity of a body. Therefore we used a very simple and easy understanding model. We defined the complexity (C) of a solid as follows:

$C(\text{solid}) = \text{number of faces}(\text{solid}) + \text{number of edges}(\text{solid}) + \text{number of vertices}(\text{solid})$

If we use the polyhedra formula of Euler  $V - F + E = 2$  we can simplify the sum for the complexity of a solid. So we get

$$C(\text{solid}) = 2 + 2 \times \text{number of vertices (solid)} \quad C(S) = 2(1 + V)$$

Looking at various studies from the field of educational psychology and psychology about the general topic of “learning with animations in computer-based learning environments” and from research of the didactics of mathematics into the specific topic of “learning of spatial-geometrical contents” and “training of spatial imagination” with or without a computer, an ambiguous picture about the impact of these learning environments transpires. Lowe (2003) for example shows in a study of students of meteorology that a multimedia learning environment does not necessarily impact positively on the learning outcome. He describes that in many cases arrows as movement indicators and illustration series would build sufficient understanding. Furthermore, it appears that the potentially supportive factor of animation is lost with that result that individuals have less control over their own learning route.

In a study similar to spatial geometrical matter, Schwan and Riempp (2004) in turn note that subjects learn the tying of nautical nodes significantly faster with interactive videos (that the subjects can themselves rewind and pause the animations) than with non-interactive videos. In connection with other investigations, they outline an ambiguous picture about the effectiveness of multimedia environments (see also Hellmich & Hartmann, 2002; Cohen, 2005).

The discussion of current research areas highlights several aspects that seem to make a detailed reflection meaningful:

- Most studies stand within the tradition of psychology and therefore hardly ask their primary questions about teaching directly.
- Generally the descriptions of the solids in rotation that are to be compared limit themselves to the angle of rotation. Other attributes such as the number of faces, angles etc. have not been extensively studied.
- Statements regarding different types of presentations and their impact - including those that relate to training impact assessments - are in parts still inconsistent especially in connection with other set variables (sex, performance level, origin, social status, age, arithmetic knowledge etc.).

Accordingly, the research questions were established.

## RESEARCH QUESTIONS

The paper presented here has its emphasis on training at schools providing secondary education in Germany. In particular we explore whether it is possible to describe solids in terms of their spatial complexity. Furthermore, it is of interest what form of presentation (illustration, model or simulation software) is the most optimal for the

individual student. For the study we choose half-regular and regular solids. With regards to that we have key research questions:

- How much will different performance groups benefit from the different representations "illustration (pictures), computer animation and model" with regards to the processing of related specific questions?
- Is it possible to describe the complexity of a solid by virtue of specific properties (number of faces, vertices and edges) and how far will such a model retain its validity for differing performance groups?

## STUDY DESIGN

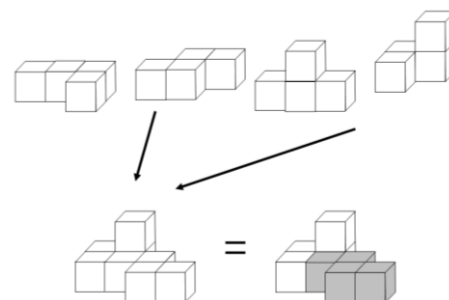
N=242 students in 11 different classes from grade 5 to 9 (students between 10 and 15 years old) were tested in five secondary schools covering the three different types: high track (Gymnasium), middle track (Realschule) and low track (Hauptschule). First, two preliminary tests were carried out: an arithmetic test with content regarding the understanding of numbers and basic arithmetic - as well as the brick-test of Birkel & Schumann (2002). Using the data of the brick-test, the student groups were scaled and allocated to the three different working environments of the structure-identification-test (SIT) resulting in comparable groups of similar performance (mean) nearly same standard deviation (SD), while attention had also been given to gender.

In the SIT, eight regular and semi-regular solid had to be worked on. Six students were tested in a laboratory in parallel. Each student worked alone at one of the three working environments (picture, model and computer).

We want to emphasize that the SIT is not a treatment in a traditional way. With this test we do not check any learning or training effects. We only want to test which working environment (illustration, real model or interactive computer-animation) supports the students better by doing spatial geometry tasks. It is just a snapshot.

### The brick-test (BST)

The brick-test of Birkel & Schumann (2002) primarily evaluates the "mental rotation" ability. The basic structure of all solids that were used is composite cubes. These composites are placed in the space more or less rotated. The students have to pick two solids from a selection of four to build the needed composite solid (see Fig. 1).



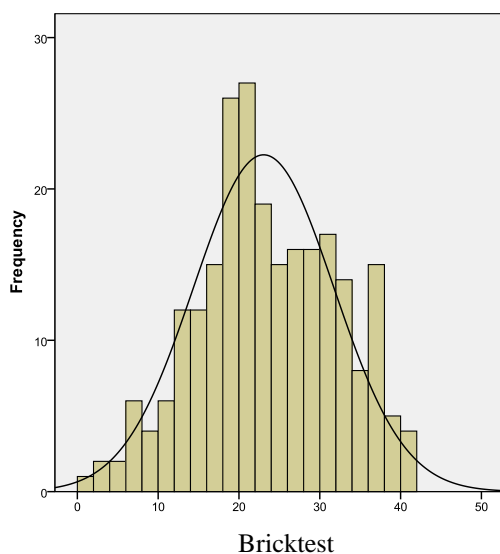
**Fig. 1: Composite Cubes of the brick-test**

The students had to assess a total of 40 composite solids in a time of 20 minutes. At the beginning two examples were

## Working group 4

assessed together with the students. The maximum the students can achieve are 40 points.

### The samples



Mean =	23,05
Std.Dev. =	8,676
N =	242

**Fig. 2: Histogram to improve the normality**

Tests of Normality						
	Kolmogorov-Smirnov <sup>a</sup>			Shapiro-Wilk		
	Statistic	df	Sig.	Statistic	df	Sig.
Bricktest	,084	242	,000	,984	242	,009

a. Lilliefors Significance Correction

**Table 1: Tests of Normality**

The presentation is similar to a normal distribution (see Fig. 2), but would not resist a statistical test (eg. Kolmogorov-Smirnov, see Table 1).

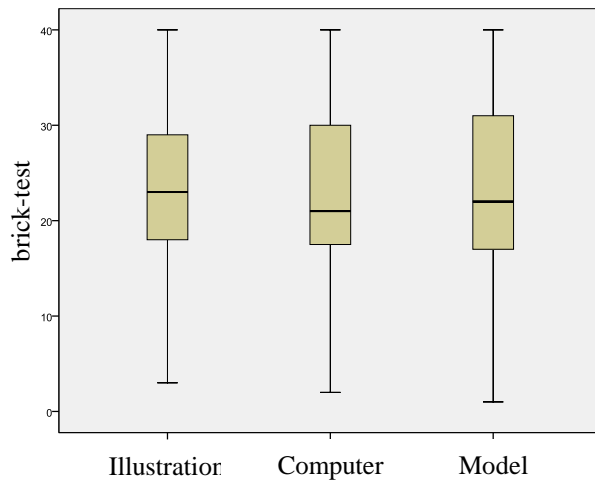
### Multiple Comparisons (Bricktest)

(I) representation	(J) representation	Mean		Sig.	95% Confidence Interval	
		Difference (I-J)	Std. Error		Lower Bound	Upper Bound
illustration Ø 23,20	computer	,445	1,357	,983	-2,83	3,72
	model	-,017	1,385	1,000	-3,36	3,32
computer Ø 22,75	illustration	-,445	1,357	,983	-3,72	2,83
	model	-,462	1,372	,982	-3,77	2,85
model Ø 23,21	illustration	,017	1,385	1,000	-3,32	3,36
	computer	,462	1,372	,982	-2,85	3,77

**Table 2: Tests for similar power distribution**



### Working group 4

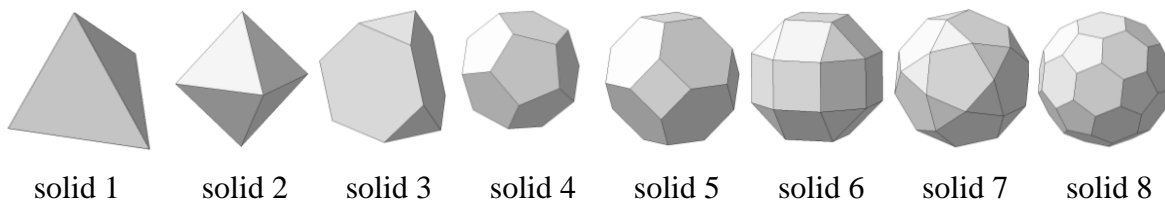


**Fig. 3: Boxplots of the groups illustration, computer and model**

To compare the three working environments (illustration, computer and model) we need three samples of students who have similar spatial ability. Therefore we created three samples from the results which the students had got from the Brick-Test. For this we divided all students into three performance groups (low, middle and high) and distributed students from these three performance groups in similar way to the three samples. Now we run this method until all students were distributed over all the working environments. The result is that all the three working environments have nearly the same mean and SD (see Table 2 and Fig. 3).

### The structure-identification-test (SIT)

The structure-identification-test of Ludwig/Steinwandel deals with a total of 8 semi-regular and regular solids (Platonic and Archimedean solids) as follows (see Fig. 4).



**Fig. 4: The Platonic and Archimedean solids used by the SIT**

For each solid 6 questions of two levels had to be answered by the students. Questions of level A (questions about the shape of the faces, the number of edges, faces which touch at one vertex) could be answered without abilities in "mental rotation" and "spatial visualization", while these abilities were necessary to answer questions of level B (questions about the number of faces, edges and vertices). The handling time was set and controlled. These periods have been empirically identified following a preliminary investigation. (e.g. solid 1 → 1 minute, solid 4 → 2:30 minutes). Thus the total duration was 22 minutes.

## FIRST RESULTS

### Comparison of working environments

In the subsequent short analysis we want to explore the question how helpful the different presentations (or working environments) are for a student in dealing with the questions. In order to be able to make specific statements we set different subsets of the data records.

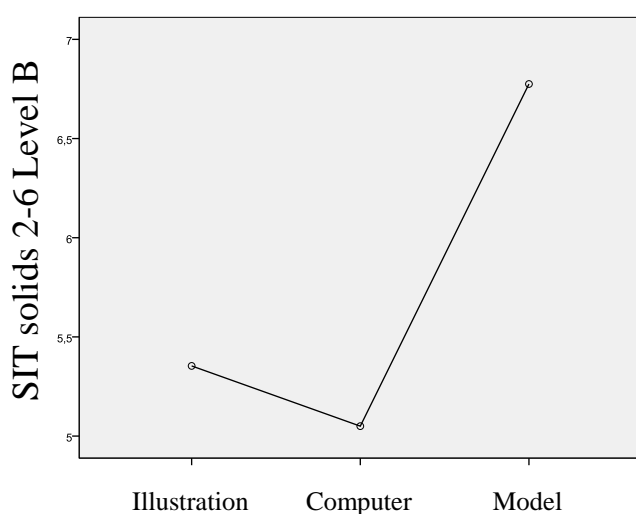
**Multiple Comparisons (SIT, only solids 1-6, Level B)**

(I) representation	(J) representation	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
					Lower Bound	Upper Bound
Illustration Ø = 5,35	Computer	,304	,453	,878	-,79	1,40
	Model	-1,421*	,455	,006	-2,52	-,32
Computer Ø = 5,05	Illustration	-,304	,453	,878	-1,40	,79
	Model	-1,725*	,440	,000	-2,79	-,66
Model Ø = 6,78	Illustration	1,421*	,455	,006	,32	2,52
	Computer	1,725*	,440	,000	,66	2,79

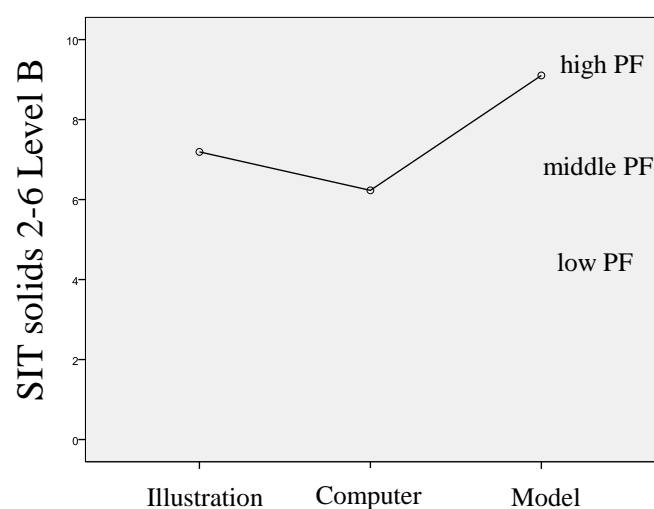
\*. The mean difference is significant at the 0.05 level.

**Table 3: Post Hoc Tests**

The illustration in Table 3 shows the correlations between performances with the various representations. For the subsequent analysis the data sets were adjusted as follows: we only evaluated solid 2 to 6 as solid 1 is considered to be a "warming-up shape" while the very complex bodies 7 and 8 differentiated poorly and we only considered absolute correct solutions (an approximation to the correct value has not been taken into account).



**Fig. 5: Mean Plots**



**Fig. 6: The performance (PF) in the different working environments of low, middle and high achievers in the BST**

## Working group 4

The ANOVA-analysis (Table 3 and Fig. 5) shows a similar performance of students working with illustrations and pupils operating with the computer. Students who worked with the model-based environment show significant better results as students who dealt with the other working environments.

In summary we can establish that the illustration-based and computer-based environments in comparison hardly bring about any advantages and lead to results on a similar level (see Fig. 5 and Table 4).

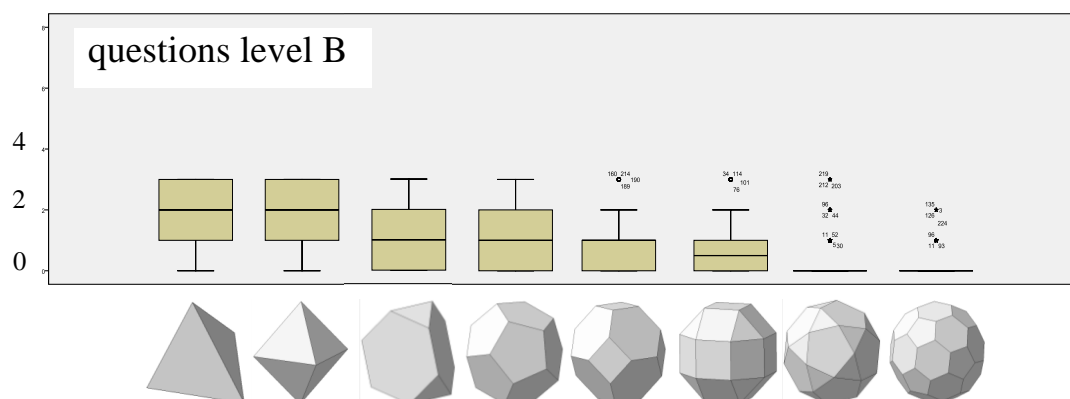
representation	representation	Significance low performance	Significance middle performance	Significance high performance
illustration	computer	,244	,119	,614
computer	model	,000	1,000	,001
model	illustration	,253	,085	,061

**Table 4: detailed significances for students with low, middle and high performances**

In Fig. 6, three ability groups (low, medium and beneficial students, related to the brick-test) are represented individually. The previous restrictions and filtering have been retained. For weak and beneficial students the interactive computer-animation was the least helpful. For medium students, there is little relevance to what environment they work. Remarkably the similar results for students with low and high performances are observed. Students with low performances show a different finding – the working environments computer and model are equally “successful”. So we can assume, that the model-based environment offers some advantages in dealing with the issues, but not always. For a more detailed view, the significances are illustrated in Table 4.

### A model to define the complexity of a solid – a first evaluation

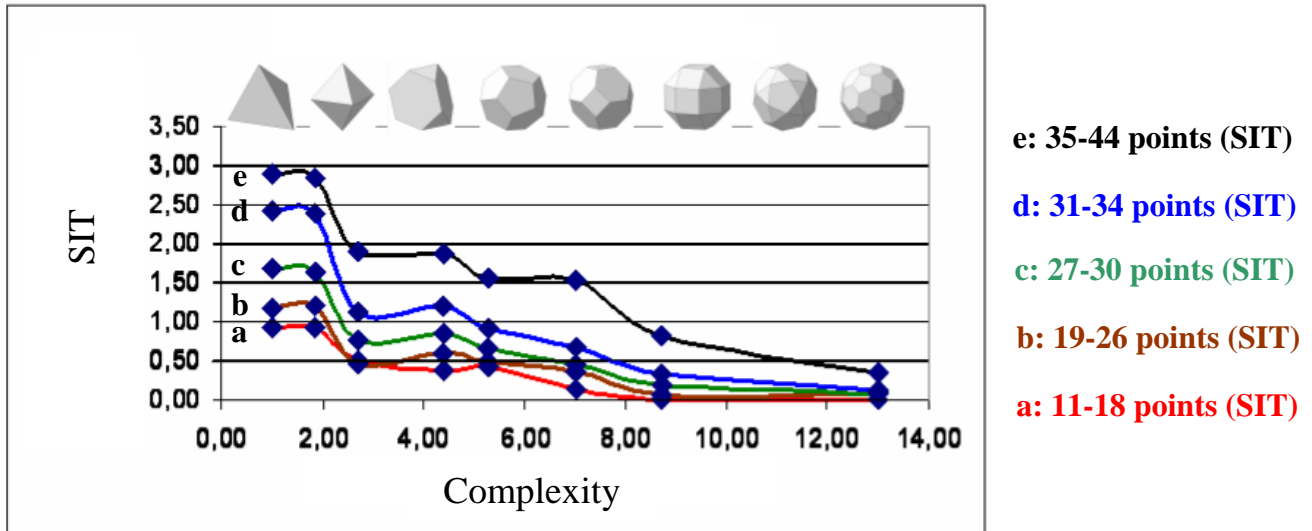
The findings for level A (questions about the shape of the faces, the number of edges, faces which touch at one vertex) show clearly that these questions were usually answered correctly by the large majority of participants.



**Fig. 7: Performance of level B questions in boxplots for each solid**

The following chart (see Fig. 7) was put together in response to questions of level B (questions about the number of faces, edges and vertices). It demonstrates clearly that the model to calculate the level of complexity does partly not differentiate pairs.

### *Reflections on performance-bands*



**Fig. 8: Results of the SIT as a function of the complexity of the solid for different performance samples**

In Fig. 8, we test the hypothesis whether a solid can be described by the applied model to determine the level of complexity. Since we do not have normal distributions – as already demonstrated – we will argue with help of five performance samples (from a = very poor to e = excellent) with regards to the structure-identification-test (SIT).

The graphical illustration shows that sometimes the necessary differentiation between solids is not confirmed by the students' performance. For example, the learners experience the level of difficulty of two different solids as of equivalent value, e.g. solid 1 and 2 or 3 and 4 or 5 and 6. As mentioned above solid 1 has to be interpreted cautiously as it is considered to be a "warming-up shape". Tests in this respect will follow. In addition, we only evaluated solid 2 to 6 while the very complex bodies 7 and 8 differentiated poorly and we only considered correct solutions (an approximation to the correct value has not been taken into account).

The depicted parallel curves suggest the following interpretation: students with better results show this performance bonus with all solids to a similar extent and regardless of the degree of complexity.

## **CONCLUSION**

Based on the analysis of these initial assessments the following two "conservative" theses are posed.

## Working group 4

- The findings confirm the rather inconsistent findings of various recent works. For example we cannot confirm that students with strong abilities in the field of spatial thinking benefit from a computer-based environment. On the contrary, this group benefits particularly strong from a model-based environment. For individual performance groups (see Fig. 4) we notice several shifts, which however due to the number of subjects studied should only be used cautiously as direction.
- The model to define the level of complexity of a solid does not differentiate appropriately. It proves a trend. However, this is not particularly surprising. This model is definitely not sufficient for a description of learning or testing environment in the field of spatial geometry. From a current point of view it appears that the description of a solid is not easy because other parameters such as the angle between two surfaces, the flat of rotation etc. may be relevant. For example a linear connection such as between the angle of rotation and the speed or correctness of processing cannot be detected (see Shepard, Metzler 1971, ter Horst & van Lier et.al, 2010).

Critical remarks and limits are as follows:

- Unfortunately, the experience in dealing with the computer could not be tested. Both an interview and a reaction test with the mouse have been unusable. It should be noted that dealing with the visualization program does not assume any computer skills.
- To estimate the complexity of a solid there are more properties necessary than these few we had considered.
- Because the SIT is not a learning environment but a working environment there are no statements possible about training effects. We only consider which working environment (illustration, model or computer) will support the students more in solving easy spacegeometry tasks.

## REFERENCES

- Ahmad, R.; Khairulanuar, S. (2009). Practising mental rotation using interactive Desktop Mental Rotation Trainer (iDeMRT). In: *British Journal of Educational Technology*, Volume 40, Number 5, 889-900(12).
- Birkel, P.; Schein, S.A.; Schumann, H. (2002). Bausteine-Test (BST), Ein Test zur Erfassung des räumlichen Vorstellungsvermögens. Göttingen u.a.: Hogrefe.
- Cohen, C.A. (2005). The Influence of Spatial Ability on the Use of Dynamic, Interactive Animation in a Spatial Problem-Solving Task. In: *Barkowsky, T.; Freksa, C.; Hegarty, M.; Lowe, R. (Ed.). Reasoning with Mental and External Diagrams: Computational Modeling and Spatial Assistance. California: Menlo Park.*

- Glück, J.; Kaufmann H.; Dünser A.; Steinbügl K. (2006). Geometrie und Raumvorstellung. Psychologische Perspektive. Wien: Fakultät für Psychologie der Universität Wien.
- Hellmich F.; Hartmann J. (2002). Aspekte einer Förderung räumlicher Kompetenzen im Geometrieunterricht. Ergebnisse einer Trainingsstudie mit Sonderschülerinnen und -schülern. In: *ZDM*, H. 34, 56–61.
- Hirnstein, M.; Bayer, U.; Hausmann, M. (2009). Sex-Specific Response Strategies in Mental Rotation. In: *Learning and Individual Differences*, H. v19 n2, 225–228.
- Jordan, K.; Wuestenberg, T.; Heinze, H.J.; Peters, M.; Jaencke, L. (2002). Women and men exhibit different cortical activation patterns during mental rotation tasks. In: *Neuropsychologia*, H. 40, 2397–2408.
- Kruger, M.; Krist, H. (2009). Imagery and Motor Processes - When Are They Connected? The Mental Rotation of Body Parts in Development. In: *Journal of Cognition and Development*, H. v10 n4, 239-26.
- Linn, M.C.; Peterson A.C. (1985). Emergence and characterization of sex differences in spatial ability: A Meta-Analysis. In: *Child Development*, H. 56, 6, 1479–1498.
- Linn, M.C.; Peterson A.C. (1986). A meta-analysis of gender differences in spatial ability. In S. Hyde & M.C. Linn (Eds.). In: *The psychology of gender*, Baltimore, 67–101.
- Lowe, R. (2003). Animation and learning: Selective processing of information in dynamics graphics. In: *Learning and Instruction*, H. 13, 157–176.
- Lowe, R. (2004). Interrogation of a dynamic visualization during learning. In: *Learning and Instruction*, H. 14, 257–274.
- Peters M.; Battista C. (4/2008). Applications of Mental Rotation Figures of the Shepard and Metzler Type and Description of a Mental Rotation Stimulus Library. In: *Brain and Cognition*, H. v66 n3, 260–264.
- Schwan, S.; Riempp, R. (2004). The cognitive benefits of Interactive videos: learning to tie nautical knots. In: *Learning and Instruction*, H. 14, 293–305.
- Shepard, R.N.; Metzler, J. (1971). Mental rotation of three-dimensional objects. In: *Science*, H. 171, 701–703.
- Souvignier, E. (Hg.) (2000). Förderung räumlicher Fähigkeiten. Trainingsstudien mit lernbeeinträchtigten Schülern. Münster: Waxmann (Pädagogische Psychologie und Entwicklungspsychologie, 22).
- ter Horst, A.C.; van Lier, R.; Steenbergen, B. (4/2010): Mental rotation task of hands: differential influence number of rotational axes. In: *Experimental Brain Research*.

# GENERATING SHAPES IN A DYNAMIC ENVIRONMENT

Sue Forsythe

School of Education, University of Leicester, UK

*The study described in this paper aims to investigate the ways in which 12 – 13 year old students conceptualise 2 dimensional shapes. Two students were observed while working together at a task in a Dynamic Geometry Environment where they were required to generate shapes by dragging 2 rigid bars which formed the diagonals within a figure. The students appeared to be attending to the symmetry of the shapes generated which helped them to position the bars, with a high degree of accuracy, to create specific shapes with the expected properties concerning equal sides and angles.*

Keywords. Dynamic Geometry Software, Dragging, Symmetry.

## HOW DO CHILDREN CONCEPTUALISE SHAPES IN 2 DIMENSIONS?

The study described in this paper was designed to investigate how 12-13 year old students reason about the geometrical properties of 2D shapes while working in a Dynamic Geometry Environment. Whilst geometry appears to be a practical area of mathematics, understanding geometrical concepts requires abstract thought. Researchers have suggested that this dichotomy leads to problems in geometry because students find it hard to appreciate the difference between the actual figure on paper and the theoretical object that it represents (Battista 2007). Students often focus on the material representation of a figure such as a drawing on paper or a figure on a computer screen. However, when solving problems in geometry it is necessary to work with the ideal perfect geometrical figure. The notion of the ‘figural concept’ describes this perfect geometric figure which, when fused with its conceptual properties, is what we manipulate when working on geometrical problems (Fischbein, 1993).

Using Dynamic Geometry Software (DGS) may act as the mediator for the figural concept and as such it can provide students with a means to understand the properties of geometrical figures. In ordinary school geometry, a theoretical object (figural concept in Fischbein’s terms) is mediated by its material representation on paper (Laborde, 1993). Laborde explains that the introduction of DGS enables us to redefine the distinction between the theoretical object and its material representation. There is now a figure on the screen (which can represent a whole class of figures) and this figure is a new kind of mediator for the theoretical object. It is different from a paper drawing in that it is dynamic. Its behaviour when dragged (when objects such as points and lines making up the figure are picked up by the dragging tool and moved on the screen) is determined by the method used to construct it, that is the geometrical properties designed into its construction.

Mariotti (1995) extends this point by claiming that drawings act as mediators between concrete and theoretical objects. Screen images of geometrical figures represent the external version of the figural concept. To construct a figure in DGS the conceptual and figural aspects must be made explicit in the construction process. In this way working in a dynamic geometry environment is useful to develop the correct interaction between the figural and conceptual aspects of geometrical reasoning. The internal logic of the geometrical figure becomes apparent when it is dragged since the geometrical relationships that defined it remain constant under dragging.

### **Important affordances of Dynamic Geometry Software**

There are a number of different ways students can use dragging to explore and conjecture in geometry (Arzarello, Olivero, Paola and Robutti, 2002). Dragging a figure can also be used as a way to test the validity of a construction. Jones (2000) noted that DGS has given us a way to validate a construction through the dragging feature and the drag test can provide the motivation for students to learn about geometrical principles. The dynamic nature of the software influences how students reason about geometrical objects. Measuring is another important affordance of DGS: measures of lengths, angles and areas continually update as the figure is manipulated by the dragging tool. Hollebrands (2007) describes two strategies that students use when dragging and measuring as reactive and proactive. When students drag in a fairly random fashion in order to see what happens and when their decision of what to do next is based on the results of the previous action then the students are using **reactive** strategies. An example (not one that Hollebrands gave) would be if students are given a quadrilateral whose diagonals are fixed length bars and they drag the diagonals in order to see which special quadrilaterals they can make. They can use the measures of sides and angles to check whether they have made the special quadrilateral and adjust the diagonals until the measurements are satisfactory.

As the students develop their understanding of the technology and the mathematics then they are able to predict the outcome of their actions and become **proactive** in their strategies. An example here would be if the students predicted that placing the diagonals so that they bisected at right angles would result in the quadrilateral being a rhombus. Hollebrands (2007) noted that encouraging students to use strategies that are more proactive may be achieved by asking students to explain and justify what happens on the computer screen in terms of geometrical properties.

Olivero and Robutti (2007) say that dragging and measuring can help students to move between the experimental or practical side of geometry (where students can measure lines and angles on geometrical figures) and the theoretical area of geometrical concepts. In my work I hope to see that students working in the practical, experimental side can progress into reasoning in the theoretical side of geometry.



## The Van Hiele Levels

The Van Hiele levels are an important model to describe the development of geometrical reasoning (Jones, 1998, Battista, 2007). Briefly, level 1 is the visual level at which children recognise a shape as a totality without considering any of its properties. Level 2 is where children start to understand shapes as being collections of properties and recognise shapes by their properties. At level 3 children are able to infer one set of properties from another and understand the hierarchical classification of quadrilaterals e.g. that a square is also a rhombus. A further two levels deal with deductive proof and advanced level geometry (Van Hiele, 1986). In England the National Framework for Teaching Mathematics gives the specific learning objectives for geometry in year 8 (12-13years), the age of the students in this study, as:

Solve geometrical problems using side and angle properties of equilateral, isosceles and right angled triangles and special quadrilaterals, explaining reasoning with diagrams and text and classify quadrilaterals by their geometrical properties (Department for Children, Schools and Families, 2007).

Thus the students in this study are expected to be working between Van Hiele levels 2 and 3.

## METHODOLOGY

The research described here is part of an ongoing study being undertaken for the author's doctoral degree. It follows a design based methodology which uses the design experiment to study and develop theories about how people think and learn, in the setting of a learning environment, and allows the researcher to study the learning process in context (Cobb, Confrey, diSessa, Lehrer, and Schauble. 2003, Barab and Squire, 2004). The design process goes through a number of iterations where the experiment is designed, trialled, the results are analysed and reflected on and the experiment is then refined to test the robustness of any observations made.



**Figure 1** A toy kite

The experiment was devised with the intention of creating a meaningful task for students to work on where they would perceive the utility of the mathematics involved, in this case geometrical concepts of 2D shapes (Ainley, Pratt and Hanson, 2006). It is based on the idea of a toy kite whose structure is formed from two sticks (or bars). In a basic kite as shown in the Figure 1 these bars are fixed at right angles and are of different lengths. The resulting structure is covered with fabric and, if we attached a long line of string, we might be able to fly it. If you imagine the shape, it will probably be as a geometric kite, i.e. the vertical bar will intersect the horizontal bar at its mid point.

Working with the idea of 2 rigid bars providing the structure for a 2D shape let's say that the bars can be moved inside the shape (the fabric is elastic and stretches to stay with the bars). This would be difficult to demonstrate using pencil and paper except for showing different positions of the bars at discrete moments in time. However it can be done in the human imagination and also in DGS.

Using the Geometers Sketchpad version 4 (Jackiw, 2001) a file was created containing a vertical bar and horizontal bar of 8cm and 6 cm respectively. The students working with the file were asked to drag one bar over the other and to use the line tool in the software to join up the ends of the bars. They then constructed the interior of the shape which fills it with colour (equivalent to putting fabric onto the kite) and helps with visualisation especially when the shape is concave.

Some researchers (eg Arzarello et al, 2002) have noted that students, working with DGS, often need to be encouraged to use the drag mode. The task in this study requires students to use dragging. Rather than asking students to construct a figure with drag proof properties, this task uses the dynamic nature of the software in a different way where the constraints are the two rigid bars inside the shape. These restrict the shape to a quadrilateral or triangle, the types of which are dependent on the lengths of the bars and the angle between them.

The task was given to pairs of students aged 12-13 years and, at the time of writing, 8 pairs had worked on the task. The students worked for two sessions each lasting fifty minutes and their on screen activity and dialogue were recorded. Dialogue and screen activity were analysed and emerging themes were identified.

Assessment of the prior knowledge of the students was carried out informally at the beginning of the task through questioning about the shapes they had made and about their properties. The students were assessed by their regular class teacher as being of average attainment with respect to their peers. Each pair were chosen as being students who would enjoy working at computer tasks and who were sufficiently confident to be able to talk about what they were doing with the researcher, whom they had not met previously. The students generally studied mathematics in the same class as each other but did not necessarily work together in class.

### **OBSERVATIONS FROM THE RECORDINGS**

At the beginning of the research I had hoped to observe that the students could classify 2D shapes in a hierarchical manner e.g. accepting that a square is a special case of a rhombus. However my observations indicated that the students were reluctant to accept this kind of reasoning. In fact De Villiers (1994) described many students' unwillingness to work with a hierarchical classification of quadrilaterals, preferring instead to use a partitional classification (e.g. where squares have equal length sides and equal angles, rhombuses have equal length sides but unequal angles) unless they can see a reason for using a hierarchical classification. Nevertheless I was able to make observations about how the students may conceptualise shapes and

the ways in which they used symmetry emerged as an important feature. Battista (2007) has conjectured that students might unconsciously perform visual transformations on shapes which help them to conceptualise properties and I argue that I have observed something akin to this.

### **Three different strategies using the affordances of Dragging and Measuring**

The students appeared to use dragging and measuring in three different ways; using the reactive and proactive strategies mentioned by Hollebrands (2007) and a third way that I noticed when they used dragging and measuring together. The reactive dragging style was observed when the students dragged bars around fairly randomly to see what different shapes they could make. This happened at the beginning of the sessions when the students did not yet know what to expect. They dragged the bars around to make shapes that they recognised from their previous experience in geometry.

Proactive dragging occurred when the students had a little more experience of working with the tasks. If the students decided to make a kite, say, then they dragged the bars straightaway to generate the shape they wanted. Several pairs of students were observed to drag the horizontal bar up and down the vertical bar treating it as a perpendicular bisector, maintaining the symmetry of the shape. In this way, using the 6 cm and 8 cm perpendicular bars, they quickly generated a kite, rhombus, isosceles triangle and arrowhead (which they did not recognise as being a concave kite).

The students attended only to the holistic shape when dragging proactively. After the shape was considered finished the students would typically check the length and angle measures to prove that they had indeed made a kite or whatever shape it was supposed to be. However the measures of sides and angles which were meant to be equal very often were not exactly equal. The students would then try to make subtle adjustments to the positions of the bars. I have named this third strategy 'refinement.' They would typically make very small dragging movements so as to get the required measurements as equal as possible. They usually got them to be close to within one degree for angles and one decimal place for lengths in centimetres and this was considered to be acceptable. Using a refinement strategy also helped the students to check and review the properties of the shapes which they already knew. For example one pair of boys decided the angles in a rhombus are equal. They tried to drag the bars to make the angles equal and still retain the shape of a rhombus but were unsuccessful. However this activity helped them to see that there are two different sizes of angles in a rhombus.

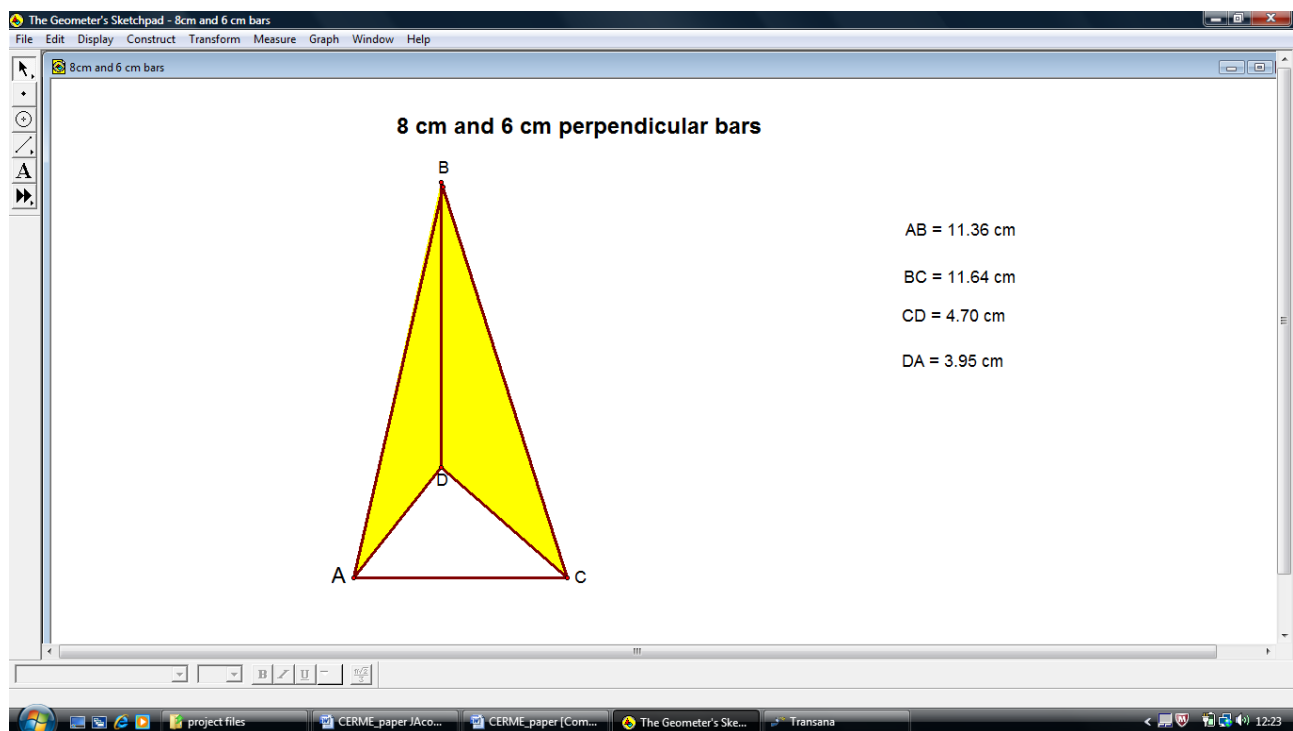
### **Using symmetry**

The most useful strategies the students used were the proactive dragging and the refinement strategy. The work of two girls will be described as an example of what the student pairs have typically done while working with the files. The girls, working in the file with the 6 cm and 8 cm bars, made an arrowhead as their first chosen

#### Working group 4

shape to explore. They proactively dragged the horizontal bar AC so that it was bisected by the (extension of) vertical bar BD. After having decided that a symmetrical arrowhead would have pairs of equal length sides, the girls used the refinement strategy to make the two pairs of adjacent edges congruent. When analysing the recording it was noted that the refinement activity took up 3: 55 minutes of the recording while the girls were trying to make an accurate arrowhead. Subsequent intervals of refinement took less than one minute, usually just a few seconds. This seems to indicate that the girls improved in their ability to position the bars using their judgement and they appeared to use the symmetry of the shape to help them to do this.

The screen shot in Figure 2 shows an instance, when Alice was making the arrowhead kite and focused on getting two of the measurements to be equal at the expense of the symmetry of the shape. Tilly pointed out that she needed to position the vertical bar in the middle of the shape and, although symmetry is not specifically mentioned, it is symmetry to which she is referring.



**Figure 2**

Tilly: Oh that was right a minute ago, it was eleven nine seven, point nine seven, you need to go higher

Res: DA and DC have changed now haven't they

Tilly: cos you've gone to that side more. You need to be in the middle and then move up. That's still a bit that side I think.

## Working group 4

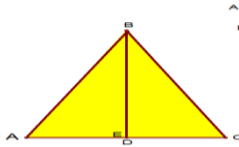
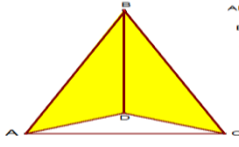
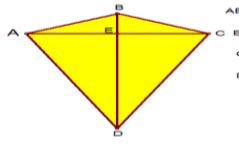
Alice: It still looks wonky to me

Tilly: So move that way a bit, no the other way. Nearly got them two. We're try and aim for D and A first then B and A and BC. If you move that a tiny bit, the other way

Res: So what are you trying to do, how are you trying to position BD?

Tilly: Yeah we're trying to get BD in the middle of the shape.

Eventually the girls decided that the best way to make a symmetrical arrowhead was to place the bars in the position of an isosceles triangle and then move the horizontal bar AC up or down from there. They moved AC so that it would be perpendicularly bisected by BD although they did not use this kind of terminology. First of all Tilly suggested they move the point D to sit on the horizontal bar AC and then move the bar BD up from that central position. Tilly had the computer mouse at this point and spent some time ensuring that the measures on the isosceles triangle were as close as she could get them (Figure 3) before moving the vertical bar BD down making an arrowhead (Figure 4) and then up (through the rhombus), to make a kite (Figure 5).

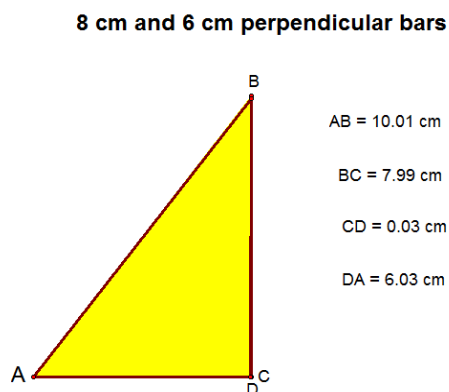
<p>8 cm and 6 cm perpendicular bars</p>  <p> <math>AB = 8.54 \text{ cm}</math>  <math>BC = 8.53 \text{ cm}</math>  <math>CD = 2.98 \text{ cm}</math>  <math>DA = 3.02 \text{ cm}</math> </p>	<p>8 cm and 6 cm perpendicular bars</p>  <p> <math>AB = 10.03 \text{ cm}</math>  <math>BC = 10.06 \text{ cm}</math>  <math>CD = 3.42 \text{ cm}</math>  <math>DA = 3.36 \text{ cm}</math> </p>	<p>8 cm and 6 cm perpendicular bars</p>  <p> <math>AB = 3.30 \text{ cm}</math>  <math>BC = 3.32 \text{ cm}</math>  <math>CD = 7.25 \text{ cm}</math>  <math>DA = 7.24 \text{ cm}</math> </p>
Figure 3 Isosceles triangle	Figure 4 Arrowhead	Figure 5 Kite

These students had realised that simply sliding the horizontal bar AC up and down the vertical bar BD such that its mid-point touched BD resulted in shapes they recognised. Later in the same session the girls observed that they simply needed to slide the horizontal bar AC down in order to make the kite (in one position) and the rhombus. “That’s a diamond and that’s a kite” was accompanied by the horizontal bar being moved down then up again keeping vertical bar AC as the perpendicular bisector.

When it was suggested they slide AC below the position for a rhombus they decided that they had generated an ‘upside down’ kite (thus focusing on the orientation of the typical representations of a kite). In each case the girls only needed to spend the smallest amount of time using refinement to make the measurements equal as they seemed to have gained experience in how to place the bars by eye in order to create a shape with symmetry. The girls visualised the shapes holistically when attending to symmetry which suggests reasoning at Van Hiele level 1. They were also focusing on moving the bars, which were the diagonals, inside the shape and noting the relative position of the bars, which indicates reasoning at Van Hiele level 2.

The girls only investigated the use of the vertical bar as a line of symmetry which may reflect vertical dominance in the natural world. When they generated the rhombus they did state that it had two lines of symmetry, i.e. both the bars. When questioned, their concept of symmetry seemed to be process based rather than coming from an esoteric understanding of the meaning of symmetry. They had clearly been taught that, if a line of symmetry exists then the shape can be folded along that line, edges and angles will coincide and therefore must be equal in size. They used this explanation as their working definition of symmetry and it helped them to decide which sides and angles in the shape need to be congruent.

### HALF A SQUARE OR HALF A RECTANGLE?



At one point the girls dragged the bars to make a right angled triangle (Figure 6). A discussion followed as to whether this shape is half a rectangle or half a square. This was interesting because the girls clearly visualised that if the right angled triangle was copied and then the copy was transformed the result would be a rectangle (or square, until they had argued it through). When Tilly said that it will make a square, Alice disagreed and had to persevere for a while until her colleague came round to her point of view.

**Figure 6**

Tilly: Cos if you get another one of them, and turn it round and make a

Alice: It would make a rectangle

Res: So you think it would make a rectangle?

Tilly: No a square

Res: If you have two of them?

Tilly: yeah

Alice: that'll make a rectangle

Tilly: Wouldn't cos they're the same

Res: Which are the same?

Tilly: er B and A, well er AC and BC

Res: Have we got those two measurements?

(A discussion took place where they decided the 2 perpendicular sides of the right angled triangle were not the same length).

#### Working group 4

Tilly: I think it'll still be a square because if you put that one there. And if you put A er D at the top, join it with the B point then put BC on the other side then it would be a same I think.

Res: What do you think Alice?

Alice: er, er, wait, I need to

Tilly: no actually it would be a rectangle

Alice: It would because, if you think about it, if you did, if you flipped that over the other side so it was like symmetry, you would get the same, and if you had both of them, then it would be a rectangle

Tilly: yeah it would be

Res: OK so why should it be a rectangle and not a square?

Tilly: because BC is longer than AD.

Although it is unclear whether Alice was thinking incorrectly that the copy of the right angled triangle would be reflected to make the rectangle (although Tilly considered that the extra triangle had to be turned round, indicating rotation) it does seem that the girls were looking at the rectangle through the lens of symmetry.

## CONCLUSION

In analysing the recordings it has become evident that the students are implicitly aware of the symmetry of 2D shapes and that this awareness is powerful for their understanding of the properties of the shapes. A preliminary conclusion from this study is that it may be more intuitive for students to focus on symmetry first and to derive other properties from it. In future iterations of the study, bars which are at different orientations to the vertical and also bars at adjustable angles will be used to investigate further.

## REFERENCES

- Arzarello, F., Olivaro, F., Paola, D., and Robutti, O.(2002). A cognitive analysis of dragging practices in Cabri environments, *ZDM* 2002 Vol.34 (3), pages 66-72
- Ainley, J., Pratt.D., and Hansen.A. (2006). Connecting engagement and focus in pedagogic task design, *British Educational Research Journal*, 32 (1): 23-38
- Barab, S., and Squire, K.(2004). Design-based research: Putting a stake in the ground. *Journal of the Learning Sciences*, 13(1) : 1-14
- Battista, M., Ed. (2007). The development of geometric and spatial thinking. *Second Handbook of Research on Mathematics Teaching and Learning* Greenwich, CT, Information Age Publishing, Inc and NCTM.
- Cobb, P., Confrey, J., diSessa, A., Lehrer. R., Schauble. L. (2003). Design Experiments in Educational Research. *Educational Researcher*, 32 (1) : 9-13.

- Department for Children, Schools and Families (2007),  
<http://nationalstrategies.standards.dcsf.gov.uk/strands/881/66/110131> accessed  
21/9/09
- De Villiers, M.,(1994) The role and function of a hierarchical classification of quadrilaterals. *For the learning of mathematics* 14 (1): 11-18
- Fischbein, E. (1993), The theory of figural concepts. *Educational studies in mathematics*. 24(2) : 139-162
- Hollebrands, K. (2007) The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education*, 38, (2): 164-192
- Jackiw,N. (2001). The Geometers Sketchpad (version 4.0) [Computer software]. Emeryville, CA: Key Curriculum Press
- Jones, K. (1998), Theoretical Frameworks for the Learning of Geometrical Reasoning, *Proceedings of the British Society for Research into Learning Mathematics*, 18 (1 & 2): 29-34
- Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics* 44 :55-85
- Laborde, C. 1993. Do the Pupils Learn and What do they Learn in a Computer Based Environment? The Case of Cabri-geometre, *Technology in Mathematics Teaching: A Bridge between Teaching and Learning*. Birmingham
- Mariotti, M. A. (1995). Images and concepts in geometrical reasoning. In R. Sutherland and J. Mason (Eds), *Exploiting mental imagery with computers in mathematics education*. Berlin: Springer
- Olivero, F., and Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving, *Int J Comput Math Learning* 12: 135-15
- Van Hiele, P. (1986) Structure and Insight. *Academic press*



# INTEGRATING NUMBER, ALGEBRA, AND GEOMETRY WITH INTERACTIVE GEOMETRY SOFTWARE

Kate Mackrell

Institute of Education, University of London

*In order to compare the potential for the integration of number, algebra and geometry using interactive geometry software, a series of tasks related to finding the area of a circle was performed using Cabri II Plus, Cinderella, GeoGebra and Geometer's Sketchpad. It was found that, while each program had the facility to perform the tasks, there were differences in the design of the programs that could lead to either facilitating or impeding the development of student understanding.*

**Keywords.** Technology, interactive geometry software, geometry, number, algebra.

## INTRODUCTION

There is an increasing awareness that the details of the design of pedagogical tools are significant and should be researched (Jackiw, 2010). This paper arises out of research on the design of four different interactive geometry (IGS) programs: Cabri II Plus (Cabri), Cinderella, GeoGebra and Geometer's Sketchpad (GSP). The focus has been on the identification of affordances and design decisions as a basis for further research on the impact of such differences on student learning.

Current IGS software provides a means by which algebra, geometry, and number can be meaningfully linked. Falcade (2007) showed that geometric construction could be used to enhance student understanding of the concept of function. Laborde (2010) explored the creation of dynamic graphs. Jackiw (2010) suggested that dynamic number provides an appropriate link between algebra and geometry.

A particular series of tasks involving the integration of number, algebra, and geometry were hence used to compare the four programs. The topic, finding the area of a circle, is universal, and each task (e.g. constructing a geometric object or an algebraic expression) involved processes common to other topics. The pedagogical approach (involving exploring and gathering information about a mathematical situation, making and testing conjectures, then generalizing and proving results) has been promoted extensively in the UK since the 1980's: the specific tasks were based upon the principle stated by Laborde (2010, p. 218) that "the teaching of mathematics must help students learn how to adequately use various representations and to move between them if needed."

As the aim of the research has been to identify rather than study the impact of design decisions, the tasks were performed only by the researcher in order to:

- a) ensure that the same tasks were performed with each program as a basis for comparison. The pedagogical approach emphasizes student choice and, in a classroom, would result in different tasks being performed by different students.
- b) address the issue of familiarity. The researcher was in communication with the developers of all the programs to ensure that any initial lack of understanding was not reflected in her conclusions.
- c) ignore the effects of any differences in task presentation necessitated by differences in the programs and in student familiarity with the programs.

## **METHOD:**

Each task was performed several times with each program as questions concerning the affordances of the programs arose and were answered by the software developers, or when more detail was required.

## **RESULTS:**

### **Task 1: Create a circle and a segment to represent its radius**

In this task, the basic mathematical situation to be explored was set up.

In Euclidean geometry, a circle is a set of points equidistant from a given point, and does not depend on location. Cabri and Cinderella each had a tool by which a circle could be created and explored simply by clicking to create a centre point and define an initial radius. Dragging the centre point of this circle moved the circle without changing its size; dragging on the circle itself changed its radius. In the other programs objects such as a point on the circumference or a segment or number giving the length of the radius needed to be chosen.

The simplest option offered by all the programs (and hence the option used in further tasks) was to create a circle given a centre point and a point on the circumference (referred to as the radius point). Dragging the centre of this circle changed both its location and its size: only by dragging the circle as a whole was its radius maintained.

In Cinderella and GSP the same motion, dragging, is used to create the circle and to move it. Making the circle by clicking at the centre, holding down and pulling felt analogous to pulling one arm of a pair of compasses away from the other. Cabri and GeoGebra required a click – release - move – click motion, a different motion from dragging, but with the same visual effect and hence potentially confusing. In GeoGebra, the algebra window needed to be hidden: otherwise coordinates of the centre point and radius point appeared after clicking and before releasing which was distracting, unnecessary, and potentially off-putting to learners who had not yet encountered coordinates.

In the geometry of Euclid, distances, areas and volumes could be compared but were never assigned a number. Introducing a number moves away from the origins of geometry, but, if the number is variable, enables a move toward algebra. In each program, a circle may be created from its centre and a number to represent its radius.

Possibilities for the radius of the circle can be ordered by variation and naming, both of which are important in developing the concept of a variable. A conjecture is that the process of manipulating numbers which are linked to objects that change may be important in developing the idea of a variable. Changing a number such as the radius of the circle and noticing the effect may be a useful introduction to the idea of variation without algebraic terminology and notation. Using names for variable numbers may be a later step in developing the idea of the variable.

In Cinderella, the number, input through a dialogue box, cannot be edited or used for any purpose other than defining the circle. GeoGebra gives the same option, although the number may be edited. In Cabri and GSP, the number is selected by clicking on any number displayed on the page, either entered directly or the result of previous measurement or calculation. Cabri allows numbers to be placed on the page without being assigned names. GSP requires a new parameter to be defined, which will be assigned a name. GeoGebra also allows previously defined numbers to be used, but only by typing in the name of the variable, such as the name of a slider, representing the number. A slider, whereby a number changes as a point is dragged, gives a visual representation of the variability of a number, and enables numbers to be changed by the same operation, dragging, that changes geometric figures. It has the potential to be an important link between geometric and algebraic representations. GeoGebra sliders have the appearance of points on segments, but the points and segments are not actual geometric objects. In the other programs, geometric sliders, which give measurements such as the relative position of a point on a segment, may be created. The numbers defined by such sliders will not necessarily have names.

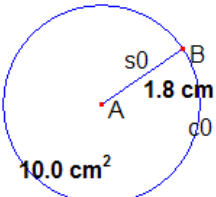
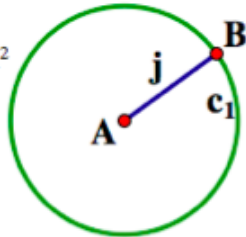
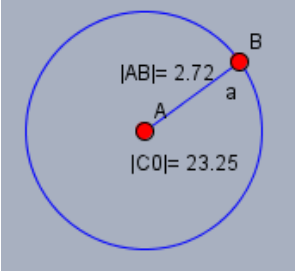
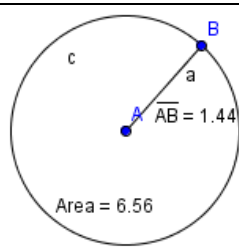
## **Task 2: Measure the area and radius of the circle.**

In this task, information is found about the mathematical object being studied.

With measurement, the introduction of numbers becomes necessary. In each program, it is straightforward to find the area of the circle and the distance between the centre and radius points.

The table below gives a screenshot for each program with a list of construction steps so far. The algebra window is also displayed for GeoGebra. Cabri objects (but not measurements) have been labeled. Other labels were supplied by the programs.

## Working group 4

	<table><tr><td>B</td><td>Point</td></tr><tr><td>A</td><td>Point</td></tr><tr><td>c0</td><td>Circle: A, B</td></tr><tr><td>s0</td><td>Segment: A, B</td></tr><tr><td>1.8 cm</td><td>Text (Distance or Length): A, B</td></tr><tr><td>10.0 cm²</td><td>Text (Area): c0</td></tr></table>	B	Point	A	Point	c0	Circle: A, B	s0	Segment: A, B	1.8 cm	Text (Distance or Length): A, B	10.0 cm²	Text (Area): c0																												
B	Point																																								
A	Point																																								
c0	Circle: A, B																																								
s0	Segment: A, B																																								
1.8 cm	Text (Distance or Length): A, B																																								
10.0 cm²	Text (Area): c0																																								
Cabri II Plus																																									
<p>AB = 2.65 cm Area <math>c_1</math> = 22.12 cm<sup>2</sup></p> 	<p>Given:</p> <ol style="list-style-type: none"><li>Point A</li><li>Point B</li></ol> <hr/> <p>Steps:</p> <ol style="list-style-type: none"><li>Let <math>c_1</math> = circle centered at A passing through B.</li><li>Let j = segment between A and B.</li><li>Let AB = distance from Point A to Point B.</li><li>Let Area <math>c_1</math> = area of Circle <math>c_1</math>.</li></ol>																																								
Geometer's Sketchpad																																									
	<table><tr><th></th><th>Who?</th><th>What?</th><th>Where?</th></tr><tr><td>●</td><td>A</td><td>Point(2.04 2.6)</td><td>(2.04 2.6)</td></tr><tr><td>●</td><td>B</td><td>Point(4.24 4.2)</td><td>(4.24 4.2)</td></tr><tr><td>○</td><td>C0</td><td>Circle(A;B)</td><td><math>(x - 2.04)^2 + (y - 2.6)^2 = 2.72^2</math></td></tr><tr><td>—</td><td>a</td><td>Segment(A;B)</td><td>—</td></tr><tr><td>A0</td><td>Area(C0)</td><td></td><td>23.25</td></tr><tr><td>Text0</td><td>Text(A;A0)</td><td></td><td> C0 = 23.25</td></tr><tr><td>C</td><td>Mid(A;B)</td><td></td><td>(3.14 3.4)</td></tr><tr><td>dist0</td><td>Distance(A;B)</td><td></td><td>2.72</td></tr><tr><td>Text1</td><td>Text(dist0)</td><td></td><td> AB = 2.72</td></tr></table>		Who?	What?	Where?	●	A	Point(2.04 2.6)	(2.04 2.6)	●	B	Point(4.24 4.2)	(4.24 4.2)	○	C0	Circle(A;B)	$(x - 2.04)^2 + (y - 2.6)^2 = 2.72^2$	—	a	Segment(A;B)	—	A0	Area(C0)		23.25	Text0	Text(A;A0)		C0 = 23.25	C	Mid(A;B)		(3.14 3.4)	dist0	Distance(A;B)		2.72	Text1	Text(dist0)		AB = 2.72
	Who?	What?	Where?																																						
●	A	Point(2.04 2.6)	(2.04 2.6)																																						
●	B	Point(4.24 4.2)	(4.24 4.2)																																						
○	C0	Circle(A;B)	$(x - 2.04)^2 + (y - 2.6)^2 = 2.72^2$																																						
—	a	Segment(A;B)	—																																						
A0	Area(C0)		23.25																																						
Text0	Text(A;A0)		C0 = 23.25																																						
C	Mid(A;B)		(3.14 3.4)																																						
dist0	Distance(A;B)		2.72																																						
Text1	Text(dist0)		AB = 2.72																																						
Cinderella																																									
	<div><div>Free Objects</div><ul style="list-style-type: none"><li>A = (2.22, 2.02)</li><li>B = (3.18, 3.1)</li></ul><div>Dependent Objects</div><ul style="list-style-type: none"><li>a = 1.44</li><li>areac = 6.56</li><li>c: <math>(x - 2.22)^2 + (y - 2.02)^2 = 2.09</math></li><li>distanceAB = 1.44</li></ul></div>																																								
<table><tr><th>No.</th><th>Name</th><th>Definition</th><th>Value</th></tr><tr><td>1</td><td>Point A</td><td></td><td>A = (2.22, 2.02)</td></tr><tr><td>2</td><td>Point B</td><td></td><td>B = (3.18, 3.1)</td></tr><tr><td>3</td><td>Circle c</td><td>Circle through B with center A</td><td>c: <math>(x - 2.22)^2 + (y - 2.02)^2 = 2.09</math></td></tr><tr><td>4</td><td>Segment a</td><td>Segment [A, B]</td><td>a = 1.44</td></tr><tr><td>5</td><td>Number areac</td><td>Area[c]</td><td>areac = 6.56</td></tr><tr><td>6</td><td>Text Textc</td><td>"Area = " + areac</td><td>Textc = "Area = 6.56"</td></tr><tr><td>7</td><td>Number distanceAB</td><td>Distance of A and B</td><td>distanceAB = 1.44</td></tr><tr><td>8</td><td>Text TextAB</td><td>"\overline{" + (Name[A]) + (Name[B]) + "}" \, = \, \, " + distanceAB</td><td>TextAB = "\overline{AB}" \, = \, \, 1.44"</td></tr></table>				No.	Name	Definition	Value	1	Point A		A = (2.22, 2.02)	2	Point B		B = (3.18, 3.1)	3	Circle c	Circle through B with center A	c: $(x - 2.22)^2 + (y - 2.02)^2 = 2.09$	4	Segment a	Segment [A, B]	a = 1.44	5	Number areac	Area[c]	areac = 6.56	6	Text Textc	"Area = " + areac	Textc = "Area = 6.56"	7	Number distanceAB	Distance of A and B	distanceAB = 1.44	8	Text TextAB	"\overline{" + (Name[A]) + (Name[B]) + "}" \, = \, \, " + distanceAB	TextAB = "\overline{AB}" \, = \, \, 1.44"		
No.	Name	Definition	Value																																						
1	Point A		A = (2.22, 2.02)																																						
2	Point B		B = (3.18, 3.1)																																						
3	Circle c	Circle through B with center A	c: $(x - 2.22)^2 + (y - 2.02)^2 = 2.09$																																						
4	Segment a	Segment [A, B]	a = 1.44																																						
5	Number areac	Area[c]	areac = 6.56																																						
6	Text Textc	"Area = " + areac	Textc = "Area = 6.56"																																						
7	Number distanceAB	Distance of A and B	distanceAB = 1.44																																						
8	Text TextAB	"\overline{" + (Name[A]) + (Name[B]) + "}" \, = \, \, " + distanceAB	TextAB = "\overline{AB}" \, = \, \, 1.44"																																						
GeoGebra																																									

**Table 1: Constructed figures and figure descriptions**

An issue with measurement is that GeoGebra and Cinderella do not display units (although this can be changed in Cinderella). Although not apparent above, trailing zeros are also not displayed. For example 3.004 to two decimal places would be written as 3 rather than 3.00, which can lead to statements such as “ $3 \times 4 = 12.02$ ”.

Unlike Cabri and GSP, Cinderella and GeoGebra distinguish names, which appear in the figure descriptions, from labels, which appear on the page. For example, the area of the circle in Cinderella has the name “A0” but the label “|C0|”, and in GeoGebra has the name “areac” and the label “Area”.

A confusion concerning objects and algebraic variables is evident in GeoGebra. Geometric objects and measurements of these objects may each be given a name, and the names may look equally “algebraic”, but only the name of the measurement refers to a variable (assuming that the measurement is not fixed), as only numerical quantities may be variables. Every program identifies the segment from A to B as a geometric object, a segment. However, GeoGebra treats it as a variable, and assigns it the value of the length of the segment: apparently “the algebraic representation of a segment is its length” (Hohenwarter, 2010, personal communication). This is also a puzzling misuse of the phrase “algebraic representation”: the length is a measurement rather than an algebraic expression, it is not the only measurement which can be made of a segment, and it does not determine the segment in the way the equation of the circle can determine the circle.

A further issue here is the naming of coordinates and equations as “values” in Geogebra. Particularly in interactive geometry, the coordinates of a free point are an indication of its (temporary) location relative to certain coordinate axes: equating a point to its coordinates is simply wrong.

There is also the issue that the information concerning coordinates and equations is displayed at all. For Cinderella, such information must be shown. In GeoGebra, such information must be hidden.

### **Task 3: Change the radius of the circle and observe the effect on its area.**

In this task, further information concerning the mathematical object was obtained, and a conjecture was made about the relationship between two variables.

Dragging is one of the chief links between geometry, number and algebra in an IGS. By dragging the radius point, a static circle, with a fixed radius and area, becomes a circle whose radius and area are now variables, capable of being related.

Unless the algebra window is hidden, GeoGebra shows the coordinates of the radius point as it is dragged: the other programs enable a focus on the way in which the area changes as the radius is changed without distraction. The measurements move with the figure in Cabri and Cinderella and can be attached to the figure in GeoGebra and GSP (although this is not straightforward). It is clear that as the radius increases, so does the area.

A conjecture is hence that the area is some multiple of the radius.

**Task 4: Test the conjecture by calculating area/radius and seeing how this changes as the radius is changed.**

In this task, the conjecture about the relationship between the two variables was tested, and refuted.

An advantage of Cabri and GSP is that the general division of area by radius could be achieved simply by using a calculator tool to divide the existing area value by the existing radius value, with numbers entered into the calculator by clicking on them. As the radius and area changed, the calculation was continually updated. It is unnecessary for the student to deal with the abstract idea of dividing one variable by another. However, the calculator can also act as an introduction to this idea. In GSP, when a number is selected on the page, its name appears in the calculator, making it clear precisely what is being calculated, and the label “area/radius” will appear next to the completed calculation. In Cabri, calculation involves more algebra; when a number is selected on the page, the number is assigned a variable name, starting with “a”. This name appears both on the page next to the number and in the body of the calculator. An expression is built up in the calculator, and the variable names on the page indicate which number will be substituted for each variable in the expression when the expression is evaluated.

The use of the function tool for calculation in Cinderella immediately made calculation seem more daunting. Numbers could be selected either by clicking on them on the screen to place their names in the calculation box, or by typing in the names. The possibility of dual input means that the user could either see the calculation as just involving numbers or as involving a relationship between variables.

In contrast, GeoGebra required the names of variables be typed in the input bar in order to perform any arithmetic operations, hence demanding the awareness that one variable might be divided by another with no means to build this awareness. The algebra window, with a large amount of distracting information, needed to be open to find the names of the variables and the text input requirement created issues with syntax, made more difficult by the confusion regarding the segment name, which behaved as a variable in the calculation.

Having performed the calculation, it is clear that the result changes as the radius changes: the conjecture concerning a linear relationship was incorrect.

**Task 5: Using the measurements to create a graph of area against radius**

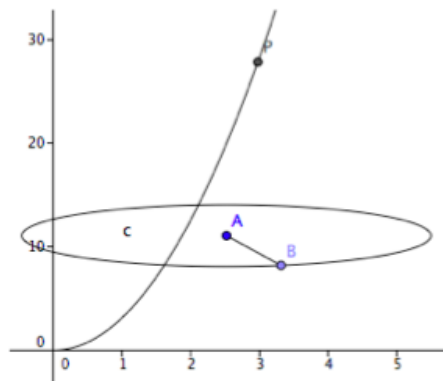
In this task, a new representation of the mathematical situation was created to give further insight on the relationship between the two relevant variables.

One of the most powerful features of IGS is the ability to visually represent the way in which measurements vary: a graph of area against radius may be constructed directly from the existing measurements.

It is possible in all programs to show inbuilt coordinate axes and directly plot the point representing (radius, area), but with Cabri or GSP the basic idea of coordinate representation may be explored. A number can be transferred to a linear object which acts as an axis. For example, the radius measurement of 2.5 cm may be used to create a point which is 2.5 units from a fixed point along a line functioning as the x axis. As the radius changes, this point will move along the axis accordingly. The corresponding area of  $19.4 \text{ cm}^2$  may be represented by a point along another axis. Parallel axes form a dynagraph (Goldenberg, 1992); axes at an angle enable the construction of the point that is reached by travelling 2.5 units along the x line followed by 19.4 units along a line parallel to the y line.

Once the point was plotted, each program enabled it to be traced, to create a visual record of the way in which area varied as radius was changed. The set of all possible points representing (radius, area) could then be obtained by creating a locus, which represented the graph of area against radius. This graph was constructed through an understanding of coordinate representation with no recourse to algebraic equations.

In order to make more of the graph visible, it would be useful to reduce the scale on the y axis. This is unproblematic in Cabri and GSP. However, although zooming in or out is permitted, Cinderella does not allow axes to otherwise be rescaled. Geogebra allows rescaling, but with the consequence that the circle changes shape, as shown below.



**Figure 2: GeoGebra circle with unequal axis scaling**

In Cabri and GSP, a fundamental design decision was to treat the screen as a simulation of a Euclidean plane, where distance is measured by a rigid ruler. Coordinate axes provide a reference frame relative to which objects such as the circle have a location and possibly an algebraic equation. When axes change, the relative location and equation of objects will change, but the objects will not.

In contrast, Cinderella and GeoGebra define all objects by using a coordinate system related to the intrinsic screen coordinate system; selecting the centre point and radius point for the circle define it by an equation referring to this coordinate system. This is why these programs give coordinates and equations in the figure descriptions.

This provides an equally acceptable model of the Euclidean plane – provided that the coordinate axes cannot be scaled independently. Hence Cinderella does not allow such scaling. In GeoGebra, when the axes change, the fixed defining equation means the circle visible on the screen needs to change shape and consequently visible area, which in an investigation likely geared to 12 or 13 year old students might be highly confusing.

**Task 6: Graph an algebraically defined curve to fit the locus and hence find a formula for the relationship between radius and area.**

In this task, a further construction is made to test the conjecture that the graph is quadratic and to find its coefficient.

Cabri is the only program that will find the equation of a locus directly (GeoGebra does not even list the locus as an object in the algebra window). All programs will fit to the locus a graph defined by means of an algebraic equation. Cabri II Plus requires an expression to be defined and applied to an axis, whereas GSP, GeoGebra and Cinderella require the definition and plotting of a function. A parameter  $p$  can be introduced to create the graph of  $y = p x^2$ . Manipulating this parameter will give the curve of best fit as  $y = 3.14 x^2$ .

**Task 7: Test the formula found**

In this task, the specific conjecture represented by the formula found in the previous task is tested.

This was achieved by editing the calculation from task 4 to  $\text{area}/\text{radius}^2$ , which now gave a constant value of about 3.14. It was also possible to create a function or expression  $3.14 * r^2$ , substitute the radius for  $r$  and compare the result with the measured area. Cabri used the simpler language of evaluating an expression, and the other programs used the language of functions, with text input needed for GeoGebra.

The final stage pedagogically would be to prove this result, or at least give some reasons why the area of a circle has this particular relationship to its radius, but this has been beyond the scope of this paper, although not beyond the scope of IGS, which could, for example, be used to compare the area of the circle to that of the square containing it, or to “unfold” the circle into an approximate parallelogram.

**CONCLUSION**

The series of tasks shown here illustrate the ways in which IGS could be used to develop links between number, algebra and geometry through representation of a mathematical situation in different ways. In particular, dynamic number has emerged



as especially important, as predicted by Jackiw (2010). Dynamic number can serve as an introduction to variation, and naming such numbers as an introduction to algebraic variables. Relationships between dynamic numbers may be explored and expressed as graphs or algebraic formulae.

A circle could be created according to its fundamental definition, without numbers, showing the primacy of geometry in this context. However, a circle could also be created by using a number to define its radius. Changing this number and noting its effect on the circle might be important in developing the idea of variation, whereas naming the number might be a move toward the idea of variable.

Measuring the area and radius of the circle involved introducing number as a description of geometry. In an environment where geometric objects may be changed by dragging, measurements are variables between which relationships may be conjectured. Such relationships may be tested by calculation, ostensibly involving just numbers, but in fact involving variables. Calculation itself may be a means of developing awareness of general expressions.

Creating a graph meant creating a visual representation of the relationship between the variables of radius and area. This was done first without algebra by using the basic definition of a graph as a locus (a unique feature of IGS environments), and then by means of an algebraic definition, showing that the relationship between radius and area could be expressed algebraically. The relationship found could be tested algebraically by substituting measurements into a formula. The relationship was not proven, however, although some justification would have been possible with IGS. Although quite different in some respects, Cabri, GSP, and Cinderella were each well suited to the tasks, giving scope for students to explore the idea of variability before requiring the use of specific variables. What was critical in these programs is that text entry involving the names of variables was not necessary. Each had specific perceived strengths and weaknesses, which will be reported in more depth in further research.

What was surprising, however, is the number of problems that arose using GeoGebra, specifically “developed as a tool to support dynamically linked multiple representations of mathematical objects” (Hohenwarter, 2010, personal communication). Jackiw (2010) pointed out a number of the problems in GeoGebra’s algebraic representation, and similar problems were found in this study. GeoGebra had mathematical errors in number (rounding of decimals), algebra (treating a segment name as a variable and assigning it a value) and geometry (not consistently representing the Euclidean plane). In addition, information was continually given which was irrelevant and distracting (the display of coordinates whenever an object was created or dragged). Its reliance on text entry made it slow to use, liable to syntax errors, and meant that students needed to understand the meaning of a variable in order to use it, an understanding that could be developed in the process of using the other programs. Text entry also meant that the algebra

window, with unnecessary and distracting coordinate information, needed to remain open in order to access the names of variables. It lacked the functionality that would enable students to understand coordinate representation from first principles. The “algebraic” representation of objects given by GeoGebra was at best irrelevant in exploring connections in the task described here. At worst it was distracting and misleading.

Future research will compare student responses to similar tasks using Cabri, GSP and Cinderella, testing conjectures made here concerning the role of dynamic numbers linked to geometric objects in facilitating the development of algebraic concepts and looking more closely at the effect of differences between these programs. The programming language available within Cinderella has been beyond the scope of this exploration. This is likely to open new possibilities for the linking of number, algebra and geometry in the context of an IGS.

## REFERENCES

- Butler, D., Jackiw, N., Laborde, J., Lagrange, J., & Yerushalmy, M. (2009). Design for transformative practices. In C. Hoyles & J.-B. Lagrange, (Eds.), *Mathematics education and technology – Rethinking the terrain*. [The 17<sup>th</sup> ICMI study] (pp. 425-438). New York: Springer.
- Falcade, R., Laborde, C., & Mariotti, M. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, 66(3), 317-333.
- Goldenberg, E., Lewis, P., & O’Keefe, J. (1992). Dynamic representation and the development of a process understanding of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: aspects of epistemology and pedagogy*. MAA Notes Number 25. Washington, DC: Mathematical Association of America.
- Hohenwarter, M. et al. (2001, 2010). GeoGebra. Linz, Austria: GeoGebra Inc.
- Jackiw N, (2009). The Geometer’s Sketchpad (Version 5) [computer software]. Berkeley, CA: Key Curriculum Press.
- Jackiw, N. (2010). Linking algebra and geometry: The dynamic geometry perspective. In Z. Usiskin, K. Andersen, & N. Zotto (Eds.), *Future curricular trends in school algebra and geometry: Proceedings of a conference*, 231-241. Charlotte, NC: Information Age Publishing.
- Laborde, C. (2010). Linking geometry and algebra through dynamic and interactive geometry. In Z. Usiskin, K. Andersen, & N. Zotto (Eds.), *Future curricular trends in school algebra and geometry: Proceedings of a conference*, 217-230. Charlotte, NC: Information Age Publishing.
- Laborde, J.-M. et al. (2003). Cabri II Plus. Grenoble, France: Cabrilog.
- Richter-Gebert, J., & Kortenkamp, U. (2007) Cinderella 2. Heidelberg: Springer.

# **INTRODUCTION TO THE PAPERS OF WG 5: STOCHASTIC THINKING**

Dave Pratt, Institute of Education, University of London, UK

and

Carmen Batanero, Universidad de Granada , Spain

Rolf Biehler, Universität Paderborn, Germany

Andreas Eichler, University of Education Freiburg, Germany

Maria Meletiou-Mavrotheris, European University, Cyprus

Efi Paparistodemou, Ministry of Education, Cyprus

## **OVERVIEW**

Our focus on stochastic thinking intentionally avoids any marginalisation of probability, which we see as a key component alongside the consideration of data in statistical analysis. Probabilistic thinking, distinct from thinking about the deterministic, involves modelling randomness and expressing subjective beliefs about uncertainty. Statistical thinking involves handling data, seeking patterns and making predictions. We see stochastic thinking as embracing both probabilistic and statistical thinking.

WG5 was attended by 35 delegates and 24 papers were accepted for the proceedings covering various aspects of stochastic thinking. This large programme stimulated much discussion. Each paper was presented for 7 minutes as a reminder of its focus with time for clarifying questions afterwards. Papers were clustered and after each cluster there was a 10-minute reaction. Following this there was an hour's discussion around that set of papers. Notes were taken during each session and these were used to create the final presentation and this report. We organized the summary of the discussion below into four themes, which roughly represent the clusters of papers:

### **(i) Curriculum and teaching**

The discussion was based around the papers by: Janet Ainley, Tina Jarvis and Frankie McKeon; Arthur Bakker, Monica Wijers and Sanne Akkerman; Maria Meletiou-Mavrotheris and Efi Paparistodemou; Anneke Verschut and Arthur Bakker.

There was also a contribution through the poster by Raquel Santos.

### **(ii) Sampling and graphs**

In this section, we discussed the papers by: Pedro Arteaga and Carmen Batanero; Dani Ben-Zvi, Katie Makar, Arthur Bakker and Keren Aridor; Helen M. Doerr and Bridgette Jacob; M<sup>a</sup> Teresa González Astudillo and Jesús Enrique Pinto Sosa; Oduor Olande.

There were further contributions resulting from the posters by: Adri Dierdorp, Arthur Bakker, Harrie Eijkelhof, Jan van Maanen; Einat Gil and Dani Ben-Zvi.

**(iii) Attitudinal and conceptual**

The discussion in this section was stimulated by papers from: Andreas Eichler and Markus Vogel; Verônica Y. Kataoka, Claudia Borim da Silva, Claudette Vendramini and Irene Cazorla; José Alexandre Martins, Maria Manuel Nascimento and Assumpta Estrada.

A further contribution was made through the poster by Assumpta Estrada and Ana Serradó.

**(iv) Probability and risk (the most popular theme).**

There were a substantial number of papers contributing to this discussion: Chiara Andrà; Egan J. Chernoff; J. Miguel Contreras, Carmen Batanero, Carmen Díaz and José A. Fernandes; M. Pedro Huerta, Fernando Cerdán, M<sup>a</sup> Ángeles Lonjedo and Patricia Edo; Antonio Orta and Ernesto Sánchez; Bernard Parzys and Michel Henry; Dave Pratt, Ralph Levinson, Phillip Kent and Cristina Yogui; Susanne Prediger and Susanne Schnell; Caterina Primi and Francesca Chiesi; Theodosia Prodromou

There was also a poster by Hasan Akyuzlu.

Of course, papers in one theme often raised issues equally relevant to another theme.

**CURRICULUM AND TEACHING**

There was considerable discussion about the role of context in teaching stochastics. In particular, the debate was around what is a ‘real’ problem or question. Was reality located in the positioning of the activity inside or outside of school, or was it related to how engaging was the task for students? Did it matter that the data were collected by students themselves? Could reality be found in the power of the statistical ideas to solve problems? Such expressions of where reality might be found seemed to indicate different communities: school vs. out of school; cultural differences across countries; statistics vs. mathematics; different curriculum layers.

There was therefore discussion about the notion of boundaries between communities that can lead to discontinuities and research that attempted to cross those boundaries. This research looked to build boundary objects (artefacts that live in both sides of the community) and to engage people such as researchers, teachers, teacher educators who tried to live in both communities, on either side of the boundary, and acted as brokers.

**SAMPLING AND GRAPHS**

Sampling was a topic that received particular attention in the discussion of the working group. Innovative technological tools such as *Fathom* and *TinkerPlots* were helping students to visualize complex statistical concepts, such as distribution, variability and sampling distribution.

Such tools were not necessarily familiar to practicing teachers and so it was recognised how important it was to provide teachers with opportunities to experience similar learning experiences as their students in order to be able to anticipate the difficulties their students might encounter.

These tools have changed the way in which students could engage with graphical information, placing more emphasis on interpretation and expression, but, at the same time, present a challenge to how skills might be assessed.

There is considerable research in recent times on informal inferential reasoning (IIR) and this approach, often based around exploratory data analysis (EDA) has put an accent on representing and interpreting data in the context of the sampling process. IIR promises new directions for bridging between statistics and probability, which had become increasingly separated with emphasis on data in EDA.

For example, one of the research studies provided an inquiry based learning environment, in which young students developed useful ideas about whether inferences could be made from samples of different sizes. Initially, they oscillated between absolutist and relativist conclusions, but they came to reason in more sophisticated ways with increasing awareness of what was at stake when making inferences from samples.

### **ATTITUDINAL AND CONCEPTUAL**

Stochastics education suffers from the negative attitudes in society in the same way as mathematics but even mathematically-minded scholars often reject the stochastic way of thinking. We need to know more about these attitudes to see how they might be changed. In contrast to the very limited research on adults' attitudes regarding the stochastic, there has been considerable research about students' concepts. For example, we heard research which aimed to identify the knowledge and thinking of students in different ages and levels of schooling to gain an empirical basis for future teaching of stochastics. This research adopted the SOLO taxonomy to structure the empirical data for different tasks, but not to classify students referring to knowledge and thinking levels.

### **PROBABILITY AND RISK**

In considering the research specifically on the conceptualisation of probability, there was discussion around the role of intuitions, representations and embodiments (research in psychological and semiotic perspectives). Enriching and refining stochastic ideas are good ways of thinking about how learning often involves coming to appreciate the power of what you already know (enriching) or its limitations when the knowledge has been over-generalised (refining). For example, knowledge that chance is unpredictable was typically over-generalised and needed to be refined to appreciate that aggregation in the long term is in a sense predictable. In contrast,

knowledge such as '*the more times, I throw the spinner, the more even is its pie chart*' needed some enrichment to realise that it applies also to dice, coins and so on.

The role of context in the process of enrichment was compared to that reported in the mathematics education literature, where contexts in word problems can present children with additional difficulties, but can provide meaning if the learners are encouraged to engage deeply with the situation. If we provide students with an artificial context, we should not be surprised if they try to make it real by bringing in personal knowledge that is not necessarily statistical. Rather than thinking of abstracting as a process of de-contextualisation, enriching and refining seemed to place emphasis on abstracting as generalising. The knowledge of teachers is key in the conceptualisation of probability by their students. Research was reported which was based on a structural model of teacher knowledge. The insufficiencies of pre-service training leave teachers with difficulties around stochastic thinking. Research has been finding evidence about what teachers do not know. To design effective teacher education, we need to know what we might expect teachers to know already, including their attitudes.

### **SOME CONCLUDING THOUGHTS**

We noted a discrepancy between research results, often taking place in particular situations, and the opportunities for large-scale implementation and dissemination.

We worried that EDA has led to isolation of probability in the curriculum and discussed the need to reconnect data and chance, perhaps through modelling tasks such as in the beta version of Tinkerplots and perhaps by placing more emphasis on subjective probability to escape the specialised uses of probability (in coins, spinners, dice etc.).

The use of subjective probability was apparent in research on risk, presented in the working group. We wondered at what age would this be appropriate? Perhaps it is possible at a younger age than we have been able to teach classical probability? Indeed, could it be the case that the teaching and learning of risk offers new opportunities for modelling with subjective probabilities?

### **FINAL COMMENTS**

The group work was seen as very useful and efficient. The group was larger than in previous years and this created new challenges. The use of a reactor had been introduced in previous years but because of the large number of papers it was felt better to react to a group of papers. Inevitably this sometimes meant the reactions were at a higher level of abstraction and perhaps less specifically useful to individual authors. Nevertheless, there were a number of strong papers and much interesting content. It is hoped that some of these authors will be able to contribute developed papers to a special issue of ZDM.

# DESIGNING PEDAGOGIC OPPORTUNITIES FOR STATISTICAL THINKING WITHIN INQUIRYBASED SCIENCE

Janet Ainley, Tina Jarvis and Frankie McKeon

School of Education, University of Leicester, UK

*We describe aspects of our work on a European project concerned with inquiry-based mathematics and science education. A framework has been developed which offers a novel perspective on integrated planning, based on deep connections between 'big ideas', and on the process of inquiry. We aim to exploit the potential of such integration to provide opportunities for mathematical and statistical ideas to be used in purposeful ways, stressing their utility. A sequence of activities embodying a possible learning trajectory concerning variability is described as an example of the application of the framework.*

## BACKGROUND: THE FIBONACCI PROJECT

This paper presents a theoretical framework developed during planning for our participation in the Fibonacci Project<sup>15</sup>; a project involving around 36 European partners that focuses on the dissemination of Inquiry-based Mathematics and Science Education. Each of the partner Centres will work with a group of teachers over a period of two years, developing materials to support inquiry-based teaching. An important feature of Fibonacci is a model for dissemination through 'twinning' Centres with different levels of experience and expertise, and other collaborative events. The project thus focuses at three levels, on Centre, teacher and pupil learning.

Centres choose whether to focus on mathematics or science, or on both subjects, and also decide the age group of the pupils involved. In Leicester we have chosen to focus on the primary age range (ages 4 – 11), and to take an integrated approach to inquiry across mathematics and science. We shall be working with a group of about 25 teachers who teach children from across the whole age range. Half of these have been selected because they have previously engaged in mathematics education or science education programmes at the School of Education, and they will each involve another colleague from their school. As primary teachers, they are unlikely to regard themselves as 'specialists' in mathematics or science, and few will have formal qualifications in either subject beyond the minimum qualification levels required to enter teaching. Their level of experience as teachers varies from some with substantial experience, to some within the first 5 years of their careers.

Starting in Autumn 2010, the teachers will come to five workshop sessions at the School of Education each year (two whole days and three shorter twilight sessions). In these sessions, the teachers will work through carefully designed sequences of

---

<sup>15</sup> The FIBONACCI Project - Large scale dissemination of inquiry based science and mathematics education, FP7-SCIENCE-IN-SOCIETY-2009-1, Grant Agreement Number 244684

activities that model the inquiry process. They will be encouraged to select appropriate activities to use in their own classrooms between the University sessions.

## **DEVELOPING A PLANNING FRAMEWORK**

As a team of teacher educators and researchers in mathematics (Ainley) and science (Jarvis and McKeon) education, our focus during the development stage of the project has been on developing a cohesive framework around which to plan the sequences of activities which we will present to the teachers. In particular, we have explored the similarities and differences in our two disciplines, what ‘inquiry’ might mean in each of them, and how the strands of content from each area might be brought together in meaningful ways. In these explorations, *handling data* quickly emerged as an area of relevance to both disciplines. We focus here on the place of statistical thinking in our planning, as an example of our overall approach.

### **Challenges and opportunities in linking mathematics and science**

As well as choosing to focus on both mathematics and science, we have also set ourselves the challenge of linking the disciplines in an integrated way which has the potential to improve and enrich learning opportunities. We recognise that this is potentially problematic, and will certainly be a way of working which is relatively unfamiliar to the project teachers. Further, we wish to avoid a superficial approach to making such links, for example by identifying where mathematical ideas might be taught ‘in preparation’ for where they are needed in science, or drawing attention to the Fibonacci patterns in sunflowers. Whilst these sorts of links might be relatively easy to make, we believe that there are potential advantages in integrating mathematics and science in a deeper way. We see at least three such advantages.

First, there is a widely recognised concern within mathematics education about the separation between ‘school mathematics’ and the ways in which mathematics is used in everyday life, which is not adequately addressed by traditional attempts to contextualise the school mathematics curriculum (e.g. Boaler, 1993, Cooper & Dunne, 2000). Ainley and Pratt have argued that this separation inhibits the development of an understanding of the *utility* of mathematical ideas, that is, why and how the ideas are useful (Ainley et al., 2006). In order to create opportunities to develop an understanding of *utility*, they emphasise the importance of pedagogic tasks that have *purpose* for learners. Scientific inquiry is a rich source of purposeful tasks in which the *utility* of mathematical ideas can be made transparent.

Secondly, mathematical ideas play an important role in the explanatory power of models in science, which is not generally exploited in the primary school curriculum. For example, in order to understand why a small child might get cold more quickly than an adult, and thus need more layers of clothing, or why large sugar crystals dissolve more slowly than regular sugar, it is important to have an understanding of how the ratio of volume to surface area might vary in different shapes.



Finally, scientific inquiry provides a purposeful context in which learners collect sets of data of various kinds. This contrasts with traditional approaches to data handling within primary school mathematics (in the UK), which tend to fall into two categories. Either ready-made sets of (clean) data are provided for children, with little consideration given to how or why the data might have been collected, or children collect data about a familiar context (such as the colours of cars in the car park), and use this to produce graphs, but without any real purpose to stimulate more than a superficial reading of the graphs produced (Pratt, 1995). In the course of a scientific experiment, decisions have to be made about which data to collect, how best to collect it, the most appropriate way to display it, and how to interpret the resulting graphs and tables. In addition, the data collected will be messy: issues about appropriate accuracy, variability, and sample sizes are both visible and of real importance. We thus see statistical reasoning, which is generally poorly developed in UK primary schools, as a key area of mathematics that can be developed through our integrated approach to mathematical and scientific inquiry in the Fibonacci Project.

### **Approaches to linking mathematics and science**

The notion of a cross-curricular approach to teaching mathematics and science may conjure up a range of different images. Czerniak et al. (1999), in a review of studies that attempted to demonstrate benefits of curriculum integration, argue that a serious impediment to such research is that

... a common definition of integration does not exist that can be used as a basis for designing, carrying out and interpreting results (Czerniak et al. 1999 p. 422)

In another review of research, Hurley (2001) identifies five categories of 'integration' reflected in a wide range of studies. These categories move from a 'sequenced' approach, in which the two subjects are planned and taught separately, but in an appropriate sequence to provide cross-subject support, to 'total' integration in which the two subjects are taught 'together in intended equality'. However, the definitions of the intervening categories are somewhat unclear.

Offer and Vasquez-Mireles (2009) offer two more specific descriptions of how cross-subject planning may be approached. They take a somewhat dismissive view of what they refer to as a 'traditional approach to integration' in which mathematics is used as a tool in science lessons, and science is used to provide a context in mathematics lessons. In their own study, Offer and Vasquez-Mireles take an approach which builds on this pattern of integration, which they describe as *correlation*.

To be a correlated mathematics and science lesson, both mathematics and science learning objectives must be clearly identified and direct the instruction (Offer and Vasquez-Mireles, 2009 p. 146)

In addition to this focus on learning objectives, correlation involves attention to the use of language, and to parallel concepts in the two disciplines.

## **MAKING DEEP LINKS THROUGH INQUIRY**

The perspectives described above offer pragmatic approaches to planning lessons, or sequences of lessons, which go some way to exploiting the opportunities offered by linking mathematics and science. Our explorations have led us to look at ways in which links of a more fundamental kind may be made by looking beyond the detail of curriculum content, and unpacking how the processes of inquiry play out in the different subject areas. The way in which we conducted our exploration was for the scientists to prepare an activity sequence which they would use to develop a particular concept in science (such as insulation and heat loss) and for the mathematician to work through this, drawing out the opportunities for developing mathematical understanding. In doing this, our aim was not only to find opportunities to use particular mathematical ideas (such as the use of place value in the scales on measuring instruments), but also to identify opportunities for mathematical thinking to offer explanatory power (such as considering the ratios of volume to surface area.)

After going through this process for a number of scientific themes, the pattern emerged of a relatively small number of ‘big’ mathematical ideas underpinning very different areas of science. One such ‘big idea’, which we take as our focus here, is that of the variability of data, including ideas about ‘signal’ and ‘noise’.

In contrast to approaches which take the content of the school curriculum as the starting point for linking the learning and teaching of mathematics and science, we have developed a model for considering links at the deeper level of ‘big ideas’, based on a consideration of the function and processes of inquiry. Because of the significant overlap between mathematics and science in the area of handling data, we have considered statistics as a distinct discipline in this model. In Table 1 we summarise our planning framework, indicating the different levels at which we need to consider the links between mathematical, statistical and scientific ideas. Whilst our planning begins with consideration of the ‘Deeper level’ of big ideas, which informs progression, we are conscious of the need to make explicit links at the ‘Surface level’ of the school curriculum which is familiar to teachers. The design of individual tasks is built upon the bottom two rows in the table, utilising the processes of inquiry, and drawing on differing purposes to determine the specific emphasis within each task.

## **EXPLORING THE USE OF DATA IN SCIENCE EDUCATION**

In scientific inquiry, children are encouraged to design and carry out experiments and to test hypotheses with the aim of finding an explanation for the observed phenomena. The need to compare the outcomes of different experiments drives the move beyond descriptive observation to measurement and quantification. The choice and use of units and measuring instruments is often ‘taken for granted’ in science teaching, and the ease with which particular instruments can be used may disguise

## Working Group 5

the complexity involved in interpreting the results. For example, digital stopwatches are accessible timing devices, but will give readings in tenths, hundredths or even thousandths of a second. The emphasis placed in science on fair testing and accuracy may engender an inappropriate respect for results given in 2 or 3 places of decimals (particularly when the meaning of such results is not well understood).

	Science	Statistics	Mathematics	
<b>Surface level: the School curriculum</b>	Electricity Forces Properties of materials Plants ...	Drawing graphs Collecting data Mean, median and mode ...	Counting Calculation Naming shapes Measurement ...	<i>Links at this level are easy to make, but may be superficial, using mathematical or statistical ideas intermittently as tools during science activities without continuity. However, teachers need the reassurance of seeing links at this level.</i>
<b>Deeper level: 'big ideas'</b>	Energy Particle theory Inheritance ...	Variability Distribution Chance Inference ...	Pattern Proportion Equivalence Ratio...	<i>Links at this level are more challenging, but offer opportunities to develop sequential learning in all subjects as the big ideas recur.</i>
<b>Nature and processes of inquiry</b>	Observation, posing questions, collecting data, analysing data to draw conclusions, predicting, hypothesising, evaluating modelling, raising further questions, ...			<i>We see strong similarities in the process of inquiry across all three disciplines, which can be reinforced with different emphases.</i>
<b>Purposes of Inquiry</b>	Generalisation that explains the world, or observed phenomena	Generalisation that informs decision making	Generalisation that holds in all cases and is internally consistent	<i>Whilst the processes of inquiry are common, we see the purposes as distinct: this provides a way of creating different emphases in task design.</i>

**Table 1:** summary of the model of integration used for planning

Such experiments provide potential opportunities to experience *variability*: a key statistical idea. Following Reading and Shaughnessy's (2004) terminology, we shall focus here on *variability* as 'the [varying] characteristic of the entity that is varying', rather than the process of describing or measuring *variation*.

In the context of a practical experiment, variability can be observed at two levels. At the macro level, the whole focus of the experiment may be on the behaviour of one (dependent) variable as another variable is deliberately changed by the experimenter. For example, children might change the gradient of a sloping surface, and measure how far a toy car travels when rolled down the slope. Here children may expect that the distance travelled will vary as the gradient is increased or decreased, and look for a pattern in their results. However, variability can also be experienced at a micro level, and this may initially be unexpected. If a particular experiment is repeated, that

is, the car is rolled again down a slope with the same gradient, the distance travelled will not be exactly the same as the first attempt. Repeating the experiment a number of times will produce a spread of results. Rather than seeing this as an opportunity to explore variability as an important statistical idea, the pressures of the curriculum tend to lead science teachers to take a pragmatic approach to the ‘problem’ of variability, using the algorithm ‘collect three measurements and then take the average (i.e. mean) value’. Observations of student teachers carrying out scientific inquiries have provided us with many examples of the uncritical use of this approach, presumably remembered from their own school experiences. In the following sections we describe a sequence of activities designed to ‘unpack’ this issue.

## PEDAGOGIC DESIGN FOR EXPLORING VARIABILITY

We now describe a learning trajectory through a sequence of inquiry activities we have designed to provide opportunities to learn about variability in the context of scientific ideas. The activities all focus on aspects of exploring flight. Our general approach has been to start with inquiry sequences which focus on the development of science concepts, and then weave across these sequences which develop the related mathematical concepts, whilst reinforcing the processes of inquiry. Any particular task or activity could then be viewed through a number of different lenses, with different aspects being emphasised as appropriate for the learners (e.g. scientific or mathematical content, aspects of inquiry, statistical thinking). The sequence of activities described here will be used with the group of project teachers, who will then try activities in their classrooms. We aim to make our planning decisions explicit, in order to support teachers’ in developing classroom activities.

### Cylinder gliders: explicit variability

The first task is based on exploring the behaviour of small gliders, made from loops of paper attached to a drinking straw (see Figure 1). The construction of the glider allows the smaller (tail) loop to be moved along the straw. In the way we structure the activity in response to this task we will focus on *making the process of inquiry visible* as a way to explore a phenomenon, on *macro-level variability* as part of that exploration, and on explicitly confronting *micro-level variability*.

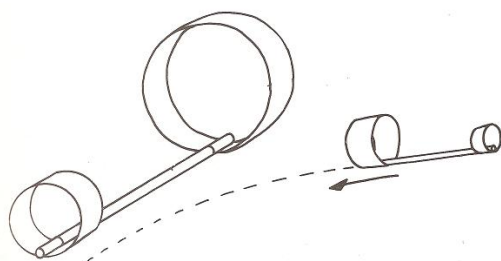


Figure 1: a loop glider

<b>Things I could change</b> Size of loops Width of loops Distance between loops Length of straw Where I fly it How I launch it <b>Things I could observe or measure</b> Length of flight Time of flight Straightness of flight	<b>I will change</b> Distance between loops  <b>I will measure</b> Length of flight  <b>I will keep these the same</b> Size of loops Width of loops Length of straw Where I fly it How I launch it
---	---

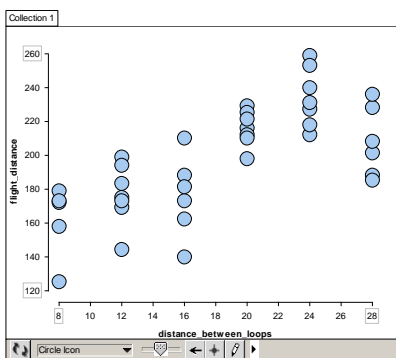
Figure 2: the discussion framework

## Working Group 5

After an opportunity to observe the behaviour of the gliders in an unstructured way, a question is posed about what affects the way in which the gliders fly. A discussion framework (Figure 2) is used to identify variables, and then design experiments to explore the affects of changes to the design (*inquiry process, macro-level*). In the example given, the focus is on the distance between the larger and smaller loops.

The variable *how I launch it* is problematic, and is likely to generate considerable discussion. It will quickly become clear that it is not possible to launch the glider in exactly the same way each time, however careful you are. This generates the need both to repeat the experiment a number of times, and to address how the resulting measurements may be interpreted. Different coloured pins are used for each distance between the two loops and placed on the floor to mark where the glider lands. This gives an immediate visual image of the variability of results (at both micro and macro level), which can be discussed. Guided questioning will focus attention on both the ‘noise’ in the array of pins, and the ‘signal’ that might be detected. However, the floor display is messy, and cannot be left in place for long, and so results are also recorded in Tinkerplots. The resulting display provides a different image of variability (see Figure 3) and may make the pattern in the results easier to see, making the utility of graphing explicit.

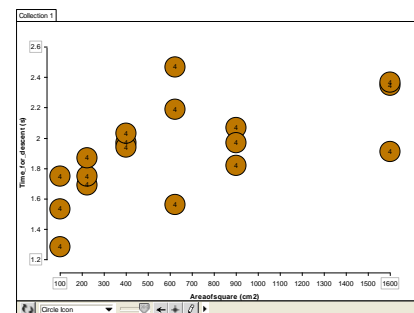
*Scientific content* will not be a focus in this task as it is not easily accessible: the emphasis needed on the *mathematical content* will vary for different age groups.



**Figure 3:** results for different gliders



**Figure 4:** parachutes being made



emphasised. As the parachutes are best dropped from a height of at least 3-4 metres, such as down a stairwell, repeats tests are time consuming, and so there is a need to consider an appropriate sample size. In the experiment recorded in the graph shown in Fig. 5, square cloth parachutes of different sizes have been used. Each has been tested 3 times, from a height of 4 metres. Note that the size of each parachute has been recorded by area, rather than by the side length: there is rich potential for mathematical discussion in this decision.

Discussion of the results, such as those shown in Fig. 5, will focus on understanding and explaining what the graph is telling us. Discussion of the signal in the results will link to *scientific* explanations concerning the effects of air resistance, fabric flexibility etc., supported by *mathematical* ideas about proportion. In discussion of the noise, emphasis might be given to a number of *statistical* ideas. One focus could be possible reasons for the *micro-level* variability, and how the experimental procedure might be improved. Another discussion might address whether three repetitions gave a big enough sample to provide reliable evidence. Through such discussions we would want to emphasise the range and spread of values, and how the signal emerges around values towards the centre of the data set for each parachute, introducing the utility of larger samples, and of a focus on the data set, rather than individual data points.

### **An Active Graphing approach: the utility of a measure of centre**

In the third task in the sequence, we make a change to the way in which the inquiry is carried out, shifting the focus to developing generalisation to inform decision making. The setting is again in the context of flight, this time using small paper spinners (see Fig. 6). Attention could be focussed on scientific or mathematical ideas, but our choice is to structure activity to emphasise the purposeful use of average values from experimental results in a way which makes the utility visible, rather than as a black-box which can be used to ‘solve the problem’ of micro-level variability.

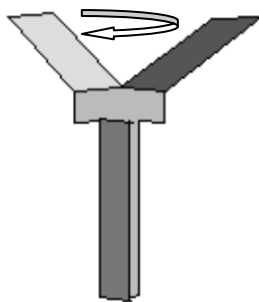


Figure 6: a spinner

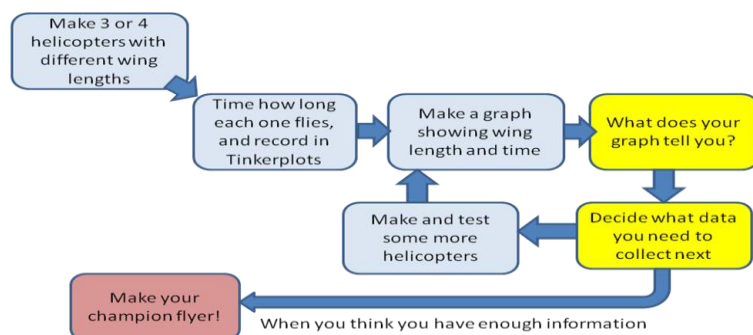
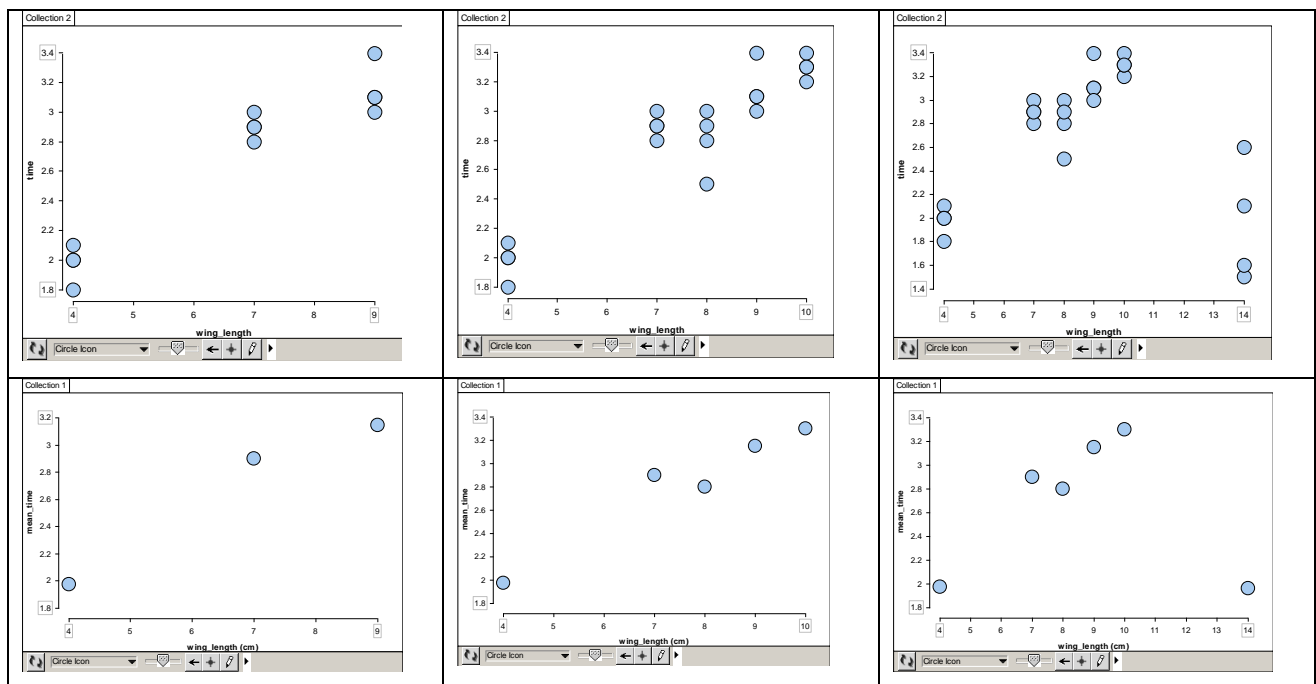


Figure 7: the active graphing approach

The task is to make a champion flyer: that is, a spinner that will fly for the longest time. Following a similar introduction to the identification of variables as that used in the previous two activities, a decision is made about which feature of the spinner

to vary (e.g. wing length). In contrast to the two previous tasks, where the whole experiment was carried out before the data was displayed on a graph, we will now use an Active Graphing approach (as shown in Fig. 7) in which a graph is produced as soon as two or three pieces of data have been collected, and then used to support decisions about further experiments and data collection (Ainley, Nardi, Pratt, 2000). Active Graphing focuses attention on the utility of graphs as analytic tools: the challenge to make a champion flyer provides a purposeful context for this.

By now the idea of repeating each test in order to obtain a set of data is well established. The need to make decisions about the optimal wing length emphasises the need for accuracy, especially as the overall pattern of results may be somewhat unexpected. In the sequence of snapshots shown in Fig. 8 there is an attempt to ‘home in’ on the best wing length. From the spread of results (top row) it is not clear whether 10cm wings perform better than 9cm ones. This provides an opportunity to introduce the *utility of using the mean* of a number of results, which produces a clearer graph (bottom row). The third pair of graphs shows that 14cm wings are not an improvement, and so the next step will be to try a wing length between 10 and 14.



**Figure 8:** three stages of activity, using full data (top row), and mean values (bottom row)

## DISCUSSION

In this sequence of tasks, and the associated activities, we have aimed to offer a learning trajectory that exploits the rich opportunities of an inquiry-based integration of mathematics and science education. There is an overt focus on variability at the macro level, as the effect that changing one (independent) variable has on another (dependent) variable relating to the flight of the glider, parachute or spinner, is measured and recorded. This relates to ‘Surface level’ curriculum content concerning measurement and fair testing. Within the experimental situations, we focus at the



level of ‘big ideas’, creating opportunities to confront and accommodate variability at the micro level, by recording and displaying the distribution of results and trying to identify patterns within the noise of a spread of results. The three tasks progressively move towards the need to see the distribution of results as an aggregate, which could be represented by its mean, rather than as a collection of individual points, which Makar and Confrey (2005) see as a key goal in learning about distributions. This provides a clear rationale for the need to repeat experimental measurements, and the expediency of working with the mean value of an appropriate sample of results. Unlike approaches to teaching about variation and distribution which may be offered in mathematics lessons, the data involved is meaningful and used in purposeful ways.

## REFERENCES

- Ainley, J., Nardi, E., Pratt, D. (2000). The construction of meanings for trend in Active Graphing. *International Journal of Computers for Mathematical Learning*, 5, 85-114. <http://www.springerlink.com/content/1382-3892/5/2/>
- Ainley J., Pratt, D., Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, 32, 23-38. DOI: 10.1080/01411920500401971
- Boaler, J. (1993). The role of contexts in the mathematics classroom. *For the Learning of Mathematics*, 13(2), 10-17.
- Cooper, C., Dunne, M. (2000). *Assessing children’s mathematical knowledge*. Buckingham: Open University Press.
- Czerniak, C.M., Weber, W.B., Sandmann, A., Ahern, J. (1999). A literature review of science and mathematics integration. *School Science and Mathematics*, 99, 421-430. <http://www.ssma.org/school-science-and-mathematics>
- Hurley, M. (2001). Reviewing integrated Science and Mathematics: the search for evidence and definitions from new perspectives. *School Science and Mathematics*, 101, 259-268. <http://www.ssma.org/school-science-and-mathematics>
- Makar, K., Confrey, J. (2005). “Variation –talk”: articulating meaning in statistics. *Statistics Education Research Journal* 4(1), 27-54. <http://www.stat.auckland.ac.nz/setj>
- Offer, J., Vasquez-Mireles, S. (2009). Mix it up: teachers’ beliefs on mixing mathematics and science. *School Science and Mathematics*, 109, 146-152. <http://www.ssma.org/school-science-and-mathematics>
- Pratt, D. (1995). Young children’s active and passive graphing. *Journal of Computer Assisted Learning*, 11, 157–169.
- Reading, C. & Shaughnessy, M. (2004). Reasoning about Variation. In D. Ben-Zvi & Garfield, J. (Eds.), *The Challenge of Developing Statistical Literacy, Reasoning, and Thinking* (201-226). Dordrecht: Kluwer Academic Publishers.



# **PRESERVICE PRIMARY SCHOOL TEACHERS' INTUITIVE USE OF REPRESENTATIONS IN UNCERTAIN SITUATIONS**

Chiara Andrà

University of Torino, Italy

*This paper focuses on primary school teachers' intuitive approaches to uncertain circumstances. In an exploratory study, data from a basic undergraduate course in probability for preservice primary school teachers were gathered and examined through semiotic lenses. A cognitive model is presented and discussed and an example is provided.*

**Key-words:** elementary probability, preservice teachers, intuition, semiotics.

It is well acknowledged that human beings have to often deal with uncertainty and they may judge, decide and behave under the sole guidance of their intuitions. Along with their intuitive reasoning, human beings may somehow represent the situation in their mind. The relationship between the representations people use and their intuitive reaction in uncertain situations is the focus of this study. Within this perspective, teachers are in charge of planning, carrying out and evaluating classroom activities. Hence, the analysis of their intuitive approaches to probability when using representations – during their training – is a relevant issue.

## **THE COGNITIVE MODEL AND ITS THEORETICAL BACKGROUND**

### **Intuitions and semiotic representations**

Intuitions in the process of learning probability have been a fruitful branch of research in Psychology and in Mathematics Education since 1970s (see for example Kahneman and Tversky, 1972; Fischbein, 1975). In the 1980s, a rich literature in Mathematics Education concentrated on (children's) probabilistic intuitions (see for example Fischbein & Gazit, 1984; Hawkins & Kapadia, 1984). Later a number of researches focused on this topic: for example, Konold *et al.* (1993) studied students' inconsistent reasoning about different aspects of the same situation. This kind of research addresses students' approaches to uncertain situations, and consequently helps to frame and analyse students' misconceptions and intuitions that can arise and be driven by such different approaches. In the subsequent decades, the discussion in the field shifted to the notion of *statistical thinking* (for a review, see Borovcnik, 2005). Taking into account both probabilistic and statistical reasoning led researchers to coin the idea of *stochastic thinking* that combines both probability and statistics. Recently, researchers have not only focused on the theoretical issues of stochastic thinking, but also on teachers' professional development, on their attitudes towards probability and statistics, on students' understanding of basic concepts (Eichler, Ottaviani, Wozniak, Pratt, 2010). In this varied scenario, the research presented in my paper addresses both intuitions and the semiotic resources

prospective teachers activate when dealing with uncertain situations. The term ‘intuition’ is used in opposition to logical knowledge, according to Fischbein:

“the process of thinking is composed of two basic interwoven aspects: the logical, analytical, discursive one (the evolution of which, in children and adolescents, has been studied by Piaget and his co-workers), and intuitive cognitions characterized by self-evidence, immediacy, globality, coerciveness.” (Fischbein, 1998, p.1)

Now, a few words need to be said about semiotic approaches. Semiotics is a powerful tool for interpreting didactical phenomena. For lack of space, I will refer solely to an example: the work of Batanero, Arteaga and Ruiz (2010) that generalizes the notion of representation by taking into account an ontology of objects that intervene in mathematical practices and analyze the graphs produced by prospective teachers through a semiotic lens. In their work, semiotic complexity is addressed. In this paper, the way teachers use representations is analyzed from Duval’s perspective (Duval, 2008): it identifies mathematical thinking and learning with the *coordination* of a semiotic system according to the following operations: *treatment* (transforming a representation into another within the same semiotic system), and *conversion* (transforming a representation into another in another semiotic system). My contention is that the coordination of different semiotic systems support/constrain intuitions in probabilistic reasoning. Depending also on the task, the semiotic resources employed, and the background of the individual, I have singled out three levels to which people can refer when estimating the probability of an event.

### **Three levels to estimate probability: experience, discrete quantities, theory**

The first level is the *level of experience*. At this level dice are rolled, playing cards are randomly selected, coins are flipped, etcetera. And at this level people win or lose. There is a huge literature in Mathematics Education focusing on and discussing the role of the context in mathematics tasks, the relationships between experience and beliefs, the goal of mathematics to prepare for citizenship, and so on (van der Kooij, 2010). Using this background, which is still the focus of several studies, I take into account experience and its role in probability learning processes for two main reasons. The first reason is that probability – as a modeling activity and more than other mathematics domains such as algebra – has a special relationship with experience. Outcomes in probability, in fact, often refer to real objects. Moreover, the frequency of such outcomes has a relationship with the estimate of the probability of an event (Piattelli-Palmarini, 1995). The other reason is that – just for its special relationship with the world of experience – probability and its learning result strictly intertwined with intuitions and misconceptions (Fischbein & Gazit, 1984).

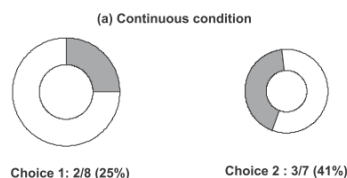
At the experience level, people have to estimate the probability of the outcomes. Hence, people start to somehow represent in their mind the situation: the representations are likely to be perceptive and approximate, and they can serve the

purpose of counting the number of possible and favorable events. Sometimes people (and, above all, teachers when teaching) represent the events using less perceptive representations such as arrows, Venn diagrams, tables, etcetera. When using representations that help counting, people are no longer at the level of experience, but stay at the second level: the *level of arithmetic thinking*. This is the level at which people shift from perception and approximation to quantities. Such quantities, at this level, may be integer numbers, and become ratios or percentages.

There is a third level, the *level of theory*. At this level lie the formulas, the axioms and the theorems of probability. The representations are formal and abstract. However, in some tasks there is no need to (know and) stay at this level, but in order to correctly solve the task a proper use of representations at the arithmetic level is enough. Outside the level of theory, in fact, it is possible to introduce probabilistic tools in a more immediate way. At each one of the three levels, in fact, it is possible to accomplish an estimate of uncertain situations, following different approaches. As a consequence, the three levels should not be regarded as in a certain hierarchy, but as different modalities for approaching and representing uncertain situations. This approach differs from the model by Pratt (2003), who “sees the phenomenalisation of randomness as providing an intuitive route towards the operationalising of chance to be taken before following a more conventional route”. The main difference is that my model does not predict an evolution, for the individual, from a *naïve* world to a theoretical one, but it concentrates on the relationship between intuitions and representations (as meaning-making activity) at each one of the three levels.

### Different approaches to uncertain situations

At the level of experience, estimating the probability of an event means avoiding the use of quantities. Semiotically speaking, it entails the use of mainly graphic and pictorial registers, related to perceptive and sensorial experience. I claim that people in general – and young children in particular – are able to say that an event is nearly impossible or almost certain, without involving any sort of computation. A confirmation of my claim has been given by Jeong, Levine & Huttenlocher (2007), who examined whether children are able to reason about proportions in the context of continuous amounts before they are able to do so in the context of discrete sets.



**Figure 1: one of Jeong, Levine and Huttenlocher (2007) tasks: donuts. Which donut has the greater percentage of its area in grey?**

The tasks they proposed used a graphic register. Figure 1 shows an example: children had not to use any sort of number, but were able to say which figure had the

largest area. Jeong's and his colleagues' interpretation of children's better performance in the continuous than in the discrete conditions was that

“It seems likely that they used a perceptual strategy similar to that used by infants and young children to code the extent of one length or region in relation to another.” (Jeong, Levine & Huttenlocher, 2007, p. 252)

In order to characterize it, two keywords related to the approach at the level of experience may be: ‘perception’ and ‘approximation’. Moreover, it is characterized by self-evidence, immediacy, globality, and coerciveness, entailed by graphical/pictorial representations. The experience level can be considered intuitive, since such characteristics – accomplished by graphical/pictorial representations – are the features that define intuitions mentioned above, according to Fischbein (1998).

The concept of proportion is, according to Fischbein (1998), another form of intuitive thinking. I claim that this kind of approach belongs to the level of arithmetic, since it is necessary to overcome the level of experience and quantify the situation to deal with it. In this case, immediacy and globality are characterized by internalized operational schemas, conveyed by arithmetical representations. At the level of arithmetic, moreover, I distinguish two approaches: one is more related to the experience level and it consists in using only percentages and proportions; the other one is linked to the level of theory and it involves the use of ratios. For a better framing of the ‘percentage approach’, I refer also to Huerta & Lonjedo (2007), who describe the *mostly arithmetical type of thinking* as

“students think in quantities but they recognize events and their associated frequencies or percentages.” (Huerta & Lonjedo, 2007, p. 735)

Hence, this level is characterized by arithmetic representations and proportional schemas: on the one hand frequencies/percentages, on the other one ratios/fractions.

The approach at the level of theory regards probability as a function that assigns a number between 0 and 1 to an event. It overcomes operational thinking and, through symbolic registers, it is characterized by functional and relational reasoning. At this level, intuition is the result of specific mathematical training. It allows students to deal with more complex situations: for example, Lecoutre and Durand (1988) showed that solely using ratios may induce misconceptions in estimating the probability of composite events. Other examples in the literature concern misconceptions in Bayesian thinking (for example Falk, Falk & Levin, 1980).

Research focusing on students' learning processes in terms of intuitions belongs to *didactics B* (didB), defined by D'Amore (2006) as an epistemology of learning. According to D'Amore, *Didactics A* (didA) concentrates on effective teaching environments (assuming implicitly cognitive transfer). But the couple *teaching environments-students* does not completely describe the learning process: the teacher, together with his role, his training and his beliefs, constitutes the third component. Research in *didactics C* (didC) (*ibidem*), grounded in Shulman's work

(1986), focuses on the teacher. It is known, in fact, that for example teachers' beliefs determine and influence knowledge (didA) and the learning process (didB). Moreover, the relationship between the resources/tools teachers choose and their epistemology belongs to the teacher's sphere: why did the teacher use that tool? According to which model did he operate such choice? Following this perspective, a more complex analysis may focus on how teachers' education and training influence the way they teach and how students learn (probability) in primary schools. This question that informs the research presented in this paper lies at the boundary of didB and didC.

The issue of teachers' training in probability and statistics is central in several studies. The work by Batanero, Arteaga, and Ruiz (2010) is interesting, since it focuses not only on teachers' education, but also on teachers' graphical competence in statistics from a semiotic perspective. Batanero, Arteaga, and Ruiz defined three levels of semiotic complexity, but (unlike my model) these levels inform a hierarchy.

### **Relations with the existing literature**

I have shown the three levels at which judging under uncertainty may occur. With respect to the aforementioned approaches, Hawkins and Kapadia (1984) distinguished four definitions for probability, and labeled them as: (1) 'a priori', (2) frequentist, (3) subjective and intuitive, and (4) formal probabilities.

The level of experience is similar to the definition provided by Hawkins and Kapadia for subjective probability: "subjective probability may rely merely on comparisons of perceived likelihood" (Hawkins & Kapadia, 1984, p.350). In fact, they also refer to the perceptive nature of this kind of thinking. It should be noticed that Hawkins' and Kapadia's (1984) definition of subjective probability does not correspond to (but has some connection with) the one provided by de Finetti (1974) as the ratio between the amount of money a person is willing to bet on the outcome of a certain event, and the amount of money he will receive in case of win, if he is willing to change his place with that of any other involved in the game. In their paper, Hawkins & Kapadia (1984) related the subjective probability not only to perception, but also to coherence. The term 'coherence' highlights that any personal judgment should be made in accordance to some 'axioms'.

Formal probability (*ibidem*) informs and characterizes the level of theory.

Some features of the 'a priori' probability (*ibidem*) lead to relate it to the arithmetic approach with ratios at the level of discreteness, and the frequentist probability has something to do with the use of percentages. On the one hand, in fact, the assumption of equal likelihood in the sample space that characterizes the 'a priori' probability may lead to the same misconceptions that arise when the classical definition of probability, as the ratio between successful and possible cases, is used (see also Lecoutre & Durand, 1988). On the other hand, the frequentist probability is obtained from observed relative frequencies of different outcomes in repeated trials

and entails the use of percentages. As the approach at the level of discreteness, the frequentist probability is still linked to the experience, but is an attempt of coming out from the real world alone.

The main differences between the model introduced in this paper and Hawkins' and Kapadia's (1984) work are two: the first difference is that the latter does not take into account any semiotic approach, the second one is that the three levels of the former depend on the kind of relations between intuitions and representations, regardless the definitions of probability.

## METHODOLOGY

During the academic year 2009/10 at the University of Torino (Italy), data from two undergraduate one-semester 30-hours basic courses in probability were gathered and are being examined. Data consist in written answers to open-ended exercises, which were administered to students all along the semester. 500 undergraduate students were involved: respectively, 150 students in Mathematics ( $M$ ), and 350 for a master course for Primary school teachers ( $P$ ). The content of the two courses differed, but in this paper data from a series of exercises that were administered at the beginning of the course are analyzed. Students' mathematical background of the two groups differed: while the  $P$  students attended one basic mathematics undergraduate course, the  $M$  students were attending the second year in mathematics undergraduate course. However, the probability backgrounds were similar in the two groups: both of them did not receive any prior teaching in probability. In Italy, probability is neither taught at lower levels of education, nor in secondary schools. I focus my analysis on the  $P$  group, while the  $M$  group is referred to for comparisons.


The example below helps illustrate the cognitive model and its applications. It is taken from a series of exercises that were given at the beginning of the undergraduate course in probability.

A company employs 75 men ( $M$ ) and 25 women ( $F$ ). 12% of  $M$  and 20% of  $F$  work in the accounts department (a.d.). Compute the probability that, drawing the surname of an employee in the a.d. it would be the surname of a  $M$ .

In the exercise it is necessary to deal with conditional probability. There is a huge literature concerning conditional probability in Mathematics Education. The reader is referred to Huerta & Lonjedo (2007). In their paper, the authors consider three versions (one with percentages, one with probabilities, and one with integer numbers) of a problem, and they show that it is possible to recognize probabilistic or arithmetic thinking when data is expressed in terms of probabilities or percentages respectively. It is well known that dealing with conditional probability is not immediate for students and needs the use of complex semiotic transformations and of possibly counter-intuitive solving strategies.

## DATA ANALYSIS

Table 1 shows the solutions provided by three *P* students for the a.d. exercise.

<p>45 uomini <math>\rightarrow</math> 12% di 75 <math>\rightarrow</math> 9  75 men <math>\rightarrow</math> 12% of 75 <math>\rightarrow</math> 9  25 donne <math>\rightarrow</math> 20% di 25 <math>\rightarrow</math> 5  25 women <math>\rightarrow</math> 20% of 25 <math>\rightarrow</math> 5</p> <p>9 + 5 = 14 (dipendenti reparto contabilità)  9+5=14 (employees in the accounts department)</p> <p><math>\frac{9}{14} \rightarrow</math> probabilità di essere uomo  <math>\frac{9}{14} \rightarrow</math> probability of being a man</p> <p><math>\frac{5}{14} \rightarrow</math> probabilità di essere donna  <math>\frac{5}{14} \rightarrow</math> probability of being a woman</p>		<p>Uomini men  Donne women</p> <p>CONTABILITÀ a.d.  NON CONTABILITÀ Not a.d.</p> <p>CONTABILITÀ a.d.  NON CONTABILITÀ Not a.d.</p> <p><math>IP[C U] = 0,12 \cdot 0,75</math>  <math>IP[U] = 0,75</math>  <math>IP[C D] = 0,2 \cdot 0,25</math>  <math>IP[D] = 0,25</math></p> <p><math>IP[\text{uomo contabile}   \text{man a.d.}] = \frac{IP[C U] \cdot IP[U]}{IP[C U] \cdot IP[U] + IP[C D] \cdot IP[D]}</math></p> <p><math>IP[U C] = \frac{0,12 \cdot 0,75}{0,12 \cdot 0,75 + 0,2 \cdot 0,25} = \frac{0,09}{0,09 + 0,05} = 0,64</math></p> <p>I collected the values of the various probabilities in the tree diagram, now I use the Bayes' formula</p>
Simona	Daniele	Emanuela

**Table 1: three representations used by *P* students for solving the a.d. exercise.**

Let us look first at the Daniele's representation in table 1: he does not solve the exercise, but uses the Venn diagrams to represent the starting point of the exercise: there is the set of M employed in the company, there is the set of employees in the a.d., and the intersection between the two sets provides the number of M in the a.d.. Even if he does not go on with the computations, Daniele is able to represent the situation in a perceptive and intuitive way, using a pictorial register. Hence, Daniele's solution belongs to the level of experience. In fact, he does not arrive to a quantitative solution, but nevertheless he is able to give an approximate estimate of a (complex) uncertain situation.

In her solution, Simona operates within the arithmetic register (table 1). No theoretical tool from the probability theory is used, with the exception – in the end of the solution – of the classical definition of probability as the ratio between the number of favorable cases (9 M) over the total number of cases (14 employees in the a.d.). This kind of solution has been taken by the majority of *P* students: they first compute the (integer) numbers of M and F working in the a.d., using the percentages and the integer numbers provided by the exercise. Then, working with integer numbers, they sum up the number of M and the number of F, obtaining the total number of employees in the a.d.: 5+9=14. In the end, they compute the ratio between the number of M in the a.d. and the total number of employees in the a.d. and obtain  $P(M|a.d.)$ . The solving process is carried out transforming representations within the

arithmetic register, and the intuitive schema of proportion is involved. This sophisticated (and correct) solution does not (explicitly) take into account either the law of alternatives, or Bayes' theorem. *M* students do not adopt these solutions, which lie at the arithmetic level, and apply the rules of probability.

*M* students, in fact, applied Bayes' theorem to solve the exercise, like Emanuela. In the *P* group, only two students (Emanuela and another one) followed this strategy. She operates at the level of theory, using a symbolic register and complementing it with a graphic register. The fact that Emanuela uses a graphic register does not imply, however, that she is working at the level of experience. Indeed, she is accomplishing a very difficult cognitive task (Duval, 2008), namely coordinating two semiotic registers. It is interesting that Emanuela uses a tree-diagram for representing and computing the probabilities, and in the end she is able to come up with the frame of a tree-diagram and apply Bayes' theorem correctly.

The solution of Simona lies at the discreteness level, since she uses ratios. An example of a solution at the discreteness level using percentages is not present in the protocols relative to this exercise. It comes from another exercise:

Which is the probability of getting an ace among 52 playing cards?

One of the *P* students, Chifan, does not make computations, but writes only a percentage: 10%. Orally interviewed afterwards, he provides this explanation

Chifan            The probability of getting an ace would be around 10%. There are, in fact, 4 aces in the pack of 52 cards. Hence, more or less the probability is 10%.

Chifan does not compute the ratio between the number of successful cases (4) and the total number of cases (52), but assigns a percentage that seemed to him to be likely (and is pretty close to  $4/52 = 0.0769$ ). Chifan's approach does not belong to the experience level only, since he resorts to the arithmetical and percentage registers to represent the situation, but it is not completely within the arithmetic level either, since approximation and perception are still part of his reasoning process.

As a side remark, this last task does not involve conditional probability. The cognitive model presented in this paper, in fact, applies to all uncertain situations, and not only to conditional probability.

## CONCLUDING REMARKS

In this paper a cognitive model to frame probabilistic reasoning within intuition and semiotic perspectives has been shown. Intuitions and semiotics may have two kinds of relationship: they can either support task solution, as in the case of Emanuela, who is able to operate a complex semiotic transformation (table 1), or impede task solution, as in the case of Daniele, who has strong intuitions, bound to pictorial representations, that do not allow him to overcome the experience level and operate a proper semiotic transformation to solve the task (table 1). Supporting or impeding



task solution depends both on the immediacy of choosing the representation that is the most proper for the task, and on the transformation(s) the subject operates starting from such representations. Hence, both the individual (together with his abilities, knowledge, etcetera) and the task contribute to determine the supporting or impeding nature of the relation between intuition and semiotic representations.

### Future perspectives

Dubinsky's APOS theory assumes that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental actions, processes, and objects and organizing them in schemas to make sense of the situations and solve problems. The underlying idea of this theory is to extend to the level of collegiate mathematics learning from the work of Piaget on reflective abstraction in children's learning. From a bird's eye view, it seems that there are many connections between this theory and the model presented in this paper. Hence, it would be interesting to go deeper into details in exploring possible integrations of the two aforementioned theoretical perspectives.

In this paper, only Duval's semiotic point of view has been taken into account. It would be interesting, in the future, to consider also other semiotic points of view, such as Radford's Objectification Theory, Arzarello's APC Space and Godino's and Batanero's Ontosemiotic Approach.

### REFERENCES

- Batanero, C., Arteaga, P., and Ruiz, B. (2010). Statistical graphs produced by prospective teachers in comparing two distributions. *Proceedings of the 6<sup>th</sup> Conference of the European Research in Mathematics Education*, 368-377.
- Borovcnik, M. (2005). Probabilistic and statistical thinking. *Proceedings of the 4<sup>th</sup> Conference of the European Research in Mathematics Education*, 485-506.
- D'Amore, B. (2006). Didattica della matematica "C". [Didactics of mathematics "C"]. In: S. Sbaragli (Ed.), *La matematica e la sua didattica, vent'anni di impegno*. Bologna, IT: Pitagora.
- Duval, R. (2008). Eight problems for a semiotic approach in mathematics education. In: L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: epistemology, historicity, classroom, and culture* (pp. 39-62). Rotterdam, The Netherlands. Sense Publishers.
- de Finetti, B. (1974). *Theory of Probability* (Vol. 1). London, UK: Wiley & Sons.
- Eichler, A., Ottaviani, M.G., Wozniak, F., and Pratt, D. (2010). Introduction on "stochastic thinking". *Proceedings of the 6<sup>th</sup> Conference of the European Research in Mathematics Education*, 344-347.
- Falk, R., Falk, R., and Levin, J. (1980). A potential for learning probability in young children. *Educational Studies in Mathematics*, 11, 181-204.

- Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht, NL: D. Reidel.
- Fischbein, E. (1998). Conoscenza intuitiva e conoscenza logica nell'attività matematica. [Intuitive and logical knowledge in mathematics activity]. *La matematica e la sua didattica*, 4, 365-401.
- Fischbein, E., and Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions? An exploratory research study. *Educational Studies in Mathematics*, 15, 1-24.
- Hawkins, C.M., and Kapadia, A.S. (1984). Children's conceptions of probability. *Educational Studies in Mathematics*, 15(4), 349-377.
- Huerta, M.P., and Lonjedo, M.A. (2007). The same problem in three presentation formats: different percentages of success and thinking processes. *Proceedings of the 5<sup>th</sup> Conference of the European Research in Mathematics Education*, 742-751.
- Kahneman, D., and Tversky, A. (1972). Subjective Probability: A judgement of representativeness. *Cognitive Psychology*, 3, 430-454.
- Konold, C., Pollatsek, A., Well, A., Lohmeier, J., and Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education*, 24(5), 392-414.
- Jeong, Y., Levine, S.C., and Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous vs discrete quantities. *Journal of Cognition and Development*, 8(2), 237-256.
- Lecoutre, M. P., and Durand, J. L. (1988). Judgements probabilistes et modèles cognitifs: Étude d'une situation aleatoire. *Educational Studies in Mathematics*, 19, 357- 368.
- Piattelli-Palmarini, M. (1995). *L'illusione di sapere*. [The illusion of knowing]. Milano, IT: Mondadori.
- Pratt, D. (2003). The emergence of probabilistic knowledge. *Proceedings of the 3<sup>rd</sup> Conference of the European Research in Mathematics Education*.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- van der Kooij, H. (2010). Mathematics at work. In: M.M.F. Pinto, & T.F. Kawasaki (Eds.), *Proceedings of the 34<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education*, 1, 121-124. Belo Horizonte, BR: PME

# **RELATING GRAPH SEMIOTIC COMPLEXITY TO GRAPH COMPREHENSION IN STATISTICAL GRAPHS PRODUCED BY PROSPECTIVE TEACHERS**

Pedro Arteaga and Carmen Batanero

University of Granada, Spain

*In this paper the graphs produced by 207 prospective primary school teachers in an open semi-structured statistical project where they had to compare three pairs of statistical variables are analysed. The graphs are classified according their semiotic complexity, and the teachers' levels of comprehension in Curcio's (1989) categorization. Most participants produced graphs with sufficient semiotic complexity to solve the task proposed; however, the correct conclusion was only reached by a minority of prospective teachers who were able to read the data produced at the "reading behind the data" level. When relating semiotic complexity of graphs to the reading level, teachers producing graphs at the highest semiotic level also reached the highest combined percentage of "reading beyond data" and "reading between data" levels.*

**Keywords:** statistical graphs, semiotic complexity, graph interpretation and comprehension.

## **INTRODUCTION**

Graphical language is an important part of statistical literacy (Watson, 2006). It is also a tool for transnumeration, a basic component in statistical reasoning consisting of "changing representations to engender understanding" (Wild & Pfannkuch, 1999, p. 227). In this work we complement our previous research on Spanish prospective primary school teachers' graphical competence (Batanero, Arteaga & Ruiz, 2010) with the aim to relate the semiotic complexity of graphs produced by prospective primary school teachers with the graph comprehension levels defined by Curcio (1989).

### **Understanding Statistical Graphs**

Previous research suggest that competence related to statistical graphs is not reached in compulsory education, since students make errors in establishing the graph scales (Li & Shen, 1992) or in building specific graphs (Pereira Mendoza & Mellor, 1990; Lee & Meletiou, 2003; Bakker, Biehler & Konold, 2004). Several authors investigated levels in graph understanding. For this particular research we are using Curcio's categorisation (1989), that consists of the following levels: (a) Reading the data, is the level of a student who is only able to answer explicit questions for which the obvious answer is right there in the graph; (b) Reading between the data involves interpolating and finding relationships in the data presented in a graph. This includes making comparisons as well as applying operations to data; (c) Reading beyond the

data involves extrapolating, predicting, or inferring from the representation to answer questions related to tendencies in the data or extrapolation from the data. In this research we are only considering these three levels, which do not imply the need to look critically at the data. This would be a new level looking behind the data, according to Friel, Bright, and Curcio (2001).

### **Graphical Competence in Prospective Teachers**

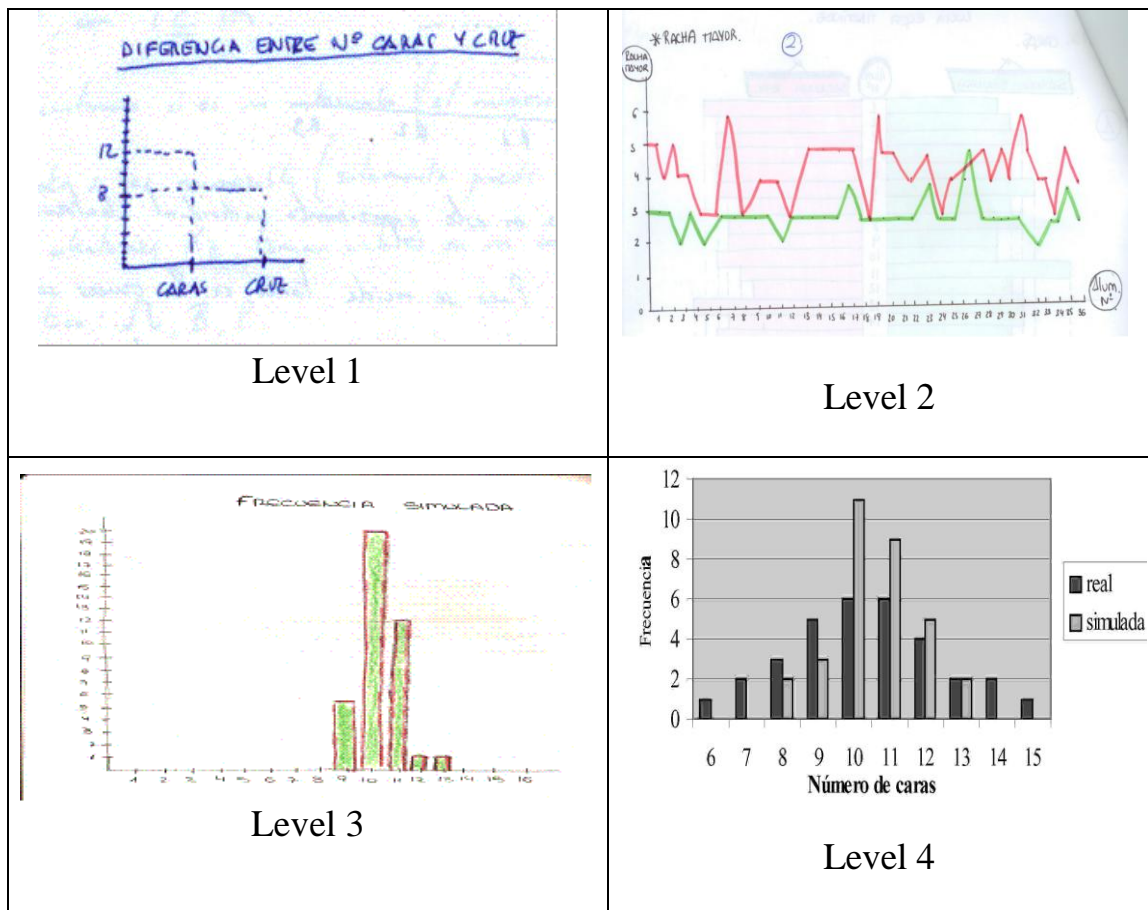
Few studies have focused on teachers' knowledge and conceptions about statistical graphs and most of these studies are related to prospective teachers (González, Espinel, & Ainley, in preparation). Results from this research highlight the scarce graphical competence in prospective teachers. For example, in a study conducted with 29 prospective primary teachers in Spain, Bruno and Espinel (2009) found frequent errors when building histograms or frequency polygons. In another study with 190 prospective primary school teachers, Espinel, Bruno, and Plasencia (2008) observed lack of coherence between the graphs built by participants and their evaluation of tasks carried out by fictitious future students. Monteiro and Ainley (2007) suggested that many Brazilian prospective teachers in a sample of 218 teachers did not possess enough mathematical knowledge to read graphs taken from daily press. In Burgess's (2002) study some teachers made graphs in their reports but were unable to integrate the knowledge they could get from the graphs with the problem context.

Batanero, Arteaga, and Ruiz (2010) analysed the graphs produced by 93 prospective primary school teachers in an open semi-structured statistical project where they had to compare two statistical variables. They defined a *semiotic complexity level* in these graphs and analysed the teachers' errors in selecting and building the graphs, as well as their capacity to reach a conclusion on the research question. Although about two thirds of the participants produced a graph with enough semiotic complexity to get an adequate conclusion, half the graphs were either inadequate to the problem or incorrect. Only one third of participants were able to get a conclusion in relation to the research question. In this paper we increase the sample size and the number of variables to be compared in the same semi-structured statistical project in order to relate the graphs semiotic complexity with the prospective teachers' level in reading the graphs.

### **Semiotic Complexity in Statistical Graphs**

In mathematical work we usually take some objects (e.g. a symbol or a word) to represent other abstract objects (e.g. the concept of average). In these situations, and according to Eco "there is a *semiotic function* when an expression and a content are in correlation" (Eco 1979, p. 83). Such a correlation is conventionally established, though this does not imply arbitrariness, but it depends on a cultural link. Font, Godino, and D'Amore (2007) suggested that all the different types of objects that intervene in mathematical practices: *problems, actions, concepts, language, properties and arguments* could be used as either expression or content in a semiotic

function. In our study we proposed an open semi-structured project to prospective teachers. To address the project, the participants had to solve some mathematical *problems* (e.g., comparing three different pairs of distributions) and perform some mathematical practices to solve these problems. The focus in our research is the statistical graphs produced by the teachers and the mathematical practices linked to the different types of graphs. When teachers produce a graph they need to perform a series of *actions* (such as deciding the particular type of graph or, fixing the scale), they implicitly used some *concepts* (such as variable, value, frequency, range) and *properties* (e.g. proportionality between frequencies and length of bars in the bar graph) that vary in different graphs. Consequently the semiotic functions implicit in building each graph also varies. We therefore should not consider the different graphs as equivalent representations of a same mathematical concept (the data distribution) but as different configurations of interrelated mathematical objects that interact with that distribution. Taking into account these ideas, Batanero, Arteaga, and Ruiz (2010) defined different levels in graphs semiotic complexity, as follow (see examples in Figure 1).



**Figure 1. Examples of graphs at different semiotic complexity level**

*L1. Representing only individual results.* When given a set of data, some students do not complete the graph for the whole data set; instead they only represent isolated data values. When the data are collected in the classroom they only represent their own data, without considering their classmates' data, for example, they represented

the number of heads in his /her individual experiment. These students do not use the idea of statistical variable or distribution when producing their graphs.

*L2. Representing all the individual values for one or several variables, without forming the distribution.* Some students do not form the frequency distribution of the variables, when they are given a data set. Instead, they produce a graph where data are represented one by one, without an attempt to order the data or to combine identical values. Consequently these students neither compute the frequency of the different values nor explicitly use the idea of distribution.

*L3. Producing separate graphs for each distribution.* When comparing a pair of distributions, some students use the idea of frequency and distribution but produce a separate graph for each variable to be compared. Often, these students use either different scales in both graphs or different graphs for the two distributions, which makes the comparison hard.

*L4. Producing a joint graph for both distributions.* This level corresponds to students that form the distributions for the two variables to be compared and represent them in a joint graph, which facilitated the comparison. These graphs are the most complex, since they represent two different variables in the same frame.

## THE STUDY

Participants in the sample were 207 pre-service teachers in Spain, in total 6 different groups (35-40 pre-service teachers by group). All of them were following the same mathematics education course and studied descriptive statistics at a secondary school level, and in the previous academic year (for 20 hours), where they had worked with another statistical project. In this paper we analyse the graphs produced by these teachers when working in a semi-structured statistical project in which participants were asked to perform a random experiment, collect data, compare three pairs of distributions and come to a conclusion about the group's intuition of randomness, basing their conclusion on the analysis of the data. The sequence of activities in the project was as follows:

1. *Presenting the problem, and experiment.* We proposed that the prospective teachers carry out an experiment to decide whether they had good intuitions on randomness or not. The experiment had two parts. In the first part (simulated sequence) each participant wrote down apparent random results of flipping a coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random. In the second part (real sequence) each participant flipped a fair coin 20 times and wrote down the results.
2. *Collecting data and instructions.* Each prospective teacher performed both experiments. After the lecturer started a discussion about how the simulated and real sequences for the whole group could be compared, some students suggested to collect data from the number of heads or number of runs. Finally the lecturer

suggested comparing the following statistical variables: number of heads, number of runs and longest run. Each prospective teacher provided his/her results in each of the variables that were recorded on a recording sheet. At the end of the session the prospective teachers were given a printed copy of the data set for the whole group of students. They were asked to produce a written report including their statistical analyses and their conclusions. There was no restriction in the report length; teachers were given freedom to use any statistical method or graph they wished and were allowed to use computers. They were given a week to complete the reports (for more details of the project, see Godino, Batanero, Roa & Wilhelmi, 2008).

## RESULTS AND DISCUSSION

Once the prospective service teachers' written reports were collected, we made a qualitative analysis of these reports. From a total of 207 students 181 (87,4%), 146 (70,5%) and 128 (61,8%) produced some graphs when analysing the number of heads, number of runs and longest run, even if the instructions given to the students did not explicitly ask them to construct a graph. These high percentages suggest that prospective teachers felt the need of building a graph to reach, through a transnumeration process (Wild & Pfannkuch, 1999) some information that was not available in the raw data. In Table 1 we present the results. These data show the relative difficulty of the variables to be analysed, as the number of heads was more familiar to the teachers than the runs.

Semiotic complexity	N. of heads	N. of runs	Longest run
L1. Representing only individual data	6 (3.3)	6 (4.1)	3 (2.3)
L2. Representing the data list	40 (22.1)	23 (15.7)	21 (16.4)
L3. Producing separate graphs for each distribution	91 (50.3)	77 (52.7)	67 (52.3)
L4. Producing a joint graph to compare both distributions	44 (24.3)	40 (27.4)	37 (28.9)
Number of participants producing graphs	181	146	128

**Table 1. Frequency (percentage) of participants producing graphs in each semiotic level and pair of variables**

Few participant produced level L1 graphs, that is, only analysed their own data and less than 25% represented the data list in the same order given in the data sheet without making an attempt to summarise the data, producing the variables distributions. Consequently the concept of distribution seemed natural for the majority of students who produced a graph, since about 75% of them built a distribution to compare each pair of variables, although the instructions did not explicitly require this. Results agree with those reported by Batanero, Arteaga, and Ruiz (2010).

## Graph comprehension

In table 2 we classify participants according to Curcio's (1989) categorization of graph comprehension, in the following way:

**R0 Do not read the graph or incorrect reading:** About 30% of the teachers in each pair of variables only produced and presented the graph in their report with no attempt to read the graph, and reached no conclusion about the problem posed. In addition, after producing the graph, between 11% to 14% of the teachers in each pair of variables failed when reading the information. Some of these failures were produced by errors in the graphs that reproduced those described in Bruno and Espinel (2009) or incorrect choice of the type of graph that was inadequate for the information represented in the graph. Other failures in reading the graph were due to incorrect understanding of a concept; for example confusing frequencies and values of the variable or misinterpreting the meaning of the standard deviation.

**R1 Reading data:** Between 22- 25% of the participants made a correct literal reading of graphs labels, scales and specific information on the graph in each pair of variables. However they only considered superficial features of the graph. For example, they compared isolated values of the two variables, provided the frequency for a given value or made a general comment about the shape of the graph with no consideration given to tendencies or variability in the data.

Graph comprehension level (Curcio)	N. of heads	N. of runs	Longest run
R0. Do not read the graph	51 (28.2)	45(30.9)	42(32.8)
R0. Incorrect reading	21(11.6)	17(11.6)	18(14.1)
R1. Reading data	41(22.6)	34(23.3)	32(25)
R2. Reading between data	44(24.3)	32(21.9)	21(16.4)
R3. Reading beyond data	24(13.3)	18(12.3)	15(11.7)
Number of students producing graphs	181	146	128

**Table 2. Frequency (percent) of participants producing graphs according to comprehension level**

**R0. *Reading between data:*** Teachers classified in this level were able to make comparisons and look for relationships in the data. They either compared averages (means, medians or modes) alone in both distributions (with no consideration of variation in the data) or else compared spread (without comparing averages).

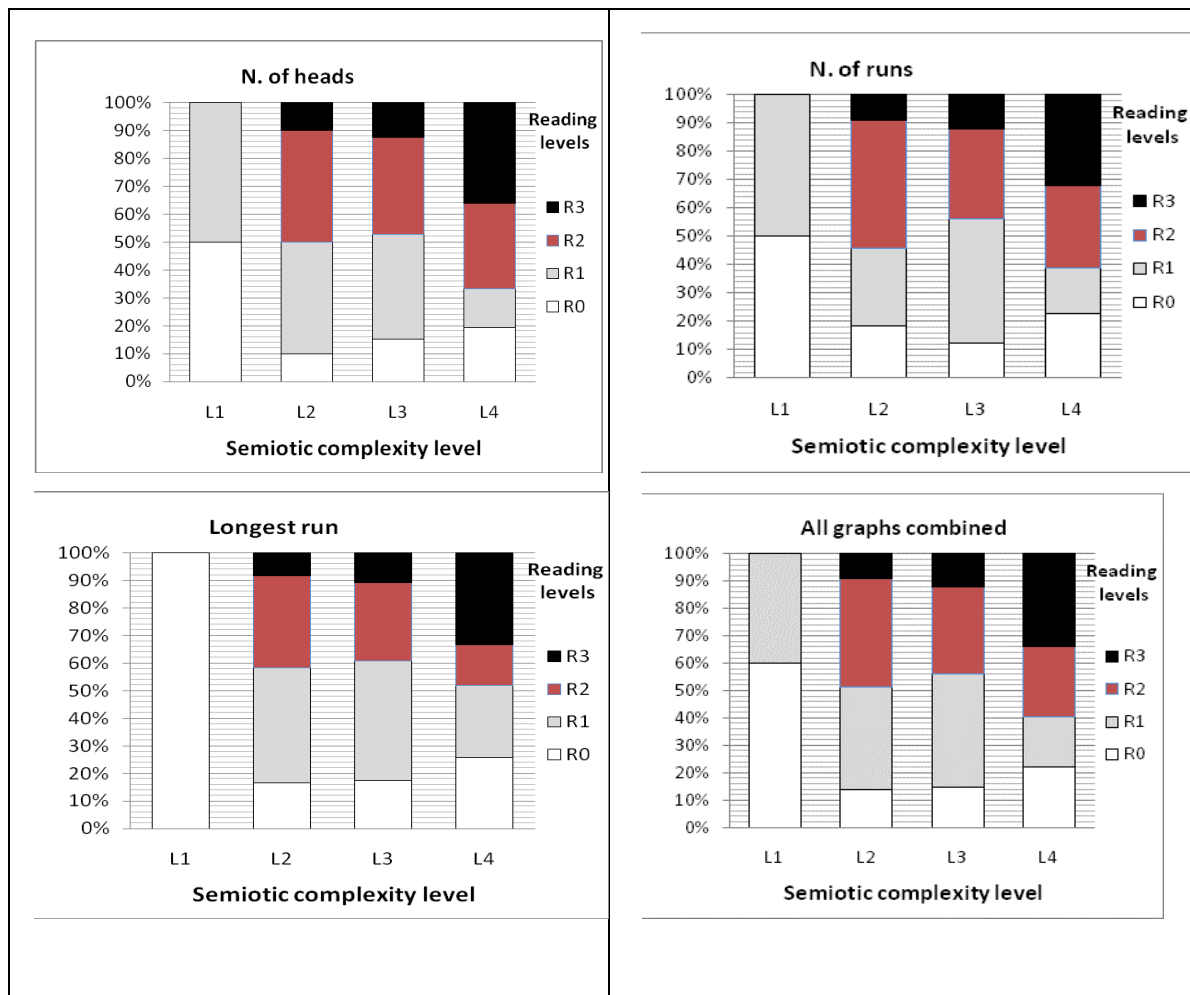
**R1. *Reading beyond data:*** Making inferences and drawing conclusions from the graph: Participants at this level were able to compare both spread and average in the distribution and conclude about the differences taking into account both data



features.

Less than 25% of the prospective teachers who built graphs reached the Curcio's (1989) intermediate level (reading between the data) and only 13% reached the upper level (Table 2). Notice that the percentage of students building graphs without interpreting them is high, which agrees with Burgess (2002). The difficulty of reading the data increased for variables related to runs that were less familiar to participants.

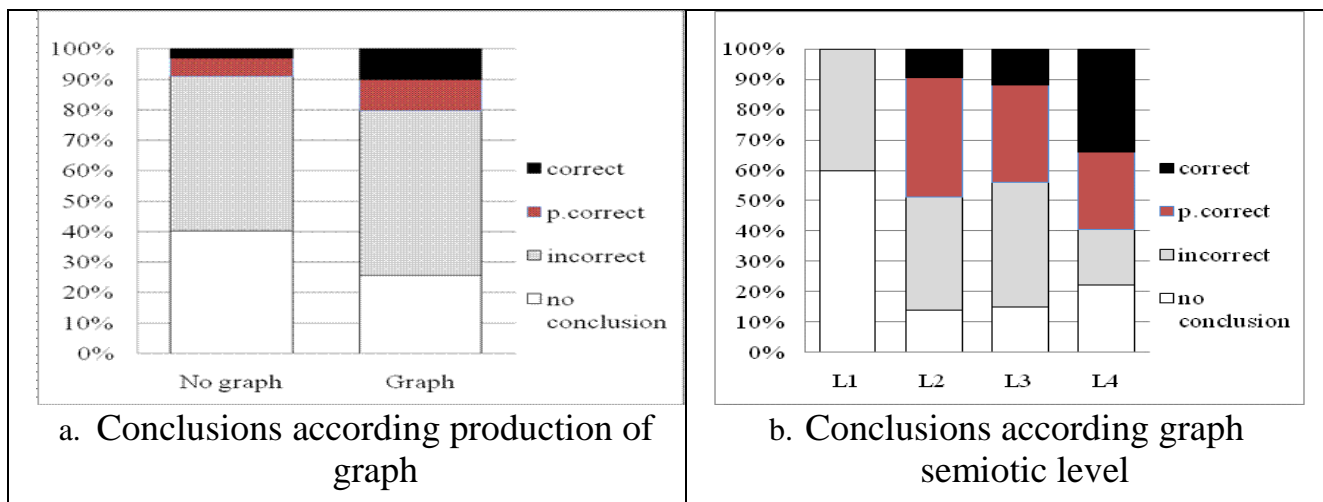
In Figure 2 we take into account only the prospective teachers who interpreted the graphs they built themselves, classifying their representations according to the graph semiotic complexity level and reading comprehension level for each pair of variables and for all the graphs combined. The Chi-square test to check independence of reading levels and semiotic complexity levels for all the graphs combined was statistically significant ( $\chi^2=40.4$ ,  $df=9$ ,  $p<0.0001$ ) and therefore we can accept that these two types of levels are related.



**Figure 2. Reading levels by semiotic complexity in the graph**

Prospective teachers producing semiotic level L1 graphs either made an incorrect reading or only reached the literal “reading data” level. The percent of incorrect reading dramatically decreased in the remaining levels; however this percent is higher in level L4 than in levels L2 or L3. This is probably because more complex

graphs were harder to be interpreted correctly by participants. Although there is no clear tendency as regards literal “reading the data” level, reading the data only is not productive to reach a conclusion on the problem posed. “Reading between data” level is more frequent in level L2 graphs, because in these graphs the data variability is very easily perceived (as compared with levels L1 or L3 graphs). The highest percentage of “reading beyond data” level, when teachers are able to analyse both the tendency and spread in the data and reach a conclusion, as well the combined percentage of “reading between data” and “reading beyond data” levels were reached in semiotic level L4 graphs because in level L4 graphs students can perceive spread and tendencies more easily. Therefore level L4 graphs provide more opportunity for students to get at least a partly correct reading. Consequently, it is important that the teacher’s educator try to promote higher reading levels when possible.



**Figure 3. Conclusion according to (a) production of graph; (b) graph semiotic level**

In the project proposed the students should reach a conclusion regarding the group intuition on randomness. Only 8.9% of those prospective teachers who produced no graph got a correct or partly correct conclusion. This percentage increased to 20.2% in those teachers who produced graphs as part of their analyses (10,2% and 10 %); the differences are significant in the Chi-square test ( $\chi^2=18,72$ , d.g.=3;  $p=0.007$ ). Therefore, building a graph helped the teachers in their conclusions (Figure 3.a). The percentage of correct conclusions increased to about 30% in teachers producing level L4 graphs (Figure 3.b), because, at these levels the teachers read the graph at a higher reading level and in these graphs the perception of both tendencies and spread is easier. The percentage of partly correct conclusions was higher at level L2 graphs, because of easy perception of variability in these graphs. The Chi-square test to check independence of type of conclusions and the semiotic complexity levels for all the graphs combined was statistically significant ( $\chi^2=40,45$ ; d.g.= 9;  $p= 0,0000$ ) and therefore we can accept that these two variables are related.

### IMPLICATIONS FOR TEACHER EDUCATION

In agreement with Bruno and Espinel (2009) and Monteiro and Ainley (2007) our research suggests that building and interpreting graphs is a complex activity for

prospective school teachers. We agree with these authors in the relevance of improving the prospective teachers' levels of competences in both building and interpreting graphs, so that they can later transmit these competences to their own students. Many participants in the study limited their analysis to producing graphs with no attempt to get a conclusion about the problem posed. Consequently, prospective teachers need more training in working with statistical projects, since working with projects is today recommended in the teaching of statistics from primary school level.

*Acknowledgment.* Project EDU2010-14947 (MCINN-FEDER), grant FPU AP2007-03222 and group FQM126 (Junta de Andalucía).

## REFERENCES

- Bakker, A., Biehler, R., & Konold, C. (2004). Should young students learn about box plots? In G. Burrill & M. Camden (Eds.), *Proceedings of the International Association for Statistical Education (IASE) Roundtable on Curricular Development in Statistics Education*. (pp. 163-173). Auckland: IASE. Retrieved from [www.stat.auckland.ac.nz/~iase/publications/](http://www.stat.auckland.ac.nz/~iase/publications/).
- Batanero, C., Arteaga, P., & Ruiz, B. (2010). Statistical graphs produced by prospective teachers in comparing two distributions. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. Lyon: ERME. Retrieved from [www.inrp.fr/editions/editions-electroniques/cerme6/](http://www.inrp.fr/editions/editions-electroniques/cerme6/).
- Bruno, A., & Espinel, M. C. (2009). Construction and evaluation of histograms in teacher training. *International Journal of Mathematical Education in Science and Technology*, 40(4), 473-493.
- Burgess, T. (2002). Investigating the “data sense” of preservice teachers. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*, Cape Town, South Africa: International Association for Statistics Education. Retrieved from [www.stat.auckland.ac.nz/~iase/publicatons](http://www.stat.auckland.ac.nz/~iase/publicatons).
- Curcio, F. R. (1989). *Developing graph comprehension*. Reston, VA: N.C.T.M.
- Eco, U. (1979). *Tratado de semiótica general* (General semiotics). Barcelona: Lumen.
- Espinel, C., Bruno, A., & Plasencia, I. (2008). Statistical graphs in the training of teachers. In C. Batanero, G. Burrill, C. Reading & A. Rossman (2008). *Proceedings of the Joint ICMI /IASE Study Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education*. Monterrey, México: ICMI & IASE. CD ROM.
- Font, J. D., Godino, J. D., & D'Amore, B. (2007). An ontosemiotic approach to representations in mathematics education. *For the Learning of Mathematics*, 27 (2), 3-9.

## Working Group 5

- Friel, S., Curcio, F., & Bright, G. (2001). Making sense of graphs: critical factors influencing comprehension and instructional implications. *Journal for Research in mathematics Education* 32(2), 124-158.
- Godino, J. D., Batanero, C., Roa, R., & Wilhelmi, M. R. (2008). Assessing and developing pedagogical content and statistical knowledge of primary school teachers through project work. In C. Batanero, G. Burrill, C. Reading & A. Rossman (2008). *Proceedings of the Joint ICMI /IASE Study Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education*. [CD] Monterrey, México: ICMI e IASE.
- González, T., Espinel, C., & Ainley, J. (In preparation). Teachers' graphical competence. In C. Batanero, G. Burrill & C. Reading (Eds.). *Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education. A Joint ICMI and IASE Study*. New York: Springer.
- Lee, C., & Meletiou, M. (2003). Some difficulties of learning histograms in introductory statistics. Paper presented at the *Joint Statistical Meeting. Section on Statistical Education*. Retrieved from <http://www.statlit.org/PDF/2003LeeASA.pdf>
- Li, D. Y., & Shen, S. M. (1992). Students' weaknesses in statistical projects. *Teaching Statistics* 14 (1), 2-8.
- Monteiro, C., & Ainley, J. (2007). Investigating the interpretation of media graphs among student teachers. *International Electronic Journal of Mathematics Education* 2 (3), 188-207. Retrieved from <http://www.iejme/>.
- Pereira-Mendoza, L., & Mellor, J. (1990). Student's concepts of bar graph: Some preliminary findings. In D. Vere-Jones (Ed.), *Proceedings of the Third International Conference on Teaching Statistics* Voorburg: International Statistical Institute. Retrieved from [www.stat.auckland.ac.nz/~iase/publications.php](http://www.stat.auckland.ac.nz/~iase/publications.php)
- Watson, J. M. (2006). *Statistical literacy at school: growth and goals*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review* 67 (3), 223-265.

# THE CHALLENGES OF TEACHING STATISTICS IN SECONDARY VOCATIONAL EDUCATION

Arthur Bakker, Monica Wijers and Sanne Akkerman

Freudenthal Institute for Science and Mathematics Education

Utrecht University, the Netherlands

*The challenges of teaching statistics in vocational education, an underresearched area, are likely to be different than in general education due to the focus on a particular occupation. The current paper addresses the question what these challenges are. Through interviews with teachers, apprentices and supervisors as well as analysis of the curriculum and classroom instruction, we identified such challenges. These include the difficulty to engage students and prepare them for diverse workplaces with different levels of mediation by technology. Moreover, competence-based projects, common in vocational education, require artful coordination of theoretical and practical knowledge, and the design of representations that assist future employees to draw appropriate conclusions.*

## CONNECTING SCHOOL AND WORK

One of the key challenges in statistics and mathematics education is to engage students by giving them a sense of purpose and utility of the concepts and techniques they learn (Ainley et al., 2006). One way, both in general education (Dierdorp et al., 2011) and vocational education, is to seek inspiration for designing engaging statistics education in professional practices and thus make a connection between disciplinary content and future work. After all, as Lave (1988, p. xiii) noted: “It seems impossible to analyze education – in schooling, craft apprenticeship, or any other form – without considering its relations with the world for which it ostensibly prepares people.” In vocational education, this relation between education and occupation is apparent and the vocational area is indeed an interesting but underresearched area of study: It is where teachers and students face the challenge of connecting abstract disciplinary knowledge to its usefulness in occupations. We therefore expect that research in general education can learn from research in the vocational and workplace domain.

However, we should also note that the differences between school and work statistics are big (cf. Noss et al., 1999) because school and work practices entail very different aims, tools, communities, rules and divisions of labour. For example, school statistics is mostly general but workplace statistics is typically context-specific and mediated by technology – which is the reason that Hoyles et al. (2010) highlight the need for Techno-mathematical Literacies (which include statistical literacy). The term Techno-mathematical Literacies (TmL) refers to the technology-mediated nature of mathematics at work and the need for communicating quantitative information.

To gain more insight into these challenges we draw on a 3.5-year research project on TmL in Dutch senior secondary vocational education (MBO), which prepares for intermediate-level occupations. This project came to focus on laboratory work. Common in industry, but also in health services and safety institutes, laboratories are statistically rich and subject to rapid changes in work organisation, hence interesting places to study the challenges that vocational teachers face when preparing their students.

In this paper ask: *What are the challenges of teaching statistics in vocational laboratory education?* Articulating these can provide a basis for improving the teaching and learning of statistics in vocational and presumably other settings. The challenge of preparing students for the technology-mediated use of statistics, for example, is a general one – just like the challenge of engaging academically less able students. We do not confine our analysis to challenges that teachers experience themselves but also include those that we inferred from discrepancies between school and workplace approaches to using statistics.

## **DUTCH SENIOR SECONDARY VOCATIONAL EDUCATION (MBO)**

To sketch the setting of our research we first provide information on the MBO school system. About 40% of Dutch senior secondary students (aged 16-17) attend general education or pre-university tracks; the remaining 60% enrol in senior secondary vocational education (MBO). Levels 3 and 4 of MBO are just above basic school qualifications but much below Bachelor level (see also Bakker et al., in press). MBO used to have attainment targets for each MBO occupation (including hair dresser, baker, electrician, lab technician). For mathematics and statistics in many technical programmes this was a list of about fifty topics. Attainment targets that were less relevant for the occupation were ignored by mathematics teachers and general subjects were generally considered separate from the occupation. Over the past ten years vocational education has become more and more competence-based (Van den Berg & De Bruijn, 2009). There are now qualification files for 237 occupations which are formulated in terms of what starting employees should be able to do. The effect has been that subject knowledge such as mathematics is taught and assessed less than about ten years ago. In the file for lab technicians, for example, references to the statistical knowledge required are scarce and broadly phrased (e.g., “basic knowledge of mathematics”; “care for quality”).

## **THEORETICAL BACKGROUND**

Hardly any statistics education research has been carried out in vocational education. Therefore hardly anything is known about the challenges of teaching this subject. Given the fact that vocational education forms a link between general education and workplaces, it is likely that we can yet draw on workplace research in this area. We can also learn from international trends of competence-based education.

The workplace research most relevant to our research is that by Hoyles et al. (2010). Key trends in workplaces such as automation and computerisation of work processes are likely to lead to particular challenges. Hoyles et al. (2010) analysed these trends and the effects on what employees need to know about mathematics and, notably, statistics. Bakker et al. (2009) described a way to draw on employees' rich context knowledge but often poor knowledge of statistics by designing alternative representations of statistics that do not draw on inaccessible symbolic language. Key artefacts (boundary objects) in work processes were reconfigured in software (technology-enhanced boundary objects) so as to allow manipulation and easy interpretation. Teaching required hybrid expertise of statistics and work processes, hence our collaboration with workplace trainers.

The main reasons for introducing competence-based education in our country were first to prepare students better for specific vocations and second to take into the changing population of students (Van den Berg & De Bruijn, 2009) – those who find general knowledge hard to learn and consider themselves doers rather than thinkers, of with language or personal problems. Like in many vocational systems in the world, projects and simulations are often used to stay close to particular work tasks. The underlying reason for such pedagogic measures is the general acknowledgement of the situatedness of cognition and problematic transfer of general knowledge to everyday situations (e.g., Lave, 1988).

## **METHODS**

To identify the challenges we analysed interviews with fourteen teachers (18:05 hours in total) at four different vocational laboratory schools (MBO) – three schools were relatively close to the university and one teacher in the fourth school was interested in the theme of our project and provided us with access to its teachers, students and supervisors. As a background to understanding the challenges mentioned by teachers we also studied their course materials and observed several lessons to get a sense of how course materials were used. We further conducted interviews with eight supervisors of apprentices in a variety of labs (in total 10:40 hrs), one school and two workplace managers (2:20 hrs), nine apprentices in the workplace (4:20 hrs), 27 apprentices at release days at school (5:40 hrs). With the help of three teachers we analysed 35 final apprenticeship reports to see what statistical technique apprentices used and how well they did so. In addition, we undertook four workplace tours in different labs (2:10 hrs), spent a day of observation and interviewing in one lab, and collected several prototypical artefacts that represented how statistical knowledge was mediated (e.g., Standard Operating Procedures including calculations, graphs, data etc.). This background information helped us derive challenges that teachers did not explicitly mention.

The interviews were transcribed verbatim and coded for challenges experienced by teachers. These were categorised into seven different but related challenges

summarised in the next section. We used the remaining data sources to offer explanations and further background to these challenges.

## **CHALLENGES**

### **1. To cope with limited resources**

All teachers complained that their hours for disciplines such as statistics had diminished due to longer apprenticeships (work placement) and the introduction of competence-based education – trends that several teachers considered to be a matter of economising on costs. In practice this meant more time on projects and learning on demand, and less on general subjects such as languages and mathematics. There is generally very little time for teaching statistics, for example one hour per week for the first two years. Yet the topics encountered in labs are numerous (Bakker et al., 2010). Moreover, both teachers and students noted that students had often forgotten many topics by the time they became apprenticed. A related challenge therefore is to keep their knowledge fresh and available over the course of the years even if not required in a particular company.

### **2. To engage students in learning statistics**

All teachers and supervisors considered it important for students to understand the how and why of statistical techniques, but they also characterised their students as “doers, not thinkers”. They found it challenging to engage them in disciplinary knowledge that is not immediately linked to what students see as their purpose: becoming a lab technician. We have interviewed students who were able to attend general education but had deliberately chosen the vocational route because they thought this was better preparation for becoming a lab technician or because they preferred doing something practical. However, most vocational students have failed in general education for whatever reason. We return to this point in the last section.

### **3. To make statistics visible to managers and colleagues**

Teachers had a hard time to convince their managers and some of their colleagues that students needed some disciplinary knowledge such as statistics in order to develop the competences formulated in the qualification files. In most cases, the number of teaching hours for subjects such as mathematics, statistics and the languages decreased considerably. The time available for teaching is often dedicated to projects that simulate some work task typically found in the workplace. Managers and even colleagues of the mathematics or science who taught statistics often thought that disciplinary content could be taught “just in time”, just before it was needed in a project. We probably do not have to convince the reader that teaching hypothesis testing (t- and F-test are common in lab work) has to be carefully prepared, especially to vocational students, who typically have not succeeded academically in general education. These observations illustrate that teachers found



it hard to make statistics visible to their managers and colleagues. From the literature we can derive an explanation.

It is well known from the research on workplace mathematics (of which a large part actually is statistical in nature) that employees tend to say they do not do any mathematics, even if mathematics educators observe them using it (Noss et al., 1999). We have also experienced this: One supervising lab technician claimed his work only involved “pluses and minuses”, but when he showed us around we saw technicians modelling chemical reaction processes, using extrapolation, slope and other mathematical concepts. When confronted with this observation, he responded this was chemistry rather than mathematics. We refer to this phenomenon as the Janus head (two-faced) nature of workplace mathematics and statistics: Where employees see working with numbers as part of their discipline (in this case chemistry), we as mathematics educators see mathematics being part of their work. It is only if we look for the use of statistics in workplaces and deliberately try to improve production processes that it becomes more visible (Bakker et al., 2006). This trend is corroborated by the omnipresence of black boxes in which most of the mathematical models and statistical techniques used at work are crystallised (Williams & Wake, 2007).

The drive to make the work error-free, one manager commented, has led to a situation where the younger generation often no longer knows what happens behind the screens. The paradoxical situation is that this hardly ever leads to problems – those have been ruled out by the system – but we did hear concerns about this situation; many lab technicians found it important for apprentices to understand the why and how of what they were doing and we have evidence from observations in one lab that blindly following procedures can lead to waste of materials and time. The tension observed can be seen as a result of the black box phenomenon that Latour (1999, p. 304) described:

... scientific and technical work is made invisible by its own success. When a machine runs efficiently, when a matter of fact is settled, one need focus only on its inputs and outputs and not on its internal complexity. Thus, paradoxically, the more science and technology succeed, the more opaque and obscure they become.

The Janus head black-boxed nature of statistical knowledge required might explain why the qualification files pay so little attention to it and why teachers found it so hard to convince their managers of its importance.

#### **4. To prepare students for the technology-mediated nature of work**

Teachers often ask the question what they should teach their students and how. What do students need to understand about more complex statistics if it is carried out by a computer system? The fact that t-tests are used in many labs does not necessarily mean that an MBO lab technician should understand the formula or be able to perform one by hand, or even with a software package.

One of the teachers' key challenges that we think are relevant to education more generally is thus how to prepare students for the technology-mediated nature of statistics usage at work. One question here is whether students should learn the background of, say, statistical process control, before they use pre-fabricated SPC charts or whether they should learn SPC in relation to spreadsheets straightaway. The common assumption among teachers seemed to be that theoretical introduction with the conventional representations is the basis for practical usage. Previous research in a car factory has shown that this assumption is problematic (Bakker et al., 2009) and our observations in laboratory education corroborate this. For most students the step from symbolic representation, whether SD, t-test or correlation, to what can be inferred from them in practical terms is simply too big to teach in limited instruction time available for each topic.

Labs vary in terms of automation and computational tools, which means that schools have to prepare students for both manual and automated computation. Computations are not always taken away from employees. In some labs (about 14%, see Bakker et al., 2010), all calculations were automated in Excel sheets or dedicated computer software (such as LIMS: Laboratory Information Management System). In most others, a mix of computational tools (calculator, Excel, software) was used. The general image from the interviews was that calculations had become easier over the years because of software and automated machines, but what lab technicians need to know has not become less, only different; for example fluency in Excel has become more important. With computations outsourced to software, it becomes important to know something about the software and what it is doing. In one lesson we indeed observed how a teacher clarified to students the difference between computing something in Excel by column or by row – something relevant to statistics education we rarely find in a textbook.

### **5. To prepare for a diversity of workplaces**

Teachers are well aware that their students may choose to work in very different labs. Should all students then learn what is required in the most advanced labs? Where chemical and clinical-chemical (medical) work involves correlation, regression and validation, the biological work often has a more qualitative nature (recognizing types of crystals etc.). Microbiological work does require good understanding of powers and logarithms, because amounts of bacteria are reported in powers of ten. Interestingly, common measures of centre easily turn out problematic when working with powers of ten. One supervisor preferred medians and geometrical means over arithmetic means but did not expect MBO level technicians to understand these alternatives.

### **6. To keep up with innovations at work**

Laboratories change rapidly due to technological innovations. For example, students learn to calibrate machines in the old-fashioned way, but modern companies have big

analysers that can be calibrated with a press on a button. The statistical background of calibration – measurements modelled by regression lines and correlation coefficients – has been completely blackboxed in such cases. Employees state they only judge the correlation coefficient to see if the measurements have been precise and accurate enough (e.g., 0.999964 was considered very good).

The aforementioned rapid developments at work raise the challenge for teachers to stay up to date. This is not easy once a teacher is ‘caught’ in a teaching job. Those who have a background as lab technicians themselves typically stay in touch with old colleagues and friends on these developments. However, those with a background in mathematics or science teaching find it harder to develop a good image of trends at work and their implications for curriculum change.

#### **7. To develop their own statistical expertise**

Most teachers we interviewed did not feel expert enough to assist students with the statistics required in their projects. Adding specialist teachers to the team was no option because there was a tendency to keep the number of teachers for each student as small as possible to make supervision easier for everybody. Workplace supervisors differed considerably in terms of statistical expertise and could not always help students either. Finally we were struck by the fact how vocation-specific statistics could be: The type of statistics taught by mathematics teachers was often considered too general to be useful for lab technicians, who thought in terms of method validation, reproducibility and stability, rather than correlation and variation coefficients.

### **TO CONCLUDE: MEETING SOME OF THE CHALLENGES**

In answer to our question what are the challenges of teaching statistics in vocational laboratory education, we identified seven challenges illustrated above. Some challenges are the consequence of competence-based education (1, 3, 7), one of the student population (2), and some of the changing nature of work (5, 6), in particular the technology-mediated nature of statistics in workplace (4). Many of the challenges are related. For example, the technology-mediated nature of using statistics at work not only raises the question of how to prepare students for it, but also contributes to the invisibility of this disciplinary knowledge at work and hence at competence-based education.

One challenge we see, but teachers did not address explicitly, is how they can help their students to develop the number sense required (Bakker et al., 2010). From the interviews with supervisors we assume this requires the integration of disciplinary knowledge typically developed at school and practical knowledge developed through experience at work. For example, whether a measurement value is judged correctly also depends on experience with the range in which that value typically falls. Judgement of correctness presumably draws on multiple resources including

disciplinary and more workplace experiential knowledge. To conclude we mention how some of the challenges are or could be dealt with in practice or in research.

### **Teacher apprenticeships to stay up to date**

One important way in which teachers can stay up to date with recent developments at work is by means of teacher apprenticeships. Teachers, in particular those with a disciplinary background in mathematics or science, get the opportunity to work in a company for a few days to see how what they teach informs apprentices' work. One teacher of statistics we interviewed was very enthusiastic about this opportunity. In his case, he felt reassured that the topics he taught, such as statistical process control, were indeed used in ways that he propagated in his lessons. However, the general picture from the limited research on teacher apprenticeships (de Schutter, 2009) is that organising them and convincing teachers to take part is not easy. Moreover, not every workplace is suitable as a learning site for teachers.

### **Diversity of labs**

The way in which lab schools address the diversity challenge is to start with a general programme for statistics and tie the statistical aspects of lab work to the variants chosen. The biggest school we investigated had six different variants from which students could choose in their last two years. However, small schools did not have the resources to offer more than two variants (e.g., chemical and medical).

Diversity is not always problematic, because students are also diverse. They have different interests and qualities, and look for laboratories in which they can flourish. Likewise, companies select students that seem to fit in their type of labs. Some are good at routine work, other function better in non-routine work.

Workplaces also demonstrate a high degree of adaptivity which is possible due to the diversity of employees and tasks in a lab. If tasks turn out too difficult or important for apprentices or beginning lab technicians, they are carried out by higher-level or more experienced lab technicians. Thus workplace systems are serving as an ecology adapting to particular gaps or weaknesses in apprentices' knowledge. Such adaptivity and division of labour also has another side: We were told about lab technicians with an affinity for statistics who were given the opportunity to develop their statistical knowledge and become the team's statistics expert.

### **Finding alternative representations**

One of the main challenges in our view is how to deal with the discrepancy between how statistical measures and techniques are typically represented at school on the one hand, and how they are used in practice on the other. In course materials standard deviation and the t-test are typically represented in a symbolic language with  $\Sigma$ -signs – a language that is inaccessible to most vocational students. Our impression from observations and previous research (Bakker et al., 2009) is that many teachers and trainers think the essence of, say, a t-test is captured by its

formula, just like the mean by its calculation, and that they see little opportunity to represent such concepts alternatively, or to emphasize their meaning in usage. However, what intermediate-level employees need to know about such techniques is what their purpose and utility (Ainley et al., 2006) are and how they should be interpreted when produced by a computer system, and some conditions of usage. To us it seems sufficient for student lab technicians who do not plan to attend higher professional education to know that a t-test is useful for comparing means of data sets (e.g., to check if a new instrument is as accurate as the standard), and what it means that there is a significant difference. The little time attributed to teaching the t-test (typically one lesson) is perhaps better spent on such insights, including how to perform a t-test in Excel, than on explaining and applying the formula.

The problem of representation of such statistical concepts and tests was dealt with by Bakker et al. (2009) in the context of process improvement in a car factory. To avoid the symbolic language about process capability indices, they designed relatively simple, visual computer tools with which employees could get a sense of what these indices conveyed, and how their indices could be manipulated by changing mean, control limits or specification limits. These tools proved to facilitate communication between employees with diverse educational background. We therefore expect that it is in principle possible to convey the practical usage and implications of many statistical concepts and techniques in the context of work without anxiety-evoking formulae. We see ample reason to continue this line of research, especially in vocational settings where we can observe the connections between school and work practices and test out ways to engage students in seeing statistics as useful for a clear purpose: work.

## ACKNOWLEDGEMENTS

The project is funded by NWO, the Netherlands Organisation for Scientific Research under grant number 411-06-205 (grant holder is prof. K. P. E. Gravemeijer). We thank Nathalie Kuijpers for correcting this manuscript.

## REFERENCES

- Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, 32, 21-36.
- Bakker, A., Hoyles, C., Kent, P., & Noss, R. (2006). Improving work processes by making the invisible visible. *Journal of Education and Work*, 19, 343-361.
- Bakker, A., Kent, P., Noss, R., & Hoyles, C. (2009). Alternative representations of statistical measures in computer tools to promote communication between employees in automotive manufacturing. *Technology Innovations in Statistics Education*, 3(2). Online: [www.escholarship.org/uc/item/53b9122r](http://www.escholarship.org/uc/item/53b9122r)

- Bakker, A., Wijers, M., Akkerman, S. F. (2010). The influence of technology on what vocational students need to learn about statistics: The case of lab technicians. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics (ICOTS8)*, Ljubljana, Slovenia. Voorburg, NL: ISI.
- Bakker, A., Wijers, M., Jonker, J., & Akkerman, S. F. (in press). The use, nature and purposes of measurement in intermediate-level occupations. *ZDM The International Journal on Mathematics Education*, 43(3). DOI: 10.1007/s11858-011-0328-3
- de Schutter, J. (2009). *Succesvol docentenstages realiseren: Ervaringen, praktijkvoorbeelden en aanbevelingen voor het mbo*. [Successfully realizing teacher apprenticeships: Experiences, examples and recommendations for Dutch MBO]. Hoorn: van Beekveld & Terpstra.
- Dierdorp, A., Bakker, A., Eijkelhof, H. M. C., & van Maanen, J. A. (in press). Authentic practices as contexts for learning to draw inferences beyond correlated data. To appear in *Mathematical Thinking and Learning*, 13(1&2).
- Hoyles, C., Bakker, A., Kent, P., & Noss, R. (2007). Attributing meanings to representations of data: The case of statistical process control. *Mathematical Thinking and Learning*, 9, 331-360.
- Hoyles, C., Noss, R., Kent, P., & Bakker, A. (2010). *Improving mathematics at work: The need for techno-mathematical literacies*. London: Routledge/Taylor & Francis.
- Latour, B. (1999). *Pandora's hope: Essays on the reality of science studies*. Cambridge, MA: Harvard University Press.
- Noss, R., Pozzi, S., & Hoyles, C. (1999). Touching epistemologies: Meanings of average and variation in nursing practice. *Educational Studies in Mathematics*, 40, 25-51.
- Van den Berg, N., & De Bruijn, E. (2009). *Het glas vult zich. Kennis over vormgeving en effecten van competentiegericht beroepsonderwijs; een review*. [The glass is filling up. Knowledge about the design and effects of competence-based vocational education. A review]. Amsterdam/Den Bosch: ECBO.
- Williams, J. S., & Wake, G. D. (2007). Black boxes in workplace mathematics. *Educational Studies in Mathematics*, 64, 317-343.

# CHILDREN'S EMERGENT INFERENTIAL REASONING ABOUT SAMPLES IN AN INQUIRY-BASED ENVIRONMENT

Dani Ben-Zvi<sup>1</sup>, Katie Makar<sup>2</sup>, Arthur Bakker<sup>3</sup>, & Keren Aridor<sup>1</sup>

<sup>1</sup>University of Haifa, <sup>2</sup>The University of Queensland, <sup>3</sup>Utrecht University

*Research on informal statistical inference has so far attended little to sampling. This paper analyzes children's reasoning about sampling when making informal statistical inferences in an inquiry-based environment. Using data from a design experiment in Israeli Grade 5 (age 11) classrooms, we focus on the emergent reasoning of two boys working with TinkerPlots on investigations with growing sample size. They turn out to have useful ideas about whether inferences can be made from samples of different sizes. Initially, they oscillate between deterministic and relativistic conclusions, but they come to reason in more sophisticated ways with increasing awareness of what is at stake when making inferences from samples.*

## INFORMAL STATISTICAL INFERENCE

One of key things that statistics allows us to do is to draw inferences from samples. Doing so with formal techniques such as estimation, confidence intervals or hypothesis testing goes well beyond what most students will have the opportunity to learn, yet it seems important to give them a sense of the power of statistics by making such inferences informally (Garfield & Ben-Zvi, 2008). For these reasons, statistics educators have studied *informal statistical inference*, characterized as a generalized conclusion expressed with uncertainty and evidenced by, yet extending beyond, available data (Makar & Rubin, 2009). Two special issues have already been dedicated to this theme (Makar & Ben-Zvi, in press; Pratt & Ainley, 2008).

In earlier work we addressed the question of how informal inferential reasoning, the reasoning processes leading to informal statistical inference, can be nurtured and supported (Makar, Bakker, & Ben-Zvi, in press). Supporting elements include statistical concepts and tools, knowledge of the problem context, inquiry drivers such as doubt, explanation, and resolution of cognitive conflicts. We proposed that an inquiry-based learning environment with suitable tasks and tools as well as teacher scaffolds is especially suitable to support students' informal inferential reasoning.

## SAMPLES AND SAMPLING

The concept of sample is a central concept in statistics yet it has received limited attention in the research literature compared to other concepts such as average, variation, and inference. Concepts and issues surrounding sampling are complex and require coordinating several ideas at once. Researchers note that students and teachers often conflate samples with their population when working with data (Lavigne & Lajoie, 2007; Pfannkuch, 2008; Pratt, Johnston-Wilder, Ainley, & Mason, 2008). Others caution that students may hold extreme beliefs about

relationships between samples and their population: Those focusing on sampling representativeness might believe that a sample provides complete information about a population, while students focusing on sampling variability might believe that a sample provides no information (Rubin, Bruce, & Tenney, 1990). Watson and Moritz (2000) found that children in their study (Grade 3, age 8-9) had fairly primitive notions of samples and were typically comfortable making claims from small samples with little concern about bias, while the older children (Grade 9, age 14-15) generally attended to both sample size and representativeness in making claims, recognizing potential problems of bias and variability of small samples. Students between these ages (Grade 6, age 11-12) held a diversity of beliefs about sample size and sampling, suggesting they are at a critical age in their development of concepts of sampling.

Recent interest has arisen about the potential of informal statistical inference as an organising principle in learning statistics. Several aspects of informal statistical inference have been addressed in the literature, but the role of sampling has received surprisingly little attention given its centrality in inference. Several researchers (e.g., Arnold & Pfannkuch, 2010; Konold & Kazak, 2008) have used hands-on activities, visualisations, and simulations in helping students coordinate the complex issues of sampling in inferential reasoning. In this paper we focus on the question: *How does children's reasoning about sampling emerge when making informal statistical inferences in an inquiry-based environment?* We use 2010 data from Ben-Zvi's Connections Project (Ben-Zvi, Gil, & Apel, 2005) to respond to this question by examining the work of two boys (aged 11) participating in a teaching experiment as they grapple with drawing inferences from samples of increasing sizes.

### **GROWING SAMPLES**

A key idea behind the design is that of growing samples—an instructional idea mentioned by Konold and Pollatsek (2002), worked out by Bakker (2004) and elaborated by Ben-Zvi (2006). Starting with small data sets (e.g.,  $n=8$ ), students are expected to experience the limitations of what they can infer from them about the whole class. They are next asked to draw conclusions from the whole class and speculate on what can be inferred about the whole grade in the school. Bakker (2004) found that such an approach is helpful in supporting coherent reasoning with key statistical concepts such as data, distribution, variability, tendency, and sampling. What-if questions proved particularly stimulating.

### **METHOD**

We address the research question by drawing on findings from a design study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) carried out in three Grade 5 classrooms in Israel. This study is part of the Connections Project—a longitudinal development



and research project (2005-2010) aiming to develop an inquiry-based environment for learning statistics in grades 4-6 using *TinkerPlots* (Konold & Miller, 2005).

### **The setting and participants**

The learning sequence was built around five cycles of extended data investigations (2-3 lessons of 90 minutes each) of a student-administered survey across several grades in their school. The survey gathered student information about dimensions of body parts, free time activities, pets ownership, etc. (33 variables,  $n=270$ ), creating a rich and interesting database for investigation. In each cycle, students posed a research question, organized their sample data using *TinkerPlots*, and made sense of it to draw informal inferences.

In line with the literature on growing samples, the design of activities evolved around the idea of starting from a sample of size 8, moving to about 30 (a whole class), then 90 (a grade level), and finally 270 cases (entire cohort) (see Figure 2 in Ben-Zvi, 2006). Starting with a small sample size was a pedagogical design decision to draw students' attention to the limitations of small samples, gradually developing their reasoning about samples, confidence level in their inferences, and "what-if" questions about larger sample sizes (e.g., "If you had a sample size of X, would the inference you just made still hold?").

Students worked in pairs through a scaffolded, open-ended inquiry of the data, with some pairs presenting their investigations in front of the class for further discussion. Independent investigations were videotaped using Camtasia to capture both their computer screen and faces. In this paper we focus on one pair of academically successful and articulate boys—Liron and Shay—in their first two investigations.

### **The episodes**

In the first episode—their first independent investigation with *TinkerPlots*—the pair studied issues about free time (e.g., what students do in their free time, preferred communication method). They were given a sample of 8 students from their class (including themselves) with eleven variables to analyze and make inferences beyond the data at hand and a handout with instructions and questions about sampling issues (e.g., "Would the conclusions you have reached apply also to half of the class? Please explain."). In the second episode, the sample was increased to 29 (whole class) and they were asked to see if their conclusions still held for the larger sample.

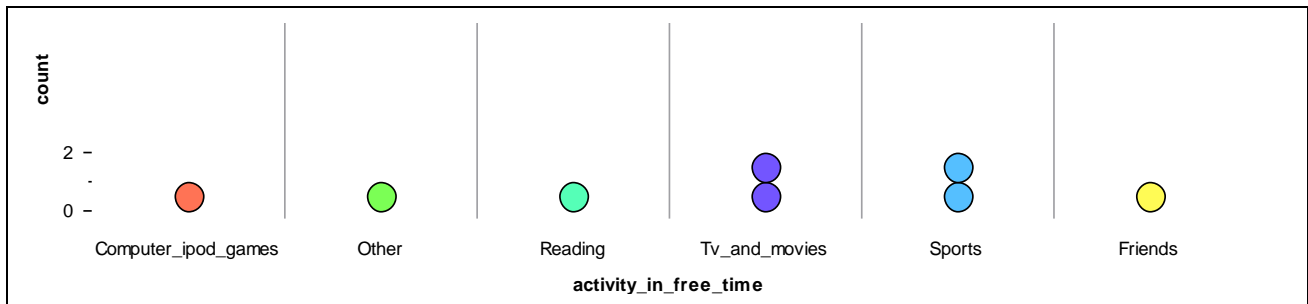
### **Data analysis**

Videos were observed, transcribed, translated from Hebrew to English and annotated for further analysis of the development of their reasoning about samples in relation to inferences. Interpretations were discussed until consensus was reached. Differences between Hebrew and English connotations of words (inference, conclusion, sample) were discussed extensively. Episodes were selected to illustrate the boys' developing reasoning about samples when making informal inferences.

## RESULTS

### First investigation, n=8

After orienting to *TinkerPlots*, Shay and Liron organized the small data set with the software. In their first graph, the eight data points spread across six categories of students' free time activities, with most categories having only one point (Figure 1).



**Figure 1: Liron and Shay's initial graph of 8 data points spread across 6 categories**

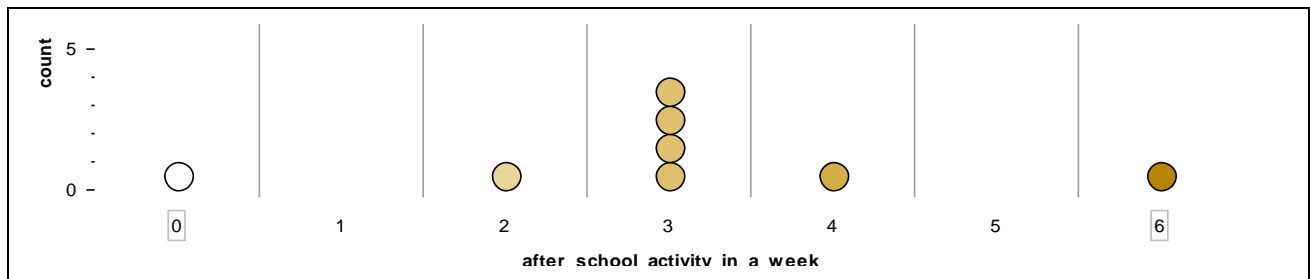
They expressed initial dissatisfaction with working from such a small set of data: “We only have 8 kids, we don't have enough data!... We don't have enough to know things properly!” (Shay, lines 9-12) and found it frustrating to draw conclusions. With only eight data points, they considered them not to be “real data”:

- 25 Shay: So let's see. Still, we don't have enough data that we can see because we have only 8 kids and it kind of spreads out. So we'll try to see something else, and then we will see if we have enough real data.
- 26 Liron: Pets is very easy. Let's check this first [scanning the pets data].
- 27 Shay: Well OK, we see it's [also] too spread out, and since we have only 8 kids, we don't have much to see. So let's try to see, err, what shall we try to see, Liron?

In looking for other variables to investigate, the boys seemed to search for an apparent pattern. With most variables having low frequencies in each category (due to the small sample), they characterized the data as too “spread out” to draw conclusions. This implies that their reluctance to draw conclusions involved both sample size and its relation to frequencies (in this case, large spread and a “flat distribution”, cf. Ben-Zvi et al., 2005) and suggests a need to acknowledge students' statistical conceptions more broadly (e.g., variability and distribution) when focusing on sampling. When exploring the number of after-school activities per week (Figure 2), the boys finally felt able to draw a conclusion from numerical data. These data were not spread out like in the variable *activity in free time*:

- 39 Liron: 3 is the biggest. It is the most common.
- 40 Shay: According to what we see, the ‘most mode one’ is 3. What we see is that it can also be said that the average here is 3. The average is 3.
- 45 Shay: ... Ok, wait, we have found something very interesting – we have the average number of children [activities] in a week is three. Let's save it.

### Working Group 5



**Figure 2: Liron and Shay's distribution with higher frequency in one value**

As they searched for interesting and sensible stories in their data, they reviewed several additional variables (e.g., methods of communication with friends). However, after having done these analyses, Shay dismissed them almost immediately:

- 56 Shay: So usually you use the telephone at home, some use the cell phone and chats on the internet. [But] as we said before, we don't have enough kids in order to have something, a proper result as needed.

Shay realized again that the small sample size was a flaw in the validity of the inference. The conflict of making valid conclusions from a small sample arose several times during the investigation. At other times, they set the issue aside and made fairly strong claims (e.g., "Boys like computers more than girls."). This oscillation between stating there is nothing to say and stating claims deterministically continued throughout this first investigation. We should cautiously note, however, that it is not clear whether Shay had changed his mind or went through the analysis with the impression that the results would be improper results. To overcome this concern, Liron proposed to collect more data.

One aim of the growing samples activities is to develop, as well as gain insight into, students' inferential reasoning about the relationship between conclusions drawn with a small sample and their inferences to larger samples (e.g., limitations of small samples, confidence about their inferences). When asked whether their claims applied more generally to the class, their initial response was quite confident that they would be. When pressed, however, their responses were qualified. "According to *these data for now*, only based on these, boys like computers more than girls and girls like to spend time with their friends more than boys" (line 198). Shortly thereafter, their responses changed again:

- 207 Int.: And what if I ask you about the whole class? What do you think your inferences look like?
- 208 Shay: It could be completely different!
- 210 Int.: ... What do you mean by "completely different"? Could it be, say, that girls like computers more than boys?
- 211 Shay: It will not be *that* different.
- 213 Int.: ... Won't it surprise you? If you see for example that the girls in the whole class, there will be more girls that like computers than boys?
- 214 Shay: Yes, it will surprise me, but there is a chance it will happen.

## Working Group 5

- 215 Int.: Why will it surprise you?
- 216 Shay: Because I know from personal experience that boys like computers more than girls, but there is a chance that my personal experience is wrong! There is a chance.

Although Shay thought the results in the entire class might be completely different than the sample results, his answer in line 211 is more nuanced. This remark may suggest that Shay anticipates that the results in the sample compared to the entire class could be similar and he sees a small chance that the results will be completely opposite. This is the first time the role of chance is mentioned in the discussion, not in explicit relation to the sample size but rather to the boy's personal knowledge of the context. In the end of this discussion, the interviewer explored the limits of what the boys were willing to conclude from different sample sizes:

- 243 Int: Can you say based on the conclusion from these data that they are true for a larger group of kids?
- 244 Liron: No. Every child has a different opinion.
- 245 Int: Shay, what do you think?
- 246 Shay: I agree. Each child has their own characteristics.
- 247 Int: How many kids in your class?
- 248 Liron: 29.
- 249 Int.: What if we ask 15 kids? Do you think then we could conclude something?
- 250 Liron: Yes.

We saw this same reasoning about the ability to infer from half of the population in discussions among other students as well. Even at the end of this activity, the boys still oscillated between not being able to say anything ("Every child has a different opinion") and being able to infer something regarding bigger samples. Their confidence in drawing conclusions oscillated during their investigation. They repeatedly asked for more data, offering even to go to the class and collect it themselves; at the same time, they often made claims based on a frequency of one (e.g., girls like to talk to friends more than boys). When probed, they were able to qualify the claims, but were uncertain whether their claims would hold more broadly.

### **Second investigation, n=29**

In the second investigation, students were given data for their entire class (29 students) as well as five additional variables (e.g., various body measurements, additional pet ownership data). Shay and Liron's immediate reaction was one of excitement "Wow! ... It's so much fun now!" (Lines 255-257). Before investigating the data, Liron anticipated that the conclusions would be different from those they made for the sample of eight. This suggests one way that the growing samples sequence stimulated students to think about sample-population relationships:

### Working Group 5

- 258 Int.: Do you think they will be similar or different for the entire class and why?
- 259 Liron: Different.
- 260 Shay: Not just different. We should say in what [way] they will be different.
- 261 Liron: Eight kids can't represent an entire class!

When the boys were given the data for the entire class, they compared their new findings (sample size 29) with those from the previous investigation (sample size 8).

- 265 Shay: I knew! Girls love to be with friends more than boys. Boys like computers more than girls. Here, here, my hypotheses are materializing!
- 266 Liron: Because?
- 267 Shay: Look, it's really beautiful. We really have conclusions!

Although the boys easily recognized the similarities, they didn't understand why.

- 289 Liron: Most of our hypotheses were confirmed. ...
- 292 Shay: ... But I don't understand. It is the same data. It is unfair. I don't get it.
- 293 Liron: Neither do I, but what do you know?
- 294 Shay: I was sure it would be entirely different data.
- 295 Liron: So was I, but here it is. You can see it.

The boys expressed surprise by the unexpected similarity then questioned its validity. Their language became more subtle, focusing on the uncertainty.

- 307 Shay: But you know it happened by chance.
- 308 Liron: It happened by chance. They didn't do it on purpose.

And later:

- 316 Shay: This quite surprised me, and I thought that if I take more kids, it will change, but who knows, apparently it is the same thing.
- 317 Liron: It must have been a coincidence.
- 318 Shay: Maybe, not for sure.

When asked for an explanation, Liron said he had no idea and Shay repeated that it happened by chance. Later, following the design idea of growing samples, they were asked again whether they expected the result to be similar for the entire 5<sup>th</sup> grade.

- 423 Shay: Now I think that if you take like, what happen is that it was [first only] 8. Now we took the entire class, it was exactly the same properties. So I think now that if you take the entire 5<sup>th</sup> grade, it will also be the same conclusions.
- 424 Int.: Why? Try to explain once more why? You just said something and I am not sure I got you.
- 425 Shay: Because from the conclusions here, we saw that once we expanded it, there were the same conclusions. Now what do I infer from this?

## Working Group 5

426 Int.: What?

427 Shay: That it is really true. That it will keep being true, also.

When probed, the boys informally quantified their level of confidence about the conclusions, both now and from the original sample of eight, but cautioned that the results would not necessarily extend to other ages in the school.

441 Int.: In the same age, if we take a scale from 1 to 10, ... how much are you certain from 1 to 10 that the results will remain the same?

442 Shay: 7.

443 Liron: I am also 7. ...

447 Int.: How sure were you from [the sample of] 8 about the entire class? ...

448 Shay: About 2. Maybe 3.

## DISCUSSION

The brief excerpts from Liron and Shay's investigations give us some insight into initial ways that children can reason about samples in an inquiry classroom designed to provoke their informal statistical inferences. The analysis of Liron and Shay's inferential reasoning about sampling issues shows a development from fairly extreme and seemingly contradictory views of what can be concluded from a small sample to more nuanced statements about the strength of their later claims and emerging quantification of confidence in making inferences. In the first investigation, the boys repeatedly expressed their lack of confidence in conclusions drawn from only eight data points while concurrently making fairly strong claims (e.g., boys like computers more than girls do) based on frequencies of only one or two data points. As they progressed, they qualified their claims as only holding for their limited data and were rightly conflicted about whether the sample would provide any information about a larger population. In the second investigation, the larger data set confirmed many of their previous conclusions, which surprised them. This confluence appeared to provoke them to question the way the data were gathered.

In addition to working with concepts of samples and informal statistical inference, strong links to other statistical key concepts arose during the investigation, such as average (average of 3 activities per week), spread and distribution, likelihood, randomness, and graph interpretation. The excerpts further underline Bakker and Derry's (in press) discussion of the importance of a holistic approach to concept development as these concepts were not encountered in isolation, but emerged collectively as relevant tools by provoking their reasoning in the context of inquiry. The opportunity to probe the students' reasoning was helpful to explore the scope, flexibility and robustness of students' concepts in action. This aligns with inferentialist approaches to have students reason within the context of complex problems and through that process find the scope and limit of their conceptions in the act of reasoning with them.

We end with a few questions for future research:

1. How can ideas about sampling in relation to informal statistical inference be further developed in the next grade?
2. What was the role of the activities design and the inquiry-based environment in the development of students' reasoning about samples and sampling? And in particular how can the instructional idea of growing samples can be further improved and used?
3. How can new tools (e.g., *TinkerPlots2*, see Konold et al., in press) probe students' inferential reasoning?

## ACKNOWLEDGEMENTS

We gratefully acknowledge Hana Manor who interviewed the two boys, and Einat Gil who supervised the activities design. The Freudenthal Institute funded part of Dani Ben-Zvi and Katie Makar's visit to Utrecht where the videos were analyzed.

## REFERENCES

- Arnold, P., & Pfannkuch, M. (2010). Enhancing students' inferential reasoning: From hands on to "movie snapshots". In C. Reading (Ed.), *Proceedings of the 8th International Conference on Teaching Statistics*. Voorburg, The Netherlands: International Statistical Institute.
- Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools*. Utrecht, the Netherlands: CD Beta Press.
- Bakker, A., & Derry, J. (in press). [Lessons from inferentialism for statistics education](#). To appear in *Mathematical Thinking and Learning*, 13(1&2).
- Ben-Zvi, D. (2006). Scaffolding students' informal inference and argumentation. In A. Rossman & B. Chance (Eds.), *Proceedings of the 7<sup>th</sup> International Conference on Teaching of Statistics* (CD-ROM), Salvador, Bahia, Brazil, 2-7 July, 2006. Voorburg, The Netherlands: International Statistical Institute.
- Ben-Zvi, D., Gil, E., & Apel, N. (2005). What is hidden beyond the data? Helping young students to reason and argue about some wider universe. In D. Pratt & J. Ainley (Eds.), *Reasoning about informal inferential statistical reasoning: A collection of current research studies*. Proceedings of the Fifth International Research Forum on Statistical Reasoning, Thinking, and Literacy (SRTL-5), University of Warwick, UK, August, 2007.
- Cobb, P., Confrey, J., diSessa, A. A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32, 9-13.
- Garfield, J., & Ben-Zvi, D. (2008). *Developing students' statistical reasoning: connecting research and teaching practice*. New York: Springer.

- Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education*, 2(1), Article 1.
- Konold, C., Madden, S., Pollatsek, A., Pfannkuch, M., Wild, C., Ziedins, I., Finzer, W., Horton, N. J., & Kazak, S. (in press). Conceptual challenges in coordinating theoretical and data-centered estimates of probability. To appear in *Mathematical Thinking and Learning*, 13(1&2).
- Konold, C., & Miller, C. D. (2005). *TinkerPlots: Dynamic data exploration* (Version 1.0) [Computer software]. Emeryville, CA: Key Curriculum Press.
- Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education*, 33(4), 259-289.
- Lavigne, N., & Lajoie, S. (2007). Statistical reasoning of middle school children engaged in survey inquiry. *Contemporary Educational Psychology*, 32, 630-666.
- Makar, K., Bakker, A., & Ben-Zvi, D. (in press). The reasoning behind informal statistical inference. To appear in *Mathematical Thinking and Learning*, 13(1&2).
- Makar, K., & Ben-Zvi, D. (Eds.) (in press). The role of context in developing reasoning about informal statistical inference. Special Issue of *Mathematical Thinking and Learning*, 13(1&2).
- Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal*, 8(1), 82-105.
- Pfannkuch, M. (2008, July). Building sampling concepts for statistical inference: A case study. Paper presented at the 11<sup>th</sup> International Congress on Mathematics Education, Monterrey, Mexico.
- Pratt, D., & Ainley, J. (Eds.) (2008). Informal inferential reasoning. Special Issue of the *Statistics Education Research Journal*, 7(2), 3-4.
- Pratt, D., Johnston-Wilder, P., Ainley, J., & Mason, J. (2008). Local and global thinking in statistical inference. *Statistics Education Research Journal*, 7, 107-129.
- Rubin, A., Bruce, B., & Tenney, Y. (1990). *Learning about sampling: Trouble at the core of statistics*. Paper presented at the annual meeting of the American Educational Research Association, Boston, MA.
- Watson, J. M., & Moritz, J. B. (2000). Developing concepts of sampling. *Journal for Research in Mathematics Education*, 31, 44-70.



# INVESTIGATING RELATIVE LIKELIHOOD COMPARISONS OF MULTINOMIAL, CONTEXTUAL SEQUENCES

Egan J Chernoff

University of Saskatchewan

*By focusing on a particular alteration of the relative likelihood task, this study contributes to research on teachers' knowledge of probability. The novel task presented prospective teachers with multinomial, contextualized sequences (possible answer keys to a 10 question multiple-choice quiz) and asked them to identify which sequence was least likely. Results demonstrate that the sample to population ratio and reflection of randomness determinants of representativeness (featured in research on binomial, platonic sequences) are present in the current situation as well. Results further suggest that the context in which tasks are presented significantly influences probability judgments. Consideration of context also provides a new lens and, concurrently, potential obstacles for analysing certain results.*

In the first *Handbook of Research on Mathematics Teaching and Learning*, Shaughnessy's seminal (1992) review of "research in learning and teaching stochastics" (p. 466) concluded with "a wish list" (p. 488) for future research. Included in the list, a call for investigation into teachers' conceptions of probability. Fifteen years later, in the *Second Handbook of Research on Mathematics Teaching and Learning*, Jones, Langrall, and Mooney (2007), who were given the "main task of reviewing and analyzing research in probability education during the period since Shaughnessy's (1992) review" (p. 910), included "Stohl's (2005) review[, which] concluded that there had been limited response to Shaughnessy's call for research on teachers' knowledge and beliefs about probability" (p. 945), in their research synthesis.

Given the dearth of research documented above, the purpose of this article, in general, is to contribute to research investigating teachers' conceptions of probability. In specific, the purpose of this article is to present results demonstrating that Kahneman & Tversky's (1972) representativeness determinants (described in detail below) extend to multinomial sequences (i.e., sequences derived from a multinomial experiment) and contextual sequences. In order to examine the relationship between representativeness and multinomial and contextualized sequences, prospective mathematics teachers are asked to compare the relative likelihood of two answer keys for a 10 question multiple choice (i.e., A, B, C, or D) math quiz.

## THEORETICAL CONSIDERATIONS

Kahneman and Tversky (1972), in examining how "people replace the laws of chance by heuristics" (p. 430), produced an initial investigation into what they called

*the representativeness heuristic*. According to their findings, an individual who follows the representativeness heuristic “evaluates the probability of an uncertain event, or a sample, by the degree to which it is: (i) similar in essential properties to its parent population; and (ii) reflects the salient features of the process by which it is generated” (p. 431). Kahneman and Tversky theorized that events are considered more probable when appearing more representative and, similarly, events are considered less probable when appearing less representative. In order to test their theory, the authors focused on some now well-known probability comparisons.

Kahneman and Tversky (1972) presented individuals with birth sequences that were considered equally likely, but were hypothesized by the authors to not be “equally representative” (p. 432). Of the three sequences presented (GBGBBG, BGBBBB and BBBGGG), the sequence BGBBBB was considered less likely than GBGBBG because BGBBBB does not reflect the ratio of boys to girls found in the parent population. Further, BBBGGG was deemed less likely than GBGBBG because BBBGGG did not reflect the random nature associated with the birthing of boys and girls in a family. While the findings supported the authors’ initial hypotheses (more representative sequences would be judged more likely) they also declared that similarity of a sample to the parent population is a necessary, but not sufficient, determinant of representativeness. As such, they further investigated the reflection of randomness determinant.

In an investigation into the appearance of randomness, Kahneman and Tversky (1972) declared that “two general properties, irregularity and local representativeness, seem to capture the intuitive notion of randomness” (p. 433). To explicate their point, the authors showed how alternating sequences, e.g., a perfect alteration of heads and tails, was too regular and thus would not correspond to the result of a random process. Local representativeness, on the other hand, is the belief that “the essential characteristics of the parent population are represented not only globally in the entire sample, but also locally in each of its parts” (p. 434). For example, individuals, when examining a short sequence of coin tosses, would expect (1) the ratio of heads to tails to be close to one, and (2) short runs, which would correspond to frequent alterations, because, as the authors indicated, “People view chance as unpredictable but essentially fair” (p. 434).

## **TASK DESIGN AND RATIONALE**

Although not the first individuals to conduct an experiment comparing the relative likelihood of sequences of outcomes, researchers of, and literature in, probability education consider – by acclamation – the early seventies research of Tversky and Kahneman (e.g., Kahneman & Tversky, 1972; Tversky & Kahneman, 1974) as *the* canonical research into comparing the relative likelihood of sequences of outcomes.

## Task developments

Kahneman and Tversky's (1972) investigation into the relative likelihood of sequences of outcomes (detailed above) was, actually, two tasks: A task was first presented that would address the sample to population ratio determinant of representativeness (i.e., are there more families with a birth order sequence of BGBBBB or GBGBBG?) and, second, a related task was presented to address the reflection of randomness determinant (i.e., are there more families with a birth order sequence of GBGBBG or BBBGGG?). These two tasks would undergo a number of changes, as they became a fixture in the field of mathematics education.

Shaughnessy's (1977) research introduced two new, important developments to the tasks. First, in comparing the chances of occurrence of different sequences, his version of the task gave students the option of choosing "(c) about the same chance" (p. 309) as one of the response items. Second, Shaughnessy asked participants to "give a reason for your answer" (p. 309). Despite these two new developments, one thing remained the same: the task remained as two tasks.

Konold, Pollatsek, Well, Lohmeier, and Lipson's (1993) version(s) of the relative likelihood task unified, for the first time, into one task, both the sample to population ratio determinant and the reflection of randomness determinant.

Which of the following is the most likely result of five flips of a fair coin?  
a) HHHTT b) THHTH c) THTTT d) HTHTH e) All four sequences are equally likely

**Figure 1.** Konold et al.'s (1993, p. 395) iteration of the relative likelihood task

As seen in Figure 1 above, three of the options presented have a ratio of 3 heads to 2 tails, while option c) has a ratio of 4 heads to 1 tail, which is consistent with earlier tasks investigating sample to population determinants. The options containing 3 heads and 2 tails present a variety of symmetrical, switch, and run considerations, which is also consistent with earlier tasks examining the reflection of randomness determinant. Further, Konold et al.'s (1993) iteration of the task adopts Shaughnessy's (1977) developments, which not only provides the equally likely option, but also "also asked subjects to provide a written justification for their answer" (p. 396). Despite Chernoff's (2009) alteration, which maintains the same ratio of heads to tails in all sequences, all other versions of the relative likelihood task found in mathematics education (e.g., Hirsch & O'Donnell, 2001; Rubel, 2007) adopt a similar framework to Konold et al.'s (1993) version of the task.

## Task consistencies

Despite developments and new versions of the task, two things have remained the same. First, the use of sequences of outcomes derived from a binomial experiment (hereafter referred to as binomial sequences) and, second, the use of what I will define below as, *platonic sequences*.

The majority of research on the relative likelihood of sequences of outcomes (e.g., Chernoff, 2009; Cox & Mouw, 1992; Hirsch & O'Donnell, 2001; Kahneman & Tversky, 1972; Konold et al., 1993; Rubel, 2007; Shaughnessy, 1977, 1981), thus far, utilizes, exclusively, binomial sequences. Although certain research does investigate sequences where the probability of success does not equal failure (e.g., Konold et al., 1993), multinomial experiments, such as the rolling a six-sided die, are reduced to binomial experiments, by, for example, painting the sides of the die two colors.

To help frame this discussion, I will distinguish between two types of sequences, which I have denoted as *platonic* and *contextualized*. *Platonic sequences*, which currently dominate research literature, are characterized by their *idealism*. For example, a sequence of coin flips derived from an ideal experiment (where an infinitely thin coin, which has the same probability of success as failure, is tossed repeatedly in perfect, independent, identical trials) would represent a platonic sequence. On the other hand, *contextualized sequences*, which are less represented in the current research literature, are characterized by their *pragmatism*. For example: the sequence derived from the severed left and right feet, which were washing up on the shores of British Columbia, Canada in late 2007 and early 2008 (e.g., LLLLLR); the sequence of six numbers obtained when buying a (North American) lottery ticket (e.g., 4, 8, 15, 16, 23, 42); the answer key to a true or false quiz (e.g., TFTTF); the answer key to a multiple choice mathematics quiz or test (e.g., ACCBDCAADB); and others, would represent contextualized sequences.

As I claim above, platonic sequences currently dominate the research literature; however, this *platonicity*, which Taleb (2007) defines as the “tendency to mistake the map for the territory” (p. xxv), of the sequences, which are devoid of context, occurred over time. For example, in the ‘beginning’, Tversky and Kahneman (1972) incorporated a frequentist or experimental perspective with their sequence of six children, when declaring, in their task, that “72 families” (p. 34) were surveyed. As the task migrated from psychology to mathematic education, platonicity began to occur. Shaughnessy’s (1977) research witnesses the first explicit move towards ideal sequences. In his version of the task, Shaughnessy declares, up front, that “the probability of having a baby boy is about  $1/2$ ” (p. 309) and, in the second half of his task states, “(same assumptions as [task 1])” (p. 309). However, the assumptions explicitly stated in Shaughnessy’s version became implicit in subsequent versions of the task. By the beginning of the 1990’s the platonification of the experiment and the sequences, in essence the entire task, was complete. For example, the research of Chernoff (2009), Cox & Mouw (1992), Hirsch & O’Donnell (2001), Konold et al. (1993), and Rubel (2007) all utilize the following phrase: *A fair coin is tossed  $x$  times*; which, in these cases, is intended to mean (from a rationalist perspective) that an infinitely thin coin, which has the same probability of success as failure, is tossed repeatedly in perfect, independent, identical trials to produce the sequences of coin

flips. In other words, the map is mistaken for the territory. Despite the platonification of the task, pragmatic responses are a mainstay in individuals' responses and are part of the motivation for the new version of the task.

## THE ANSWER KEY TASK

Given the domination of binomial and platonic sequences found in the literature, the new task contributes to existing research by utilizing multinomial, contextualized sequences. The new version of a probability comparison task, as seen in Figure 1, asked individuals to compare the relative likelihood of two different answer keys to a 10 question multiple choice quiz and to explain or justify their answer.

Which of the following, answer key 1 or answer key 2, is *least likely* to be the answer key for a 10 question multiple choice math quiz? Explain your answer

**Answer key 1:** A C C B D C A A D B **Answer key 2:** C C C B B B B B B B

**Figure 1. Answer key version of the comparative likelihood task**

## PARTICIPANTS AND RESULTS

Data for this study was gathered by asking participants, 59 prospective teachers, to respond, in writing (with no time limitations), to the task presented in Figure 1. Results show that the majority (48 / 59) of the participants in the study (81%) chose answer key 2 (hereafter refereed to as AK2 and, similarly, answer key 1 as AK1) as least likely to be the answer key for a 10 question multiple-choice quiz. More specifically, 23 participants (74%) in Class A and 25 participants (89%) in Class B chose AK2 to be least likely. Given similar themes in the identified response justifications, the impending analysis of results does not distinguish between the two classes.

## ANALYSIS OF RESULTS

In this section, exemplary response justifications from the 48 individuals who chose AK2 as least likely to be the answer key for a 10 question multiple choice quiz are analyzed. Responses justifications are organized into two main sections: (1) multinomial considerations and (2) contextual considerations.

### Multinomial considerations

As noted, Kahneman and Tversky (1972) declared two determinants of representativeness as sample to parent population and reflection of randomness. As such, the analysis of responses for determinants of representativeness extending to multinomial sequences is similarly broken into (1) sample to parent population and (2) reflection of randomness sections.

**Sample to population.** The response justifications for 9 particular individuals demonstrate that representativeness – more specifically “the features that determine the similarity of a sample to its parent population” (Kahneman & Tversky, 1972, p. 33) – extend from binomial to multinomial sequences of outcomes. While all 9 individuals declared the proportion of multiple choice answers presented in AK2 (i.e., 3 C’s and 7 B’s) does not reflect the proportion of multiple choice answers for the population, different individuals presented the notion in different ways, as seen in the responses of Adam and Ben.

Adam: AK2 because there is too little variety of answers.

Ben: AK2 is least likely because it has only C’s and B’s!

While implicitly stated in certain responses, Frank is explicit in his use of percentages and expected frequencies when declaring that the sample population should have an equal distribution of available answers.

Frank: There are 4 possible letters, so each should show up around 25% of the time. This is true of AK1 (A:30% B:20% C:30% D:20%). So AK2 (C:30% B:70%) is least likely.

Ike’s response epitomizes the similarity of sample to population determinant of representativeness extended to multinomial sequences.

Ike: AK2 upon first impression because it doesn’t appear to be random, since each question has 1 in 4 chances of either being A, B, C, or D. The likelihood/probability of that occurring is low.

For Ike, AK2 is less likely than AK1 because AK2, by not having an even distribution of answers A, B, C, and D, “doesn’t appear random.” In other words, AK2, with only B’s and C’s, is not representative of the sample to parent population and is influencing Ike’s (and Adam, Ben, and Frank’s) appearance of randomness, which, in turn, is influencing his probabilistic judgment, because, according to Kahneman and Tversky’s (1972) (confirmed) hypothesis, less representative sequences are deemed less likely.

**Reflection of randomness.** As with the sample to parent population determinant, it will be shown that the reflection of randomness determinant, more specifically local representativeness, extends from binomial to multinomial sequences of outcomes. Kahneman and Tversky’s (1972) notion of local representativeness (which states “a representative sample is one in which the essential characteristics of the parent population are represented globally in the entire sample, but also locally in each of its parts” (p. 36)) influences the relative likelihood of the answer keys for two participants, whose answers are featured in the responses below.

Jen: AK2 [...] because the probability of sequential answers being identical is low.

## Working Group 5

Kate: AK2 is the least likely because there would not be so many “B” answers in a row.

For Jen, the chances of “sequential answers being identical” (i.e., 3 C’s and then 7 B’s) is not representative for part of the answer key and, thus, not likely to be the answer key. Similarly for Kate, “the so many ‘B’ answers in a row” is not locally representative and, thus, she concludes AK2 is least likely to be the answer key. Although stated differently by Jen and Kate, for both, the long run of 7 B’s found in the latter section of AK2 is not locally representative. That is, the essential characteristics of the parent population are not found in the BBBBBBBB section of the answer key. As such, the entire sequence, which, for them, does not appear random, is not representative of a 10 question multiple-choice answer key. Given the answer key is not (locally) representative, AK2 is considered less likely to be the answer key.

### Contextual considerations

According to Kahneman and Tversky (1972), “As is true of the similarity of sample to population, the specific features that determine apparent randomness *differ depending on context*” (p. 35, my italics). Taking into account certain contextual considerations associated with the answer key task, however, demonstrates that an innate answer key structure (not only the determinants of representativeness) can account for probability comparison responses. To get a sense of the innate structure of answer keys, a variety of perspectives – including: answer key, personal experience, teacher, student, and combined or multiple perspectives – will be presented with the main goal of presenting the subject under investigation, that is, innate answer key structure, in a greater context.

**Answer key perspective.** The responses from six individuals took the answer key into consideration in their response justifications and, in doing so, they provided insight into the properties associated with an innate structure of answer keys. For example Mary (like Jen and Kate above) references the length of runs for answers C and B, which are found in AK2. However (unlike Jen and Kate), the respondent qualifies that long runs are not a feature found with answer keys.

Mary: answer keys usually do not have a constant answer for consecutive questions in a row. They are usually mixed up upon each other and will occasionally be in a row.

Similarly, the justification of Quinn, like that of Adam and others above, makes reference to the lack of variety of answers found in AK2. However, unlike Adam and others above, Quinn also qualifies that the lack of variety is not a feature found in answer keys.

Quinn: #2 because it seems there’s only two possible answers for this quiz – not multiple choice.

## Working Group 5

For all six respondents, the long run of B's and the lack of variety amongst the available answers found in AK2 are reasons why AK2 is deemed less likely. However, the features they mention are inherent, for these individuals, to an innate structure associated with the answer key and answer keys in general.

**Personal experience.** Instead of qualifying responses with references to the structure of answer keys, six individuals made reference certain properties associated with their personal experiences with answer keys. For example, the reference to a lack of variety of answers, seen below in the response Tara, and the reference to the long run of the one answer, see below in the response of Uma, exemplify responses seen from other individuals who also referred to their personal experience with answer keys.

Tara: From my experience with multiple-choice exams, the answers never line up one after the other, like in AK2. The multiple choice exams I studied for such as math, have always looked more like AK1, where there is a variety of answers such as ACCBD instead of CCCBBB.

Uma: there are too many answers that are the same ex)cccbbb. This (as a student) always made me confused. If the answers are all in a line like that, it makes the student feel like they did something wrong.

However, for Tara and Uma, as was the case with those individuals who referenced answer key perspective, the features they describe, i.e., long runs and a lack of variety are inherent, for these individuals, to their personal experiences with answer keys and answer key structure.

**Teacher perspective.** Certain individuals, three in total, projected themselves into a teacher's perspective for their response justifications. Wendy's response provides insight into how she perceives teachers' use of and experience with answer keys.

Wendy: Normally teachers or instructors who set up answer keys tend to highlight a number of letters and use variation. A teacher would very rarely chose B to represent an answer 7 times in a row, as they usually seem to make random answer keys according to correct letters.

In her response, Wendy references both a lack of variation (i.e., sample to parent population) and the answer B appearing 7 times in a row (i.e., local representativeness) as reasons for AK2 being least likely. She notes that there would not be a of consecutive answers because this is not something that instructors who set up answer keys would do because teachers would make more "random" answer keys. Seen again, the respondent references a lack of variety and a long run of one answer as features in AK2; however, the justifications for these individuals are based on how they inherently perceive a teacher would set up the answer key.

**Student perspective.** A focus on the presence of a pattern was apparent in the responses justifications of 6 individuals who projected themselves into a student's perspective. Similar to previous perspectives analyzed, their responses provide



## Working Group 5

insight into how they perceive students' use of and experience with patterns and answer keys. For example, and as seen in the response justification below, the presence of a pattern for AK2 and the absence of a pattern for AK1 led to the declaration that AK2 is less likely than AK1.

Doug: I think AK2 is least likely to be the answer key because there is only 2 lines going straight down. AK1 has a zig-zag and it just seems better to have the answers all over rather than a boring pattern. Everyone knows the answers don't follow a pattern, if they did, everyone would get the answers right.

Doug's response deviates from previous perspectives because it does not, at least explicitly, reference the small variety of available answers presented and the long run of B's. Instead, the respondent qualifies that patterns are a feature not inherent to multiple-choice exams because students will be able to pick up any pattern that exists. Therefore, multiple choice quizzes containing a pattern are less likely than quizzes containing a pattern, because of students' acute ability to pick up on the pattern and, thus, the integrity of the test is compromised. However, Doug's response does not deviate from previous perspectives because it, too, is based on how the respondent inherently perceives a student would interact with an answer key.

**Multiple Perspectives.** The response justifications from 13 individuals, who also chose AK2 as least likely, combined many of the perspectives detailed above. For example, the following response combines all previously detailed perspectives.

Mike: AK2 is least likely. Answer keys go something like a rhyming scheme: ABBACC etc. A "teacher" would never give so many consecutive correct answers under the same letter. It would both corrupt the integrity of the test and play mind games with the student.

Mike's response, like those of the other 12 individuals, references the long string of one answer in a row and, further, with his notion of a "rhyming scheme," references a pattern or lack thereof inherent to answer keys. However, Mike justifies his stance according to interplay of: personal experience, the teacher's perspective, student perspective, and the answer key. His response exemplifies an answer-key ethos and, for Mike, is the basis for why he considers AK2 as least likely to be the answer key.

## DISCUSSION AND CONCLUSION

Theoretically speaking, the answer keys presented to participants are equally likely to occur. In fact, each of the 1 048 576 possible answer keys are equally likely to occur. Despite this fact, the majority, over 80%, of participants in this study indicated that AK2 was least likely to be the answer key for a 10 question multiple-choice exam.

On the one hand, for individuals who referenced the multinomial nature of the sequences they were presented, the little variety of answers (i.e., no use of A and D) and the long run of answer B justified their responses of AK2 being least likely. As

such, it appears that certain representativeness determinants (specifically and respectively: sample to parent population and the local representativeness component of reflection of randomness) extend to multinomial sequences.

On the other hand, for individuals who referenced the contextual aspects of the sequences they were presented, the little variety of answers, the long run of B responses, and the presence of a pattern in AK2 and lack of a pattern in AK1 were all used as justifications for why AK2 was deemed less likely. As such, it *may* also be argued that the representativeness determinants (i.e., sample to parent population and both components – local representativeness and irregularity – of perceived randomness) extend to contextual sequences. However, the innate structure of answer keys, also revealed in the analysis of results, provides a new way to account for these response justifications and, simultaneously, provides a potential obstacle for research(ers).

According to the representativeness heuristic, people expect sequences of outcomes to accurately reflect the parent population, to have frequent switches and short runs and irregularity. However, as shown in the analysis of results, specifically for answer keys, people already expect a variety of answers coupled with frequent switches and short runs and irregularity, as they feel these features are inherent to answer keys. As such, it becomes difficult to declare whether, for certain contextual sequences, like in the answer key task, participants are employing representativeness determinants or, in this instance, the representativeness lens is rendered moot due to contextual considerations. To better understand whether the representativeness heuristic influences innate answer key structure or whether answer key structure influences the representativeness heuristic, further research in this area will need to be conducted.

## REFERENCES

- Chernoff, E. J. (2009). Sample space partitions: An investigative lens. *Journal of Mathematical Behavior*, 28(1), 19-29.
- Cox, C., & Mouw, J. T. (1992). Disruption of the representativeness heuristic: Can we be perturbed into using correct probabilistic reasoning? *Educational Studies in Mathematics*, 23(2), 163-178.
- Hirsch, L. S., & O'Donnell, A. M. (2001). Representativeness in statistical reasoning: Identifying and assessing misconceptions. *Journal of Statistics Education*, 9(2). [Online: <http://www.amstat.org/publications/jse/v9n2/hirsch.html>]
- Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, (pp. 909-955). New York: Macmillan.

### Working Group 5

- Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, 3, 430-454.
- Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education*, 24(5), 392-414.
- Rubel, L. H. (2007). Middle school and high school students' probabilistic reasoning on coin tasks. *Journal for Research in Mathematics Education*, 38(5), 531-556.
- Shaughnessy, J. M. (1977). Misconceptions of probability: An experiment with a small-group, activity-based, model building approach to introductory probability at the college level. *Educational Studies in Mathematics*, 8, 285-316.
- Shaughnessy, J. M. (1981). Misconceptions of probability: From systematic errors to systematic experiments and decisions. In A. Schulte (Ed.), *Teaching Statistics and Probability: Yearbook of the National Council of Teachers of Mathematics* (pp. 90-100). Reston, VA: NCTM.
- Shaughnessy, J. M. (1992). Research in probability and statistics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: Macmillan.
- Taleb, N. N. (2007). *The black swan: The impact of the highly improbable*. New York: Random House.
- Tversky, A., & Kahneman, D. (1974). Judgment under uncertainty: Heuristics and biases. *Science*, 185, 1124-1131.

# PROSPECTIVE TEACHERS' COMMON AND SPECIALIZED KNOWLEDGE IN A PROBABILITY TASK

J. Miguel Contreras<sup>1</sup>, Carmen Batanero<sup>1</sup>, Carmen Díaz<sup>2</sup>, and José A. Fernandes<sup>3</sup>

<sup>1</sup>University of Granada, Spain; <sup>2</sup>University of Huelva, Spain,

<sup>3</sup>University of Minho, Portugal

*The aim of this research was to assess the common and specialized knowledge of elementary probability in a sample of 183 prospective primary school teachers in Spain, using an open-ended task. Common knowledge of probability was assessed in the first part of the task, where teachers had to compute simple, compound and conditional probability from data presented in a two-way table. The specialized knowledge of probability was assessed in the second part of the task, where teachers were asked to identify and classify the mathematical content in the problem proposed. Results suggest participants' poor common and specialized knowledge of elementary probability in this task and point to the need of reinforcing the preparation of prospective teachers to teach probability.*

## INTRODUCTION

The reasons for including probability in schools have been repeatedly highlighted over the past years (e.g., Gal, 2005; Jones, 2005): usefulness of probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions, and the important role of probability reasoning in decision making. Consequently, probability has recently been included in the primary school curriculum in many countries, where changes do not just concern the age of learning or the amount of material, but also the approach to teaching (Franklin et al., 2005). The success of these curricula will depend on the extent to which we can convince teachers that probability is an important topic for their students, as well as on the correct preparation of these teachers. Unfortunately, several authors (e.g., Franklin & Mewborn, 2006; Chick & Pierce, 2008) agree that many of the current programmes still do not train teachers adequately for their task to teach statistics and probability. The above reasons suggest to us the need to reinforce the specific and didactic preparation of primary school statistics teachers, and also the relevance of assessing the teachers' difficulties and errors in learning the topic.

## Components in Teachers' Knowledge

An increasing number of authors have analysed the nature of knowledge needed by teachers to achieve truly effective teaching outcomes. Shulman (1987) described "pedagogical content knowledge" (PCK) as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 6). Ball and her colleagues (Ball, Lubienski, & Mewborn, 2001; Hill, Ball, & Schilling, 2008) developed the notion of

“mathematical knowledge for teaching” (MKT) in which they distinguished six main categories (see Ball et al., 2001 for a comprehensive description). Our research was intended to assess two of these components in prospective primary school teachers in relation to elementary probability. More specifically we were interested in the following components of teachers’ knowledge:

- *Common content knowledge* (CCK) or the mathematical knowledge teachers are responsible for developing in their students; that is, the mathematical knowledge that is typically known by competent adults (Hill et al., 2004). In this research we assess common knowledge of elementary probability with a task where teachers are asked to compute single, compound and conditional probability from a two-way table.
- *Specialized content knowledge* (SCK). In addition to common knowledge, teachers need to know the content they teach in ways that differ from what is typically taught and learned in mathematics courses. SCK is the mathematical knowledge that is used in teaching, but not directly taught to students (Hill et al., 2004). We include here the ability to recognise what probabilistic concepts or properties can be addressed in the teaching tasks and resources (that was considered by Chick & Pierce, 2008 as a part of PCK). To assess this knowledge, in this research participants are asked to identify and classify the mathematical content they used to solve the first part of the task.

Below we first summarise related previous research and then describe the method and results in this study.

## PREVIOUS RESEARCH

### Two-Way Tables and Conditional Probability

A two-way or contingency table serves to present in a summarised way the frequency distribution in a population or sample that was classified according to two statistical variables (an example is included in the task presented in Figure 1). Research on contingency tables, started with the pioneer study by Inhelder and Piaget (1955), and focused on students’ strategies and conceptions when assessing association between the variables in rows and columns from the data presented in a two-way table (e.g., Batanero, Estepa, Godino, & Green, 1996). More recently research has focussed on students’ performance when computing probabilities with data presented in a two-way tables (see Huerta, 2009, for an analysis of the structure of these problems).

Also relevant for this study is research related to conditional probability, such as that by Falk’s (1986) who remarked that many students do not adequately discriminate between the two different conditional probability, that is,  $P(A/B)$  and  $P(B/A)$ . Falk termed this confusion as *fallacy of transposed conditional*. Einhorn and Hogarth (1986) observed that some students confused joint and conditional probability because they misinterpreted the conjunction “and”.

## Teachers' Probabilistic Knowledge

The scarce research related to primary school teachers' understanding of probability indicates this understanding is weak. For example, Begg and Edward (1999) found that only about two-thirds of the in-service and pre-service primary school teachers in their sample understood equally likely events and very few understood the concept of independence. Batanero, Cañizares, and Godino's (2005) found three widespread probabilistic misconceptions in a sample of 132 pre-service teachers related to representativeness (Tversky & Kahneman, 1982), equiprobability (Lecoutre, 1992) and the outcome approach (Konold, 1991). Fernandes and Barros (2005) study with 37 pre-service teachers in Portugal suggested the teachers' difficulties to formulate events and to understand compound and certain events. In addition, these teachers frequently used additive reasoning to compare probabilities.

In relation to knowledge needed to teach probability, Stohl (2005) suggested that few teachers have prior experience with conducting probability experiments or simulations and many of them may have difficulty implementing an experimental approach to teaching probability. In Lee and Hollebrands's (2008) research, although the participant teachers engaged students in investigations based on probability experiments, their approaches to using empirical estimates of probability did not foster a frequentist conception of probability. Teachers almost exclusively chose small samples sizes and rarely pooled class data or used representations supportive of examining distributions and variability across collections of samples so they failed to address the heart of the issue.

Estrada and Díaz (2006) asked 65 prospective primary school teachers, who had followed a 60 hours long course in statistics education at the University of Lleida, in Spain, to compute simple, compound and conditional probability from data presented in a two-way table and analysed the solutions provided by these teachers. The authors found a large proportion of participants who were unable to provide any solution to the problems. There were a variety of errors, including confusion between compound and conditional probability, confusion between an event and its complementary, confusion between probabilities with possible cases (absolute frequencies), and assuming independence in the data. The aim of the present paper is to expand the research by Estrada and Díaz (2006) with a bigger sample of prospective teachers, who had not followed a specific course in statistics education. In addition, the second part of the task is intended to assess the SCK of probability that was not taken into account in Estrada and Díaz's research.

## METHOD

The sample in the study consisted of 183 prospective teachers at the Faculty of Education, University of Granada, Spain. The task analysed in this paper was answered individually by each participant as a part of the final assessment in a course of Mathematics Education. In this course (60 teaching hours), the prospective

## Working Group 5

teachers are introduced to the primary school mathematics curriculum, didactic resources, children's difficulties, and technological tools for teaching elementary mathematics. Most sessions are devoted to practical work, in which participants performed didactic analyses (including identification of mathematical content) of curricular guidelines, school textbooks, assessment items and children responses to these items, and teaching episodes. Several sessions of the course are devoted to probability and statistics education. The previous year all these prospective teachers took a Mathematics course (90 teaching hours) with about 10 hours of in-classroom work and 40 additional hours of extra-classroom work devoted to statistics and probability (data, distribution, graphs, averages, variation, randomness and probability, including some exercises of compound and conditional probability).

The task given to participants is presented in Figure 1 and is similar to another task used by Estrada and Díaz (2006), although the statement was simplified, in order to avoid the use of negative statements in the wording of the item and the use of inequalities in the definition of the events in the sample space. The three questions in the first part of the task, were aimed to assess the prospective teachers' CCK in relation to elementary probability. More specifically we were interested in the prospective teachers' ability to read the table and identify the data needed to compute a simple probability (question a), a compound probability (question b) and a conditional probability (question c). The second part was aimed to assess the participants' SCK knowledge of probability; more specifically we were interested in their ability to identify the mathematical problems, concepts, properties, language, procedures and language implicit or used to solve the task.

A survey in a small school provided the following results:			
	Boys	Girls	Total
Liking tennis	400	200	600
Disliking tennis	50	50	100
Total	450	250	700
Part 1. Providing that we select one of the school students at random:			
a. What is the probability that the student likes tennis?			
b. What is the probability that the student is a girl and likes tennis?			
c. The student selected is a girl. What is the probability that she does like tennis?			
Part 2. Identify the mathematical content you used to solve the above tasks (specify the types of problems; concepts, procedures; properties, mathematical language and mathematical arguments you used to solve the task).			

**Figure 1. Task given to participants in the study**

## RESULTS

### Common mathematical knowledge

The written reports produced by the participants in the study were analysed and the answers to each question were categorized, taking into account the correctness of the response, as well as the type of errors, in case of incorrect response, as follows:

*Basically correct answers:* We group in this category answers that showed students correctly read the two-way table, identified the probability required and provided a correct solution to the problem. We also include here responses that provided a correct numerical result, with incorrect symbolization of probabilities, such as for example: “*The probability of liking tennis is  $P(600/700)$* ” (Student 70). The percentage of basically correct responses is low, except for the first question (65,6%), in agreement with what was reported by Estrada and Díaz in their sample.

*Confusing probabilities:* Some participants confused simple, compound and/or conditional probabilities. The most frequent confusion (13,7%) was between conditional and compound probability: “*Probability of liking tennis assuming the student is a girl is  $200/700$* ” (Student 73). This is an error described by Einhorn and Hogarth (1986) in university students and also found in 17% of prospective teachers in Estrada and Díaz’s research. A few participants confused  $P(A/B)$  and  $P(B/A)$ , an error that was described by Falk (1986): “*There is 33% probability that a girl likes tennis*” (Student 71). In the following example, instead of computing a simple probability, the student computed two conditional probabilities; we observe the student’s inability to read the data in the two-way table as he did not reach the “reading between data” level (Curcio, 1989): “*Probability of liking tennis is:  $4/6=66,6\%$  for boys and  $2/6=33,3\%$  for girls*” (Student 36). Other students confused simple probability with the probability of an elementary event: “*Probability of liking tennis if you select a student at random is  $1/700$ , since there are 700 students*” (Student 82). The percentage of pre-service teachers confusing different probabilities was slightly lower than that reported by Estrada and Díaz, possibly because the task was simplified.

*Confusing events:* A few prospective teachers identified the probability but confused an event and its complement, an error described by Estrada and Díaz (2006); “*Probability of liking tennis is  $\frac{50}{250} = 20\%$* ” (Student 102), which again suggest the pre-service teachers’ inability to read the two-way table. Additionally some prospective teachers confused other different mathematical objects; such as probability and frequency (or number of favourable cases) and for this reason obtained a probability higher than 1.

*Confusing formulas:* A small number of pre-service teachers identified correctly the probability to be computed and used correct symbols, but did not remember the formula, so that the final result was wrong. Other errors consisted in computing



means of frequencies, or assuming independence in the data and applying directly the product's rule for independent events when computing compound probabilities.

**Table 1. Frequency (and percentage) of responses to the three questions**

Teacher's answer	$P(A)$	$P(A \cap B)$	$P(A/B)$
Basically correct	120 (65,6)	75 (41,0)	80 (43,7)
Confuse probabilities	8 (4,4)	46 (25,1)	30 (16,4)
Confuse other objects	9 (4,9)	10 (5,5)	5 (2,7)
Confuse formulas	3 (1,6)	2 (1,1)	2 (1,1)
Confuse events	0 (0,0)	7 (3,8)	8 (4,4)
Other errors	1 (0,5)	4 (2,2)	11 (6,0)
Do not provide an answer	42 (23,0)	39 (21,3)	47 (25,7)
Total	183 (100)	183 (100)	183 (100)

The students' responses are presented in Table 1, where we use the following abbreviations:  $A$ ="the student likes tennis";  $B$ ="the student is a girl". Although the majority of participants correctly computed simple probability, less than 45% of responses when computing compound and conditional probabilities were correct. Also, similarly to Estrada and Díaz's research, an important percent of participants in our study did not provide any solution. There were a variety of errors reported in previous research, in particular confusion between different probabilities, and at the same time we found other mistakes which have not been described in the literature, such as confusing a simple probability with the probability of an elementary event.

### Specialized knowledge of content

In the second part of the task, we asked the participants to identify the probability content needed to solve the task. We included the following categories of objects:

- *Problems*: We expected the students to identify the three different specific problems in the task: A simple probability problem in part (a), a compound probability problem in part (b), and a conditional probability problem in part (c).
- *Language*: Verbal, numerical and tabular mathematical language appears in the task statement; depending on the solution, some students would also use symbolic and graphical language.
- *Concepts*: Implicit in the task we can identify the concepts of random experiment (selecting a school student at random); simple and compound events; sample spaces, favourable and unfavourable cases for each question; simple, compound and conditional probability, fraction, ratio and proportion, frequency and percentage, integer numbers, operations with integer numbers (division).

## Working Group 5

- *Properties* (or relationships between concepts). Some properties implicit in this task are: The probability axioms; the relation between the probability for an event and that of its contrary; the fact that sample space is restricted in computing conditional probability; equivalence of two fractions when dividing the two terms of the fraction by the same number; the Laplace rule; the relation between the total sample size and the totals in rows or in columns; the relation between double, marginal and conditional frequencies.
- *Procedures* (or algorithms). Possible procedures that can be used in solving these tasks include doing numerical operations, such as division or addition, operating or simplifying fractions, reading a table; transforming a probability in percentage; applying the formulas for computing simple, compound and conditional probability, and computing percentages or proportions.
- *Arguments*. The main correct type of argument used to solve the task is *deductive* argument, which was identified by many students.

Many students were able to identify and correctly classify some of the above mathematical objects in the problem; although, in general, the number of objects identified was quite small, and an important proportion of students were unable to give examples in some categories. Other examples provided by the students were considered incorrect, due to some of the following reasons:

- Some responses were too imprecise, for example, answering that a mathematical problem was “*replying the questions that appear after the data table*” (Student 95) or that “*there are three different mathematical problems in the task*” (Student 125); these responses do not specify the type of problems (simple, conditional or compound probability problem).
- Some students confused the different types of mathematical objects; for example, some of them considered the procedures “*interpreting the table*” or “*performing a division*” to be concepts. Other students confused procedures with their solution or confused phenomenological elements with mathematical objects. For example, some students suggested “*girls*” or “*liking tennis*” instead of “*event*” as examples of mathematical concepts.
- Other students included in their responses some mathematical objects that were not needed to solve the task, such as, for example, “*median, mode, standard deviation*”.

The number of correct and incorrect examples of mathematical objects provided by each participant in each category varied, ranging from not being able to identify a mathematical object in a given category to including several examples (in table 2 we present the mean and standard deviation). Results suggest that identifying the mathematical objects implicit in the task was not easy for the participants in the sample. On average, only half the students correctly identified a mathematical

problem (even when three different problems were proposed in the task) and only a third identified correctly a property or the use of deductive argument. The easiest elements for the prospective teachers were concepts (2-3 concepts correctly identified per participant), procedures and language (1-2 correctly identified). Anyway, although some prospective teachers suggested incorrect mathematical objects in all the categories, the average number of correct responses was higher than the number of incorrect responses in all the categories and the differences were statistically significant, except for properties and arguments that were hardest to be identified in the task by participants.

**Table 2. Mean and standard deviation for the number of correct and incorrect mathematical objects identified in the task**

Objects	Correct		Incorrect		p-value in the t-test of differences
	Mean	Std. Dev.	Mean	Std. Dev.	
Problems	0,54	0,70	0,17	0,47	0,004*
Language	1,37	1,91	0,57	1,23	0,007*
Concepts	2,22	2,01	0,87	1,58	0,002*
Procedures	1,40	1,71	0,38	0,89	0,003*
Properties	0,32	0,77	0,26	0,62	0,076 N.S.
Arguments	0,37	0,62	0,26	0,62	0,066 N.S.

\* Differences statistically significant at 0,05.

## IMPLICATIONS FOR TRAINING THE TEACHERS

Our results suggest that computing simple, compound and conditional probabilities from a two-way table was not easy for participants in the sample who showed a weak common knowledge of probability to solve this task. Many teachers were unable to provide an answer to the problems, in agreement with Estrada and Díaz' (2006) research, or made errors reported in previous research, particularly by Einhorn and Hogarth (1986) and Falk (1986). We agree with Falk that the everyday language we use to state a conditional probability problem lacks precision and is therefore ambiguous. However, a future teacher should master both the concept and the language used in teaching, particularly the language which today is part of statistical literacy, which is important for their students, and which they should transmit them.

Participants also had difficulty in identifying and classifying mathematical objects in this task coinciding with Chick and Peirce's (2008) results, which suggest that the specialised knowledge of elementary probability was also poor. These results are cause for concern, since prospective teachers in our sample are likely to fail in future teaching of probability in some professional activities, such as "figuring out what

students know; choosing and managing representations of mathematical ideas; selecting and modifying textbooks; deciding among alternative courses of action” (Ball, Lubienski, & Mewborn, 2001, p. 453). These activities involve mathematical reasoning and thinking, which were weak for these teachers when dealing with probability. To conclude these results suggest the need to reform and improve the probability education these future teachers are receiving during their training in the schools of education.

**Acknowledgements:** Research supported by the project EDU2010-14947 (MCINN-FEDER), grant FPI BES-2008-003573 (MEC-FEDER) and group FQM126 (Junta de Andalucía).

## REFERENCES

- Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers’ mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (pp. 433-456). Washington, DC: American Educational Research Association.
- Batanero, C., Estepa, A., Godino, J. D., & Green, D. R. (1996). Intuitive strategies and preconceptions about association in contingency tables. *Journal for Research in Mathematics Education*, 27(2), 151-169.
- Batanero, C., Godino, J. D., & Cañizares, M. J. (2005) Simulation as a tool to train Pre-service School Teachers. In J. Addler (Ed.), *Proceedings of ICMI First African Regional Conference* [CD]. Johannesburg: International Commission on Mathematical Instruction.
- Begg, A., & Edwards, R. (1999, December). Teachers’ ideas about teaching statistics. Paper presented at the *Annual Meeting of the Australian Association for Research in Education and the New Zealand Association for Research in Education*. Melbourne, Australia.
- Chick, H. L., & Pierce, R. U. (2008). Teaching statistics at the primary school level: beliefs, affordances, and pedagogical content knowledge. In C. Batanero, G. Burrill, C. Reading, & A. Rossman (Eds.), *Joint ICMI/IASE Study: Teaching Statistics in School Mathematics. Challenges for Teaching and Teacher Education. Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference*. Monterrey: ICMI & IASE.
- Curcio, F. R. (1989). *Developing graph comprehension*. Reston, VA: NCTM.
- Einhorn, H. J., & Hogart, R. M. (1986). Judging probable cause. *Psychological Bulletin*, 99, 3-19.
- Estrada, A., & Díaz, C. (2006). Computing probabilities from two way tables: an exploratory study with future teachers. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics*.

Salvador (Bahia, Brazil): International Association for Statistical Education.

- Falk, R. (1986). Conditional probabilities: insights and difficulties. In R. Davidson & J. Swift (Eds.), *Proceedings of the Second International Conference on Teaching Statistics* (pp. 292-297). Victoria, Canada: International Statistical Institute.
- Fernandes, J. A., & Barros, P. M. (2005). Dificuldades de futuros professores do 1º e 2º ciclos em estocástica (Pre-service primary school teachers difficultis in stochastics). In *Actas do V Congresso Ibero-Americano de Educação Matemática (CIBEM)*, Porto (Portugal): Faculdade de Ciências [CD].
- Franklin, C., & Mewborn, D., 2006. The statistical education of PreK-12 teachers: A shared responsibility. In G. Burrill (Ed.), *Thinking and reasoning with data and chance – NCTM 2006 Yearbook* (pp. 335-344). Reston, VA: NCTM.
- Gal, I. (2005). Towards “probability literacy” for all citizens: building blocks and instructional dilemmas. In G. Jones (Ed.), *Exploring probability in schools: challenges for teaching and learning* (pp. 39-63). New York: Springer.
- Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers’ topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Huerta, P. (2009). On conditional probability problem solving research – structures and contexts. *International Electronic Journal of Mathematics Education*, 4(3), 163-194.
- Inhelder, B., & Piaget, J. (1955). *De la logique de l’enfant a la logique de l’adolescent* (From the logic of children to the logic of adolescent). Paris: PUF.
- Jones, G. (2005). Introduction. In G. Jones (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning* (pp. 1-12). New York: Springer.
- Konold, C. (1991). Understanding students' beliefs about probability. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 139-156). Dordrecht: Kluwer.
- Lecoutre, M. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics*, 23, 557-568.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1-22.
- Stohl, H. (2005). Probability in teacher education and development. In G. Jones (Ed.), *Exploring probability in schools: Challenges for teaching and learning* (pp. 345-366). New York: Springer.
- Tversky, A., & Kahneman, D. (1982). Judgement of and by representativeness. In D. Kahneman, P. Slovic y A. Tversky (Eds.), *Judgement under uncertainty: Heuristics and biases* (pp. 84 – 98). Cambridge, MA: Cambridge University Press.

# INVESTIGATING SECONDARY TEACHERS' STATISTICAL UNDERSTANDINGS

Helen M. Doerr & Bridgette Jacob

Syracuse University

*Most secondary teachers are familiar with the procedures of basic descriptive statistics, but they have not necessarily been prepared to interpret graphical representations of data or to reason about sampling distributions. In this exploratory research study in the United States, we investigate the understandings of eleven teachers who participated in a semester-long course, using Fathom, to develop their understanding of these concepts. We present the analysis of a pre- and post-test of content knowledge, the teachers' performance on two tasks, and their use of Fathom as a tool to simulate and represent sampling distributions.*

## INTRODUCTION

Over the last twenty years, researchers have made substantial progress in understanding students' conceptions of probability and statistics. Over this same time frame, there have been calls for a greater emphasis in schools on the inclusion of statistics topics throughout the curriculum (NCTM, 2000; Pfannkuch & Begg, 2004). However, a vision of improved teaching and learning of school statistics relies heavily on the knowledge and skills of teachers to enact instruction that engages students in developing statistical reasoning. Unfortunately, considerably less progress has been made in understanding teachers' statistical reasoning. As Shaughnessy pointed out in his recent review of the research on statistics learning and reasoning, "More research is needed on teachers' conceptions of statistics. Teachers have the same difficulties with statistical concepts as the students they teach." (2007, p. 1000). The broad goal of the research reported in this paper is to contribute to the research base on teachers' understandings of statistics, with an eye towards characterizing the knowledge that teachers bring to tasks involving graphical representations of data distributions, including sampling distributions. These understandings are key to developing a robust understanding of statistical inference, a topic that is taught at the upper secondary level in the United States. In particular, we are interested in two questions: (1) how do secondary teachers interpret graphical representations of data and (2) how do secondary teachers reason about sampling distributions?

## BACKGROUND

The statistics background for most secondary mathematics teachers in the United States is very limited (Shaughnessy, 2007). Some pre-service teachers will have had limited formal coursework in statistics, and often their experience of this coursework is somewhat removed from the statistical content that they will need to teach in the

secondary school. Such formal coursework in statistics would rarely address the specialized kinds of statistical knowledge that is needed for teaching that is different from just more statistical content, such as the potential misunderstandings that arise from students in the classroom (c.f., Shaughnessy & Chance, 2005). While many secondary teachers will be fluent with the procedures of descriptive statistics such as those in the study by Makar and Confrey (2004), teachers are likely to struggle with using graphical interpretations to data distributions. These researchers found that teachers were able to develop a robust understanding of distribution when working with data in a meaningful context, namely, interpreting test results for students, but the teachers encountered difficulty in distinguishing between the variability in a data set from that of a related sampling distribution. In their study with secondary teachers using *Tinkerplots*, Rubin and colleagues (Rubin et al., 2005) found that the shape of the distribution influenced the strategies that teachers used when comparing two distributions and that the teachers were more confident in their conclusions about symmetric distributions than about skewed distributions. As Pfannkuch (2006) points out, reasoning about data sets from their graphical representations is a complex task, requiring one “to attend to a multiplicity of elements within and between box plots, and to make judgements.” (p. 29). For the teacher, the complexity of this task is further layered with the necessity of generating the kinds of talk that will communicate the concepts represented in box plots in ways that will build towards informal inference.

While the most widely available technological tool for data analysis in the secondary classroom in the United States is the graphing calculator, this tool is more limited in its capabilities for learning data analysis than currently available software. Software designed for the learning of statistics (such as *Tinkerplots* or *Fathom*) provides the learner with opportunities to flexibly explore the data. The “landscape-type” design (Bakker, 2002) of *Fathom*, in contrast to “route-type” software tools, does not assume a particular learning trajectory for the teachers, but provides many routes for exploration. In her study of secondary teachers' comparing distributions, Madden (2008) argues that route-type tools can scaffold teachers' learning of both statistical content and new technology environments by moving from physical experiment to route-type tools to landscape-type tools. In addition, landscape tools have the potential to support an “expressive” approach to data modeling (Doerr & Pratt, 2008) that would allow teachers to create meaningful representations and interpretations of data and sampling distributions. In this study, we assume the reciprocal relationship between representations and models described by Rubin et al. (2005): “Not only does the model of data a person currently holds influence the representation she chooses to use, but the representation in turn influences the model of data she is developing.” In this sense, the simulation and representational capabilities of *Fathom* can reveal the person's current way of thinking and support the development of that thinking.

## DESIGN AND METHODOLOGY

This exploratory study was designed to gain insight into secondary teachers' knowledge about the graphical representation of data and sampling distributions. To this end, authors designed and taught a one-semester course that would engage teachers with a range of tasks involving the investigation and exploration of statistical concepts using the software package *Fathom* (Finzer, 2001). The statistical content of the course consisted of investigations into variation and distribution, sampling distributions, confidence intervals, and inferential statistics. In addition, the course included various readings and discussions about (a) the nature of statistical reasoning and how it compares to other forms of mathematical reasoning and about (b) secondary students' learning and statistical reasoning.

The choice of *Fathom* was intended to support the teachers' learning by providing an interface that would allow them to flexibly explore multiple graphical representations (e.g. shifting between box plots, dot plots and histograms) while being able to easily compare data sets and to make changes to the data so as to explore conjectures. *Fathom* also provided the simulation tools necessary to create sampling distributions and representations of the population, the sample, and the sampling distribution. We saw this as critical to developing the teachers' knowledge of sampling.

There were 11 subjects who participated in this study. Eight of the participants were pre-service teachers, two were in-service teachers, and one was a doctoral student in mathematics education. Eight of the participants were female and three were male. All participants had completed the equivalent of an undergraduate major in mathematics, with all but one having had at least one course in statistics. All participants completed a 20 item pre- and post-test of their statistical knowledge in six categories: graphical representations, sampling variation, inference, data collection and design, bivariate data and probability. These items were drawn from the Comprehensive Assessment of Outcomes in a First Statistics course (CAOS, <https://app.gen.umn.edu/artist/caos.html>). These items have been used with college students, and this study extends those results to this group of teachers.

All participants completed two paper-and-pencil tasks prior to specific instruction that (a) required them to compare the standard deviation of two distributions, based on their graphs (delMas & Liu, 2005), and (b) to analyze the relationship between a population and a sampling distribution (Chance, delMas, & Garfield, 2004). As we anticipated, the teachers had many of the same difficulties as college students did with these tasks. Because of the difficulties, we designed a third task (described in more detail below) using *Fathom* that engaged the participants in investigating confidence intervals and how the number of samples and the size of samples affect the sampling distribution. The multiple display windows and the animation features in *Fathom* provided an opportunity for participants to build displays that showed the population, a particular sample, and the sampling distribution as it was being built.



## RESULTS

In this section, we first report on the results of the pre- and post-test. We then present the findings on two tasks on graphical interpretations. This is followed by a brief analysis of one participant's use of *Fathom* to represent her understanding of sampling distributions.

### Post-Test Results

The post-test results suggest that there was an overall improvement in the teachers' understanding of the statistical concepts, as measured by the 20 items on the test, and this gain was significant ( $p < 0.05$ ), as shown in Table 1,  $n = 11$ . The only sub-area of the test where there was a significant difference from the pre-test to the post-test was in the area of graphical representations with six items. This result likely reflects the emphasis given to graphical representations and the extensive use of *Fathom* within the course.

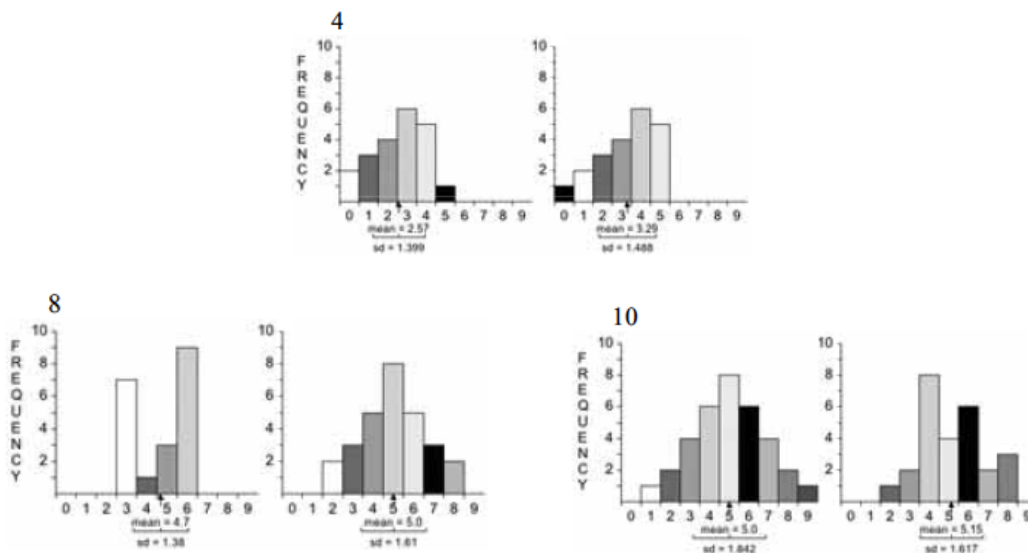
	pre-test mean (SD)	post-test mean (SD)	p-value
Overall $n=11$	11.09 (4.23)	12.54 (3.96)	0.027
graphical representation $n=11$	4.09 (1.45)	5.00 (1.79)	0.033

**Table 1: Pre-post test results for overall concepts and graphical representations.**

The first item among the six graphical representation items addressed the ability to describe and interpret a distribution displayed in a histogram. There was no change in this item from the pre- to the post-test, with 82% correct. The persistent error was misinterpreting the magnitude of the standard deviation of a near normal distribution as too small. Performance on the second item increased from 64% correct to 73% correct; the most common error was tending to select a normal distribution that did not make sense in the context of the problem. Items three and four addressed the ability to interpret the median and the quartiles in a box plot; these two items went from 73% to 91% correct and 55% to 82% correct. The error made on item four was incorrectly reasoning that the boxplot with a longer upper whisker would have a higher percentage of data above the median. On item five, which tested the understanding that a distribution with the median larger than the mean is likely skewed to the left, the correct response rate went from 64% to 82%. All participant errors on item five (both pre- and post-test) were incorrectly selecting a somewhat symmetric, mound-shaped graph. On item six, which addressed the ability to estimate standard deviations for different histograms, there was a change from 73% to 91% correct. Taken together, these results seem to suggest a tendency for some teachers to have more difficulty interpreting skewed distributions than symmetric distributions and a tendency to incorrectly choose a normal distribution.

## Graphical Interpretations of Standard Deviation

Early in the course, we gave the participants the task from delMas and Liu (2005) on interpreting standard deviations graphically to the participants. Three of these items (#4, 8, and 10) are shown in Figure 1. The mean and the standard deviation are displayed for the graph on the left, but only the mean is displayed for the graph on the right. The participants were asked to determine whether the standard deviation for the graph on the right was greater than, less than, or equal to the standard deviation for the graph on the left. As delMas and Liu point out, test item 4 “was specifically designed to see if students understood that given the same frequencies and range, a distribution with a stronger skew tended to have a larger standard deviation” (2005, p. 63). According to delMas and Liu, test items 8 and 10 were “designed to challenge the belief that a perfectly symmetric and bell-shaped distribution will always have a smaller standard deviation. Students were expected to find these items more difficult than the others.” (2005, p.63). Unlike the students in delMas and Liu’s study, the participants in our study had more difficulty with item 10 than item 8.



**Figure 1: Comparing standard deviations graphically.**

Of the ten participants who completed this task, four incorrectly answered that the standard deviations of the two distributions shown in item #4 were equal. This suggests that these participants are not attending to how skew affects the standard deviation. For item 8, four of our participants incorrectly concluded that the graph on the right had a smaller standard deviation, despite the fact that the graph on the left has a smaller range and represents a smaller number of values. This suggests that these participants might have reasoned that the symmetry of the bell-shaped curve with a large portion of the density centered about the mean would result in a smaller standard deviation.

## Working Group 5

For item 10, the graph on the right appears to have less density around the mean, but at the same time, it has a smaller range and represents a smaller number of values. As delMas and Liu point out, a reasonable response to both items 8 and 10 could be that the standard deviations needed to be calculated in order to determine how they differ. In their study, many of the students used calculations to come to a correct answer. Those who did not calculate came to the same incorrect conclusion as nearly all of the participants in this study. Only one of the ten participants answered this item correctly. Seven concluded that the distribution on the right had the greater standard deviation. One of the remaining two students concluded that the standard deviations were equal and the other answered with a question mark. These results suggest that these participants might be assuming that a symmetric normal distribution minimizes the standard deviation. The results also point to the difficulties in determining the standard deviation from a graph when having to interpret combined effects of density about the mean, range, and frequency.

### Graphical Interpretations of Population Distributions and Samples

We asked our participants to compare the shape and the variability of a sampling distribution to a population, based on a task described in Chance, delMas and Garfield (2004). Prior to specific instruction, we asked our participants which of the graphs in A through E represented a distribution of sample means for 500 samples of size 4 and of size 16, based on the population distribution shown in the upper left in Figure 2. We asked them to state whether these sampling distributions would have less, more or the same variability as the population and as each other. The results of this task are shown in Table 2 and Table 3.

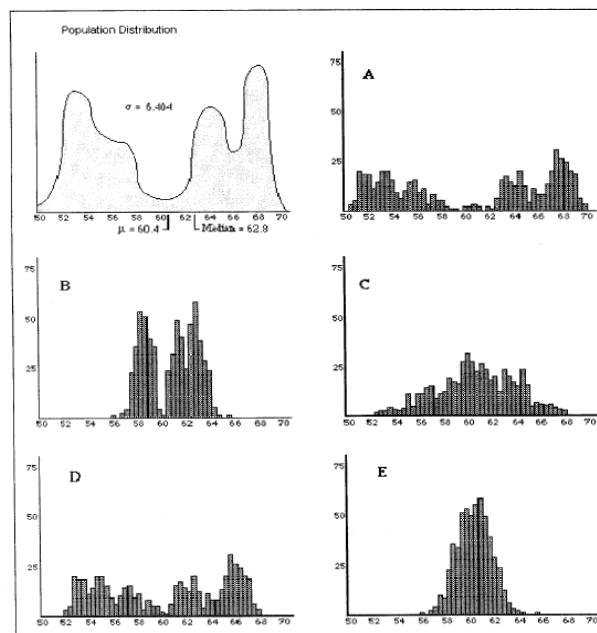


Figure 2: Population distribution from Chance, delMas, & Garfield (2004), p. 321

## Working Group 5

Only 4 of the 11 participants chose the correct response (*C*) when asked for the distribution of sample means for 500 samples of size 4. When asked for the distribution of sample means for 500 samples of size 16, only 5 of the 11 participants chose the correct response (*E*). For both items, the majority of participants chose distributions that indicate a belief that the sampling distribution should look like the population. This is a common misconception held by students (Chance, delMas & Garfield, 2004). The choice of response *B* for the second item would indicate that some participants believed that the sampling distribution continued to look like the population as the sample size increased, but with reduced the variability.

responses	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
sample means of size 4	2	0	4	5	0
sample means of size 16	2	4	0	0	5

**Table 2: Possible distributions of sample means drawn from a population**

As shown in Table 3, most participants (7 out of 11 and 10 out of 11) correctly compared the variability of the samples of size 16 to both the population variability and the variability of the samples of size 4. However, when comparing the variability of samples of size 4 to the variability of the population, 5 of the participants incorrectly expected the samples of size 4 to have *more* variability. This misconception is particularly interesting given that nearly all of the participants correctly compared the variability of a sample of size 4 to a sample of size 16. We speculate that these participants might be confusing the variability of a single sample with the variability of the sampling distributions. This needs further investigation.

responses	<i>less</i>	<i>same</i>	<i>more</i>
samples of size 4 compared to population	6	0	5
samples of size 16 compared to population	7	3	1
samples of size 16 compared to samples of size 4	10	0	1

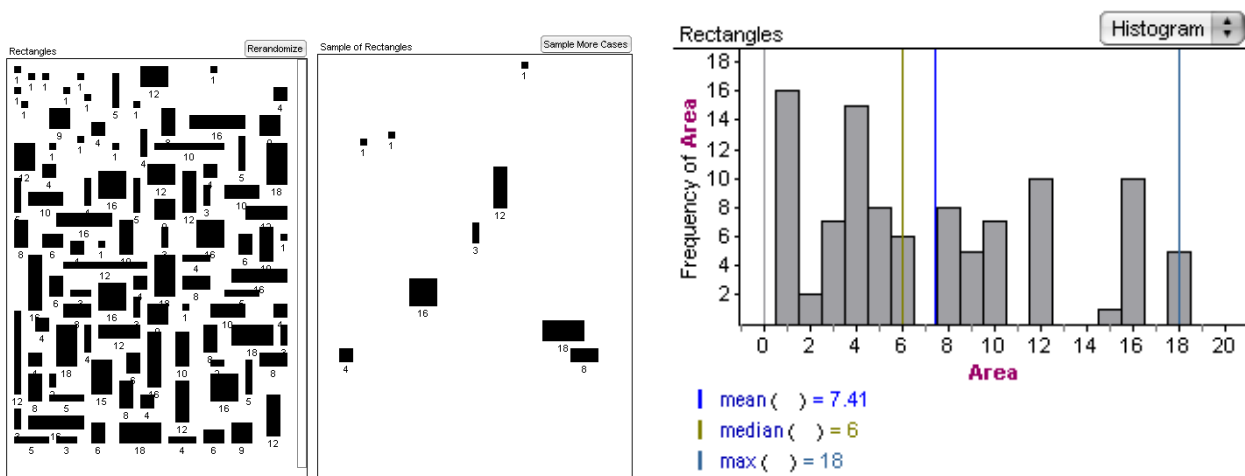
**Table 3: Comparing the variability of the sample distributions**

### Representing the sampling distribution using *Fathom*

Given the difficulties that a number of participants had with the previous task, we designed a task using *Fathom* to help participants better understand the three levels of abstraction that are present in a sampling simulation: (1) the population; (2) a particular sample from the population; and (3) the collection of measures that result from repeated sampling. This task consisted of two parts: the first part of the task asked the participants to set up a simulation in *Fathom* and investigate how the

sampling distribution of a summary statistic compared to the population distribution and how changing the number of samples and the sample size affects the sampling distribution. The participants successfully completed this first part of the task using a collection of 100 random rectangles (Key Curriculum Press, 2002, p. 127).

The second part of the task asked the participants to create a display that could be used to help students understand the difference between the population distribution, the sample, and the sampling distribution. This creation of a display that could be understandable to a learner was a substantial shift in the nature of the task that the participants were engaged in. Rather than convincing themselves about the relationships involved, the participants now shifted their attention to create a representation that could be used to illuminate the relationships among the population, a sample and the sampling distribution of a particular statistic. The work of one pre-service teacher is shown in Figures 3 and 4 and is representative of the kind of displays most participants created.

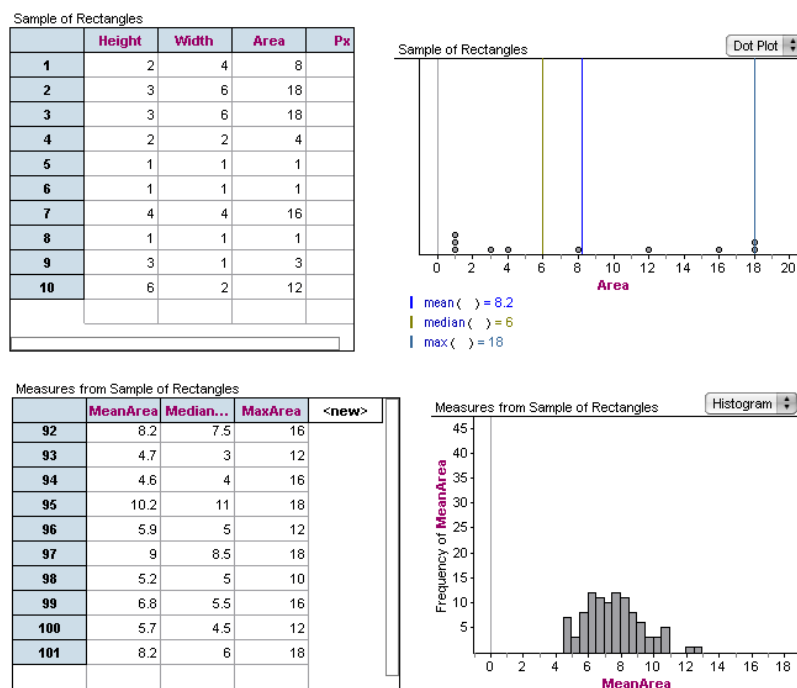


**Figure 3: Juxtaposing the population, the sample and the population distribution**

This pre-service teacher displayed the entire population of 100 rectangles, next to a randomly chosen sample of 10 rectangles along with a histogram showing the distribution of the areas along with the summary statistics for the entire population (as shown in Figure 3). The next section of her display (shown in Figure 4) juxtaposed a table that summarized the dimensions of each of the ten rectangles in the randomly drawn sample and the distribution of the areas for that sample. Just below this, she positioned a table (lower left in Figure 4) that showed the mean, median and maximum area for each sample of 10 rectangles up to 101 such samples. Next to the table is the graph of the distribution of the sample means. The pre-service teacher created this display with the animation feature of *Fathom* turned on so that she could see the sample distribution changing with each sample of 10 rectangles, while the sampling distribution was slowly being built and taking on the shape of a normal distribution. All participants introduced dynamic elements of animation into their displays, which are not fully captured by static snapshots.

## Working Group 5

This display suggests that this pre-service teacher was separating the three levels (or tiers as Madden (2008) refers to them) of abstraction that one needs to understand in order to grasp the concept of the sampling distribution. By using the simulation capabilities of *Fathom*, along with its flexibility in selecting representations, this pre-service teacher has clarified the distinction between the population, the sample, and the collection of sample means. The animation feature seemed to make visible how the sampling distribution is built over time as samples are collected. This particular display has the potential to help the pre-service teachers avoid confusing the population distribution with the sampling distribution as seen in the second task reported above. On a related item on the pre- and post-test, that asked participants to select an appropriate sampling distribution for a particular population and sample size, we found that the response rate went from 45% to 64% correct.



**Figure 4: A snapshot of a dynamic display building the sampling distribution**

## DISCUSSION AND CONCLUSIONS

The results of this study provide some evidence that, as Shaughnessy (2007) argued, teachers have some of the same difficulties with statistics as do students. We note that a limitation of this study is the small number of participants and that while there was a statistically significant gain on the post-test, this gain was small and largely in the area of graphical representations. Consistent with the findings of Makar and Confrey (2004) and Rubin et al. (2005), we found that some teachers had difficulty interpreting skewed distributions and tended to inappropriately choose symmetric normal distributions. This was evidenced when having to interpret how the combined changes in density about the mean, range and frequency affected the standard deviation. It is likely that these secondary teachers' prior learning of statistics

focused largely on the computational algorithm for standard deviation. Extending the results of Chance et al (2004), we found some teachers, like students, had a tendency to see the sampling distribution as having the same shape as the population distribution. Most teachers correctly reasoned about the variability of the distribution of larger samples, but incorrectly expected greater variability in the sampling distribution of smaller samples when compared to the distribution of the population. We found nearly all participants were able to create animated displays with *Fathom* that brought clarity to the three levels of abstraction that are present in any sampling distribution. This suggests that a land-scape tool such as *Fathom* has the potential to make visible teachers' models of statistical concepts.

## REFERENCES

- Bakker, A. (2002). Route-type and landscape-type software for learning statistical data analysis. In B. Phillips (Ed.), *Developing a statistically literate society: Proceedings of the Sixth International Conference on Teaching Statistics*. Voorburg, The Netherlands: International Statistical Institute.
- Chance, B., delMas, R., & Garfield, J. (2004). Reasoning about sampling distributions. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking*, (pp. 295-323). Dordrecht, Netherlands: Kluwer Academic Publishers.
- delMas, R., & Liu, Y. (2005). Exploring students' conceptions of the standard deviation. *Statistics Education Research Journal*, 4(1), 55-82.
- Doerr, H. M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M. K. Heid & G. W. Blume (Eds.), *Research on technology in the teaching and learning of mathematics: Syntheses and perspectives: Vol. 1. Mathematics Learning, Teaching and Policy*, pp. 259-285. Charlotte, NC: Information Age.
- Finzer, W. (2001). *Fathom Dynamic Data Software*. (Version 2.1) [Computer Software]. Emeryville, CA: Key Curriculum Press.
- Key Curriculum Press. (2002). *Exploring statistics with Fathom*. Emeryville, CA: Author.
- Madden, S. (2008). Dynamic technology scaffolding: A design principle with potential to support statistical conceptual understanding. Paper presented at the *11th International Congress on Mathematical Education*, 6-13 July. Monterrey, Mexico. Retrieved from <http://tsg.icme11.org/tsg/show/15>
- Makar, K. & Confrey, J. (2004). Secondary teachers' statistical reasoning in comparing two groups. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking*, (pp. 353-373). Dordrecht, Netherlands: Kluwer Academic Publishers.

### Working Group 5

- Pfannkuch, M. (2006). Comparing box plot distributions: A teacher's reasoning. *Statistics Education Research Journal*, 5(2), 27-45.
- Rubin, A., Hammerman, J. K., Campbell, C., & Puttick, G. (2005). The effect of distributional shape on group comparison strategies. In K. Makar (Ed.), *Reasoning about distribution: A collection of current research studies* (SRTL-4), University of Auckland, New Zealand, 2-7 July. Brisbane: University of Queensland.
- Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 957-1009). Charlotte, NC: Information Age Publishing
- Shaughnessy, J. M. & Chance, B. (2005). *Statistical questions from the classroom*. Reston, VA: NCTM.



# MENTAL MODELS OF BASIC STATISTICAL CONCEPTS

Andreas Eichler, Markus Vogel

University of Education Freiburg, University of Education Heidelberg

*This report focuses on a research programme that aims to identify the ability to cope with basic statistical concepts of young students (primary school and secondary school) without schooling in statistics, and to identify the students' mental models when acting within simple statistical situations. Firstly, the basic elements of the theoretical framework of the research programme will be outlined. Afterwards the method and exemplary results of a pilot study will be discussed. Finally, these results will be evaluated with regard to follow-up studies.*

Keywords: mental models, basic statistical concepts, statistical knowledge

## INTRODUCTION

About 60 years ago, Piaget and Inhelder (1951) published their seminal work referring to the development of thinking with probabilities. Nowadays, a huge amount of subsequent research exists that, on the one side, replicates the well known levels of the student's development of probabilistic thinking but, on the other hand, modifies, enhances or contradicts the findings of Piaget and Inhelder (Jones et al., 2007). Overall, however, we have a substantial knowledge about the development of the (naïve) probabilistic thinking of young students which is a basis to decide on the time to introduce probabilistic concepts in schools (Fischbein, 1975).

In recent years statistics instead of probability seem to become both the essential part of the stochastics curriculum in many countries, and a crucial part of the research into stochastics education (Shaughnessy, 2007). The majority of the existing research approaches in this field are intervention studies, trying to investigate how instruction promotes the students' statistical thinking (Ben-Zvi & Garfield, 2004; Shaughnessy, 2007). In addition, researchers developed theoretical frameworks towards statistical acting (e.g. Wild & Pfannkuch, 1999). However, in contrast to the research on students' probabilistic thinking, Mokros & Russel (1995) suggested 15 years ago that we knew little about students' development of a statistical thinking without statistical schooling and the situation has not improved today. Nonetheless, empirical knowledge about students' naïve knowledge of basic statistical concepts and statistical situations is important in order to adequately design a statistics curriculum, and to avoid students' misunderstandings referring to statistical concepts or statistical methods (Fischbein, 1975). For this reason in this paper we will discuss the first phase of a research programme that has the main aim:

- to identify the naïve knowledge represented by mental models of young students (aged 9 to 18) regarding basic statistical concepts and situations, and
- to investigate the development of this knowledge depending on age.

To discuss our research approach, firstly, we will clarify basic elements of our theoretical framework. Afterwards, we will discuss the method and provide some examples of the main results of a pilot study of our ongoing research. The pilot study itself consists of the following aims:

- to prove several tasks with regard to their adequateness for investigating students' naïve knowledge towards different statistical concepts and situations,
- to get information about whether students actually show different mental models by working with different statistical tasks, and
- to explore whether it is possible to identify differences in the students' ability to cope with basic statistical concepts and situations depending on age.

## **THEORETICAL FRAMEWORK**

The theoretical framework involves two components. The first part deals with the identification of possible subjects and competencies that may represent students' basic statistical knowledge. The second part involves the discussion of students' development of thinking concerning statistical concepts.

### **Statistical Reasoning and Prior Statistical Knowledge**

Wild and Pfannkuch (1999) suggested five aspects to be the main thinking processes in acting statistically. They considered them as being independent from the expertise of the actor. These five processes are

- the recognition of the need for data,
- transnumeration,
- the consideration of variation,
- reasoning with statistical models, and
- integrating the statistical and contextual information, knowledge, conceptions.

Except for the second type of reasoning, these components are not necessarily based on the students' prior statistical knowledge gained through schooling. Thus, it should be possible to identify them as parts of young students' naïve knowledge concerning statistical concepts.

While the five aspects of the statistical thinking represent an individual's statistical acting beyond specific statistical topics, several topics were declared to be central referring to statistical knowledge, i.e. sampling, central measurement and variation, distribution, graphical representation of data, or regression and correlation (e.g. Curcio, 1989; Mokros & Russel, 1995, Bakker 2007).

### **Students Development Concerning Statistical Thinking**

A main assumption for this research is that students' competencies in basic statistical concepts develop with age. Unlike Piaget and Neo-Piagetians like Case (1992) we

assume that this development cannot be described adequately by a staircase model. We follow Siegler (1996) who postulated the development of children's thinking had to be considered as "overlapping waves". The fundamental idea of his approach is making allowance for the observable variability of children's thinking. This variability depends on different factors, like specific circumstances, requirements and available knowledge, which influence a child's concrete actions within different situations. Based on several empirical studies Siegler provided evidence for high inter-individual as well as intra-individual variability when using complex strategies for solving problems (e.g. Siegler, 1995; Schauble, 1990).

The theory of mental models (Johnson-Laird, 1983) incorporates the situation's impact on cognitive processes, too. This theory suggests that, when interacting with (statistical) demands of a specific situation, the learner builds a mental model in order to simulate relevant aspects of the situation to be cognitively mastered (cf. Seel, 2001). Mental models are of dynamic nature: They are not to be seen as being fixed structures of memory; mental models are constructed according to a task and its requirements within a situation representing the structure or the function of the modelled object (Schnotz & Bannert, 1999). Thus, mental models concern the situation and they also facilitate to differentiate the students' cognitive development (or maturation) in using basic statistical concepts. Furthermore, tasks concerning statistical thinking and statistical knowledge contain information a student could proceed and internalize in a mental model being specific with regard to his or hers individual abilities, pre-knowledge and apperception of the task's representation. According to the information process model of Schnotz & Bannert (1999), the elements of a mental model may be changed, enriched or modified during the persistent mutual processes of internalizing and externalizing, when a student is working on a statistical task but they do not disappear at all. Thus, information about statistical mental models, and hence statistical thinking and statistical knowledge, could be made available by analyzing the tasks (content, representation), the learners specific situation (experience and pre-knowledge with regard to statistical content, statistical methods and statistical context) and the learners outcomes (written or spoken responses) after working on the tasks.

Not the statistical mental models themselves are claimed to be captured, but the researchers' reconstruction of these models based on learning outcomes. It is obvious that different students will show different grades of performance. A categorization system describing different performances that is often used in statistics education is based on the SOLO model of Biggs and Collis (1982). In the adaptation of Watson and Moritz (2003), this model has four levels or rather modes, i.e.

- the prestructural mode: students solve a task using irrelevant information,
- the unistructural mode: students solve a task using an isolated information,
- the multistructural mode: students solve a task using a set of information,

- the relational mode: students solve a task using a set of information and considering an interconnected knowledge of context and statistical concepts.

Although alternative models exist to describe students' performances in acting with specific statistical tasks (e.g. Watson et al., 2003), in the pilot study we used the model outlined above only, but consistently, since the scope of statistical tasks in the pilot study encompassed different basic statistical concepts.

In summary, using this theoretical framework the underlying aim of the pilot study was to get information about students' ability to cope with basic statistical concepts within different situational requirements. A further aim was to see if and how it could be done to reconstruct students' potential mental models when working on different statistical situations.

## METHOD

To investigate different aspects of students' naïve knowledge concerning basic statistical concepts and situations we have designed tasks referring to

- central measurement,
- variability of statistical data,
- proportional reasoning involving the students consideration of variability,
- interrelation of bivariate data, and
- simple random experiments.

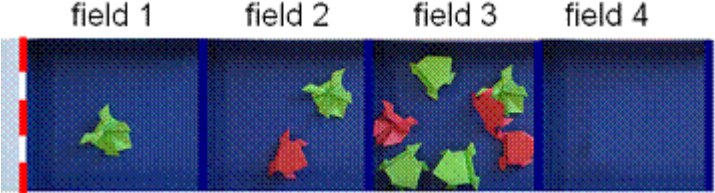
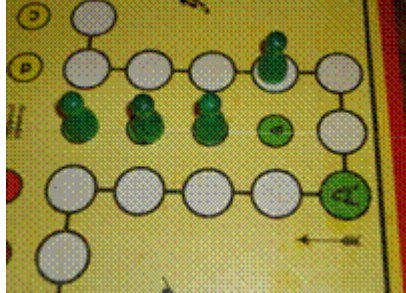

The design of the tasks was based on the research literature as mentioned above. Some of these tasks were adopted without changes, some other tasks were modified, and some new tasks were developed. Every task contains a decision-making process in a statistical situation and an open-ended item in which the student had to justify the decision. The particular aim of the pilot study was to investigate whether the students of different age were able to work with these tasks and whether these tasks could be useful to identify students' mental models when acting with different statistical situations.

Figure 1 shows two of the tasks. The first task, called the *frog-task*, deals with the students' mental models referring to proportional reasoning and variability of statistical data. The second task, called *the die-task*, deals with the students' mental models referring to a simple random experiment (Figure 1).

Both items have adequate normative solutions. For instance, estimating 100 frogs in field 3 after 100 jumps or choosing the ordinary die in the second task are inadequate solutions. Our main focus in analysing students' solutions, however, was to code students' different performances in justifying their solution according to the four-level-model of Watson and Moritz (2003). In this study we coded no solution with 0, a solution that matches the prestructural mode with 1, a solution that matches the

## Working Group 5

unistructural mode with 2, a solution that matches the multistructural mode with 3, and, finally, a solution that matches the relational mode with 4. We also tried to identify different mental models that were, potentially, the basis of the students' justifications of their solutions.

 <p>Andrea lets the frog jump ten times from the starting line. The frog ends in field 1 once, in field 2 twice, in field 3 seven times, and never in field 4.</p> <p>Make an estimation:</p> <p>How often will the frog get to field 3 after 100 jumps? Justify your answer.</p> <p>How often will the frog get to in field 3 after 1000 jumps? Justify your answer.</p> <p>How often will the frog get to field 4 after 1000 jumps? Justify your answer.</p>	 <p>You will win a game if one die shows 3 the first time you throw.</p>  <p>Which of the two dice would you choose to win this game?</p> <p>Why have you chosen this die!</p>
---	---

**Figure 1: The frog-task (left side) and the die-task (right side)**

To identify possible differences in the performances of the students' solutions depending on both the students' age and the students' specific situation (experience and pre-knowledge with regard to statistical content, statistical methods and statistical context; see above), we selected samples of students in different grades. The samples answering the two tasks we discussed above are shown in Table 1.

die-task		frog-task	
Grade [age]	Sample size	Grade [age]	Sample size
2 [7-8]	19	4 [9-10]	21
4 [9-10]	25	10 [15-16]	45
8 [13-14]	65	12 [17-18]	23
11 [16-17]	17		
n = 126		n = 89	

**Table 1: Sample sizes in the pilot study**

All primary students and most of the students in higher grades had never learned about statistics in school before taking the test. There were only a few students who had gained mostly little statistical experience along their schooling.

As we proceed, we begin with the discussion of students' solutions of the die-task. Afterwards, we will discuss the results in a broader sense including open questions that have arisen from the results of our first research step, and including possible next research steps.

## **SOME RESULTS OF THE DIE-TASK**

The die-task provides the following question (see Figure 1): "Why have you chosen this die?"

A grade 4 student who chose the ordinary die justified his solution as follows: "Because, it will spin better". This student provided an inadequate solution on the one side and, on the other side, referred to a physical feature of the die, which is irrelevant in the statistical situation of choosing one of the two dice. For this reason, we coded his solution with 1 (representing the prestructural mode).

Another grade 4 student justified his selection of the cubic-die as follows: "Because the side is bigger. Thus, you get the three faster." The student referred to a single physical feature of the die which is relevant for an adequate choice of one of the two dice. The student was not able, however, to describe the relationship between the different sizes of the sides of the cuboid-die, and, respectively, the different sizes of the side showing the 4 on both dice. For this reason, we coded this justification with 2 (unistructural mode).

A third student (grade 8) justified his selection of the cuboid-die as follows: "Because the first die has sides of the same size, which is not the case for the second one". This student compared the two dice according to a relevant physical feature, and, thus, used a set of relevant information in the statistical situation. For this reason, we coded this justification with 3 (multistructural mode).

Finally, one student (grade 8) justified her selection of the cuboid-die as follows: "The sides showing the 3 and the 4 are bigger than the others. For this reason, the chance of getting a 3 is about  $\frac{2}{8}$  (two eighth), thus  $\frac{1}{4}$  (one fourth). This chance is bigger than  $\frac{1}{6}$  (one sixth) in the other die." This student considered a lot of relevant information. She compared the symmetrical areas of the cuboid-die showing the 3 and the 4 and estimated a probability by comparing all the areas of the die. She also showed knowledge about the probability of the ordinary die and was able to compare these probabilities. Thus, we coded this justification with 4 (relational mode).

In Table 2 the absolute numbers of students' justifications in the whole sample ( $n=126$ ) that were coded from 0 to 4 is shown. The justifications are divided into those that are given concerning the cuboid-die (r: right) and the ordinary die (w: wrong).

Three results of the descriptive analysis can be pointed out:

- The students' ability to select adequately the cuboid-die seems to increase by age.

### Working Group 5

- The students' performance in formulating a justification for their choice of one of the two dice seems to increase by age, as well.

If the students choose the inadequate die, they show a low performance in justifying their choice.

Code	Grade 2 n = 19		Grade 4 n = 25		Grade 8 n = 65		Grade 11 n = 17	
	r	w	r	w	r	w	r	w
0	0	0	0	0	0	0	0	0
1	5	14	4	12	8	8	1	1
2	0	0	6	0	44	0	12	2
3	0	0	2	1	4	0	1	0
4	0	0	0	0	1	0	0	0
Sum	5	14	12	13	57	8	14	3

**Table 2: Absolute numbers of codes concerning the students' justifications**

### SOME RESULTS OF THE FROG-TASK

The coding of the students' justifications in the frog-task was based on considerations similar to the die-task. Although we coded the justifications to the four answers of the frog-task as a whole, we illustrate the codes from 1 to 4 by presenting only justifications to the last question ("How often will the frog get to field 4 in 1000 jumps?"):

- 1: "250 times. The chance that the frog will get to each of the four fields is 25%."
- 2: "0 times. 1000 is a hundred times larger than 10 and, thus, we have to multiply 0 (number of frogs having reached field 4 by jumping 10 times) by 100."
- 3: "8 times. Because getting to field 4 seems to be more difficult."
- 4: "0-10 times. The more often the frog will jump, the bigger is the chance that the frog will get to field 4."

The justification of the first student (grade 10) describes an equal-probability model, which is an inadequate model in the statistical situation of the frog-task (which would imply a distribution of jumps according to the normal distribution). The justification of the second student (grade 12) describes a fixed proportional model without consideration of variation. In contrast, the third student (grade 4) seems to grasp the idea of variation implicitly and in an unsophisticated way. Finally, the fourth student (grade 12) showed a more sophisticated understanding of variation and of the relationship between variation and the number of jumps. Table 3 shows the absolute numbers of students' justifications in the whole sample (n = 89) that were coded from 0 to 4.

Regarding the results of the descriptive analysis, it is important to state that:

## Working Group 5

- the performance of the students' justifications seems to increase with the student's age. However the increase is low in comparison to the die-task and, for instance, one student of grade 4 (see above) showed a higher performance as a student of grade 10 and even a student of grade 12;

the performance of the students' justifications is higher compared to the justifications referring to the die-task.

Code	Grade 4, n = 21	Grade 10, n = 45	Grade 12, n = 23
0	0	1	0
1	7	5	3
2	10	19	11
3	4	17	6
4	0	3	3

**Table 3: Absolute numbers of codes concerning the students' justifications**

## DISCUSSION

The results of the first step of our on-going research have to be interpreted carefully. These results, though, should facilitate the finding of appropriate questions and the phrasing of adequate hypotheses referring to students' ability to cope with basic statistical concepts. Taking into account these constraints the following aspects seem, nevertheless, to be worthwhile being considered in follow-up studies.

Firstly, the results concerning the two tasks as well as for other tasks gave evidence that students' ability to cope with basic statistical concepts and to justify decisions in different statistical situations are dependent on age (see the results in both tasks). This hypothesis will be investigated in a follow-up study involving randomised samples of students of different grades. Although there seems to be a correlation between the students' performance concerning different tasks, students of the same age also justify their acting in a statistical situation in considerably different ways (see for example the two students of grade 12 concerning the frog-task). Further, there is some evidence that, partially, students formulate different justifications in tasks that are structurally equivalent. Both results meet the Siegler's theory of "overlapping waves" in children's thinking (Siegler, 1995).

There is some evidence that students use different mental models coping with statistical situations. Although it is, theoretically, not possible to identify the mental model a student actually uses, research results seem to reveal some indication of equivalent inter-individual mental models. For instance, some of the students neglected in both tasks the randomness of the data's genesis or, respectively, the randomness of future data (e.g., the student we coded with 0 in the die-task). Another pattern concerns the omnipresence of the fair chance: Neglecting the information given in the frog-task, these students estimate future events consistently based on the model of an equal-probability (see the student, we coded with 0 in the frog-task).



According to the theory of mental models, we found great differences in the students' ability to cope with statistical situations depending on the representation of the task. For instance, students' seemed to use significantly different mental models when the die-task was represented by two real dice, the picture of the dice (c.f. the die-task discussed above), or the description of the two dice. We also hypothesise that different representations of other statistical situations will have a great impact on the students' ability to cope with these statistical situations. For instance, the frog-task using the picture of the statistical situation shown in Figure 1 could be compared to the same task involving a picture of a real experiment or more abstract graphical representations of the situation, e.g. a dot plot.

Two important questions were not investigated in our pilot study: Do we measure the students' ability to cope with statistical situations or do we only measure the students' ability to communicate their ideas? Do the differences in communicating decisions and their justifications in a statistical situation contrast with a student's acting in the same situation? The outward appearance of the written justifications of students of different age varies considerably (c.f. Figure 2 to 5). At the current stage of our research programme, however, we postulate that a student's ability to cope with statistical situations is similar to the student's ability to verbalise his decisions and his decisions' justifications in a statistical situation.

Two follow-up-studies are being undertaken currently on the dependence of students' ability to cope with basic statistical concepts from age and on the students' different mental models by coping with statistical situations that will be—in the first step— structured using the SOLO taxonomy but will be described by a more differentiated systems inductive developed categories. From both studies which will include quantitative and qualitative parts we expect to get a better understanding of students' (naïve) knowledge concerning statistical concepts and students' (naïve) statistical thinking.

## REFERENCES

- Bakker, A. (2007). *Design research in statistics education*. Utrecht: Freudenthal Institute.
- Ben-Zvi, D. & Garfield, J. (2004) (Eds.). *The challenge of Developing statistical literacy, reasoning and thinking*. Dordrecht: Kluwer Academic Publishers.
- Biggs, J.B. & Collis, K.F. (1982). *Evaluating the quality of learning: The SOLO taxonomy*. New York: Academic Press.
- Case, R. (1992). *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge*. Hillsdale: Erlbaum.
- Curcio, F. R. (1989). *Developing graph comprehension*. Reston, VA: N.C.T.M.

- Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht: D. Reidel Publishing Company.
- Johnson-Laird, P. N. (1983). *Mental models*. Cambridge: University Press.
- Jones, G.A., Langrall, C.W., & Mooney, E.S. (2007). Research in probability: Responding to classroom realities. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909-956). Charlotte, USA: Information Age Publishing.
- Mokros, J. & Russel, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, 26, 1, 20-39.
- Piaget, J. & Inhelder, B. (1951). *La genèse de l'idée de hasard chez l'enfant*. Paris: PUF.
- Schauble, L. (1990). Belief revision in children: The role of prior knowledge and strategies for generating evidence. *Journal of Experimental Child Psychology*, 49(1), 31-57.
- Schnotz, W. & Bannert, M. (1999). Einflüsse der Visualisierungsform auf die Konstruktion mentaler Modelle beim Text-und Bildverstehen. *Zeitschrift für Experimentelle Psychologie*, 46(3), 217-236.
- Seel, N.M. (2001). Epistemology, situated cognition, and mental models: "Like a bridge over troubled water". *Instructional Science*, 29 (4-5), 403-427.
- Shaughnessy, M. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 957-1010). Charlotte, USA: Information Age Publishing.
- Siegler, R. S. (1995). How does change occur: A microgenetic study of number conservation. *Cognitive Psychology*, 28, 225-273.
- Siegler, R.S. (1996). *Emerging minds. The process of change in children's thinking*. New York: Oxford University Press.
- Watson, J.M., & Moritz, J.B. (2003). Fairness of dice: A longitudinal study of students' beliefs and strategies for making judgments. *Journal for Research in Mathematics Education*, 34, 270-304.
- Watson, J.M., Kelly, B.A., Callingham, R.A., & Shaughnessy, J.M. (2003). The measurement of school students' understanding of statistical variation. *International Journal of Mathematical Education in Science and Technology*, 34, 1-29.
- Wild, C. & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review* 67(3), 223-248.

# INSTRUCTIONAL REPRESENTATIONS IN THE TEACHING OF STATISTICAL GRAPHS

**M<sup>a</sup> Teresa González Astudillo**

**Jesús Enrique Pinto Sosa**

*University of Salamanca*

*Autonomous University of Yucatan*

*Abstract: This work is focused on the knowledge that two teachers of Statistics have about instructional representations and strategies in the teaching of statistical graphs. The instructional representations they use for teaching are definitions and explanations, questions, examples and exercises, and instructions; all of them centered on explanations, the use of examples and prototypic exercises, and the construction of graphs. Occasionally, the manipulation of data and graphs is used to explore the meaning of previous concepts and explore the characteristics and elements of the graph.*

*Key Words: pedagogical content knowledge, instructional representation, statistical graphs, teaching statistics*

## INTRODUCTION

Pedagogical content knowledge (PCK) emerges from Lee Shulman's theoretical framework entitled "*the knowledge base for teaching*". This knowledge base consists of seven categories, three of which are content related: (a) content knowledge (CK), (b) pedagogical content knowledge (PCK), and (c) curriculum knowledge. The other four categories refer to general pedagogy, learners and their characteristics, educational contexts, and educational purposes (Shulman, 1987).

Within the category of PCK, Shulman (1986) included three closely connected key elements or domains of knowledge: the subject matter knowledge for teaching, the knowledge of instructional representations and strategies, and the understanding of specific learning difficulties and students' conceptions (Pinto & Gonzalez, 2006).

The purpose of this research is to explore the PCK of two teachers who teach Statistics focusing on the topic of statistical graphs in schools of Psychology and Education. In this paper we present exclusively what corresponds to one of the three elements of PCK, i.e. the knowledge of instructional representations.

### *Instructional representations*

Instructional representations are "ways of talking, showing, enacting, or otherwise representing ideas so that the unknowing can come to know, those without understanding can comprehend and discern, and the unskilled can become adept" (Shulman, 1987, p.7). Therefore, the teacher must bear in mind both to the possible

meanings within the subject, and the comprehension that his or her students are able to reach.

McDiarmid, Ball & Anderson (1989) apply the term *instructional representation* to “a wide range of models that may convey something about the subject matter to the learner: activities, questions, examples, and analogies, for instance”. The sources from which instructional representations derive are two; those originated from the teacher himself and those originated from outside. The forms of instruction in the first group are: worksheets, activities, explanations, questions and answers that teachers provide their students with. The second group of representations derives from the curricular materials (textbooks, guidelines for the teacher, equipment, software, films and videos), courses or workshops for teachers and experiences shared by colleagues in the school context.

Regarding the type of representation there are different forms. Shulman (1986) mentions some: *analogies, illustrations, examples, explanations and demonstrations*. Llinares, Sanchez & Garcia (1994) suggest additionally *drawings* and the *graphical representations* found in books or those provided by the teacher on the board. Even and Tirosh (1995) present *questions, activities and discussions* linked with the mathematical content as examples of representations. McDiarmid, Ball & Anderson (1989) also talk about *verbal expositions, diagrams, simulations, dramatizations, and content analysis*; verbal, symbolic, graphic or concrete representations stand out as inherent to mathematical content. Furthermore, Putnam & Borko (2000) discuss *models and metaphors*.

Shulman (1986) states that a teacher should select and use a wide repertoire of representations, according to the specific content in question, which highlights the importance of the teacher’s knowledge of instructional representations linked to the topics being taught and the way the teacher uses them (Llinares, Sanchez & Garcia, 1994).

### *Statistical graph*

We will focus specifically on knowledge of the instructional representations of a specific topic: statistical graphs. A statistical graph is a “construct which was developed in specific cultural contexts to mediate interpretation of data... an activity which is related to a complex range of elements and processes” (Monteiro & Ainley, 2006, p.1) and it is considered a reasoning tool to learn something new about the context it represents, gain new information or learn from the data (Pfannkuch, 2006). For this reason, teachers should develop PCK that will help them design activities to increase the conceptual understanding of statistical graphs in their students, based on the relationships between the main components of the graph and the necessary process for its interpretation (Friel, Curcio & Bright, 2001).

Each instructional representation for teaching and learning statistical graphs favours certain conceptions that are transmitted to the students. For example, Meletiou &

Stylianou (2003) considers that when students *use technology* this enables them to engage in interpretation and translation of graphically represented information. Thompson (1994) proposes a strategy to *incorporate significant data* from the student, Curcio (1989) sequences a set of *activities* to develop graphical understanding, and Peden (2001) shows an *activity* from four data sets to capture and select, analyze and interpret the best graph. Moreover Cazorla & Santana (2006) show examples of *problem situations* of interest either for quantitative or for qualitative analysis to understand the characteristics of graphs. Moore (2000) suggests using *appropriate software* to make graphs, *group problem solving* and *discussion, written and oral presentations* and *projects*.

However, several studies in statistics education (Burgess, 2002; Monteiro & Ainley, 2006) have concluded that the difficulties for the teaching and learning statistical graphs as well as a limited repertoire of instructional representation, clearly exemplify the training that statistics teachers have.

## THE RESEARCH

This research focuses on the representation of statistical data, specifically graphical representation, which is one of the first topics in any introductory course in statistics at any school level, as well as the point of departure for the statistical analysis process in a group of data and a basic element in statistical thinking (Pfannkuch & Rubick, 2002).

Considering this view, the critical issues that we wanted to establish were: How does a teacher plan his/her classes? What kind of knowledge does he/she have about the instructional representations of the content that is being taught? What kind of activities does he/she use in teaching? How does he/she use them? We did not attempt to characterize or typify teacher's knowledge, but to understand and go deeper into his/her knowledge, to know how it is used to teach statistical content in his/her classes.

In order to achieve this, two cases were selected (Alicia and Luis) on the basis of the following criteria: knowledge about the topic of statistical graphs (SG), different initial training (a mathematician and a psychologist), a teacher who taught Statistics in a school of Psychology and the other in a school of Education, different teaching experience in Statistics, willingness and motivation to collaborate and share their thoughts and knowledge, as well as a positive attitude towards learning. Both teachers teach at a public university located in Mérida, México.

The following instruments were used for data collection: the syllabus of each Statistics course (Sy), a contextual and biographical interview (I1), a didactics questionnaire with four hypothetical teaching-learning situations about SG (Q1 to Q4), each of them with different items (1a,1b..), an in-depth interview about their answers to the questionnaire (I2), and the materials used for the teaching of SG which included class notes (Not), exercises (Exe), Exams (Exa), course programs

(Pro), textbooks (Txt), student notebooks (Sno), presentations (Pre) and web pages (Web).

## RESULTS

From the answers to the questionnaire and the interviews and from the materials used in their classroom we have classified the instructional representations used by Luis and Alicia to teach SG in their classroom in five types: definitions and explanations, questions, examples and exercises, instructions and data and graphical manipulation. Alicia uses the first four, and although Luis uses other types of instructional representations (project based learning and case study) in his classroom, the only ones linked to SG are questions and data and graphical manipulation. These instructional representations are discussed by Shulman (1986) (*explanations and examples*), Even & Tirosh (1995) and Peden (2001) (*questions, activities*), Curcio (1989) (*activities*) and Meletiou & Stylianou (2003) (*using technology*).

We will describe the characteristics of each of them as follows using only certain excerpts regarding each of the teachers as illustrations considering that they repeated the same ideas though out the documents we have analyzed.

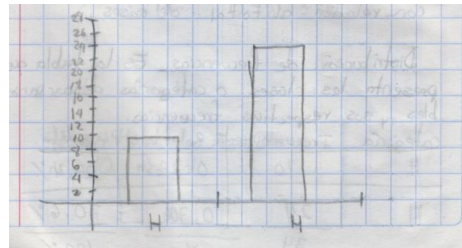
### Definitions and explanations

Definitions are a description of topics that appear at two points of the explanations: *as a closure of an inductive process* after an explanation as a synthesis or conclusion, or *to begin a new topic*. For instance, to start to work with the histogram, Alicia presents its definition showing some of its characteristics, indicating the kind of variable associated to the graph, and then provides an example with the criteria for its construction specifying the sequence of steps that the students should follow.

#### Graph

Bar Graphs allows a simple visualization of the distribution of qualitative data for absolute frequencies and relative ones. On the horizontal axis the categories are represented and on the vertical axis, in a continuous line, the values from zero to the maximum reached by the frequencies.

The frequencies are shown by the height of the bars, and all of them have the same width.



**Figure 1. Translation of a definition used by Alicia (Sno.01)**

Definitions come, generally, from textbooks (Txt) or course notes (Not) that Alicia uses in her classrooms and she dictates them to the students or writes them up on the board for them to copy. These definitions are supplemented with explanations where every characteristic of the graphic is commented, clarified, described and supported, for example indicating what kind of a variable each graph is useful for.

## Questions

Questions are instructional representations posed to the students in order to help them reflect on the characteristics of SG. They vary according to the context in which they are asked: those related to the teacher's knowledge of the teaching of SG or those generated from the hypothetical situations that were proposed in the questionnaire.

In the *first context* Alicia and Luis used questions as a resource to develop statistical reasoning, to find out if the concept was understood, to reinforce acquired knowledge or to identify mistakes.

- 38 Alicia: In this sense they are quite familiar with their grades average. It is appropriate to ask them: "Well, when you get your grade average, what meaning does it have? Why do you only manage to get a particular grade? Let's say that while you were taking a subject, how did you get the final score?" and then they start saying: "well, [because] there were some partial exams, they were graded this way and then we get that er... final grade," and "why do they only consider that grade?" (I2)
- 06 Luis: Well what graph do you think is better? "I am going to do a pie chart", Ok what are the advantages that this has compared to this other one?... (I2)
- 87 Alicia: Then, when I see that something is... like they are just writing it or they are copying it from somewhere, then I start asking them, I tell them: "Well er... why did you write this? What are you going to answer?". I start asking questions and that's the way I can see more or less who are the ones who have some difficulty and what is happening (I2)

In the *second context*, questions are posed with a clear intention to make the students realize, in an inductive way, what the characteristics of each graph are and the nature of the data represented or to develop a critical sense to help them learn.

- Alicia: What is the meaning in the graph of each one of the axis? (Q3b.1)
- 02 Luis: What does the stem in a stem-and-leaf plot represent? (I2)
- Alicia: First I would ask them: "What is the variable to be studied?" (Q4a.1)
- 09 Luis: Is it a continuous variable or not? (I2)
- Alicia: Is it clear for the reader what the cars positions represent? (Q4a.5)
- Luis: What are the parts of the graph whose relationships are incorrect? (Q4bii.3)
- Alicia: What could be included in the graph to make it is easier for any reader to understand? (Q3b.2)

The questions associated to each context differ from those of the others. In the first one we find questions more related to reinforcing knowledge and to clarifying the students' doubts, while in the second one, they are centered mainly on the exploration of the structural components of the graph.

## Exercises and examples

*Examples* are learning activities that have a direct relationship with the knowledge and understanding of the concepts or topics of the program and are generally provided immediately after a definition or explanation of the topic. *Exercises*, however, are learning activities whose main objectives are to develop practice, and to confirm that the concepts have been understood. Despite this difference, Alicia mentioned both terms repeatedly and indistinctly so they are included within the same category.

This instructional strategy was used to *assess the students' knowledge* of graphs construction or to enable the student to *discriminate between one graph and another*:

82 Alicia: (she is thinking) They did some exercises in class where they had to do, from their frequency distribution, then some graphs, we handled histograms, mm... some pictograms. Then I asked them for exercises in which I gave them some data and they had to do their distributions and their graphs. (I1)

39 Alicia: ...In the exercises where they are figuring out their distribution of frequencies, I ask them to make a graph.... I tell them to make all the graphs without paying attention to which one is the best or which one would be more suitable for that type of information. (I2)

These are activities in which, starting from a data set, the student must organize them, prepare a frequency table and, finally, build up one or more selected graphs (Figure 2). All the exercises come from textbooks.

2.4. In example 1.6, the situation proposed is that of a teacher who applied a mathematics diagnostic test to his students. Suppose that the 53 results obtained by the teacher were:

STUDY ACTIVITY
-------------------

4	1	5	6	4	2	4	3	5	2	4
8	6	6	0	6	10	6	8	7	7	5
6	9	6	7	3	7	7	7	10	5	8
4	9	11	6	9	6	8	9	6	10	
7	5	3	7	8	5	1	5	4	11	

Organize this data in a frequency distribution table and build up the corresponding bar graphs, frequencies and relative frequencies.

Figure 2. Representative example of the type of exercises (Exe.20)

In relation to the context of the exercises, most cases provide a set of data from an educational context: psychological or intelligence test scores, students' height measurements, exams, means of school transportation, preference for a school subject, and values from a students' opinion poll. However, exercises far removed from the educational context were also found: bulbs, salaries, stocks and shares sold, enterprises, currencies, daily consumption of water and measurements of earthquakes. Regarding this, Alicia stated that "it is difficult to find examples related to the area of education" (I1) although she recognized the need to find appropriate examples for these students.



## Working Group 5

In another set of exercises, in addition to drawing the corresponding graph, the student is asked to *calculate and/or interpret some statistical value* (e.g. mean, median, and mode).

- 40 Alicia: I only worked with the histogram as the exercise I showed you, where I give the histogram and from there they have to give a distribution, for example, of frequencies and obtain their central or dispersion measurements starting from that graph. (I2)

Only 3 out of the 22 exercises that were analyzed asked for computing certain statistical values, and all of them were notes in Alicia's own handwriting, that is, she added these tasks to the exercises. In another 3 exercises, the idea was to *explore the graph(s)* and to answer certain questions related to their interpretation.

### Instructions

Instructions are indications, activities, signals or orders given to help students understand different aspects of GR. Apart from exercises which are done on paper; instructions are in general oral, sometimes supported by some concrete object.

- Alicia: I would also ask them, based only on their diagrams (without looking at their data), to tell me how many people took exactly 23 minutes to get to school... (Q1a.2)

- Alicia: and that if these were not that clear to them, what they could include in the graph so that it would be more comprehensible for any reader (Q3b.2)

These instructions are often supported by questions.

### Data and graphical manipulation

Data and graphical manipulation is a modification of data and graphs, with the help of different resources such as technology, with some particular aim. Specifically, to teach SG is a way to explore what happens along the axes of the graph, as well as the similarities and differences between graphs, for example with the stem and leaf and the histogram:

- 01 Luis: For example, when we have worked with graphs, I ask them to manipulate the graph itself in their minds. For example, in this class, I generally do that in the computer lab because we can handle the graphs with the projector and things, for example, suddenly, "What would happen if you turn the axis and change the graph?" because in the case of the stem and leaf graph... " in other words, if we turn it and leave it ... or turn the graph of the histogram, what happens? in other words, what similarities do you find between the two? Yes? And at that moment to start to see what particular elements are found in both of them. (I2)

Luis makes reference to the manipulation of data to generate the graph, or to the rotation of the axes or the graph itself. This strategy is not a systematic or habitual activity in his classrooms, but circumstantial, depending on what occurs during the lesson and it is aimed at the analysis of the graph during SPSS practice.

## DISCUSSION

We have identified and characterized the instructional representations used by Alicia and Luis to teach SG, their types, purposes, points at which they are used and the role of teachers and students. Four of them are oral: definitions and explanations; questions; instructions and data and graph manipulation; and one of them is a concrete representation: examples and exercises (McDiarmid, Ball & Anderson, 1989). The teacher is the source of the questions, instructions, and data and graphics manipulation; while definitions, examples and exercises are taken from textbooks.

The instructional representations used by Alicia are teacher-centered whereas the ones used by Luis are thought to actively engage the students. Both of them resort to representations that allow interaction with the students like questions and data and graphical manipulation. Nevertheless, they are not previously planned activities but they arise from the dynamics of the classroom and the students' needs. These activities are only directed to reading data and reading between the data because they only aim to uncover the characteristics of the graphs and the relationship with the context.

Instructional representations used for teaching do not adequately nourish the development of statistical literacy in students and support materials reflect a passive approach to the teaching of graphing, and are quite different from: the recommendations of Meletiou & Stylianou (2003) to improve the comprehension of graphs; the developmental process of the cognitive levels of graph comprehension through interactive activities in the real contexts discussed by Curcio (1989); the information about general graph techniques and broad bibliography about graphs proposed by Pittenger (1995); the inclusion of updated texts with recent novelties regarding the way to approach the study of SG from Moore (2000) and Salkind (2000), and the practical applications presented in the *Teaching Statistics*, to mention just a few examples.

The results obtained suggest the need to better understand the role and effect of teachers' practices on students' learning statistics, such as the instructional representations. It is important to know and understand the nature and origins of instructional representations with the aim of designing training programs to help teachers to develop strategies to correct inappropriate beliefs and conceptions and to identify their prior knowledge on SG. These instructional representations can be a useful referent to train other teachers. Also, curriculum developers should consider teachers' limited knowledge and experience of data handling, and adopt an active graphing perspective that emphasizes the learners' role as interpreters and users of graphs in different moments and opportunities as analytical tools in solving problems in real contexts.

## REFERENCES

- Burgess, T. (2002). Investigating the 'data sense' of preservice teachers. En B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics*. Ciudad del Cabo, Sudáfrica: International Statistical Institute and International Association for Statistical Education. Recuperado el 25 de junio de 2005, de [http://www.stat.auckland.ac.nz/~iase/publications/1/6e4\\_burg.pdf](http://www.stat.auckland.ac.nz/~iase/publications/1/6e4_burg.pdf)
- Cazorla, I. M. y Santana, E. S. (2006). *Tratamento da informação para o ensino fundamental e médio*. Itabuna: Via Litterarum.
- Curcio, F. (1989). *Developing Graph Comprehension. Elementary and Middle School Activities*. Reston: NCTM
- Even, R. & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students a sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, 29(1), 1-20.
- Even, R. y Tirosh, D. (1995). Subject-matter knowledge and knowledge about students a sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, 29(1), 1-20.
- Friel, S., Curcio, F. y Bright, G. (2001). Making sense of graphs: Critical factors influencing. Comprehension and Instructional Implications. *Journal of Research in Mathematics Education*, 32(2), 124-158.
- Graeber, A. (1999). Forms of knowing mathematics: What preservice teachers should learn. *Educational Studies in Mathematics*, 38(1-3), 189-208.
- Llinares, S., Sánchez, V. & García, B. M. (1994). Conocimiento del contenido pedagógico del profesor. Tareas y modos de representación de las fracciones. *Revista de Educación*, 304, 199-225.
- McDiarmid, G. W., Ball, D. L. & Anderson, Ch. W. (1989). Why Staying One Chapter Ahead Doesn't Really Work: Subject-Specific Pedagogy. In M. C. Reynolds (Ed.), *Knowledge Base for the Beginning Teacher* (pp. 193-205). Nueva York: Pergamon Press.
- Meletiou, M. & Stylianou, D. A. (2003). Graphical Representation of Data: The Effect of the Use of Dynamical Statistics Technological Tool. In *Proceeding Sixth International Conference on Computer Bases Learning in Science* (pp. 296-306). Praga: Universidad de Zilina.
- Monteiro, C. y Ainley, J. (2006). Student teachers interpreting media graph. En A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics*. Salvador, Brasil: International Statistical Institute e International Association for Statistical Education. Recuperado el 14 de abril de 2009, de <http://www.stat.auckland.ac.nz/~iase/publications>
- Moore, D. (2000). *The basic practice of Statistics*. 2d. ed. New York: W. H. Freeman and Company.

## Working Group 5

- Moore, D. (2000). *The basic practice of Statistics*. New York: W. H. Freeman and Company.
- Peden, B. F. (2001). Correlational analysis and interpretation: Graphs prevent gaffes. *Teaching of Psychology*, 28(2), 129-131.
- Pfannkuch, M. & Rubick, A. (2002). An exploration of students' statistical thinking with given data. *Statistical Education Research Journal*, 1(2), 4-21. Recovered on 23 June 2005, de <http://fehps.une.edu.au/serj>
- Pfannkuch, M. (2006). Comparing box plot distributions: A teacher's reasoning. *Statistics Education Research Journal*, 5(2), 27-45. Recuperado el 14 de noviembre de 2007, de [http://www.stat.auckland.ac.nz/~iase/serj/SERJ5\(2\)\\_Pfannkuch.pdf](http://www.stat.auckland.ac.nz/~iase/serj/SERJ5(2)_Pfannkuch.pdf)
- Pinto, J. & González, M. (2006). Sobre la naturaleza conceptual y metodológica del conocimiento del contenido pedagógico en matemáticas. Una aproximación para su estudio. *Actas del X Simposio de la Sociedad Española de Investigación en Educación Matemática* (pp. 237-255). Huesca: Universidad de Huesca.
- Pittenger, D. J. (1995). Teaching students about graphs. *Teaching of Psychology*, 22(2), 125-128.
- Putnam, R. & Borko, H. (2000). El aprendizaje del profesor: implicaciones de las nuevas perspectivas de la cognición. In B. J. Biddle, T. L. Good e I. F. Goodson (Eds.), *La enseñanza y los profesores I* (pp. 219-309). Barcelona: Paidós
- Salkind, N. (2000). *Statistics for people who (think they) hate statistics*. London: Sage Publications, Inc.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Research*, 15 (2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: foundations of new reform. *Harvard Educational Review*, 57 (1), 1-22.
- Thompson, B. W. (1994). Making data analysis realistic: Incorporating research into statistics courses. *Teaching of Psychology*, 21 (1), 41-43.

# ASSESSING DIFFICULTIES OF CONDITIONAL PROBABILITY PROBLEMS<sup>1</sup>

M. Pedro Huerta, Fernando Cerdán, M<sup>a</sup> Ángeles Lonjedo, Patricia Edo

Universitat de València (Spain)

*In this paper we define some difficulties of conditional probability problems and their measurements. We also show some results about the measurements of those difficulties in relation to the structure of problems, the contexts in which they are formulated and the sample of students that solved them. Finally, we also comment on the possible influences of these factors on the difficulties of problems.*

## INTRODUCTION

Some research reports inform us of the difficulties and the low success rate students have in solving tasks or problems involving Bayes' rule, or in general in solving conditional probability problems (Diaz & De la Fuente, 2007). These difficulties, it is said, depend on multiple factors. One of them is the cognitive complexity of the concept. Sometime this complexity is related to data format and the presentation of the data in problems (Estrada & Diaz, 2006; Huerta & Lonjedo, 2006; Lonjedo & Huerta, 2007), the context in which problems are put forward (Ojeda, 1996; Carles & Huerta, 2007) and the particular language with which conditionality and events are expressed (Maury, 1984). However, in general, these studies do not pay attention to the mathematical structure of problems and the context in which problems are put forward, at least not in a systematic way, factors that could be the reason why these problems are so difficult for students. Therefore, in the main, these factors are not taken into account in research results. Nevertheless, for us, they are influential factors on the students' success and on their difficulties in solving problems.

Due to their structure, we already know that we can consider a particular world of problems that is generated by means of a limited set of events and probabilities and the relationships between them (Huerta, 2009). Concretely, with two basic events, 16 probabilities and 18 ternary relationships<sup>2</sup> between these probabilities it is possible to generate this world of problems, which we call the *world of the ternary problems of conditional probability* (Cerdán & Huerta, 2007; Huerta, 2009). Many school problems belong to this world, as do a considerable number of tasks used in research, and they can be seen as instances of more general realistic situations (Carles & Huerta, 2007). For this reason their study, we think, is relevant.

Two particular pieces of research will be briefly examined in this paper. Apart from problems and the students' sample, both pieces of work share the same objectives and research methodology. Thus, we will firstly show the shared objectives and methodology and, in continuation, the results we have obtained up to now. Because the research is still not finished and due to the limitation number of pages available

for this report, only the global results of the research is shown along with a few comments on them.

## OBJECTIVES

Our general objectives in this report are as follows:

1. To identify difficulties of conditional probability problems.
2. To introduce a measure of these difficulties and to carry out these measurements.
3. To consider to what extent the structure and context variables, considered as independent variables, are influential factors on the difficulty of the problems.

## BACKGROUND: ELEMENTS OF THE FRAMEWORK

Several years ago, we began the study of the conditional probability problems in a systematic and methodical way (Lonjedo & Huerta, 2007; Huerta & Lonjedo, 2006; Cerdán & Huerta, 2007; Carles & Huerta, 2007; Lonjedo, 2007; Huerta, 2009; Edo, 2010). In these works the main object of research is the problems themselves. We identified a particular family of problems that we call *ternary problems of conditional probability*<sup>3</sup>, and with the help of a tool we call *trinomial graph* we carried out analytical readings of the problems (Huerta, 2009). An analytical reading of a problem is one that only pays attention to known and unknown data and their relationships but ignoring other elements of the problem that we know are also important. The analytical readings of problems on a trinomial graph allow researchers and teachers to determine the complexity of problems by means of their minimal graph, that is to say, the graph containing the minimum number of required relationships between known and unknown data that solves the problem. At the same time, we use the graph in order to identify problems in which the analytical reading is either arithmetical or algebraical<sup>4</sup>.

Because of the structure of data, ternary problems of conditional probability have been classified into four families and twenty sub-families (Lonjedo, 2007). The L-family of problems is characterized because every problem belonging to it has either 0 or 1 or 2 or 3 known conditional probabilities as a maximum. We call this Level of problems (Huerta, 2009). Related to each L-family, the  $C_iT_j$ -subfamily of problems is characterized by having 0, 1 or 2 known absolute (marginal) probabilities (the characteristic  $C_i$  of a problem) and one unknown probability, which is asked for in problem (the type T of the problem: Type  $T_1$  if a conditional probability is asked for,  $T_2$  if a marginal and Type  $T_3$  if an intersection probability). Therefore, analytically speaking, for each family and sub-family, there is at our disposal a problem representing each one of the classes of problems. All these previous theoretical studies about problems permit us to construct questionnaires of problems to be administered (look at the examples in the annex).

All the above-mentioned are elements of the framework in which we place our research on conditional probability problem solving. This framework is still in progress but some of its components can be studied in Huerta (2009).

## METHODOLOGY FOR ASSESSING DIFFICULTIES OF PROBLEMS

Bearing in mind all the above considerations, in order to construct questionnaires for assessing difficulties of problems we first must to decide on what independent variables they might depend. Thus, as task variables (Kulm, 1979) they are, and in order to investigate difficulties depending on structure and context variables, we fixed data format in every problem we put forward. All data referring to probabilities, either known or unknown was expressed in a percentage format, in the case of conditional probability. The reason for taking this decision can be found in various studies (e.g. Watson & Kelly, 2007; Lonjedo & Huerta, 2007). There is a general agreement that students think better using conditional frequencies than with conditional probabilities (see for example, Jones, Langrall & Mooney, 2007).

Thus, for each problem in the questionnaires, we define the following independent variables:

- *Structure variable*, given by the L-family and the C, T sub-family.
- *Context variable*, taking the following values: *Stat-Social*, *Stat-Health* and *Diag-Health*. (See the examples in the annex)

Depending on these two variables, we analyse students' resolutions paying attention to two main dependent variables: process and product variables. Difficulties of problems will be measured throughout the product variable, which takes the following values:

- *Tackled*. Number of students who tackle a problem. We say that a student tackles a problem if we can recognize that student undertakes the problem in some way.
- *Answer*. Number of students that answer the question posed in the problem. The answer to the question may be anything that students declare is the answer to the question posed.
- *Number*. Number of students that respond to the question of the problem by means of a correct number (a percentage in this case).
- *Description*. Number of students that attach an expression to the numerical answer, describing what this number is measuring. For a given number, the attached expression may be correct or incorrect whether the number is correct or not.

By means of these variables we define the following difficulties of problems:

- *Appreciated difficulty (ADP)* of a problem. This tries to measure students' difficulties before the process of solving the problem starts. Therefore, we are

supposing that students, after reading the problem, decide whether to tackle the problem or not. If they do not, we suppose it is because he/she appreciates that the problem is too difficult to be solved.

- *Problem's difficulty (PD)*. This difficulty will inform us to what extent it is difficult to give an answer to the question in problem.
- *Problem solution difficult (PSD)*. In relation to the previous difficulty, this one indicates to us how difficult is to give a correct answer to the problem.
- *Difficulty of Correct Description of the solution of the problem (CorrectDescD)*. In this case, we will obtain information about students' difficulties in giving a correct description of the event which is measured by the given number as a solution.

Differences between difficulties may be also appreciated through their measures and the way we measure them. Indeed, each one of these difficulties is measured in percentages, as a result of applying the following formulae:

$$ADP = 100 - \left( \frac{\text{tackled}}{\text{number of students}} \right) \times 100$$

$$PD = 100 - \left( \frac{\text{answer}}{\text{tackled}} \right) \times 100 \quad PSD = 100 - \left( \frac{\text{number}}{\text{tackled}} \right) \times 100$$

$$DescD = 100 - \left( \frac{\text{Description}}{\text{answers}} \right) \times 100 \quad CorrectDescD = 100 - \left( \frac{\text{CorrectDesc}}{\text{Description}} \right) \times 100$$

These difficulties are ranked in an  $[0, 100]$ -interval. If a difficulty is measured by one 0, then this means that the problem does not pose that particular difficulty for any student whereas, on the opposite side of the interval, a difficulty which is measured by one 100 is present in the problem for all students.

## RESULTS FROM TWO INDEPENDENT, BUT RELATED, RESEARCHES

### 1. Research on difficulties of problems from $L_0$ -family in students 15-16 aged without previous instruction on conditional probability.

The  $L_0$ -family of problems can be divided into three sub-families (Huerta, 2009; Edo, 2010):  $C_0T_1$ ,  $C_1T_1$  and  $C_2T_1$ . There are 6 *basic-options* of problems representing to this family: one from  $C_0T_1$ ; two from  $C_1T_1$ ; and finally three from  $C_2T_1$ . Accordingly, we constructed one basic questionnaire containing 6 problems. Owing to the fact we finally consider three contexts, in order to explore difficulties of problems depending on contexts, we prepared a total of 18 problems (one of them is given in the annex). All those problems were distributed into 6 questionnaires each containing 3 + 3 problems structurally isomorphic in pairs, but put forward in different contexts. For each questionnaire, and from students' resolution of problems, the possible influence of the context in the difficulties of problems becomes observable for researchers.



### Working Group 5

Furthermore, at the same time, the influence of the structure of problems on these difficulties is also observable.

Questionnaires were administered to 165 students aged 15-16 years. Students solved problems during their mathematics class. They received no teaching on conditional probability.

The 990 students' resolutions provided the following global results (Table 1):

L <sub>0</sub> -Problems' resolutions	Tackled	Answer	Number	Descriptions	Correct Desc
990	712	585	214	320	139
Difficulties (%)	ADP (28.1)	PD (17.8)	PSD ( <b>69.9</b> )	DescD (45.3)	CorrectDescD ( <b>56.6</b> )

**Table 1: Global results en frequencies and difficulties of problems from L<sub>0</sub>-family, in percentages.**

If we take into account contexts and structures, the following tables (Tables 2 & 3) show us to what extent these are influential factors:

Structure/ Difficulty	ADP	PD	PSD	CorrectDes D
C <sub>0</sub> T <sub>1</sub>	8.5	13. 5	59.5	<b>69</b>
C <sub>1</sub> T <sub>1</sub>	27.2	17. 8	<b>64.9</b>	49.6
C <sub>2</sub> T <sub>1</sub>	<b>41.9</b>	21. 4	<b>65.5</b>	51.4

**Table 2: Difficulties of problems from L<sub>0</sub>-family, depending on their structure (%)**

Context/difficult y	ADP	PD	PSD	CorrectDes D
Stat-Social	20.9	13. 4	50	41.2
Stat-Health	23	18. 1	<b>75</b>	<b>75.7</b>
Diag-Health	<b>40.3</b>	23. 3	62.9	55.1

**Table 3: Difficulties of problems from L<sub>0</sub>-family, depending on contexts (%)**

A deeper analysis of data suggests that the structure variable is an influential factor on the ADP and PD but not on the PSD. At the same time, the context variable is also an influential factor but not in the same manner as the structure variable does. Indeed, the Diag-Health context is more influential on every difficulty than the Stat-Social context, but the Stat-Health is the most influential context on PSD.

## **2. Research on difficulties of ternary problems (problems belonging to all families) in student teachers of high school mathematics.**

In this research we studied the difficulties of ternary problems in a sample of 54 students doing a Professional Master in Secondary Mathematics Teaching at the Universitat of València (Spain). Not all of them were Mathematics graduates, as there were also graduates in Economics, Engineering and Architecture. However all of them had taken courses in probability during their regular studies.

In order to construct a questionnaire for assessing difficulties of problems, and given that we did not have prior experience on problem resolutions in students belonging to this sample, many previous decisions had to be taken into account. Only the experience with  $L_0$ -problems could help us in doing this. In addition to this, previous to deciding what problems were going to be included in the questionnaire, we carried out a theoretical study of problems at every level, in order to determine *basic-options* of problems representing all families and sub-families, as had been done for the  $L_0$ -family. Based on this study, among others, the following decisions were taken:

- The questionnaire consists of 7 ternary problems distributed as follows: One problem from  $L_0$ -family, two problems from  $L_1$ -family, three problems from  $L_2$ -family, and one problem from  $L_3$ -family. Among them, two problems have algebraical readings (one from  $L_2$ -family and one from  $L_3$ -family) and five arithmetical readings (some examples are given in the annex).
- In every problem, the data format is percentages.
- *Stat-social*, *Stat-Health* and *Diag-Health/man* are the contexts in which problems are put forward. Differences between Diagnostic Test in Health or Manufacture must be found in the elements of the context: people or manufactured pieces that are tested.
- If a problem has a complex analytical reading, either arithmetical or algebraical, we decided to avoid extra difficulties by contextualizing the problem in a context that we know is not very influential in the difficulties (see the above study results). This is the case, for example, in the problem that belongs to the  $L_3$ -family. We already know that all problems from this family have an algebraical reading. On the other hand, from the previous research, we also know that Stat-social is a less influential context on the difficulties than the other two we considered. Therefore, we decided to formulate the problem from  $L_3$ -family in a

### Working Group 5

Stat-social context. Conversely, the problem from L<sub>0</sub>-family is formulated in the most influential context, the Stat-Health context.

The 378 students' resolutions produced the following global results (Table 4):

Resolutions	Tackled	Answer	Number	Descriptions	Correct Desc
378	369	269	109	183	138
Difficulties (%)	ADP (2.4)	PD (27.1)	PSD (70.5)	DescD (32)	CorrectDescD (24.6)

**Table 4: Global results en frequencies and global difficulties of problems in %**

In a similar way, as was done in the above example, the following table (Table 5) provides us with information concerning the difficulties of the problems depending on the family to which they belong:

Level	ADP	PD	PSD	DescD	CorrectDesD
L <sub>0</sub>	1.8	16.7	68.5	28.9	28.1
L <sub>1</sub>	1	26.2	68.2	29.1	25
L <sub>2</sub>	4.3	28.4	69.8	<b>38.7</b>	27.9
L <sub>3</sub>	0	<b>37</b>	<b>79.6</b>	20.6	11.1

**Table 5: Difficulties of problems depending the family they belong to, in (%)**

These results suggest that the structure of problems belong to L<sub>3</sub>-family is a more influential factor on PD and PSD than the other structures of problems. Perhaps, this influence is due to the fact that problems belong to L<sub>3</sub>-family have algebraical readings.

And, finally, the following table (Table 6) shows the difficulties of problems depending, this time, on the contexts in which the problem was presented:

Context	ADP	PD	PSD	DescD	CorrectDesD
Stat-Social	1.8	35.2	71.7	28.1	14.9
Stat-Health	4.6	31.1	<b>74.6</b>	33.8	27.7
Diag-Health/Man.	1	11.2	64.5	34.7	33.9

**Table 6: Difficulties of problems depending on contexts, in (%).**

As in the above study, for this sample of students, the Stat-Health context is the most influential context on the PSD, but not in a meaningful way. The students, however, do not appreciate difficulties in problems because almost all students tackled the problems (ADP<5%).

## DISCUSSION AND CONCLUSIONS

From a global point of view, we can conclude that ternary problems of conditional probability are difficult for students. Finding a correct answer to a problem, which is basically finding a conditional probability when three known data are given in a context, is a difficult task for every student, whether they are from secondary school or graduated ( $PSD \geq 70\%$  for both samples of students). But, in general, at the beginning of the process, it seems that these difficulties are not especially appreciated by students (ADP between 28%, in the first study, and 2.4% in the second one) when they tackle the problems and try to give an answer to the question posed in them (PD between 17.8% and 27.1%). However, the answers given are usually incorrect in a very high percentage of cases (PSD of 69.9% in the first study, 70.5% in second one). Among those students who give an answer, some of them do not describe the number, given as the answer, by means of an expression representing the event and the number is its measure (45.5% in first, 32% in second). That is to say, for these percentages of students, the answer is simply a number, neglecting what this number is measuring and why. Among the students that describe the number with an expression, there is a percentage of students that do it incorrectly (56.6% in the first one, 24.6% in the second one). In some of these incorrect expressions it is usual to recognize one of the most common misconception in conditional probability: the conditional probability by intersection probability (Lonjedo & Huerta, 2007; Lonjedo, 2007), even if the given number is a correct number and the students are either at secondary school or graduated.

Structures and contexts appear as influential factors on the difficulty of problems, as we have shown above. Thus, for example, in the second study, we can see how the difficulty of giving an answer to problems increases as the number of known conditional probabilities in text of problems also becomes greater. However, when an answer is given, the difficulty of giving the correct answer is not as sensitive to this factor, except in the  $L_3$ -problem. It seems that there are other structure-based factors that are influential on this difficulty. Indeed, every problem from  $L_3$ -family has an algebraical reading.

Finally it is reasonable to think that contexts are an influential factor on the difficulties of problems. But difficulties are not sensitive to these influences in the same measure. Thus, while the Diag-Health context is an influential factor on the appreciated difficulty of the problem for secondary school students, this is not the case for future teachers. This could mean that the context is not a well-known context for the former but that it is for the latter. On the contrary, if we pay attention to the problem solution difficulty, it is common to both samples that the Stat-Health context is the most influential. Considered in an isolated way or in combination with structure, difficulties are very sensitive to it. This context is characterized because it belongs to a more general situation called *Les situations causalistes* (Henry, 2005), that particularly in the Health context is very influential: every time someone treats

himself/herself with a medicine, then he/she is cured of the illness, but if this is not so then he/she does not. Obviously, there are no conditional probabilities involved.

Although the results we just show in this report are global and their analysis is not yet finished, some suggestions can be made for the future, both for researchers in probabilistic education and for mathematics teachers. Researchers could take into account that tasks used to investigate students' behaviour have different difficulties depending on the structure of their data and the context chosen for putting forward the task. It is a hypothesis to suppose that varying structure and context in tasks imply that students' behaviour may also be different in each case, or not. This hypothesis might be contrasted.

On the other hand, teachers should be aware that there are families of conditional probability problems that contain problems with different difficulty degrees, again depending on structures of data and context. Perhaps a teaching model could be considered which was based on solving problems belonging to these families, organized sequentially, and exploring through them as contexts as possible. This could potentially be a good way to improve students' competence in solving conditional probability problems.

## NOTES

[1] EDU/2008-03140/EDU Project, funded by Ministerio de Ciencia e Innovación, España.

[2] These are examples of them:  $p(A) + p(\bar{A}) = 1$ ;  $p(A | B) \times p(B) = p(A \cap B)$ . They may be either additive or multiplicative.

[3] *Ternary problems of conditional probability* are defined as problems that fulfil the following conditions: (1) One conditional probability is involved, either as known data or as unknown data or both; (2) Three probabilities are known; (3) All probabilities, both known and unknown are connected by ternary relationships.

[4] If a problem has an algebraical reading then it means that, in order to solve it, an extra-data will be required (an unknown). If it has an arithmetical reading, then it does not.

## REFERENCES

- Carles, M.; Huerta, M. P. (2007). Conditional probability problems and contexts. The diagnostic test context. In D. Pitta-Pantazi & G. Philippou (Eds.) *Proceedings of the CERME 5*, 702-710.
- Cerdán, F.; Huerta, M. P. (2007). Problemas ternarios de probabilidad condicional y grafos trinomiales. *Educación Matemática*, 19 (1), 27-62.
- Díaz, C.; De la Fuente, I. (2007). Assessing students' difficulties with conditional probability and Bayesian reasoning. *International Electronic Journal of Mathematics Education*, 2 (3), 128 -143.
- Edo, P. (2010). *Estudios sobre la familia de problemas de probabilidad condicional de nivel  $N_0$*  (Studies about the family of conditional probability problems of level

- L<sub>0</sub>). Unpublished Memory of degree in Master of Research in Didactic of Mathematics. University of Valencia.
- Estrada, A; Díaz, C. (2006). Computing probabilities from two-way tables: an exploratory study with future teachers. *Proc. ICOTS-7*. Retrieved June 7, 2009, from <http://www.stat.auckland.ac.nz/~iase/publications/17/C413.pdf>.
- Henry, M. (2005). Modélisation en Probabilités conditionnelles. In M. Henry (Ed.) *Autour de la modelisation en probabilités*. IREM de Franche-Comté, (6) 173 - 185.
- Huerta, M. P. (2009). On Conditional Probability Problem Solving Research —Structures and Context. In M. Borovcnik & R. Kapadia (2009), Special issue on “Research and Developments in Probability Education”. *International Electronic Journal of Mathematics Education*, 4 (3), 163-194.
- Huerta, M. P.; Lonjedo, M. A. (2006). The nature of the quantities of in a conditional probability problem solving behaviour. In M. Bosch (Ed.) *Proceedings of the CERME 4*. Sant Feliu de Guíxols (Spain).
- Jones, G. A.; Langrall, C. W.; Mooney, E. S. (2007). Research in Probability (Responding to Classroom Realities). In F. Lester (Ed.), *Second Handbook of Research on mathematic teaching and learning*, 909-956. NCTM.
- Kulm, G. (1979). The classification of Problem-Solving Research Variables. In G. A. Golding & C. E. McClintock (Ed.), *Task Variables in Mathematical Problem Solving*, 1-22. ERIC
- Lonjedo, M. A.; Huerta, M. P. (2007). The same problem in three presentation formats. Different percentages of success and thinking processes. In D. Pitta-Pantazi & G. Philippou (Eds.) *Proceedings of the CERME 5*, 732-741.
- Maury, S. (1984). La quantification des probabilités : analyse des arguments utilises par les élèves de classe de seconde, *Recherches en Didactique des Mathématiques*, 5, (2), 187-214.
- Ojeda, A. M. (1996). Contextos, representaciones y la idea de probabilidad condicional. In F. Hitt (Ed.), *Investigaciones en Matemática Educativa*. México: Grupo Editorial Iberoamericana, 291-310.
- Watson, J. M.; Kelly, B. A. (2007). The development of conditional probability reasoning. *International Journal of Mathematical Education in Science and Technology*, 38:2, 213-235.

### **Annex: Example of problems used in the researches**

Problem in Level L<sub>0</sub>, L<sub>0</sub>C<sub>0</sub>T<sub>1</sub>-subfamily. Stat-social context. 15-16 years old students. The 4<sup>th</sup> grade class is made up of 30 students between boys and girls. Among the students, there are 7 boys who wear glasses, 10 girls who do not use them, and 8

### Working Group 5

boys who do not wear them. Among boys in the class, what percentage wears glasses?

Problem in Level  $L_1$ ,  $L_1C_1T_1$ -subfamily. Diag-Health context. Graduated students.

A population at risk for tuberculosis is subject to the tuberculin test. Different studies show that 57% of the population suffers from tuberculosis, and among those with tuberculosis 59.6% give positive in test. In addition, it is known that 13% did not have tuberculosis but is positive in the test. Among those who are positive in the test, what percentage is suffering from tuberculosis?

Problem in Level  $L_2$ ,  $L_2C_1T_1$ -subfamily. Stat-Health context. Graduated students.

A population suffers from eye infection. Of these, 42% are treated with a new antibiotic. The results show that, of those treated with the antibiotic, 83.3% were cured, and people who have not cured 14.9% were treated with the antibiotic. Among those who have been cured, what percentage has not been treated with the new antibiotic?

Problem in Level  $L_3$ ,  $L_3C_0T_2$ -subfamily. Stat-social context. Graduated students.

Of the girls in a high school, 37.5% wear glasses. Of the boys, 28.6% wear glasses. Of those who do not wear glasses, 50% were boys. Among high school students, what percentage are girls?

# USING A RASCH PARTIAL CREDIT MODEL TO ANALYZE THE RESPONSES OF BRAZILIAN UNDERGRADUATE STUDENTS TO A STATISTICS QUESTIONNAIRE

Verônica Y. Kataoka<sup>1</sup>, Claudia Borim da Silva<sup>2</sup>, Claudette Vendramini<sup>3</sup>, Irene Cazorla<sup>4</sup>

<sup>1</sup>University Bandeirante of São Paulo, <sup>2</sup>University São Judas Tadeu, <sup>3</sup>University São Francisco, <sup>4</sup>State University of Santa Cruz

*The aim of this study was to analyze the degree of difficulty of questions included in a statistical literacy questionnaire answered by 1343 Brazilian undergraduate students who were studying a statistics course for the first time. The responses to the questionnaire, which was made up of multiple-choice items, were classified according to the structure of the observed learning outcome (SOLO) taxonomy, and the results were analyzed using item response theory. Questions that involved the calculation of measures of central tendency with an outlier and variability associated with reading a graph presented the highest degree of difficulty, implying that these areas are not explored in basic education. Considering the importance of measures of central tendency and spread for statistical literacy, it is important to continue promoting the improvement of statistics education in Brazilian schools.*

Keywords: SOLO taxonomy; item response theory; statistics questionnaire

## INTRODUCTION

In Brazil, at the end of the 1990s, following the publication of the National Curricular Parameters (Parâmetros Curriculares Nacionais – PCN), basic statistical concepts were formally incorporated into the mathematics curriculum in primary and middle schools (Brazil, 1997, 1998) and in high schools (Brazil, 2002, 2006). With regard to teaching statistics, these curricular guidelines are in line with Watson's (2003) suggestion that, in order to attain statistical literacy in adults, it is important that students, whilst still in school, are exposed to statistical and probabilistic concepts. Wallman (1993, p. 1) was one of the first authors to define statistical literacy as:

[...] the competence to understand and to critically assess statistical results that permeate our daily lives, together with the capacity to recognize the contribution that statistical thought can provide to public, private, professional, and personal decisions.

This idea of statistical competence forms the cognitive component of Gal's (2002) definition of statistical literacy. This author supplements his definition with the component of disposition, which relates to the competence of individuals to discuss or communicate their reactions to statistical information, such as their understanding of the meaning of information, their opinions on the implications of such information or their considerations in relation to the acceptance of the conclusions provided (Gal, 2002, pp. 2-3).



According to Gal (2002), the cognitive elements of statistical literacy are: (a) mathematical knowledge; (b) contextual knowledge; (c) literacy skills (i.e., being able to understand written or oral text and tabular or graphic displays); and (d) the competence of developing critical questions. The author suggests five areas of basic statistical knowledge that together make up statistical literacy: (a) knowing why data are needed and how data can be produced; (b) familiarity with basic terms and ideas related to descriptive statistics; (c) familiarity with graphic and tabular displays and their interpretation; (d) understanding the basic concepts of probability; and (e) knowing how statistical conclusions or inferences are reached.

Considering the literacy model suggested by Gal (2002) and keeping in mind the Brazilian context, which, for over 10 years, has included curricular guidelines for teaching statistics in basic education (consequently, the Brazilian students who entered university after 2008 should have finished their basic education with a minimum level of statistical literacy), this research aims to answer the following questions: do university students learn about these five topics and therefore attain a basic knowledge of statistics during their school education? Which topics are easier or harder for students? The answers to these questions may guide pedagogical initiatives that will contribute to the improvement of the teaching and learning of statistics in Brazilian schools.

In order to answer the questions above, the objective of this study was to use item response theory (IRT) to analyze the degree of difficulty of questions answered by undergraduate students who were studying statistics for the first time. The questions covered the statistical content included in the PCN and were fully in line with the components of Gal's (2002) model.

## **METHOD**

This study was conducted with 1343 undergraduate students from five Brazilian universities in three different states. The following instruments were applied in the first semester of 2009: a statistical questionnaire and a profile questionnaire which included questions relating to the students' socio-educational characteristics, knowledge and utilization of statistics in their daily lives, both personal and professional. The data were collected during the first two weeks of the semester with the objective of minimizing the influence of formal learning on the results of both questionnaires.

The age of the participants ranged from 17 to 58 ( $M = 22.3$ ,  $SD = 5.3$ ), and 52.0% were male. In terms of specialty, 35.4% were studying an exact area of science, 14.5% were studying biology/health and 49.7% were studying human sciences (0.4% did not answer). The students were distributed across two private universities in São Paulo (40.6% and 24.3%), a state university in São Paulo (7.2%), a private university in Rio Grande do Sul (4.2%) and a federal university in Pernambuco (23.6%).

## Working Group 5

Of the students involved in this survey, 41.7% and 31.0% considered statistics to be very important for their personal capabilities or as a basic requirement for other undergraduate courses respectively. Only 21.0% of the students could not imagine other areas for which statistics would be important. In addition, 35.7% of students considered statistics to be moderately relevant to their daily lives.

### Statistical questionnaire

The statistics questionnaire included seven multiple-choice questions involving the following content: measures of central tendency and spread; probability; reading and interpreting two-way tables and confidence intervals. All of this content, except confidence intervals, is included in the Brazilian curricular guidelines for schools. Despite the fact that confidence intervals are not a part of the Brazilian school curriculum, one question was included in order to assess whether or not this concept, which appears frequently in the media, should be part of students' extra-curricular knowledge.

The first question (Q.1), covering measures of central tendency, was adapted from the work of Watson and Callingham (2003) and Garfield (2003), and required the students to choose the most accurate measure to represent the weight of the an object measured by nine students using the same instrument. The sample contained an outlier. The expected response was that the outlier should be removed and the mean of the other remaining values should be calculated. There were three questions relating to probability. The second question (Q.2) was taken from the work of Watson and Callingham (2003) and Garfield (2003) and required the interpretation of a simple probability (a bottle of medicine has the following notice: "For application to areas of skin, there is a 15% chance of developing a rash."). The third question (Q.3) was suggested by Garfield (2003), and asked studentes to analyse the accuracy of a weather forecast (the meteorologist identified in his reports the days on which there was a 70% probability of rain). The expected response was the selection of an option that included the value of 70% in the interval.

Statistics from the State Highway Dept. are presented in the table below, providing information about the number of accidents (fatal or otherwise) with victims, and the condition of the driver (sober or drunk):

Driver/ victim	Non-fatal	Fatal
Sober	1228	275
Drunk	239	86

Would you say that the condition of the driver (sober or drunk) affects the occurrence of fatalities? What is this probability? (a) Sober,  $P(\text{sober}|\text{fatal}) = 275/361$  (b) Drunk –  $P(\text{drunk}|\text{fatal}) = 86/361$  (c) Sober –  $P(\text{fatal}|\text{sober}) = 275/1503$  (d) Drunk –  $P(\text{drunk}|\text{fatal}) = 86/325$  (e) None of the above

**Figure 1: Fifth question from the statistics questionnaire**

## Working Group 5

The fifth question (Q.5) required the interpretation of conditional probability in a problem suggested by Kataoka et al. (2008) (Figure 1). For this question, we expected the students to answer that the condition is the driver being drunk, and to calculate  $P(\text{fatal}|\text{drunk}) = 86/325$ .

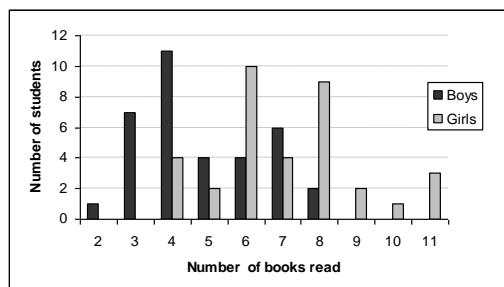
A research agency held a survey in order to assess voters' intentions in one State Government election. A total of 6000 voters in 26 cities were randomly selected. Based on this research, in which 41% of voters in the poll said that they intended to vote for candidate A, 23% for candidate B, 21% candidate C, and 15% for one of the remaining candidates, and considering that the margin of error was 2% with a 99% degree of confidence, is it possible to predict who will dispute the run-off election with candidate A? Choose one answer:

- (a) Yes, candidate B      (b) Yes, candidate C      (c) No, it could be candidate B or C  
(d) No, more information is required      (e) None of the above

**Figure 2: Excerpt from fourth question from the statistics questionnaire**

The fourth question (Q.4), which was developed by the authors of the present work, covered the confidence interval for proportion, and required the student to analyze the results of an electoral poll regarding the state government (Figure 2). For this question, we expected that the students would add the margin of error to the percentage of votes for candidates B and C, and that a technical tie occurs between the two candidates, and it did not allow the students to choose between them.

The principal of School X was interested in assessing students' reading skills. Therefore, he asked for the number of books read by the girls and boys at the school during the first semester. The results are presented in the bar chart below. Based on the graph, the principal concluded that (choose one answer only): (a) The variability of the number of books read by boys is higher, because they read between two and eight each; (b) The variability of the number of books read by girls is higher, because there are many girls who read six to eight books; (c) The variability of the number of books read by girls is higher, because on average they read more books; (d) The variability of the number of books read by girls and boys is almost equal; and (e) None of the above.



**Figure 3: Excerpt from sixth question from the statistics questionnaire**

## Working Group 5

Questions six and seven were developed by the authors and were related to the concept of variation in the context of a group comparison. This concept is considered by Wild and Pfannkuch (1999) to be a central element of statistical thinking. In both questions, data about the groups were presented graphically, with one question about variability and another about standard deviation. In question six (Q.6), a double bar chart was presented in order to allow the students to compare variability between the groups (Figure 3).

### Data analysis

The responses of the students were classified using content analysis (Bardin, 1995) and reclassified based on hierarchical levels of the structure of the observed learning outcome (SOLO) taxonomy devised by Biggs and Collis (1991) (Table 1).

Q.	Description of categories
<b>1</b>	<ul style="list-style-type: none"> <li>0. Blank response or use of an inadequate method</li> <li>1. Calculates the mean with all of the values or notices the outlier</li> <li>2. Uses a method in which the result is a good representation of the data e.g. mode, median etc.</li> <li>3 Removes the outlier (15.3) and calculates the mean</li> </ul>
<b>2</b>	<ul style="list-style-type: none"> <li>0. Blank or incorrect response</li> <li>1. Answers considering only informal interpretation (small or good chance)</li> <li>2. Answers considering only one numerical interpretation of 15% (approximately 15 in 100) or a numerical and informal interpretation</li> </ul>
<b>3</b>	<ul style="list-style-type: none"> <li>0. Intervals that do not contain 70%</li> <li>1. Notices that for all of the days, the forecast was 70% and, of these, 95% and 100% of these days had 70%.</li> <li>2. Interval that contains 70%</li> </ul>
<b>4</b>	<ul style="list-style-type: none"> <li>0. Blank response or incorrect interpretation</li> <li>1. Chooses one of the probable candidates</li> <li>2. Notes that the margin of error does not allow selection between candidates B and C</li> </ul>
<b>5</b>	<ul style="list-style-type: none"> <li>0. Blank reply or calculates simple probability</li> <li>1. Calculates the conditional probability, but uses the incorrect sample space</li> <li>2. Calculates the correct conditional probability</li> </ul>
<b>6</b>	<ul style="list-style-type: none"> <li>0. Blank reply or reads the graph with only one measure of central tendency</li> <li>1. Reads the total range only</li> <li>2. Correct comparison, taking into consideration variation (total range and density frequency)</li> </ul>
<b>7</b>	<ul style="list-style-type: none"> <li>0. Blank reply, or reads the graph with one measure of central tendency only</li> <li>1. Reads the total range only</li> <li>2. Correct comparison, taking into consideration variation (total range and density frequency)</li> </ul>

**Table 1: Description of the categories of each question according to the SOLO taxonomy**

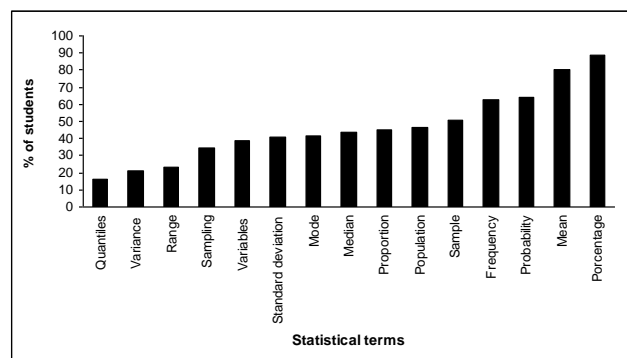
## Working Group 5

According to Biggs and Collis (1991), the SOLO taxonomy consists of: the prestructural level (code 0), in which the student makes basic mistakes and provides insufficient answers; the unistructural level (code 1), in which the student presents several conclusions that could be correct, but which are not mutually consistent and focus on only one relevant aspect of the problem; and the multistructural level (code 2), in which the student notices more than one relevant aspect, but does not integrate them, meaning that their answers may be inconsistent. The relational level (code 3) was not used in the analysis, with the exception of the question about the measure of central tendency, in which one of the options required the student to suggest a method with which to define a measure that would represent the set of weight measures that included an outlier (Table 1). In this level (code 3), the student presents a conclusion that could potentially relate to all of the relevant aspects, showing an overall degree of consistency.

Each response category for each question was denominated as an item and is represented by the number of the question followed by the category value; for example, the notation 2.1 means question 2 and the response which was classified as category 1. After classifying the responses, the items included in the instrument were analyzed based on a Rasch partial credit model (Rasch, 1980; Masters, 1982), which is an extension of the Rasch model for dichotomous items. This model is suitable for the analysis of responses obtained from two or more ordinal categories (polytomic items), such as the categorization of responses based on the SOLO taxonomy. The quality of the items was analyzed using the difficulty/adjustment measure of the model (*infit* and *outfit*) and the correlation between the question and the Rasch measure.

## RESULTS AND DISCUSSION

In the profile questionnaire, the students were asked if they had already studied statistics and probability in basic education; 30.5% of them said yes, 38.3% said that they had studied these two subjects but that they did not remember the content and 31.2% said that they had never studied statistics or probability. In another question, when asked which terms they knew and were able to recall, the most familiar terms were: percentage; mean; probability; frequency and sample (Figure 4).



**Figure 4: Percentage of statistical terms that students knew and could interpret**

These responses relate to the students' educational backgrounds at school, as the questionnaire was conducted during first two weeks of the semester. As previously stated, of the content covered in the statistics questionnaire, the students were not expected to be (formally) aware of confidence intervals.

The IRT results indicate a good fit to the Rasch model, because the infit/outfit rates of questions were within acceptable limits (between 0.5 and 1.5) and the correlation was higher than 0.2 (from 0.39 to 0.52). The difficulty of the questions ranged from -0.93 (Q.4) to 0.65 (Q.7). The skill level of the 1343 students involved in the present survey varied from -2.74 to 2.90 ( $M = 0.14$ ;  $DP = 0.78$ ), indicating that the difficulty level of the questionnaire ( $M = 0.00$ ;  $DP = 0.62$ ) was slightly below the skill level of the participants.

The two questions with a greater level of difficulty were related to variation: Q.6 (with a difficulty index of 0.58) and Q.7 (with a difficulty index of 0.65), despite the fact that 21.4% and 41.1% of the students confirmed that they knew and were able to interpret the terms "variance" and "standard deviation" respectively (Figure 4). In this case, what probably happened is that students know how to compute a standard deviation, but the interpretation of a standard deviation associated with reading a graph is not taught in basic education. Silva and Coutinho (2008), in their study, noted that even though many mathematics teachers reported having an understanding of standard deviation, the majority only presented a informal level of variation reasoning, and this affected the correct interpretation of this measure.

	Item	Logit	% response		Item	Logit	% response
Increasing difficulty ↓	1.0	-1.14	7	Increasing difficulty ↑	1.3	1.52	4
	2.0	-0.89	16		5.2	0.71	27
	4.0	-0.88	13		6.2	0.71	25
	3.0	-0.61	27		7.2	0.69	20
	5.0	-0.32	45		3.2	0.49	47
	7.0	-0.24	45		4.2	0.44	61
	6.0	-0.22	49		2.2	0.38	67
	4.1	-0.22	26		1.2	0.31	37
	2.1	-0.05	17		7.1	0.20	35
	1.1	0.02	52		5.1	0.19	28
	6.1	0.14	26		3.1	0.15	26

**Table 2: Order of item difficulty (according to logit) and % of response for each item in the questions**

According to the analysis of item difficulty (Table 2), we noted that, among the categories with a SOLO unistructural level, item 4.1 (question 4, category 1) presented the least difficulty (without considering the answers in the prestructural level – code 0), despite the fact that the response was associated only with reading a percentage value in a simple table, without considering the margin of error presented in the question (Table 1). In addition, 61% of the responses (item 4.2) were classified

at the highest level expected for this question. This result makes sense, as 88.5% of the students stated that they knew about and were able to interpret the term "percentage" (Figure 4).

The most difficult item (logit 1.52) with the lowest percentage of answers (4.0%) was the first question in category 3 (item 1.3), even though 80.6% of the students stated that they knew and were able to interpret the term "mean" (Figure 4). A probable explanation is that measures of central tendency are taught in basic education without the presence of outliers, which requires an extended grasp of the subject by the student. This difficulty was also identified by Garfield (2003) and Watson and Callingham (2003).

With regard to the three questions about probability (Q.2, Q.3 and Q.5), the students had difficulty with the items corresponding to categories 1 and 2, varying from -0.05 to 0.71. The highest proportion of answers in category 2 was 67.0% for the second question (Table 2). The results of Q.2 were as expected, as 64.1% of the students said that they knew about and were able to interpret the term "probability", but not conditional probability, which was the concept which was required in order to discern the correct reply to the fifth question (Figure 4). Students' difficulty with questions involving the concept of conditional probability has already been identified by several researchers, such as Estrada and Díaz (2006), Díaz and de la Fuente (2007) and Díaz (2010).

## **FINAL CONSIDERATIONS**

The IRT analysis helped to improve the interpretation of these results, taking into account that this method not only considers the students' total score, but also the questionnaire response vectors that can vary for the same total score. In addition, it is important to consider that in accordance with IRT, the items' level of difficulty and the skill level of the students were measured on the same scale, thus allowing a more detailed analysis of the results.

In a global analysis, the results seem to indicate that Brazilian students enter university with a basic knowledge of statistics and probability. However, the questions that involved the calculation of measures of central tendency with an outlier and variability associated with reading a graph had apparently not been explored in basic education, and for this reason they presented the highest degree of difficulty. According to this point of view, it is probable that the principles of statistical literacy have not yet been fully incorporated into the teaching and learning of statistics at this educational level. The results of the present survey are an indication that a change is needed in Brazilian schools in order to put into practice the PCN recommendations as well as the statistical literacy model suggested by Gal (2002).

The next phase of the present research will be to investigate the results of a post-questionnaire that was applied to the same students after their experience with the

statistics course at undergraduate level, thus checking whether or not these difficulties persist after a formal university education in statistics.

## REFERENCES

- Bardin, L. (1995). *Análise de conteúdo (Content analysis)*. Lisboa: Edições 70.
- Biggs, J. B., & Collis, K.F. (1991). Multimodal learning and the quality of intelligent behavior. In H. A. H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement*. New Jersey: Lawrence Erlbaum Associates, 57-76 p.
- Brazil, Ministério da Educação. Secretaria de Educação Fundamental. (1997). *Parâmetros Curriculares Nacionais: Matemática (National Curricular Parameters: Mathematics)*. Brasília: Ministério da Educação/Secretaria de Educação Fundamental.
- Brazil, Ministério da Educação. Secretaria de Educação Fundamental. (1998). *Parâmetros Curriculares Nacionais: Matemática (National Curricular Parameters: Mathematics)*. Brasília: Ministério da Educação/Secretaria de Educação Fundamental.
- Brazil, Ministério da Educação. Secretaria de Educação Média e Tecnológica. (2002). PCN Ensino médio: *Orientações educacionais complementares aos parâmetros curriculares nacionais – Ciências da natureza, matemática e suas tecnologias (PCN secondary education: Complementary educational guidelines to the national curricular parameters – Natural sciences mathematics and technology)*. Brasília: Ministério da Educação.
- Brazil, Ministério da Educação. Secretaria de Educação Média e Tecnológica. (2006). *Orientações curriculares nacionais para o ensino médio – Ciências da natureza, matemática e suas tecnologias (National curricular guidelines for secondary school – Natural sciences mathematics and technology)*. Brasília: Ministério da Educação.
- Díaz, C., & de la Fuente, I. (2007). Dificultades en la resolución de problemas que involucran el Teorema de Bayes. Un estudio exploratorio en estudiantes de psicología. *Educación Matemática*, 18(2), 75-94.
- Díaz, C. (2010). Psychology students' understanding of elementary bayesian inference. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics* (pp: 1-6). Auckland, New Zealand: International Association for Statistical Education.
- Estrada, A., & Díaz, C. (2006). Computing probabilities from two way tables. An exploratory study with future teachers. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics* (pp: 1-4). Auckland, New Zealand: International Association for Statistical Education.



## Working Group 5

- Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistical Review*, 70(1), 1-25.
- Garfield, J. (2003). Assessing statistical reasoning. *Statistics Education Research Journal*, 2, 22-38.
- Kataoka, V. Y., Souza, A. A., Oliveira, A. C. S., Fernandes, F. M. O., Paranaíba, P. F., & Oliveira, M. S. (2008). Probability teaching in Brazilian basic education: Evaluation and intervention. Paper presented at the *11th International Congress on Mathematical Education*, Monterrey, México.
- Masters, G. N. (1982). Rasch model for partial credit scoring. *Psychometrika*, 47, 149-174.
- Rasch, G. (1980). *Probabilistic models for some intelligence and attainment tests*. Chicago: University of Chicago Press.
- Silva, C. B. da, & Coutinho, C. Q. S. (2008). Reasoning about variation of a univariate distribution: A study with secondary mathematics teachers. In C. Batanero, G. Burril, C. Reading & A. Rossman (Eds.), *Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference* (pp: 1-6). Auckland, New Zealand.
- Wallman, K. K. (1993). Enhancing statistical literacy: Enriching our society. *Journal of the American Statistical Association*, 88, 1-8.
- Watson, J. M. (2003). Statistical literacy at the school level: What should students know and do? In *Proceedings of the 54<sup>th</sup> Bulletin of the International Statistical Institute (ISI)* (pp. 1-4). Voorburg, Netherlands: ISI.
- Watson, J. M., & Callingham, R. A. (2003) Statistical literacy: A complex hierarchical construct. *Statistical Education Research Journal*, 2(2), 3-46.
- Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67(3), 223-265.

# **ATTITUDES OF TEACHERS TOWARDS STATISTICS: A PRELIMINARY STUDY WITH PORTUGUESE TEACHERS**

José Alexandre Martins<sup>1</sup>, Maria Manuel Nascimento<sup>2</sup> and Assumpta Estrada<sup>3</sup>

Instituto Politécnico da Guarda (IPG) and UDI-IPG, Portugal<sup>1</sup>, Universidade de Trás-os-Montes e Alto Douro (UTAD) and CM-UTAD, Portugal<sup>2</sup>, Universitat de Lleida, Spain<sup>3</sup>

*In this paper, we will discuss teachers' attitudes towards statistics, as we believe that these attitudes have a key role in the teaching and learning process. We will examine the attitude concept, review the literature regarding scales for the measurement of attitudes towards mathematics and statistics, and present and analyse some of the results of a questionnaire/scale applied to a sample of Portuguese teachers.*

## **INTRODUCTION**

This work is part of a preliminary study of the attitudes of teachers from the second (ages 10 and 11) and third (ages 12 to 14) cycles of basic education in Portugal towards statistics. It could be argued that without the full commitment of teachers to the teaching and learning process, it would be very difficult to implement any significant changes in the ways in which statistics is taught. Nóvoa (1992, 30) stated that “the schools can not change without the participation of the teachers”. In addition, Gal and Ginsburg (1994, 1) wrote:

While statistics educators have focused on improving the cognitive side of instruction, i.e., the skills and knowledge that students are expected to develop, little regard has been given to non-cognitive issues such as students' feelings, attitudes, beliefs, interests, expectations, and motivations. (...) We believe that further attention to such factors is warranted, as they may contribute to students' difficulties in learning basic concepts in statistics and probability.

Therefore, it is necessary for this approach to include aspects such as attitudes, beliefs, interests, expectations and motivations which are associated with statistics, because teachers must go beyond the transmission of knowledge, positive attitudes, both in terms of the discipline and of its contents, and in relation to students, schools and education in general.

In this work, we will provide a brief summary of the current understanding of attitudes towards statistics and the Scale of Attitudes Towards Statistics (Escala de Actitudes hacia la Estadística de Estrada EAEE) (Estrada, 2002). We will also begin to analyse the results of a survey of in-service Portuguese teachers in the second and third cycles of basic education. The main purpose of this survey is to analyse Portuguese teachers' attitudes towards statistics.

## ATTITUDES TOWARDS STATISTICS

Theoretical and empirical issues related to attitudes have received a great deal of attention over recent years, and different perspectives regarding what attitudes are have emerged. In conceptualising the affective domain in relation to mathematics education, McLeod (1992) distinguished between emotions, attitudes and beliefs and conceptualised attitudes as learned predispositions to respond positively or negatively to given objects, situations, concepts or people. As such, according to Aiken (1980, p. 2), the attitudes comprise three dimensions: cognitive (beliefs, knowledge), affective (emotional, motivational), and performance (behaviour, active tendencies). More recently, Phillip (2007) described attitudes as ways of acting, feeling or thinking, which (in this case) indicate the disposition or opinions of a person with regard to statistics.

Attitudes are intense feelings which are relatively stable, and which result from positive or negative experiences encountered whilst learning a subject (in this case statistics) over a period of time. Students may have these experiences of statistics at school, or as part of an informal extra-curricular learning process. In other cases, students transfer their negative feelings towards mathematics onto statistics (Gal & Ginsburg, 1994).

Within the field of statistics education, the level of interest in the beliefs, attitudes, and expectations that students bring to the classroom is increasing, as “such factors can impede learning of statistics, or hinder the extent to which students will develop useful statistical intuitions and apply what they have learned outside the classroom” (Gal & Ginsburg, 1994, p.1). These authors have suggested that statistics teachers should not focus solely on the transmission of knowledge and skills, because if students encounter difficulties whilst learning about statistics, this experience might impede their desire to receive further instruction. They may also fail to appreciate the potential usefulness of statistics in their professional and personal lives. These theories are particularly relevant in terms of the training which teachers receive.

In the last two decades, a large number of tools with which to measure attitudes and anxiety levels relating to statistics have been developed (Gal & Ginsburg, 1994; Carmona, 2004). Three of the most widely used instruments with which to measure attitudes towards statistics are Wise’s (1985) Attitudes Towards Statistics Scale (ATS), Roberts and Saxe’s (1982) Statistics Attitude Survey (SAS) and Schau, Stevens, Dauphine and del Vecchio’s (1995) Survey of Attitudes Towards Statistics (SATS).

These scales have been validated using samples of students at college or university, but not among teachers or future teachers. For this purpose, Estrada (2002) proposed and developed a Scale of Attitudes Towards Statistics (EAEE), which was applied to prospective and in-service teachers.

The EAEE is a combination of three scales: the SAS (Roberts & Bilderback, 1980) and the ATS (Wise, 1985), which are the most commonly used in an international context, and the Spanish scale proposed by Auzmendi (1992).

Based on these three scales, a list of 36 items was developed, consisting of both positive (affirmative) and negative items in order to avoid the problem of acquiescence (Morales, 1988), and including various pedagogical and anthropological components, as described by Estrada (2002) and Estrada et al. (2004). These items were submitted to a panel of expert judges, and, following their evaluation, a final scale consisting of 25 items (14 affirmative and 11 negative) was proposed. The distribution of items according to the various components is provided in Table 1.

Teaching component	Anthropological component		
	Social	Educational	Instrumental
Affective	1, 11, 25	7, 12, 23	10, 13, 16, 20
Cognitive	2, 19, 21	4, 6, 17	3, 24
Behavioural	9, 18	8, 15, 22	5, 14

**Table 1: Components of attitudes as assessed in the EAEE**

In this study, attitudes were measured using a 25-item Likert scale. The independent variables we considered were: gender, group (prospective vs. in-service teachers), the number of previous courses taken in statistics, specialty (the subject in which the prospective teachers were specialising or the subject which the teachers taught) and the number of years of teaching experience in mathematics (for in-service teachers).

## METHOD, RESULTS AND DISCUSSION

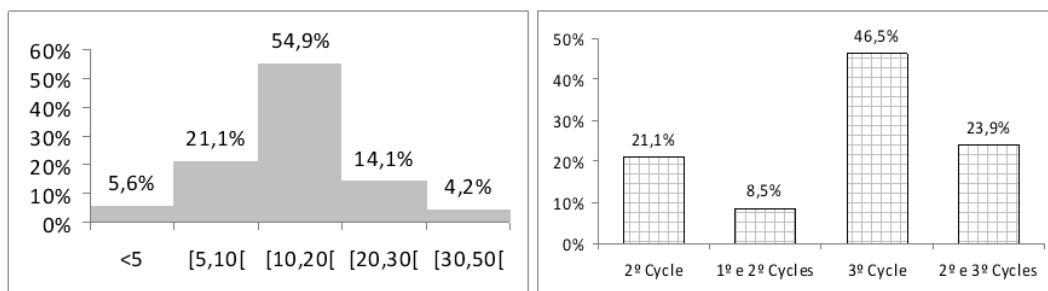
In this study, we analysed the results of a survey of 71 teachers in the second and third cycles of basic education in Portugal. The survey was distributed in the teachers' annual meeting (ProfMat 2008, 2-4 September) and in some of those cycles schools in Seia, S. Romão, Oliveira do Hospital and Lisbon. This allowed us to obtain a sample of the 212 teachers who were involved in these teaching cycles in Portugal at the time. The EAEE (Estrada, 2002) was translated and adapted to the Portuguese language by the authors, and two items were added. These two new items were related to the use of technology in statistics classes. In Table 1, these items were included in the section in which the behavioural components of teaching overlap with educational and anthropological components (items 26: "In statistics classes, I need to use the computer and the Internet in order to perform the tasks that I have set", and 27: "I do not need to use a calculator in order to perform the tasks that I set in statistics classes"). All of the items comprised statements, to which the

## Working Group 5

respondents marked their level of agreement or disagreement on a five-point Likert type scale (from 1: strongly disagree, through 3: neither agree nor disagree, to 5: strongly agree). Of the 27 items, 15 were positively worded (for instance, question 26) and 12 were negatively worded (for instance, question 27). For the 12 negatively worded items, the scale was reversed when the responses were analysed (from 1: strongly agree, through 3: neither agree nor disagree to 5: strongly disagree), meaning that the teachers' attitudes towards statistics could be measured in terms of the total score for all of their answers. The minimum score was 27, and the maximum was 135, with a mid-point of 81. In the data analysis, statistical software and a spreadsheet were used.

Of the 71 respondents, 61 (86%) were women. With regard to the teachers' years of experience, we found a mean of approximately 15 years, with a standard deviation (SD) of 7.8 years. The distribution is shown in Figure 1 (left), in which we can see the number of new teachers and those with far more experience.

The cycles in which the teachers were working are also shown in Figure 1 (right). Of the 71 respondents, only 51 indicated the subject in which they had graduated (we think that the 20 missing answers may have been a result of the presentation of the question). Of these respondents, around 78% were mathematics graduates (in teaching or other areas) and around 6% were economics graduates. These 51 respondents had graduated from a total of 13 different subject areas.



**Figure 1: Years of experience of teaching (left) and teaching cycles (right)**

Table 2 provides an analysis of each item. The mean value and SD were computed for a positive scale, meaning that all of the scores are comparable. The nearer to five they are, the stronger the indication of a positive attitude towards statistics. In contrast, if the scores are closer to one, this indicates a negative attitude towards statistics.

	Item	Mean	SD
1	Some statistical information transmitted in television programmes bothers me	2.7	1.2
2	Statistics helps me to understand today's world	4.2	0.9
3	Through statistics, one can manipulate reality	1.6	0.7

### Working Group 5

4	Statistics is fundamental to the basic training of future citizens	4.3	0.6
5	I solve day-to-day problems using statistics	3.6	0.7
6	We should not teach statistics in schools	4.9	0.3
7	I have fun in classes in which I teach statistics	3.8	0.7
8	I find that statistical problems are easy	3.8	0.9
9	I do not understand the statistical information that appears in the media	4.1	1.0
10	I like statistics, because it helps me to fully understand the complexity of certain issues	3.7	0.9
11	I feel intimidated by statistical data	4.4	0.8
12	I find the world of statistics interesting	4.0	0.7
13	I like serious work which involves statistical analysis	3.9	0.8
14	I do not use statistics outside of school	3.4	1.0
15	When I attended statistics classes, I did not fully understand what was said	3.7	1.1
16	I am passionate about statistics because it helps me to view problems objectively	3.3	0.8
17	Statistics is easy	3.5	1.0
18	I find it easier to understand the results of elections when they are shown using graphics	3.7	1.0
19	Statistics are only good for people in scientific areas	4.7	0.5
20	I like to solve problems using statistics	3.8	0.7
21	Statistics is worthless	4.8	0.7
22	If I could eliminate part of the syllabus, it would be statistics	4.6	0.9
23	I usually explain statistics problems to my colleagues if they do not understand	2.7	0.9
24	Statistics helps people to make better decisions	3.8	0.7
25	When I read, I avoid statistical information	4.4	0.6
26	In statistics classes, I need to use the computer and the Internet in order to perform the tasks that I have set	3.2	1.1
27	I do not need to use a calculator in order to perform the tasks that I set in statistics classes	3.7	1.1

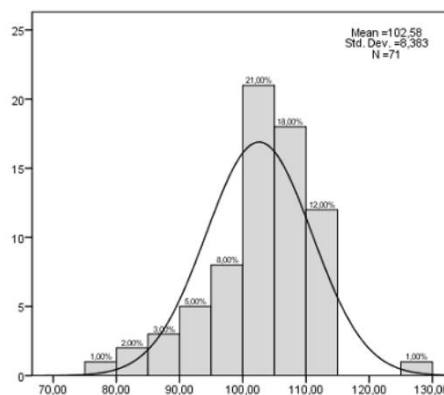
**Table 2: Mean and SD for each item**

The items with the highest mean scores ( $> 4.5$ ) were 6, 19, 21 and 22 (all of them are negatively worded, we note again). In addition to indicating positive attitudes towards statistics, these results reinforce our idea that these mathematics teachers seem to be aware of the importance of statistics as a part of the mathematics curriculum, as well as its importance for citizens in the modern world.

The items with the lower mean scores ( $< 3$ ) were 1, 3 (both of them are negatively worded, we note again) and 23. The result for item 23 seems to indicate a lack of collaboration between colleagues. With regard to items 3 and 1, we suspect that the respondents “wrongly” understood the meaning of the statements.

It is important to note that the two items which were related to the use of technology in statistics classes (26 and 27, this last one is negatively worded) had mean scores of 3.24 and 3.68 and SD values of 1.08 and 1.09 respectively. These results may indicate that there is still room for improvement, as teachers do not make use of technology in their classrooms.

Figure 2 indicates the frequencies of the total scores. From the graph, we can see that most of the respondents had a mean total score which was higher than 81 (the level of indifference). The modal score is 104, and the mean is 102.58, with a SD of 8.38 and a slight negative skewness (-0.53).

**Figure 2: The frequencies of the total scores**

Teaching component	Anthropological component			Total
	Social	Educational	Instrumental	
Affective	3.83	3.67	3.70	3.67
Cognitive	4.55	3.99	2.73	3.99
Behavioural	3.94	3.77	3.48	3.77
Total	4.13	3.80	3.40	3.80

**Table 3: Mean scores according to the components of attitudes**

Table 3 presents the respondents' mean scores according to the components of attitudes. According to a briefly analysis of these scores, none of the components – teaching or anthropological – presented high mean scores ( $> 4.5$ ) or low mean scores ( $< 3$ ). Nevertheless, a high mean score arises when the cognitive component of teaching is combined with the social anthropological component, and a low mean score can be obtained by crossing the cognitive component of teaching with the instrumental anthropological component.

We also studied the reliability of this survey; we obtained a Cronbach's alpha of 0.749 as a coefficient of internal consistency, and, taking into account the small number of respondents for this preliminary study, we believe this to be a reasonable value. Furthermore, in order to test the effect of each item on the internal consistency of the scale, we ran 27 more tests, using only 26 items at a time. For these 27 tests, the values for Cronbach's alpha varied from 0.725 to 0.768.

We analysed the influence of gender on attitudes towards statistics, and we did not find any statistically significant differences ( $F(1, 69) = 1.582$ ,  $p$ -value: 0.213). However, we found that the mean total score for women, which was 103.8 (SD 1.09), was higher than the mean total score for men, which was 99.5 (SD 2.31). This result is in accordance with the results of Estrada et al. (2004).

With regard to the effect of their years of experience on teachers' attitudes towards statistics, despite a generally positive attitude, no statistically significant differences were found ( $F(4, 66) = 0.367$ ,  $p$ -value: 0.831). Nevertheless, we found that the mean total score was higher and the SD was lower if the respondents had been teaching mathematics for fewer years, and vice versa. This result may be related to specific training in statistical and pedagogical approaches and teaching methods which the teachers received during their initial training.

Regarding the influence of the cycles of education on the teachers' attitudes towards statistics, we noticed that the mean total score was relatively high, but with no statistically significant differences ( $F(3, 67) = 0.963$ ,  $p$ -value: 0.415). Slightly higher scores were observed in the case of teachers who taught in the second and third cycles and the first and second cycles (the smaller group of respondents), who also had the highest SD values. Nevertheless, the question of whether or not a more positive attitude towards statistics may be associated with the cycles of basic education needs further research.

Regarding the different areas of training and their relationship with attitudes towards statistics, no statistically significant differences were found ( $F(1, 49) = 0.021$ ,  $p$ -value: 0.885). However, we noted that the respondents with basic training in mathematics had slightly lower mean scores (102.55 and SD 1.36), while those with training in other areas had a mean score of 103, but a greater SD (3.28). This result is interesting and somewhat unexpected; suggesting once again that further analysis in this area is required.



## **PRELIMINARY CONCLUSIONS**

The sample size and the characteristics of this survey mean that our results cannot be generalised. We must state that the purpose of this study was not to make any generalisations, but to test the validity of the items and the performance of the scale used in the similar study carried out by Estrada et al. (2004).

Distributing this survey through the ProfMat 2008 organising committee did not prove to be a good option, since there the teachers had several other requests (workshops, conferences, team groups, and so on) which may explain the low response rate and possibly even some of the “wrong” answers.

For this preliminary study, Cronbach’s alpha (the coefficient of internal consistency) was 0.749. In our view, this value is reasonable, given that the sample size was very small. For the purposes of the survey, this initial study suggests that teachers’ attitudes towards statistics are generally positive, with a mean total score of 102.58 (and SD 8.38), which is well above the mid-point (81), which would indicate their indifference. These results are in line with the results of Estrada et al. (2004). We did not find statistically significant differences in the respondents’ attitudes towards statistics in terms of some of the variables studied, but we should emphasise the following points: women had a slightly higher mean total score than men; the mean total scores were slightly lower in the groups of teachers with fewer years of teaching experience; the mean total scores were lower for the teachers who taught in only one of the cycles of basic education, and teachers with training in mathematics also had a higher mean total score than teachers with other forms of training. The mean scores of the two additional items were only slightly above three (indicating indifference), which leads us to infer that (in terms of these respondents at least) the use of technology in statistics classes needs further study.

These results must be read with great caution and with full knowledge of the (very) small size of the studied sample. Nevertheless, this preliminary study has provided us with some directions for further research and action; for instance, in terms of rewriting some of the items and the need to validate this Portuguese translation of the EAEE (Estrada, 2002).

Having followed on from the study by Estrada et al. (2004), we have not addressed all of the issues which were raised during the survey, but we intend to do so in further analysis. In order to continue this study of attitudes towards statistics, we feel the need to collect more answers using this survey in order to validate the scale when studying teachers in the second and third cycles of Portuguese basic education, and in order to improve the graphical presentation of the first section: the questions regarding sample characteristics of teachers. We also intend to conduct further statistical analysis, including multivariate analysis.

## REFERENCES

- Aiken, L.R. (1980). Attitudes measurement research. In D.A. Payne (Ed.), *Recent developments in affective measurement* (pp. 1-24). San Francisco: Jossey-Bass.
- Auzmendi, E. (1992). *Las actitudes hacia la matemática estadística en las enseñanzas medias y universitarias*. Bilbao: Mensajero.
- Carmona, J. (2004). Una revisión de las evidencias de fiabilidad y validez de los cuestionarios de actitudes y ansiedad hacia la estadística. *Statistics Education Research Journal*, 3(1), 5-28. Retrieved from [http://www.stat.auckland.ac.nz/~iase/serj/SERJ3\(1\)\\_marquez.pdf](http://www.stat.auckland.ac.nz/~iase/serj/SERJ3(1)_marquez.pdf)
- Estrada, A. (2002). *Análisis de las actitudes y conocimientos estadísticos elementales en la formación del profesorado*. Ph.D. Universidad Autónoma de Barcelona. Retrieved from <http://www.tesisenxarxa.net/TDX-0502103-191818/>
- Estrada, A., Batanero, C., & Fortuny, J.M. (2004). Un estudio comparado de las actitudes hacia la estadística en profesores en formación y en ejercicio. *Enseñanza de las Ciencias*, 22(2), 263-274.
- Gal, I., & Ginsburg, L. (1994). The role of beliefs and attitudes in learning statistics: towards an assessment framework. *Journal of Statistics Education*, 2(2). Retrieved from <http://www.amstat.org/publications/jse/v2n2/gal.html>
- McLeod, D.B. (1992). Research on affect in mathematics education: A reconceptualization. In D.A. Grows (Eds.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York, NY: Macmillan & NCTM.
- Morales, P. (1988). *Medición de actitudes en psicología y educación*. San Sebastián: Universidad de Comillas.
- Nóvoa, A. (1992). Formação de professores e profissão docente. In Nóvoa, A. (Ed.) *Os professores e sua formação* [Teachers and their training] (pp. 15-33). Lisbon: Publicações D. Quixote.
- Phillip, R.A. (2007). Mathematics teachers' beliefs and affects. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing & NCTM.
- Roberts, D.M., & Bilderback, E.W. (1980). Reliability and validity of a statistics attitude survey. *Educational and Psychological Measurement*, 40, 235-238.
- Schau, C., Stevens, J., Dauphine, T., & Del Vecchio, A. (1995). The development and validation of the survey of attitudes towards statistics. *Educational and Psychological Measurement*, 55(5), 868-875.
- Wise, S.L. (1985). The development and validation of a scale measuring attitudes toward statistics. *Educational and Psychological Measurement*, 45, 401-405.

# **DEVELOPING AN ONLINE COMMUNITY OF TEACHING PRACTITIONERS: A CASE STUDY**

Maria Meletiou-Mavrotheris, Efi Paparistodemou

European University Cyprus, Frederick University Cyprus

*The article presents an overview of the main experiences gained from a case study which investigated the forms of collaboration and shared knowledge building undertaken by a multinational group of teachers participating in online professional development. This study took place during the pilot delivery of Early Statistics, an online professional development course in statistics education targeting European elementary and middle school mathematics teachers. A central conviction underlying the course design is that learning is a social act best supported through collaborative activities, and thus learning as part of a community of practitioners can provide a useful model for online teacher professional development.*

## **INTRODUCTION**

Numerous initiatives in online teacher training serving large numbers of educators are underway (e.g. Garfield & Everson, 2009; Cady & Rearden, 2009). Several of these programs exploit the richness of interactions fostered by the Web to build and study network-based services with the aim of fostering online communities of teaching practitioners. Communities of practice is a construct grounded in an anthropological perspective that examines how adults learn through social practices (Gray, 2004). A community of practice consists of a group of individuals with a shared domain of expertise, who engage in a process of collective learning about practices that matter to them (Wenger, 1998). A promise of new web-based technologies is that they can enable geographically dispersed teachers to engage in online communities, in which they can exchange ideas with other teachers and garner support as they try new strategies in their classrooms (Cochran-Smith & Lytle, 1999).

This chapter focuses on the question how the information and communication tools made available by modern internet technologies could be effectively utilized in order to build and study network-based services with the aim of fostering online communities that promote statistics teachers' learning and development. It first provides an overview of the existing literature on online communities of practice. It then reports on some of the experiences from an exploratory study designed to investigate the forms of collaboration and shared knowledge building undertaken by a multinational group of teachers participating in online professional development in statistics education. The main insights gained from the study regarding enabling and constraining factors to the successful implementation of an online community of practice are discussed. Based on the analysis of these data, some recommendations for mathematics educators involved in pre-service and/or in-service teacher training

who wish to incorporate online communities of practice in their work are provided.

## **BACKGROUND**

Online communities of practice are constantly evolving into many forms and styles as they embrace new and evolving technologies. While, however, they proliferate in cyberspace, little is still known about best practices for their effective design, as empirical research on this topic is still at an initial stage. Conducted studies indicate that online communities of practice are, indeed, a promising model for both pre-service and in-service teacher training (Cady & Rearden, 2009; Dalgarno & Colgan, 2007). They have a great potential to support teacher professional development through placing educators at the center of their learning (Kayler & Weller, 2007), thus promoting their independence and self-directed learning. Online communities of practice facilitate not only communication, but also the collaborative finding, shaping, and sharing of knowledge among teachers.

Despite the potential of online communities of practice, several studies have found their introduction in educational contexts to be less successful than anticipated (e.g. Kennard, 2007). These studies highlight several difficulties in building and maintaining online communities involving shared professional learning. A study by ten Cate (2007), for example, has identified the following obstacles to participation in an online community of practice: limited time, limited access, limited opportunities to meet face-to-face, and language barriers. Language barriers are a particularly serious challenge for international communities of practice, where members come from different countries and time zones, and communicate with other teachers in a foreign language (Trayner, Smith, & Bettoni, 2007).

Timely postings by group members are considered to be a necessary component in building a functional community of practice (Kayler & Weller, 2007). However, studies of participation demographics in online communities and social networks have found that between 46 percent and 82 percent of users are invisible observers who rarely or never participate (Preece, 2000). There is strong research evidence indicating that many distance learners join discussion forums, read messages, but do not contribute to discussions (Simpson, 2002).

The design of cognitive tools to promote learner participation in online communities of practice involves many inter-related considerations (e.g. moderator involvement, reliability and stability of the technology, etc.), most of which are not yet well understood (Stahl, 2006). More research is still needed to shed light into how to best support the development of healthy and sustaining online communities of teaching practitioners. Below some experiences related to educating statistics teachers at a distance are analyzed.

## METHODOLOGY

### Context and Participants

The research comes from *EarlyStatistics*, a 3-year project (2005-2008) funded by the European Union under the Socrates-Comenius Action, which exploited the affordances offered by distance learning technologies to help improve the quality of statistics instruction offered in European schools. The project consortium, comprised of five universities in four countries (Cyprus, Spain, Greece, Norway), developed and pilot tested an intercultural online professional development course in statistics education. The course aims at helping European teachers improve their pedagogical and content knowledge of statistics through exposure to innovative web-based educational tools and resources, and cross-cultural exchange of experiences and ideas. A central conviction underlying the course design is that learning as part of an international community of practice can provide a useful model for teacher professional development (Wenger, 1998).

A case study design with mixed methods was employed in the study. The case studied consisted of the group of fourteen in-service teachers that participated in the pilot delivery of *EarlyStatistics*, which took place during the final year of the project (in spring 2008) in three of the partner countries – Cyprus, Spain and Greece. Participants voluntarily enrolled in the course. They did not gain any extrinsic rewards such as compensation, or academic credit incentives. A prerequisite for participation in the course was proficiency in English, since English was the language of instruction and of online communication.

Nine of the course participants were female and five male. Seven were aged between 31-40, while three were younger (21-30 years old) and four older (41-50 years old). The majority of the participants had been teaching for more than ten years. Since they originated from three different European countries, teachers were geographically, culturally, and linguistically heterogeneous. They came from different educational systems, and had varied educational backgrounds. They were either elementary or secondary school teachers (9 elementary school teachers, 5 secondary school level teachers), and differed considerably in their mathematical and statistical knowledge, and in their confidence and experience in teaching statistics. There was also variety in teachers' experience and comfort with internet technologies, and in their previous experience in taking online courses.

The *EarlyStatistics* course lasted 13 weeks, and was made up of six Modules. In Modules 1-3 (Weeks 1-6), the focus was on enriching the participants' statistical content and pedagogical knowledge by exposing them to similar kinds of learning situations, technologies, and curricula to those they should employ in their own classrooms. To help teachers go beyond procedural memorization and acquire a well-organized body of knowledge, the course emphasized and revisited a set of central statistical ideas. Through participation in authentic educational activities

## Working Group 5

such as projects, experiments, computer explorations with data, group work and discussions, participating teachers learned where the “big ideas” of statistics apply and how, and developed a variety of methodologies and resources for their effective instruction. In Modules 4-6, the focus shifted to classroom implementation issues. Teachers customized and expanded upon provided materials (Module 4; Weeks 7-9), and applied them in their own classrooms with the support of the design team (Module 5; Weeks 10-11). They wrote up their experiences, including a critical analysis of their work and that resulting from their pupils. Once the teaching experiment was completed, they reported on their experiences to the other teachers, and also provided video-taped teaching episodes and samples of their students’ work, for group reflection and evaluation (Module 6; Weeks 12-13).

The course was delivered through a blended learning approach. There were a few face-to-face meetings with local teachers, but the biggest part of the course was delivered online, by utilizing the project information base for teaching, support and coordination purposes. To offer teachers flexibility and to accommodate different time zones, the largest portion of the course was conducted asynchronously through online discussion and e-mail groups. There was also some synchronous communication through use of technologies such as chat rooms, audio/video streaming, and videoconferencing.

Each module involved a range of activities, readings, and contributions to discussion forums, as well as completion of group and/or individual assignments. Both the dialogue and the assignments were structured so as to explicitly make ties among theory and practice. Reflective questions created situations for the participating teachers to critically examine the subject matter through additional personal research or reading of the course material, thus giving them the opportunity to make new connections between theory and their personal and professional experiences.

Concurring with Roseth, Garfield, and Ben-Zvi (2008), *EarlyStatistics* was built on the premise that statistics instruction ought to resemble statistical practice, which is an inherently cooperative enterprise. A number of strategies were employed by the project consortium to promote online dialogue and transnational collaboration among participating teachers, and to ensure that all teachers actively contribute to course activities and discussions, including the following: (i) Monitored discussion forums allowed teachers to discuss content and help each other; (ii) Discussion questions were assigned bi-weekly. These were conceptual questions keyed to a major theme, and addressing content as well as pedagogical concerns; (iii) Participation in discussion forums and other collaborative activities was a compulsory element of the course; (iv) Participants were assigned to small groups, and each group was facilitated by a tutor. Groups received periodic milestone group assignments which they jointly completed using social software tools such as wikis.

Members of the *EarlyStatistics* consortium with expertise in statistics education facilitated the course. Since the course was designed to be a community-based

learning experience, their role was to guide discussions, encourage full, thoughtful involvement of all participants, and provide feedback. Facilitators helped to deepen the learning experience for course participants by encouraging productive interaction and critical reflection on workplace practices.

### **Instruments, Data Collection and Analysis Procedures:**

Documenting online interactions is a multifaceted phenomenon that requires complementary methods of data collection and analysis in order to understand how learning is accomplished through interaction, how learners engage in knowledge building, and how designed media support this accomplishment. Consequently, to increase understanding of the research setting, the current study employed a variety of both qualitative and quantitative data collection techniques, including: (i) The contents of the online discussion boards in which teachers participated during the course; (ii) Group assignments completed by teachers throughout the course; (iii) Quantitative statistics automatically collected by the system (e.g. number of teachers participating in a discussion forum, or successfully completing group assignments, number of postings by each participant, etc.); (iv) An open-ended web-based survey administered at the course completion, aimed at determining teachers' perceptions, opinions, feelings and motives regarding their participation in collaborative course activities and the impact these might have had on their professional development; and (v) Semi-structured interviews of a selected group of teachers that surveyed their views on the effectiveness of the online communication during the course.

## **RESULTS**

The overall feedback from the target user groups from all partner countries participating in the *EarlyStatistics* course pilot delivery, as well as from external experts regarding the course content, services, and didactical approaches was generally very positive. Key conclusions from the analysis of the user feedback were that *EarlyStatistics* was quite successful in helping teachers to improve their pedagogical and content knowledge of statistics by offering interactive, technology-rich instructional materials and services that enhance the teaching and learning process, and by providing course participants the opportunity to collaborate with other teachers and begin the construction of a community of practice. Moreover, data obtained from the teaching experimentations in the course participants' classrooms suggest positive gains in student learning outcomes and attitudes towards statistics (for more details see Chadjipadelis & Andreadis, 2008).

In the survey administered at the completion of the pilot course delivery and the follow-up interviews, teachers were asked to indicate "*what they liked the most about the EarlyStatistics course*". The promotion of communication and collaboration among teachers was an aspect of the *EarlyStatistics* course that was considered by all course participants to be an important strength of the program: "*I*

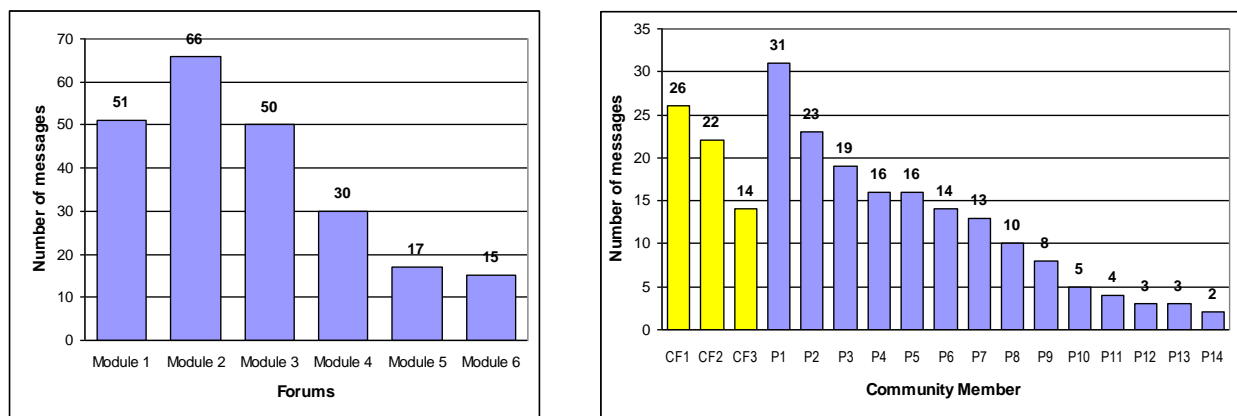
*liked the interaction with the other teachers. It is useful to share your ideas and problems with other teachers from different educational levels”; “It was very useful to be able to communicate with teachers of different levels and perspectives. This direct communication with everyone has helped to continue the hard work of self-learning”.*

In particular, teachers praised the fact that *EarlyStatistics* had allowed them, through computer-mediated communication, to share content, ideas, and instructional strategies with teachers from different countries and educational systems:

*The international cooperation among teachers having the same agony on how to teach their students better statistics; the fact that I also learned a lot of things about statistical syllabus, pedagogical aspects, and how education is implemented in other countries.*

*Distance training has helped me to understand that the problems that I have when teaching statistics are also common in other European Countries.*

*It is good to ‘hear’ colleagues from other countries that face similar problems like you and sometimes, because of a different view on a point, suggest ideas you didn’t think of.*



**Figure 1: Distribution of messages in the *EarlyStatistics* Discussion Forums per Module (Figure 1-left), and community member (Figure 1-right)**

Recently, *EarlyStatistics* won, ex-aequo with *Maths4Stats* (a joint project coordinated by Statistics South Africa), the 2009 Best Cooperative Project Award in Statistical Literacy. This award is given every two years by the International Association of Statistics Education (IASE) “*in recognition of outstanding, innovative, and influential statistical literacy projects that affect a broad segment of the general public*”.

Despite the overall success of the *EarlyStatistics* project and the very positive feedback from the groups of teachers participating in the pilot delivery and from external experts in statistics education, a number of shortcomings have also been identified. The biggest difficulty experienced by the consortium was its limited success in establishing a functional online community of practice, which was a main objective of the project. Despite the fact that the course team employed several



strategies to promote teacher dialogue and collaboration, and that the course facilitators tried their best to ensure that all teachers actively contributed to discussion forums, there was often a lower than anticipated learner-to-learner interaction.

Figure 1-left shows the distribution of messages in *EarlyStatistics* per forum. We can see that while at the beginning of the course there was big enthusiasm and very high participation in the discussion forums, interaction dropped off over time. In contrast to the vibrant interaction and rich dialog characterizing the earlier part of the course, often what happened towards the end of the course was that only 3-4 teachers would actively participate in the discussion forums, while the rest would make minimal or no contributions.

Figure 1-right shows the distribution of messages written in the forums per community member (P1-P14 stand for the course participants, and CF1-CF3 for the course facilitators). As we can see, there was a huge variation in the degree of participation among community members. There were a few teachers whose level of engagement was very high. At the same time, several other teachers participated only sporadically in the forums and, as a result, wrote a very small number of messages. These teachers exhibited a silent manner of participation. Checking their records of participation, we discovered that despite them not being active in the discussion forums, they continued to join the forums and read the messages posted by other members. Based on the analysis of the data collected during the study, we have identified a number of factors that adversely affected online participation of course participants:

**Course Workload:** A reason that might have contributed to our limited success in building an online community of practice is the pilot course workload, which proved to be overwhelming. When asked, in the end-of course survey and in the follow-up interviews, to indicate “what they liked the least about the *EarlyStatistics* course”, most participants mentioned the course workload which made it very difficult for them to keep up with the course requirements due to their overburdened schedules: *“The papers we had to read in the first modules were too many and our time to work on them limited, because of our jobs”*; *“There was too much studying involved.”*

**Duration of Discussion Forums:** The short duration of the discussion periods was another aspect of the course criticized by participants. Several participants noted that the study pace had been too high for them, and that the time allocated to each discussion forum (two weeks in average) was not adequate: *“The course had too many assignments on the theoretical part, and there was not enough time for working on them and posting on the forums.”*

**Lack of physical proximity:** During the pilot delivery there were a few face-to-face meetings with local teachers, but not with the group as a whole. Course

participants got the chance to virtually meet teachers from the other countries through video-conferencing, this however cannot be as effective as face-to-face interaction. Moreover, planning of videoconferencing and other activities that required synchronous communication (e.g. chat sessions) proved very difficult to schedule, as it was almost impossible for all of the teachers to be available at the same time. As a result, teachers built strong local groups but had more limited than desired interaction with teachers from other countries.

**Language barriers:** Researchers that have studied the dynamics of international online communities of practice (e.g. Trayner, Smith, & Bettoni, 2007) have found communication among people who speak different first languages to be a very serious challenge for such communities. In this study, language of communication also proved to be an obstacle to participation for some of the course participants. While most of the teachers did not seem to have any problems reading and writing in English, for a few of them language was a barrier that prevented them from fully participating in online discussions: *“It was a bit difficult and time consuming for us to read bibliography in English and to post our thoughts in the forum.”* Teachers with language difficulties did not post as often as their peers who had better English writing skills, and their contributions tended to be shorter.

**Limited experience of course facilitators in online instruction:** The important role of facilitators and moderators is a main theme emerging from research studies examining online communities of practice. In *EarlyStatistics*, the limited experience of the course facilitators in distance learning was a drawback of the pilot course. The team members that facilitated *EarlyStatistics* are very experienced statistics educators who have been involved for several years in teacher training. Nonetheless, this was the first time they were offering professional development online. Consortium partners with extensive previous expertise in distance education acted as mentors and provided hands-on training on a number of topics relating to distance learning. This undoubtedly helped course facilitators to improve their instructional skills in distance education. However, they still faced some difficulties in running the pilot course, and particularly in leading the discussion forums. While in guiding discussions, they tried to encourage full, thoughtful involvement of all participants, and to provide constructive feedback, they did not always manage to achieve productive interaction and critical reflection of the participants.

Insights gained from the *EarlyStatistics* pilot delivery have informed the revision of the course to better support online community building. The heavy course workload was corrected in the revised version of the course, and the length of time allocated to each discussion forum was increased from two to three weeks, in order to allow teachers more time for reflection and for online communication. Similarly, the technical difficulties experienced in the pilot delivery have been resolved. Additionally, the lack of a face-to-face meeting with participants from other countries experienced by the teachers participating in the pilot delivery, will not be

an issue in future offerings of the *EarlyStatistics* course, which will have a blended-learning format. At the beginning of the course, teachers from all over Europe will gather together to attend a one-week intensive seminar (they can finance their expenses by applying for a grant under the Lifelong Learning/Comenius In-Service Training Program). During this meeting, they will get familiarized with the course and its objectives, and with the facilities offered by the course e-Learning system. More importantly, they will get the chance to meet and interact with one another, and with the course facilitators. We believe that this initial in-person meeting will reinforce teacher online engagement by helping mitigate the problem of trust and social presence online (Ardichvili, Page, & Wentling, 2003). Finally, the limited experience of the *EarlyStatistics* course facilitators in online instruction will not be such a big issue in future offerings of the course. Undoubtedly, however, the valuable experiences they gained from this pilot delivery will allow them to employ much more effective moderating strategies in future offerings of the course.

## DISCUSSION

A common thread emerging from educational research is the direct relationship between improving the quality of teaching and improving students' learning. Thus, the provision of high-quality, ongoing professional development for teachers has become a paramount issue in school reform efforts. The need for the training of large numbers of teachers makes distance learning an attractive option. The traditional approach is to provide teacher training through a well-designed course package. The *EarlyStatistics* project has adopted a different approach, guided by contemporary visions of web-based instruction which support "learning" and "community" rather than "instructional" models of professional development.

Unlike traditional, individualistic approaches to teacher professional development, properly designed online communities of practice can foster a culture of sharing and sustained support for teachers. By allowing geographically dispersed teachers to interact, communities of practice can enable them to connect and learn from each other in ways not possible in more traditional, face-to-face professional development programs. However, despite the potential of online communities of practice, the existing research literature indicates that their introduction in educational contexts is often less successful than anticipated. Findings from the current study concur with the research literature, indicating that successful building of an online community of practice, particularly in a cross-national context, is very challenging.

While online teacher professional development courses share many features with face-to-face programs, the experience gained from the *EarlyStatistics* program, suggest that they also present some unique challenges. Teachers participating in such a course are likely to be characterized by diversity in a number of parameters (pedagogical and content knowledge of statistics and mathematics, educational level and grade they teach, cultural and/or professional backgrounds, comfort with

technology and with distance learning, etc.). Several pedagogical and technical issues should be taken into account in the design of an online professional development course, in order to provide an effective environment that motivates teachers and supports the development of a functional online community of practice:

- Access to multiple distance collaboration tools that promote interaction with peers and with course facilitators;
- Careful scheduling of course activities to offer teachers flexibility, and to accommodate different time zones;
- Adoption, whenever possible, of a blended learning approach, to allow teachers to personally meet and interact with each other.
- Setting of realistic work expectations so as not to overburden teachers;
- Provision of adequate time for teachers to formulate and articulate contributions to online discussions;
- Prompt and effective moderation of online interactions by course facilitators;
- Support, in the case of international online communities, of members experiencing difficulties in oral and/or written communication in English.

Research in the area of online communities of practice is still at an embryonic stage. More research is needed to advance our understanding of how to best take advantage of computer-mediated communication tools to support the development of effective virtual communities that can act as vehicles for teacher learning and growth. Despite the tentative and non-generalizable nature of the current case study's findings, it does contribute some useful insights into the factors that may facilitate or impede the successful implementation of an online community of teaching practitioners, suggesting possible methods for improving their implementation in distance education. These insights have helped to further enhance the quality and effectiveness of the EarlyStatistics course, and sketch a road map for our future research work and for other similar online community building endeavors.

## REFERENCES

- Ardichvili, A. Page, V., & Wentling, T. (2003). Motivation and Barriers to Participation in Virtual Knowledge-Sharing Communities of Practice. *Journal of Knowledge Management*, 7(1), 64–77.
- Cady, J., & Rearden, K. (2009). Delivering online professional development in mathematics to rural educators. *Journal of Technology and Teacher Education*, 17, 281-298.
- Chadjipadelis, T. & Andreadis, I. (2008). *EarlyStatistics Evaluation Report* (Internal document. Project: 226573-CP-1-2005).
- Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of Knowledge and Practice: Teacher Learning in Communities. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of*

## Working Group 5

*Research in Education* (pp. 249–305). Washington, DC: American Educational Research Association.

- Dalgarno, N., & Colgan, L. (2007). Supporting novice elementary mathematics teachers' induction in professional communities and providing innovative forms of pedagogical content knowledge development through information and communication technology. *Teaching and Teacher Education: An International Journal of Research and Studies*, 23, 1051-1065.
- Garfield, J., Everson, M. (2009). Preparing Teachers of Statistics: A Graduate Course for Future Teachers. *Journal of Statistics Education*, 17(2). Retrieved November 15, 2010, from <http://www.amstat.org/publications/jse/v17n2/garfield.html>.
- Gray, B. (2004). Informal Learning in an Online Community of Practice. *Journal of Distance Education*, 19(1), 20-35.
- Kayler, M., & Weller, K. (2007). Pedagogy, Self-Assessment, and Online Discussion Groups. *Educational Technology & Society*, 10(1), 136-147.
- Kennard, C. (2007). Wiki Productivity and Discussion Forum Activity in a Postgraduate Online Distance Learning Course. In C. Montgomery C., & J. Seale (Eds.), *Proceedings of World Conference on Educational Multimedia, Hypermedia and Telecommunications 2007* (pp. 3564-3569). Chesapeake, VA: AACE.
- Preece, J. (2000). *Online communities: Designing usability, supporting sociability*. Chichester, United Kingdom: John Wiley & Sons.
- Roseth, C.J., Garfield, J.B., & Ben-Zvi, D. (2008). Collaboration in Learning and Teaching Statistics, *Journal of Statistics Education*, 16(1). Retrieved September 5, 2010, from [www.amstat.org/publications/jse/v16n1/roseth.html](http://www.amstat.org/publications/jse/v16n1/roseth.html).
- Simpson, O. (2002). *Supporting Students in Online, Open and Distance Learning*. London and New York: Taylor and Francis Group.
- Stahl, G. (2006). *Group Cognition: Computer Support for Collaborative Knowledge Building*. Cambridge, MA: MIT Press.
- ten Cate G. (2007). *Supporting a Community of Trainers: Can a Community of Practice help address trainers' need for continuous, easily accessible, and context-appropriate support?* International Institute for Communication and Development, The Hague. Retrieved October 15, 2010, from [www.ciera.org/library/reports/inquiry-3/3-014/3-014.pdf](http://www.ciera.org/library/reports/inquiry-3/3-014/3-014.pdf).
- Trayner, B., Smith, J. D., & Bettoni, M. (2007). *Participation in International Virtual Learning Communities: A Social Learning Perspective*. Lecture Notes in Business Information Processing, 1(Part IV), 402-413.
- Wenger, E. (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge, UK: Cambridge University Press.

# **RISK TAKING AND PROBABILISTIC THINKING IN PRESCHOOLERS**

Zoi Nikiforidou, Jenny Pange

University of Ioannina

*Decision making under uncertainty involves the underlying concepts of risk and probabilities. By getting involved in risk taking experiences preschoolers develop stochastic thinking and skills that allow making risky or riskless choices depending on the outcome and related information. In the current study, children (N= 80), at the age of 4-6 participated in the 'Cups' task (Levin et al, 2003) through concrete manipulatives versus pc. Findings allow implications for the design of the probabilistic tasks in relation to the cognitive capacities of young children.*

## **INTRODUCTION**

Risk challenges and undergoes most human activities and decisions. Through risky contexts children and adults understand and confront the dangers and uncertainties of life at a personal and more general level (Slovic, 1999). There is no one definition of risk thus, according to Harding (1998) risk refers to “*a combination of probability or frequency of occurrence, of a defined hazard and the magnitude of the consequences of the occurrence*” (p.167). The terminology and methods used for dealing with risk and uncertainty vary a lot and make it difficult to communicate across the various areas of applications and disciplines (Aven, 2009).

Methodologically, risk-taking has been analysed not only through actual rates (i.e., with the use of interviews or questionnaires) but also through more artificial, laboratory based risk-taking tasks. The major goal in this later approach has been to provide analogues to the risks children and adolescents actually engage in, while controlling extraneous variables and assessing the effects of key variables (Boyer, 2006). One such key variable is the estimation and apprehension of probabilities and the likelihood of events.

Recent studies support that preschoolers may exploit and develop probabilistic thinking, as for instance they have been found to possess characteristics of probabilistic thinking (Jones et al, 1997; Nikiforidou & Pange 2010) and make use of probabilistic evidence in order to produce causal relations (Kushnir & Gopnik, 2005). The development of such skills in the preschool context, under appropriate pedagogical practices, is a matter in progress (Pange & Talbot, 2003) and strongly complements the field of decision making under risk and uncertainty by implying what is called ‘*risk literacy*’ (Gigerenzer et al, 2007).

Research in the area of risky decision making by young children usually consists of an adaptation of tasks that aim at older ages and have already been tested on adolescents and adults. The ‘Cups-task’ developed by Levin et al, (2003) is one example, among others, that supports that risk is age-related and that children get


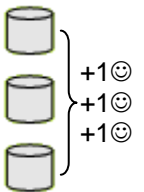
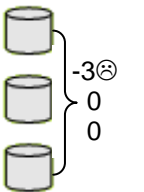

affected by the loss or gain domain and the expected value (Schottmann, 2001; Levin et al, 2007). Among these lines, young children tend to select more risky choices in order to avoid losses than to achieve gains. This constitutes the preference shift as proposed by the Prospect Theory (Kahneman and Tversky, 1979) and analysed mainly with older participants. The aim of the current study is to examine whether preschoolers may or may not participate in a risky decision-taking task (similar to the ‘Cups-task’) by assessing the likelihood of an event in terms of immediate gain and loss. Can they estimate the probability of winning and losing under risky and riskless options? Will they show differences in the gain and loss domains?

## MATERIALS AND METHODS

The study took place in two Greek kindergartens, during 2010. Children ( $N = 80$ ), aged 4-6, participated in mixed age-groups consisted of five members. The between-subject study was realised after parents gave their written consent and as soon as teachers expressed their willingness to collaborate.

The material used in the experiment included six small round metallic boxes and 12 cards of 4.5 x 5.5 cm each: six depicted happy faces and six sad faces. Children would record by themselves the outcome of their choice in specially designed sheets. There were two domains; the domain of gain and the domain of loss. In the gain domain children would win the equivalent number of stamps. On the contrary, in the loss domain they would cross out respective happy faces from a given bank of faces.

The experiment took place within the participants’ schools, thus in a separate room from their classroom in order to avoid disturbance. At a first point, children were seated around a table and were presented with the stimulus. After counting the boxes, the researcher would divide them in two sections composed by 3. Each section would be placed on each side of the table (right and left) by producing the risky and the riskless sides, depending on the number of cards that would be placed under each box. Therefore, children would observe the researcher separate the boxes in triads and place cards underneath them. In the riskless side, there would be one card per each box and in the risky side randomly 1 box would get 3 cards while the remaining two boxes would get no card at all (Table 1).

Material	Design of gain domain		Design of loss domain	
	risky	riskless	risky	riskless
concrete manipulative (metallic boxes and cards)				

**Table 1: Design of the risk-taking task**

## Working Group 5

As the researcher explained orally the whole procedure in both domains, children would follow the shuffling of the boxes per each side; the risky boxes within them and the riskless accordingly within them. Then, each child was asked to select one box and this implied a two-level decision.

Firstly, children were expected to choose between the risky and riskless option and afterwards they were expected to point at a particular box in order to find out whether and what they had won or lost. In the gain domain if children selected from the risky side they were prone to win 3 stamps by 33% chance, otherwise they would come up with none by 66%. If they chose a box from the riskless side they would gain one stamp for sure. In the loss domain, depending on their choice, children would either erase for sure one happy face from their bank of faces if they went for the riskless option or if they went for the risky option they would either erase all 3 happy faces by 33% or none by 66%.

As soon as one domain ended children went on with the other domain. The domains were counterbalanced and every time instructions were provided orally. Each trial was repeated 3 times and participants marked down their gains or losses. Children with more stamps or less crossed out faces were considered to be the winners, thus there was no final prize.

## RESULTS

Children preferred the risky option compared to the non-risky option by 74.3% in the gain domain and by 91.3% in the loss domain. By conducting a  $\chi^2$  chi-test it can be seen that in both domains there are statistically significant differences in children's responses as, in the gain domain,  $\chi^2(1) = 54.5$ ,  $p < 0.00$  and in the loss domain,  $\chi^2(1) = 157.9$ ,  $p < 0.000$ .

In the following table (Table 2) the frequencies are reported per domain and trial.

Responses				
		trial 1	trial 2	trial 3
Gain	Risky	57	54	60
	riskless	19	23	17
Loss	Risky	70	70	71
	riskless	7	7	6

**Table 2: Overall responses.**

A binomial test revealed that there is a significant preference for the risky option in every trial of both domains:  $z_{1g} = 4.07$ ,  $p < 0.05$ ,  $z_{2g} = 3.887$ ,  $p < 0.05$ ,  $z_{3g} = 4.21$ ,  $p < 0.05$ ,  $z_{1l} = 4.67$ ,  $p < 0.05$ ,  $z_{2l} = 4.67$ ,  $p < 0.05$ ,  $z_{3l} = 4.7$ ,  $p < 0.05$ .



The trend of selecting the riskless option instead of the risky was more evident in the gain domain. In the loss domain children were more likely to go for the risky boxes. In the gain domain the odds ratio in selecting the riskless/risky option in the 1<sup>st</sup> trial was 4.16, in the 2<sup>nd</sup> trial 7.3 and in the 3<sup>rd</sup> trial 7.4. In the loss domain, in the 1<sup>st</sup> trial children were more likely to select the riskless response by 1.77, in the 2<sup>nd</sup> by 2.16 and in the 3<sup>rd</sup> by 2.22.

## DISCUSSION

Children participated actively in this child-oriented risk-taking task. They were excited with this game and expressed patterns in their choices that comply with the preference shift phenomenon (Kahneman and Tversky, 1979). Under this perspective, choices involving gains are risk averse and choices involving losses are risk seeking. One of the basic features of decision under risk and uncertainty is that losses loom larger than gains. Risk seeking is prevalent when people must choose between a sure loss and a substantial probability of a larger loss and alternatively, people often prefer a small probability of winning a large prize over the expected value of that prospect.

Young children at the age of 4-6 would select the risky option so as to avoid losing happy faces rather than maximizing their stamps. The preference shift phenomenon related to this age group is in compliance with Levin's et al (2003) and Levin's et al (2007) studies. If children had no indications of probabilistic thinking then they wouldn't show such patterns of choice; they would either be risk-seeking in all trials or risk-averting. On the contrary, children seemed to estimate and try to avoid in the loss domain not only the -1 but also the -3, whereas in the gain condition they would be satisfied with a +1. Findings in this direction support that children possess a limited understanding of probabilistic notions (Nikiforidou & Pange, 2010).

Risk-taking tasks addressing to young children have been developed during the last years, usually as an adjustment of tasks designed for older participants. Within the laboratory-based tasks external variables are expected to be controlled and participants are encouraged to express their risky or riskless choices. Thus, the act of choosing, for instance one box, based on prior information is different from the reasoning and inferring that accompany this choice. In this direction further research should take in consideration children's justifications and explanations concerning these choices not only in experimental tasks but also in tasks that relate to real life situations and children's interests.

To sum up, matters of design and ways of measuring young children's decisions under risk and uncertainty are a field of great interest that has emerged during the last years. By taking into account that risk-taking is very complicated and multidimensional, research investigates aspects of children's cognitive and probabilistic thinking. How can children at the age of 4-6 get engaged at a first point and learn at a more long-termed level to manipulate decisions under risk that entail

probabilistic concepts? How can concrete stimuli provide incentives that allow young children to express stochastic ideas and take risky decisions and finally, how can they be perceived in a useful manner for individuals' personal and social life within the framework of 'risk literacy'?

## REFERENCES

- Aven, T. (2009). Perspectives on risk in a decision-making context - Review and discussion, *Safety Science*, 47 (6), 798 – 806.
- Boyer, T. (2006). The development of risk-taking: A multi-perspective review, *Developmental Review*, 26 (3), 291-345.
- Gigerenzer, G., Gaissmaier, W., Kurz-Milcke, E., Schwartz, L. M. & Woloshin, (2007). Helping doctors and patients make sense of health Statistics. *Psychological Science in the Public Interest*, 8 (2), 53-96.
- Harding, R. (1998) *Environmental Decision-making: the roles of scientists, engineers and the public*, The Federation Press, Sydney.
- Jones G., Langrall C. Thornton C. Mogill T. (1997). A framework for assessing and nurturing young children's thinking in probability, *Educational studies in Mathematics*, 32, 101–125.
- Kahneman, D. and Tversky, A. (1979). Prospect theory: an analysis of decision under risk, *Econometrica*, 47, 263–91.
- Kushnir, T., & Gopnik, A. (2005). Children infer causal strength from probabilities and interventions, *Psychological Science*, 16 (9), 678–683.
- Levin I., Hart S. (2003). Risk preferences in young children: early evidence of individual differences in reaction to potential gains and losses, *Journal of Behavioral Decision Making* 16, 397 – 413.
- Levin, I. P., Hart, S. S., Weller, J. A., & Harshman, L. A. (2007). Stability of choices in a risky decision making task: A 3-year longitudinal study with children and adults, *Journal of Behavioral Decision Making*, 20, 241–252.
- Nikiforidou Z. & Pange J. (2007). Sample space and the structure of probability combinations in preschoolers, *Proceedings of CERME*, 782-790. Larnaca.
- Nikiforidou, Z. and Pange, J. (2010). The notions of chance and probabilities in preschoolers. *Early Childhood Education Journal*, 305-311.
- Pange, J., & Talbot, M. (2003). Literature Survey and Children and their perception of Risk, *ZDM*, 35, 182-186.
- Slovic, P. (1999). Trust, Emotion, Sex, Politics, and Science: Surveying the Risk-Assessment Battlefield, *Risk Analysis*, 19(4): 689-701.

# INFLUENTIAL ASPECTS IN MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF STATISTICS VARIATION<sup>1</sup>

Antonio Orta, Ernesto Sánchez

Department of Maths Education, Cinvestav. México

*This study explores the way in which certain statistics notions influence the understanding of variation among junior high students. A teaching experiment with 50 13-year-old students was undertaken, in which the concept of statistical variation was studied in the context of temperature measurements. After the teaching, a problem of comparing wait-time for a movie of different cinema chains was administered to the students. Some of them learnt to identify which set of data have more variation by reading their frequency distributions; others also did it by calculating and comparing the ranges. However, students are unable to respond to a question related to choosing a cinema chain to go by considering the uncertainty that should have been seen through variation. This observation brings a reflection on the understanding of the relationship between variation and uncertainty.*

## INTRODUCTION

It is acknowledged among the statisticians and statistics educators that variation is the core of statistics. Moore (1990) emphasized the omnipresence of variation and the importance of modelling and measure variation in statistics; Wild and Pfannkuch (1999) proposed the perception of variation as one fundamental kind of statistical thinking and Watson, Kelly, Callingham, and Shaughnessy (2003: 1) pointed out that “statistics requires variation for its existence”.

Variation is a very complex concept, complex in such way that understanding it requires connecting it with many other notions. For example, Watson, Callingham and Kelly (2007) suggest that understanding variation involves perceiving uncertainty, anticipated change, unanticipated change and outliers. Konold and Pollatsek (2002) emphasized the importance of jointly considering variability (noise) and centre (signal), because both ideas are needed to find meaning when analyzing data. Garfield and Ben-Zvi (2008: 203) remarked that “understanding the ideas of spread or variability of data is a key component of understanding the concept of distribution, and is essential for making statistical inferences”. Wild (2006: 11) suggested that the notion of distribution “underlies virtually all statistical ways of reasoning about variation”.

The absence of the notion of variation in the curricula in mandatory education and the lack of research on students' understanding of it, in contrast to its importance in statistics, was pointed out by Shaughnessy (1997). Since then, an increasing number of studies on students' understanding of variation have been published. Recent research results indicate the possibility of developing intuitive notions of variation from earlier grades. In the present study the way in which the notions of mean, range

and uncertainty influence middle school students' understanding of variation is explored.

## CONCEPTUAL FRAMEWORK

The notion of *understanding* used in this paper is described by Hiebert and Carpenter (1992: 67); they point out that a concept “is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections”. This means that a certain degree of understanding of a mathematics or statistical concept involves recognizing at least some of its relevant relationships with other concepts and procedures.

Variation of data is connected with many concepts, but in comparing tasks three basic connections are the mean, the range and the graph of data. Two other concepts considered in this study are the mean deviation as a more precise measure of variation and the notion of uncertainty as a bridge to interpret variation in some contexts. All these concepts are related each other and they have connections with other more elemental notions.

Variation in data can be seen through the frequency distribution of the data. Line and bar graphs, and histograms are some of the ways of representing the frequency distribution of data. Since these representations are spread across the curriculum of middle school it becomes important to investigate how students learn variation through them.

As Konold and Pollatsek (2004) suggested it is convenient to consider signal (in our case, mean) connected with noise (variation) to have a concise but meaningful description of data. Since in calculating the mean deviation the mean itself is necessary as a reference point and, in comparing sets of data, it is convenient to consider their means, the knowledge of mean cannot be disregarded in achieving an understanding of variation in data.

Although range is the most elemental measure of variation and it is easily understood by students, range is insufficient to describe variation. Standard deviation is the most convenient measure, but this is a very difficult notion. An alternative could be the mean deviation since it can be more comprehensible to middle school students.

Watson et al. (2007) point out the relation between variation and uncertainty. In this sense, it is convenient to consider the idea of Tal (2001) about that variation can be information or uncertainty; it is information when it can be explained and uncertainty when it cannot. In contexts where variation is uncertainty, perceiving it as risk could lead to making good decisions. Kahneman and Tversky (2000) point out that people are risk-averse; this means they tend to choose situations where the risk is minimal. The problem however is whether students see properly uncertainty and risk in the situations where variation is uncertainty.

## METODOLOGY

*Participants.* The participants in this study were 50 students, one teacher and the authors of this paper. The students were distributed in two groups of the eighth grade (13 years old) from a public middle school in Mexico City. The teacher has 10 years experience and he recently received a master degree in a professional development program in maths education.

*Instruments.* In this study, three different instruments were designed and implemented in order to obtain data: a diagnostic test, a set of worksheets filled by the students during a teaching sequence and a post-test. This last one contained three tasks, one of which is about comparison of the waiting time in theatres of different chain cinemas (adapted from Shaughnessy, et al 2004); it is presented in Figure 1 below. Only the responses to this task are analyzed in this paper.

A recent trend in movie theatres is to show commercials along with previews before the movie begins. The *wait-time* for a movie is the difference between the advertised start time (like in the paper) and the actual start time for the movie. A class of 10 students investigated the wait-times at three popular movie theatre chains in Mexico: *Cinemex*, *Cinépolis* and *Multicinemas*. Each student attended three movies, a different movie in each theatre. The class's results are shown in the charts below. (Times were rounded to the nearest half-minute.)

Cinemex	Cinépolis	Multicinemas
12.0	15.0	15.5
21.0	15.5	17.0
15.0	16.0	18.0
15.0	16.0	16.5
13.0	16.5	16.0
16.0	16.5	16.5
16.0	18.0	16.5
16.0	16.0	15.0
20.0	15.5	15.0
18.0	17.0	16.0

- In the attached sheet, make the graph corresponding to each chart.
  - Calculate the average waiting time for each movie chain and draw a line along the mean in the graph that you made.
- [Incises c-h are presented below]

**Figure 1. Task on comparing wait-times in movie chains**

*Procedures.* A teaching experiment of eight sessions, of 100 minutes each, was designed with the purpose of providing conditions for the learning of the variance notions. The students were instructed in the handling and organization of data through the eight sessions, in each they filled out a worksheet. In sessions 1 to 7 the body temperature context was used. In session 1, the students responded a questionnaire about body temperature. Every student measured his or her

## Working Group 5

temperature with a mercury thermometer. Data were gathered in a list. The mode and mean were calculated. In session 2, students made graphics freely with data gathered in the previous session. In session 3, students were taught about the correct way to make frequency graphs; they were also given a squared sheet with axes on it for the purpose of more accurate graphics. Students were asked to locate the mean in the graph. In session 4, data belonging to the other group were considered. Students were taught how to calculate the range and how to associate it with the variability. They discussed the accuracy of measurement in each of the groups and their relation with the range. In session 5, new data were gathered, but this time, it was done with digital thermometers. The new data obtained were processed in the same way as previously. Students were asked to compare the data groups in terms of their accuracy and variability; to do this they used the range. In session 6, the concept of mean deviation was introduced as a more accurate measure of variance. The students calculated the mean deviation from the data obtained in previous sessions. In session 7, the accuracy of the temperature measurements obtained with the mercury thermometer were discussed and analyzed compared to those obtained with the digital one; to do this they used the range and the mean deviation of their data. Finally, in session 8, some problems with new data given were solved where range and mean deviation had to be calculated to obtain the solutions.

*Analysis procedure.* After the teaching, the post-test was administered to the students. The responses to the task on waiting time were analyzed by ranking them in different categories inspired by the SOLO model of Biggs and Collis (1991). These authors define a PUMR learning cycle as follows.

*Prestructural*, The task is engaged, but the learner is distracted or misled by an irrelevant aspect [to the task] [...]. *Unistructural*, The learner focuses on the relevant domain and picks up one aspect to work with. *Multistructural*, the learner picks up more and more relevant or correct features, but does not integrate them. *Relational*, The learner now integrates the parts with each other, so that the whole has a coherent structure and meaning (p. 65).

For the sake of analysis, the post-test activity was divided in four sub-tasks: Graph construction, mean and variation, reading variation in the graph, and interpretation of variation. The responses of the first three tasks were analyzed by defining levels analogous to PUMR levels.

## RESULTS

### Instructions observations

The students showed familiarization with the context of body temperature. They knew that temperature varies from person to person and that such variation is low, just around 36.5°C. Data were obtained and organized without difficulty and its mean and median were calculated.

However, students had some difficulties representing data in frequency graphics. Even when they chose the X axis to represent the temperature values, most of them

neither grouped data nor organized them according to the magnitude of values but rather by the order in which they were obtained. Some others grouped and arranged data in order, but did not place them according to their relative magnitude. In order to overcome these difficulties, a session to instruct students in constructing frequency graphs was implemented. With the support of the teacher's instructions, students had no difficulties in constructing the graphs of the data obtained in the previous sessions.

It must be highlighted that, when requiring one value of the temperature of the group to inform another person (the principal, for instance) about the measures taken, students agreed on proposing the mean. Some of them say mean because they see in this a precise value resulting from operating the data, and others for considering it a representative value.

When students were asked to compare the accuracy of measurements in the two groups of data, many of them proposed using the mean and/or the graph or comparing against clinical data; only three students mentioned the range as a measure for accuracy in data. When they were asked to compare the variability of the groups of data, they identified variability in the graphs, especially focused on outliers, and spontaneously looked for causes of variation: illness of some students, errors in the measurement process, and the quality of thermometers. After that, students were formally instructed in calculating the range and mean deviation of data and in using them to measure variation.

The instructions sessions provided the students with knowledge and language that permitted undertaking observations about their thinking about statistical notions related to variability. Although the performance in group tasks seemed high, many difficulties persisted in the individual performance on post-test tasks.

### **Graph construction**

One way to perceive variation in data is by looking at its frequency graph. To define the PUMR levels the following aspects were considered relevant to complete the task:

1. Select the X axis to distribute data on it in an ordered way
2. Calculate the frequencies of each value and represent them in Y-axis of the graph
3. Respect distances between values, this mean considering the X-axis as a numerical straight line and not only a place to put labels.

In the *prestructural level* none of these aspects were considered. In the *unistructural level* the responses are focused only on one of these aspects. In the *multistructural level* the responses are focused on two aspects (mostly 1 and 2), but failed in the third. Finally, in the *relational level* responses focused on the three aspects mentioned. The distribution of the responses is show in the table1.

SOLO Categories	Pre-structural	Uni-structural	Multi-structural	Relational	No answer
Number of students	30	5	7	3	5

Table 1

### Mean and variation

Based on the graphs they made, the students had to answer the following questions.

- c. Is there any difference in the average waiting time in the three movie chains? Explain your answer.
- d. How can you know and determine the variation in the waiting time from the three movie chains?
- e. Which of the three movie chains has more variation?

The responses to these questions are jointly considered in order to catch the students' understanding about mean and variance. First, they had to verify that the waiting-time mean is the same in the three Cinemas; second, they should know that can use the range or the mean deviation to determine the variation and, finally, they should apply this criterion to say that Cinemex has more variation than the others.

In an analogous way to PUMR levels, the responses can be classified in different levels taking into account some statistical clue aspects, namely the mean, the range or the mean deviation, as well as the identification of the cinema with greater variation.

In *level 0* (analogous to prestructural), the responses are not based on any statistical clue aspects nor indicate the cinema with greater variation; for example, the answers of Anahí to the three questions are as follows, respectively.

- Anahí:
- c) Both are too different from the last one.
  - d) By going to the movies just like the other teenagers but to three movie theatres.
  - e) There is more variation in the third one than in the second one.

In a *level 1*, the responses show evidence that only one of these clue aspects was considered, without indentifying the cinema with greater variation; for example, the answers of Alejandro are the following.

- Alejandro:
- c) No.
  - d) By getting the average or range.
  - e) Any of them.

In *level 2*, the responses show evidence that the mean and the range or mean deviation from the data were considered. Sometimes the chain with greatest variation is identified, however they present some error or inconsistency, as in this example.



## Working Group 5

- Mónica:
- c) No because the average of the three movies is the same.
  - d) By adding the minutes and dividing them by the number of data.
  - e) At Cinemex because the range is bigger.

In a *level 3*, the responses show evidence that the mean, the range or media deviation from the data were considered and that the chain with greatest variation is identified.

- Miguel:
- c) No
  - d) Range.
  - e) In the first one, Cinemex.

The frequencies resulting from classifying the responses are shown in Table 2

Categories	Level 0	Level 1	Level 2	Level 3	No answer
Number of students	19	17	3	3	8

Table 2. Analysis of the answers on questions of points f, g and h

In some responses can be seen that there is confusion between the notions of mean and variation. For example, a student says that the variation can be determined “by getting the average or the range”, and another says that “by adding up the minutes and dividing them by the number of data”. Even when the students are acquiring and incorporating terms such as “mean” and “variation” or synonyms of these, they have not yet acquired their corresponding meanings.

### Reading variation in the graph

In a second part of the same task, the students were given the corresponding well made graphs of data (Figure 2) and they were told the mean and median of each movie theatre data set. The idea was to give the opportunity to answer the questions about variation avoiding mistakes made when making the graph or calculating the mean.

[A graph of data is presented]	
f.	Which of the three movie theatre chains has more variation in waiting time? Why?
g.	In which of the movie theatre chains is there more variation, Cinépolis or Multicinemax? Explain.
h.	If the three theatres are equidistant from your home, which one would you choose to watch a movie?

Figure 2: Graphs given to students

To answer question f, the variation can be seen at first sight and argued by comparing the ranges. To answer question g, it is necessary to calculate the mean deviation and therefore use the mean.

## Working Group 5

To analyze the responses to questions f and g, only the justifications or explanations are considered. Again, the responses can be classified at different levels taking into account the presence of some correct aspects in their justifications.

At *level 0*, students focus on idiosyncratic aspects or on their experiences without using statistics arguments:

Josue: f) At Cinemex because it takes more time to go in.

g) Cinépolis, because there are more people waiting all that time for the movie to start.

At *level 1*, the arguments focused on visual perceptions in the graph

Itzel: f) At Cinemex because of data are more dispersed

g) Cinépolis and Multicinema because of data are equally dispersed.

At *level 2*, responses are supported by considering the range

Garay: f) At Cinemex because its range is that from 12 to 21 and at Cinepolis and Multicinemmas it is 18.

g) Cinepolis, because at Cinepolis the 16.5 is 2 and at Multicinemmas it is 3.

At *level 3*, responses are supported by considering the range or mean deviation in incise a) and mean deviation in b). None of responses fell in this category.

The frequencies of responses in each level are presented in Table 2.

Categories	Level 0	Level 1	Level 2	Level 3	No answer
Number of students	22	14	7	0	7

Table 3. Frequencies of responses in each level

### Interpreting variation in the context of the task

Question h takes us to considering how the problem of variation in the context of cinema-chains should be interpreted, how the question should be answered taken into account the data given? In a previous analysis students were expected to relate variation to uncertainty. This may require that students make a ‘mental experiment’, imagining the consequences of variation in waiting time in different cinemas when they go there: the greater the variation in the waiting time, the greater the uncertainty about when the movie will start. If someone likes watching previews or waiting for the movie, it does not help choose the cinema with most variation because the movie may start too early. However there was no response where students related variation to uncertainty.

In fact most justifications were in terms of the personal experiences of the respondents, for example, seven students gave reasons similar to that given by Juan: “Cinemex, because it is very near to my home”. The students that attained higher levels in other questions did not discern that the best theatre to go to is Multicinemmas

(or Cinépolis) because it has less variation and therefore less uncertainty. These are some of the answers:

Miguel: h) Cinépolis. Because I like how they have the things and I do not mind how many previews they show.

Estefany: h) Cinépolis because the waiting time is less there and it is more famous.

In a later analysis it was discovered that the context is not the best to promote interpreting variation as uncertainty. The justifications of a selection of a theatre in terms of personal experiences are reasonable although participants didn't draw conclusions from the data. The reason is that in this case uncertainty does not bring adverse or grave consequences since most people do not mind to wait for long or short time in a cinema.

## DISCUSSION AND CONCLUSIONS

It is important that, during the process of understanding the notion of variation, students be able to make and read graphs as well as to calculate and interpret the mean of data. Acquiring these competences implies different levels of complexity. Most students that show a good level of competence when making graphs and who are able to compare sets of data based on the mean can identify which cinema chain has the greatest variation.

This study shows that it may be convenient to teach the graphing of distributions and central tendency measures (in particular the mean) by presenting them through problem-situations in which variation is also part of the problem. The meanings of those notions are mutually binding and the understanding of each of them is stronger when they are conceived as part of a concept web rather than as isolated notions. However, it must be expected that the understanding of each of these notions goes through different levels so that they progressively join into more complex concept webs.

It is particularly convenient to rethink the role of uncertainty within this concept web. The intuition of "avoiding uncertainty as much as possible" can indeed be applied to many situations in which data variance can be understood in terms of uncertainty. This can help when making decisions or when making informal inferences.

It becomes frustrating to observe that students were not able to infer that it is more convenient to go to the movie theatre which has less variability in its wait-time. One of the students, whose answers are classified in the higher levels, chooses the movie theatre with less variability but her reason is that less variability means "less wait-time". This is one way mean may be confused with variation. Even when this confusion was also present among other students, this particular case is interesting because she gave good responses in the other questions. This leads to the question of what was missing in order to achieve better understanding of variation. Our answer

is that she was unable to associate variation with uncertainty. This difficulty may not be surprising because the teaching episodes did not help students to see that relationship since, in the context of body temperature, variation is interpreted as information, not as uncertainty. Also, in the problem of wait-time in cinemas waiting for long or short time is not really important for people. The authors are trying to investigate the same research question using another problem or context where variation brings more serious or grave consequences.

1. This work was sponsored by grant 101708 from the National Council of Science and Technology (CONACYT).

## REFERENCES

- Biggs, J.B., Collis, K.F. (1991). Multimodal learning and the quality of intelligent behaviour. In H. A. H. Rowe (Ed.), *Intelligence. Reconceptualization and measurement* (57-76). Hillsdale, NJ., USA: Erlbaum.
- Garfield, J., and Ben-Zvi, D. (2008). *Developing students' statistical reasoning. Connecting research and teaching practices*. New York: Springer.
- Hiebert, J., Carpenter, T.P. (1992). Learning and teaching with understanding. In D. A. Grows (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (66-97), Reston, Va, USA: National Council of Teachers of Mathematics.
- Kahneman and Tversky (2000). Prospect theory. An analysis of the decision under risk. In D. Kahneman and A. Tversky (Eds.), *Choices, Values and Frames*. New York, USA: Cambridge University Press.
- Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education*. 33(4), 259-289.
- Moore, D.S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: new approaches to numeracy* (95-137). Washington DC., USA: National Academy Press.
- Shaughnessy, J.M., Ciancetta, M., Best, K., and Canada, D. (2004). Students' attention to variability when comparing distributions. *Paper presented at the Research Presession of the 82nd Annual Meeting of the National Council of Teachers of Mathematics*. Philadelphia, PA.
- Shaughnessy, J.M. (2007). Research on statistical learning and reasoning. In F. K. Lester Jr. (Ed.) *Second handbook of research mathematics teaching and learning* (pp. 957-1010). Charlotte, NC: Information Age Publishing and National Council of Teachers of Mathematics.
- Tal, J. (2001). *Reading between the Numbers. Statistical thinking in everyday life*. New York, NY, USA: McGraw-Hill.

### Working Group 5

- Watson, J. M.; Callingham, R.A.; and Kelly, B.A. (2007). Students' appreciation of expectation and variation as a foundation for statistical understanding. *Mathematical Thinking and Learning*, 9(2), 83-130.
- Watson, J., Kelly, B., Callingham, R., and Shaughnessy, M. (2003). The measurement of school students' understanding of statistical variation. *International Journal of Mathematical Education in Science and Technology*, 34 (1), 1-29.
- Wild, C. (2006). The concept of distribution. *Statistics Education Research Journal*, 5(2), 10-26. On line: [www.stat.auckland.ac.nz/serj](http://www.stat.auckland.ac.nz/serj).
- Wild, C., and Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, 67 (3), 223-265.

# CARRYING OUT, MODELLING AND SIMULATING RANDOM EXPERIMENTS IN THE CLASSROOM.

Michel HENRY<sup>\*</sup> & Bernard PARZYSZ<sup>\*\*</sup>

*Today, the teaching of randomness in schools makes frequent use of technology – for example, random experiments that are simulated by the use of a random generator. In fact, what is actually being simulated is a probabilistic model of the experiment. This is a problem when one must teach probability through both a classical and frequentist approach, as is the case in French high schools. We propose a solution for overcoming this issue and approaching the concept of a probabilistic model by on the one hand preserving the ‘isomorphism’ between the protocol put into play in the ‘concrete’ experiment and the simulation process, and, on the other hand, by making the experimental hypotheses explicit and taking an interest in the analogies between the tables produced by the software when simulating different experiments (which means that they implicitly refer to the same model). More generally, making use of computers is of didactical interest because they can help students to grasp and better understand the modelling process in random experiments.*

Keywords: random experiments, simulation, spreadsheet, model, modelling process.

## INTRODUCTION

Today, many countries include statistics and probability in their math curriculum at the high school level. This change is obviously linked – at least in part – to the increasing amount of statistical data published in the media. But it is also linked to the evolving development of technology, especially with regard to software that can process vast numerical data. Teaching probability at the high school level is indeed a delicate issue, since students are confronted with many difficulties – difficulties that often compound one another. These include: the introduction of a new concept – one of its meanings being related to a specific idea of limit (stabilized frequency); language of sets; logic of events; and, above all, modelling processes.

At any rate, the didactic answer to such a challenge cannot be to overlook this process. It is, rather, to have students become familiar with and experiment with – as soon as possible – real-life random situations, as is now the case in many countries at the junior high – and even elementary – school levels. It is also necessary to give students the linguistic tools associated with the description of such situations.

Additionally, various instruments (calculators, computers) and forms of software (spreadsheets) have become important tools in the domain of probability study [Batanero *et al.* 2005]. They include a ‘random’ generator to simulate random

---

<sup>\*</sup> Université de Franche-Comté.

<sup>\*\*</sup> Université d’Orléans & LDAR (université Paris-Diderot).

experiments – which is much quicker than performing the actual experiment. For this reason, it is necessary that research in didactical engineering at the high school level should emphasize the study and proposition of various situations that can easily be simulated and are, of course, efficient from an educational point of view [Pratt 2005].

In France – and in many other countries – probability curriculum develop an approach of the concept of probability through relative frequency, in which the probability of an event linked with a random experiment can be estimated by a large number of iterations of that experiment. This is opposed to calculating it from symmetry considerations, using the classical formula (which cannot always be done):

$$\frac{\text{number of favourable issues}}{\text{total number of issues}}$$

Even if the genuine experiment can be repeated, this is difficult and time-consuming. Therefore, genuine experiments are often replaced by simulations using a spreadsheet. But this is not as simple for students as it is for us ([Pichard 1998], [Girard 2005]) and frequently, students do not acknowledge the similarities between several experiments that look different but implicitly refer to the same theoretical model (of which they have no idea at that time). In this presentation, we want to address these questions and propose some solutions that make use of spreadsheets.

## A RANDOM EXPERIMENT

In order to simulate a random phenomenon with a spreadsheet, one must decide how to use the random generator in accordance with the phenomenon. To illustrate this, we shall use a famous historical example: a game called ‘heads or tails’ (*croix ou pile* in French), which was widely played in France in the 18<sup>th</sup> century and which comprises two players and the use of one coin. The first player tosses the coin: if he gets “heads,” he wins; if he gets “tails,” he tosses the coin again. This time, if he gets “heads,” he wins, and if he gets “tails,” the other player wins.

About that game, the French mathematician Jean le Rond d’Alembert (1717-1783) [D’Alembert 1754] wrote in the great Encyclopedia that two solutions could be given to determine each player’s chances. First solution: there are four combinations – heads/heads, heads/tails, tails/heads and tails/tails. Since the first player wins in three of these four combinations, his odds are three times better than the second player’s. But d’Alembert suggests that one can reduce the number of total combinations to three, “*since when heads has come at first, the game is over and the second tossing is of no use.*” These combinations are: heads, tails/heads and tails/tails. In this case, the first player’s odds are only two times better than the second player’s.

D’Alembert’s text has been discussed in several high school classrooms. However, it causes great confusion, and students are generally split into three groups of differing opinions [Parzysz 2007]. To get over this puzzling situation, students quickly think of playing a certain number of games, and, by doing so – by chance – they soon

become convinced that the first player has more chances to win than the second player. But what are the correct odds? Three against one or two against one? To be ‘almost’ sure, a great number of games must be played. In some cases, because not enough games are played (around 1,000 are necessary), students find a frequency of about 0.7 and cannot choose between  $2/3$  and  $3/4$ . This is typically the problem posed by a frequentist evaluation of probability: even if it allows for the dismissal of some solutions, it does not necessarily ascertain that one of them is correct.

### EXPERIMENT, SIMULATION, MODEL

Since playing a number of ‘real’ games is time-consuming – and, moreover, is not always practical – another option is to simulate the game. But, for many students, even replacing a dice by another one is dubious, so replacing a genuine experiment by a ‘fake’ one (a simulation) – particularly with a spreadsheet – poses similar difficulties. In these conditions, can a simulation really replace the experiment to get an answer of probabilistic nature?

Let us now see what French 10<sup>th</sup> grade textbooks say about what it means to “simulate” a random experiment. You can sort the ideas into four primary points where the fourth corresponds to our point of view [Parzysz 2008]:

- First idea: a simulation is a *substitute* of the experiment
- Second idea: there must be an *analogy* between the experiment and its simulation
- Third idea: a simulation is more *economic* than the experiment
- Fourth idea: a simulation is a *model* of the experiment

These ideas can be synthesized by saying that simulation consists of replacing a given random experiment by another one, which is easier and/or faster to put into play – provided it can be ascertained that the simulation statistically reflects the characteristics of the genuine experiment. In fact, its appropriateness is ensured by the probabilistic model which underlies the simulation. On the whole, the process is as follows: being given an experimental protocol and the possible tools, one suggests a model for the experiment and implements a simulation of this model. Afterwards, the experimental data will be confronted with the simulation and the model is then either accepted or rejected (fig. 1).

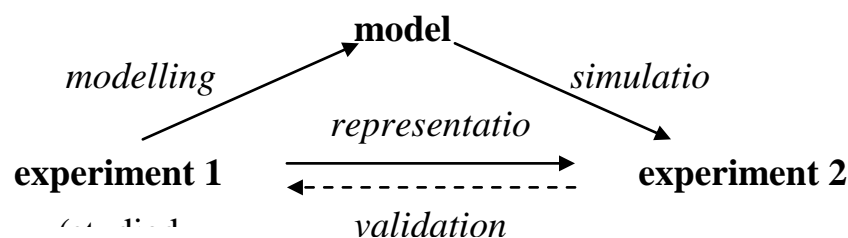


Figure 1



## Working Group 5

This may be a real problem for younger students who do not know what a probabilistic model is. For them, in those cases, the above ternary diagram becomes binary (the ‘model’ vertex disappears). In our example, the class could be split into two groups: in one group, students will carry out the genuine random experiment (exp 1), while in the other group, students will simulate it with a spreadsheet (exp 2).

Then, a comparison between the two experiments may be carried out, leading to reflection that aims to make explicit some hypotheses and the ways in which they correspond to features of the simulation.

For instance, let us consider the elementary example of tossing a coin, for which we want to study the distribution of heads and tails in a designated number of tosses. We can make the following hypotheses:

- (H1) There can only be two results: ‘heads’ or ‘tails’ (according to the visible side of the coin).
- (H2) If the coin falls on its edge, the try is nullified.
- (H3) The coin is well-balanced (*i.e.* heads and tails have the same chance of occurring).
- (H4) The coin is tossed so that the result cannot be foreseen.

This set of hypotheses obviously constitutes a simplified abstraction of reality [Henry 1999], in which some aspects of the experiment are neglected (*e.g.* H2: coin falling on edge) or assumed (*e.g.* H3: well balanced coin) and some details of an experimental protocol are made explicit (H1, H4).

Now the hypotheses must be transformed into issues of the random generator. This will be attained by defining a transformation and associating an issue, H (heads) or T (tails), to any number  $x$  ( $0 \leq x < 1$ ) produced by the random generator, such as:

- (H1) + (H2)  $\rightarrow$  the issue H is associated with a value  $x < .5$ , and the issue T with  $x \geq .5$ ;
- (H3) and (H4) correspond to the randomness of the sequence of numbers.

To make this last point clear, let us recall that a so-called ‘random’ generator is in fact a deterministic generator, since the numbers produced are successive terms of a recurrent sequence given under the form of decimal fractions with a fixed number of digits [Parzysz 2005]. Nevertheless, we may assume that the generator has been tested by the manufacturer and can consider it a fair simulator of a random sequence.

## **SIMULATION PROCEDURES AND MODELLING**

When simulating a random experiment with a spreadsheet, it frequently happens that several possibilities can be considered *a priori*. To illustrate this, let us go back to our example: the ‘heads or tails’ game described above, in which we shall suppose that the coin is well balanced. Two procedures (at least) are possible:

*Procedure 1.*

Produce a random variable (alea1) with two equally distributed issues H and T:

- if H happens you win;
- if T happens, produce a random variable in the same conditions as above (alea2), then: if H happens, you win;
- if T happens you lose.

This procedure leads to table 1, in which H is associated with a random number smaller than .5.

<i>Game</i>	<i>Alea 1</i>	<i>Result</i>	<i>Alea 2</i>	<i>?</i>
1	.732	again	.813	lost
2	.307			won
3	.042			won
4	.967	again	.274	won
5	.766	again	.819	lost

**Table 1***Procedure 2.*

Produce two independent random variables (alea1 and alea2) with two equally distributed values H and T:

- if both show T you lose;
- otherwise you win.

This procedure leads to table 2, in which H is associated with a random number smaller than .5.

<i>Game</i>	<i>Alea 1</i>	<i>Alea 2</i>	<i>?</i>
1	.732	.813	lost
2	.307	.785	won
3	.042	.654	won
4	.967	.274	won
5	.766	.819	lost

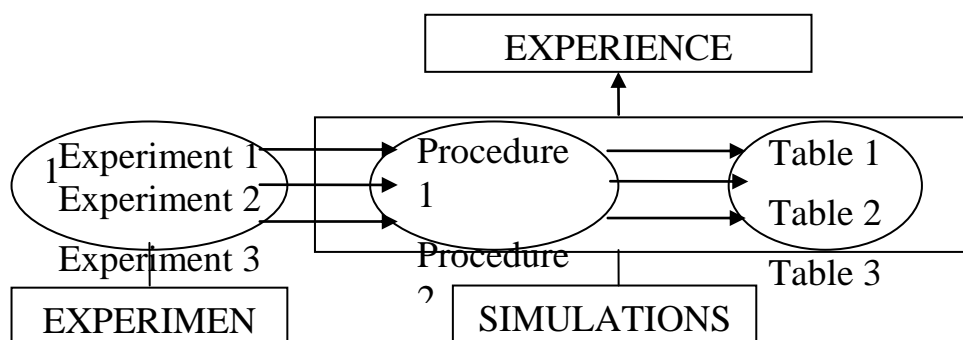
**Table 2**

The second simulation is easier to carry out than the first, which contains two interwoven aleas, but it introduces a second tossing that does not always occur in the real game. Although very useful, this virtual toss may cause some students to only reluctantly accept the second procedure as a simulation of the real experiment. The same opposition was expressed by the French mathematician Roberval (1602-1675), as Pascal mentioned in his famous exchange of letters with Fermat in 1654 about the now-famous ‘share-out problem’ [Parzysz 2009]. Therefore, it proves necessary to make students understand that this kind of change does not modify the results (as Fermat himself did).

But this difficulty can be overcome. Let us consider the two above tables (intentionally, the same numerical data appears in both tables). We can observe that in table 1, the “Result” column can be removed without any consequence and that in table 2, suppressing some numbers (namely .978 and .724) has no effect on the outcome of the game, since the player has already won.

After these changes are made, the two tables become identical, showing that the two implemented procedures (hence the two associated experiments) yield the same results and that one can replace *procedure 1* by *procedure 2* without any disruption to the data.

More generally, comparing several experiments through their simulations by looking both at the associated procedures (in order to make the underlying hypotheses explicit) and at their implementations on a spreadsheet makes it possible to convince students that some simulations (and their corresponding experiments) may be ‘equivalent’ in spite of any apparent differences. When carried out, they behave similarly from a probabilistic point of view. This is because they correspond to the same ‘experience scheme,’ which can be considered a first step toward the notion of a probabilistic model to be built later on (fig. 2).



**Figure 2**

Thus, through the idea of experience scheme, such examples of simulations lead students to a first idea about probabilistic models and the modelling process [Garfield 2008]:

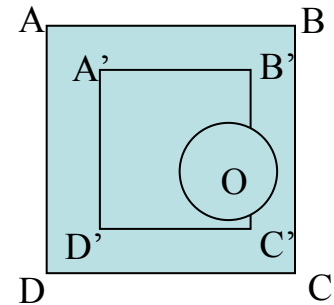
- describe and analyze the random experiment at stake
- work out an experimental protocol, i.e. the set of features defining the experiment from a probabilistic point of view: list of observable issues, characteristics of the events that will be considered (corresponding to each issue), actions to perform in order to ascertain that the *same* experiment is repeated
- make work hypotheses explicit in order to be able to control the relevance of the model in construction
- interpret the characteristics of the real experiment in terms of model hypotheses (especially the probabilities representing randomness)
- translate these hypotheses into computer instructions, in order to solve problems that seem, at first sight, inaccessible via mere calculation
- theoretically exploit the model to draw properties accounting for phenomena which can be observed in reality
- interpret the results of the simulation by comparing them to the initial model hypotheses

## SIMULATION AS A TOOL FOR SOLVING PROBLEMS

With the d'Alembert's problem, we can verify that the result of a simulation is in accordance with the probability calculated a priori in the model – assuming the same probability to all ordered pairs of possible issues. This reinforces student confidence in simulation (but, in fact, this confirms that the computer and software makers satisfied the schedule of conditions with regard to the random generator). Nevertheless, the didactical interest of simulation must not be neglected, since it allows for finding solutions to problems that are too difficult or even impossible to solve by hand [Biehler 1991].

Let us now consider the following example inspired by the 'free tile' game (*franc-carreau*, in French). This game was very popular in France during the 18<sup>th</sup> century, and the probability problem associated with it was first analyzed by the mathematician and naturalist Buffon (1707-1788) [Buffon 1733]. In this game, a coin is thrown on a floor covered with square tiles and players bet on its final position: will it stop on a single tile ('free tile') or land on a line adjoining two tiles?

Let us consider the tile where the centre  $O$  of the coin is fallen. With the hypothesis (implicit for Buffon) that all locations of the tiling had the same chance to be reached, the answer to this – now classical – problem can easily be found by comparing the areas of the two squares  $ABCD$  and  $A'B'C'D'$  to obtain this geometrical probability :  $A'B'^2 / AB^2$ .



In a simulation of this problem, with  $AB = 1$  and the radius of the coin equal to  $\frac{1}{4}$  ( $A'B' = \frac{1}{2}$  and the probability of 'free tile' is .25), we obtained for several series of 10,000 successive throws relative frequencies equal to .2564; .2515; .2445; .2497...

On the one hand, the simulations will give similar results from a probabilistic point of view. On the other hand, a simple calculation will show that the ratio of the 'good' area ('free tile') to the total area is quite close to the values obtained by simulations. Thus, students can interpret these values as approximate measures of the probability of 'free tile' and, convinced by quality of this simulation, gain confidence in this tool.

In 1733, Buffon also studied the case of a small stick thrown on a parquet floor (this is the well-known 'Buffon's needle problem' [Klain 1997], [Aigner 2006] or [Henry 2003]). These historical examples lead us to a more difficult problem than the free tile game, which can be solved by a simple simulation on a spreadsheet. Let us now assume that a rod – and not a coin – is thrown on a square-tiled floor. The probability of 'free tile' cannot be calculated so easily. However, made confident by the good result given by the computer simulation in the case of the coin, we infer that it will be the same for the rod, *i.e.* that the stabilized frequency will be close to the probability which could – perhaps – be calculated by hand (such a calculation

implies advanced mathematical knowledge). Thus, our problem is as follows: throwing a rod on a floor covered with square tiles, what is the probability of it landing on a ‘free tile’

Let us represent the rod by a segment  $[AB]$  (fig. 3), the length of which is  $a$ , and consider the tile on which the end  $A$  fell. This tile is represented by the unit square in a cartesian coordinate system.

Let  $x$  and  $y$  be the random coordinates of  $A$  ( $0 \leq x < 1$ ,  $0 \leq y \leq 1$ ) and  $t$  the random angle between the  $x$ -axis and vector  $AB$  ( $0 \leq t \leq 2\pi$ ). The rod makes ‘free tile’ when these two conditions are satisfied:

$$0 \leq x + a \cos t < 1 \text{ and } 0 \leq y + a \sin t < 1$$

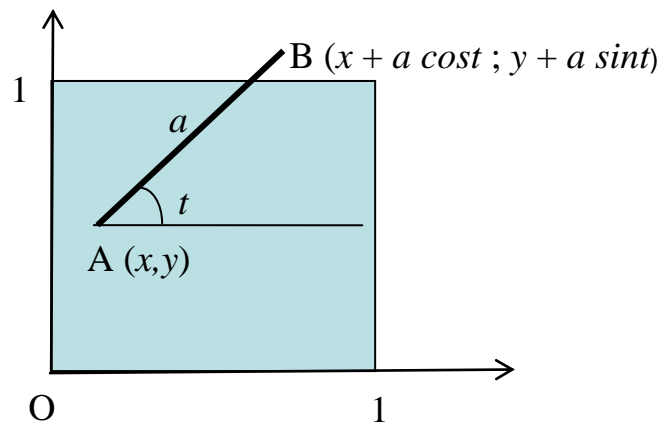


Figure 3

The volume in  $x, y, t$  delimited inside the cartesian product  $[0,1] \times [0,1] \times [0,2\pi]$  by these conditions is not easy to calculate, but the simulation of the experiment is within reach of a senior high school student with the model hypothesis of the uniform distribution of  $(x, y, t)$  in the above product. Here is a possible carry-out of this simulation with a spreadsheet.

Length of AB	(cell) A1	.44	Comment	Example
$x(A)$	B1	=ALEA()	Alea 1	.306
$y(A)$	C1	=ALEA()	Alea 2	.477
angle $t$	D1	=2*PI()*ALEA()	Alea 3	5.786
$x(B)$	E1	=B1+A1*COS(D1)		.692
$y(B)$	F1	=C1+A1*SIN(D1)		.267
Free tile = 1	G1	= IF((E1>0)*AND(E1<1)* AND(F1>0)*AND(F1<1);1;0)		1

For instance, by implementing this simulation, we had got the following results for a rod of length  $a = .44$

Frequency of free tiles in 250 tries	Frequency of free tiles in 1000 tries	Frequency of free tiles in 5000 tries	Frequency of free tiles in 10,000 tries	Confidence interval at 5% risk for 10,000 tries	
.496	.496	.493	.499	.489	.509

Thus, we have here a tool for solving problems that plays the exact same role as graphic calculators, which can draw curves to represent functions – for finding the approximate graphical solution of a system of equations, for instance.

## **DIDACTICAL SIGNIFICANCE OF SIMULATION**

Despite the obvious utility of simulation, our interest is fundamentally didactic. First, computers are capable of working on vast statistical series, which gives meaning to statistical summaries (parameters, graphs) and demonstrates their relevance. They also provide a dynamic presentation of how the notions of relative frequency and probability work and interact – notions that are encountered in France in the 9<sup>th</sup> grade, either theoretically, in an equiprobabilistic context, or through practical experiments with numbers large enough to foster an understanding of the law of large numbers in actuality.

Of course, the computer does not make use of the notion of probability when it performs a simulation: it just exhibits the effects of the equirepartition principle on the random numbers that it generates, (a principle which is part of its specifications.) But even so, the use of computer simulations in the classroom – as a pseudo-random generator – fosters a better understanding of the notions of relative frequency, sample fluctuation, variability of empirical parameters and, finally, probability.

Making the most of the power and speed of computers to generate pseudo-random numbers and introduce them in relevant formulas – which realizes a simulation of the model of a studied experiment – introduces students to a great richness of new virtual random experiments. This is not the only interest of computers, however, for the didactical significance of simulation is fundamentally to lead students to grasp the modelling process as described above. From a didactical point of view, we stress the importance of posing problems in terms of modelling, which implies that one must make the model hypotheses explicit and choose them in relation to the studied problem [Henry 2001]. This allows students to avoid paradoxes and overcome obstacles linked with their own preconceptions, since probability is a domain in which intuition is often misleading. Identifying such insufficient or erroneous conceptions among students, investigating their origins, spotting and analyzing such difficulties (which are cultural as well as epistemological and didactical) and designing didactical situations in order to overcome them – these are all necessary tasks for future research in math education.

To conclude, we quote a paper presented in CERME 6 [Papaieronymou 2009], which reports these recommendations of four U.S. professional organizations that place considerable emphasis on experimental vs. theoretical probability and simulations:

“Secondary mathematics teachers need to be able to plan and conduct experiments and simulations, distinguish between experimental and theoretical probability, determine experimental probabilities, use experimental and theoretical probabilities to formulate and solve probability problems, and use simulations to estimate the solution to problems of

chance. Secondary mathematics teachers should be able to provide a model which gives a theoretical probability that can be compared to experimental results, which in turn is essential when studying the concept of relative frequency. In order to help students develop an understanding of probability as a long-run relative frequency, secondary mathematics teachers need to understand the law of large numbers and be able to illustrate it using simulations”.

## REFERENCES

- Aigner, M. & Ziegler, G. M. (2006). *Raisonnements divins : quelques démonstrations mathématiques particulièrement élégantes*. Springer Verlag, 2d ed.
- Batanero, C., Bieler, R., Carmen Maxara, C., Engel, J., Vogel, M. (2005). Using Simulation to Bridge Teachers' Content and Pedagogical Knowledge in Probability. In *15<sup>th</sup> ICMI Study Conference: The Professional Education and Development of Teachers of Mathematics*. Available on line <http://stwww.weizmann.ac.il/G-math/ICMI/strand1.html>
- Biehler, R. (1991). Computers in Probability Education. *Chance Encounters: Probability in Education* (Kapadia, R. & Borovcnik, M., eds), 169-211. Dordrecht, Kluwer Academic Publishers.
- Buffon, G. L. Leclerc (1733). *Solution de problèmes sur le jeu du franc-carreau*, Mémoire présenté à l'Académie Royale des Sciences. In *Essai d'arithmétique morale, Histoire Naturelle, générale et particulière, Supplément, tome quatrième*, Paris, Imprimerie Royale, 1777, 46-148.
- D'Alembert, J. Le Rond (1754). *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, Diderot et d'Alembert eds, tome IV, article Croix ou Pile : <http://diderot.alembert.free.fr/C.html>
- Garfield, J. B. & Ben-Zvi, D. (2008). Learning to Reason about Statistical Models and Modeling, in *Developing Student's Statistical Reasoning*, part. II, chap. 7, 143-164. Springer.
- Girard, J. C. & Henry, M. (2005). Modélisation et simulation en classe, quel statut didactique ? *Statistique au lycée*, vol. 1 (Chaput, B. & Henry, M., eds), 147-159. Paris : APMEP, brochure n° 156.
- Henry, M. (1994). L'enseignement du calcul des probabilités dans le second degré : perspectives historiques, épistémologiques et didactiques. *Repères-IREM* **14**, 69-104.
- Henry, M. (1999). L'introduction des probabilités au lycée : un processus de modélisation comparable à celui de la géométrie. *Repères-IREM* **36**, 15-34.
- Henry, M. (2001). Notion de modèle et modélisation dans l'enseignement, in *Autour de la modélisation en probabilités*, Besançon, Presses Universitaires Franc-Comtoises, 149-159.

## Working Group 5

- Henry, M. (2003). Des lois de probabilité continue en terminale S, pourquoi et pour quoi faire ? *Repères-IREM* **51**, avril 2003, 5-25.
- Klain, D. A. & Rota, G.-C. (1997). *Introduction to Geometric Probability*, Accademia Nazionale dei Lincei, Cambridge University Press.
- Papaieronymou, I. (2009). Recommended Knowledge of Probability for Secondary Mathematics Teachers. *Proceedings of CERME 6* (Working Group 3 on “stochastic thinking”). On line : <http://www.inrp.fr/editions/editions-electroniques/cerme6/>.
- Parzysz, B. (2005). Quelques questions à propos des générateurs aléatoires. *Statistique au lycée* vol. 1 (Chaput, B. & Henry, M., eds), 181-199. Paris : APMEP, brochure n° 156.
- Parzysz, B. (2007). Expérience aléatoire et simulation : le jeu de croix ou pile. Relecture actuelle d’une expérimentation déjà un peu ancienne. *Repères-IREM* **66**, 7-44.
- Parzysz, B. (2009). Des expériences au modèle, via la simulation. *Repères-IREM* **74**, 91-103.
- Pascal, B. (1963). Correspondance avec Fermat, in *Œuvres complètes*, 43-49, ed. Seuil, Paris (Original letter 1654).
- Pichard, J.-F. (1998). Approche épistémologique et diverses conceptions de la probabilité. *Repères-IREM* **32**, 5-24.
- Pratt, D. (2005). How do teachers foster Students’ understanding of probability. *Exploring Probability in School* (Jones G. A. ed.), 171-189. Springer.



# **RISK-BASED DECISION-MAKING BY MATHEMATICS AND SCIENCE TEACHERS**

Dave Pratt, Ralph Levinson, Phillip Kent & Cristina Yogui

Institute of Education, University of London

*In modern society, the notion of risk is drawn upon to convey much human decision-making. In fact, risk is variously depicted and the psychological literature is divided in how people make sense of risky situations. At a time when curricula, especially in the UK, are recognising the need for students to be sensitised to risk-based decision-making, we have been studying how mathematics and science teachers make sense of risk, focussing on activity around a specific personal dilemma. We analyse this data through the lens of the priority heuristic. We identify possible limitations in the priority heuristic. We conclude that these teachers needed greater support for coordinating the dimensions of risk and offer one software response.*

## **RISK**

In this paper, we consider how a mathematics and science teacher engage in risk-based decision-making with a view to evaluating whether recent psychological research adequately describes that process and what additional resources might be offered to support such decision-making.

In modern society risk permeates decision-making at both personal and policy levels, a fact now being recognised in curricula. The exponential increase in talk around risk has not ironically resulted in its clear definition and the epistemological basis for risk continues to be a subject of debate (Adams, 1995; Stirling, 1999).

In the media, risk is typically portrayed as identical to likelihood, a problem when hazards with differing severity are compared. In such circumstances, both likelihood and impact need to be addressed simultaneously in some sort of trade-off. Indeed, standard decision-making theory formulates risk as the product of the probability (as a measure of likelihood) and disutility (as a measure of loss) and proposes that decisions should aim to minimise the total calculated risk. However, the psychological literature has demonstrated that individuals often do not make risk-based decisions by minimising total risk as in standard theory.

## **HEURISTICS FOR MAKING JUDGEMENTS ABOUT RISK**

Risk-based decision-making is especially complex and so people draw on intuitive heuristics. It has been demonstrated that such heuristics are vulnerable to bias (see, for example, descriptions of the availability and representativeness heuristics in Kahneman et al., 1982). Recent research (Brandstätter et al, 2006) has focussed on trying to specify what people attend to and the information-search procedures used when making risk-based decisions. According to Brandstätter et al, in the case of losses, the decision-making process is described by the *priority heuristic* as follows:

First, compare the minimum losses of the alternative decisions. If the difference between the two minima is at least 10% of the maximum loss, choose the decision associated with the lesser of the two minimum losses. Otherwise, compare the probabilities of the minimum losses of the alternative decisions. If the two probabilities differ by more than 0.1, choose the decision associated with the higher probability of loss. Otherwise, compare the maximum losses of the alternative decisions and choose the decision associated with the lower maximum loss. (*Our paraphrasing of the heuristic.*)

It is claimed that, under certain conditions, the priority heuristic predicts the decision most people will make *and* the decision-making process that they will undertake. In the literature, this heuristic has been demonstrated by reference to situations in which people are making decisions about different gambling situations with clearly specified profits (or losses) and probabilities.

In this paper, we use the priority heuristic to analyse the teachers' decision-making. We have two main research questions in conducting this exercise. First, we ask whether the priority heuristic is an adequate model in a complex situation where the losses and likelihoods are not clearly specified and may even not be easily quantifiable. We hope through asking this question that we might gain some insights into the scope of the priority heuristic. Second, we ask whether, through such an analysis, what sort of resources might support teachers (and perhaps others) in deploying more sophisticated strategies.

## METHOD

Through an iterative design process, we developed a computer-based scenario, Deborah's Dilemma (DD), in which mathematics and science teachers were required to respond to the fictitious Deborah's difficulty in deciding whether to have an operation that could cure a painful spinal condition. The operation might result in a number of complications described through various, and at time conflicting, sources of information. Should Deborah choose not to have the operation, she would need to manage her pain level through changing her daily routines of work, domestic and leisure activity.

Information about Deborah's condition was set out within the software in a deliberately personal way, to offer different perspectives with varying levels of authority. Two software tools accompanied the information about the condition. First, a probability simulator allowed the teachers to model the possible complications to gain a sense of how often the operation might be successful, and how often complications of varying degrees of severity might occur. Second, a 'Painometer' offered a quantified experience of Deborah's pain in relation to a "tolerable" level, as the pain was influenced by the activities in which Deborah engaged. The teachers modelled which activities Deborah would undertake to infer the effect of these on Deborah's pain level.

Three pairs of teachers (one science and one mathematics from the same school in each pair) worked through DD to arrive at a specific response. A researcher sat with each group but only intervened to demonstrate relevant aspects of the software, to address any technical points and to ask questions for clarification. Video screen capture software recorded the teachers' dialogues and on-screen actions. The session lasted approximately 2 hours. Data for the analysis consists of an audio transcript and a video record of their interactions with the software for each pair of teachers.

We present below the findings through the case of Peter (science) and Emma (mathematics). Analysis of the data from the other teachers' is ongoing.

### **PETER (P) AND EMMA (E) ENGAGE WITH DEBORAH'S DILEMMA**

We present the activity by P and E in two stages. The first stage represents the process by which P and E came to a decision based on their interpretation of the information given in DD. Later, we present their ongoing activity after an intervention from the researcher (the first-named author).

#### **Stage 1: Before the researcher's intervention**

P and E read the introductory information about DD and formed an initial reaction:

P: If I was Deborah I think I'd have the operation.

E: I agree – so we'll go for operation first.

They then began to look more closely at the information available about the operation and expressed some concern about the reliability of the data, especially in relation to Deborah's personal research:

P: Now her own research. Reliability, source.

E: Yes, that is questionable – one list from any old website you don't know, could be one person.

As they read the information, they discussed the chance of the operation being successful as well as the possible severity of the complications and their likelihoods.

P: Right that's saying 95% of the time it works, but if it doesn't you have to have the operation again.

E: 1 in 1000 of nerve root or spinal cord damage. 5[referring to the fifth listed complication] is temporary and happens 1% of the time...

P: Number 4 sounds scary [referring to nerve root/spinal cord damage]... it might mean a bit of tingling, pins and needles, which is a different level from being in a wheelchair. But you don't know, and of course they can't tell you because, when you're in the operating theatre, different things happen to what was expected.

P and E discussed which complications should be included in their model:

E: Shall we put the other one – the 1 in 500, what was that? [checks web page] damage to trachea/oesophagus - only possibly permanent, but if you can't eat that is significant I would say – it would worry me! [laughs]

## Working Group 5

P: That sounds really horrible. Should happen less than 1 in 500 cases. Let's add that one *[returning to the software tool]*... call it 'trachea damage'.

35 minutes into the investigation, P and E ran their model, starting with one case, then extending it to 10, 100 and 10010 trials.

E: That's not bad, 17 [failures] out of 10,010. I'd take those odds.

P and E continued to modify and run their model, adding complications related to anaesthetic and infection from a superbug.

E: 9010 successful. And anaesthetic is 1, which doesn't necessarily mean dying *[laughter]*. And 33 superbugged – slightly horrible, but they should have gone to a better hospital. So that was the biggest. Nerve damage was pretty low *[referring to the number of cases of nerve damage out of 1000, which was 8]*. The no eating no drinking thing, 17.

P: But you don't know how severe that is; it could be anything from a sore throat up to no eating.

E: Only one had a problem with the general anaesthetic.

P: I think that probably means death, or severe brain damage, something pretty awful.

E: And superbug can be awful. But again, out of 990, which have failed, only 49 people, which is 50 in 1000, which is tiny.

P: And the rest just had the pain they had before.

E: If you had the operation without success, you had the uncomfortable experience, but you haven't lost much else apart from time. At least you haven't gone backwards. I think she should have it.

E confirms in line 0 their original intuition (line 0) that Deborah should have the operation. At this point (49 minutes into the investigation), the researcher proposed that they began to consider Deborah's lifestyle through the second software tool. They discussed which activities to include and what levels to set the amount of activity and consequent pain incurred.

E: I don't believe her, that she does that much sport. She didn't seem that upset about not being able to do sport.

P: It did impact on her life though.

E: Shall we leave it as, a fair bit?

P: May not be higher than that, just 'more pain'.

After 56 minutes, P and E ran their model of Deborah's lifestyle.

E: Look! *[Laughter]* She's always above the tolerance, apart from once in a blue moon.

P: Yes... oh look though, it's painful to look at isn't it?

P: If it was like that you would stop doing your sport.

The researcher proposed that they experiment with different settings.

P: OK, take her sport, like I can't do all these things.

E: But even if she does, she's still above it *[Sport slider is moved to zero]* – she can do lots and lots of work and no sport and she is OK *[pain level always under the tolerance level]*

## Working Group 5

P: Yes, though she's always up to tolerance level.

E: If I increase this [*gradually increasing the Sport slider*], watch for the first point...

P: There!

E: See, that is not much sport she is allowed to do.

P: You'd want the operation immediately.

Having added the sorting activity, P and E felt even more confident that Deborah should have the operation. They continued by adding further activities to model Deborah's lifestyle, and remained convinced that having the operation was the better option.

P: As soon as you look at this one it would make children think she should have the operation, with the impact on your life.

E: We were undecided until we started looking at the pain.

P: Yes, because then you are thinking about what it does to your life. Every day it always hurts, and when she does sport, it always hurts when she shops. The risky bits of the surgery might not happen to her, but she knows every day 'when I go shopping it's going to hurt me'. With the surgery lot of things are short-term, even if you got worse for a while then you know the end point is going to be better than you were in the first place.

E: When we were looking at the surgery, successful outcome, we did not really, it wasn't conclusive until we looked at the pain threshold.

P & E often thought about the problem through the eyes of their students in school (see line 0). Lines 0 and 0 give some indication of what influences P and E in coming to this conclusion as does E's later comment (line 0):

E: Yeah, you forget about all the numbers and think, "Bloody hell!"

### Stage 2: After the researcher's intervention

After 85 minutes of the investigation, the researcher wished to probe P & E's basis for their position – wondering to what extent was it sensitive to the parameters in the problem. He asked P and E consider how far the probabilities would need to change for them to reverse their decision that Deborah should have the operation. As a result, P and E reviewed the complications and, after running the model 1000 times, found complications on 50 occasions i.e. a worse position than prior to the operation on 1 in 200 occasions.

E: 1 in 200, that's actually not - it's not as successful as we thought! [*laughter*] Are you mortified now? 1 in 200, that's still pretty good though, I think. You're not convinced are you?

P: I don't think they'd be keen on surgery – that's it's not working?

Noting that P & E were now aggregating the complications as 'being in a worse situation', the researcher pressed by asking what they would think if all 50 occasions involved death or impairment.

E: I'm slightly – if I was Deborah, and there was me dying, that would be better than being still alive with something horrible – do you know what I mean, it sounds stupid, but if I'm dead I don't care, but if I'm alive and feeling pain, obviously it

## Working Group 5

depends - another thing is what her family situation was, if she's got young children, with a 1 in 200 chance, you'd rather be there for your kids, whereas being by yourself, you know you might have a slightly... I think I probably wouldn't, I dunno, I'm a bit of a...

P: Because the pain she has doesn't stop her leading a normal life.

E: She leads a restricted life, but she's not bedridden or anything.

After 100 minutes, P and E reviewed their position.

E: Oh, but she's still got to live with that pain every day, I'd still go for... I don't know if I'd change my mind...

P: She's got a 1 in 200 chance of being worse off.

E: But she's in pain for most of her life.

P: That is partly under her control; she could stop sport for example.

E: Yeah, I think I'm changing my mind, but she couldn't stop her work; she could stop driving, but she wouldn't be able to carry things. Oh, we should just have stopped when we were happy! *[laughter]*

P: She could change her job. Probably she's been through some of those thoughts already. She didn't go straight to the doctor. She's lived with it quite a long time.

E: You'd try to make adjustments, you wouldn't be considering the operation if you hadn't thought about adjustments.

The intervention apparently led to P and E being less confident about what decision Deborah should make. They finally wrote:

...she can to a certain degree control the pain by not doing certain activities like sport but this lowers her quality of life. If she has the operation, there is a 1 in 200 chance of her having horrible complications plus there are other alternatives with the exercises and the neck brace. Her personal home life would also be a significant factor, depending on children etc or if she is a carer...

## DISCUSSION

Brandstätter et al (2006) claim that the priority heuristic not only predicts the decision but also describes the decision-making process. We examine both of these claims.

### The priority heuristic as a predictor of Peter and Emma's decision

Table 1 shows the possible outcomes and their likelihoods as entered by P and E into their model of the decision to have the operation.

<i>Outcome</i>	<i>Likelihood as entered</i>	<i>Likelihood as n in 10000</i>
Operation successful	450 in 500	9000
Nerve damage	1 in 1000	10
Trachea/oesophagus soreness	1 in 500	20

## Working Group 5

General anaesthetic	1 in 1000	10
Superbug infection	1 in 400	25

**Table 1: Complications in P & E's model for the decision to have the operation**

In modelling Deborah's lifestyle, P and E entered sport, work and shopping as three activities that impacted on her pain level. They formed the view that, with these activities in place, Deborah would suffer almost constant above-threshold pain.

Following the priority heuristic, as set out in the subsection on *Heuristics for making judgements about risk*, the minimum losses that could be incurred are zero for a successful operation and constant above-threshold pain if no operation. The lower loss is zero and so priority heuristic predicts an initial decision that Deborah should have the operation. However, that is only the final decision if the minimum losses differ by at least 10% of the maximum loss, death or paralysis. It is unclear whether the difference between zero loss and constant above-threshold pain is more than 10% of the loss incurred by death or paralysis. If not, then the priority heuristic may predict a different decision, based on the probabilities of the minimum losses, 0.9 (for the successful operation) and 1, or perhaps slightly below 1, for the constant ongoing pain. The difference between these two probabilities is less than 0.1 and so the priority heuristic refers next to the maximum loss. This is presumably death or paralysis and so the prediction is a decision not to have the operation.

In conclusion, the priority heuristic makes the correct prediction for Stage 1 provided constant above-threshold pain is regarded as an order of magnitude below the loss associated with death or paralysis, which is perhaps reasonable, though not entirely clear, given E's view in line 0. Insofar as P and E were, after the intervention, wavering towards a decision not to have the operation, it is unclear why the intervention might have changed any of the considerations above, other than possibly by placing 'unnatural' emphasis on the complications associated with the operation.

### **The priority heuristic as a description of the decision-making process**

One criticism might be that P and E took an early view that Deborah should have the operation (lines 0-0), arguably signalling an end to the search for a decision, and yet continued beyond the apparent stopping condition, in contradiction of the priority heuristic's description of the process. We think this criticism would be unfair. The priority heuristic assumes that all information is available at the point of decision; it is certainly the case that, in taking their early view, P and E had not yet assimilated all of the information, so we take continued activity as part of the process of reaching a point where a decision might be made that could follow the priority heuristic (activity such as making sense of the likelihoods and judging to the severity of the complications in lines 0-0, and such as deciding which activities in Deborah's life were significant and how they should be measured in lines 0-0).

The heuristic indicates that the decision-making process will place first priority on perceived losses. Certainly there is substantial focus on the harms that might result from complications arising out of the operation (lines 0-0; 0-0) and in Deborah's lifestyle (lines 0-0). However, there is also considerable discussion about likelihoods and these often take place alongside discussion of losses. More pertinently, P and E make specific reference to losses when describing how they were making their decision. Thus, in line 0, E explicitly articulates how her focus is on the possible zero loss outcome of having the operation and uses that fact to argue for Deborah having the operation. Similarly in lines 0-0, both P and E clarify that the operation became increasingly the better option when they considered the constant pain of not having the operation, presumably in comparison to the possible zero loss when having the operation. Perhaps line 0 captures the sentiment when E refers to forgetting about the numbers (by which we think she means the probabilities).

These articulations strengthen the notion that in the end, after assimilating all of the information through reading, discussion and modelling, P and E did in Stage 1 seek to minimise the minimum loss, in accordance with the priority heuristic.

After the intervention, P and E were encouraged to consider that extreme complications were rather likely. According to the priority heuristic, the decision should remain the same since the minimum losses were not affected by this re-evaluation. So, why might P and E show signs of changing their mind? Our interpretation is that the intervention, by focussing on likelihoods, artificially pushed P and E into the later steps of the priority heuristic, resulting in the consideration of maximum losses and a decision not to have the operation.

### **Limitations of the priority heuristic**

Broadly speaking, we think the priority heuristic provides a good prediction of both the decision and the decision-making undertaken by P and E. In conducting this analysis, we have become aware of some interesting limitations. The findings reported in the Brandstätter et al study were based on responses to situations that were already quantified both in terms of loss (or gain) and likelihood. Rarely in real-world problems is it possible to quantify very precisely, if at all, either likelihoods or losses. This raises a question about how individuals cope with such uncertainty. Perhaps the priority heuristic is still relevant when there are clear and distinct differences, such as between the zero and constant above-threshold losses. But when the teachers were artificially pushed beyond that step in the heuristic, they found decision-making much more difficult. Perhaps their hesitancy was partly because the differences were far less self-evident, such as the differences in the probabilities of the two minimum losses. We wonder what the teachers would have done if the dilemma had involved two decisions with similar unquantified minimum losses.

The uncertainty in real world decision-making is further accentuated in practice by concerns about the source of data. P and E regularly referred to these concerns (such



as in lines 0 and 0). In such circumstances, people are likely to go with the judgement of what they see as the highest authority.

It is also important to recognise that the judgements of loss (and to a lesser extent perhaps likelihood) are essentially subjective and so differences in decision across individuals do not necessarily reflect a failure in the priority heuristic but might instead represent differences in individuals' judgements. This is highlighted by E in line 0, when she acknowledges that death might under certain circumstances be preferable to constant pain, and, in line 0 and in the final report, where she refers to the relevance of Deborah's family situation to making such judgements.

## CONCLUSION

The above analysis demonstrates that the two teachers placed highest priority on losses (rather than probabilities) and that they decided according to the lesser of the minimum losses that would be incurred by the two possible decisions. It is also evident that their use of the priority heuristic is not robust. A simple intervention seemed to push them to focus on elements of the heuristic that would normally not have been triggered. Such a lack of robustness is not surprising when the teachers were dealing with a complex scenario with many aspects unquantified and in mutual conflict. Nevertheless, we believe that this uncertainty reflects common scenarios for personal decision-making. Perhaps this is indicated by Cokely and Kelly (2009), when they claim from their experimental evidence that more precise process modelling of risk choices with the priority heuristic would require at least one parameter that creates variation in the search and stopping rules.

We have presented the activity of one pair of teachers in detail. Analysis of the remaining data is continuing. This further analysis might reveal differences across different subject disciplines but any such findings would need to be treated with caution. Already, we see in our analysis reasons to be sceptical about the easy transfer of psychological research findings from straightforward situations to complex scenarios. These initial conjectures will be tested in the ongoing analysis.

Furthermore, the software is a response to another distinction between educational and some psychological research. As educators (rather than psychologists), we need to ask what does this mean for schools, and in particular for mathematics and science classrooms. The above activity demonstrates the complexity of risk-based decision-making, especially in rich scenarios such as DD. The evidence that people avoid trade-offs and apply heuristics such as the priority heuristic is compelling. As educators, should we be satisfied with increasing our understanding of how such decisions are made or should we take this evidence as a pedagogic challenge to find ways to support thinking that engages more explicitly with trade-offs by facilitating the co-ordination of the various dimensions of risk? Perhaps inspired by the progress made in statistics education in supporting students' meanings for randomness and inference (for example: Pratt, 2002; Konold, 2007; Ben-Zvi, 2004), we are aiming to

respond by designing new tools, with the explicit aim of supporting coordination of risk. We imagine tools that can list and order hazards by size of risk, that consolidate harm, likelihood as well as ethical and moral dimensions.

In the latest design of DD, we have incorporated a concept-mapping tool, which was not available at the time of the above activity. The teachers would be encouraged to keep an ongoing map by connecting boxes containing information they have entered about possible hazards. Later, the teachers could press a 'Show Risk' button and the hazard boxes would change colour. Boxes towards the left of the screen would become darker while those to right would become lighter on a continuous scale. The teachers would be told that the darker the hazard, the greater its risk. Inevitably, the teachers would now judge that some of the boxes were in the wrong position on the screen. They would be able to drag the boxes to what they would judge to be the correct relative position according to their estimation of the risks. We conjecture that such tools might provide an educational intervention that would enable teachers, and later non-teachers such as students, to coordinate the dimensions of risk into a single construct with the promise that they might, under certain circumstances, use thinking about trade-offs rather than strategies, such as the priority heuristic.

## REFERENCES

- Adams, J. (1995). *Risk*. London: UCL Press.
- Ben-Zvi, D. (2004). Reasoning about variability in comparing distributions. *Statistics Education Research Journal*, 3.2, 42-63.
- Brandstätter, E., Gigerenzer, G & Hertwig, R. (2006). The priority heuristic: Making choices without trade-offs, *Psychological Review*, 113.2, 409-432.
- Cokely, E. & Kelly, C. M. (2009). Cognitive abilities and superior decision making under risk: A protocol analysis and process model evaluation, *Judgement and Decision Making*, 4.1, 20-33.
- Kahneman, D., Slovic, P. & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.
- Konold, C. Harradine, A & Kazak, S. (2007). Understanding distributions by modelling them, *International Journal of Computers for Mathematical Learning*, 12.3, 217-230.
- Pratt, D. & Noss, R. (2002). The Micro-Evolution of Mathematical Knowledge: The Case of Randomness, *Journal of the Learning Sciences*, 11.4, 453-488.
- Stirling, A. (1999). *On Science and Precaution in the Management of Technological Risk (A synthesis report of case studies)*. Seville: Institute for Prospective Technological Studies (European Commission Joint Research Centre). Retrieved December 18, 2008 from: <ftp://ftp.jrc.es/pub/EURdoc/eur19056en.pdf>.

# INDIVIDUAL PATHWAYS IN THE DEVELOPMENT OF STUDENTS' CONCEPTIONS OF PATTERNS OF CHANCE

Susanne Prediger & Susanne Schnell

Technical University of Dortmund, Germany

*An essential property for distinguishing random from haphazard events is the existence of patterns in the long term. Its inclusion into the individual repertoire of conceptions counts as a prerequisite to developing adequate conceptions of chance and probability. This paper exemplifies results from a teaching experiment designed to investigate 11 to 13 year-old students' individual pathways of constructing, enriching and refining their conceptions of patterns of chance*

Students' individual conceptions of chance and probability have often been investigated empirically. The construction of conceptions that match the underlying stochastic theory (shortly called *intended* or *mathematically-appropriate* conceptions) seems to be a major challenge for stochastic education (e.g. Shaughnessy, 1992) and is deeply influenced by students' initial everyday conceptions (Fishbein, 1975; Konold, 1989). While early conceptualisations of these initial conceptions labelled them as misconceptions (e.g. overview in Shaughnessy, 1992), stochastic education researchers with a constructivist background have taken them seriously as starting points for individual learning processes (e.g. Konold, 1989; Pratt & Noss, 2002) and studied their development. In this tradition, this paper aims at contributing to a deeper understanding of students' individual *pathways of constructing, enriching and refining their conceptions of patterns of chance* as observed in design experiments.<sup>16</sup>

## THEORETICAL BACKGROUND

### **Context-differentiated activation of constructs as an aim for processes of horizontal conceptual change**

The relevance of individual initial (mis-)conceptions for the construction of conceptions has been explained in constructivist terms: individual, active constructions of mental structures always build upon the existing prior mental structures by accommodation to experiences with new phenomena, while the initial structures serve as “both a filter and a catalyst to the acquisition of new ideas” (Confrey, 1990, p. 21). According to the conceptual change approach (Posner et al., 1982; first applied to probability by Konold, 1989), learning thus has to be conceptualised as “re-learning, since prior conceptions and scientific conceptions are often opposed to each other in central aspects” (Duit & von Rhöneck, 1996, p. 158). For many years, conceptual change approaches have (implicitly or explicitly) guided

---

<sup>16</sup> The design experiments are embedded in the long-term project KOSIMA that conducts design research for a complete middle school curriculum (cf. Hußmann, Leuders, Barzel, & Prediger, 2011).

the design of learning situations by providing means to overcome initial conceptions and develop them into intended mathematically appropriate conceptions. These means concern, for example, the relevance of concrete experiences, the confrontation of predictions with real outcomes and the generation of cognitive conflicts (see Posner et al., 1982). However, the far reaching aim of “overcoming” individual prior conceptions in mathematics classrooms that guided early views on conceptual change is not universally applicable. Empirical studies show that it is not always realisable, as individual conceptions often continue to exist next to the new conceptions and are activated situatively (cf. Tyson et al., 1997; for probability e.g. Shaughnessy, 1992; Konold, 1989). Rather than a substitution of initial conceptions, the more adequate aim is the *shift of contexts* in which initial and intended conceptions are to be activated. “Successful students learn to utilize different conceptions in appropriate contexts.” (Tyson et al., 1997, p. 402). Pratt & Noss (2002) emphasise changes in priority between initial and intended conceptions as one pathway of a conceptual change.

Prediger (2008) called this modified perspective on conceptual change with persisting co-existence of initial and intended conceptions a *horizontal view*; in contrast to the *vertical* view on conceptual change, which aims at overcoming initial conceptions. The horizontal view considers students’ initial conceptions as legitimate ideas that can persist if they are weaved into a new framework (similar to Abrahamson & Wilensky, 2007) and can be refined by knowledge of their context-specific scope of validity. Thus, the question guiding the design and analysis of a learning situation for facilitating *horizontal* conceptual change transforms into the following: *How can a learning situation support the extension of individual repertoires of conceptions* (constructing and enriching), *and how can learners be enabled to choose adequate conceptions in varying contexts* (refining and generalising)?

For terminological clarification, we mention that in line with the conceptual change approach, the notion ‘conception’ here refers to all subjective mental structures used by learners to explain their experiences. Conceptions may range on different epistemological levels of complexity from concepts, intuitive rules up to local theories that connect different concepts (Gropengießer, 2001, p.30ff.) and can vary in the degree to which they match the underlying mathematical theory. Although the conceptual change approach is suitable to describe the macro-structures in individual pathways of development of conceptions (see for example Prediger & Rolka, 2009), the fine-grained analysis of micro-structures in the processes of constructing, enriching and refining conceptions require a further operationalization on the micro-level (similarly in diSessa, 1993; Pratt & Noss, 2002). For this purpose, we adopted the notion ‘construct’ as the smallest empirically-identifiable unit of conceptions from Schwarz et al. (2009) and their methodology of reconstructing them by means of three observable *epistemic actions*: Conceptions are seen as webbing of

constructs. An epistemic action of *constructing* is defined as (re-)creating a new knowledge construct by *building with* existing ones. This is identified when a construct is first verbalised or shown by action in the analysed learning situation (although sometimes being constructed before the observed situation). Previous constructs can be *recognized* as relevant for a specific context and used for *building-with* actions in order to achieve a localized goal.

Due to our horizontal view, two major adaptations of the notions were necessary: 1. As we consider idiosyncratic conceptions to be legitimate building blocks, we extended the normatively-guided focus from mathematically (partially) correct constructs (Ron et al., 2010) to all individual constructs, being in line with mathematical conceptions or not. 2. Our descriptions of horizontal learning pathways are mainly focused on the epistemic actions of constructing and required the distinction of two special cases of constructing, namely enriching and refining. A construct is identified to be *enriched*, when a complementary construct is put into relation to it which means there are connections to other constructs identifiable. A construct is said to be *refined*, when it is enriched by conditions of applicability; in our study mostly as narrowing the range of applicable situations from a broad initial one. In other situations, initial constructs are *generalized* and transferred to new contexts (as reconstructed e.g. by Pratt & Noss, 2010, p. 94).

### **of patterns and deviations distinguishing long-term and short-term Conceptions contexts as precondition for context-adequate choices**

The existence of patterns in long series of chance experiments can be identified as a crucial insight for developing adequate conceptions of chance and probability (Prediger, 2008). This focus is strengthened by Moore's definition of random as "phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions" (Moore, 1990, p. 97). This includes the important distinction between short-term and long-term contexts which is central since Konold (1989) described many people's "different understanding of the *goal* in reasoning under uncertainty" (p.61, emphasis added) as an important source of deviant conceptions. Whereas probabilistic conceptions only apply to long-term contexts, many people intend to predict single outcomes of chance experiments in a short-term perspective (Konold, 1989). Deviant conceptions — like betting on numbers that have a specific significance such as birthdays — can be experienced as unsuccessful in long-term contexts, but they prove just as (un)suitable — for single outcomes — as the intended probabilistic conceptions. Therefore, the well-known empirical law of large numbers is crucial for horizontal conceptual change since it explains why one can adopt probabilistic conceptions in a successful way (in long-term contexts), although randomness cannot be predicted for single outcomes (the short-term context). The empirical law of large numbers explains the *sense* and *preconditions*, but also the *limits* of probabilistic considerations and offers thus the conceptual base for a context-adequate choice of conceptions.

Borovcnik (2006) emphasised that the learning process while experimenting with dice etc. is hindered by the fact that chance, and therefore the produced data does not only have *patterns*, but also many *deviations*. That is why students have to include these experiences into their conceptions. Therefore, developing context-adequate probabilistic conceptions does not only include the important shift of attention from short-term contexts to long-term contexts (cf. Pratt & Johnston-Wilder, 2007), but also the construction of conditions *when* regularities are visible: whereas patterns are visible in sufficiently long series of outcomes, they can be disturbed by many outliers in short series, and single outcomes might not conform to an expected pattern at all (see Table 1). In this paper, we describe a case of successful development while constructing, enriching and refining constructs of patterns and their deviations in relation to the context.

## DESIGN OF THE TEACHING EXPERIMENTS

### The learning situation based on ‘Betting King’

To facilitate the differentiation between short-term and long-term contexts in the sense of a horizontal view of conceptual change, a learning situation for 11 to 13 year old students has been designed by Prediger & Hußmann (2012) to provide opportunities for experiences with the empirical law of large numbers. The core element of the learning situation is the board game “Betting King” (Fig. 1), which challenges students to bet on one of four coloured animals in a race. Betting activities refer to making predictions which animal will be the fastest and on which position each animal will end up. The four coloured animals are powered by throws of a coloured 20-sided die (red ant: 7, green frog: 5, yellow snail: 5, blue hedgehog: 3), so that the red ant is theoretically the fastest with a chance of  $7/20$ . Most children quickly notice the red ant to be a *good bet*. Soon, they activate a fruitful ordinal conception of chance, relating the expected order of animals to the number of coloured faces on the die. In this way, the students’ initial resources to link the empirical pattern to the colour distribution are taken into account. Beyond that, the learning situation aims to refine these initial conceptions into an understanding of when this pattern can be predicted more confidently according to the long-term or short-term context. For this purpose, the context attribute “total of throws” is materialised in the game by a STOP sign for the throw counter. By setting the STOP sign for each game, students can deliberately define the total of throws between 1 and 40 for the board game, and between 1 and 10000 throws for the computer simulation (Fig. 2). In order to lead students from

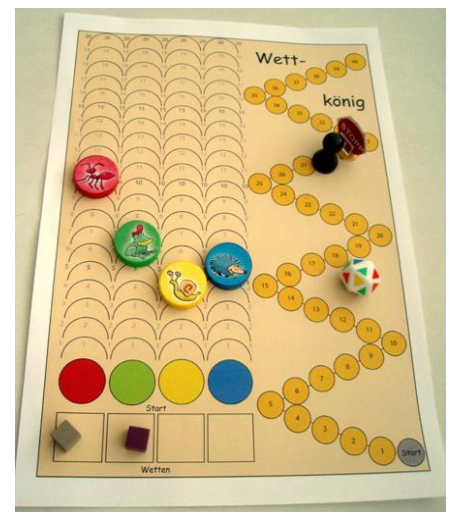


Fig.1 The Game ‘Betting King’

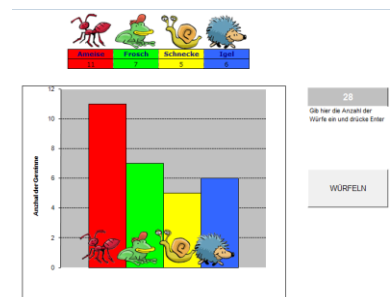


Fig. 2 Screen of corresponding computer simulation

unsystematically playing the game into systematically investigating the situation, protocol sheets guide the collection of game result data for various predefined throw counts (1, 10, 100 and 1000, later 2000). For refining constructs by the conditions of their applicability, it is important to become aware of the role of the total number of throws.

	<b>Short-Term Context:</b> Single games with small total of throws	<b>Long-Term Context 1:</b> Series of games with small total of throws	<b>Long-Term Context II:</b> Series of games with large total of throws
<b>Pattern</b>	<b>S-pattern</b> non-existent	<b>L1-pattern</b> red ant mostly wins, frog & snail are in similar positions, hedgehog loses mostly	<b>L2-pattern</b> red ant always wins, frog & snail are second, hedgehog last
<b>Quality of prediction</b>	<b>S-predictability</b> difficult to bet, but red ant is still the best	<b>L1-predictability</b> red ant is a good bet, but not a secure bet	<b>L2-predictability</b> red ant is a good and secure bet
<b>Relevance of disturbance</b>	<b>S-disturbance</b> some single outcomes completely differ from any expected pattern	<b>L1-deviation</b> pattern difficult to see due to lot of disturbances	<b>L2-deviation</b> pattern strongly visible, still some disturbance
<b>Explanation of appearing pattern</b>	no adequate explanation for the outcome itself	e.g. L1-theoretical explanation L1-empirical explanation	e.g. L2-theoretical explanation L2-empirical explanation L2-law of large numbers

**Table 1: Intended probabilistic constructs on patterns of chance in Betting King**

From a probabilistic point of view, the distinction between different long-term contexts and a short-term context is crucial as was elaborated above. But whereas this distinction oriented our data-guided systematisation of intended constructs in Table 1, students first have to construct this distinction between contexts by themselves. In our learning situation, this construction of differences is facilitated by the following leading question: “Apparently, the red ant is a *good bet*. But when is this bet not only the best bet but also a *mostly secure bet*?” The question challenges students to enrich their conceptions by context-differentiated constructs of predictability which are linked to individual constructs of relevance of disturbance, but in many different ways, as we learned in the ongoing data analysis.

## RESEARCH QUESTIONS AND DESIGN OF THE STUDY

Following the paradigm of design research (Gravemeijer & Cobb, 2006), the learning situation was tested and improved cyclically over three courses of evaluation in six classes (grade 5 and 6, students aged 11 to 13). The empirical analysis of classroom learning processes in Prediger & Rolka (2009) showed that most students could indeed find better and more secure betting strategies and learned to differentiate between long-term and short-term contexts. However, for gaining a deeper understanding on the detailed *processes* of the development of conceptions,

classroom data was too incomplete. For that reason, a further series of teaching experiments was conducted in a laboratory situation by the second author of this paper.

The teaching experiments (following Gravemeijer & Cobb, 2006) based on the presented learning situation were conducted in a series of game interviews with ten couples of students of grade 6 (age 11-13) in a German comprehensive secondary school. The semi-structured interviews of 4x 45-90 minutes were guided by an intervention manual that defined the role of the interview with the attitude of giving as little help as possible but also to provide guidance in situations that were crucial for the continuation of the interview sessions. Each session was videotaped and transcribed in detail for the analysis. The data corpus also included the record of the computer screen and written products. Though the underlying research interest of the ongoing analysis addresses a range of different questions concerning a more detailed description of the processes of conceptual change, this article focuses on the following questions:

26 How are students constructing, enriching and refining constructs for patterns, deviations of patterns and predictability in relation to short- and long-term contexts?

27 Which constructs do students use for explaining these patterns?

## THE CASE OF RAMONA AND SARAH – FIRST RESULTS

The case of Ramona and Sarah exemplifies how relating and enriching different constructs can provide students with a tool to make sense of the different observations of patterns of chance in relation the specific short-term or long-term context. This case was chosen as the girls show a broad range of constructs and are highly able to verbalize their ideas. Due to limited space, comparisons with other couples are restricted to the concluding remarks.

### Episode 1: Refining by differentiating the L1-context from S-context

When introduced to the learning environment, Ramona and Sarah are eager to find a strategy to win as often as possible. For this reason, they keep looking for patterns in single throws of the die or in results of games. Episode 1 below starts after 15 minutes of playing. All four games so far with totals of throws between 25 and 37 have been won by the red ant, with the first game tied with the green frog. Having the outcome of the fourth game on the board as documented in Fig. 1 (red ant on 11, frog on 7, snail on 5, blue hedgehog on 6), the students express their ideas on the found patterns.

- |     |        |  |
|-----|--------|--|
| 422 | Ramona | <i>(points to red ant on the board)</i> This one is the fastest. Then, the hedgehog should come, then [green] frog, then [yellow] snail <i>(points to animals)</i> |
| 423 | Sarah  | Why?   |



## Working Group 5

424 Ramona I don't know, because the- the ant has won almost every time so far.

As a first construct, Ramona describes in line 424 the pattern that the red ant wins more often (L1-pattern winning ant) and relates this to her empirical observation (L1- empirical explanation). She apparently refers to the series of four games by expressing “almost every time so far” in 424. In the (not printed) turns following the above episode, Sarah tries to find an explanation for the empirical pattern and comes up with the idea that not all faces of the die are equal, which prompts Ramona to count. After counting twice, they find the correct colour distribution of 7,5,5,3.

481 Interviewer Now you have counted [all colours on the die]. What does that mean?

482a Sarah That red, well, more- well, that red wins actually, because it has more and then you get it more often, when you throw the die. And then green and yellow, because they-

482b Sarah Well, you two- That is why they are again so- Green and yellow (*points on yellow snail and then blue hedgehog on the board*)

482c Sarah Eh, green and yellow (*points to yellow snail and then green frog, then to both simultaneously*) are sometimes far apart, but.

483 Ramona Blue has good chances, too, because-

484 Sarah Yes.

485 Ramona You also have- blue has sometimes a lot of luck and then it gets the three faces sometimes very often.

486 Sarah You see it here (*points on the board to snail and blue hedgehog*).

In 482a, Sarah enriches the pattern-construct that was so far only empirically explained with an additional theoretical explanation of the colour distribution (L1-theoretical explanation). While the observation and also the empirical explanation of the pattern of the red ant as best animal come from a *series* of games (with totals between 25 and 37, L1-context), she switches in 482b to the *single* result of the game that is still displayed in front of her (see Fig.1) and tries to transfer the L1-pattern to the single game. By pointing to the board, she is possibly trying to demonstrate the theoretically expected pattern, but her use of half sentences and her pointing to the wrong animals in 482b seem to indicate that she is experiencing a conflict between the deviant S-pattern and the expected L1-pattern. In this moment, the constructed L1-pattern is possibly already starting to get refined implicitly as Sarah experiences a problem in its scope of applicability for the single short game. In 482c, she corrects herself by pointing to green frog and yellow snail, but seems not to be describing a pattern anymore, as she uses the term “sometimes” (S-deviation). Sarah seemingly does not solve the problem between L1-pattern and the deviant S-pattern here, as she ends her sentence with a “but” in 482c, even though she is not interrupted.

### Episode 2: Constructing luck as S-explanation for deviation

Ramona expresses in 483 to 485 a new construct that had not been mentioned before. She explains this situation that differs from the L1-pattern by the “luck” that the blue hedgehog must have had (S-explanation for the deviation). Keeping the term

“sometimes”, she is seemingly still speaking about single outcomes as opposed to a series. Sarah concurs with this explanation by demonstrating it on the board. Here, the girls seem to have found an explanation by excluding this and possibly other single outcomes from the scope of applicability of the L1-pattern and therefore making the difference between short-term and long-term context explicit. Still, the construct of luck is only brought up in relation with the notion of the distribution of colours.

### Episode 3: Building with the luck-construct for S-explanation for deviation

Over the course of all interviews, they again build with this construct to explain single outcomes of games being not in accordance with the theoretically expected pattern. One example is Episode 3 (about 35 min. later). So far, Ramona and Sarah have filled in several protocol sheets while playing more than 25 further games with a total of throws between 1 and 20 and have written down their strategy for betting. The interviewer’s question leads Ramona to clarify the distinction between pattern and luck further:

- 1203 Interviewer Could you read out loud what you have written, Sarah?  
 1204 Sarah Always stay on the ant-(...)  
 1206 Sarah As it has the most faces on the die and therefore you roll it more often.  
 1207 Interviewer Hm, you put that very well. What I don’t get completely yet: I bet on the hedgehog and won, for example. Or – well, not only ant has won-  
 1208 Ramona That is just luck.  
 1209 Interviewer It’s only luck?  
 1210 Ramona It is not a strategy, it is truly luck.

Here, the previously constructed S-explanation for deviation is *recognized* as being usable in a situation, in which the interviewer seems to point to single games. By emphasising the difference between luck and strategy, Ramona *builds with* it by referring it to the unpredictability of the single (lucky) outcomes in single short games (S-Prediction) and the more predictable L1-pattern (L1-predictability). This contributes to refining the distinction of S- and L1-context.

### Episode 4: Constructing the L1-L2-distinction

In the second interview, Ramona and Sarah start to focus on the long-term context L2 of games with high totals of throws, which is supported by using the computer simulation and protocol sheets that include total of throws up to 1000. Ramona and Sarah address the question, when the red ant is a good bet without an interviewer’s stimulus. Having filled in a protocol sheet and a series of 16 games with increasing totals of throws, they realize that their consequent bet on ant has won the first game, lost for the next four and won every game from the sixth one on (with totals of throws of 10, 100 and 1000):

- 975 Sarah (*points to sixth game on the protocol sheet; total of throws: 10*) From here on, you only always win with the ant.

This utterance could be an indication that she is constructing a notion of the predictability of the pattern ant-winning in relation to the context (distinguishing L2-predictability from L1-predictability). Although not marking exactly those games with at least 100 throws, her formulation “from here on” clearly addresses a series of games and seems vaguely to refer to the larger total of throws as they increase in the bottom of the sheet. While filling in a summary sheet, the girls become aware of their results showing clear patterns: If the total of throws was one or ten, all animals won, while the red ant was the only winner in all games with throw totals of 100 and 1000. Asked to formulate their strategy now, the following dialogue begins:

1079 Sarah At 100 and 1000, the ant always wins. At 10 and 1, it's always different-

...

1085 Ramona [At 10 and 1], mostly winning are-

1086 Sarah (*points to upper part of protocol sheet*) snail, frog and sometimes ant, too.

1087 Ramona There, ant is not winning as often and here (*points to lower part of protocol sheet*) you can see it, only ant.

The girls refine their construct of L2-pattern by contrasting it to the L1-pattern. Both seem to accept that ant is the only one winning in long games which is in accordance with Sarah's statement in 975. Referring to the series of short games (i.e. the L1-context), Sarah revises her previous statement and remarks in 1079, that at a throw total of one or ten “it's always different”. It is possible that she emphasises the distinction between the L1 and L2 context and focuses the L1-deviation more than the L1-pattern itself. Furthermore, she might relate the absence of a pattern in L1 to the previous construct S-disturbance in the context of short games, which was then explained by the construct “luck”. Here, both girls do not mention luck as a possible explanation, but point out the shift of context as the explanation for the discrepancies in the observations of patterns (L1-explanation). When Ramona starts to mention winning animals in 1085, Sarah points out three of four animals, relativising the red ant by adding the adverb “sometimes”. In contrast to 1079, she is possibly now pointing out a L1-pattern, which is refined by Ramona in 1086. She emphasises the words “as often” and “always” and makes the distinction between her construct for L2-pattern (ant wins always) and L1-pattern (ant wins sometimes) very explicit. Furthermore, she hereby constructs the notion of both L1-deviation and L2-deviation.

## CONCLUSION

Like the girls' pathway of developing conceptions, all ten interview-pairs create complex networks of constructs while trying to make sense of several, partially-conflicting experiences. In each case, a shift of focus between short-term and long-term context can be reconstructed. Beyond that, the individual pathways are highly individualised.

Ramona and Sarah are able to enrich pattern constructs with explanations not only in a long-term perspective, but also refine these patterns regarding the absence of patterns in a series and single outliers in short-term contexts. Their individual constructs “luck” and “pattern” seem not only to be connected to each other, but also to the distribution of colours. The deviation of patterns is only mentioned in relation with “pattern” and while mentioning explicitly the total of throws as being low. Though the girls don’t compare this whole network of constructs and test its coherence, it seems from an outside point of view that by defining the scopes of applicability, their constructs are not contradictory, but in coexistence with each other. This gives evidence to the horizontal view on conceptual change and provides a short but deep insight into how the individual pathways of students can lead into conceptions consisting of networks of constructs, in which even rather idiosyncratic constructs such as luck have a scope of applicability that does not seem to obstruct the intended mathematical constructs. For some students, the negotiation of ranges of applicability of constructs is more complicated than for Ramona and Sarah. Further steps of data analysis include the identification of common conceptions for many participants and sharpen the description of the character of the network of constructs.

## REFERENCES

- Abrahamson, D. & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning*, 12, 23-55.
- Borovcnik, M. (2006). Probabilistic and statistical thinking. In M. Perpinan & M.A. Portabella (Eds.), *Proceedings of CERME 4* (pp. 484-506). Sant Feliu de Guixols, Spain: ERME.
- Confrey, J. (1990). A Review of the Research on Student Conceptions in Mathematics, Science and Programming. *Review of Research in Education*, 16(1), 3-56.
- diSessa, A. (1993). Towards an epistemology of physics. *Cognition and Instruction*, 10 (2–3), 105-225.
- Duit, R. & von Rhöneck, C. (1996). *Lernen in den Naturwissenschaften*. Kiel: IPN.
- Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Reidel: Dordrecht.
- Gravemeijer, K. & Cobb, P. (2006). Design research from the learning design perspective. In J. van den Akker et al. (Eds.), *Educational Design Research* (pp. 45-85). London: Routledge.
- Gropengießer, H. (2001). *Didaktische Rekonstruktion des Sehens. Wissenschaftliche Theorien und die Sicht der Schüler in der Perspektive der Vermittlung*. Oldenburg: DIZ.
- Hußmann, S., Leuders, T., Prediger, S. & Barzel, B. (2011, in press). Kontexte für

## Working Group 5

sinnstiftendes Mathematiklernen (KOSIMA) – ein fachdidaktisches Forschungs- und Entwicklungsprojekt. In: *Beiträge zum Mathematikunterricht 2011*.

- Konold, C. (1989). Informal Conceptions of Probability. *Cognition and Instruction*, 6, 59-98.
- Moore, David S. (1990). Uncertainty. In L. Steen, *On the shoulders of giants: New approaches to numeracy* (p. 95-137). Washington, DC: National Academy Press.
- Posner, G., Strike, K., Hewson, P.W. & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, 66(2), 211-227.
- Pratt, D. & Johnston-Wilder, P. (2007). The relationship between local and global perspectives on randomness. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the CERME 5* (pp. 742-752). Larnaca, Cyprus. Retrieved from <http://ermeweb.free.fr/CERME5b/WG5.pdf>
- Pratt, D. & Noss, R. (2002). The Micro-Evolution of Mathematical Knowledge: The Case of Randomness. *Journal of the Learning Sciences*, 11(4), 453-488.
- Pratt, D. & Noss, R. (2010). Designing for Abstraction. *International Journal of computers for mathematical learning*, 15(2), 81-97.
- Prediger, S. (2008). Do you want me to do it with probability or with my normal thinking? Horizontal and vertical views on the formation of stochastic conceptions. *International Electronic Journal of Mathematics Education*, 3(3), 126-154.
- Prediger, S. & Rolka, K. (2009). Using betting games for initiating conceptual change. *Asian Journal of Educational Research and Synergy*, 1(1), 61-71.
- Prediger, S. & Hußmann, S. (2012, in press). Spielen – Wetten – Voraussagen. Den Zufall im Griff?. To be published in S. Prediger, B. Barzel, S. Hußmann & T. Leuders (Eds.), *Mathewerkstatt 6*. Berlin: Cornelsen.
- Ron, G., Dreyfus, T. & Hershkowitz, R. (2010). Partially correct constructs illuminate students' inconsistent answers. *Educational Studies in Mathematics*, 75(1), 65-87.
- Schwarz, B., Dreyfus, T. & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In *ibid.* (Eds.), *Transformation of Knowledge through Classroom Interaction* (pp. 11-41). London & New York: Routledge.
- Shaughnessy, J. M. (1992). Research in probability and statistics. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 465-494). New York: MacMillan.
- Tyson, L.M., Venville, G.J., Harrison, A.G. & Treagust, D.F. (1997). A multi-dimensional framework for interpreting conceptual change in the classroom. *Science Education*, 81(4), 387-404.

# THE ROLE OF RELEVANT KNOWLEDGE AND COGNITIVE ABILITY IN GAMBLER FALLACY

Caterina Primi & Francesca Chiesi

Department of Psychology – University of Florence, Italy

*A new model by Stanovich et al. (2008) specifies the ways in which knowledge and cognitive capacity might interact in shaping reasoning performance. This model proposes that normative performance relies on knowing relevant rules and procedures (called “mindware”), detecting the need to implement them, and holding of the necessary cognitive capacity to acquire/use them appropriately. The aim of the present study was to test these assumptions investigating gambler fallacy inside probabilistic reasoning. Participants were primary (N=251) and college students (N=151). Results provide support for the claim that mindware plays an important role in probabilistic reasoning, and there is an interplay with cognitive ability. Theoretical and educational implications of results are discussed.*

## INTRODUCTION

According to dual-process theories, mental functioning can be characterized by two different types of process which have different strengths and weaknesses (e.g., Brainerd & Reyna, 2001; Epstein, 1994; Evans 2006; Sloman, 1996; Stanovich, 1999). Type 1 processes<sup>17</sup> are rapid and mandatory when the triggering stimuli are encountered, they do not require much cognitive effort, and they can operate in parallel. Type 1 processing is cognitively economical, its output is not consciously generated but seems to “pop” into consciousness (Sloman, 1996), and people “feel” intuitively that the responses are right (Epstein, 1994). Whereas Type 1 processing often leads to correct responses, in some cases they lead to systematic biases and errors. By contrast, Type 2 processes are relatively slow and computationally expensive, they are available for conscious awareness, serial, and often language based. Type 2 processes are also often associated with the use of normative rules and logical responding.

To exemplify the role of the two types of process in reasoning, imagine that in order to win a prize you have to pick a red marble from one of two urns (A and B). Urn A contains 20 red and 80 blue marbles, and Urn B contains 1 red and 9 blue marbles. When you respond to the task, you can compare the ratio of winning marbles in each urn (20% vs. 10%) which requires some time, mental effort and computations, or you can simply rely on the feeling/intuition that it is preferable to pick from the urn with more “favourable events”. In this example, both processes cue the normatively correct answer (that is, Urn A). On the other hand, it is possible to set up a task

---

<sup>17</sup> Several terms have been used to refer to these two aspects of cognitive functioning (e.g. heuristic vs analytic, experiential vs rational), here we follow Evans (2006) in using the terms Type 1 and Type 2 processes.

where Type 1 and Type 2 reasoning cue different responses. For example, if you can choose between picking a marble from an urn containing 10 red and 90 blue marbles, or from an urn containing 2 red and 8 blue marbles, the feeling/intuition that it is preferable to pick from the urn with more “favourable events” results in a normatively incorrect choice.

When Type 1 and Type 2 processes do not produce the same output, Type 1 processes usually cue responses that are theoretically incorrect and, according to dual-process theorists one of the most critical functions of Type 2 processes in these cases is to interrupt and override Type 1 processing. However, this does not always happen. In the case of a conflict between intuitions and normative rules even educated adults will predominantly produce heuristic responses.

In a recent paper Stanovich, Toplak and West (2008) outlined how people can reach a correct solution when the task besides the normative solution elicits competing response options that are intuitively compelling. They stated that people have to possess the relevant rules, procedures, and strategies, they have to recognise the need to use them, and then they have to have the necessary cognitive capacity to inhibit competing responses. In their model of reasoning, Stanovich and colleagues (2008) referred to rules, procedures, and strategies derived from past learning experiences as “mindware” (Perkins, 1995). If the relevant mindware can be retrieved and used, alternative responses became available to engage in the override of the intuitive compelling answers.

Errors can arise when we have a *mindware gap*. Indeed, when relevant knowledge, procedures, and strategies are not available, i.e. they are not learned (or poorly compiled), we can not have an override since to override the intuitive response a different response is needed as a substitute. Instead, when the relevant knowledge, procedures, and strategies can be easily retrieved, and a normative solution becomes available, errors are termed *override failures*: different alternatives are produced and there is the attempt to take the intuitive response offline, but this attempt fails since beliefs, feelings and impressions seem to be right beside rule-based considerations. So we have an override failure when people hold the rule but they do not base their answer on it.

Finally, Stanovich and colleagues (2008) addressed the role of several factors that might affect reasoning and, among them, particular attention was paid to cognitive ability. Kahneman and Frederick (2002) pointed out that “intelligent people are more likely to possess the relevant logical rules and also to recognize the applicability of these rules in particular situations [...] that enable them to overcome erroneous intuitions when adequate information are available.” Thus, in both children and adults, reasoning errors are expected to be related to cognitive ability (Evans, Handley, Neilens, & Over, 2009; Kokis, MacPherson, Toplak, West & Stanovich, 2002; Morsanyi & Handley, 2008).

Starting from these premises, the aim of the present study was to test the Stanovich and colleagues' model inside probabilistic reasoning investigating gambler fallacy (Kahneman, Slovic & Tversky, 1982). Indeed, the model of Stanovich and colleagues provides a theoretical framework for integrating the educational and dual-process approaches emphasizing the role of both relevant knowledge and cognitive capacity in the development of reasoning skills. As the rules of probabilistic reasoning are very hard to derive from personal experiences (e.g., Fischbein, 1987) – that is, the actual patterns of probabilistic outcomes are “messy” or even resemble more what could be predicted based on the fallacies of probabilistic reasoning than on the relevant normative rules (see Hahn & Warren, 2009) - normative probabilistic knowledge mostly stem from what learned at school.

In details, we aimed (a) to investigate when sound probabilistic reasoning could be prevented by the lack of relevant knowledge (that we call “mindware” following Stanovich et al.'s terminology) for dealing with probability comparing different educational levels, and (b) to take into account the role of individual differences in cognitive ability and the interactions between mindware and cognitive capacity.

Committing gambler fallacy, people tend to estimate the likelihood of an event by taking into account how well it represents its parent population, i.e. a sequence of the same outcome (given two possible options) must be followed by the other outcome in order to equilibrate the proportion. In this way they do not take into account base-rates along with the independence notion. In the present study gambler fallacy was investigated in primary students since these basics of probability are taught to the fourth and fifth graders following the Italian national curricular programs<sup>18</sup>. Then, we compared primary school students probabilistic reasoning to college students in order to better explore the role of mindware starting from the assumption that relevant knowledge should be consolidated through education as well as the ability to recognize the need to use it in specific situations. In sum, we included three groups: students before they were taught probability issues (third graders), students who had been taught probability issues (fifth graders) and college students who had encountered issues related to probability throughout primary to high school years.

We predicted that probabilistic reasoning performance strongly relied on relevant mindware. We expected that younger primary students should perform worse due to their *mindware gap*. This difference should be observed even when individual differences in cognitive ability are partialled out. We predicted that college students

---

<sup>18</sup> Specifically, curricula include statistical surveys and their representations, some linguistic/conceptual issues related to possible, impossible, improbable events, and the development of judgment under uncertainty and estimation of odds through games of chance, inside the classical definition of probability (see <http://www.rhoda.it/programm.htm>). These issues were firstly included in the 1985 reform, and more recently the importance of teaching these topics was stressed in the revised government program (Moratti, 2004).



would generally perform better due to their more consolidated knowledge of the relevant rules of probability, and their higher cognitive ability. Whereas no difference should be observed among older primary students and college students once individual differences in cognitive ability are taken into account.

## METHOD

### Participants

The participants were 251 primary school students attending Grade 3 ( $n=133$ , 68 males; mean age: 8.3 yrs) and Grade 5 ( $n=118$ , 65 males; mean age: 10.5 yrs) and 151 college students (30 males; mean age: 20.3 yrs). The primary schools students were enrolled in Italian primary schools that serve families from lower middle to middle socioeconomic classes. Primary students were invited to participate. Their parents were given information about the study and their permission was requested. The college students were all students in Psychology degree program at the University of Florence (Italy). They were volunteers, and did not receive any reward for their participation in this study.

### Measures and Procedure

*Gambler Fallacy Task:* A preliminary version of this task was used in a previous study run with children and college students (Chiesi & Primi, 2009). It consists in a marble bag game in which different base-rates in combination with two different sequences of outcomes were used. Using marbles - compared to the tossing of a regular coin traditionally employed to test gambler fallacy – the present task allows for testing this bias with both equally likely and not equally likely proportions. In detail, the task was composed of 3 different trials in which proportion of Blue (B) and Green (G) marbles varied (15B & 15G; 10B & 20G; 25B & 5G).

Before the actual task was presented, all children were shown a video in order to exemplify the concept of sampling with replacement. The bag shown in the video has a see-through part and instead of drawing a marble from the bag, the marble is pushed into the corner and then moved back inside the bag. Since the bag is closed it's a way to make clear that the number of the marbles stays always the same.

After the video, each participant received a sheet where it was written the following instruction: “15 blue and 15 green marbles have been put into a bag such as the one shown in the video and one ball has been pushed in the see-through part” The first question was “It's more likely it is...”. The following instruction explained that: “The game was repeated with same bag and a sequence of 5 green marble was obtained”. The second question was: “The next one is more likely it is....”. The following instruction told that: “The game was repeated again with same bag and a sequence of 5 blue marble was obtained”. The third question was: “The next one is more likely it is....”. After this first trial, the other two trials with different marbles proportion were presented, and for each one the three questions were asked. In sum, each participant had to answer 9 questions, three for each trial.

We formed two composite scores summing correct answers. One represents the necessary knowledge to tackle the task, i.e. how the probability of a single event changes referring to base-rates, and it was called Mindware score (range 0-3). The other, represents normative reasoning, i.e. higher the score, higher the respondent's ability to avoid gambler fallacy, and it was called Probabilistic Reasoning score (range 0-6).

After the Gambler Fallacy task, cognitive ability was measured using two short forms of the Raven's Matrices, one suitable for children, the other for adults.

*Set I of the Advanced Progressive Matrices (APM-SET I):* To measure children's cognitive abilities the APM-SET I (Raven, 1962) was administered as a short form of the Raven's Standard Progressive Matrices (Raven, 1941) as suggested by Nathaniel-James et al. (2004). The Set I of APM is composed by 12 matrices increasing in their difficulty level, and the items covered the range of difficulty of SPM (Raven, 1962). These items are composed of a series of perceptual analytic reasoning problems, each in the form of a matrix. The problems involve both horizontal and vertical transformation: figures may increase or decrease in size, and elements may be added or subtracted, flipped, rotated, or show other progressive changes in the pattern. In each case, the lower right corner of the matrix is missing and the participant's task is to determine which of eight possible alternatives fits into the missing space such that row and column rules are satisfied. Test adaptation to the Italian children population was done using IRT analysis procedure (Ciancaleoni, Primi & Chiesi, 2010).

*Advanced Progressive Matrices Short Form (APM-SF, Arthur & Day, 1994):* College students were administered the Advanced Progressive Matrices Short Form (Arthur & Day, 1994). The APM-SF is composed by 12 matrices derived from the APM. Matrices characteristic are described above. Test adaptation was done using IRT analysis procedure (Primi, Galli, Ciancaleoni, & Chiesi, 2010).

## RESULTS

As expected, a differences between Grade 3 and Grade 5 was found in Mindware ( $t(247) = -4.3, p < .001, d = -.55$ ) and Probabilistic Reasoning ( $t(247) = -4.92, p < .001, d = -.62$ ). Older children performed better (Mindware:  $M = 1.90; SD = .86$ ; Probabilistic Reasoning:  $M = 2.83; SD = 1.52$ ) than younger children (Mindware:  $M = 1.45; SD = .80$ ; Probabilistic Reasoning:  $M = 2.07; SD = .84$ ).

In order to control the effect of cognitive ability, two ANCOVAs were run in which Raven Matrices score was used as a covariate, Grade as the independent factor, and Mindware and Probabilistic Reasoning score as the dependent variables. The results showed that once the significant effect of cognitive ability was partialled out ( $F(1,246) = 15.16, p < .001, \eta_p^2 = .06$ ), the main effect of educational level on Mindware was still significant ( $F(1,246) = 6.25, p < .01, \eta_p^2 = .03$ ). In the same way, once the significant effect of cognitive ability was partialled out ( $F(1,246) = 31.24, p < .001$ ,

$\eta_p^2=.13$ ), the main effect of educational level on Probabilistic reasoning remained significant ( $F(1,246)=6.53, p<.01, \eta_p^2=.03$ ).

Starting from these results, we aimed to identify the relative weight of the two factors related to correct reasoning, that is cognitive ability and relevant knowledge. So, we conducted a hierarchical regression analysis - separately for third graders, fifth graders and college students - in which the criterion variable was the Probabilistic Reasoning, and predictors were Cognitive Ability, entered first into the analysis, and Mindware.

<b>Third Grade</b>				
Step		Multiple R	$R^2$ Change	$F$ Change
1	<b>COGNITIVE ABILITY</b>	0.03	/	4.64*
2	<b>MINDWARE</b>	0.00	0.00	<i>ns</i>
<b>Fifth Grade</b>				
1	<b>COGNITIVE ABILITY</b>	0.19	/	26.24**
2	<b>MINDWARE</b>	0.30	0.11	18.27**
<b>College</b>				
1	<b>COGNITIVE ABILITY</b>	0.05	/	4.31*
2	<b>MINDWARE</b>	0.47	0.42	68.14*

\* $p < .05$ , \*\* $p < .01$

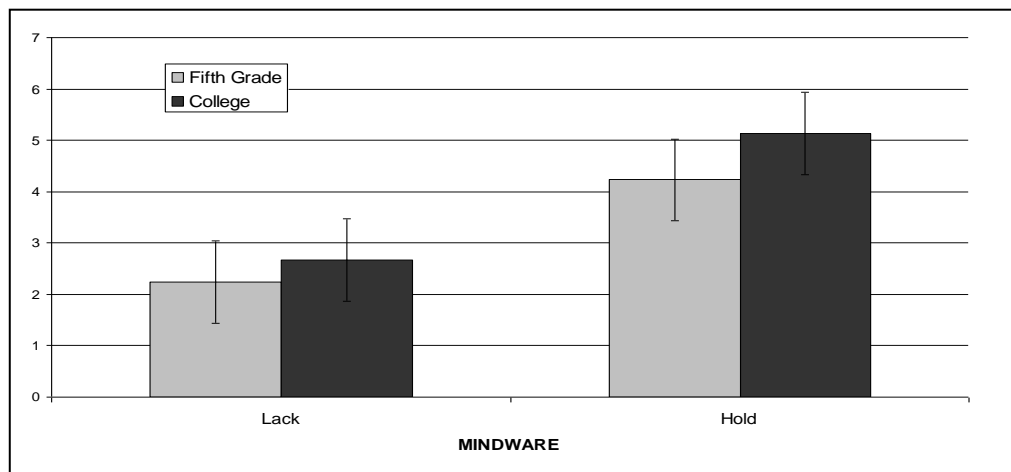
**Table 1: Hierarchical regression for each student group with Cognitive Ability and Mindware entered as predictors of the Probabilistic Reasoning score.**

Different patterns of results were observed for the three groups (Table 1). In third graders probabilistic reasoning was totally unrelated from mindware, and cognitive ability accounted for a little part of it (less than 5%). In fifth graders both cognitive ability and relevant mindware predicted probabilistic reasoning: cognitive ability accounted for 19% of the variance, and mindware accounted for an additional 11%. Finally, mindware explained in large part college students' reasoning accounting for the 42% of the variance.

Results show that both cognitive ability and relevant mindware would lead to increase students' reasoning performance and the relative weight of the two factors depend on educational level. Moreover, we can argue that students who correctly answered the first question not only hold the relevant mindware but the question makes them aware of the need to use it in answering the following questions. In other word, mindware was retrieved and made available to substitute the compelling intuitive response likely elicited by the sequence of five identical outcomes (i.e., all green/blue marbles).

In order to further ascertain the role of mindware on probabilistic reasoning, we compared who hold a well-learned rule and who lack or hold a poorly compiled rule. To do that, we created two groups (Hold vs Lack) using as a criterion the maximum

Mindware score (that is, who always answer correctly, even when the correct answer was “blue and green are equally likely” and not a dichotomous choice between blue and green). In this way third graders were excluded from the analysis since only few students (as expected) were found to belong to Hold group. A 2X2 ANCOVA was run in which cognitive ability was used as a covariate, Educational Level (Fifth Grade vs College) and Mindware (Lack vs Hold) as independent variables, and Probabilistic reasoning as dependent variable. The results showed that once the significant effect of cognitive ability was partialled out ( $F(1,199)=15.58$ ,  $p<.001$ ,  $\eta_p^2=.07$ ), the main effects of Educational Level ( $F(1,199)=5.03$ ,  $p<.05$ ,  $\eta_p^2=.03$ ) and Mindware ( $F(1,199)=113.07$ ,  $p<.001$ ,  $\eta_p^2=.36$ ) were significant, as well as the interaction between them ( $F(1,199)=4.54$ ,  $p<.05$ ,  $\eta_p^2=.02$ ) (Figure 1). Looking at the effect sizes, we can observe that the stronger difference depends on mindware, i.e., in both groups students had a low performance when they lack, or lack to retrieve and apply, the relevant mindware.



**Figure 1: Means of Probabilistic Reasoning score in function of Mindware and Educational Level.**

## CONCLUSION

In this work we investigated the effects of relevant knowledge on primary and college students' probabilistic reasoning ability, and we examined the interactions between relevant knowledge and cognitive ability. According to Stanovich et al. (2008), the present study suggests that in solving gambler fallacy tasks the correct solution can be reached holding the relevant mindware and recognising the need to use it. Moreover, we found that individual differences in cognitive ability can be accounted for explaining sound reasoning in both primary and college students but, once the effect of cognitive ability has been taken into account, if the relevant knowledge is hold, retrieved and applied primary and college students perform equally well. In the same way, when they do not possess it or use it, their performance is equally bad.

In sum, correct probabilistic reasoning relies strongly on knowledge about rules. Since these rules are very hard to derive from personal experiences, we may conclude that normative probabilistic reasoning mostly stem from what learned at school. In this way, it becomes relevant define methods to make students aware of the need for rules even when they “feel” that these rules do not work, that is when conclusions derived from the theory are counterintuitive.

This study offers some cues to cross the bridge from a psychological approach to an educational approach. Psychological theories on reasoning assert that people are prone to rely on intuitions and that in dealing with probability intuitions seem to be right beside rule-based consideration. Nonetheless, rules are needed to reason normatively and to avoid biases. Didactical interventions have to focus on solving this puzzle.

## REFERENCES

- Arthur, W., & Day, D. (1994). Development of a short form for the Raven Advanced Progressive Matrices test. *Educational and Psychological Measurement*, 54, 395-403.
- Brainerd, C. J., & Reyna, V. F. (2001). Fuzzy-trace theory: Dual processes in memory, reasoning, and cognitive neuroscience. *Advances in Child Development and Behavior*, 28, 49-100.
- Chiesi, F. & Primi, C. (2009). Recency effects in primary-age children and college students using a gaming situation. *International Electronic Journal of Mathematics Education, Special issue on “Research and Development in Probability Education”*, 4, 3, [www.iejme.com](http://www.iejme.com).
- Ciancaleoni, M., Primi, C. & Chiesi, F. (2010). Exploring the Possibility to Use Set I of Advanced Progressive Matrices As a Short Form of Standard Progressive Matrices. *Testing, Psychometrics, Methodology in Applied Psychology, TPM*, 17, 2, 91-98.
- Epstein, S. (1994). An integration of the cognitive and psychodynamic unconscious. *American Psychologist*, 49, 709-724.
- Evans, J. St. B. T. (2006). The heuristic-analytic theory of reasoning: extension and evaluation. *Psychological Bulletin*, 13 (3), 378-395.
- Evans, J. St. B. T., Handley, S.J., Neilens, H., & Over, D. E. (2009). The influence of cognitive ability and instructional set on causal conditional inference. *Quarterly Journal of Experimental Psychology*, 63, 892-909.
- Fischbein, E. (1987). *Intuition in Science and Mathematics*. Dordrecht, The Netherlands: Reidel.
- Hahn, U., & Warren, P.A. (2009). Perceptions of randomness: Why three heads are better than four? *Psychological Review*, 116, 454-461.

- Kahneman, D., & Frederick, S. (2002). Representativeness revisited: attribute substitution in intuitive judgement. In *Heuristics and Biases: The Psychology of Intuitive Judgment*, ed. T Gilovich, D. Griffin, D Kahneman, (pp. 49-81). New York: Cambridge University Press.
- Khaneman, D., Slovic, P., & Tversky, A. (1982). *Judgment under uncertainty: Heuristics and biases*. New York: Cambridge University Press.
- Kokis, J. V., MacPherson, R., Toplak, M. E., West, R. F., & Stanovich, K. E. (2002). Heuristic and analytic processing: age trends and associations with cognitive ability and cognitive styles. *Journal of Experimental Child Psychology*, 83, 26-52.
- Moratti, L. (2004). Decreto Legislativo 59/04 applicativo della Riforma Moratti, Supplemento Ordinario, n. 31, *Gazzetta Ufficiale*, n. 51 (2.03.2004).
- Morsanyi, K., Handley, S.J. (2008). How smart do you need to be to get it wrong? The role of cognitive capacity in the development of heuristic-based judgment. *Journal of Experimental Child Psychology*, 99, 18-36.
- Nathaniel-James, D. A., Brown, R. G., Maier, M., Mellers, J., Toone, B., & Ron, M. A. (2004). Cognitive abnormalities in schizophrenia and schizophrenia-like psychosis of epilepsy. *Journal of the Neuropsychiatry and Clinical Neurosciences*, 16, 472-479.
- Perkins, D. N. (1995). *Outsmarting IQ: The emerging science of learnable intelligence*. New York: Free Press.
- Primi, C., Galli, S., Ciancaleoni, M. & Chiesi, F. (2010, July). *Applying IRT to the Advanced Progressive Matrices –Short Form*. Paper presented at the 7th International Test Commission, Hong Kong.
- Raven, J. C. (1941). Standardization of progressive matrices. *British Journal of Medical Psychology*, 19, 137-150.
- Raven, J. C. (1962). *Advanced progressive matrices*. London: Lewis & Co. Ltd.
- Sloman, S. A. (1996). The empirical case for two systems of reasoning. *Psychological Bulletin*, 119, 3-22.
- Stanovich, K. E. (1999). *Who is Rational? Studies of Individual Differences in Reasoning*. Mahwah, NJ: Erlbaum.
- Stanovich, K. E., Toplak, M. E., & West, R. F. (2008). The development of rational thought: A taxonomy of heuristics and biases. *Advances in child development and behavior*, 36, 251-285.

# CONNECTING EXPERIMENTAL AND THEORETICAL PERSPECTIVES

Theodosia Prodromou

University of New England, Australia

*This paper presents a case study of a group of pre-service teachers (age 21- 52) as they work in a domain of stochastic abstraction to reason about “experimental” and “theoretical” perspectives. I am particularly interested in investigating whether pre-service teachers could construct a bidirectional link between the data-centric and modelling perspectives on distribution, similar to the tentative model I introduced elsewhere for coordinating the two perspectives on distribution. In this study, we have seen echoes of these ideas in relation to experimental and theoretical probabilities. The results show students’ movement between probabilities at the micro level and the shape of histograms at the macro level.*

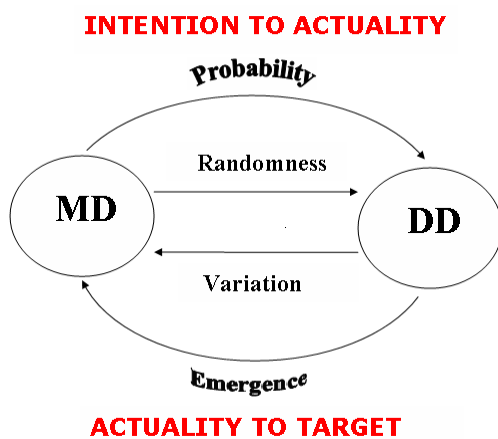
## INTRODUCTION

The concept of probability distribution lies at the heart of statistics at university level (Cohen & Chechile, 1997). Students at university level are expected to have a sophisticated understanding of theoretical statistical principles. This requires an appreciation of probability theory, an ability to describe the variation due to sampling and calculate and judge the variation pattern of a random variable and its respective probabilities.

Konold and Kazak (2008) claim that there are four fundamental ideas which are central to the domain of data and chance: model fit, distribution, signal-noise, and the Law of Large numbers. Model Fit involves the development of expectations about certain characteristics of the data with regard to the possible kind of a “model” that the data follows (Konold & Kazak, 2008). For instance, if we examine the outcomes of tossing a fair coin repeatedly, we expect to get about an equal number of sequences of heads and tails. When we look at the data, we evaluate our data with regards to our “model”. However, there are two approaches to model fit. Exploratory data analysis (EDA), introduced by Tukey (1972), is an approach that places emphasis on looking at data sets in order to form hypotheses worth testing, instead of putting forward hypotheses concerning the possible kind of probability model the data follows. EDA, therefore, allows the data to reveal an underlying structure and suggest admissible models that best fit. EDA seems to have two working principles: Firstly, to look at the data, valuing: i) graphical displays, ii) numerical summaries, and iii) the natural pattern-recognition capabilities that humans possess. Secondly, to look at the data with a fundamental respect for real data, and a profound distrust in modelling that is not related to real data, because there is always an element of uncertainty whether the data that is being generated from some assumed theoretical

distribution sufficiently imitates the fundamental features of the process that generated the data.

EDA approach is taking a data-centric perspective (Prodromou, 2008; Prodromou and Pratt, 2006) on distribution in the underlying sense that the primary focus of the principal techniques and principles used in EDA is on data from which a pattern or model may or may not be discerned. In the data-centric perspective on distribution, data will spread across a range of values. From this perspective, variation generates distribution since “without variation, there is no distribution” (Bakker and Gravemeijer, 2004, p. 149). This perspective is compared to that of the statistician, who accounts for unexplained variation as that part of a hypothetical model which is not apparently associated with a main effect. Here the emphasis is on mathematical models that are called theoretical distributions (e.g. Normal, Binomial), in which we attribute probabilities to a range of possible outcomes (discrete or continuous) in the sample space. Prodromou and Pratt (2006) and Prodromou (2007; 2008) refer to this approach as the modelling perspective on distribution. The modelling perspective on distribution pays attention to randomness and the shape of the probabilities that determine the outcomes. In this modelling approach the model gives rise to variation that is portrayed as a random movement away from the main effect. In the modelling perspective, data distributions are seen as variations from the ideal model, the variations being the result of noise randomly affecting the signal, as reflected in the model itself. Prodromou (2007; 2008) introduced the tentative model (Fig1) for the coordination of the data-centric and modelling perspectives on distribution.



**Fig1: A tentative model for the connection of the data-centric and modelling perspectives on distribution.**

Prodromou (2008) first considered the connection from modelling distribution (MD) to data-centric distribution (DD), in the top half of Fig1. Her 15 year old students typically tended to gravitate towards simple causal explanations when they observed the modelling distribution which in some sense generates the data. They invented probability (Piaget and Inhelder, 1975) to operationalise randomness. She considered the connection for DD to MD in the bottom half of Fig1. Many of her students first



recognised variation in data and made the connection from data-centric to modelling perspectives, the target interpretation. Her students had great difficulty however in operationalising variation to explain this connection, and turned to relatively vague references to emergence, as if emergence were a causal agent by which the data-centric distribution targets the modelling distribution.

Prodromou (2007; 2008) claimed that the target and intention models probably are not dissimilar from how experts appreciate the co-ordination of the two perspectives on distribution. She suggested that, experts add to this image a co-ordinated understanding of how the Law of Large Numbers which appears to be the principal aspect of synthesizing the two perspectives on distribution, relates the probabilistic and emergent mechanisms.

## **APPROACH OF THE STUDY**

At an Australian regional university, one hundred pre-service teachers, who were destined to deliver the content of the K-6 Mathematics Syllabus (NSWBOS 2002) in the state of New South Wales, were attending an online unit called 'Numeracy Enrichment for Primary Teachers'.

The main purpose of this course was to provide primary pre-service teachers, with access to studies in selected topics in mathematics that would enhance their enjoyment and knowledge of mathematics and help them to become more positive about their ability to understand some of the structures of mathematics, its applications and relationships. It is noteworthy to point out that this course was designed for pre-service teachers' personal mathematical understanding, rather than about teaching mathematics. Many of the activities could, however, be used in the primary classroom.

The content of this unit of study was organized into eight topics. These topics address all the content strands [1] of the Australia mathematics curriculum: number, space, measurement, data and patterns and algebra.

This course was available online and located on the Blackboard Learning System. The online site of this unit included features such as a message board, a discussion forum, an online white board and a virtual classroom, and electronic downloads of teaching material. For each topic there was a discussion forum set up on UNEonline. The discussion was monitored regularly, and the unit co-ordinator contributed as well as students in the course. Pre-service teachers were expected to attempt all investigations and activities outlined in the Topic Notes for this unit and write up their solutions, explain, justify and reflect on their thinking processes throughout the unit in a journal or logbook format that would be compiled into a portfolio. This portfolio constituted the assessment for the unit and submitted into parts.

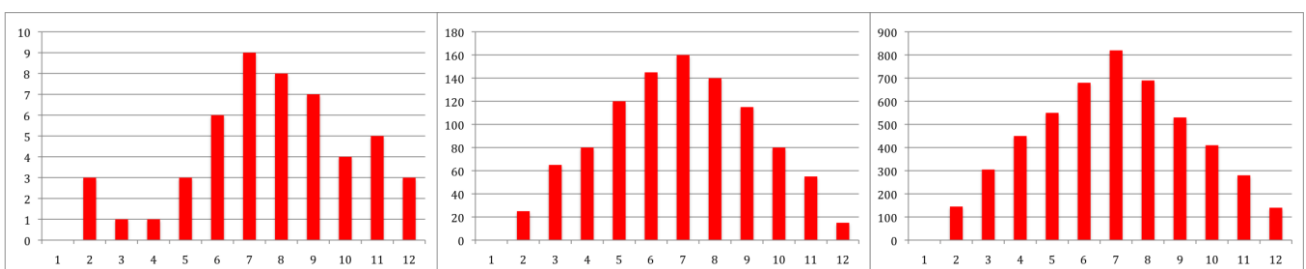
The eighth topic of this unit was about probability investigations, experiments with random number generators, simulations and mathematical models. The Game of

## Working Group 5

Craps described below, was one of the computer-based simulations [2] that pre-service students used. It simulates a gambling game played by rolling two dice. Players take turns rolling two dice and make bets with chips on different areas on a craps table that displays the various different areas which pay out according to numbers rolled. The simulation displays the odds for each different bet when the cursor is hovered over the spot where the bet would be placed. The user of the Craps simulation could set the amount of his wager and clicks on the spot where he wants to place his bet. When the first roll total of both dice is seven (7) or eleven (11) is an instant win and the roll is called “natural”. This outcome pays \$1 for a \$1 bet. A first roll total of two (2), three (3) or twelve (12) which is called “craps” loses the bet. If a roll total of (4, 5, 6, 8, 9, 10) is rolled, that number becomes the “point” and the “point” must be rolled again before a seven (7) is rolled in order to win. There are different payoffs for each point. A point of four (4) or ten (10) will pay pre-service teachers 2:1 (\$2 for a \$1 bet); 5 or 9 pays 3:2; 6 or 8 pays 6:5. If a seven (7) is rolled before that next “point” the player loses the bet (Crapped out) and the two dice must be passed to the next player.

Pre-service were asked to create a strategy for playing the craps game. The assignment task encouraged pre-service teachers to look at this game and see what the chances of winning were by finding out how often a number came out when they added the faces of two dice. The problem involved rolling two dice 50, 1000 and 5000 times, adding the numbers together and recording the sum of the rolled dice outcomes on the respective positions from 1 to 12.

Pre-service students decided to create a spreadsheet that would simulate randomly rolling two dice and add their faces. After they simulated 50 rolls of two dice they drew a histogram for the sum of two dice (Fig2). Moreover, they drew histograms to show the sum of the rolled dice, 2-12 after 1000 rolls and 5000 rolls (Fig2).



**Fig2: Histograms for the sum of two dice after 50, 1000 and 5000 rolls.**

Pre-service teachers were asked to observe the three graphs very carefully and : 1) describe the pattern they could see emerging, and discuss the occurrence of the outcomes from rolling two dice; 2) explain which numbers occurred more often and which occurred least often; 3) determine the percentage for each outcome when reading the ‘5000 rolls’ graph.

The notes drew pre-service teachers' attention to the fact that not every probability was calculated using simulation and experimental probability. It therefore, introduced another type of probability called theoretical probability, that was a method which did not perform a single experiment, but instead used theory and reason to determine the chance of an event occurring.

Pre-service students were asked to 4) complete a chart by adding the different possibilities together when two dice are rolled and (5) complete a table for each sum of the two dice, the frequencies and % of occurrence. (6) The sixth task was to draw a histogram of the data and compare the histogram they constructed to the '5000 rolls' histogram. (7) They were also asked to compare the percentage of the occurrence of the theoretical probabilities to the experimental probabilities calculated earlier when pre-service teachers constructed the '5000 rolls' histogram and make connections to the Craps simulation. The questions required pre-service teachers to justify their answers based on their knowledge about empirical and theoretical distributions, experimental and theoretical probabilities.

The series of the seven tasks was selected because it was approachable, both through conducting physical trials and through theoretical analysis via the sample space. When pre-service teachers were asked to determine the optimal sample space for an event, they were restricted to contexts where the sample space was relatively small, such that pre-service students were capable of generating all possible cases, being the only source of probability.

One hundred pre-service teachers e-submitted their portfolio. To illustrate the ideas emerged from pre-service teacher' work, I present excerpts taken from the portfolios of ten pre-service teachers (PT 1, PT 2, PT 3, PT 4, PT 5, PT 6, PT 7, PT 8, PT 9, PT 10) who were ranging in age from 21 to 52 years old. The data included excerpts from pre-service teachers' explanation and justification of the assignment tasks and their reflections on mathematical processes. At the first stage, I wrote extensive field notes, during and immediately after reading each portfolio. Excerpts were coded and compared across portfolios to infer explanations for pre-service students' reflections. In this sense, the analysis was one of progressive focusing (Robson, 1993).

In this study, my overarching aim is to observe how pre-service students 1) justify their answers based on their knowledge about experimental and theoretical probabilities, and 2) whether they can construct a bidirectional link between the data-centric and modelling perspectives on distribution, similar to the tentative model for coordinating the two perspectives on distribution (Fig1).

## RESULTS

Pre-service teachers were first challenged to describe the pattern they could see emerging from the three distributions for the sum of two dice after 50 rolls, 1000 rolls and 5000 rolls.

### Working Group 5

- 1 PT 1: The frequency starts out low at the beginning of the graph, leading up to a peak in the middle and then descending back again towards the end of the graph, similar to a mountain. I have also noticed that certain numbers have partners on either side of the peak, however 7; the most frequent number has no partner. For example 8 and 6 are partners, as is 2 and 12. In that their frequencies are similar numbers.
- 2 PT 2: After rolling 2 dice 50 times, 7 is the most occurring number, followed by 8 and 9. Apart the clear 7 leader, the rest of the numbers are fairly scattered however, as the dice is rolled more times, 1000 and then 5000, the graph is peaking at 7, the leader and coming down evenly to 2 and 12.

PT 1 was inclined to resort to a causal explanation for the histogram by comparing its shape to a mountain and anthropomorphizing the occurrence of its underlying bars. PT 2, in his attempt to understand the emerging pattern, he assumed centralized control. In particular, he saw the histogram as a centralised system where a centralised controller – 7 the leader – impressively coordinated the pattern.

Pre-service teachers were also inclined to use geometrical shapes to describe the shapes of the histograms.

- 3 PT 3: The bar graph is triangular in shape with the outside bar being shortest and the inner ones becoming taller.
- 4 PT 4: The pattern is a pyramid, with the apex being number 7.
- 5 PT 5: I noticed the bell curve that is evident from the graphs.
- 6 Pt 6: A pattern is emerging whereby a bell curve is forming centred on the outcome of number 7.

Many pre-service teachers (PT 3) described the histogram as triangular in shape. PT 4 saw the histogram as a three-dimensional object. In contrast, PT 5 referred to the emerging pattern of a bell curve.

After comparing the histograms of 1000 and 5000 rolls, pre-service teachers noticed that the result of performing the same simulation a large number of times tended to provide more stable outcomes.

- 7 PT 2: As the dice is rolled more times, 1000 and then 5000, the graph, we see increasingly stable and symmetrical orderings of likeliness reaching a crescendo at 7 and falling away equally on either side.
- 8 PT 5: The more rolls that are undertaken, the more constant and smooth the results are becoming.
- 9 PT 8: ... the more rolls that are recorded, the more even the distribution becomes when 7 is rolled most frequently, then 6 and 8 then after that 5 and 9, then 4 and 10 then 3 and 11, then 2 and 12 in even steps.

Pre-service teachers were challenged to work out the theoretical probabilities of each of the sums occurring, determine the percentage of the occurrence of each outcome (Table 1) and compare those percentages to the scores they calculated for the histogram 'Rolling two dice 5000 rolls' (Table 1).

## Working Group 5

Outcome	% of occurrence	Frequency
2	$1/36=2.78\%$	1
3	$2/36=5.56\%$	2
4	$3/36=8.33\%$	3
5	$4/36=11.11\%$	4
6	$5/36=13.89\%$	5
7	$6/36=16.67\%$	6
8	$5/36=13.89\%$	5
9	$4/36=11.11\%$	4
10	$3/36=8.33\%$	3
11	$2/36=5.56\%$	2
12	$1/36=2.78\%$	1

Outcome	% of occurrence
2	$150/5000=3\%$
3	$310/5000=6.2\%$
4	$425/5000=8.5\%$
5	$560/5000=11\%$
6	$675/5000=13.5\%$
7	$830/5000=16.6\%$
8	$690/5000=13.8\%$
9	$520/5000=10.4\%$
10	$410/5000=8.2\%$
11	$290/5000=5.8\%$
12	$140/5000=2.8\%$

**Table 1: Theoretical probabilities (left hand side) and percentages of occurrence of roll total of two dice when ‘rolling two dice 5000 rolls’ (right hand side).**

Pre-service teachers seemed to articulate a notion of signal with noise:

- 10 PT 7: The theoretical will always be the same and never fluctuate or change but experimental will deviate lightly above or below the theoretical probability.
- 11 PT 9: The more times a die is rolled... the histogram for the experimental probabilities will be above or below the histogram of the theoretical probability of an event occurring.

Pre-service teacher (PT 7) expressed noise as “above or below” the theoretical probability and saw the theoretical probability as a central value that might in a sense direct the outcome. PT 9, in turn, seemed to have a sense of there being a signal as the theoretical distribution and noise as a discrepancy between the data observed and the smooth bell curve.

When pre-service teachers referred to the tendency of individual values of experimental probability towards the expected values, they attributed this tendency to “chance”.

- 12 PT 10: It is evident that as the number of trials 1, wrote: “The percentage of occurrence I calculated using Theoretical probability is very close to the percentage of occurrence what we found using experimental probability, however the numbers are not exactly the same. For example for the outcome 7 in experimental probability I got 16.4% as the occurrence percentage, however in theoretical the answer is 16.6%. The numbers however are very close and I am sure that if further experiments were conducted, one such experiment would emerge in which the numbers would be exactly the same as the theoretical probability percentages, this is all up to chance.

As we have witnessed from the previous excerpt, pre-service teachers did not appear to perceive a continuum tendency of the data-centric distribution towards a target. The element of randomness appeared as a distractive factor for the successful coordination of the two perspectives on distribution. I wondered whether pre-service

## Working Group 5

teachers might have developed an understanding (at the global level) of the distribution targeting the modelling perspective on distribution. When the question required them to compare the two perspectives on distribution, pre-service teachers appeared to make comparisons locally.

- 13 PT 6: It is evident that as the number of trials increases, the experimental probability approaches the theoretical probability.
- 14 PT 4: The experimental probability figures will, the more dice rolls that are completed, eventually get closer and closer to the theoretical probability figures...we used experimental probability to 'prove' theoretical probability.

There was, however, a reference (line 14) of using the experimental probability to prove theoretical probability. Some of pre-service teachers separated the theoretical and experimental probability.

- 15 PT 2: Theoretical probability is the knowledge that is calculated of what will happen under ideal conditions. Experimental is the action of going and doing a chance experiment and recording the results for analysis.

PT 2 appeared unable to suggest any connection for the co-ordination of the experimental and theoretical probabilities. Most of the pre-service teachers, however, preferred the security provided by the theoretical probability methods.

- 16 PT 5: While the experimental data, with a high number of trials, is quite successful at producing probability results, I believe that it is easier and more valuable and reliable to us the theoretical probability methods, as this ensures that every possible outcome paths is addressed.
- 17 PT 1: Experiments are not entirely necessary if you have theoretical probability, which also gives you an even more precise answer to the chance of something happening.
- 18 PT 7: This means that you can predict how many times a 4 will appear as a combination of rolling two standard die, without having to actually roll them. The answer you find in this theoretical probability will be the same if you actually perform the experiment physically. Thus, theoretical probabilities can provide us with the ability to predict the outcome of an experiment that may take a very long time to perform or one that it is difficult to perform due to reasons such availability of resources etc.

Pre-service teachers appeared to acknowledge that the theoretical probabilities have the power to predict the experimental probabilities (16, 17, 18). Although, those excerpts have provided an insight into the resemblance of the two perspectives on probability, a strong connection has not been suggested to be made from the modelling perspective on distribution towards the data-centric perspective on distribution, in which the modelling distribution in some sense generates the data.

The role of large numbers (though probably finite numbers) in this process became very obvious.

## DISCUSSION

Pre-service teachers resort to causal explanations to describe the shape of the emerging pattern (1). They gravitate towards adopting a centralised mindset to describe the organization of the histogram (2) by a centralised controller. It is interesting how pre-service teachers use terminology from geometry to describe the shape of the histogram as triangular or a three-dimensional object (3, 4). Pre-service teachers, who distinguish the theoretical from experimental probabilities, were not always able to coordinate those two different perspectives on probability. When they tried to make the connection from the experimental to theoretical probability there was an element of chance (or uncertainty) in the occurrence of the data, out of which the theoretical probability emerges (12). This element caused uncertainty for pre-service teachers, as the connection of data-centric to modelling perspective on distribution caused uncertainty for the 15 year old students in a previous study (Prodromou, 2008). In both cases there was variation omnipresence in the data out of which both entities emerge. The only difference is that the theoretical probability emerges at the local level and the distribution emerges through self organisation at the global level. In both cases students and pre-service teachers articulate their emerging sense of how both entities emerge, by using situated abstractions (Noss and Hoyles, 1996), informal heuristics that capture generally expressed in term of the structuring resources within the settings.

## NOTES

1. The Australian *mathematics K-6 Syllabus* is organised into six strands — one process strand, Working Mathematically, and the five content strands, Number, Patterns, and Algebra, Data, Measurement, and Space and Geometry.
2. The game of Craps is available online from [http://www.ildado.com/free\\_craps.html](http://www.ildado.com/free_craps.html)

## REFERENCES

- Cohen, S., & Chechile, R. A. (1997). Probability distributions, Assessment and Instructional Software: Lessons learned from an Evaluation of Curricular Software. In I. Gal and J.B. Garfield (Ed.), *The Assessment Challenge in Statistics Education* (pp. 253-662). Voorburg Netherlands: International Statistics Institute.
- Bakker, A., & Gravemeijer, K. (2004). Learning to reason about distribution. In D. Ben-Zvi & J. Garfield (Eds.), *The Challenge of Developing Statistical Literacy, Reasoning and Thinking*, (pp.147-168). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Fischbein, E. (1975). *The Intuitive Sources of Probabilistic Thinking in Children*. Holland: Reidel Publishing Company.

## Working Group 5

- Kahneman, D., Slovic, P., & Tversky, A. (1982). *Judgement Under Uncertainty: Heuristics and biases*. Cambridge: Cambridge University Press.
- Konold, C., & Kazak, S. (2008). Reconnecting Data and Chance. *Technology Innovations in Statistics Education*, Vol. 2: No. 1, Article 1. <http://respositories.cdlib.org/uclastat.cts/tise/vol2/iss1/art1>
- Lecoutre, M. P. (1992). Cognitive models and problem spaces in "purely random" situations. *Educational Studies in Mathematics*, 23(557-568).
- New South Wales Board of Studies (2002). *Stage 6 Syllabus General Mathematics*. Sydney, Australia: New South Wales Board Of Studies.
- Noss, R., & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. London: Kluwer Academic Publishers.
- Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children*. (L. Leake Jr., P. Burrell, & H. D. Fishbein, Trans.). New York: Norton. (Original work published in 1951).
- Pratt, D. (1998). The construction of meanings in and for a stochastic domain of abstraction. *Unpublished PhD Thesis*, University of London, London.
- Prodromou, T. (2007). Making connections between the two perspectives on distribution. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 801-810). Larnaca, Cyprus.
- Prodromou, T. (2008). Connecting thinking about distribution. *Unpublished PhD Thesis*, University of Warwick, UK.
- Prodromou, T., & Pratt, D. (2006). The role of causality in the Co-ordination of the two perspectives on distribution within a virtual simulation. *Statistics Education Research Journal*, 5(2), 69-88.
- Robson, C. (1993). *Real World Research*. Oxford: Blackwell.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Tukey, J. W. (1972). Data analysis, computation and mathematics. *Quarterly for Applied Mathematics*, 30, 51-65.



# IMPLEMENTING A MORE COHERENT STATISTICS CURRICULUM

Anneke Verschut and Arthur Bakker

Freudenthal Institute for Science and Mathematics Education, Utrecht University,  
The Netherlands

*Coherence is one of the objectives of a new statistics curriculum for grades 10-11 (age 15-17) in the Netherlands. Since coherence has different meanings for different curriculum representations, this paper evaluates to what extent the initial attempts to design and implement a coherent statistics curriculum were successful for different curriculum representations. Data were collected from curriculum developers (ideal curriculum), authors of teaching materials (written curriculum), teachers (perceived curriculum) and classrooms (operational curriculum). The results suggest that the implementation of a coherent curriculum in classrooms requires a clear statement of this objective in the ideal curriculum, worked out into guidelines for curriculum authors and concrete classroom activities.*

*Keywords: coherence, curriculum implementation, statistics education, teacher support.*

## INTRODUCTION

A new statistics curriculum for the high school level (grades 10-11, ages 15-17) is being developed in the Netherlands. The design and implementation process is part of a broader reform movement that should lead to more coherent science and mathematics curricula in general. In this paper we evaluate the initial attempts to implement the ideal of a more coherent statistics curriculum into the classroom.

What is new in this curriculum is that students learn the concepts of statistics through working with real data sets. The idea is that this helps them to see the relevance of the statistical concepts and techniques they learn. The reform follows similar movements in other countries (e.g., USA, Germany, and New Zealand) and is inspired by international research in the field of statistics education. Teaching students a list of statistical recipes is not enough to make them statistically literate. Students also need to see the coherence between the concepts they learn and the basic principles underlying data analysis (Moore, 1997; Tarr & Shaughnessy, 2007). In the new curriculum, to be implemented nationally in 2015, teachers are encouraged to let students work with real data sets and information technology. Furthermore, the curriculum has both a theoretical and a practical strand. In the practical strand students do research projects where they can apply to real-world problems the theoretical concepts they have learnt in the theoretical strand. The experts in statistics education whom we interviewed thought that these characteristics can indeed help to improve the coherence of the new curriculum (Verschut & Bakker, 2010).

However, it is a well-known problem that there is often a huge gap between the original ideas and intentions of a new curriculum and the curriculum actually enacted in classrooms (Begg, 2005; Van den Akker & Voogt, 1994). In this paper we describe the initial experiences with the development and implementation of the new curriculum, based on the experiences of a small group of schools that are piloting the exemplary teaching materials. The question we seek to answer is: *to what extent do curriculum developers, writers of teaching materials and teachers indeed succeed in making statistics education at the high school level more coherent?*

## **THEORETICAL BACKGROUND**

When people in a reform process speak about a coherent curriculum, what do they actually mean? The first issue we have to deal with is that at least three representations of a curriculum are commonly distinguished: the intended, implemented and attained curriculum (Goodlad, 1979). In a more refined version of this typology the intended curriculum contains both the ideal curriculum (the vision or basic philosophy underlying a curriculum) and the written curriculum (intentions as specified in curriculum documents and/or teaching materials). The implemented curriculum includes both the perceived curriculum (interpretations of intended curriculum by users, particularly teachers) and the operational curriculum (teaching and learning activities actually enacted in classrooms). The attained curriculum is represented by students' learning experiences and learning results.

In this paper we focus on the intentions of policy-makers to have a coherent curriculum in 2015. However, their power is confined to the ideal curriculum. Many questions arise in such a situation: How has the policy-makers' ideal of a coherent curriculum been transformed into the written curriculum by the writers of the exemplary teaching materials? How did teachers perceive the new curriculum?

Curriculum materials play a role in encouraging or supporting new curriculum goals (Herbel-Eisenmann, 2007). So, those who develop curriculum materials need to carefully attend to their discursive choices so that they do not undermine their own intentions. Furthermore, additional support for teachers to make pedagogical choices in line with the new curriculum goals may be needed within the teaching materials (Herbel-Eisenmann, 2007). Teachers have a considerable impact on the transformation process from the written curriculum to the attained curriculum as they decide how to interpret the written curriculum (Stein et al., 2007). Teachers intend to match the written curriculum concerning the content, but their inclinations to match the innovative goals differ (Eichler, 2010). The differences result from teachers' different attitudes and beliefs towards statistics, their knowledge of statistics, and their professional identity (Eichler, 2010; März et al., 2010; Stein et al., 2007).

## **What people mean by coherence**

Another issue that comes up when we talk about a coherent statistics curriculum is what stakeholders mean by coherence. In other words: what constitutes a coherent curriculum? From interviews with national and international experts (Verschut & Bakker, 2010), we inferred the purpose of a coherent curriculum basically is to provide students with coherent knowledge and that coherent statistical knowledge includes:

- conceptual understanding of statistical concepts and their connections
- knowing when, why and how to use what statistical concept or technique
- statistical reasoning, and
- transfer to subjects other than statistics.

Furthermore, we found that building curriculum materials around a central theme such as the key concepts of statistics was assumed to advance coherence of students' statistical knowledge. Another way is to use a concentric method around ever recurring topics, for instance by recognizing that there are two basic types of statistical questions: either you want to know whether a certain variable is correlated with another, or you are comparing two or more groups. Emphasizing the purpose of what is learnt, as happens in an inquiry-based or problem-based learning curriculum, could also improve coherence of students' knowledge. A further point that was mentioned is the importance of making the relation between chance and statistics more explicit; in the old curriculum these were two separate worlds. In other countries (e.g., United States) researchers have also observed this problem (cf. Konold & Kazak, 2008).

Active learning and students' motivation are both recognized as promoters of coherent knowledge (Bransford et al., 2000; Kali et al., 2008). The experts we interviewed underlined this by mentioning motivating learning activities such as doing real-life research projects and working with real data, and active learning activities like discussion and reflection, when asked what teachers could do in the classroom to promote coherent knowledge (implemented curriculum). They also mentioned that the use of computer software for visualization of data could support development of conceptual understanding.

Thus, the characteristics of our new curriculum, working with real data sets and information technology, and the research projects in the practical strand, indeed have the potential to make statistics education more coherent. The question remains to what extent it works.

## **METHOD**

In order to answer our research question we collected data from all people involved in the implementation of the new curriculum at different curriculum representations.

We read the report of the committee that had the task to write down the basic philosophy of the new curriculum (SKACA, 2007), and interviewed all of its four members plus the person who initiated the curriculum reform together with one of the SKACA members, to better understand the *ideal* curriculum. For a better insight into the development process of the *written* curriculum, we attended the meetings of the authors of the exemplary teaching materials, made notes during these meetings and observed the evolution of the different versions of the instructional materials.

Fourteen teachers of five different schools have tried out three chapters of the exemplary teaching materials in their classrooms in the school year 2009/2010. The classrooms were all of grade 10 of the so-called general education track [1]. The teachers who implemented the materials did so because their schools had signed a contract with the Dutch reform committee for mathematics education (cTWO) to participate in the evaluation procedure of the new curriculum. Two of the teachers (of two different schools) had some knowledge of the ideas behind the new curriculum before they started piloting the materials, as they had been involved in a small try-out project of the initial ideas in the year before these pilots took place. They both initiated the participation of their schools in the evaluation procedure. The rest of the teachers were more or less charged by their schools to participate. We interviewed nine of the participating teachers (at least one per school) a few weeks after they started to work with the materials and attended at least one lesson per school. We also asked them to complete a short questionnaire on their educational and professional background, asking for information on years of experience in teaching mathematics, and statistics in particular, educational background, experience in doing statistical inquiries. This information is relevant since teachers' backgrounds have an influence on their attitudes and beliefs towards statistics education, and thus on their perceptions of a new curriculum (März et al., 2010).

In the interviews we sought a clarification of the *perceived* curriculum by questioning the teachers on their experiences with the former statistics curriculum, how they perceived the attempts to reform, and whether they were aware that coherence was one of the ideas behind the reform attempts. We further asked them what in their opinion coherence meant, and what type of teaching or learning activities they thought could help students to develop coherent knowledge. A first impression of what the operational curriculum would look like emerged by the answers to questions such as in what respect they thought they would teach differently in comparison to the former curriculum, and whether they felt capable of teaching in this new way. Furthermore we asked them what type of support they would need to implement this curriculum properly.

After each chapter all teachers were asked to send us evaluation materials such as teacher questionnaires, annotations with the materials, log-books and student questionnaires. We received materials from nine different teachers (at least one per school), whereas some of them stated that they had discussed their findings with

their colleagues (who did not send any materials) and that the materials reflected their shared views. At the end of the school year we had a group discussion with six of the teachers in order to evaluate their initial experiences of the past school year. This meeting, combined with the evaluation materials gave us a clearer picture of the preliminary *operational* curriculum.

All data collected were analysed according to a coding scheme based on the general evaluation criteria for a coherent curriculum that we developed in an earlier paper (Verschut & Bakker, 2010) and summarised in the previous section of this paper.

## **RESULTS**

In order to answer our research question we first need a short description of the ideal curriculum.

### **Ideal curriculum**

Although one of the objectives of the curriculum reform was that the learning strands should be coherent, which presumably helps students to develop more coherent knowledge, the SKACA committee did not explicitly mention the notion of coherence in their report. They did offer some ideas that could help to improve coherence of students' statistical knowledge, such as building the curriculum around the investigative cycle, developing concepts by making use of real world problems (problem-oriented approach), making the relationship between chance and data more explicit, and teaching probability as being at the service of statistics so as to avoid the common problem of probability and statistics being two separate areas in the mathematics curriculum.

### **From ideal to written curriculum**

Based upon our notes on the meetings of the curriculum authors that we attended, it became clear that the writers of the exemplary teaching materials did not pay much attention to coherence of the materials. From the discussions about the new educational materials it transpired that the authors did not think it was necessary to clearly highlight the connecting thread in the materials for students or teachers, or to offer teachers additional material to explain the ideas behind the materials. An illustrative remark often made by one of the authors in this respect is: "We should not pamper the teachers too much." Furthermore, they did not explicitly try to incorporate teaching or learning activities that could advance coherent knowledge. Their main concern was to cover all of the attainment targets [2], which – admittedly – was already a puzzle in itself.

### **From written to perceived curriculum**

The questionnaire on the educational and professional backgrounds of the participating teachers gave us a rather diverse picture: years of experience ranged from one to over thirty. Two of the teachers had an econometrist background rather than a mathematical one, and thus presumably had learned more about statistics than

the other teachers. One of them even carried out statistical enquiries in the past for her profession. The other teachers had no specific interest in statistics or statistics education.

What came to the fore from the interviews with the teachers was that at the time they started working with the exemplary teaching materials most of them were not aware that coherence was one of the goals. When asked what they thought was important for this type of students, they mainly mentioned things that are related to what we can summarise as statistical literacy. For instance, they mentioned goals such as being critical readers of newspaper articles on statistical results, being able to critically evaluate the outcome of statistical reports written by other people, and understanding that statistical knowledge is useful in their daily life or other subjects at school.

When asked, they appeared to have an image of what constitutes coherent knowledge and how to improve it, but the notions they had were rather vague, and were not the same as the notions we mentioned in the theoretical background section. They mainly thought of a connecting thread or a clear structure in the materials, or they thought of coherence with other school subjects, such as geography or biology and applications of statistics outside the mathematics curriculum. When we explained the rationale behind a more coherent curriculum, they all considered it an objective worth striving for. However, teachers also complained that they did not know what the attainment targets were of the new curriculum, and thus could not see what the things they were teaching in this first chapter should lead to in the next chapters. This was also due to the fact that at the time they were working with Chapter 1 the rest of the chapters were still under development.

### **From perceived to operational curriculum**

In the lessons we observed we saw hardly any classroom activities that were mentioned in the interviews or literature as stimulating coherent knowledge. An explanation of this disappointing result is that teachers had very little time to prepare their lesson series since the exemplary teaching materials were finished just before the lesson series started. We assume many of our observations could be different in a next year when teachers have worked with more chapters and have studied the attainment targets.

One of the teachers piloting the materials is a member of cTWO, and thus is more aware of the underlying ideas and intentions of the materials. In her lesson we could see some attempts to motivate students to seek for coherence in the things they have learnt. For example she told her students, while doing their exercises, to ask themselves the question “How can I make use of the knowledge I learn from doing this exercise, when I (in another context) am trying to find an answer on a research question?” We consider this an attempt to stimulate reflection on the applicability of acquired knowledge.

Another positive example was a lesson we observed of a teacher with an econometrist background. She tried to give students a feeling for the role of statistics within the investigative cycle. For example, in one of her lessons she gave her students a group work assignment to invent a research question related to pocket-money and think about a possible research set-up that could lead to an answer to this research question. The results were discussed by the entire class. We think this activity stimulated coherent knowledge as it emphasized the central role of the investigative cycle in statistics, it motivated and activated students because of the real-life context and group work, and the class discussion at the end stimulated the students to reflect on the pros and cons of their research questions and methods.

### **Teachers' evaluations of the operational curriculum**

From the evaluation materials collected during the rest of the school year and the evaluation meeting we had at the end of the school year, we learnt that teachers found the new teaching materials promising, but difficult to implement. They thought the new approach could lead to more coherent statistics education, as one of them stated:

Last year we basically taught them [the students] some tricks, now we also talk with them about statistics and they can get a clearer picture of what statistics is used for and how it works.

However, the first results were not promising, so the teachers felt they failed in utilizing the opportunities of the new curriculum. For instance, some teachers gave their students an assignment to do a small research project on their own or in groups of two or three students, but were disappointed by the results. Only a small proportion of the students showed that they got a grip on the aims and ways of thinking in statistics.

One of the reasons the teachers gave for the disappointing results was that the intentions of the authors of the exemplary teaching materials were not always clear. The teachers said they had difficulty in finding the connecting thread in the materials, how the different chapters were connected and what the entire curriculum was aiming at. They asked for more structure in the materials as a way to support teachers in implementing the curriculum.

The teachers recognized that they have a central role in advancing coherence of their students' knowledge because in their view this type of students has no natural inclination to seek for coherence in what they learn. The teachers understood that for instance having more classroom discussions or promoting reflection by asking a lot of questions could help their students to develop coherent knowledge. However, they did not have a lot of experience with this type of activities and were concerned that it would take up too much time of their lessons, since the entire mathematics curriculum is rather overloaded and statistics is part of the school exams, not of the national exams. Suggestions in the materials for classroom activities that stimulate

coherent knowledge and make efficient use of time would be highly appreciated: “When they think it is important, it should be in the materials!”

## DISCUSSION

Our results indicate that the first year of piloting the new curriculum was not very successful. The original ideas and intentions of the *ideal* curriculum were not worked out neatly: what is meant by coherence within a statistics curriculum at different curriculum representations and how could it be promoted. This made the implementation process even more complex than could be expected on the basis of the literature (Begg, 2005; Van den Akker & Voogt, 1994).

It is hopeful that for the *perceived* curriculum we found that teachers indeed recognized the potential of the new teaching materials to provide students with a more coherent knowledge base of statistics: a better understanding of statistical reasoning processes, and a better understanding of when, why and how the statistical techniques they have learnt can be of use. However, the teachers felt they failed in transferring this rich knowledge to their students in the *operational* curriculum. This feeling was intensified by the disappointing results of a research project assignment.

The teachers complained that their failure was partially due to the *written* curriculum: The authors of the teaching materials had not highlighted the structure and connecting thread in the materials. Teachers asked for more guidance and support in implementing this curriculum, for instance by making the intentions of the authors of the materials more distinct, or by offering suggestions for classroom activities that stimulate coherent knowledge. Although the group of teachers piloting the materials consists of teachers of diverse educational and professional backgrounds, they were unanimous in their desire for more implementation support in the teaching materials. Our findings confirm the suggestions by Herbel-Eisenmann (2007) for more support for teachers within the teaching materials. An issue that remains for future research is to what extent differences appear in teachers’ inclinations to match the innovative goal of this curriculum, i.e. coherent knowledge, following their different educational and professional backgrounds as has been reported by others (Eichler, 2010; März et al., 2010; Stein et al., 2007).

It is easy to ask for a coherent curriculum – this is done in policy documents worldwide. However, without specifying what coherence means in terms of connecting threads, recurring themes, a concentric approach, the repeated use of an investigative cycle or problem types (e.g., group comparison versus correlation questions), the concept remains empty to most curriculum authors, let alone teachers. Our impression is that more specific measures to promote coherent curriculum strands should be mentioned, trialled and investigated.

In the first place the objective should be clearly stated in the *ideal* curriculum, otherwise authors of teaching materials and teachers are not aware of the objective, but stating the objective of coherent statistical knowledge is not enough. The authors



of teaching materials should include concrete measures and activities that can promote coherence of students' statistical knowledge in the *written* curriculum, and thus translate the broad concept of coherence, typical of policy documents into some concrete and easy to apply measures. One might think of indicating the connecting thread or structure in the materials, give suggestions for classroom activities such as discussion or reflection, and include exemplary items for assessment.

In the next stage of our research we will develop and evaluate concrete implementation support materials, such as suggestions for classroom activities that may contribute to coherent knowledge. The design of these implementation support materials will be based on teachers' initial experiences with the curriculum and literature on ways to promote coherent knowledge. Inspired by the notion of *educative curriculum materials* (Davis & Krajcik, 2005), and remarks made by the teachers of our research group, we argue these materials should be included in the teaching materials.

So far we only looked at the intended and implemented curriculum. In the next stage of our research we will also investigate students' results to see if the invented activities indeed lead to more coherent statistical knowledge. For that purpose we need to develop more insight into how coherent knowledge can be measured.

### NOTES

1. In Dutch secondary education 60% of the students attend pre-vocational education, 20% are in the general educational track and 20% are in the pre-university track.
2. Attainment targets are the statutory objectives of a school subject as formulated by the Dutch government such as: the candidate can interpret statistical data that are represented and/or summarised in diverse ways, and critically appreciate their relevance.

### REFERENCES

- Begg, A. (2005). Statistics curriculum and development: New ways of working. In G. Burrill & M. Camden (Eds.), *Curricular development in statistics education: International Association for Statistical Education 2004 Roundtable* (pp. 10-20). Voorburg: International Statistical Institute.
- Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.) (2000). *How people learn: Brain, mind, experience, and school*. Washington D.C.: National Academy Press.
- Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher*, 34(3), 3-14.
- Eichler, A. (2010). The transformation process from written curricula to students' learning. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics (ICOTS8), Ljubljana, Slovenia*. Voorburg, The Netherlands: International Statistical Institute, [www.stat.auckland.ac.nz/~iase/publications.php](http://www.stat.auckland.ac.nz/~iase/publications.php).

- Goodlad, J. I. (1979). *Curriculum inquiry: The study of curriculum practice*. New York: McGraw-Hill.
- Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the “Voice” of a mathematics textbook. *Journal for Research in Mathematics Education*, 38(4), 344-369.
- Kali, Y., Linn, M. C., & Roseman, J. E. (Eds.) (2008). *Designing coherent science education: Implications for curriculum, instruction, and policy*. New York: Teachers College Press.
- Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education*, 2 (1), Article 1, <http://repositories.cdlib.org/uclastat/cts/tise/vol2/iss1/art1>.
- März, V., Vanhoof, S., Kelchtermans, G., & Onghena, P. (2010). De vernieuwing van het statistiekonderwijs in Vlaanderen [The innovation of the statistics curriculum in Flanders: An analysis of implementation]. *Pedagogische Studiën*, 87 (2), 134-151.
- Moore, D. (1997). New pedagogy and new content: The case of statistics. *International Statistical Review*, 65, 123-165.
- SKACA (2007). *Aanbevelingen voor de inhoud van statistiek en kansrekening binnen wiskunde A en C vwo en wiskunde A havo [Recommendations for content of statistics and probability within math A en C at vwo-level and math A at havo-level]*. Downloaded from [www.ctwo.nl](http://www.ctwo.nl), 10 January 2008.
- Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.
- Tarr, J. E., & Shaughnessy, J. M. (2007). Data analysis, statistics, and probability. In F. Lester & P. Kloosterman (Eds.), *Results from the 2003 National Assessment of Educational Progress*, Reston, VA: National Council of Teachers of Mathematics.
- Van den Akker, J., & Voogt, J. (1994). The use of innovation and practice profiles in the evaluation of curriculum implementation. *Studies in Educational Evaluation*, 20, 503-512.
- Verschut, A., & Bakker, A. (2010). Towards evaluation criteria for coherence of a data-based statistics curriculum. In C. Reading (Ed.), *Proceedings of the Eighth International Conference on Teaching Statistics (ICOTS8)*, Ljubljana, Slovenia. Voorburg, The Netherlands: ISI, [www.stat.auckland.ac.nz/~iase/publications.php](http://www.stat.auckland.ac.nz/~iase/publications.php).

## INTRODUCTION TO THE PAPERS OF WG 6: APPLICATIONS AND MODELLING

Chair: Gabriele Kaiser

Team members: Susana Carreira, Thomas Lingefjård, Geoff Wake

The starting point of the working group was a panel discussion of the understanding of the nature of modelling and applications and its theoretical description. One central aspect, namely the distinction between applications and modelling, was intensively discussed by Blum, who referred to the ICMI Study on Applications and Modelling (Blum et al., 2007). He distinguished between modelling and applications as follows: modelling tends to focus on the direction “reality to mathematics” and emphasises the *processes* involved. When *modelling* we are standing outside mathematics looking into mathematics and asking, ‘where can I find some mathematics to help me with this problem?’ Applications tend to focus on the direction “mathematics to reality” and emphasises the *objects* involved. With *applications* we are standing inside mathematics looking out and looking for a particular piece of mathematical knowledge, we can use.

Furthermore, the differences and commonalities that exist between problem solving and modelling were intensively discussed by Doerr who described problem solving as inward looking, dealing with pre-mathematised word problems, in which the givens and goals are static and the goal is a particular solution. Modelling she described as outward looking with the phase of transition from real-world to mathematics as central and the givens and goals dynamic and with the goal being a tool that can be re-used. Both approaches share one important commonality, namely the fundamental question, whether modelling is for mathematics learning or mathematics learning is for modelling. She proposed an integrative view, in which the development of mathematical content and the development of modelling abilities are joint processes, in which the learning of mathematics happens in the context of modelling real world problems and modelling real world problems happens in the context of learning mathematics. An understanding of this joint development will be a key element in advancing a research agenda on modelling and problem solving.

Amongst other aspects the role of technology in applications and modelling was discussed intensively by Carreira. She argued that the inclusion of technology into modelling processes can allow the usage of more complex mathematics, especially the inclusion of simulation or experimentation, where investigation would be possible. Furthermore, the visualization of images and results would contribute to a better understanding of the real situation and of mathematical ideas.

The working group discussed various strands concerning modelling and applications: the majority of the papers dealt with results of empirical research, which displayed a large variety of themes. Several papers dealt with theoretical aspects such the

distinction of hypotheses and assumptions within modelling processes by Grigoros or the usage of realistic Fermi problems from the perspective of the anthropological theory of didactics by Ärleback. Two reports were of psychologically oriented studies: the study on the connections between the mathematical thinking styles of teachers and their interventions by Borromeo Ferri and Blum and the analysis of the problems solving process by Johannes Gross. Frejd described a curricular oriented study of modelling in Swedish national course tests. A subject-bound study of the understanding of average rate of change was proposed by Doerr and O'Neil. Other papers dealt with theoretical aspects such as the role of the teacher's identity in a psychologically oriented study by Jensen and an interdisciplinary oriented study of modelling in an integrated mathematics and science curriculum by Wake. Teacher education was researched in a curricular oriented study of a teaching sequence for future mathematics teachers by Lingefjård, the role of modelling as a big idea within the beliefs of future teachers was researched by Siller. The use of digital tools within modelling problems in German centralised examinations was discussed by Greefrath.

The sessions closed with reflections on the discussion in the light of the panel debate at the beginning and identified the following main research areas for the future:

- How to move forward in the scientific debate on applications and modelling?
- How to make progress concerning the inclusion of applications and modelling in curriculum and school practice?

These two areas give rise to questions that can be differentiated as follows:

- Where are research deficits? Can we identify them?
- What kind of empirical research is necessary? Large-scale studies versus case studies? Role of replication studies?
- What kinds of measures are necessary and can be constructed? Is the development of teaching units or learning materials sufficient? What is the role of comprehensive learning environments?
- What can we say about the role of culture in the teaching and learning of mathematical modelling? Is it culture-free or what do we need to consider as cultural effects in our debate?

### REFERENCE:

Blum, W., Galbraith, P., Henn, H.-W., & Niss, M. (2007). *Modelling and applications in mathematics education. 14<sup>th</sup> ICMI study*. New York: Springer.

# ARE INTEGRATED THINKERS BETTER ABLE TO INTERVENE ADAPTIVELY? – A CASE STUDY IN A MATHEMATICAL MODELLING ENVIRONMENT

Rita Borromeo Ferri and Werner Blum

University of Hamburg and University of Kassel, Germany

*A lot of empirical studies have shown that the learning and teaching of mathematics is highly complex and influenced by many factors. The teacher, in particular, can promote high-quality lessons with a substantial learning outcome. In previous studies, preferred mathematical thinking styles and preferred interventions of teachers were reconstructed as important factors when teaching modelling in the mathematics classroom. In the present empirical study, the main question was whether there is a connection between certain thinking styles and certain intervention types of teachers. In this paper, first results and hypotheses of this quantitative-qualitatively oriented study will be presented.*

## THEORETICAL BACKGROUND

First, we will briefly give a theoretical background regarding the two basic elements of our study, mathematical thinking styles and teacher interventions.

A mathematical thinking style is “*the way in which an individual prefers to present, to understand and to think through mathematical facts and connections by certain internal imaginations and/or externalized representations. Hence, a mathematical style is based on two components: 1) internal imaginations and externalized representations, 2) the wholist respectively the dissecting way of proceeding.*” (Borromeo Ferri 2003)

Within the theory of mathematical thinking styles the term “preference” is important, because a *style*, speaking in the sense of Sternberg (1997), refers to how someone likes to do something, whereas an ability refers to how well someone can do something. So styles are preferences in using abilities. Sternberg’s general theory of styles influenced also the construct of mathematical thinking styles. On the basis of empirical studies (see Borromeo Ferri 2003, 2004) three mathematical thinking styles could be reconstructed so far with students from grades nine and ten and with teachers of secondary schools:

- *Visual thinking style*: Visual thinkers show preferences for distinctive pictorial imaginations and representations as well as preferences for the understanding of mathematical facts and connections in a holistic way. The internal imaginations are mainly effected by strong associations with experienced situations.
- *Analytical thinking style*: Analytic thinkers show preferences for formal imaginations and representations. They are able to comprehend mathematical

facts preferably through symbolic or verbal representations and prefer to proceed rather in a sequence of steps.

- *Integrated thinking style:* These persons combine visual and analytic ways of thinking and are able to switch flexibly between different representations or ways of proceeding.

In a further empirical study (see Borromeo Ferri 2010), one of the main questions was whether there is an influence of mathematical thinking styles particularly on the *modelling* behaviour of students and teachers. Summarising the results shortly only regarding the teachers' behaviour, it became clear that teachers who differ in their mathematical thinking styles have preferences for focusing on different parts of the modelling cycle while helping or discussing with students. More concretely: Analytic thinkers preferred the "mathematical part" of the modelling cycle and were more interested in formal solutions of modelling tasks, whereas the validation of the real results was not so important for them. Visual thinkers tended to think more in terms of the "real world" (the given situation in the task) and to enrich the reality with their own associations, whereas looking on formal solutions was not in their focus. Integrated thinkers found a balance between reality and mathematics for themselves and in their behaviour with their students. So mathematical thinking styles seem to have a substantial influence on the way teachers deal with modelling problems in the classroom.

Of particular interest is now whether and in which way certain types of teacher interventions are linked with certain mathematical thinking styles. In the following, we will refer to Leiß' (2007) theory of teacher interventions. Leiß characterises *teacher interventions* as all verbal, para-verbal and non-verbal interferences of a teacher in the solution process of students. *Adaptive* teacher interventions are characterised as supporting individual students in a minimal way, so that students can work as much on their own as possible (realising a balance between students' independence and teacher's guidance, in the spirit of Maria Montessori: "Help me to do it by myself").

Leiß (2007) carried out his empirical study for reconstructing teacher interventions within the DISUM project (see Blum/Leiß 2007b). He created, on the basis of existing theories of interventions (e.g. Zech 1996) and his own classification (Leiß/Wiegand 2005), a coding scheme for analysing his observations. He distinguished levels, activators and aims of interventions. These will be explained in the following tables into more detail, because we also used this coding scheme for our analyses.

## Working Group 6

Levels of interventions	Aims of interventions
<i>with regard to the content</i> interventions of the teacher referring to the content, here: the modelling process and the corresponding mathematics	<i>diagnose</i> teacher asks students about the current state of their solution process
<i>strategic</i> interventions concerning the meta-level, that is general aspects of the modelling and problem solving process	<i>evaluation/feedback</i> teacher gives feedback to students' solution process without further information or correction
<i>affective</i> interventions trying to influence the mental state of students	<i>indirect advice</i> subtle hints of the teacher in order to help students to find the "best way of solving" according to the teacher's opinion
<i>organizational</i> interventions concerning the basic conditions of students' working including group interactions or presentations	<i>direct advice</i> teacher gives relevant explanations and information to students explicitly
	<i>conscious non-intervention</i> no intervention of the teacher although students may have problems

**Table 1: Level and aims of interventions**

Activators of intervention	
invasive	own initiative of the teacher to intervene in the solution process of the students
responsive	students ask the teacher explicitly for advice

**Table 2: Activators of interventions**

Main results of Leiß' study were, among others, that strategic interventions are included in the intervention-repertoire of the observed teachers only very marginally and that the teachers often choose indirect advice where students have to find only one step by themselves in order to come over the difficulty.

More sophisticated results were obtained in the so-called main study of the DISUM project (see Blum 2011). Altogether 25 grade 9 classes participated in this study with a pre-, post- and follow up test design and with ten lessons devoted to

modelling tasks. The research focus was the comparison between two teaching styles: a more teacher guided “directive” style and an “operative-strategic” style focussing more on students’ independent work in groups. The results showed a significantly higher progress of modelling competency of students taught in the “operative-strategic” style. The best results concerning progress in modelling competency could be reconstructed in those classes where the balance between students’ independence and teachers’ guidance was realised best, based on experts’ ratings. This corresponds to a constructivist view of learning where students have to build their knowledge and competencies as actively and independently as possible, supported by the teacher. More generally, the background model for our studies is a view on “*quality mathematics teaching*” characterised by a demanding orchestration of teaching the mathematical subject matter (by giving students vast opportunities to acquire mathematical competencies and making connections within and outside mathematics), by a permanent cognitive activation of the learners and by an effective and learner-oriented classroom management (see Blum/Leiß 2007b and Blum 2011).

## RESEARCH QUESTION AND METHOD

The central question of our study was: *Do connections exist between teachers’ mathematical thinking styles and types of teacher interventions in mathematical modelling environments?*

In particular the results of the main study of the DISUM project, mentioned before, raised new questions. In fact, the “operative-strategic” style was more successful, but big differences among the teachers concerning the quality of teaching and the progress of their students could be observed, although all teachers got a special training for teaching the ten lessons modelling unit. So an open question is whether these differences can be explained more deeply with certain individual attributes of the teachers. The procedure of our study was as follows. Firstly, regarding data collection: we chose four of the 25 teachers of the main study of the DISUM project (for whom a particular lot of video material was available) for analysing the same two videotaped lessons with the tasks “Filling up I and II” (see Blum/Leiß 2007a). Additionally, we also conducted focussed interviews (Flick 1990) with all four teachers in order to reconstruct their mathematical thinking styles. The guideline for these teacher interviews was developed and used in the recent study Borromeo Ferri (2010). Examples of interview questions are: What does mathematics mean for you? Do you think that your attitude concerning mathematics changed through your teaching life? What is your preferred way of solving tasks (concretely: the “Filling up” task)? What is important for you when teaching mathematics?

The interviews were transcribed and analysed in combination with the actions of the teachers in the lessons for reconstructing their mathematical thinking styles. A basic instrument for the classification of the thinking styles was the coding scheme



developed by Borromeo Ferri (2003). In order to get more validity, two persons analysed the data independently, in the sense of so-called concurrent coding (Schmidt 1997, 559). With the same method, using the coding scheme of Leiß (2007), teacher interventions were reconstructed directly from the videos. For analysing the interventions, a structure was created so that interventions of the teachers could be better compared in several phases of the lessons. The following six phases were considered during the whole analysing process: 1. Introduction of the lesson (task “Filling up I”); 2. Group work; 3. Plenum (discussion about students’ results); 4. Improving students’ own results; 5. Plenum (reflections on the solution process); 6. Transfer to the second task “Filling up II”.

## RESULTS

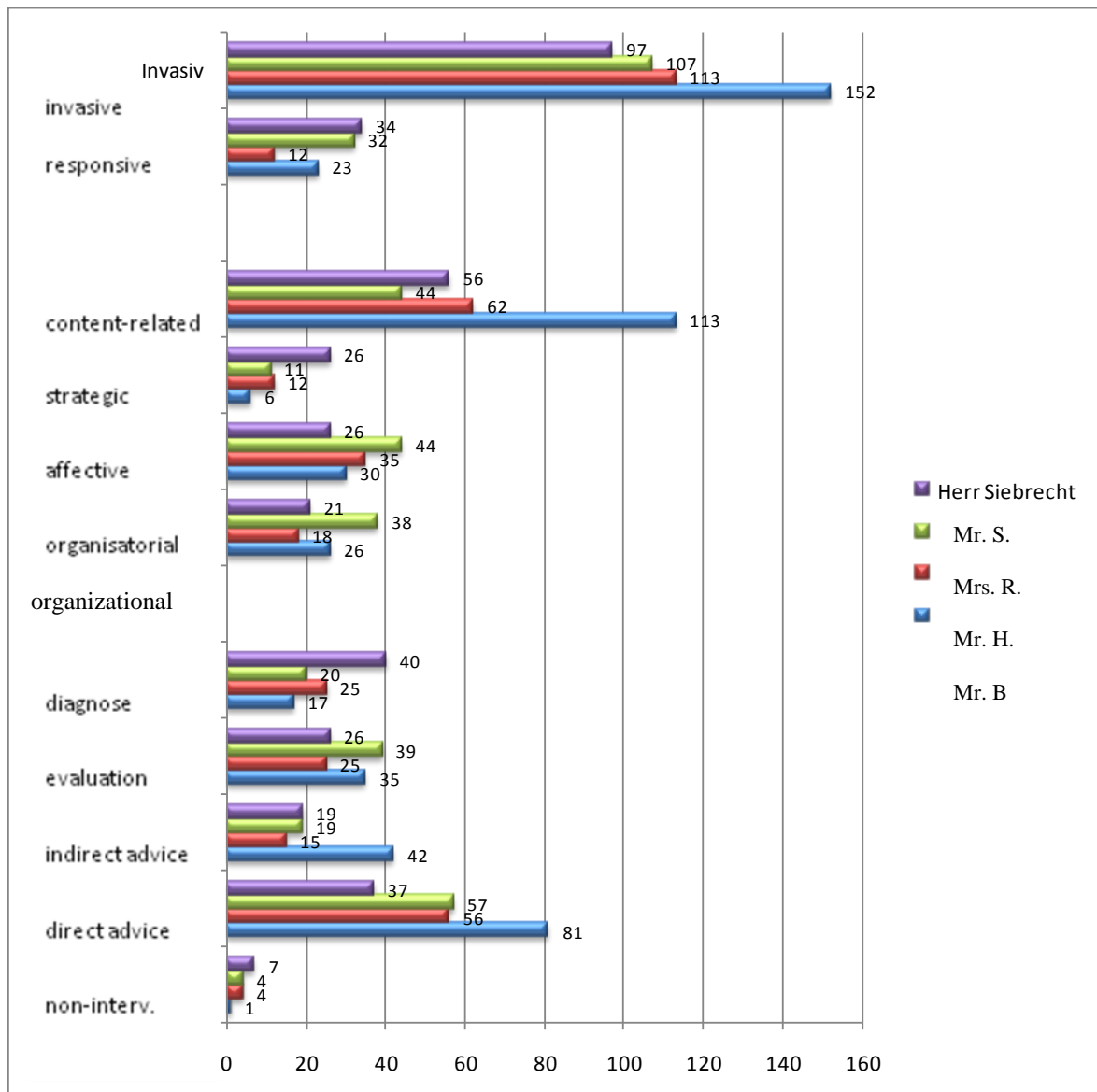
On the basis of the interviews and the videos, three of the teachers, Mr. H., Mr. B. and Mrs. R., could be reconstructed as analytic thinkers and one, Mr. S., as an integrated thinker. So a visual thinker was not in this (small) sample.

### **a) Quantitative analyses of teacher interventions (lessons “Filling up I and II”)**

The duration of the six phases was different in the four classes. A deeper look at phases 2 and 3, which means the group work and the first plenum, offered interesting results how the teachers acted. In Mrs. R.’s and Mr. S.’ lessons the pupils worked in the sense of the “operative-strategic” style independently in their groups. Mr. H. and Mr. B. were teaching in the “directive” style which is also their preferred way in everyday teaching: students sitting head-on, solving the tasks in the whole class. Mr. H. got the highest number of interventions namely in particular invasive, context-bound and directive interventions. During the plenum phase, only Mrs. R. and Mr. S. intervened. Because Mr. H. worked on the modelling problem with the whole class the students only got “one right result” that was not discussed further. Mr. S. asked every group of students to explain their results, which explains his high number of interventions. Mrs. R. also discussed the different results of the students.

In the following table 3, all interventions of all teachers during all phases are presented. Concerning the activators of the interventions, Mr. B. (152) has the the biggest number of invasive interventions, but also Mr. H. (115) and Mrs. R (107) intervened strongly invasively, in contrast to Mr. S. (87). Concerning the level of interventions, Mr. H. intervened 50% and Mr. S. 43% content-related, and Mr. B. even 65% Strategic interventions can be observed for Mr. S. twice as much as for the other teachers. Concerning affective and organisational interventions, Mrs. R. has the highest rate with 32% and 27%. Results of t-tests (confidence-interval 95%) showed that all these differences are significant.

## Working Group 6



**Table 3: Interventions in total**

### b) Qualitative analyses of teacher interventions (lessons “Filling up I and II”)

In the following, interventions and connections to mathematical thinking styles only of Mrs. R and Mr. S (both within the “operative-strategic” design) are reconstructed because of the restricted space. We do that firstly with the help of a table summarising the main characteristics of each teacher and secondly with a comparison of some concrete actions of the teachers. In the left column one can find the actions within the lessons and in the right column statements from the interview. We start with Mrs. R., reconstructed as an analytic thinker.

<i>Interventions and actions (lesson)</i>	<i>Reflections about own teaching (interview)</i>
<ul style="list-style-type: none"> <li>- lesson is structured</li> <li>- students can work independently in their groups</li> <li>- different interventions in the groups: less interventions in groups which work on the path preferred by Mrs. R., other groups are more or less guided during the solving process; altogether many affective and organizational interventions</li> </ul>	<ul style="list-style-type: none"> <li>- she introduces new themes in maths by a concrete example and then mainly algorithms are important for her</li> <li>- she prefers a clear structure of lessons</li> <li>- she does not like some mathematical contents such as riddles</li> <li>- she personally likes to solve tasks formally</li> </ul>

**Table 4: Mrs. R. – some characteristics**

*Interpretation:* Mrs. R. prefers and also shows well structured maths lessons. Analysing the two lessons as well as the other lessons in that teaching unit, it becomes clear that Mrs. R. intervenes regularly and consciously in the thinking and solving process of the students:

“I went from one group to the other and I always said think about it.”

Mrs. R. intends to have their students learn from mistakes:

“I always tell my students that it is not bad making mistakes, but learning from mistakes is good.”

However, this statement is in a certain contrast to her real actions in classroom. In all analysed lessons Mrs. R. was trying to prevent students from making mistakes and to guide them to her preferred solution. We interpret her well structured lessons in combination with her analytic thinking style as a kind of aplomb while teaching and a sign of her intention to avoid [ambiguity](#) regarding mathematical contents. She likes to have the control of as many aspects during lessons as possible, especially concerning the way of solving tasks:

“We have a certain ritual of solving tasks and I exercise this with my students.”

For Mrs. R. visualising is also relevant, but her preferred way of thinking becomes evident when she talks about solving a linear equations task:

“Personally I do not see the straight line, I see the actual data. Of course I know that behind is the system of equations.”

Altogether, according to our interpretation, Mrs. R. tries to compensate a certain one-sidedness, caused by her mathematical thinking style, by planning carefully each step during a lesson. She emphasised in the interview that acting this way is keeping her from ambiguity.

## Working Group 6

Mr. S. was reconstructed as an integrated thinker

<i>Interventions and actions (lesson)</i>	<i>Reflections about own actions (interview)</i>
<ul style="list-style-type: none"> <li>- lesson is structured</li> <li>- students can work independently in their groups</li> <li>- indirect interventions for getting partial solution of students without guiding them much</li> <li>- new ideas of students are picked up with extra information</li> <li>- helping the students to do things by themselves; he gives the same attention to all students and praises solutions not only when they are correct</li> <li>- aim of intervention is predominantly diagnosis or indirect, also often non-interventions</li> </ul>	<ul style="list-style-type: none"> <li>- mathematics is for him “everything”, training thinking processes as well as applications</li> <li>- he likes diversity in teaching maths and developed special methods for several phases within a lesson</li> <li>- during university he had problems with the formal exactness of mathematics</li> <li>- he sees a balance for himself between formal and visual acting</li> <li>- he welcomes different solutions of students</li> </ul>

**Table 5: Mr S. – some characteristics**

*Interpretation:* The main intentions and actions of Mr. S. can be summarised in one sentence: he tries to help the students to do it by themselves. For many students that is enough for being active during the whole lesson. Students are not afraid asking something. For him “the main principle is to inspire students for mathematics.” So that is why he created various methods. At the beginning of lessons he gives students sometimes a riddle or a logical story.

“Pupils should see that one can solve problems with the help of mathematics and that mathematics is training thinking processes.”

His own problem with the exactness of mathematics during his university studies has influenced his way of teaching and assessment at school. This becomes clear while analysing the two lessons:

“Students who understood the problem will not get a worse mark if they don’t do it totally formally correct.”

For Mr. S. it is important to support sustainable knowledge so he creates his lessons usually very open and uses several methods. He also welcomes individual solution processes of the students, for instance in the case of linear equations the whole spectrum from formal equations over tables to graphs. So we reconstructed in his lessons that Mr. S. is listening to students’ ideas and is doing some kind of

“juggling” with these ideas. He is encouraging every group to think in different directions and also to think about solutions of other groups.

### c) Reconstructed connections

Connections between interventions and mathematical thinking styles were reconstructed on the basis of the cases shown here, both with respect to qualitative and quantitative aspects. The teachers’ mathematical thinking styles were clearly correlated with their actions in lessons. The integrated thinker, Mr. S., had a more flexible repertoire of intervening, including strategic interventions, he let students work much more independently and develop more individual solutions. The analytic thinkers tried a lot more to guide students according to the teachers’ preferred ways of solving the tasks.

Now what was the effect of these ways of teaching? Comparing the learning progress of the students (see the remark in section 1; for more details see Schukajlow et al., 2009), Mr. S.’s students learned significantly more during this ten lesson teaching unit (progress: half a SD) than the students of the other three teachers (progress: quarter of a SD), in particular concerning modelling.

So our central hypothesis on the basis of our case study is:

*Integrated thinkers are better able to intervene flexibly and minimal-adaptively and thus get better results concerning students’ learning progress.*

Of course, this result ought to be replicated with a bigger sample of teachers before implications for teacher education can be drawn.

An open question is how teachers’ mathematical thinking styles and interventions are connected with other factors such as their subject-related professional knowledge and competencies or their epistemological convictions and beliefs. Perhaps the guiding forces behind both thinking styles and interventions are the same. This should also be subject of further studies on teachers.

## REFERENCES

- Blum, W. (2011). Can modelling be taught and learnt? Some answers from empirical research. In: Kaiser, G.; Blum, W.; Borromeo Ferri, R.; Stillman, G. (Eds), *Trends in teaching and learning mathematical modelling, ICTMA 14*. New York: Springer.
- Blum, W.; Leiß, D. (2007a). „Filling Up“– the problem of independence-preserving teacher interventions in lessons with demanding modelling tasks. In: Bosch, M. (Ed.), *Proceedings of CERME 4*. Guixol, pp. 1623-1633.
- Blum, W.; Leiß, D. (2007b). Investigating Quality Mathematics Teaching – the DISUM Project. In: *Developing and Researching Quality in Mathematics*

*Teaching and Learning* (Eds: C. Bergsten/ B. Grevholm), Proceedings of MADIF 5. Linköping: SMDF, pp. 3-16.

Borromeo Ferri, R. (2003). Mathematical thinking styles – an empirical study. In: *Proceedings of CERME 3*. Bellaria.

Borromeo Ferri, R. (2004). *Mathematische Denkstile. Ergebnisse einer empirischen Studie*. Hildesheim: Franzbecker.

Borromeo Ferri, R. (2010). On the influence of mathematical thinking styles on learners' modelling behaviour. In: *Journal für Mathematik-Didaktik*, 31 (1), pp. 99-118.

Flick, U. (1999). *Qualitative Sozialforschung, eine Einführung*. Rowohlt: Reinbek.

Leiß, D.; Wiegand, B. (2005). A classification of teacher interventions in mathematics teaching. In: *Zentralblatt für Didaktik der Mathematik*, 37, 3, pp. 240-245.

Leiß, D. (2007). *Hilf mir es selbst zu tun. Lehrerinterventionen beim mathematischen Modellieren*. Hildesheim: Franzbecker.

Schmidt, C. (1997). "Am Material": Auswertungstechniken für Leitfadeninterviews. In: Friebertshäuser, B.; Prengel, A. (Eds.): *Handbuch qualitativer Forschungsmethoden in der Erziehungswissenschaft*. Weinheim: Juventa, pp. 544-568.

Schukajlow, S.; Blum, W.; Messner, R.; Pekrun, R.; Leiss, D.; Müller, M. (2009). Unterrichtsformen, Emotionen und Anstrengung als Prädiktoren von Schüler-Leistungen bei anspruchsvollen mathematischen Modellierungsaufgaben. In: *Unterrichtswissenschaft* 37 (2), pp. 164-186

Sternberg, R. (1997). *Thinking Styles*. New York: Cambridge University Press

Zech, F. (1996). *Grundkurs Mathematikdidaktik*. Weinheim: Beltz

# **A MODELLING APPROACH TO DEVELOPING AN UNDERSTANDING OF AVERAGE RATE OF CHANGE**

Helen M. Doerr and AnnMarie H. O'Neil

Syracuse University

*Students' difficulties in reasoning about change are well documented in the research literature. In this paper, we describe a modelling approach to developing students' abilities to create and interpret models of phenomena where change is occurring and developing their concepts of rate of change. We describe the design of a model development sequence and its classroom implementation with 33 students just prior to beginning their university studies. The pre/post-test results show that the students made significant gains in their understanding of the concept of average rate of change. We present an analysis of the areas of greatest gains and of the difficulties some students encountered in their interpretations of changing phenomena.*

Much research over the last twenty years has documented the difficulties that students encounter when creating and interpreting models of changing phenomena. Several studies have examined the role that students' concepts of function and rate of change play in students' abilities to represent and to reason about dynamic events (Carlson et al., 2002; Monk, 1992; Thompson, 1994). As Oerhtman, Carlson and Thompson (2008) argue, in order to reason dynamically, students must be able to simultaneously attend to both the changing values of the output of a function and the rate of that change as the input values vary over intervals in the domain. The complexity of such reasoning has proven difficult even for high achieving undergraduate students (Carlson, 1998). A related line of research (Confrey & Smith, 1994; Michelsen, 2006; Shternberg & Yerushalmy, 2003) has emphasized the role of context in the development of students' reasoning about changing phenomena. Michelsen (2006) argues that one source of difficulty for students in applying functions in context is that students fail to treat variables as related quantities that change and hence have difficulties in recognizing that functions are tools for describing, explaining and predicting the relationships among changing quantities.

The broad goal of our research is to understand the inter-related development of students' abilities to create and interpret models of phenomena where change is occurring and the development of students' concepts of rate of change. We also wish to move beyond descriptions of single model eliciting activities to characterize more fully the nature of model development sequences (Lesh et al., 2003; Doerr & English, 2003). In this paper, we describe a sequence of model development tasks that was designed to support the development of students' concept of rate of change by engaging them in creating and interpreting models of physical phenomena that change. In particular, we are interested in two research questions: (1) how did the sequence of modelling tasks support the development of students' concepts of rate of

change and (2) to what extent were students able to interpret rates of change in context?

## **THEORETICAL BACKGROUND**

Modelling approaches to the teaching and learning of mathematics encompass a wide range of theoretical and pragmatic perspectives (Kaiser & Sriraman, 2006). As Kaiser and Sriraman point out, modelling research based in the “contextual modelling” perspective draws on the design of activities that motivate students to develop the mathematics needed to make sense of meaningful situations. Much work done within this perspective draws on model eliciting activities (MEAs) developed by Lesh and colleagues (Lesh et al., 2003). Model eliciting activities confront the student with the need to develop a model that can be used to describe, explain or predict the behavior of familiar or meaningful situations. Such MEAs encourage students to engage in a cyclic process where they express, test, and refine their own ways of thinking about meaningful situations.

However, a single MEA in isolation is seldom enough for a student to develop a generalized model that can be used and re-used in a range of contexts. To achieve this goal, we argue that students need multiple opportunities to explore the relevant mathematical constructs and to apply their model in new settings. Sequences of structurally related model exploration activities and model application activities are needed, accompanied by discussions and presentations that focus on the underlying structure of the model and on the strengths of various representations and ways of using them productively. Each stage of this sequence engages students in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students. Thus, in this study, we have designed an instructional sequence (described in more detail in the next section) that begins by engaging students with meaningful problem situations that elicit the development of significant mathematical constructs. Students then explore and apply those constructs in other situations leading to the development of a model that is usable in a range of contexts. This study was designed to examine students’ learning of the concept of average rate of change when engaged in a sequence of model development tasks.

## **DESIGN AND METHODOLOGY**

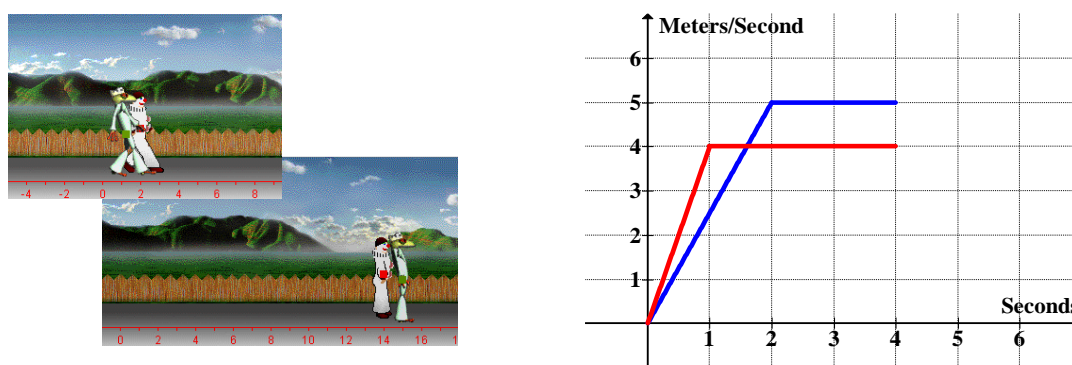
### **The Model Development Sequence**

Model development sequences are structurally related activities that encourage student exploration and are accompanied by teacher-led discussions, student presentations, and summaries so as to focus attention on the structural similarities among the tasks and on the use of representations across the activities. As in our prior work, we began this sequence with a model-eliciting activity, using a familiar and meaningful situation, in this case, motion along a straight line. Students were



asked to create graphs using a motion detector and their own bodily motion and to generate descriptions of that motion. The graphs included comparative situations of faster and slower constant speed, changing speed and changing direction, and graphs where the motion was not physically possible.

Following the model-eliciting activity, the students engaged in several model exploration activities. An important goal of these activities was to engage students in using everyday language to make sense of the average rate of change in two different contexts and to develop their understanding of the representational systems for describing change. These activities were designed to help students to think with the conceptual system (or model) and to think about the underlying structure and generalizability of the model. The first model exploration activity used SimCalc Mathworlds (Kaput & Roschelle, 1996). This activity reversed the representational space of the model-eliciting activity where bodily motion created a position graph and extended that space as the students made velocity graphs that generated the motion of a simulated character. From this motion, the students created position graphs, thus developing an understanding of how the position graph could be constructed by calculating the area between the velocity graph and the  $x$ -axis. In exploring this linked relationship between the velocity and position graphs, students began to reason about the position of characters solely from information about the velocity of the characters. For example, in one task, the students were given written descriptions of the motion of two characters and asked to create appropriate velocity graphs, such as that shown in Figure 1.



**Figure 1. Interpreting position from a description of velocity.**

The students were then asked to determine which character had walked farther and who was walking faster at various points in time. The students were asked to justify whether or not the two characters would ever be at the same position at the same time. This model exploration activity provided an opportunity for students to develop their abilities to interpret position information from a velocity graph and velocity information from a position graph.

The second model exploration activity used the “Gym” task from the interactive mathematics textbook by Yerushalmy (2005). This applet (<http://www.cet.ac.il/math/function/english/line/rate/rate10.htm>) was designed to help students understand

how the rate of change is expressed in table values, graphs, and equations. Using the context of training plans on a weight-lifting machine in a gym, the students explored the difference between constant and non-constant rates of change. Specifically, they investigated the graphical and numerical representations of weight training plans where the weight lifted increased or decreased at a constant rate, at an increasing rate or at a decreasing rate. The students were asked to explore the applet in order to make generalized observations about how setting the initial weight, the first change in weight and the change of the change affected the shape of the corresponding graph and the table of values.

The third component of the model development sequence consisted of two model-application activities that focused on applying their model to new problem situations. This was intended to lead to a generalized understanding of average rate of change. In the first task, students were asked to create a model of the intensity of light with respect to the distance from the light source, to analyze the average rates of change of the intensity at varying distances from the light source and to describe the change in the average rates of change as the distance from the light source increased. The second model application task investigated the rate at which a fully charged capacitor in a simple RC circuit discharged with respect to time. The students built the circuits, charged the capacitor, and then measured the voltage drop across the capacitor as it discharged. Students were given a set of resistors and capacitors and were asked to develop a model they could use to answer these three questions:

- (1) How does increasing the resistance affect the rate at which a capacitor discharges?
- (2) Compare the rates at which the capacitor is discharging at the beginning, middle and end of the total time interval. How does the average rate of change of the function change as time increases?
- (3) How does increasing the capacitance affect the rate at which a capacitor discharges?

Taken together, these two model-application tasks focused the students' attention simultaneously on the quantity that was measured and on how that quantity was changing with respect to some other quantity (i.e., distance or time). A coordinated understanding of these two measurements is at the crux of representing and reasoning about dynamic events (Oerhtman et al., 2008).

### **Rate of Change Concept Inventory**

To measure students' understanding of average rate of change, we designed a "Rate of Change Concept Inventory" consisting of items in four categories: algebraic expressions (5 items), graphical interpretation (8 items), symbolic interpretation (3 items), and purely contextual (1 item). Fourteen of the items on this test were drawn from the research literature (described earlier) on students' conception of rate of change. Three items were developed to test the students' mastery of the algebraic

representations involved in expressing and computing average rates of change. The development of this concept inventory is part of our on-going work.

### **Setting, Participants, Data Collection and Analysis**

This model development sequence formed the basis for a six-week course, taught by the second author, to students who were preparing to enter their university studies. There were 33 subjects from this course who volunteered to participate in the study. Eleven of the participants were female and 22 were male. All but one participant had completed four years of study of high school mathematics; 18 students had studied calculus in high school and 15 had not studied any calculus. The participants worked in groups of three or four to complete the model-eliciting tasks and model-application tasks. The model exploration tasks were done individually at a computer; however, the participants were encouraged to discuss their work with each other. Following each task in the sequence, there was a whole-class discussion which usually involved students in presenting the results of the work produced during the model-eliciting and model-application tasks. The class discussion following the model-exploration tasks focused on the structural features of the model and on the relationships among different representational systems. The students worked in pairs to complete final reports on their findings for each of the model-application tasks. All participants completed all of the tasks in the model development sequence described above. The written work from these collaborative tasks and from individual course examinations was collected and analyzed. All participants completed the 17 item pre- and post-test of “Rate of Change Concept Inventory.” The overall pre- and post-test scores and the four sub-scores were analyzed using t-tests.

## **RESULTS**

### **Change in students’ understanding of average rate of change**

The post-test results show that there was a significant improvement in the students’ understanding of the concept of average rate of change from an overall score of 52% correct to 75% correct, as measured by the 17 items on the test. There was a significant improvement in three of the sub-score areas: algebraic expressions, graphical interpretations, and symbolic interpretation. Table 1 displays the overall performance on the pre- and post- test and the performance on each of the sub-scores for  $n = 33$  students.

While the overall scores improved by 23%, there were seven items on the concept inventory for which the improvement was greater than 30%. Two of these were algebraic expression items, four were graphical interpretation items, and one was a symbolic interpretation item. The two algebraic expression items asked the student to find the equation of the line joining two points and to calculate the average rate of change between two points on a parabola.

	<b>Pre-test</b> mean (sd), percent correct	<b>Post-test</b> mean (sd), percent correct	p-value
Overall	8.87 (3.47), 52%	12.69 (2.81), 75%	<0.001
Algebraic	2.64 (1.27), 53%	3.70 (1.13), 74%	<0.001
Graphical	4.96 (1.82), 62%	6.78 (1.31), 85%	<0.001
Symbolic	0.94 (0.90), 31%	1.70 (1.02), 57%	<0.001
Contextual	0.33 (0.48), 33%	0.52 (0.51), 52%	n.s.

**Table 1. Pre-test and post-test results for overall concepts and subscores.**

There were substantial gains on two items that measured students' proficiency in being able to express algebraically two basic ideas about average rate of change. On the item that asked for the equation of a function with a constant rate of change  $n=16$  (48%) students answered the item correctly on the pre-test and  $n=26$  (79%) answered correctly on the post-test. Similarly, on the item that asked students to compute the average rate of change for a function with a non-constant rate of change the number correct went from  $n=8$  (24%) on the pre-test to  $n=22$  (67%) on the post-test.

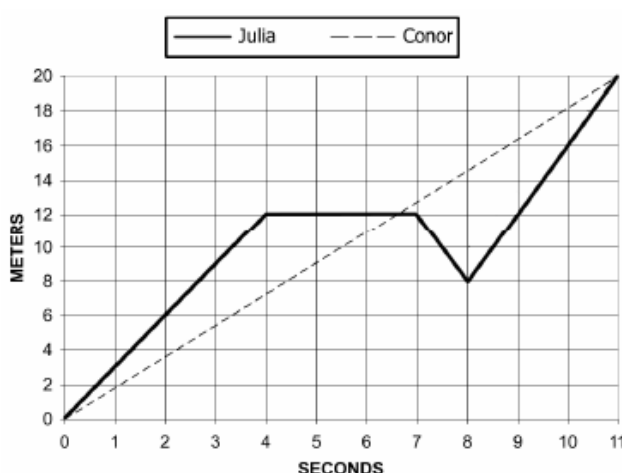
The symbolic interpretation item required the student to interpret the meaning of the parameters in an exponential growth function: "The model that describes the number of bacteria in a culture after  $t$  days has just been updated from  $P(t)=7(2)^t$  to  $P(t)=7(3)^t$ . What implications can you draw from this information?" There was a substantial gain on this question of 40 percentage points from the pre-test ( $n=12$ , 36%) to the post-test ( $n=25$ , 76%). This likely reflects the emphasis in the model development sequence on making meaningful interpretations of data and on giving descriptions of the average rate of change in various contexts, including the exponential change in the circuit task.

There were substantial gains on four items that measured students' proficiency at interpreting rate of change when given graphical information (see Table 2).

<b>Graphical Items</b>	<b>Pre-Test</b>		<b>Post-Test</b>			
	n	%	n	%	$\Delta n$	$\Delta\%$
Q4 C	19	58%	29	88%	10	30%
Q5 C and	22	67%	32	97%	10	30%
Q5 D	20	61%	30	91%	10	30%
Q8 A	18	55%	28	85%	10	30%
Q10	5	15%	19	58%	14	43%

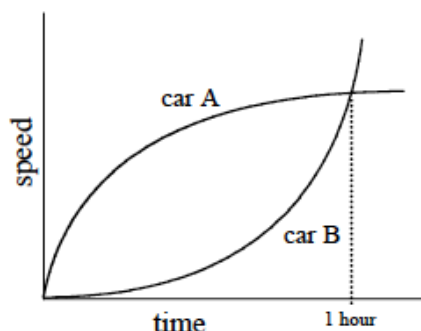
**Table 2. Improvement on graphical interpretation items**

Questions Q4 C, Q5 C and D, and Q8 all addressed interpreting information about velocity when given a position graph. Question Q10, on the other hand, involved interpreting position information when given a velocity (or speed) graph. This coordination of representational systems (shifting between the velocity or rate graph and its associated position graph) was the main focus in the model-exploration tasks described earlier. Due to space limitations, we will illustrate these results with Q5 C and D and Q10.



**Figure 2. A graphical representation of a 20-meter race between two runners.**

Item Q5 C and D asked for a description of the runner whose graph is shown by the solid line in Figure 2 over two time intervals: from 4 seconds to 7 seconds and from 7 seconds to 8 seconds. On the post-test, 97% and 91% of the participants were able to correctly describe the runner as standing still and as moving back towards the starting position. In other words, the participants were able to correctly reason about the velocity (or average rate of change) over an interval when given a position (or quantity) graph.



**Figure 3. Interpreting the relative position of two cars given their speed.**

Item Q10 represents an important reversal of the above problem and one that is a well-known source of difficulty for calculus students (Monk, 1992). This item requires an understanding of how to reason about position when given a velocity

graph. We found a 43% improvement in the number of students who were able to correctly interpret the relative position of two cars, starting from the same position and travelling in the same direction, when given the speed graph shown in Figure 3.

### Interpreting rate of change in context

In this section we report on our analysis of the students' responses to a written examination item. The students were given a data set of the voltage drop across a capacitor for 50 seconds and asked to compute the average rate of change over three subintervals, from  $t = 5$  to  $t = 10$  seconds,  $t = 20$  to  $t = 25$  seconds and  $t = 40$  to  $t = 45$  seconds. After computing these average rates of change, the students were asked to "Write two or three sentences interpreting this data." Since this data is modelled by an exponential decay function, the voltage is decreasing over each sub-interval but the average rate of change is increasing across the sub-intervals. Describing such functions as decreasing at an increasing rate presents both conceptual and contextual challenges. The conceptual challenges reside, in part, in coordinating one's understanding of the change in function values with the rate of that change over various subintervals; the difficulty of the comparisons becomes more complex when the rates are negative but increasing. The contextual challenges arise from the difficulties in natural language in describing magnitude of the voltage drop, since this magnitude decreases, while the signed average rate increases as it becomes less negative. Everyday language for describing the change in rate appears in conflict with formal mathematical language for describing that change.

We categorized students' responses to this item into four levels. These levels and the number of student responses are shown in Table 3. Approximately 61% of the students correctly described both the behavior of the function and its average rate of change with half of those responses (10 out of 20) including a reference to the context.

Category	Number of Responses
(3) Correctly described both function and rate of change, referring to context	10
(2) Correctly described both function and rate of change, minimal or no reference to context	10
(1) Correctly described function, but not rate of change	1
(0) Incorrect description	12

**Table 3. Interpretations of rate of change**

An example of a response in category (3) is: "The average rate of change increases (gets less negative) as time progresses, meaning the voltage on the capacitor is decreasing at an increasing rate." An example of a response in category (2)

expressed that the function is decreasing, while “the average rate of change is going from a more negative number to a less negative number, meaning the average rate of change is increasing over time.” Although this response does not specifically refer to the context of the problem, namely the voltage drop across the capacitor, the student correctly distinguishes between the change in the function and the change in the average rate of change.

Of the 12 responses in category (0), six of those responses indicated that the student conflated the changes in the function with the changes in the average rate of change. An example of such a response is: “The average rate of change is decreasing at an increasing rate. This is because the numbers are getting closer to zero.” It is not the average rate of change that is decreasing, but rather the function values (or voltages) are decreasing. The values that the student correctly calculated for the average rate of change were negative and increasing since those numbers were “getting closer to zero.” This student is not using the values of the average rates of change to infer the characteristics of the function (decreasing at an increasing rate), but rather the student incorrectly concludes that it is the average rate of change itself that is decreasing at an increasing rate. This error points to the difficulty of simultaneously attending to changes in the function values and changes in the average rate of change.

## DISCUSSION AND CONCLUSIONS

The results of this study provide some evidence that the model development sequence had a positive impact on the students’ understanding of average rate of change, as measured by the statistically significant overall gain on the “Rate of Change Concept Inventory.” The gains on the graphical items may be due to the model exploration tasks that focused on the coordination between representational systems. This coordination included shifting between position and velocity graphs, thus distinguishing between the function’s graph and the graph of its rate of change, and shifting between numerical and graphical data, thus linking the value of the average rate of change with the graph of the function. Overall, the majority of students were able to give meaningful interpretations of data and descriptions of change in contexts. At the end of the study, a small number of students still had difficulty in distinguishing between changes in the function values and changes in the average rate of change. The difficulty in expressing this distinction suggests the need for further research on the development of students’ language for describing changing phenomena.

## REFERENCES

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *CBMS*

*Issues in Mathematics Education: Research in Collegiate Mathematics Education III*, 7, 114–162.

- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *JRME*, 33(5), 352–378.
- Confrey, J. & Smith, E. (1994). Exponential functions, rates of change and the multiplicative unit. *ESM*, 26, 135-164.
- Doerr, H. M., & English, L. D. (2003). A modeling perspective on students' mathematical reasoning about data. *JRME*, 34(2), 110-136.
- Kaiser, G. & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM*, 38(3), 302-310.
- Kaput, J. & Roschelle, J. (1996). *SimCalc: MathWorlds*. [Computer software].
- Lesh, R. A., Cramer, K., Doerr, H. M., Post, T., & Zawojewski, J. (2003). Model development sequences. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp. 35-58). Mahwah, NJ: Lawrence Erlbaum.
- Michelsen, C. (2006). Functions: A modelling tool in mathematics and science. *ZDM*, 38(3), 269-280.
- Monk, S. (1992). Students' understanding of a function given by a physical model. In E. Dubinsky & G. Harel (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175-194). Washington: MAA.
- Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-41). Washington: MAA.
- Shternberg, B. & Yershalmy, M. (2003). Models of functions and models of situations: On the design of modelling-based learning environments. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning and teaching* (pp. 479-498). Mahwah, Erlbaum.
- Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *ESM*, 26, 229-274.
- Yerushalmy, M. (2005). Functions of interactive visual representations in interactive mathematical textbooks. *International Journal of Computers for Mathematical Learning*, 10, 217-249.



# AN INVESTIGATION OF MATHEMATICAL MODELLING IN THE SWEDISH NATIONAL COURSE TESTS IN MATHEMATICS

Peter Frejd

Department of Mathematics, Linköping University

*Mathematical modelling is one competence assessed in the Swedish national course tests in mathematics (NCT) in upper secondary school. This paper presents, with empirical data from the NCT, a content analysis about what aspects of mathematical modelling are assessed and how it is assessed. The main conclusion is that only fragments (intra-mathematical aspects) of the modelling process are being assessed.*

## INTRODUCTION

Supported by curricula that stress the use of ‘real world problems’ in mathematics teaching, researchers in the field of mathematical modelling and applications in mathematics education have argued for more mathematical modelling activities into mathematics classrooms. However, the presence of modelling activities in day-to-day teaching is still limited at many places (Blum, Galbraith, Henn & Niss, 2007). The present situation in Sweden is quite similar, where the role of mathematical modelling in teaching and learning mathematics has been made more and more explicit in official curriculum guidelines for upper secondary school since 1965 (Ärlebäck, 2009). However, Frejd and Ärlebäck (in press) found that only 23% of the upper secondary students from a sample across Sweden (n=381) stated that they had used or heard about mathematical models or modelling in their education. In addition, the meaning of the notion mathematical modelling is only described in implicit terms in the curriculum, which opens up for various interpretations of the notion and how it should be implemented into school activities (Ärlebäck, 2009). A possible way to make modelling become a more wide spread activity in the classroom is to use more modelling activity in assessment tasks, according to Niss’ (1993) premise: “What is not assessed in education becomes invisible or unimportant” (p. 27). One activity assessing mathematical modelling for all students in Swedish upper secondary school is the national course tests in mathematics (NCT).

The aim of this paper is to investigate *what* is assessed in the national course tests about mathematical modelling and *how* is it assessed. It also investigates what interpretations of the notion of mathematical modelling have been done by its authors of the NCT and what reasons they provide for those interpretations.

## THE ‘MODELLING CYCLE’

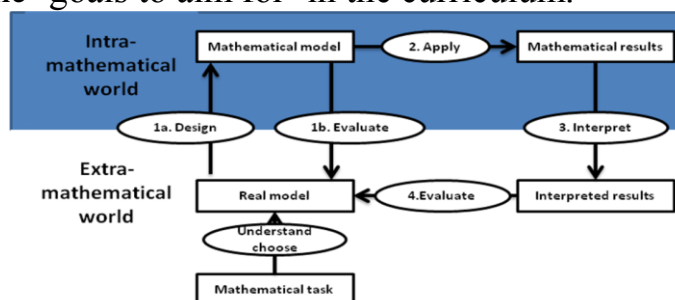
The notion of mathematical modelling is not unambiguously defined and is depending on the theoretical perspective adopted (Frejd, 2010). The theoretical

perspective used in this paper refers to a conception of mathematical modelling known as the ‘modelling cycle’. The ‘modelling cycle’ is a general mathematical modelling process (Kaiser, Blomhøj, & Sriraman, 2006) described in different ways depending on the research aim (Borromeo Ferri, 2006). One such cyclic process is described in terms of six phases and transitions between these, see for example Borromeo Ferri (2006). The starting point of the modelling process is called the *real situation (RS)*, which might be a description of the problem in everyday knowledge in the domain of an ‘extra-mathematical world’ or ‘real world’. A *mental representation (ME)* of the situation is made to understand the task. Then the information from the mental representation is idealized, filtered and simplified/structured to get a *real model (RM)*, which might be an external representation of the situation. This real model is then mathematized into a *mathematical model (MM)* in the domain of a ‘mathematical world’. Finally the modeller or the modellers work mathematically with the mathematical model to get solutions, *mathematical results (MR)*, which are interpreted in terms of *real results (RR)* and validated in the ‘extra-mathematical world’. The validation may show that the real result is not satisfactory and that other aspects have to be taken into consideration; the modeller or the modellers then have to do the process over again and make another lap in the cycle. In educational research literature one can find researchers both in favour of as well as critical to this cyclic perspective on modelling (cf. Jablonka & Gellert, 2007; Ärleback, 2009; Frejd, 2010). In this study the general modelling process was used, because the construction of national course tests in mathematics (NCT) is based on an interpretation of the curriculum guidelines by Palm, Bergqvist, Eriksson, Hellström and Häggström (2004), drawing such a view on modelling.

### **The modelling cycle and the National Course Tests (NCT)**

Palm et al. (2004) have done an analysis and an interpretation of the present national curriculum guidelines, for upper secondary school in Sweden, as a basis for the construction of the NCT in mathematics courses B-D [1]. They state that “[...] this interpretation is one of many possible interpretations of the official curriculum guidelines” (Palm et al., 2004, p. 1, my translation). The result of their interpretation is described in terms of six mathematical competences (problem solving, algorithm, concept, modelling, reasoning, and communicating) similar to those adopted in PISA, TIMSS2003, NCTM (USA) and KOM (Denmark) (Palm et al., 2004). According to Palm et al. (2004) the modelling competence is needed to solve ‘real world’ problems. Some situations may be familiar to the students and the modelling process is then a routine procedure related to the algorithm competence. Other situations may be new to the students and demands a “*genuine modelling process*” (Palm et al., 2004, p. 18), related to the problem solving competence (no discussions are made if the modelling competence has connections to reasoning-, concept- or communication-competences). The *genuine modelling process* is defined as the entire process displayed in Figure 1. The process is cyclic and similar to the

‘modelling cyclic’ involving two domains as described in the previous section and is interpreted from the ‘goals to aim for’ in the curriculum.



**Figure 1.** The structure of the modelling process from Palm et al. (2004, p.18, my translation).

The ‘goals to aim for’ in the curriculum declare that mathematics teaching should ensure the students to “develop their ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models” (Skolverket, 2000). How this statement has been turned into a cyclic process may not be obvious, but can be seen as one interpretation (cf. above). In relation to this interpretation, Palm et al. (2004) have identified five different types of modelling items that test; 1. The entire modelling process, 2. Parts (essential, 1a, 1b and 3; see Figure 1.), 3. Real life problems, 4. Open problems and 5. Items with too much or too little information.

The *genuine modelling process* is a holistic view about the modelling process. The competence to master this process similar to Blomhøj and Højgaard Jensen’s (2003) definition “[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context” (p. 126). I have adopted Blomhøj and Højgaard Jensen’s definition, also used in Frejd and Ärleback (in press), and used it in the next section.

## METHODOLOGY

To address the aim and based on the discussion in the previous section, the research questions are: Which transitions of the modelling process are presented and what are put aside in the national course tests? What types of problems are being used to assess modelling? How is it assessed in relation to the ‘official guidelines for assessment’ (a guideline for teachers on how to assess students’ solutions of items in the NCT)?

For examining course documents Robson (2002) suggests the method of content analysis. I have followed Robson’s guidelines for content analysis which are: start with the research question; decide on a sampling strategy; construct categories for the analysis which are to be mutually exclusive (categorized in one way), exhaustive (everything relevant should be categorized) and operational (explicit specifications); test the coding on samples of items and assess reliability.

The sample used is the last ten years of national tests in mathematics D (a total of 19 tests). It is the last mandatory course in upper secondary school (science programme) and it concludes “an in-depth knowledge of concepts and methods learned in earlier courses” (Skolverket, 2000); the main content is comprised by trigonometry, derivative and integrals, which may be used to assess mathematical modelling in the NCT. The strategy used in the analysis is to first identify items in the test that have an extra-mathematical context and then to scrutinize them.

The extra-mathematical domain is related to what is often called the ‘*real world*’ (Blum, Galbraith, Henn & Niss, 2007). To delimit the possibility to miss any modelling items with an extra-mathematical context an exclusion of texts related to pure mathematics has been done. Examples of excluded texts begin with “*Arrange the following numbers in increasing order*” and “*A triangle has the sides...*”, in contrast to “*water flows with a constant speed into an empty container*” and “*a tone sounds different if it is played on an organ or a violin*”(my translations).

In the definition adopted from Blomhøj and Højgaard Jensen (2003) the modelling competence is divided into sub-competencies (describing a more atomistic view of the process) to create exhaustive categories to answer the research questions. To capture significant aspects about the transitions in the modelling process, 11 *modelling categories* were developed. The categories or guiding questions used in the analysis, see the Appendix, were developed from: I) the curriculum guidelines (develop their ability to design, fine-tune etc.); II) Palm et al. (2004) (the five identified items, etc.); III) a research tool from Hains, Crouch and Davis (2000) designed to measure students’ ability in sub-competencies, IV) Other research literature discussing modelling competencies such as Mass (2006).

One reason for the choice of the material (I-IV) is that both the curriculum guidelines and the interpretation by Palm et al. (2004) are the base for the construction of modelling items in the NCT. Another reason is that the research tool (a multiple choice test) from Hains, Crouch and Davis (2000) has been used in several research studies (for details see Frejd and Ärlebäck, in press) and is measuring sub-competencies. The research literature focusing on modelling competencies was a help to strengthen and fortify the 11 modelling categories.

The process of linking together the material (I-IV) and to get as mutually exclusive categories as possible was done based on the transitions between the *phases* such as real model to mathematical model. To illustrate the construction process of the categories I will use an example, starting with the transition from the real situation to the real model. I identified aspects from: I) Skolverket (2000), the student should develop their ability to interpret a problem situation; II) Palm et al. (2004), open problem and understand the problem; III) Hains, Crouch and Davis (2000) the sub-competence *to make simplifying assumptions concerning the real world problem*; and IV) Maass (2006) a competence “to make assumptions for the problem and

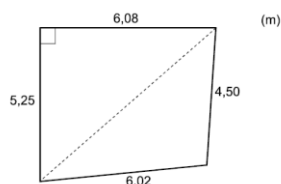
simplify the situation” (p. 116). These aspects turned into the modelling category/ guiding question nr 1 “Do students need to make simplifying assumptions about the problem situation in order to solve the problem?” To make the guiding questions more operational by the use of explicit specifications the example, *How long time does it take to evacuate your school?*, is discussed for each category (see Appendix).

To answer what types of problems being used the following questions were asked to each item: From where in the modelling process is the item originating, i.e. the starting point of investigation? What type of model do the students need to use? Is it a ‘realistic question’, i.e. is it stated in the text from where the data arise, such as real places (names of cities, countries etc), time (year, date) and is it possible to find similar data (e.g. on the web)?

To analyse the last research question, what do the ‘official guidelines for assessment’ outline (emphasize) in order to assess the item, have I developed *assessments categories*. These categories are based on frequently used words in the official guidelines for assessment and they are *motivation* (explanations, insight etc), *solution strategy* (general investigation, a general approved solution etc), *result* (correct answer, approved determination of something, approved solution with a specified answer etc), *to set up a mathematical expression* (to set up a equation, model etc) and *math language* (students use of math symbols, conventions etc).

I have analysed all 315 tasks (items) in the 19 national course tests (D-course) from the years 2000-2009. To illustrate the method for analysing I will present two examples (see Figure 2) from the test in spring 2005. First some information about the NCT (D-course) is provided. The NCT (D-course) is divided in two parts and the limitation of time is four hours. Part one is without access to calculators and part two with access to a calculator. Part two includes an item of *performance assessment* (i.e. an essay answer task that invites students to write a more extensive solution in some paragraphs).

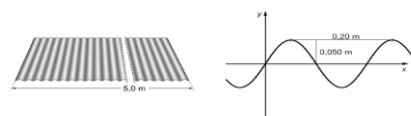
12. Daniel och Linda tittar på en lägenhet. Enligt uppgift är vardagsrummet  $31,2 \text{ m}^2$ . De vill kontrollera om detta stämmer och mäter väggarna och ritat en skiss över rummet. De vet att ett hörn i rummet är rätvinkligt. Så här ser deras skiss ut.



Vilken area har vardagsrummet enligt Daniels och Lindas skiss?

(2/2)

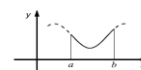
15. En korrigerad plåt tillverkas genom att en plan plåt veckas. Sedd från sidan har den korrigerade plåten på bilden formen av en sinuskurva med perioden  $0,20 \text{ m}$  och amplituden  $0,050 \text{ m}$ .



- a) Bestäm en formel för "plåtkurvan" på formen  $f(x) = A \sin kx$  (0/1)

Det finns en formel för beräkning av kurvlängd. Enligt denna gäller att längden  $s$  av en kurva  $y = f(x)$  från  $x = a$  till  $x = b$  kan beräknas som:

$$s = \int_a^b \sqrt{1 + (f'(x))^2} dx$$



- b) Hur lång plan plåt ska man utgå ifrån för att den korrigerade plåtens längd ska bli  $5,0 \text{ m}$ ? (0/3/0)

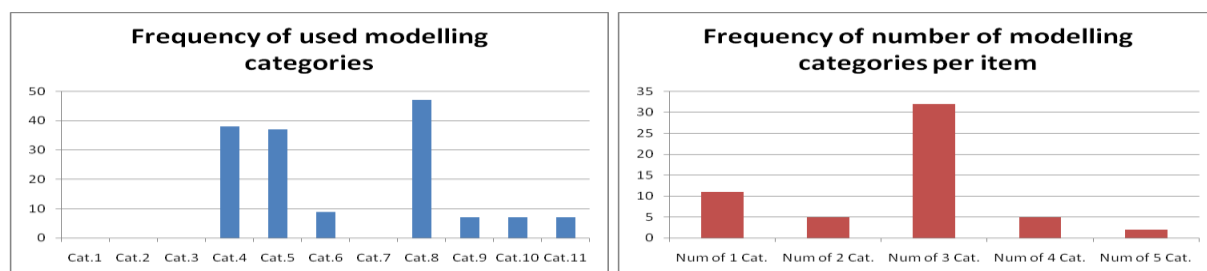
**Figure 2. Two tasks (items) from the NCT (D-course) spring 2005 retrieved from: <http://www8.umu.se/edmeas/np/>.**

The item to the left in Figure 2 is about Daniel and Linda who want to evaluate if the size of a living room in a flat is correct. The outset for investigation may appear to be the *real situation*, but the focus is on *the real model* which is given (a sketch). It may

also look realistic with a reasonable size of a living room, but it is not described where it is and it is not described how the angle is measured. The modelling categories used are nr 4, 5, 8 and 9 (see Appendix), because the students are supposed to assign the diagonal as a variable, set up a trigonometry model, calculate a result and compare to 31,2 m<sup>2</sup>. No assumptions, no clarifications etc. are needed (i.e. category 1 and 2). The guidelines for assessment and marking emphasize the assessment category result (calculations of diagonal +1point, calculation of any angel +1p, approved solution +1p, approved answer (29,4 m<sup>2</sup>) +1p marked [*result, result, solution strategy, result*]). The aim of the right item in Figure 2 is to decide a formula for a “platecurve” on  $f(x)=Asinkx$  and to decide the length of a flat plate needed to create a 5 m long “platecurve”. The starting point of investigation is both the real model and the mathematical model. It seems not to be realistic: why do you want to have the shape of the curve in  $Asinkx$ ? Why a curve in the first place? And in the second question a plater never makes a 5 m wide plate as it would be impossible to work with. The modelling categories used are nr 4, 6 and 8, because the students are expected to assign a variable in the integration (let  $x$  go from 0 to  $z$ ), fine tune the model (search for  $A$  and  $k$ ) and calculate a result (7,3 m). However, no formulations of any new models are needed, only to use the given ones. The mathematical models used here are the trigonometry model and the integration model (which includes differentiation). The guidelines for assessment emphasize to set up a math expression and to find a result. For assessing reliability of the coding and the additional questions, Holsti’s method (Holsti, 1969) was used for two independent coders, with a satisfactory result ( $0.89 > 0.8$ ).

## RESULTS

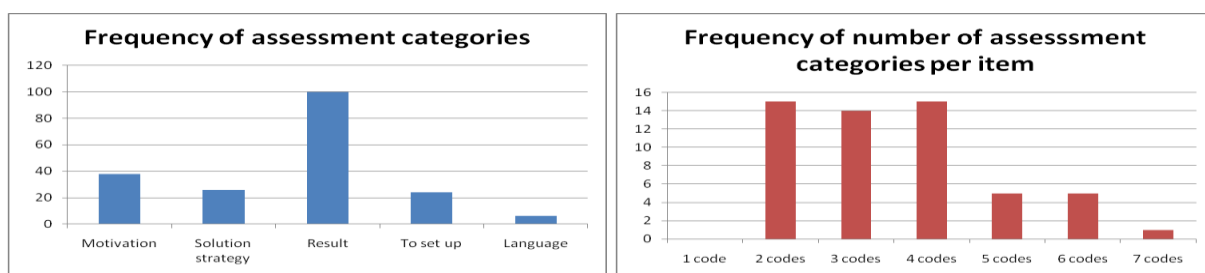
Extra-mathematical contexts were found in 55 items (four items in part one of the test and 51 items in part two including five performance assessments). The starting point of the modelling process in the scrutinized items was the mathematical model (in 54% of the items), the real model (29%), both the real model and the mathematical model (15%) and both the mathematical model and the mathematical result (2%). The frequency of investigated modelling categories from the content analysis and the frequency of number of modelling categories per item are displayed in Figure 3.



**Figure 3. Frequency of used modelling categories and frequency of number of modelling categories per item**

The left diagram in Figure 3 shows that the modelling categories 8 (calculate a result), 4 (assigning variables) and 5 (formulate a mathematical statement) are the most frequently appearing categories in the coding. Categories 6 (alter the mathematical model), 9 (relate the solution to the real world), 10 (interpret a given model) and 11 (critically assess the conditions) are together used in less than 20% of the items. Notable is that categories 1 (simplifying assumptions), 2 (clarify), 3 (simulations) and 7 (select) were not visible in the data. The right diagram in Figure 3 show that three modelling categories used per item are most common.

The results of the content analysis referring to the assessment guidelines are illustrated in Figure 4 below.



**Figure 4. Frequency of assement codes and Frequency of number of assessment codes per item.**

The left diagram in Figure 4 shows that the *result* (correct answer, correct derived function etc.) is most frequently used in the assessment guidelines: more than every second assessment code is referring to the result (100 codes out of a total of 193). The right diagram in Figure 4 shows that two codes to four codes are the most frequently used number of codes to code an item in the assessment guidelines and five codes to seven codes are found in 20% of the items analysed.

The different types of models that the students were supposed to use were based on trigonometry (45%), integration (16%), polynomial functions (15%), exponential functions (11%), differential equations (7%), and a mix of these models (5% of the total number of items analysed). The items were realistic or close to realistic in 29% of the extra-mathematical items. The official assessment guidelines to the NCT also provide a list of items for every test referring to Skolverkets' (2000) statement that students should develop their ability to design, fine-tune and use mathematical models etc. The lists include in total 95 items (two tests did not have a list) but only 27 of those items had an extra-mathematical content according to my analysis.

## CONCLUSION AND IMPLICATIONS

The designed research tool has been useful to pinpoint exactly what parts of the modelling process the NCT (D-course) focuses on and what has been left out. The primary aspects being assessed concerning the modelling process relate to the intra-mathematical world, such as to use an already existing model to calculate a result or assign variables to formulate a mathematical statement to calculate a result. Aspects

left out or occurring sparsely are related to extra-mathematical parts (the real situation and validation), such as to do simplifying assumptions about the problem, to clarify what facts are most important, critically assess conditions, and interpret the result and relate to the real situation.

From a holistic view of modelling the conclusion is that there have not been any modelling items in the NCT (D-course) during the last decade, because not all aspects of the modelling process were represented in a single item. The models that the students were supposed to use were oriented towards trigonometry (45% of the models), which is not odd as new content in the D-course is trigonometry. Most of the items were not based on realistic data (71% of the investigated items). However, in this study the result is based on a web search for possible comparative data.

The outcome that *result* (correct answer, correct derived function etc) is most frequently used in the guidelines for assessment is not surprising, because one of the goals of the NCT is reliable assessment, meaning that it should be equivalently assessed all over Sweden. According to Niss (1993), standardized schemes that are not time consuming are traditional requirements for treating large populations of students, and in this case the focus on *result* may be seen as a fast marking scheme (correct or incorrect).

One may question the impact of Palm's et al. (2004) interpretation of the curriculum on the construction of the NCT. Most of the listed modelling items in the assessment guidelines do not refer to extra-mathematical content and no items refer to the entire modelling process. Neither were open problems (open for different interpretations, assumptions and solutions) nor items with too much or too little information found. However, items questioning parts of the modelling process were found and maybe a further investigation of NCT from other mathematics courses will give another result.

In Frejd and Ärlebäck (in press) most of the students expressed that they had not worked with problems similar to Hains, Crouch and Davis (2000) problems before. This study can conclude that the students have not been assessed on such problems in the NCT (D-course) either. The discrepancies between the *sub-competencies* found in Frejd and Ärlebäck (in press) can therefore not be accounted for by the NCT. Many of the *sub-competencies* are invisible in the NCT such as categories 1 (simplifying assumptions), 2 (clarify), 3 (simulations) and 7 (select) and there is no difference in frequency between categories 4 (assigning variables is used in 69% of the items) and 5 (formulate a mathematical statement is used in 67% of the items).

Finally, some researchers argue that if only a limited set of aspects of mathematics is assessed it may create a disordered view of what mathematics really is (Niss, 1993). The result from this study that only fragments (intra-mathematical aspects) of the modelling process are assessed suggests further research. Questions investigating the issue may be: What conceptions do teachers in upper secondary school express about



the notion of mathematical modelling? To what extent do they describe mathematical modelling activities as part of mathematics/ mathematics education?

## NOTES

1. Mathematic B-probability theory, second order equations; Mathematic C-differential calculus; Mathematic D-Trigonometry, etc. for more info see [http://www3.skolverket.se/ki/eng/nv\\_eng.pdf](http://www3.skolverket.se/ki/eng/nv_eng.pdf)

## REFERENCES

- Blum, W., Galbraith, P. L., Henn, H., & Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education. The 14th ICMI study*. New York: Springer.
- Blomhoj, M., & Jensen, T. H. (2003). Developing mathematical modelling competence: conceptual clarification and educational planning. *Teaching Mathematics and its Applications*, 22(3), 123-139.
- Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM*, 38(2), 86-95.
- Frejd, P. (2010). Revisiting perspectives on mathematical models and modeling. In C. Bergsten, E Jablonka, & T. Wedege (Eds.), *Mathematics and mathematics education: Cultural and social dimensions: Proceedings of Madif 7, Stockholm, 26-27, January, 2010* (pp. 80-90). Linköping: SMDF.
- Frejd, P., & Ärleback, J. B. (in press). First results from a study investigating Swedish upper secondary students' mathematical modelling competencies. To appear in Proceedings of ICTMA 14, *The 14th International Community of Teachers of Modelling and Applications, Hamburg, July 27-31, 2009*.
- Jablonka, E., Gellert, U. (2007). Mathematisation – demathematisation. In U. Gellert & E. Jablonka (Eds.), *Mathematisation and demathematisation: social, philosophical and educational ramifications* (pp. 1-18). Rotterdam: Sense Publishers.
- Kaiser, G., Blomhøj, M., & Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *ZDM*, 38(2), 82-85.
- Haines, C., Crouch, R., & Davis, J. (2000). *Mathematical Modelling Skills: A research instrument*. Technical Report No. 55. University of Hertfordshire: Department of Mathematics.
- Holsti, O. R. (1969). *Content analysis for the social sciences and humanities*. Reading, Mass.: Addison-Wesley.
- Maaß, K. (2006). What are modelling competencies? *ZDM*, 38(2), 113-142.
- Niss, M. (1993). Assessment in mathematics education and its effects: An introduction. In M. Niss. (Ed.), *Investigations into assessment in mathematics education: An ICMI Study* (pp. 1-30). Dordrecht: Kluwer.

## Working Group 6

Palm, T., Bergqvist, E., Eriksson, I., Hellström, T., & Häggström, C. (2004). *En tolkning av målen med den svenska gymnasiematematiken och tolkningens konsekvenser för uppgiftskonstruktion* (Report No. 199). Umeå: Enheten för pedagogiska mätningar, Umeå universitet.

Robson, C. (2002). *Real world research: a resource for social scientists and practitioner-researchers* (2. ed.). Oxford: Blackwell.

Skolverket (2000). *Natural science programme, programme goal, structure and syllabuses*. Retrieved from: [http://www3.skolverket.se/ki/eng/nv\\_eng.pdf](http://www3.skolverket.se/ki/eng/nv_eng.pdf)

Ärlebäck, J. B. (2009). *Mathematical modelling in upper secondary mathematics education in Sweden. A curricula and design study*. Dissertation. Linköping: Linköpings universitet.

## APPENDIX

Guiding questions/ Modelling categories	How long time does it take to evacuate your school?
1. Do students need to make simplifying assumptions about the problem situation in order to solve the problem? [RS to RM]	Could be to do assumptions about number of students, locations of classrooms, number emergency exits, etc
2. Based on the real situation do students need to clarify what is to be accomplished by the mathematical model? (Is it unclear from the outset what kind of facts are most relevant to formulate a mathematical model and what relations are needed to formulate the mathematical model?) [RS to RM/MM]	Could be to clarify what of the following facts are important to include in the mathematical model: the width of emergency exits, number of students in a classroom or the velocity of some students walking in a line, etc? What are the relations between the facts?
3. Do students need to focus on exact arrangements that can provide a basis for the intended simulation? (Is the student supposed to do or evaluate some simulation method to investigate a problem?) [RM to MM]	How can we set up a simulation? Has a simulation of the problem been done, and if so could another method give a better result?
4. Are students expected to assign variables, parameters, or constants to solve the problem? [RM to MM]	Could be set $x$ as length, $v$ as angle, $y'$ as velocity, $n$ as number of students etc ( But not just to use a given model for instance to set $t=0$ for evaluate reaction time)
5. Do students need to set up a mathematically formulated statement (a mathematical model) describing the problem addressed? [RM to MM]	Could be set up a new model or a function ( $t = t_0 + x y' + \dots$ ) and maybe included to integrate or differentiate some function. ( But, not just to use a given model)
6. Do students need to adapt the mathematical model in order to improve (fine-tune) the result? [MM to RM]	A model is created (or given) for instance $t = t_0 + x y' + \dots$ and $t_0$ is supposed to be decided to fit the given data.
7. Do students need to select one mathematical model out of several possibilities and motivate such a choice (the one that fits the data best)? [RM to MM]	Based on the assumptions made, several types of mathematical models are possible. (more than one model is given in the official guidelines for assessment)
8. Do students need to use the model to calculate a mathematical result? [MM to MR]	Example: Calculate the evacuation time for your school building? Could be $t = 11,3$ min.
9. Do students need to interpret and relate the mathematical solution to the (real world) context? (Is it explicitly given in the item text a value/ a statement that is possible to compare to mathematical solution?) [MR to RM]	It could be stated in the text that during the last 10 years of emergency drills the average evacuation time is 13, 4 min.
10. Are students asked explicitly to explain the meaning of a given mathematical model or a mathematical result? [MM/ MR to RM]	It could be stated in the text that a model is found for evacuation time $t = \dots$ could you explain what it means?
11. Do students need to critically assess the conditions, opportunities and limitations of the mathematical model? [MM to RM]	The mathematical model might be valid only between 8 am to 2 pm during weekdays under certain conditions, factors not discussed might be panic etc.

# MODELLING PROBLEMS AND DIGITAL TOOLS IN GERMAN CENTRALISED EXAMINATIONS

Gilbert Greefrath

University of Münster, Germany

*In Germany many centralised mathematics examinations consist of modelling tasks or nearly realistic tasks. A further aspect is added by the usage of digital tools, e. g. a computer algebra system, in examinations. In this article we discuss the simultaneous use of realistic tasks and digital tools in examinations. Some criteria for good examination tasks with modelling problems and the use of technology are presented.*

## INTRODUCTION

The use of digital tools (i. e. spread sheet, dynamic geometry environments, function plotter and computer algebra systems) in centralised examinations influences many aspects of mathematics education. For example it was found that students working with a computer algebra system (CAS) tended to give shorter written solutions than non-CAS students (Ball 2003). Especially the use of digital tools in centralised mathematics examinations showed effects on the content of teaching, teaching methods, types of tasks and other questions concerning mathematics education. It becomes obvious that the question about suitable centralised examinations with digital media and tools does not only refer to examinations, but also to the teaching preceding the examinations.

Other competencies that are relevant in mathematical education are not automatically connected with the use of digital media and tools (s. e. g. Maaß 2006). Tasks relying on the use of digital tools do, however, require and promote certain process-related competencies such as problem solving, interpreting data or modelling. Moreover, it is possible to give the students CAS-tasks which require them to choose between several possible solution strategies by themselves.

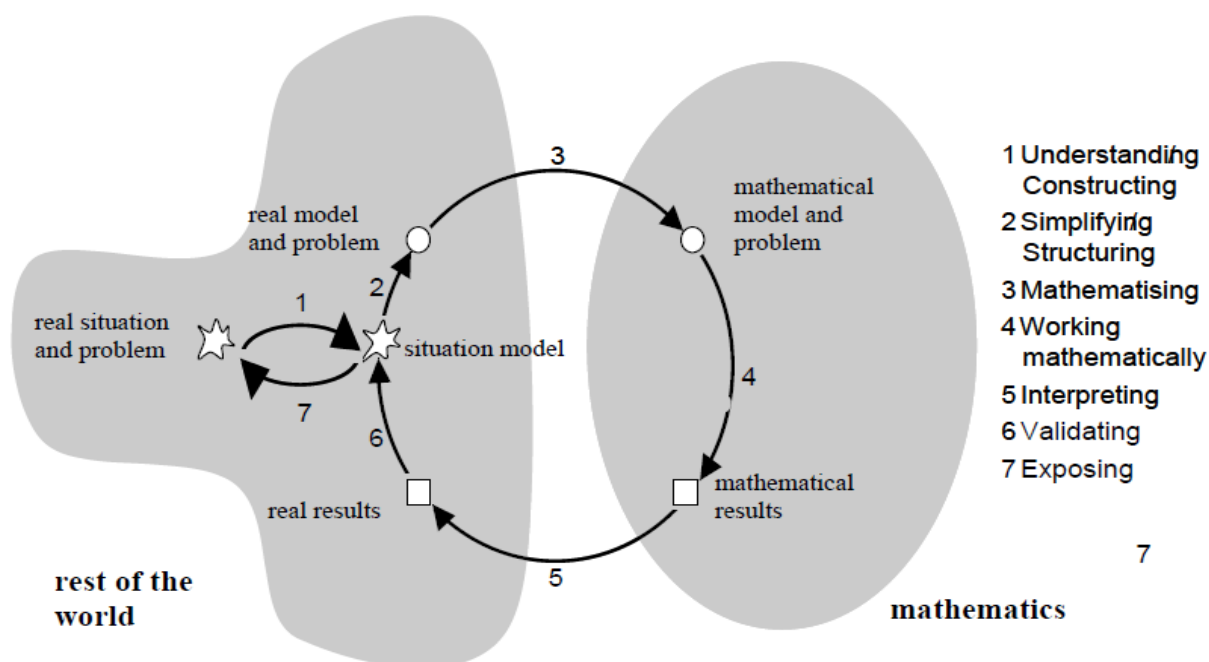
The introduction of CAS into examinations has the potential to provide more solution strategies for the students. It allows the student to move on from an examination where the examiner controls the solution strategy to one in which the student controls the solution strategy (Brown 2003a). Thus there is a potential shift in tasks away from the ones in which only algorithms are trained to more realistic ones in which the students are required to do more modelling. So in addition to possible technical difficulties we have to consider possible blockages during transitions in the modelling process (s. e. g. Galbraith & Stillman 2006). In particular centralised examination questions containing modelling problems and digital tools require special attention. Based on the potential use of digital tools in the modelling cycle and the German situation in upper secondary school examinations we

formulate some criteria for good examination tasks with modelling problems and the use of technology.

## MODELLING AND DIGITAL TOOLS

Realistic mathematical modelling problems (cf. for example Blum et al.) can be well-established in many classrooms, when we have adequate examination questions to not only test the modelling competencies but also the use of the necessary digital tools.

The solution of modelling problems using a digital tool requires that two important translation processes take place. Firstly, the real problem (cf. Blum & Leiß 2006) of the task has to be understood and translated into mathematical language. The digital tool, e. g. the computer algebra system calculator, though, cannot be used before the mathematical expressions have been translated into the language used by the digital tool. The computer results then have to be translated back into mathematical expressions. Finally the task can be solved by relating the mathematical results to the given real situation.

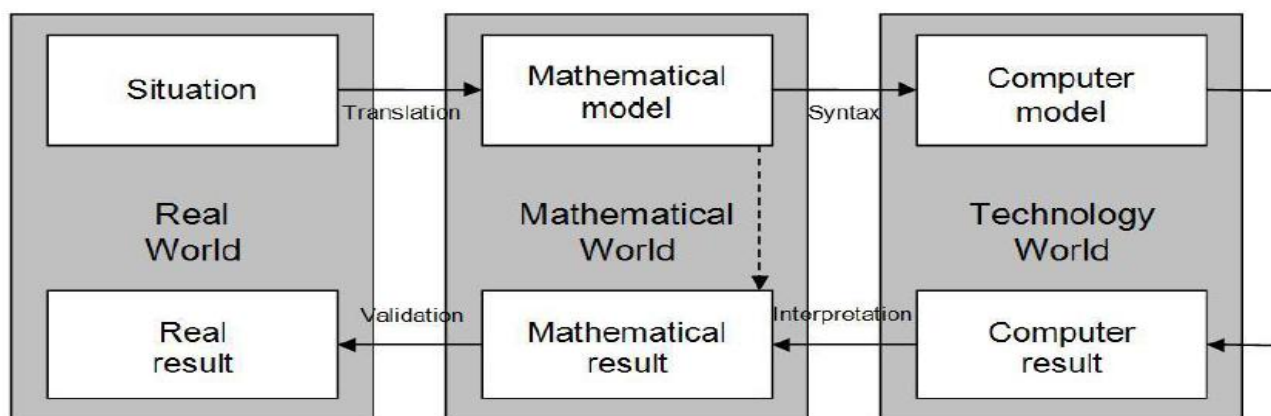


**Figure 1. Modelling cycle (Blum & Leiß 2006)**

As an illustration of the potential use of digital tools in the modelling process we integrate the use of digital tools as the technology world in a simplified modelling cycle. Compared with the modelling cycle of Blum & Leiß (figure 1), we use a less detailed characterisation of the rest of the world, because we focus on the other part of the cycle. The *situation* in our model (cf. figure 2) describes the *real situation* and the *real model* in Modelling cycle of Blum & Leiß. Here the *technology world* has been added to the mathematical world (mathematics) and real world (rest of the

world) in the common models of modelling (e.g. Blum & Leiß 2006, Galbraith & Stillman 2006).

The separation in three different worlds shown in figure 2 is artificial. For example the development of a mathematical model depends on the mathematical knowledge on the one hand and is on the other hand affected by the possibilities given in the technology world. Using technology provides additional possibilities to work with certain mathematical models, which would not be used and lead to a solution if technology was not available (Siller & Greefrath 2010, Geiger 2011).



**Figure 2. Modelling cycle concerning technology (Greefrath & Mühlenfeld 2007, cf. also Savelsbergh et al. 2008)**

When creating examination questions many aspects should be considered. As a basic principle (following Winter), applications in education should only be used in sense-making real situations or if they bring advantages in understanding the problem (Henn 2007). But due to the complexity factor examination tasks containing a whole modelling process are not possible in most cases. Consequently we have to construct examination tasks with parts of the modelling cycle.

Examination questions can be categorized according to the potential use of digital tools when solving them. *Firstly*, the question can be deliberately structured such that digital tools cannot or may not be used to solve the problem. *Secondly*, questions are thinkable where digital tools can be used, but have no potential to contribute to the solution. *Thirdly*, there are questions where digital tools could contribute to the solution of the examination question but their use is not required and *fourthly* there are questions which cannot be solved without the use of digital tools (Brown 2003b).

Looking at the German situation mostly questions of the types two and three are used in examinations with digital tools (cf. Weigand & Bichler). In the following we illustrate the German situation of examination questions with modelling problems and use of digital tools. One aspect concerns problems resulting from the real situation described in the task; a second aspect concerns the use of digital media, i.e. the computer model (cf. figure 2).

## THE GERMAN SITUATION

In Germany there are specific practical regulations for the usage of digital tools in upper secondary school examinations. Additionally, there are different regulations in all of the 16 federal states in Germany. Several federal states started with centralized examinations in the last years and permit the use of computer algebra systems. For example in the federal state of North Rhine-Westphalia (NRW) two calculator-specific versions of the mathematics examination have been employed since 2007. One group of examination questions is for standard or graphing calculators (non-CAS group) and another group for computer algebra systems (CAS-group). The mathematical content of the examination questions is (nearly) identical for both groups, but some interesting aspects are different. There are only minimal differences between these two groups of tasks, but a positive trend can be seen in the CAS-group (cf. Pallack 2008). Tasks with CAS are more open-structured. For example the described real model of a situation has no given equation of the function or no given coordinate system. Hence the tasks are a bit more realistic and the students have to take more steps (e.g. simplifying, mathematizing, cf. Figure 1) in the modelling cycle.

Another aspect when using technology in examinations is the diversity in the computer models (cf. figure 2) used. For example with a CAS-calculator we can determine a questioned polynomial of degree 3 with a linear equation system or a statistic regression. In addition the solution could be calculated numerically or algebraically. This is of course a good starting point for an interesting discussion in a mathematics lesson, but the evaluation in centralised examinations becomes more difficult. So the diversity of possible computer models complicates the marking in centralised examinations.

An analysis of current examinations in the NRW part of Germany shows that there are characteristic types of tasks depending on the treated mathematical domain (Greefrath, Siller & Weitendorf).

- In stochastics the tasks for examinations are application-oriented. The stochastic models used, for example, the binomial distribution, are well known. So the applications are not really modelling problems in the sense of ISTRON (Blum et al.), but standard tasks. The students in both groups (CAS and non-CAS) have nearly or even exactly the same examination questions, so the use of CAS does not change tests in stochastics. The main problem in stochastics is – apart from the CAS usage – the missing connection between reality and mathematical models. The fit of the stochastic model is one of the most important questions. But most of the final secondary-school examinations in Germany are lacking this part.
- Tasks in analytic geometry usually are unrealistic. A typical question, for example, is situated in the context of an excessive simplified tower. The question

is, if the temperature sensor in a special point is in the shadow of the tower or not. No one will think that this is a meaningful use of mathematics. There is no improvement imaginable due to the use of CAS. Examination tasks should show sense-making use of Mathematics. So tasks in this mathematical domain can only be improved by making them inner-mathematical with or without CAS. As a consequence, we have no modelling problems in this part of an examination, just a relevant use of Mathematics (in the sense of Winter, cf. Henn 2007, BE 2).

- The most common mathematical field for modelling problems in examinations is analysis. Here, tasks should include real-world problems. The changes due to the use of CAS show the right tendency, but actual tasks contain too many standard parts. For example in 2008 there was a problem on a model for concentration of a certain medicine in blood. The main part of the task was to calculate interesting points of the graph (e.g. maximum and inflexion point). The blood concentration context was not really necessary to solve the task. Considering all analysis tasks in the upper secondary school examinations for CAS in NRW the proportion of parts that really need a CAS (the forth role of digital tools in examination questions) and not only a graphing calculator is less than 5 per cent. Obviously there is high potential for improvements.

## **CRITERIA FOR EXAMINATION TASKS**

On the Basis of the experiences in Germany some criteria for good examination tasks with modelling problems and use of technology can be pronounced. The first is the sense-making use of Mathematics. If necessary – like in analytic geometry – it is better to have relevant use of Mathematics rather than simple word problems with unrealistic contexts. The second is a good choice of interesting and relevant real world problems to treat an essential part of the modelling cycle in exams. The third is an essential use of CAS in tasks especially for CAS. That means the examination question for CAS should not be solvable without use of digital tools and the students have a real choice between different computer models (cf. the modelling cycle concerning technology above).

But not all parts of examinations need to require a digital tool. For the examination of mathematical competencies, which were acquired with digital tools, it is not necessarily needed, that the digital tools are present in the test situation (Greefrath, Leuders & Pallack, 2008).

## **CLOSING REMARKS**

Centralised examinations with digital media and tools have to be prepared carefully. The use of digital tools can be discussed for e.g. from a socio-political or didactic point of view. Thus it is necessary to say why it is feasible to use digital media and tools in examinations, which can be done at different levels. On the other hand, the use of digital media and tools in class can be promoted by their use in examinations.

When we use digital tools in centralised examinations, certain mathematical operations that have so far been performed by the students themselves, such as differentiating or solving simple equations or systems of equations, will be performed by the digital tool. It is thus necessary to describe special competences for the use of digital tools (of CAS in particular) and to publish these descriptions before the examinations take place. It is necessary to develop clear guidelines for the use of calculators in examinations – in particular for the centralised ones. Especially, it has to be examined whether the examinations can be solved in a comparable way with all permitted digital tools, the comparability referring both to the time-frame and to the required competences.

Many examples show that technology can be helpful at any step of the modelling cycle (Greefrath, Siller & Weitendorf). Therefore the use of technology does not only create an important appendix to the modelling cycle (see Figure 2), but influences each part of the cycle (cf. Geiger 2011). So the technology world is relating to the real world and mathematical world. This multiple influence of technology in solving modelling problems can also be found and integrated in examination questions. The role of technology in modelling activities is as important in tests as in process-related educational situations.

## REFERENCES

- Ball, L. (2003): Communication of mathematical thinking in examinations: A comparison of CAS and non-CAS student written responses, *CAME 2003 - The Third CAME Symposium Learning in a CAS Environment: Mind-Machine Interaction*, Curriculum & Assessment, IUFM, Reims, France, June 23-24, 4-12
- Blum et al. (1994 et seqq.): Materialien für einen realitätsbezogenen Mathematikunterricht, Schriftenreihe der ISTRON-Gruppe, Hildesheim, Franzbecker
- Blum, W. & Leiß, D. (2006): How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum and S. Khan (Eds.): *Mathematical Modelling (ICTMA12): Education, Engineering and Economics*. Chichester, Horwood Publishing, 222-231
- Brown, R. (2003a): Comparing system wide approaches to the introduction of Computer Algebra Systems into examinations, *CAME 2003 - The Third CAME Symposium Learning in a CAS Environment: Mind-Machine Interaction*, Curriculum & Assessment, IUFM, Reims, France, June 23-24
- Brown, R. (2003b): Computer Algebra Systems and Mathematics Examinations: a comparative study. *The international Journal of Computer Algebra in Mathematics Education*, 10, 155-182



- Galbraith, P. & Stillman, G. (2006): A Framework for Identifying Student Blockages during Transitions in the Modelling Process, *ZDM, Zentralbl. Didakt. Math.* Vol. 38 (2), 143-162
- Geiger, V. (2011, in press): Factors affecting teachers' adoption of innovative practices with technology and mathematical modelling, in: G. Kaiser, W. Blum, R. Borromeo Ferri, G. Stillman (Eds.): *Trends in teaching and learning of mathematical modelling*, ICTMA 14, Springer New York Dordrecht Heidelberg London
- Greefrath, G., Leuders, T. & Pallack, A. (2008). Gute Abituraufgaben - (ob) mit oder ohne Neue Medien, *Der Mathematische und Naturwissenschaftliche Unterricht*, 61 Bd. 2, 79–83.
- Greefrath, G., Mühlenfeld, U. (2007): Realitätsbezogene Aufgaben für die Sekundarstufe II, *Bildungsverlag Eins*, Troisdorf
- Greefrath, G., Siller, H.-St., Weitendorf, J. (in press): Modelling Considering the Influence of Technology, in: G. Kaiser, W. Blum, R. Borromeo Ferri, G. Stillman (Eds.): *Trends in teaching and learning of mathematical modelling*, ICTMA 14, Springer New York Dordrecht Heidelberg London
- Henn, H.-W. (2007): Modelling pedagogy – overview, in: W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.): *Modelling and applications in mathematics education*. The 14th ICMI study. Springer New York
- Maaß, K. (2006): What are modelling competencies? *ZDM, Zentralbl. Didakt. Math.* Vol. 38 (2), 113-142
- Pallack, A. (2008): Abitur: Solving by clicking? Verständnis- und prozessorientierte Aufgaben in Klausuren, *Math. Lehren* 146, 54-58 (2008).
- Savelsbergh, E.R., Drijvers, P.H.M., Giessen, C. van de, Heck, A., Hooyman, K., Kruger, J., Michels, B., Seller, F., & Westra, R.H.V. (2008): Modelleren en computer-modellen in de  $\beta$ -vakken: advies op verzoek van de gezamenlijke  $\beta$ -vernieuwingscommissies. Utrecht: Freudenthal Instituut voor Didactiek van Wiskunde en Natuurwetenschappen.
- Siller, H.-St., Greefrath, G. (2010): Mathematical Modelling in Class regarding to Technology, in: V. Durand-Guerrier, S. Soury-Lavergne, F. Arzarello (Eds.): *Proceedings of the CERME 6*. January 28th - February 1st 2009, Lyon (France)
- Weigand, H.-G. & Bichler, E. (2010): Symbolic Calculators in Mathematics Lessons – The Case of Calculus, *International Journal for Technology in Mathematics Education*, 17, 3-15

# ANALYSIS OF THE PROBLEM SOLVING PROCESS AND THE USE OF REPRESENTATIONS WHILE HANDLING COMPLEX MATHEMATICAL STORY PROBLEMS IN PRIMARY SCHOOL

Johannes Groß<sup>1</sup>, Katharina Hohn<sup>1</sup>, Siebel Telli<sup>1</sup>, Renate Rasch<sup>2</sup>, Wolfgang Schnotz<sup>3</sup>

<sup>1</sup>DFG Graduate School “Teaching and Learning Processes”, University of Koblenz-Landau, Germany

<sup>2</sup>Institute for Mathematics, University of Koblenz-Landau, Germany

<sup>3</sup>Department of Educational Psychology, University of Koblenz-Landau, Germany

*The aim of this study was to analyze the structure of the problem solving process and the use of different external representations while working on complex story problems in primary school. 17 2<sup>nd</sup> grade students and ten 4<sup>th</sup> grade students participated in our study. Different influence factors on the process of solving a story problem were measured via questionnaires. In a maximum of 40 minutes, the students had to work on five complex story problems individually. They were allowed to use different [auxiliary material](#). If the students couldn't continue on their own, they received help from the researcher. The students were videotaped individually and interviewed right after every story problem. The problem solving processes was analyzed according to a newly developed system of categories for this study. Our findings show that the ability groups and the grade levels differed in terms of representations they used, as well as in terms of the time they needed to solve the problems. Also, the total word problem score differed between grade levels as well as between ability groups.*

Keywords: complex story problems, problem solving process, system of categories, external representations

## THEORETICAL BACKGROUND

The term “complex story problems” specifies a group of problems, that are different from “regular story problems” (Rasch, 2001). Unlike regular story problems, these complex story problems are based on very complex mathematical structures, and cannot be solved by arithmetic operation models, with which students are usually familiar (Rasch, 2001). In order to solve these specific tasks, students have to restructure their prior knowledge (Winter, 1992). The positive effects of complex story problems on learning have already been shown by former research. For instance, Rasch (2001) reported that the use of these tasks can improve students' “problem solving competences” (Klieme, 2005) in primary school classes. Yet, there are very few studies on the use of complex story problems (Verschaffel, 2000). This group of tasks also plays a minor role in mathematical didactics. To convince didacts of the advantages of complex story problems and to get teachers to implement these tasks in their lessons, further research on the use of complex story problems in

primary schools is needed (Verschaffel, 2000). Our study makes a contribution to this goal.

The problem solving process can be observed from different perspectives. The field of mathematical didactics focuses on the modeling process (Verschaffel, 2000), while cognitive psychology has a special interest in the problem solving process (Franke, 2003). Every perspective has its pros and cons (Verschaffel, 2000). Moreover, solving story problems is a complex process. In order to be able to research this process, it is important to combine these different scientific perspectives (Franke, 2003). The presented study is working towards this aim.

The theoretical framework underlying our study is a combination of two models concerning the course of students' problem solving processes with regard to “story problems” (Verschaffel, 2000). The first model (Reusser, 1993) is called “Student-Problem-Solver” (SPS), which was developed for primary school. According to the SPS model, the story problem is recoded and gradually transformed by the student in stages (Textbase, Episodic Situation Model, Mathematical Problem Model and Numerical Equation) into a solution. The second model is the “cognitive-metacognitive model of mathematical problem solving” (Montague & Applegate, 1993). This model identifies seven cognitive processes (read, paraphrase, visualize, hypothesize, estimate, compute, check) as essential for effective and efficient problem solving. According to Montague and Applegate (1993), a student must be capable of selecting or developing the adequate cognitive processes with regard to the specific requirements and characteristics of the story problem. Both models focus on different aspects of the problem solving process on story problems and complement one another. By combining these two models we developed a new system of categories for analyzing the problem solving processes. This system (for details, see Groß, Hohn, Telli, Rasch & Schnotz, 2010) consists of 9 categories and 30 facets, which cover the whole problem solving process. The system can be used to analyze the problem solving processes and the use of different external representations while working on complex story problems in primary school mathematics.

The development of “problem solving competences” (Klieme, 2005) is an important aspect of the curriculum for mathematics (NCTM, 1989). The TIMSS-study indicated that German students show a great deficiency regarding mathematical problem solving competences (Baumert et al., 1997). Now, research is needed to find out where these deficits come from and what can be done to solve this matter. The present study is situated in the field of “problem solving” and makes a contribution to the goal mentioned above.

A representation is a mental or physical occasion, which stands for something else (Schnotz, Baadte, Müller & Rasch, 2010). Students were allowed to use different auxiliary material to solve the complex story problems (sheet of paper, colorful

pencils, 1-unit cubes, 10-unit rods). According to Schnotz et al. (2010) these materials are external representations. Representations are very important elements for the problem solving process in mathematical education (Goldin, 2007). The positive effects of representations on learning in the problem solving process in primary schools have already been shown in various studies (Stern, 2005). Yet, studies on the configuration of the problem solving process and the use of different external representations on complex story problems are scarce (Goldin, 2007). That is especially unfavorable, because the positive effects of complex story problems on learners' problem solving competences in primary school classes have already been shown in studies (e.g. Rasch, 2001). By closing this gap in research, an effort is made to help students in their struggles with problem solving in mathematics at school (Baumert et al., 1997). In this study we are trying to reach this goal.

### RESEARCH QUESTIONS

The underlying study is supposed to analyze the structure of the problem solving processes of primary school students when working on complex story problems in general, as well as in terms of integrating external representations into the problem solving process. The aim of this study is to find out if there are specific differences in the research question 1), 2), 3) and 4) concerning the two grade levels and ability groups:

- 1) Which external representations are applied in the problem solving process regarding complex story problems?
- 2) What are the differences in the persistence and total word problem score when working on complex story problems?
- 3) How is the problem solving process organized concerning complex story problems?
- 4) Which stages of the problem solving process are connected to which external representations with regard to complex story problems?

### METHODS

#### Participants

A total number of 27 students from a public primary school participated in our study, which was conducted in April 2010. 17 students were from 2<sup>nd</sup> grade; nine of them were girls, and eight of them were boys. The other ten students attended the 4<sup>th</sup> grade; six of them were girls, and four of them were boys.

#### Instruments and Procedures

Firstly, students provided information on gender and age. Secondly, the influence factor cognitive ability on the processes of solving a story problem (Renkl, A. & Stern, E., 1994) was explored via a well-established questionnaire. Verbal as well as

visual spatial intelligence was measured with scales in this test (Kognitiver Fähigkeitstest [KFT], Heller & Geisler, 1983; Heller, Schön-Gaedike & Weinläder, 1976). Different test versions were applied with regard to the two different grade levels. Moreover, scales from the Project for the Analysis of Learning and Achievement in Mathematics (Projekt zur Analyse der Leistungsentwicklung in Mathematik [PALMA], Pekrun, Götz, Zirngibl, v. Hofe & Blum, 2002) were used to assess students' mathematical self-efficacy as well as their mathematical self-concept. Afterwards students were asked to solve five complex word problems that were selected from a book that contains complex story problems (Rasch, 2008). They had been used in former studies (e.g. Rasch, 2001) in math classes in primary school. The problems were different from each other regarding the mathematical areas they were based on and the subject areas they dealt with. In our study, students were successively given the following five complex story problems:

- 28 Two bandits discover a hidden treasure, 2 bags of gold coins. They count the coins. In one bag there are 34 coins, in the other there are 52 coins. They want to share the prey fairly. How many coins do they have to take out of the fuller bag and put into the other bag, until the coins are equally distributed between the bags?
- 29 The devil said to a poor man: "If you pass the bridge, I will double your money, but you have to drop 8 coins into the water when you come back." When the man returned for the third time, he didn't have any pennies. How much money did he have at the beginning?
- 30 A snail in a 20-meter deep well wants to go up to the meadow. She always crawls up 5 meters during day time and glides 2 meter down when she is asleep. On what day does she reach the top of the well?
- 31 Streblinde, Quicki and Murks want to buy ice cream. Every child has got money for two scoops of ice cream. The iceman offers 3 flavors of ice cream: chocolate, vanilla and raspberry. What dish of ice cream could Quickie buy? Find all different possibilities!
- 32 Mummy, Daddy and Murks take a steamboat. For children it is only half price. Altogether they pay €30. How much does a ticket cost for an adult and how much does it cost for a child?

In a maximum of 40 minutes, the students could individually solve these five tasks. The problems were read out by the researcher and then given to each child in written form. The complex story problems were arranged randomly with regard to a Latin Square Design to avoid sequence). First, students had to try solving the problems on their own. They were given different [auxiliary material](#) they were allowed to use at all times. If the students couldn't continue on their own, they received help from the researcher who used standardized questions. If a child still was not able to solve a problem, the researcher turned to the next task. The students were videotaped

individually during the solution process and interviewed with a “structured task-based interview” (Goldin, 2007) right after every task about the heuristics they used to solve the story problem.

## **Analysis**

The participants were divided into two groups, according to their results in a German standardized cognitive abilities test (KFT). Depending on these test results, there was a group of higher (HC) and a group of lower maths competence (LC) in both grade levels. The groups were put together through median split. In the 2<sup>nd</sup> grade there were nine students in the LC group: four of them were girls and five of them were boys. Eight Students in the 2<sup>nd</sup> grade were in the HC group: five of them were girls and three of them were boys. In 4<sup>th</sup> grade there were five students in both ability groups. In the LC group there were four girls and one boy. In the HC group there were two girls and three boys.

The videos were analyzed by two raters. An evaluation of the inter-rater reliability (Wirtz & Caspar, 2002) was scheduled and afterwards the training of independent raters was organized. The problem solving processes will be analysed according to a system of categories (Groß, Hohn, Telli, Rasch & Schnotz, 2010) which was newly developed for this study.

## **RESULTS**

### **External representations which are applied in the problem solving process**

The results of our descriptive analysis show that the ability groups of both grade levels differ in terms of the representations that were used to solve the story problems.

The results of the 2<sup>nd</sup> grade are listed in Table 1. At this grade level only members of the HC groups used “fingers” and “logical pictures” as kinds of representation.

In both ability groups, “spoken language” was the kind of representation that was used most. Members of the HC group used this kind of representation 7.70 times on average, the LC group 8.00 times. The external representation “realistic pictures” was not used in any of the ability groups of 2<sup>nd</sup> grade.

The results of the 4<sup>th</sup> grade are listed in Table 2. At this grade level members of the LC Group used all kinds of external representation. Meanwhile, “working materials” and “fingers” as kinds of representation were not used at all by members of the HC group. The kind of representation “number, arithmetic” was used the most in both ability groups. The HC group used this kind of representation 8.40 times, the LC group 7.80 times on average.

Looking at each grade level on its own, there was no big difference between the two ability groups: If a kind of representation was used, members of both ability groups used it similarly on average.

## Working Group 6

	HC-group 2nd grade		LC-group 2nd grade	
	M	SD	M	SD
<b>spoken language</b>	7.70	4.79	8.00	4.93
<b>numbers, arithmetic</b>	4.50	0.71	3.00	.00
<b>words, sentences</b>	2.50	.71	4.50	.71
<b>working materials</b>	4.00	.00	2.67	1.53
<b>fingers</b>	3.00	.00		
<b>logical pictures</b>	5.00	.00		
<b>realistical pictures</b>				

**Table 1: Average number of occasions individual representations are used (2<sup>nd</sup> grade)**

	HC-group 4th grade		LC-group 4th grade	
	M	SD	M	SD
<b>spoken language</b>	6.80	5.07	4.60	3.44
<b>numbers, arithmetic</b>	8.40	4.83	7.80	5.63
<b>words, sentences</b>	2.80	1.79	5.20	2.59
<b>working materials</b>			3.33	1.52
<b>fingers</b>			1.00	.00
<b>logical pictures</b>	3.00	1.41	3.50	3.54
<b>realistical pictures</b>	1.00	.00	1.00	.00

**Table 2: Average number of occasions individual representations are used (4<sup>th</sup> grade)**

The comparison of the two grade levels (Table 3) showed that students of the 4<sup>th</sup> grade use the external representation “realistic pictures” in contrast to students of the 2<sup>nd</sup> grade. All the other kinds of representation occurred in both grade levels.

There is no big difference in the total number of uses of representation when comparing the two grade levels. Only “numbers, arithmetic” was used more often in fourth grade (8.10) than in second (3.60).

	2nd grade		4th grade	
	M	SD	M	SD
<b>spoken language</b>	7.80	4.69	5.70	4.24
<b>numbers, arithmetic</b>	3.60	.89	8.10	4.95
<b>words, sentences</b>	3.50	1.29	4.00	2.45
<b>working materials</b>	3.00	1.41	3.33	1.53
<b>fingers</b>	3.00	.00	1.00	.00
<b>logical pictures</b>	5.00	.00	3.25	2.22
<b>realistical pictures</b>			1.00	.00

**Table 3: Average number occasions individual representations are used (2<sup>nd</sup> and 4<sup>th</sup> grade)**



### *Differences in the persistence and total word problem score*

Table 4 shows the differences in the persistence which was measured in seconds via time sampling and the total word problem score. The total word problem score is the probability of students to solve a word problem correctly – it is the mathematical average of correct solutions for all word problems.

On average, members of the HC group (2<sup>nd</sup> grade) needed less time to solve the story problems. That was not the case for the 4<sup>th</sup> grade students. At this grade level, the members of the LC-group needed less time for working on the story problems.

On comparison of the two grade levels, it becomes obvious that students of the 2<sup>nd</sup> grade need more time to solve the story problems than students of the 4<sup>th</sup> grade. On average, 2<sup>nd</sup> graders needed 105.85, 4<sup>th</sup> graders 69.86 seconds.

At both grade levels, students of the HC group achieved a higher total word score than students of the LC group. HC students of the 2<sup>nd</sup> grade reached a score of .24, HC students of the 4<sup>th</sup> grade a score of .36. The 2<sup>nd</sup> grade LC students only reached a total word score of .03 and 4<sup>th</sup> graders only scored .16.

Comparing the two grade levels, it becomes obvious that students of the 4<sup>th</sup> grade achieve a higher total word score (.26) than students of the 2<sup>nd</sup> grade (.15).

	HC-group 2nd grade		LC-group 2nd grade		HC-group 4th grade		LC-group 4th grade		2nd grade		4th grade	
	M	SD	M	SD	M	SD	M	SD	M	SD	M	SD
persistence	102.38	59.36	110.80	93.86	71.08	28.29	68.64	18.09	105.85	72.83	69.86	22.42
total word problem score	.24	.27	.03	.08	.36	.26	.16	.17	.15	.24	.26	.23

**Table 4: Average persistence and total word problem score (2<sup>nd</sup> and 4<sup>th</sup> grade)5**

## OUTLOOK

The design and development of the system of categories has been completed. Using this system, the remaining research questions will be answered. At the moment, the observers are analyzing the video material of our study using the system of categories. Further results of the analysis will be presented later on.

## REFERENCES

- Baumert, J., Lehmann, R., Lehrke, M., Schmitz, B. Clausen, M., Hosenfeld, I., Köller, O., & Neubrand, J. (1997). *TIMMS – Mathematisch naturwissenschaftlicher Unterricht im internationalen Vergleich. Deskriptive Befunde*. Berlin: Leske & Buderich.
- Franke, M. (2003). *Didaktik des Sachrechnens in der Grundschule*. Heidelberg: Spektrum.



- Goldin, G.A. (2007). Representation in School Mathematics A Unifying Research Perspective. In J. Kilpatrick (Ed.), *A research companion to principles and standards for school mathematics* (pp.275-285). Reston: National Council of Teachers of Mathematics.
- Groß, J., Hohn, K., Telli, S., Rasch, R., & Schnotz, W. (2010, September). *Analyse des Problemlöseprozesses bei der Bearbeitung von problemhaltigen Textaufgaben durch Grundschul Kinder*. Poster session presented at the 74th meeting of AEPF, Jena, Germany.
- Heller, K., & Geisler, H. J. (1983). *Kognitiver Fähigkeitstest für 1. bis 3. Klassen (KFT 1-3): Manual*. Weinheim: Beltz.
- Heller, K. A., Schön-Gaedike, A.-K., & Weinläder, H. (1976): *Kognitiver Fähigkeits-Test für 4. bis 13. Klassen (KFT 4-13+) Manual*. Weinheim: Beltz.
- Klieme, E., Leutner, D., & Wirth, J. (2005). *Problemlösekompetenzen von Schülerinnen und Schülern. Diagnostische Ansätze, theoretische Grundlagen und empirische Befunde der deutschen PISA-2000- Studien*. Wiesbaden: VS Verlag für Sozialwissenschaften.
- Montague, M., & Applegate, B. (1993). Middle School Students` Mathematical Problem Solving. An Analysis of Thinking-Aloud Protocols. *Learning Disability Quarterly*, 16 (1), 19-32.
- National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston. National Council of Teachers of Mathematics.
- Pekrun, R., Götz, J., Zirngibl, A., v. Hofe, R., & Blum, W. (2002). *Skalenhandbuch PALMA: 1. Messzeitpunkt (5. Klassenstufe)*. München: Universität München, Institut Pädagogische Psychologie.
- Rasch, R. (2001). *Zur Arbeit mit problemhaltigen Textaufgaben im Mathematikunterricht der Grundschule*. Hildesheim: Franzbecker.
- Rasch, R. (2008). *42 Denk- und Sachaufgaben Wie Kinder mathematische Aufgaben lösen und diskutieren*. Seelze-Velber: Ehrhard Friedrich Verlag.
- Renkl, A., & Stern, E. (1994). Die Bedeutung von kognitiven Eingangsvoraussetzungen und Lernaufgaben für das Lösen von einfachen und komplexen Textaufgaben. *Zeitschrift für Pädagogische Psychologie*, 8, 27–39.
- Reusser, K. (1993). Tutoring systems and pedagogical Theory. Representational Tools for Understanding, Planning, and Reflection in Problem Solving. In S.P. Lajoie & S.J. Derry (Eds.), *Computers as Cognitive Tools*. Hillsdale: Lawrence Earlbaum Associates.

## Working Group 6

- Schnotz, W., Baadte, C., Müller, A., & Rasch, R. (2010). Creative Thinking and Problem Solving with Depictive and Descriptive Representations. In L. Verschaffel, E. De Corte, J. Elen, & T. de Jong (Eds.), *Use of External Representations in Reasoning and Problem Solving*. Amsterdam: Elsevier.
- Stern, E. (2005). Kognitive Entwicklungspsychologie des mathematischen Denkens. In van Aster, M. & Lorenz, J. (Hrsg.), *Rechenstörungen bei Kindern. Neurowissenschaft* (S.137-149). Göttingen: Vandenhoeck & Ruprecht.
- Verschaffel, L. Greer, B. & De Corte, E. (2000). *Making Sense of Word Problems*. Lisse: Swets & Zeitlinger Publishers.
- Winter, H. (1992). Zur grundsätzlichen Problematik des Sachrechnens. *Sachunterricht und Mathematikunterricht in der Primarstufe*, 8, 350-369.
- Wirtz, M. & Caspar, F. (2002). *Beurteilerübereinstimmung und Beurteilerreliabilität*. Göttingen: Hogrefe.

# APPLICATION AND THE IDENTITY OF MATHEMATICS

Kasper Bjerling Soby Jensen, IMFUFA, Roskilde University, Denmark

*In the paper a conceptual framework for discussing the identity of mathematics as a school subject is constructed with particular emphasize on application of mathematics. The framework is used to analyze the identity of mathematics, as it appears on two different kinds of domains: the political system and the teachers. At the end it is discussed whether this framework gives us new insights into mathematics teaching. It is concluded that the framework can articulates important aspects.*

## INTRODUCTION

This paper presents a theoretical element of a larger research project on the *identity of mathematics* as a subject in the Danish general upper secondary school (the Gymnasium). The term *identity* is borrowed from the subject specific regulations. Every subject of the Gymnasium has its own regulation, starting with a paragraph named “identity”. An analysis of those paragraphs shows that an identity of a subject seems to consist of three aspects: 1) A general description of the objects studied in the subject, 2) specific descriptions of methods, theories, contents, etc. in the subject and 3) external justifications of the existence of the subject as an independent entity.

Those three aspects are used to compare different subjects, by highlighting their principal differences. They are also used to declare how the political system officially wants the subject to be identified. For many of the subjects, such identification may be uncontroversial. But in the case of mathematics, this is not so. The identity of mathematics as a discipline and especially as a school subject is in general disputed.

I define an *identity of mathematics* as a holistic view of what kinds of tasks, contents, knowledge, actions, etc. that can be recognized as belonging to the field of mathematics. It is my claim that an identity of mathematics as a school subject can be described as a vector in (at least) three dimensions: 1) A view on the role of theory, 2) a view on the role of application and 3) a view on the role of meta-issues. In short they can be named the *in-*, *with-* and *about dimension* (with inspiration from Jankvist 2008). A dimension can be described in several ways. For this purpose, a set of levels ordered by inclusion will be convenient.

Identities of a school subject are not well described animals, living in a well defined territory. It is blurred creatures living on qualitatively different domains and in different states. The domains can be the political system, the individual teacher, a textbook, a student, etc. The different kinds of states found on a domain depend on the characteristics of this.

A math teacher is an example of a domain. Here we can find an identity of mathematics as a school subject in different states, e.g. *intended identity*, *practiced*

*identity* and *principal identity*. While the first two have to do with teaching, the last one is the persons more general identification of mathematics as a discipline and academic field.

My overall research question(s) is: »*Which identities dominate the mathematics subject in the Danish gymnasium today and what consequences does it have for the possibilities of making general changes in the identity?*«. By this I want to grab the struggle between identities focusing narrowly on mathematics as a theoretical field versus identities allowing mathematics to be a tool for application outside its own world.

To answer this question, I am investigating four categories of domains: 1) The political system, 2) the textbook systems, 3) the math teachers and 4) the academic environment around the subject of mathematics in the Gymnasium. In this paper I will present some of the theoretical considerations about the *with*-dimension. I will follow this up, by giving some examples on the analysis of two of those four mentioned domains, the political system and the math teachers.

I should underline here, that the discussion about the identity of mathematics as a subject in the Danish Gymnasium is relevant, because of a larger reform that was fully implemented in 2008. The reform has moved the subject of mathematics toward a stronger role for application. This can be seen in two ways:

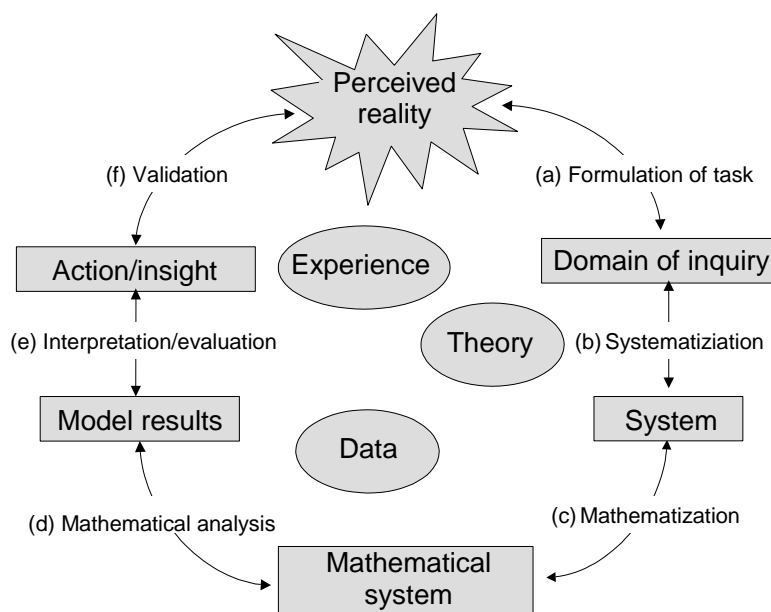
Firstly, the reform has emphasized cooperation between different subjects. This change forces mathematics to think in terms of application. This change has been discussed in several articles (e.g. K. B. Jensen (2010) and Andresen & Lindenskov (2008)). I will not go deeper into this in the paper. Secondly, there are a larger emphasize on modeling and application in the new regulations of mathematics as a single subject. It is this change that I will discuss here. What role does application and modeling play, when mathematics is on its own? This role has earlier in the pre-reform era been discussed in e.g. T. H. Jensen (2007).

## **THE ROLE OF APPLICATION**

As claimed above, the view on the role of application is an independent dimension in an identity of mathematics. The simplest “value” for this dimension would be zero, corresponding to the view, that application of math shall not play any role at all. This is not the same as saying that mathematics can’t be applied, but that the applications belong elsewhere. This viewpoint can be found among the math teachers in the Danish Gymnasium (that typically holds a master degree in mathematics from a university), but in this very radical form, it will probably be rare.

So to describe the existing viewpoints as parts of identities, I have chosen to formulate a suitable number of inclusively ordered levels, i.e. that a lower level can be seen as contained in a higher level. In order to construct those levels, it will be necessary to choose a set of notions about application of mathematics.

Application is a matter of working with models in a more or less unfolded way. A model is an object, that can be described as a triple  $(S, M, R)$ , where  $S$  is a real-world situation,  $M$  a collection of mathematical objects and  $R$  a relation between  $S$  and  $M$  (Blum and Niss, 1991). Modelling is a process, in which a model is constructed. The modelling process can be described as six sub-processes, as in Blomhøj and Kjeldsen (2006). The sub-processes are shown in figure 1.



**Figure 1: The modelling cycle consisting of six sub-processes**

A good entrance to the discussion of the role of application is to talk about different kinds of tasks based on the application of mathematics. From the modelling-cycle, five kinds of such tasks can be defined:

- *Modelling task.* A task that can only be solved by going through sub-process (b), (c), (d) and (e), and eventually also (a) and (f).
- *Model task.* A task involving sub-process (c), (d) and (e).
- *Mathematization task.* A task involving sub-process (c) and eventually (d).
- *Interpretation task.* A task involving sub-process (e) and eventually (d).
- *Wrapped task.* A task that in any practical sense only involves sub-process (d).

I will then define roles of application on five inclusively ordered levels:

1. *Illustration.* The role of application is narrowly to illustrate the pure theory.
2. *Motivation.* The role of application is to motivate work with the pure theory.
3. *Service function.* Math has a service function in other subjects and areas.
4. *Personal tool.* Math is a tool, one carries around to use in the real world.

5. *Critical inquiry.* Math is a field for investigating a wide range of problems.

Level 1 and 2 will be found in identities that see mathematics as a field for only pure theoretical activities. Applications must serve theory. On the illustration level, it is not important that the application has anything to do with the real world, while on the motivation level the cases must have some sort of real world relevance. That gives the order of the two levels. On these two levels *wrapped tasks* are sufficient.

On level 3, mathematics is recognized as a field of theory that other subjects and fields may borrow with great advantage. For instance when biologists and physicists needs a differential equation to be set up, solved and interpreted, or a carpenter needs to calculate the dimensions of a roof. On this level mathematics should deal with real world problems, but not before they have a clear mathematical form. On this level *wrapped tasks* are found, but also *mathematization*, *interpretation* and *model tasks*.

On level 4, it is important to be able to solve real world problems, when you meet them. Math should be a tool that you carry with you to use on appropriate problems. An economist must know how to handle problems like »what is the actual taxation as a function of income tax and VAT«, and a physicist must know how to handle problems like »with what speed does a parachute land« and »how early does Venus rise«. Therefore those questions are relevant to ask in mathematics. *Wrapped tasks* play a minor role on this level and the *modeling task* are introduced.

On level 5, mathematics is recognized as a field where a lot of open questions can be examined critically. It can be tasks like »what is the best means of transport«, »how many bosses should a company hire«, »how many elevators are needed in a warehouse with many floors«, and »should we trust the polls«. On level 5 a major part of the tasks worked with should be of the *modeling* kind.

Those five levels will be a part of my framework to analyze what identities are dominating in the Danish Gymnasium. In the two following sections, I will give examples on the use on two different categories of domains.

## THE POLITICAL SYSTEM

In the political system, the identity of mathematics as a subject in Danish Gymnasium lives primarily in different kinds of documents. Therefore document analysis is the most important method to uncover it. The political system is an aggregation of many peoples' individual viewpoints and interests, so it is not expected that a clear well defined identity is found.

The documents to analyze are first of all the *regulation*, especially its appendices on mathematics. Secondly it is the *guidelines* following the regulation. And then it is the annual written examinations. The written examination is taken by all students with mathematics on medium or high level. It consists of typically 16-17 tasks, which must be answered in 5 hours without communication with others.

The regulation presents mathematics as a subject in three steps: 1) Identity and purpose, 2) Mathematical aims and 3) Core- and extension material. The purpose says:

One of the aims of the teaching is to give the pupils knowledge of some of the important parts of mathematics' interactions with culture, science and technology. In addition, the aim is to give the pupils an insight into how mathematics can contribute to understanding, formulating and solving problems within a number of different subjects, as well as an insight into mathematical reasoning... (EVA 2009, p. 59)

This text doesn't focus narrowly on mathematical theory. Actually one can barely say that it mentions that mathematics as a theoretical activity should play an independent role. So we must be above the illustration- and motivation level. The talk of interaction and solving problems in other subjects, points at a *service function*- or *personal tool* level. So let us look at just one of the following nine mathematical aims:

Pupils should be capable of using functions and their derivatives in setting up mathematical models based upon data or knowledge from other subject areas. They should also be able to have an opinion about the idealizations and range of such models, be able to analyze given mathematical models, and undertake simulations and extrapolations. (ibid)

This text presents theoretical elements as tools to apply in real world. So again we are above illustration and motivation, and the talk of pupils "being capable of" and "have an opinion about" draws towards the *personal tool*-level. But then the core material is presented. The core material is ten dots presenting the "syllabus". The content that every student are expected to learn. Here are three of the 10 dots:

- the definition and interpretation of the derivative, hereunder growth rate and differentials, the derivatives for elementary functions and the rules for the differentiation of  $f + g$ ,  $f - g$ ,  $k \cdot f$ ,  $f \cdot g$ ,  $f \circ g$ , proof of selected derivatives
- monotonic functions, maxima, minima and optimization along with the connections between these concepts and the derivative
- fundamental properties of mathematical models (EVA 2008, p. 60)

The first two dots present pure theoretical contents. Application and real world are not mentioned. The other seven dots are of the same character. And then the 10th last dot, talks about models. So in the core material, mathematics is presented as a large collection of theoretical concepts and rules, and models as a little additional aspect. From this perspective, applications are drawn towards something serving the theory. Finally, if we look at the Guidelines, they say:

To demonstrate knowledge about application of mathematics means, that you in a reflected way can present some content that you have worked with. In that do not lay the


idea that students independently can take care of a mathematical problem and modeling of a material or problem, which has not been prepared. (UVM 2008, p.22., my translation)

In this text, the talking about unassisted applications is laid dead. Instead application is something taking place in continuation of work done by others. So this text places the role of application around the level of *motivation* or *service function*.

The four pieces of text draws together a blurred picture of the systems *declared identity*. The relation between theory and application is unclear. According to the general declarations of aims and purposes, application should be very central. But if one looks at the list of mandatory contents, it is the pure theory that is in focus. It is also unclear on what level applications are to be presented. But again, the general parts draw up, the concrete parts draw down.

It is therefore my claim that the regulation leaves the teacher with a broad range of choices of what identity he or she will practice in the daily teaching. Therefore it is very important to examine the identities held by the individual teachers. But the system has one important tool left: tasks for the written examination. Even though the rules are unclear, the teacher still has to take the written examination into account.

The written examination is a collection of typically 16-17 tasks, that every student must answer in 5 hours. The first five tasks must be answered and handed in, in the first hour, without any aid. The remaining tasks must be answered with the use of calculators, tables of formulas, computer programs and other means, not including communication with others. The tasks are very similar from year to year. They are therefore a clear message to teachers and students about what kind of task they should be training to answer. Out of the 11-12 tasks with aids, 5-6 are formulated in an applied way (i.e. by referring to some extra-mathematical context). It is my assumption, that these tasks are the strongest declaration from the system about, what role application should play. Here I will just give two examples of typically applied tasks from written examination on the highest level (UVM 2009, *my translation*):

Example A	Example B
<p>In a model, the weight of a certain fish as a function of the fish's age, is given by:</p> $w = 20 \cdot (1 - 0.89 \cdot e^{-0.17t})^3,$ <p>Where <math>w</math> is the weight (measured in kg), and <math>t</math> is the age (measured in years).</p> <ol style="list-style-type: none"> <li>Use the model to decide the fish's weight, when it is 3 years.</li> <li>Determine the age of the fish, when the fish's weight is 13 kg.</li> </ol>	<p>In a garden a flower bed is landscaped with the shape of a circle sector (se the figure).</p> <p>It is informed, that the area of the flower bed as a function of the angle <math>v</math> (measured in radians) is:</p> $A(v) = \frac{200v}{(v+2)^2}$  <ol style="list-style-type: none"> <li>Determine <math>v</math>, so the area of the flowerbed becomes as big as possible</li> </ol>



Both of these tasks are of the *wrapped* kind. In both tasks are given an explicit expression and asked questions that it is very simple to unwrap as pure mathematical questions. Said in another way, it is possible to formulate both tasks as pure theoretical tasks. Therefore the term “wrapped”, because it is pure tasks wrapped in an extra-mathematical context.

Example A is a task of the kind *given  $y = f(x)$ , find  $f(x_0)$  and  $x$  so that  $f(x) = y_0$* . Example B is of the kind *given  $y = f(x)$ , find the  $x$  where  $f(x)$  has its maximum at the interval  $[a;b]$* . It is my claim, that those tasks focuses mathematics on the theoretical dimension, while it places what we could call the systems *practiced identity* on the lower levels of the application dimension. The tasks do illustrate and motivate theory through application, but they do not train how to apply math.

## THE TEACHERS

In the case of a teacher, the identity lives as a collection of viewpoints, believes, habits, abilities, etc. inside the teachers mind. Therefore more sophisticated methods than document analysis are needed, to uncover it. In the singular case, the best method would probably be a combination of deep conversations and observation of teaching practice. But if we want an overview of the entire population of teachers, we need to do a survey on a well chosen sample.

Here the main problem is how to screen a person’s mind. If the person is asked directly about the identity, you will probably not get complete and accurate answers. Therefore it is necessary to ask questions where the answer builds more or less unconsciously on the teacher’s identity of mathematics. Examples of such a method are to present a number of different kinds of tasks to the teacher, and ask him or her to place it in relation to their ideas about the subject (e.g. central, supplementary or not belonging), and to show the teacher four proposals for the identity-paragraph in the regulation and ask which one the person would vote for in a referendum.

It is not the purpose of this paper to justify my methodology, present results or draw any conclusions. But I will exemplify the use of the identity concept on the domains of teachers, by referring to interviews made with four math teachers, as a pilot study before designing a survey.

The teachers were asked to comment on the task »How early does Venus rise? «. This task can be answered in many different ways. Venus is placed between the Earth and the Sun. Therefore it must rise relatively close to sunrise. One way to answer the question, is to estimate the greatest time difference between the rise of Venus and sunrise. By making a plane geometric model one can convince him- or herself, that this happens when the Sun-Venus line is perpendicular to the Earth-Venus line. In terms of Sun-Earth ( $r_E$ ) and Sun-Venus ( $r_V$ ) distances, this gives the following time difference ( $T$ ):

$$T = \frac{\arccos\left(\frac{r_V}{r_E}\right)}{2\pi} \cdot 24\text{h} \approx 3\text{h}.$$

The interviewed teachers were not presented for this or any other ideas to a solution. Two of the teachers refused the task, claiming that it does not belong to the field of mathematics. One of these two teachers explained this in the following way:

- T1 [...] as the question stands, you can't calculate it unless you have some preknowledge, unless you have been taught astronomy. And of course there is some mathematics in it [...] It would be a good fourth question in a report, where a lot of other questions leads up to it.

According to this teacher, the question is first relevant, when another subject has been on work. Then there will be something to do for mathematics. This indicates a *service function*-level. The other refusing teacher says:

- T4 It depends on where you see it from. And it requires different astronomy software at your disposal... I really don't think that has something to do with mathematics.

This teacher doesn't recognize the task as something where mathematics can play a role. This does as well indicate that we are not at the highest level of application. A teacher at the two highest levels would be expected to spontaneously being open towards investigating the possibilities of a question like this.

The two other teachers didn't refuse the task, but declared the task to be "supplementary", i.e. useful but not central.

- T2 [...] the description of the celestial bodies is traditionally handled by physics. But fundamentally it is also a mathematical question and can be modeled with mathematics [...] so of course it can be supplementary, but as the task stands here... it demands to much additional knowledge.

Here the teacher does not refuse the task. Instead he talks about modeling. He is though critical to the missing informations. So to him, mathematics is not a field of *critical inquiry*, where open questions are examined. On the other hands, it is more a lack of information, than a lack of another discipline that bothers the teacher. This indicates a teacher on the *personal tool*-level.

So what I try to indicate here is that by asking those kinds of questions, it is possible to state a first impression of a teacher's position on the *with*-dimension of identity.

## DISCUSSION

In this paper I have presented a conceptual framework of the identity of mathematics as a school subject. The part of the framework describing the application of mathematics was particularly developed. The framework has been used to discuss the identity of mathematics living on two different kinds of domains. The political

system and the teachers. Examples have been given on how to use the conceptual framework on those two domains.

Whether a conceptual framework is useable or not, must be decided in its ability to articulate relevant problems from reality. In this case it is the overall discussion on the conflict between a pure theoretical approach to mathematics versus an applied approach and the discussion between different approaches to application.

The identity concept adds a way of addressing the disputes on what should characterize the mathematics subject. The concept describes different holistic views on the subject. Differences in such views can be used to address special challenges, when the aims and contents of the subjects are changing. At the same time it can be used to give an overall quantitative description of “the state of the art”, though this is methodological complicated. So the conceptual framework presented, seems to be useful to the Danish context, but may very well be so for other countries as well.

## REFERENCES

- Andresen, M. & Lindenskov, L. (2008). New roles for mathematics in multi-disciplinary, upper secondary school projects. *ZDM*, **41**, 213-222.
- Blomhøj, M & Kjeldsen, T.H. (2006). Teaching mathematical Modelling through project work. *ZDM*, **38**, 163-177.
- Blum, W. & Niss, M. (1991). Applied Mathematical Problem Solving, Modelling, Applications, and links to other subjects – State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics*, **22**, 37-68
- EVA – The Danish Evaluation Institute. (2009). *The subject of Mathematics from an international perspective*. Retrieved from <http://www.eva.dk>.
- Jensen, K.B. (2010). Tværfaglige samspil mellem matematik og historie I gymnasiets studieretningsprojekt (SRP). *MONA*, 2010(1), 32-53.
- Jensen, T.H. (2007). Udvikling af matematisk modelleringskompetence som matematikundervisningens omdrejningspunkt – hvorfor ikke? (doctoral dissertation). Roskilde University, Denmark. *IMFUFAtekst*, **458**.
- UVM – Undervisningsministeriet (2008). Matematik A – Stx, Undervisningsvejledning Juli 2008. Retrieved from <http://www.uvm.dk>
- UVM – Undervisningsministeriet (2009). Matematik A – Studentereksamen - Skriftlig prøve (5 timer), STX093-MAA, December 2009. Retrieved from <http://www.uvm.dk>

# STUDENTS CONSTRUCTING MODELING TASKS TO PEERS

Thomas Lingefjärd

Department of Education, University of Gothenburg

*In a mathematics education course at the University of Gothenburg, students were asked to develop modelling tasks or modelling situations to each others. There are many reasons for encouraging the development of peer tutoring among students. When explaining something for a class mate, students must clarify their own thinking in order to give an explanation and must be prepared to have misconceptions confronted and corrected through discussion and listening. In general students learn much more if engaged in the teaching of a course. But will the receiving students learn what was intended? In this paper I will discuss how this activity was carried out by two groups of students and what they thought they learned from it.*

## INTRODUCTION

To teach matters of learning and teaching principles to prospective teachers is a huge task. There is so much stuff and information to cover in order to prepare prospective mathematics teacher for the upper secondary school level and regularly rather few teaching hours in which we can teach it. In a mathematics education course for prospective mathematics teachers, we decided to try to combine several course objectives such as: Learning how to use a wide spectrum of technological resources, learning how to engage and challenge students' mathematical thinking, and learning how to create learning situations which are relevant for most students in upper secondary schools.

### Technological resources

We wanted them to learn different aspects of GeoGebra since it is such an excellent tool when teaching and learning many different branches of mathematics. It is free, portable (runs on any machine who runs Java), and combines Algebra, Geometry and spreadsheet number theory. We also wanted them to learn how to use digital cameras and free software for splitting a film into images. All students had access to cameras and/or cellular telephones which they could film with. We also introduced them to the software VirtualDub, which enables you to split avi files into images. All films are made of many images, from 25 – 150 images per second. It is not surprising that there exist software that can separate the images from each other again.

### Students

There were 24 students in this mathematics education course, they had all been studying mathematics at the university level for two semesters, and some of them (although not all) also had studied another teaching subject such as physics or chemistry. In our course, they have read and discussed article about theories on learning as well as papers on mathematical modelling (namely, Sfard, 1991;

Lingefjård, 2006). We decided to split them up in six groups with four students in every group. Some of the students were known to be good in mathematics, some of them already knew about GeoGebra, some of the students knew about software for editing and handling movies, and so forth. We tried hard to make the groups equally strong, with respect to these variables, as possible.

Three groups were chosen as teaching groups (presenting tasks) and three groups were chosen as students groups (receiving and solving tasks). One teacher group and one student group were paired.

### **Student activity**

All three teaching groups developed interesting teaching materials. What I will present in this paper is just one task out of three and perhaps not the best or most accurate task. Nevertheless, it is an interesting activity in the sense that it combines most, if not all, of the course objectives mentioned above.

## **THEORETICAL FRAMEWORK**

Studies on student learning (Ramsden and Entwistle, 1985; Johnson *et al.*, 1989, Johnson and Johnson, 1991, Ross and Cousins, 1993, Marton and Booth, 1997) suggest that students adopt at different times and in different circumstances, different approaches to learning. In groups of two or more individuals, students work together, share and clarify ideas. Through talking to each other about subject matter, students can discover what they know and what they do not understand and ‘make sense’ of what they are learning. Some studies show that peer or co-operative learning forces students to engage in higher order thinking, which includes application, analysis, synthesis and evaluation (Ellis and Whalen, 1990). Of course there is no certain way to declare that peer or co-operative learning is the best way to learn. It seems that the more we learn about different ways of learning, the more complex and vague the picture becomes.

The reader might find that it has been somewhat bewildering, our bringing together findings on qualitative differences in the way in which learning is experienced that originate from studies of learning in widely differing educational contexts, that if infants, preschool children, secondary school pupils, and university students, and moreover from such widely differing cultural contexts as Britain, Sweden, China, Uruguay. Our assumption is that a phenomenon, such as learning as experienced, can be described in the terms of the complex of differing dimensions of variation identified. (Marton & Booth, p. 54, 1997)

In order to learn something certain conditions must be met. Learning situations are always experienced within a framing of different circumstances, such as a context, a certain time, a place, and so forth – while a phenomenon is experienced as abstracted from or transcending such an anchorage. But learning is also closely connected to the possibility of variation.

Variation theory makes it possible to analyse teaching and learning in commensurable terms, which implies that ‘what the teacher intends the students to learn’, ‘what is made possible to learn in a lesson’ and ‘what the students learn’ are connected and described in a similar way. From a variation theory position, learning is defined as a change in the way something is experienced, seen, or understood. A fundamental assumption is that the learner, in one way or another, experiences what is learned. The educational system aims at developing the learners’ capability to handle various situations, to solve different problems, and to act effectively according to one’s purposes and the conditions of the situation. However, the possibility of acting on, or handling, a situation depends upon how we make sense of it. We act in accordance with how we perceive the situation. This is affected by our previous experiences, but the experiences we see as relevant are also affected by how we experience the situation. ‘We try to achieve our aims, not in relation to the situation in an objective sense but in relation to how we see it’ (Marton et al., 2004, p. 5).

When working with prospective teachers, it is of interest to regard what they see as possible to learn from a situation both in the sense of their own learning, but also in their role as future teachers. In variation theory, learning is seen as becoming able to discern critical features of an object of learning. The object of learning is the definition of a competence or understanding of something, for example a particular content taught in a mathematics lesson. The object of learning is consequently not the same as learning objectives in a course; it is not the subject or the content taught and learned but rather the capability connected to that particular knowledge.

### **Teacher group task**

The students in this group went to a gym with a basketball hall and while one of them were practicing distance shots with a ball against the basket, the others in the group practiced to film the shots by different cameras and cellular telephones from several distances. When they were satisfied with the shooters performance and the content in the films, they went to a computer lab at the university and transferred all the films to a computer, watched all the short films and selected the best film in terms of quality. After that they downloaded the free software VirtualDub and used it to split the three second long film into frames. See figure 1 and figure 2 for two such frames. There are several more frames between these two.



**Figure 1&2: The start of a basket shot and the same basket shot a moment later**



Researcher: So why did you choose this specific activity? What do you see as the possible object of learning? What will your peers understand or learn from this activity?

Student 1: Our first objective of learning is of course that our peer students should learn more about mathematical modelling. But we also think that they will learn about specifics in the modelling process, specifics such as that every point in the graph has a certain value which defines the point's position in a Cartesian coordinate system. When you look at the different representations of the same object in GeoGebra, you understand more about that object.

Researcher: Please elaborate on that.

Student 1: I am referring to Sfard's theory on dual nature of mathematical conceptions. It must be better to see different representations and be able to vary between them in order to understand more about them...

Student 2: We also think that when you look at that ball's position, thereby sketching the trajectory of the ball, then you understand something about how the single images are part of a whole film. This is represented for thirteen positions of a ball. So you understand more about the connection between reality and the images and that connection is exactly the mathematical model.

It is obvious that the teaching group chiefly was focused on the mathematical modelling activity as their object of learning. For them this whole complex process with many different specifics became reification into the mathematical model. Besides being software for teaching geometry and algebra, GeoGebra also contains several other possibilities suited for teaching different branches of mathematics. One such possibility is to insert an image anywhere in the coordinate system and then use it as a background. Another one is to mark a point at that image and then select "copy the coordinates to the input line", e.g.  $A = (-2.02, -0.56)$ . From the input line, you can easily relabeling the coordinates to  $A1 = (-2.02)$   $B1 = (-0.56)$  and thereby make them part of the spreadsheet. Once in the spreadsheet, the set of points can be used to create a List of points, which then becomes an object. In our example, the ball's centre act as the marking part in every picture and with every frame put in the same place, GeoGebra present us with a set of data points as in figure 3.



	A	B
1	-2.02	-0.56
2	-1.74	-0.1
3	-1	0.78
4	-0.7	1.06
5	-0.38	1.28
6	-0.08	1.46
7	0.2	1.54
8	0.68	1.6
9	0.92	1.58
10	1.14	1.5
11		
12		

**Figure 3: The motion of the ball represented in two ways.**



Student: And if you take any of the points (in GeoGebra) and drag it a little bit off, then you get another trajectory. See here, the fitted curve moves. So, by varying the position of just one single point, the mathematical model can indicate that the shooter made a goal or maybe missed to make a goal. It will teach you even more about mathematical modelling and how careful you must be when you measure data. (See figure 4).

$$f(x) = -0.31x^2 + 0.38x + 1.48$$

Correlation coefficient = 0.91



**Figure 4: Observe that the trajectory of the fitted curve, the correlation coefficient and the formula for the polynomial will all change if one point is moved.**

It seems as if the students hold an intuitive sense of the variation possibilities offered by the technology. It is important to acknowledge that variation can be represented in many different ways. In the student statement above, it is the concept of position which is varied in different representational forms. It is also notable that for the students in the teacher group the reality now was transferred into the film, with part of that reality expressed by the frames inside GeoGebra.

As indicated, the students in the corresponding student group were given 13 frames, illustrating the shot from the start and up to the peak of the balls trajectory towards the basket. They were asked to construct a mathematical model based on the balls position in the frames and then use the model to predict if the shot was a goal or not.

### Student group solving

After some initial discussion about the procedures and command structure of VirtualDub and GeoGebra with regard to this specific problem, they started to insert the images into GeoGebra and once done with this and with creating the set of data points, they also started to analyse the situation:

Student 1: Now when we have all the points in the spreadsheet and have created the List of point, we can use the regression command `FitPoly[Listal, 2]`. If we do that, with a polynomial of grade 2, we should get a curve fit that shows a hit or a fail. Do you follow?

Student 2: Yes, this is such an excellent way of introducing and using the concept of equations of second degree. This is actually the first time that I understand



why we actually teach the students about them in the upper secondary school. Amazing!

Researcher: And what do you think that you are learning while you do this? What do you see as the object of learning?

Student 1: There are several things of course, I was not aware of the possibility to just go away and make a film and the split it into images and just say to the students: Analyse this situation. It is just so cool!

Researcher: So what did you learn?

Student 2: I also learned much more about second degree equations and I will definitely use part of this when I teach introduction to second degree functions next time, it is awesome.

What we see here is that the students group assume a different object of learning compared to the teacher group. The object of learning can obviously be seen from three different points of view: the “teacher’s”, the “student’s” and the researcher’s. The object of learning seen from the point of view of what the teacher or, in this case, the students who acted as teachers, is what is called the intended object of learning. Compare this to the “intended curriculum”. The intended object of learning was connected to mathematical modelling and the correlation between data points and regression curves.

Any teacher hopefully always has a particular goal and intention about what the students should learn, for instance that students are able to understand that a function must be continuous in order to have a derivative everywhere. On the other hand, what students (the learners) actually learn is the experienced (lived) object of learning and that can be observed during the learning situation by the students’ expressions, or after the lesson in an exam. Obviously, it is not necessary that the intended and the experienced (lived) object of learning coincide. In this activity, the students in the students group, experienced another object of learning, namely an example of how second degree equations can describe ball trajectory paths.

The object of learning seen and analysed from the researcher’s point of view, implying ‘what it is made possible to learn’, is the enacted object of learning. The enacted object of learning describes what features of the content are possible to experience during a “lesson”. The variation used to elicit the features, the values within a dimension of variation, is the range of change (Watson & Mason, 2006) and the way it is possible to discern the variation is through different patterns of variation.

For me, as a researcher, it was most obvious that the students now experienced a way of working with technology that can enable their own teaching in their future teacher professions. Some subset of their object of learning was of course related to mathematical modelling and the selection of suitable curves for the curve fitting

process. For me that was in the details, while the overall picture was about new challenging working methods and opportunities when technology is at hand. Altogether, these results created an opportunity for a challenging seminar with all the prospective teachers where we discussed what they had learned and what the intentions were. That was probably the most vivid discussion I had ever seen in a seminar.

## DISCUSSION

It seems that this is an excellent example of how modern technology enables us to construct interesting mathematical problems directly from situations around us. It seems hard to imagine that this problem could have been constructed and presented by one group of prospective teachers to another group of prospective teachers without technology at hand.

Nevertheless, the experiment with students constructing problems for others enables someone to study the learning outcome and thereby observing the difference between the groups acknowledge of the object of learning and between my own expectations of the learning outcome.

The more complex a teaching and learning situation gets the harder it is to analyse the situation in stereotypes or easy commensurable terms. Whenever we try to teach something, we should be aware of the fact that students or humans never perceive reality, since here is no reality outside our notion.

We cannot separate our understanding of the *situation* or our understanding of the *phenomena* that lend sense to the situation. Not only is the situation understood in terms of the phenomena involved, but we are aware of the phenomena for the point of view in the particular situation. Furthermore, not only is our experience of the situation molded by the phenomena as we experience them, but our experience of the phenomena is modified, transformed, and developed through the situations we experience them in. (Marton & Booth, p. 83) (Italics in original)

## REFERENCES

- Ellis, A. and Whalen, S. (1990). *Co-operative learning*, Richmond Hill: Scholastic Press.
- Johnson, D.W., Stanne, M.B. and Garibaldi, A. (1989). Impact of group processing on learning in independent learning, *Journal of Social Psychology*, **130**, 507-516.
- Johnson, D.W. and Johnson, F.P. (1991). *Joining together - group theory and group skills*. Englewood Cliffs, NJ: Prentice Hall.
- Lingefjård, T (2006). Faces of mathematical modeling. *ZDM – The International Journal on Mathematics Education*, **38(2)**, 96-112.

- Marton, F., & Booth, S. (1997). *Learning and Awareness*. New Jersey: Lawrence Erlbaum Associates.
- Marton, F., Runesson, U. & Tsui, B. M. (2004). The space of learning. In F. Marton, A. B.M. Tsui with P. P. M. Chik, Y. P. Ko, M. L. Lo, D. F. Ng, M. F. Pang, W. Y. Pong & U. Runesson, *Classroom discourse and the space of learning*. Mahwa, NJ: Erlbaum.
- Ramsden, P. and Entwistle, N. (1985). Student Learning research: retrospect and prospect, *Higher Education Research and Development*, **4**, 51-69.
- Ross, L and Cousins, B (1993) Self reports and observed behaviour: seeking and giving help to peer, Paper presented at the *Annual Meeting of the American Research Association*, Atlanta.
- Sfard, A.(1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, **22**, 1-36.
- Watson, A., & Mason, J. (2006). Seeing exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Teaching and Learning*, **8(2)**, 91-111.

# **MODELLING AS A BIG IDEA IN MATHEMATICS WITH SIGNIFICANCE FOR CLASSROOM INSTRUCTION – HOW DO PRE-SERVICE TEACHERS SEE IT?**

Hans-Stefan Siller<sup>1</sup>, Sebastian Kuntze<sup>2</sup>, Stephen Lerman<sup>3</sup>, Christiane Vogl<sup>1</sup>

<sup>1</sup>University of Salzburg, Austria; <sup>2</sup>Ludwigsburg University of Education, Germany;

<sup>3</sup>London South Bank University, U.K.

*There is not only a consensus that modelling is a big idea for mathematics as a scientific discipline, but also that this big idea should have an impact on the mathematics classroom. Consequently, teachers should be aware of this big idea and know how modelling relates to a variety of curricular contents. However, empirical research into views of pre-service teachers related to big ideas in general and to modelling as a big idea in particular is scarce. Hence, this study concentrates on views of Austrian and German pre-service teachers about the significance of modelling as a big idea. For this purpose, we focus on the role of modelling as a big idea, and report about results concerning the pre-service teachers' perceptions.*

## **INTRODUCTION**

Modelling is considered as a big idea by the scientific communities in mathematics and mathematics education (cf. Blum et al., 2007; Lesh et al., 2007; Stillman et al. 2008) – it remains very important for mathematics and its development, and there is a consensus on its importance for mathematics instruction and mathematical literacy. In contrast, empirical findings have suggested that mathematics teachers might lack awareness of the significance of modelling in everyday instruction. However, there is still a need of broadening the empirical base in this field, and teachers' perceptions about the significance of modelling in comparison with other big ideas have hardly been the subject of research so far.

This paper responds to this need of research by investigating views of Austrian and German pre-service teachers about the significance of modelling. These views about modelling are also compared to views about other big ideas, and they are, moreover, examined with respect to specific content areas. The results indicate that modelling is perceived as a significant big idea among others. However, low evaluations by a subgroup of pre-service teachers suggest that the awareness of modelling as a mathematics-related big idea should be fostered. Intercultural differences could stem from differences in declarative and procedural professional knowledge about modelling.

In this paper, we will first give an introduction to the theoretical background of considering modelling as a big idea. After specifying the research questions, we will give information about sample and methods in a corresponding section. We will then present results and finally discuss them in a concluding section.

## THEORETICAL BACKGROUND

### The “idea” of big ideas

Learning opportunities can be particularly rich when they address mathematical concepts holistically (cf. e. g. English, 2003). In mathematical content knowledge, the importance of overarching concepts has been identified by Bruner (1960) and Schweiger (2010), in whose work the notion of “fundamental idea” is central. According to Schweiger (1992, p. 203), a fundamental idea is „a bundle of actions, strategies or techniques, [...] that

1. can be made visible in the historical development of mathematics, that
2. appear as sustainable for structuring curricular conceptions vertically, that
3. are appropriate as ideas for giving answers to the question, what mathematics is, and for speaking about mathematics, that therefore
4. can make mathematics instruction more flexible and at the time more transparent.

Furthermore, [...] an anchoring in everyday language and thinking, like a corresponding archetypical thought, expression or action [is] necessary.“

The idea behind the approach of putting such fundamental, overarching, or simply “big” ideas in the foreground of mathematics instruction is to facilitate building up links across contents, to foster the organisation of mathematical knowledge, to encourage reflection and to help learners to make sense of mathematics. Seen from a moderate constructivist perspective, these aspects can be used to describe what can be seen as a big idea. Such a characterisation affords integrating prior approaches to overarching concepts into a pragmatic understanding, and offering an openness towards the reflections about mathematics by teachers as it may take place in their professional development. Accordingly, by referring to the notion of big ideas in mathematics (s. also Kuntze et al., accepted), we think of ideas that should

- have a high potential of encouraging learning with understanding of conceptual knowledge (including orientation, linking and anchoring of knowledge)
- be relevant for building up meta-knowledge about mathematics as a science
- support abilities of communicating meaningfully about mathematics
- encourage reflection processes of teachers connected with designing rich and cognitively activating learning opportunities as well as with accompanying and supporting learning processes of students.

Hence we do not consider big ideas as belonging to a pre-defined catalogue, but we emphasise the potential of these ideas for making mathematics meaningful and rich for conceptual learning according to the aspects above. Examples of big ideas the project ABCmaths ([www.abcmaths.net](http://www.abcmaths.net)) focuses on are “using multiple

representations”, “dealing with infinity”, “doing-undoing/inverting”, or “generalising/specialising” (for more details, see ABCmaths team, in preparation). Of course, these ideas may overlap, but they have different emphasis.

Consequently, for profiting from rich learning opportunities, learners and also their teachers should be aware of big ideas linked to mathematics. We will exemplify this by focusing on the big idea of modelling.

### **Modelling as a big idea in mathematics**

Indeed, “mathematical modelling” not only clearly satisfies well the meta-scientific, curricular, historical and learner-oriented criteria of Schweiger (1992), but also affords reflecting on mathematical concepts and their roles for solving real-world problems as well as creating rich learning opportunities. As Pollack (1979, p. 240) states, modelling “requires an understanding of the situation outside mathematics and of the process of mathematisation as well as of the mathematics itself”. And, conversely, the role of mathematics can be recognised particularly clearly when reflecting about modelling processes. The contribution of mathematical ideas and concepts (Lakoff et al., 2000) can be made transparent for learners through their use in modelling real-world problems. A prototypical example of the impact of modelling on mathematics with implications also for mathematics instruction can be seen in De Lange’s (1996) approach of using real world problems as an opportunity to develop mathematical concepts. Further, the approach of developing mathematical knowledge by using and refining models not only allows curricular conceptions but also helps to anchor modelling in everyday thinking (cf. Blum et al., 2007; Lesh et al., 2007; Stillman et al. 2008).

### **Implications for the classroom and desiderata for the professional knowledge of mathematics teachers**

Although mathematical modelling has had a history as long as mathematics itself, the same cannot be said of the history of mathematical modelling in school curricula. A good deal about modelling as classroom teaching and learning strategy has been written e.g. by Swetz et al. (1991) or White (1994). Today the importance of mathematical modelling is acknowledged in current curricula and standards.

Consequently, modelling is given a key role in the assessment of mathematical competency. For instance, it is an important constituent in ‘Mathematical literacy’, which is defined by the OECD / PISA (2003, p. 24): “Mathematical literacy is an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen.” This definition highlights the relevance of modelling and underpins its role as a big idea. Consequently, competencies in modelling play an important role in national standards (cf. KMK, 2005; AECC, 2008), too.

Hence, developing modelling competencies is a fundamental goal for mathematics instruction as described by Klieme et al. (2001, p. 186): “In the centre of basic education the ability of mathematical modelling should be found.” In the classroom, students should be encouraged and supported to engage in meaningful modelling tasks and modelling activities (Blum, 1996; Maaß, 2004). As a precondition it is acknowledged that students learn mathematical modelling by (actively) participating in the experience of constructing a model and working to correct and improve their models. This requires teachers to allow students to formulate, test, discuss and adjust their thinking. The role of teachers is to provide opportunities for this learning to take place and to guide the students through a modelling process, while allowing them freedom within each stage.

As a prerequisite for teachers to design corresponding learning opportunities with rich modelling activities, teachers need to be aware of the idea of modelling and its relations to a variety of contents. Hence, teachers’ views about the significance of modelling appear as crucial for the development of mathematics instruction.

### **Teachers’ views about modelling**

Studies into teachers’ views about modelling have to be seen as embedded in more general research focusing on professional knowledge and teachers’ beliefs or convictions (e.g. Shulman, 1986; Ball, Thames & Phelps, 2008). The corresponding theoretical background is developed in more detail in Kuntze (accepted). Leading to the research questions of this study about the significance of modelling as a big idea, we will, in the following, focus on specific findings. A recent study about teachers’ views about modelling has been realised in the framework of the LEMA project (Maaß et al., 2009). In this study, teachers demonstrated little knowledge about modelling, e.g. about the modelling cycle. Kuntze and Zöttl (2008) showed that pre-service teachers favoured tasks with low modelling relevance to tasks with higher modelling requirements. In a study by Kuntze (accepted) in-service teachers showed a tendency of similar patterns of answers in a task-specific survey centred on the idea of modelling. Again, the in-service teachers were not very confident with respect to their knowledge about the modelling cycle, which suggests a lack of professional knowledge related to modelling. We consequently expect that teachers might attribute not the highest significance to modelling, as a consequence of a non-optimal professional knowledge related to modelling. These findings correspond to evidence from other studies suggesting that modelling in classrooms has to compete for attention (cf. Kaiser-Meßmer, 1985; Maaß, 2004). As there is probably a dilemma for many teachers between different instructional goals in concurrence with the aim of developing and fostering modelling competencies of the students, research should focus on the importance teachers associate with modelling in comparison with other important mathematics-related ideas like those presented in the previous section. As views about the significance of modelling as a big idea have hardly been subject to quantitative empirical research so far, this study on comparisons with the

significance attributed to other big ideas is exploratory. For linking the results with other findings in the ABCmaths project, we focus on the set of big ideas in the scope of ABCmaths.

## RESEARCH QUESTIONS

Against this theoretical background, we focus on the following research questions:

- What views about the significance of modelling as a big idea in mathematics do pre-service teachers hold?
- How do these views compare to the significance attributed to other big ideas?
- In which content domains do pre-service teachers root the importance of modelling as a big idea?
- Are there differences between pre-service teacher groups of different school cultures?

## SAMPLE AND METHODS

In order to find out about the research questions above, a test was administered to 117 German pre-service teachers (78 female, 35 male, 4 without data) and 42 Austrian pre-service teachers (27 female, 15 male) before the beginning of a university course, respectively. The German pre-service teachers had a mean age of 22.33 years ( $SD = 3.56$  years) and had been studying on average for 2.19 semesters ( $SD = 1.12$ ). 61 pre-service teachers were preparing for being teachers in primary schools, 35 in secondary schools for lower-attaining students, and 15 for working in schools for students with special needs. The 42 Austrian pre-service teachers had a mean age of 22.5 years ( $SD = 3.34$  years) and had been studying on average for 4.93 semesters ( $SD = 2.10$ ). All of them were preparing to be teachers in secondary schools. The choice of the sub-samples was not meant to be the base of a representative international comparison. However, comparing the answers in these sub-samples can give insights into how stable under a change of culture (internationally or related to school type) the considered teachers' views are.

The exploratory approach of this study and the research questions highlight the need of gaining an overview on the pre-service teachers' views. This suggests the use of quantitative methods based on a questionnaire survey. Accordingly, the pre-service teachers were asked to complete a questionnaire unit on their views about the significance of modelling and other big ideas. This questionnaire unit was part of a longer instrument, which was used in an analysis of needs study in the research project ABCmaths ([www.abcmaths.net](http://www.abcmaths.net); cf. also Kuntze et al., accepted). The questionnaire unit contained a rating of the overall significance of modelling among other big ideas (see Fig. 1), and a rating of their importance for specific content areas, respectively. The pre-service teachers were asked to rate the significance of the big ideas with numbers between 0 (low significance) and 5 (high significance). Moreover, the questionnaire unit provided short explanations on the big ideas

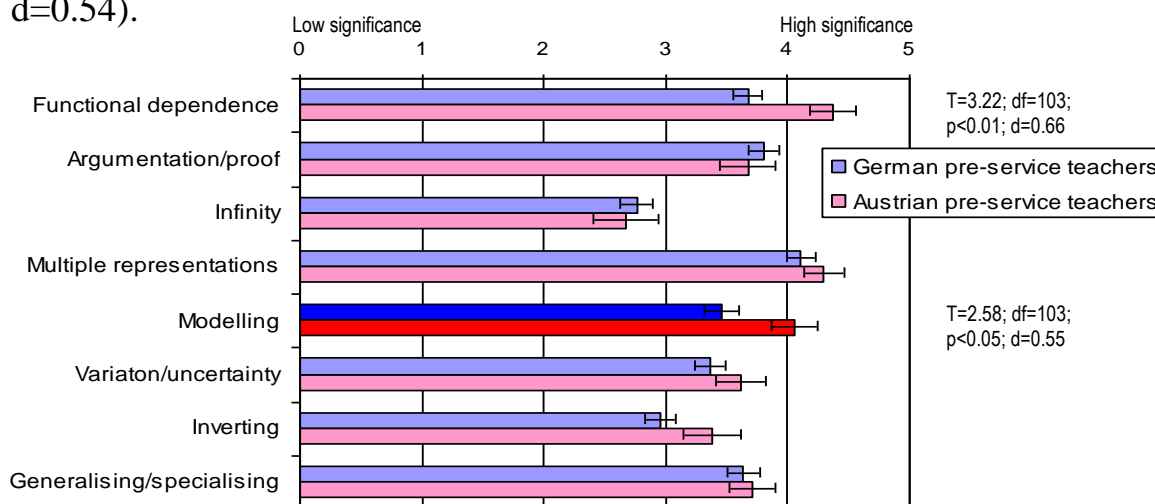


appearing in the questionnaire in order to prevent the participants from misunderstandings. In the following analyses, pre-service teachers with complete data are considered.

## RESULTS

In this section, we present results in the order of the research questions given above. As an exception, we will refer to the last research question about inter-cultural differences constantly while reporting the results, in order to support easy access to the data. The pre-service teachers held on average a positive view about the significance of modelling as a big idea in mathematics. The average values of the ratings are in the upper half of the scale, 3.46 (SD=1.28) for the German and 4.06 (SD=1.10) for the Austrian pre-service teachers. The difference is significant ( $T=2.58$ ;  $df=103$ ;  $p<0.05$ ;  $d=0.55$ ), indicating a possible inter-cultural effect.

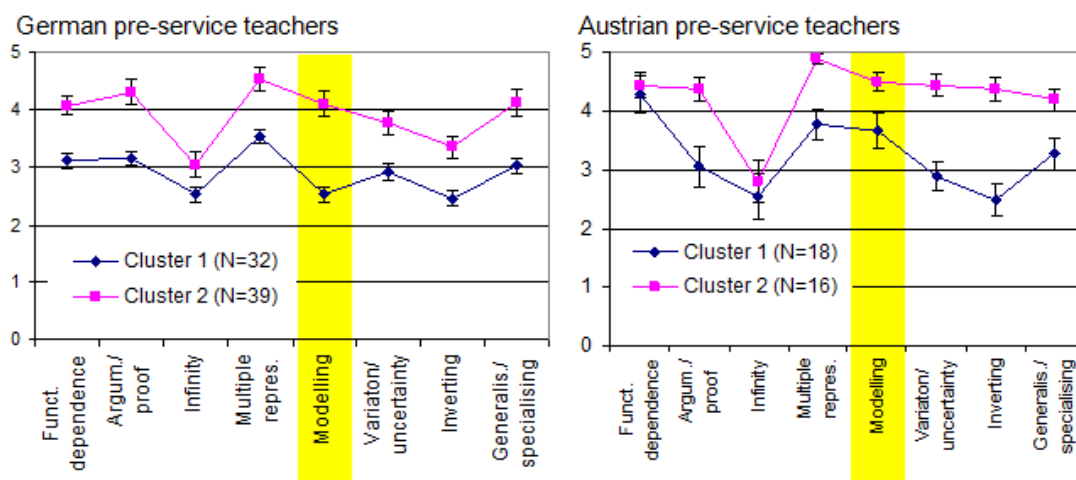
Figure 1 shows the importance assigned to modelling compared to the average significance attributed to other big ideas. As can be seen, modelling was not the most important big idea in the eyes of pre-service teachers. For German participants, the importance given to modelling was significantly lower than the importance given to “argumentation/proof” ( $T=2.08$ ;  $df=70$ ;  $p<0.05$ ;  $d=0.33$ ), “multiple representations” ( $T=4.38$ ;  $df=70$ ;  $p<0.001$ ;  $d=0.59$ ) and higher than “infinity” ( $T=2.94$ ;  $df=70$ ;  $p<0.01$ ;  $d=0.49$ ) and “inverting” ( $T=2.41$ ;  $df=70$ ;  $p<0.05$ ;  $d=0.37$ ). For the Austrian pre-service teachers, no other big idea was rated significantly higher than modelling. However, the only big ideas to have been rated significantly lower than modelling were “infinity” ( $T=4.54$ ;  $df=33$ ;  $p<0.001$ ;  $d=1.03$ ) and “inverting” ( $T=2.89$ ;  $df=33$ ;  $p<0.01$ ;  $d=0.54$ ).



**Figure 1: Ratings of the significance of big ideas: Means and standard errors**

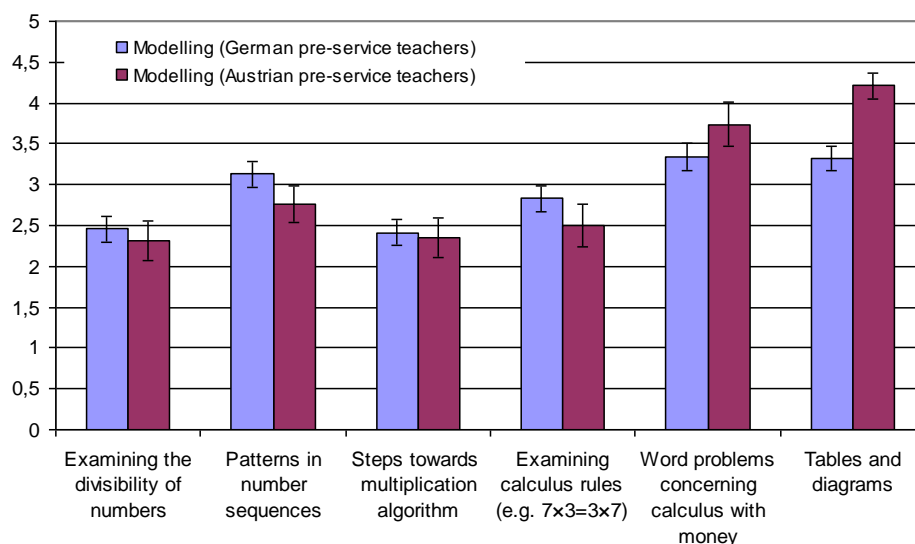
Beyond considering mean values, we focused on differences within the sub-samples by calculating (separate) cluster analyses on the base of the ratings of the big ideas (Ward method). The results are presented in Figure 2, showing a cluster with lower and one with higher ratings, splitting the sub-samples approximately by half. In the

case of the German pre-service teachers, these clusters show the largest difference in their views of modelling among all big ideas, whereas the ratings of modelling appear more homogeneous for the Austrian clusters.



**Figure 2: Ratings of subgroups of pre-service teachers based on a cluster analyses**

Correlations between the ratings of the significance of big ideas can give further insight into how these were interrelated in the eyes of the pre-service teachers. In the German sub-sample, the significance of “modelling” correlates with the ratings of “functional dependence” ( $r=0.45^{**}$ , correlation significant with  $p<0.01$ ), “multiple representations” ( $r=0.75^{***}$ ,  $p<0.001$ ), “inverting” ( $r=0.42^{*}$ ,  $p<0.05$ ), and “generalising/specialising” ( $r=0.40^{*}$ ,  $p<0.05$ ). In the Austrian sub-sample, the rating of “modelling” correlates with the perceived importance of “multiple representations” ( $r=0.37^{***}$ ,  $p<0.001$ ) and “variation/uncertainty” ( $r=0.31^{**}$ ,  $p<0.01$ ).



**Figure 3: Significance of modelling related to content areas**

In order to stay not only on the level of very general evaluations of the significance of modelling as a big idea, the third research question focused on making a link

between the pre-service teachers' general views and selected content areas. Figure 3 shows the average rating of the significance of modelling related to six content areas. The values concerning "patterns in number sequences", "word problems concerning calculus with money" and "tables and diagrams" were somewhat higher than the perceived significance related to the other content areas. With the exception of "tables and diagrams", there are no significant differences between the Austrian and German sub-samples.

## DISCUSSION AND CONCLUSIONS

Views about the significance of modelling and other big ideas related to mathematics can be seen as an indicator of the awareness teachers have towards ideas linked to mathematics when defining learning goals and designing learning opportunities for their classrooms (cf. Shulman, 1986; Kuntze, accepted). Even if the results should be interpreted with care, given the exploratory approach of the study, modelling appears to have been perceived on average as a significant big idea. However, in the cluster analysis, it became apparent that a large portion of the pre-service teachers saw modelling as relatively insignificant compared to other big ideas. The Austrian pre-service teachers put more emphasis on this idea than did the pre-service teachers in the German sub-sample. This might be not, or not only, an effect of countries but it could also be an effect of the school types the teachers were preparing for and the corresponding cultures. Hence, school culture may play an important role for the professional development process of prospective teachers with respect to modelling. Against this background, the results do not contradict prior findings (e.g. Maaß & Gurlith, 2009; Kuntze, acc.), but they broaden the scope of possible explanations.

An interesting question related to these results is whether differences in professional knowledge may have caused the differences in the evaluations of "modelling" and "functional dependence". Possibly, prior professional learning of the Austrian teachers could have caused this effect. The more general question for further research would be whether and how views concerning the significance of big ideas can be developed and influenced by teacher education programs.

Even though there are some intercultural differences, it is interesting that some of the Austrian and German pre-service teachers' views were relatively similar, e.g. those related to the mathematical content areas. However, despite of these resemblances, teacher education should increase the awareness of modelling also related to a broad spectrum of other content areas, including areas that may be evaluated as less relevant in the first instance. This raises not only the question of further research about content domain-related views about the idea of modelling, but also to what extent such views may tend to evolve with increased instructional experience, in short terms: What are the views of in-service teachers related to the significance of modelling as a big idea? Moreover, the findings call for empirical research into the

role of the views examined in this study for the teachers' choice of learning opportunities in the classroom.

Considering the structure of professional teacher knowledge itself, research with the goal of identifying relationships with other aspects of professional knowledge could open up ways of effective mathematics teacher professionalisation. For the later research question, we expect results from ongoing studies and analyses in ABCmaths.

## ACKNOWLEDGEMENTS

The project ABCmaths is funded with support from the European Commission (503215-LLP-1-2009-1-DE-COMENIUS-CMP). This publication reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein. We acknowledge the cooperation of Elke Kurz-Milcke, Anke Wagner, Claudia Wörn (PH Ludwigsburg) and Karl-Josef Fuchs, Michael Schneider (Universität Salzburg) as well as Peter Winbourne (London South-Bank University) and Bernard Murphy (Mathematics in Education and Industry, UK) who collaborate as team members in the project ABCmaths.

## REFERENCES

- AECC (2008). *Bildungsstandards im Fach Mathematik*. Österr. Kompetenzzentrum für Mathematikdidaktik, Klagenfurt: IDM.
- Ball, D., Thames, L., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Blum, W. (1996). Anwendungsbezüge im Mathematikunterricht – Trends und Perspektiven. In *Schriftenreihe Didaktik der Mathematik*, Band 23, 15-38.
- Blum, W., Galbraith, P., Niss, M., & Henn, H.-W. (Eds.). (2007). *Modelling and applications in mathematics education*, New ICMI Studies Series no. 10. New York: Springer.
- Bruner, J. (1960). *The process of education*. Cambridge, Mass.: Harvard University Press.
- De Lange, J. (1996). Real problems with real world mathematics. In L. Alsina & al. (Eds.), *Proceedings of the 8<sup>th</sup> ICME*, Sevilla: S.A.E.M. Thales, 83-110.
- English, L. (2003). Mathematical modelling with young learners. In S. Lamon, W. Parker & K. Houston (Eds.), *Mathematical modelling: A way of life*. ICTMA. Chichester: Horwood Publishing, 3-18.
- Kaiser-Meßner, G. (1985): *Anwendungen im Mathematikunterricht. Band 2 – Empirische Untersuchungen*. Hildesheim: Franzbecker.
- Klieme, E., Neubrand, M. & Lüdtke, O. (2001). Mathematische Grundbildung: Testkonzeption und Ergebnisse. In P.-K. Deutschland (Ed.), *PISA 2000: Basiskompetenzen von Schülerinnen und Schülern im internationalen Vergleich*. Opladen: Leske + Budrich.

- KMK. (2005). *Bildungsstandards im Fach Mathematik für den Hauptschulabschluss*. Neuwied: Luchterhand.
- Kuntze, S. (accepted). In-service and prospective teachers' views about modelling tasks in the mathematics classroom – Results of a quantitative empirical study. [Proceedings Book of ICTMA 14].
- Kuntze, S., Lerman, S., Murphy, B., Kurz-Milcke, E., Siller, H.-St., & Winbourne, P. (accepted). Professional knowledge related to Big Ideas in Mathematics – an empirical study with pre-service teachers. In *From a study of teaching practices to issues in teacher education* (CERME 7).
- Kuntze, S. & Zöttl, L. (2008). Überzeugungen von Lehramtsstudierenden zum Lernpotential von Aufgaben mit Modellierungsgehalt. *math. did.*, 31, 46-71.
- Lakoff, G. & Núñez, R. (2000). *Where Mathematics Comes From*. Basic Book.
- Lesh, R., & Zawojewski, J. (2007). Problem solving and modelling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Charlotte, NC: IAP.
- Maaß, K. (2004). *Mathematisches Modellieren im Unterricht – Ergebnisse einer empirischen Studie*. Hildesheim: Franzbecker.
- Maaß, K., & Gurlith, J. (2009). Designing a teacher questionnaire to evaluate professional development in modelling. In *Application and Modelling*, CERME 6, WG 11, 2056-2066.
- OECD / PISA (2003). *The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Paris: OECD.
- Pollak, H.O. (1979). The interaction between mathematics and other school subjects. *The int. commission on math. instruction*, IV. Paris: ICMI, 232-248.
- Schweiger, F. (1992). Fundamentale Ideen. Eine geistesgeschichtliche Studie zur Mathematikdidaktik. In *JMD*, 13(92), 199-214.
- Schweiger, F. (2010). *Fundamentale Ideen*. Aachen: Shaker.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Stillman, G.A., Brown, J. P., Galbraith, P. L. (2008). Research into the teaching and learning of applications and modelling in Australasia. In H. Forgasz, A. Barkatsas, A. Bishop, B. Clarke, S. Keast, W-T. Seah, P. Sullivan (Eds.), *Research in mathematics education in Australasia 2004-2007*. Rotterdam, The Netherlands: Sense Publishers, 141-164.
- Swetz, F., & Hartzler, J. S. (1991). *Mathematical Modelling in the Secondary School Curriculum*. Reston, Virginia: N.C.T.M.
- White, A. L. (1994). Managing the Modelling Process in the Secondary Classroom. In D. Rasmussen & K. Beesey (Eds.), *Mathematics without Limits*. Brunswick Melb: M.A.V., 442-446.

# MODELLING IN AN INTEGRATED MATHEMATICS AND SCIENCE CURRICULUM: BRIDGING THE DIVIDE

Geoff Wake

University of Manchester

*This paper explores the theoretical rationale behind a new approach to developing integrated mathematics and science curriculum experiences using the construct of bridging concepts. Such interdisciplinary approaches provide challenges for teachers and learners as they need to develop new practices in their classrooms and communicate across traditional subject boundaries whilst ensuring curriculum objectives in both subjects are met. Experiences from students across case study schools suggests that they need considerable time and space in which to develop conceptual understanding within and across mathematics and science and the vocabulary with which to communicate their thinking. The development to-date points to an important role for applications and modelling in mathematics in such cross-curriculum learning.*

## BACKGROUND:

The place of mathematical modelling in curricula of compulsory schooling is often contested and even with the pressure of international assessment such as PISA giving priority to applications of mathematics it may struggle to find a place (Ofsted, 2008). Equally scientific enquiry, as part of what might be considered a ‘reform’ agenda in science, that prioritises scientific method and consequently sees the need for enquiry as central, prioritises approaches that are not always prominent in day-to-day teaching (for example, Sadeh and Zion (2009), Wilson et al (2010)). The work described here relates to an EU (Comenius) funded project COMPASS (Common problem solving strategies as links between Mathematics and Science<sup>19</sup>) which seeks to tackle these issues promoting an integrated approach to the learning of mathematics and science by developing a range of classroom materials and professional development support for teachers. Fundamental to this project is the development of new approaches to teaching that motivate learning both within and across the disciplines. Pragmatically, given that there is no official demand for this (at least in England), this requires some understanding of how such innovation can be motivated in ways that ensure that current curriculum demands are being met.

Here, therefore, I present a theoretical rationale behind a new approach to interdisciplinary teaching and learning in mathematics. The outcomes of an initial adoption of materials based on this are reflected upon so as to inform a next iteration

---

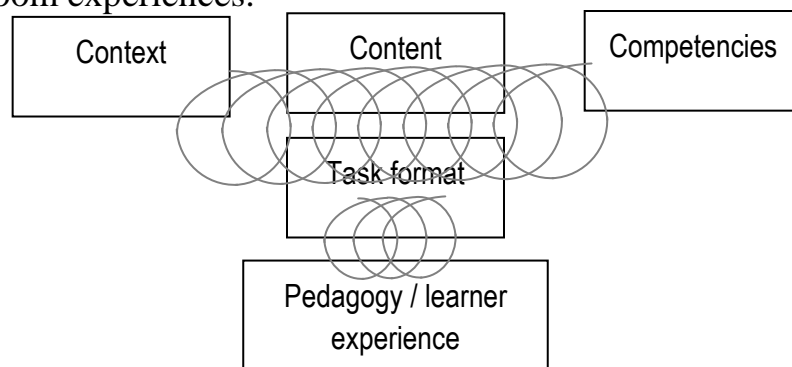
<sup>19</sup> Project: 503635-LLP-1-2009-1-DE-COMENIUS-CMP. This is funded under the Lifelong Learning Programme. For further details see <http://www.compass-project.eu/project.php>.

of their further development. I firstly explore how such aims might be achieved by considering, albeit briefly, the nature of the different domains of mathematics and science and exploring different ways that the demands of each, and indeed the demands of teachers and learners of each, might be met. I present a theoretical overview and a rationale of the design of interdisciplinary teaching and learning that, therefore, is also cognisant of these analyses of the subject domains of mathematics and science informed by an understanding of interdisciplinary learning approaches. Additionally, the resulting approach also has to meet the aims of the COMPASS project that seeks to see mathematical and scientific activity situated in contexts that are meaningful to critically informed European citizens of the future. The approach adopted is exemplified in this paper by outlining a set of materials that have gone through a first cycle of design and improvement in collaboration with mathematics and teachers working in a small group of schools in England. Finally, I draw on case study data and particularly focus group interviews with teachers to reflect further upon the design principles and implications for the integration of mathematics and science and the potential importance of applications and modelling in such curriculum design.

## **DEVELOPING A THEORY INFORMED DESIGN APPROACH**

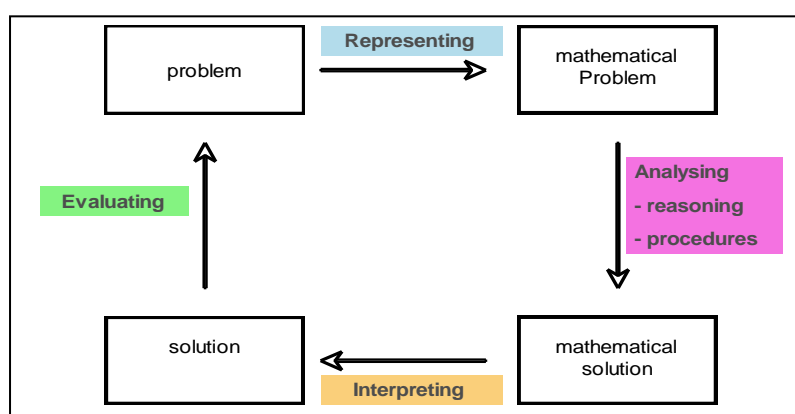
Nikitina (2006) identifies, on the basis of empirical classification, three basic approaches to the development of interdisciplinary approaches: contextualising, conceptualising and problem – centering. Her classification suggests that, in general, underlying differences and similarities in pedagogies and epistemologies in different disciplines are likely to result in one or other of these different approaches being adopted when school subjects are considered in common patterns of interdisciplinarity. For example, in the case of interdisciplinary approaches involving (1) disciplines in the humanities the common approach is often one of *contextualising*, that is one of developing tasks and materials that pay attention to issues of time, culture and personal experience. This contrasts with likely approaches to interdisciplinary work (2) in the scientific disciplines that is likely to focus on *concepts* that are central to constituent disciplines and work in quantifiable ways in making connections and (3) that takes a critical *problem-solving* approach to issues of social concern and which may require a more eclectic combination of knowledge and skills across disciplines. This would suggest that an interdisciplinary approach to study of mathematics and science is commonly one that focuses on concepts. It is perhaps not surprising that Nikitina's classification is reflected in other more philosophical analyses of mathematics and science knowledge domains that attempt to capture the essence of these in their enactment as school subjects. For example, Figure 1, adapted from the PISA overview of the mathematical literacy domain (OECD, 2003), draws attention to important components to take into account when considering this: concepts, competencies and contexts or situations in which these might be situated. These in turn inform formulation of tasks and ultimately learner

experience. The looping connections in the diagram suggest that there is no straight forward heuristic that can be adopted in developing tasks and approaches to lessons, but rather that the process is inevitably fuzzy as the different factors are brought to bear on classroom experiences.



**Figure 1. School mathematics domain (Wake, 2010) based on PISA analysis (OECD, 2003)**

The influence of the OECD international comparative study (PISA) that has over recent years measured student performance at age 15 in mathematics, science and literacy has been important in informing curricula design across nations (Grek, 2009). This is the case in England where in 2007 (QCA, 2007) a new national curriculum for all subjects was introduced. In mathematics this gave greater prominence to competencies / process skills than had hitherto been the case reflecting the greater emphasis given to these by the PISA tests. The key processes of *representing*, *analysing* (using procedures and reasoning), *interpreting* and *evaluating* have been emphasised in relation to a problem solving cycle (see Figure 2) which is based on the modelling schema included in the OECD framework (OECD, 2003).



**Figure 2. Problem solving / modelling cycle in the English mathematics curriculum**

The problem solving cycle as represented in the curriculum is more general than in PISA as it is expected that students could be working in the world of mathematics itself rather than working towards solution of a problem in a non-mathematical reality (as in PISA). The corresponding framework for the scientific literacy domain



equally identifies the importance of the three components; (knowledge /) *concepts*, (processes /) *competencies* and (situations /) *contexts*. As in the case of mathematics, science in the school curriculum tends to focus on content and competencies at the expense of context with the emphasis on the former of these components. In science, discussion of concepts identifies thirteen major and diverse themes ranging from the property of matter to genetic control. Scientific processes are grouped under three main themes: (1) describing, explaining and predicting scientific phenomena, (2) understanding scientific investigation, (3) interpreting scientific evidence and conclusions (OECD, 2003, p137).

In summary, school mathematics and science practices may, therefore, be considered as being culturally and historically situated primarily in the *concept* component of the domain with recent curriculum formulations in England having the intention of shifting the focus to the *competency* component. This shift in focus of curriculum priority suggests a potential way forward in interdisciplinary curriculum design would be to focus on common mathematical and scientific competencies. However, there remains one component of the subject domain that as yet has not been discussed in relation to its potential to support interdisciplinarity: that of *contexts*. This is often neglected in both mathematics and science which, as school disciplines, may be seen as rather abstract areas of study with only hints of their applications and usefulness in contemporary contexts. There have been recent attempts in science to remedy this with courses such as 21<sup>st</sup> Century Science (Millar and Osborne, 1998) attempting to situate scientific understanding more realistically in the lives of the students who are expected to study this. It should be noted that such courses are not without their critics who see them as diluting true / ‘academic’ scientific knowledge and understanding. The role of *contexts* in the interdisciplinary approaches to be developed is considered important by the COMPASS project that seeks to provide teachers with rich, motivating materials that allow young people to explore meaningful problems in a European context. In other words an important aim of the project is to motivate students by posing problems that can immediately be seen to have importance to young people as citizens of Europe with a concern for their environment and the population that inhabits this.

Overall, then, in designing materials for classroom use in ways that bring together meaningful learning in both mathematics and science as separate disciplines, as well as in ways that makes powerful interconnections, it was the COMPASS group’s desire to pay due attention to each of the components (contexts, content and competencies) of the discipline domain without privileging any one at the expense of the other. In an attempt to do this whilst also addressing the pragmatic issue of providing neutral ground on which mathematics and science teachers might jointly work together we developed the construct of the “bridging concept”. This is an organising structure that draws on mathematical and scientific thinking in ways that provide a way of visualising and understanding how measurable and therefore

quantifiable phenomena (interact or) behave. Importantly, Nikitina's analysis suggests that this is likely to meet with least resistance from mathematics and science teachers who appear to be most comfortable with interdisciplinary approaches of a type that gives prominence to concepts. However, before going on to describe bridging concepts more generally, and exemplify in some detail one such concept, I wish to emphasise that the materials that we eventually produced pay due attention to the other important components of the discipline domains. The bridging concept, therefore, attempts to take a new epistemological approach to knowledge across science and mathematics by looking for ways in which understanding phenomena might be considered as being general and consequently having applications across many situations / contexts. For example, the need to understand how outcomes at the micro-level are scaled up and have implications at the macro-level appears important in many situations. Consider, for example, issues (1) of inoculation against disease: how does an individual's decision to be inoculated or not impact on the likely spread of disease at the level of the population? (2) of the impact of a switch to energy saving devices / equipment such as light bulbs: what are the implications for the individual and for society as a whole? To investigate these problems deriving from context driven issues clearly requires the use of both concepts and competencies from mathematics and science but it is the intention that at a meta-level students also have opportunities to consider how micro-level actions have implications and outcomes at a macro-level. It is the task formulation (Figure 1) that determines how concepts and competencies in mathematics and science will interact. At this stage we have not identified a substantive number of such bridging concepts but the examples and teacher and student reactions that I go on to discuss focus on a further bridging concept: that of *flow*. Again there are multiple needs to understand flow across the different sub-disciplines of science and indeed more widely in technology-based settings: flow is a central characteristic of contexts ranging from electric circuits to ecosystems, from the human heart to tidal estuaries, from solar emissions to traffic management. When linked with flow, the concept of equilibrium is a key to understanding stability in systems as varied as electricity supply, transport, geothermal activity and ecosystems. In summary, therefore, bridging concepts seek to value concepts from both the science and mathematics domain in making sense of phenomena and have general applicability at a meta-level. Importantly they provide a driver to facilitate cross-disciplinary thinking in ways that are deeper / richer than would otherwise be the case. The intention is that the bridging concept gives new insights to teachers and learners alike at how they can use mathematics and science to make sense of a range of issues of importance to European citizens of today and the implications of these for the future.

### **BRIDGING CONCEPT: FLOW**

The materials developed and exemplified here focus on the bridging concept of flow and supported students towards a contextual problem based on flooding. "How can

further disastrous flooding and environmental damage be prevented? – a case study of Lynmouth in Devon, UK.” However, this task comes as the culmination of a substantial programme of lessons in both mathematics and science that focused on key underpinning conceptual understanding. The flow of liquid was central throughout this particular set of materials although at the beginning the need to understand flow as an important concept was motivated by video clips that allowed students to consider flow in the context of traffic and crowd movement. Here due to restrictions of space I will exemplify the teaching materials with reference to mathematics materials only, before returning to consider implementation issues that arise from case studies of classrooms in a number of schools.

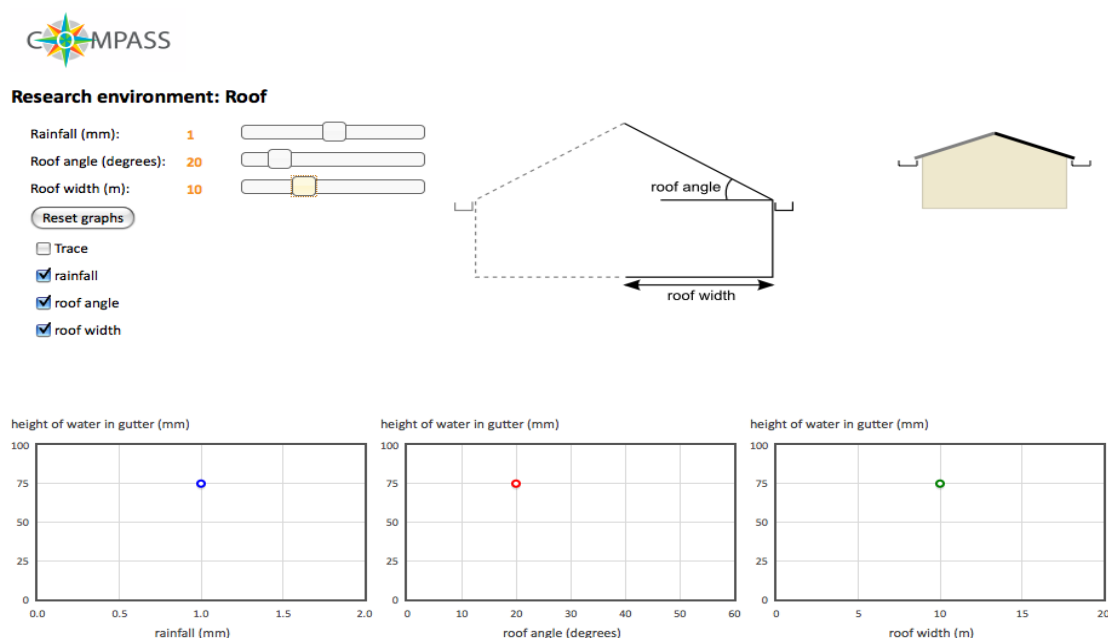
The mathematics is organised around two web-based *research environments* or *applets*<sup>20</sup> which students are encouraged to use to explore in a systematic way how different parameters affect aspects of measuring / quantifying and controlling flow. These applets model, in an idealised way, situations that the students will need to understand when they come to consider flow in science (for example, how flow is affected by the permeability of materials) and eventually the flow of rainwater that may give rise to flooding. In working with the applets students are expected to *predict – test – explain*, that is, to pose a question and predict the outcome before they use the applet to test this, and then explain why their prediction was correct or not, following the problem solving cycle with questions posed and predictions of outcomes important in concept development.

The first applet “Roof” simulates rain falling on a roof and draining into a blocked gutter. This provides an opportunity for students to come to an understanding that water, being a liquid, can flow from one place to another, and that we can measure / calculate its volume when it collects in a container. This provides an idealised scaled down model of a hillside with rain draining into a river that students will need to consider at a later stage. The applet model assumes that all of the water on the roof drains into a gutter on each side of the roof (Figure 3), and that the gutters are blocked so that all of the water collects there and is distributed evenly throughout the gutter. The gutter is assumed to be horizontal and that the depth of water in all parts of the gutter is the same throughout. The *applet* allows students to vary the total amount of rainfall, the angle of slope of the roof and the width of the house with graphs showing how each of these affect the height of water in the gutter. Using a trace facility allows the user to see how each variable is affected as they alter, using the slider, any one of the parameters between its minimum and maximum values. Although it is possible for students to carry out calculations of area, volume and capacity it is also possible for them to pose questions such as, “What would happen if the width of the house was doubled, multiplied by three, halved and so on?”

---

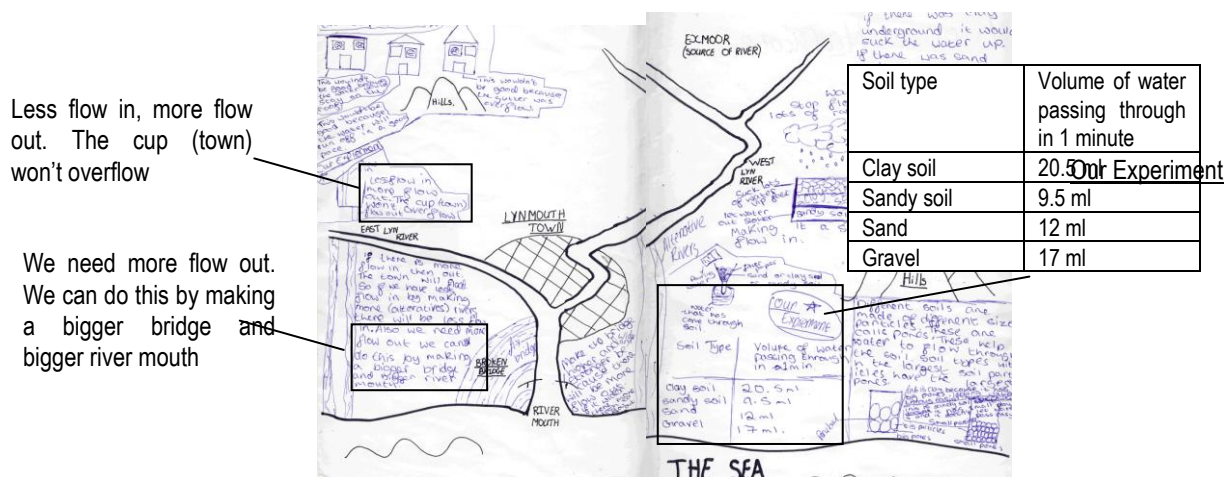
<sup>20</sup> Currently available at: <http://130.88.43.249/compass/>

A second applet simulates water flowing into and out of different cuboid and trapezoidal containers. In these simulations it is assumed that flow rates are constant throughout and that the containers are horizontal so that depths of water are constant throughout the container at all times. Having selected either a rectangular or trapezoidal container of volume 10, 20 or 30 units, and selected flow rates for water entering and leaving the container, graphs, on one set of axes, can be selected to be plotted of (i) total water flow in, (ii) total water flow out, (iii) volume of water in the container and (iv) total volume of water overflow from the container against time. A separate additional graph of height of water in the container against time is plotted. Again students are expected to explore in a systematic way, and again “predicting – testing and explaining”, how the flow through a ‘container’ is affected by important parameters. This simulation can be used to model water flow through guttering from a roof or from a hilly landscape through a river or drainage channel.



**Figure 3. Applet used to investigate the collection of rainwater from a roof**

As a final part of the sequence of activities students were asked to engage with a substantial problem that asked them to think about how flooding might be prevented in a valley leading to the sea: this scenario was based on an event in Lynmouth in Devon in 1952 where flash flooding caused substantial damage. Students were asked to use the results of some of the experiments they had carried out in maths and science to give the best possible advice to (a) farmers and others using and draining the land on the moors above Lynmouth; (b) bridge engineers planning to rebuild bridges in the valley; (c) engineers planning how the river should flow through the town; (d) people wanting to rebuild their houses and homes in Lynmouth. It was noticeable that much of the output of the students was descriptive rather than involving calculations. Some of this is shown in Figure 4.



**Figure 4. Student accounts in response to flooding problem**

In England the approach and materials explained in this paper have been used in a first cycle of development by a mathematics and science teacher from each of eight different schools. The teachers' work and accounts of their experiences<sup>21</sup> with their students in using the materials has been facilitated by two days of professional development workshops generously funded by an IMPACT grant from the Science Learning Centre North West: additionally over a period of two weeks each school had the support of a newly qualified mathematics and science teacher. In general each of the schools worked with pupils in both mathematics and science using the materials. The following reflections on the outcomes of this implementation are informed by focus groups that were convened as part of the second of the workshops during which teachers were asked to reflect on nine key areas that explored the design approach and their implementation of this.

## REFLECTIONS AND THE ROLE OF MODELLING

In considering the students' work in mathematics many of the teachers drew attention to ideas of modelling although the applets effectively present them and their students with ready-made models that they were able to use to explore issues that would eventually relate to a more complex reality. This type of exploratory environment, therefore, might be classified as being of the perspective "Educational modelling type (b) conceptual modelling" in the classification system as proposed by Kaiser et al (2007). The applets allow quantifiable aspects of the situation to be varied and the effects of doing so to be explored systematically: however, other potential variables are fixed to ensure a relative simplicity of the model and consequently provide for a didactical focussing on key concepts and competencies. For example, in the use of both applets students were encouraged to "predict-test-explain" with teachers reporting that this enquiry based activity resulted in a much

<sup>21</sup> These accounts can be accessed at: <http://www.education.manchester.ac.uk/research/centres/Ita/LTAResearch/compass/>

more dialogic classroom with students taking the lead in directing activity and working in collaborative groups. The conceptual understanding of issues relating to flow encapsulated in the applets appears to have been of fundamental importance to students learning to use these to inform their work in science and in tackling the substantial modelling task at the end of the sequence of lessons. Perhaps it is not surprising that students did not focus primarily on quantitative work when a more qualitative conceptual understanding had been sought by asking them to predict outcomes and graphical representations of these when working with the applets.

A crucial question for the design of the interdisciplinary approach I report here is the role that the bridging concept (in this case “flow”) played. In focus group interviews teachers reported that this was important (i) for themselves as teachers because it provided them with a new way of thinking / working together as a team developing a common frame of reference for participants, (ii) for students because it provided a new starting point and setting in which to work with familiar ideas to see things differently, (iii) in mathematics as it involved students in modelling and applications, and (iv) in science because it provided a common overview by which links and connections across the curriculum could be made. It appears, therefore, that the bridging concept provides a boundary object (see for example, Tuomi-Grohn and Engestrom, 2003) at the intersection of the usual formulations of the mathematics and science curricula. It seems that this allows students and teachers means by which they might develop a new epistemology that brings together knowledge and competencies in mathematics and science in ways that interplay with, and reinforce, each other. Our work with teachers and students in schools to date has highlighted that this requires that students have time and space in which they can explore key ideas (in this case related to flow) that allows them to develop conceptual understanding and gives them language and terminology by which they can start to communicate with each other about relatively complex ideas. The applets we developed in the case reported here (and which we have developed for other units of work) appear to provide a vehicle by which students might be able to develop this necessary understanding. However, we note that the understandings developed are indeed at a more conceptual level than merely being quantifiable / calculable. For example, students have considered what might occur if both flow-in and flow-out were doubled without worrying about specific rates of flow. We also note that the modelling work produced by students drew substantially on this conceptual understanding rather than leading naturally to an approach that resulted in careful calculation. We are lead to conclude, at this stage, that in developing an interdisciplinary approach to mathematics and science the construct of ‘bridging concept’ is useful as it allows both teachers and students new approaches to developing knowledge that synthesises mathematics and science. In such approaches to learning we have found that students need support in understanding new concepts and language with which to discuss these. We have found purpose built applets that



students can use to explore how varying different parameters can affect a mathematical model of a situation particularly useful. These applets involve students in developing a range of modelling sub-competencies as well as engaging with important mathematical content, providing a playful environment in which a pre-constructed model can be used with purpose to explore potential outcomes relatively quickly. It is to be hoped that these new concept focused approaches in mathematics may ultimately have spin-off as and when students come to build their own, and quantifiable models of situations. In general, therefore, early indications are that bridging concepts have the potential to inform the design of materials that may support an interdisciplinary approach to the learning of mathematics and science content and competencies in meaningful contexts. For mathematics in particular they give a motivating purpose for more conceptual, dialogic classrooms with mathematical models central.

## REFERENCES

- Grek, S. (2009). Governing by numbers: the PISA ‘effect’ in Europe. *Journal of Education Policy*, 24(1), 23-37.
- Kaiser, G., Sriraman, B., Blomhøj, M., Garcia, F. J. (2007). Report from the working group modelling and applications. In D. Pitta-Pantazi, G. Philippou (Eds.), *Proceedings of the 5th CERME* (pp. 2035–2041). Nicosia: University of Cyprus.
- OECD. (2003). *Framework for Mathematics Assessment*. Paris: OECD.
- Ofsted (2008). *Mathematics: understanding the score*. London: Ofsted.
- Sadeh, A. and Zion, M. (2009). The Development of Dynamic Inquiry Performances within an Open Inquiry Setting: A Comparison to Guided Inquiry Setting, *Journal of Research in Science Teaching*. Vol. 46, No. 10, pp. 1137 – 1160.
- Wilson C. D., Taylor J. A., Kowalski S. M., Carlso, J. (2010) The Relative Effects and Equity of Inquiry-Based and Commonplace Science Teaching on Students’ Knowledge, Reasoning and Argumentation, *Journal of Research in Science Teaching*. Vol. 47, No. 3, pp. 276 – 301
- Nikitina, S. (2006). Three strategies for interdisciplinary teaching: Contextualizing, conceptualizing, and problem-centering. *Journal of Curriculum Studies*, 38(3), 251–271.
- QCA (2007). *The National Curriculum*. London: QCA.
- Millar R, Osborne J (1998) *Beyond 2000: Science Education for the Future*. London, UK: King’s College
- Tuomi-Grohn, T. and Y. Engestrom 2003. *Between School and Work: New Perspectives on Transfer and Boundary-Crossing*. The Netherlands: Elsevier.
- Wake, G. (2010). *Connecting mathematics with reality - connecting reality with mathematics*. QCDA: London. (<http://www.qcda.gov.uk/resources/4466.aspx>).

# **EXPLORING THE SOLVING PROCESS OF GROUPS SOLVING REALISTIC FERMI PROBLEM FROM THE PERSPECTIVE OF THE ANTHROPOLOGICAL THEORY OF DIDACTICS**

Jonas Bergman Ärleback

Department of Mathematics, Linköping University, Sweden

*Abstract: This paper reports on a first attempt to use the notions of 'Research and Study Course' (RSC) and 'praxeologies' within the Anthropological Theory of the Didactics (ATD) to analyse groups of students engaged in the mathematical activity of solving realistic Fermi problems. By considering so called realistic Fermi problem as a generating question in a RSC the groups' derived sub-questions are identified and the praxeologies developed to address these are discussed.*

## **INTRODUCTION AND BACKGROUND**

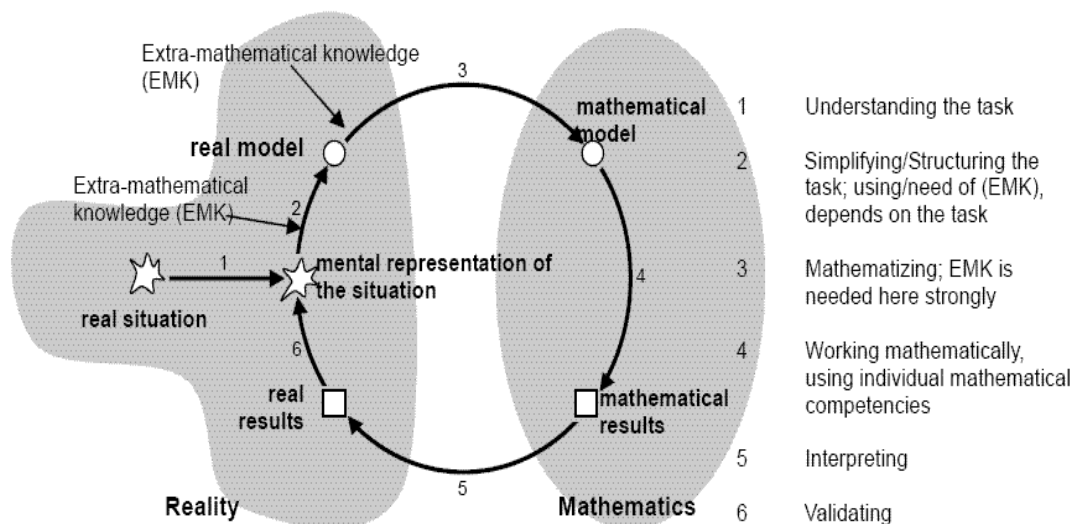
Working with mathematical models and modelling is a central part of the national intended written curriculum for the Swedish upper secondary mathematics education (Skolverket, 2000). Indeed, although without any definitions provided and only implicitly described, the notions of mathematical modelling and models have successively been explicitly more emphasised in the last two curricula reforms from 1994 and 2000 respectively (Ärleback, 2009a). Generally however, mathematics education research involving mathematical models and modelling at the Swedish upper secondary level is largely under-researched, and interest in research with this particular focus is just in its infancy. Nevertheless, one of the most palpable conclusions from this initial research is the big discrepancy found between the intended written curriculum and what the students actually attain (Ärleback, 2009b). For example, in a study of 381 third year upper secondary students 77 % stated that they never had encountered the notions during their upper secondary education (Frejd & Ärleback, accepted). Part of the problematique might be that teachers have difficulties in formulating and explaining their conceptions of these notions (Ärleback, 2010), or that mathematics teaching at this educational level in Sweden strongly is influenced by 'traditional' textbooks (Skolverket, 2003) with little or no discussions about models and modelling. However, it has been suggested and concluded that the introduction of, and students' initial conceptualisation of, mathematical modelling (interpreted in line with the written curriculum documents) at the upper secondary level adequately and efficiently can be done using so called *realistic Fermi problems* (Ärleback, 2009c; Ärleback & Bergsten, 2010). The aim of this paper is to continue, deepen and extend this line of investigation.

## **MATHEMATICAL MODELLING AND REALISTIC FERMI PROBLEMS**

In the research literature in mathematics education there are many different perspectives on and ways to approach mathematical modelling (e.g. Blum, Galbraith,



Henn, & Niss, 2007; Lesh, Galbraith, Haines, & Hurford, 2010). Concepts and notations used are for instance those of *competencies* (Blomhøj & Højgaard Jensen, 2007; Maaß, 2006); *modelling skills* (Berry, 2002); and, *sub-processes* or *sub-activities* (Blomhøj & Højgaard Jensen, 2003). These all focus on the descriptions of, relations between and/or the transitions of phenomena in the real world and their mathematical representations. From a Swedish perspective, the intended written curriculum (e.g. Skolverket, 2000) permits a broad interpretation of the meaning and content of the notions of mathematical models and modelling (Ärlebäck, 2009b). One such interpretation with a real influence on the school practice has been presented by Palm et al. (2004), which is used for the construction of national test items focusing on mathematical modelling [1]. In principle, the interpretation by Palm et al. concord with the view on modelling illustrated in Figure 1.



**Figure 1: The modelling cycle as presented by Borromeo Ferri (2006, p. 92) after adaption from Blum and Leiß (2007)**

Based on this view on modelling Ärlebäck (2009c) and Ärlebäck and Bergsten (2010) explored the idea that so called realistic Fermi problems are ‘modelling problems in miniature’, which potentially could be useful and time effective for introducing some of the typical features of mathematical modelling; see Ärlebäck (2009c) for details. Realistic Fermi problems are characterized by (1) their *accessibility*, meaning that they can be approached by all individual students or groups of students as well as be solved on both different educational levels and on different levels of complexity. Normally, any specific pre-mathematical knowledge is not required to provide an answer; (2) their clear real-world connection, to be *realistic*; (3) the need to *specify and structure the relevant information and relationships* to be able to tackle the problem. In other words for the problem formulation to be open in such a way that the problem solvers not immediately associated the problem with a know strategy or procedure on how to solve it, but

rather urge the problem solvers to invoke prior experiences, conceptions, constructs, strategies and other cognitive skills in approaching the problem; (4) the absence of numerical data, that is the *need to make reasonable estimates* of relevant quantities; and (5) their inner momentum to *promote discussion*, that as a group activity they invite to discussion on different matters such as what is relevant for the problem and how to estimate physical entities (e.g. respectively (3) and (4) above) [2].

The realistic Fermi problem the groups of students solved used in Ärlebäck (2009c) and Ärlebäck and Bergsten (2010) was *the Empire State Building problem*:

***The Empire State Building problem (ESB-problem):***

On the street level in Empire State Building there is an information desk. The two most frequently asked questions to the staff are:

- *How long does the tourist elevator take to the top floor observatory?*
- *If one instead decides to walk the stairs, how long does this take?*

Your task is to write short answers to these questions, including the assumptions on which you base your reasoning, to give to the staff at the information desk.

The data from three groups of students working on the ESB-problem using only paper and pencils for approximately 30 minutes was analysed using a developed analytical tool called *Modelling Activity Diagram (the MAD framework)*. This framework is inspired by Schoenfeld's 'graphs of problem solving' (Schoenfeld, 1985), the view adapted on mathematical modelling described above, and the five characteristic features of realistic Fermi problems. It pictures the different types of sub-activities the groups engage in during the problem solving process in terms of the categories *Reading*, *Making model*, *Estimating*, *Verifying*, *Calculating*, and *Writing* (see Figure 2 and Ärlebäck (2009c) for details). However, although this macroscopic schematic representation clearly shows that the sub-activities are dynamically and richly represented in the solving process of the Fermi problem (cf. Figure 2), it does not provide any detailed information about what kind of discussions, topics and questions the groups addressed and investigated. In order to get a more nuanced and circumstantial picture of the problem solving process involving realistic Fermi problems in these respects this paper aims to provide a more microscopic analysis focusing on what actually is discussed within the groups, especially in connection to mathematical topics and content. Hence, the research question studied in this paper can be states as follows: *What are the questions the students address during the problem solving process of the EBS-problem? What mathematics do the students use in the problem solving process and how is the use of this mathematics motivated and justified?*

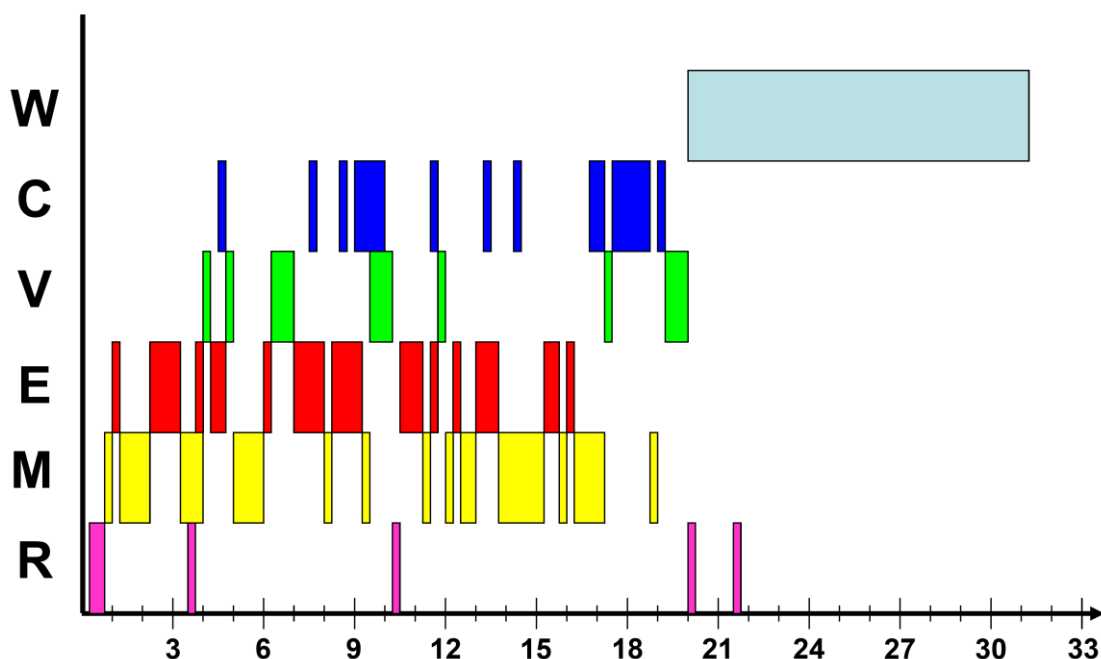


Figure 2: An example of a Modelling Activity Diagram of one of the groups solving the ESB-problem (Ärleböck, 2009c, s. 346).

## THERETICAL FRAMEWORK, METHODOLOGY AND METHOD

This paper uses the notions of *praxeologies* and *Research and Study Course* (RSC) from ATD. Within this framework *praxeologies* are used to describe any human activity in terms of two ‘blocks’: a *praxis* block (‘know-how’ or ‘practical-part’) containing both a designated type of *tasks* and the *techniques* used/needed to complete/perform these; and a *logos* block (‘know-why’ or ‘knowledge-part’) containing the *technologies* that explain, justify and describe the techniques as well as the formal justification of these technologies, the *theory*. As the name *praxeologies* suggests the praxis- and logos blocks are to be regarded as inseparable (Barbé, Bosch, Espinoza, & Gascón, 2005; Rodríguez, Bosch, & Gascón, 2008).

The notion of *Research and Study Course* (RSC) introduced by Chevallard (2004; 2006) is a general model that can be used for both designing and analyzing learning and study processes. A main emphasis of a RSC is on the *generating question*,  $Q_0$ , which should be intriguing and of genuine interest to the students as well as ‘rich enough’ to encourage the students to derive, pursue and answer dynamically raised and related (sub-)questions,  $Q_i$ , in the quest of trying to arrive at an answer to the question  $Q_0$ . In addressing these questions the students have to invoke, use and/or develop one or more *praxeologies*. The derived sequence of sub-questions  $Q_i$  and their respective answers  $R_i$  are often represented and illustrated in a ‘tree-diagram’ which illustrates the relationships between the different studied questions  $Q_i$ ; see Figure 3 for an example.

In terms of ATD the study reported on in Ärleback (2009c) and Ärleback and Bergsten (2010) can be conceptualised as an investigation of the *didactical praxeology* with the *task* to introduce mathematical modelling to students at the upper secondary level using the suggested *technique* presented by realistic Fermi problems. The issues addressed in these papers, as well as in this one, are concerning the (underdeveloped) *logos block* of this *praxeology*, especially the *technology* part. The ATD concepts of RSC and praxeologies provide theoretical constructs focusing both on what questions the students tackle when solving the ESB-problem as well as how and why. In the notion briefly introduced above the research question(s) studied in this paper can be reformulated as: *Given the ESB-problem as a generating question in a RSC, what sub-questions are addressed by the participating groups of students and what (mathematical) praxeologies are used and developed?* Due to space limitation the main emphasis in this paper will be on the first of these questions.

From an ATD perspective García et al. (2006) have presented a conceptualisation of mathematical modelling which basically equates all mathematical activity with mathematical modelling. In this paper however, the view of modelling is inherited from Ärleback (2009c) as briefly described and argued for in the previous paragraph.

To address the research question, widening and deepening the analysis of the groups of students solving realistic Fermi problems, the recorded video and transcribed data from one of the groups used in Ärleback (2009c) was revisited and re-analysed. The basic idea was to consider the students' work on the realistic Fermi problem in the context of a SRC as the generative question  $Q_0$  and to see what (sub-)questions  $Q_{i,j}, \dots$  this led the students to investigate, and in addition to link these questions the MAD representation of the problem solving process of the studied group. Note that in the ESB-problem the generating question,  $Q_0$ , actually is two questions:

$Q_{0,1}$ : *How long does the tourist elevator take to the top floor observatory?*

$Q_{0,2}$ : *If one instead decides to walk the stairs, how long does this take?*

The approach taken was in line with Hansen and Winsløw (2010) who make use of the RSC as an analytic model. Focus of the analysis was on the group activity as whole and thus firstly on explicit questions uttered by the members of the group, and secondly on how these questions were addressed in terms of the constituents of one or more praxeologies. Although there exist an a priori analysis in Ärleback (2009c) identifying some of the questions the problem solvers need to address in order to solve the problem, this paper only focus on the empirical questions actually addressed by one of the groups of students during their problem solving session.

## RESULTS

The questions  $Q_{i,j}, \dots$  the students derived from the generative questions and examined are presented below in the order in which they were raised and posed during the

problem solving session. The formulations below are in principle the students' own wording; some minor alterations have been made in order make the actual question intelligible and more concise. Basically the questions  $Q_{1...}$  are concerned with the ESB's physical appearance,  $Q_{2...}$  address  $Q_{0,1}$  (taking the elevator), and  $Q_{2...}$  address  $Q_{0,2}$  (taking the stairs):

$Q_1$ : How high is the Empire State building?

$Q_{1,1}$ : How many floors are there in the Empire State Building?

$Q_{1,1,1}$ : How high is a floor?

$Q_{1,2}$ : How high can a general building be?

$Q_{1,3}$ : How high was the World Trade Centre?

$Q_2$ : How fast is an elevator?

$Q_{2,1}$ : What is the weight of the elevator?

$Q_{2,1,1}$ : How much work is being done by the elevator?

$Q_{2,1,1,1}$ : Given the work done by the elevator, can we then calculate its velocity?

$Q_{2,2}$ : How long does it take to ride the elevator to Michael's apartment [a friend]?

$Q_{2,2,1}$ : On what floor is Michael's apartment?

$Q_3$ : How tired does one get from walking the stairs?

$Q_{3,1}$ : How longer does it take for one floor?

$Q_{3,2}$ : How much longer does it take for each floor?

$Q_{3,2,1}$ : How long does it take to walk up the first floor?

$Q_{3,2,1,1}$ : How fast is normal walking pace?

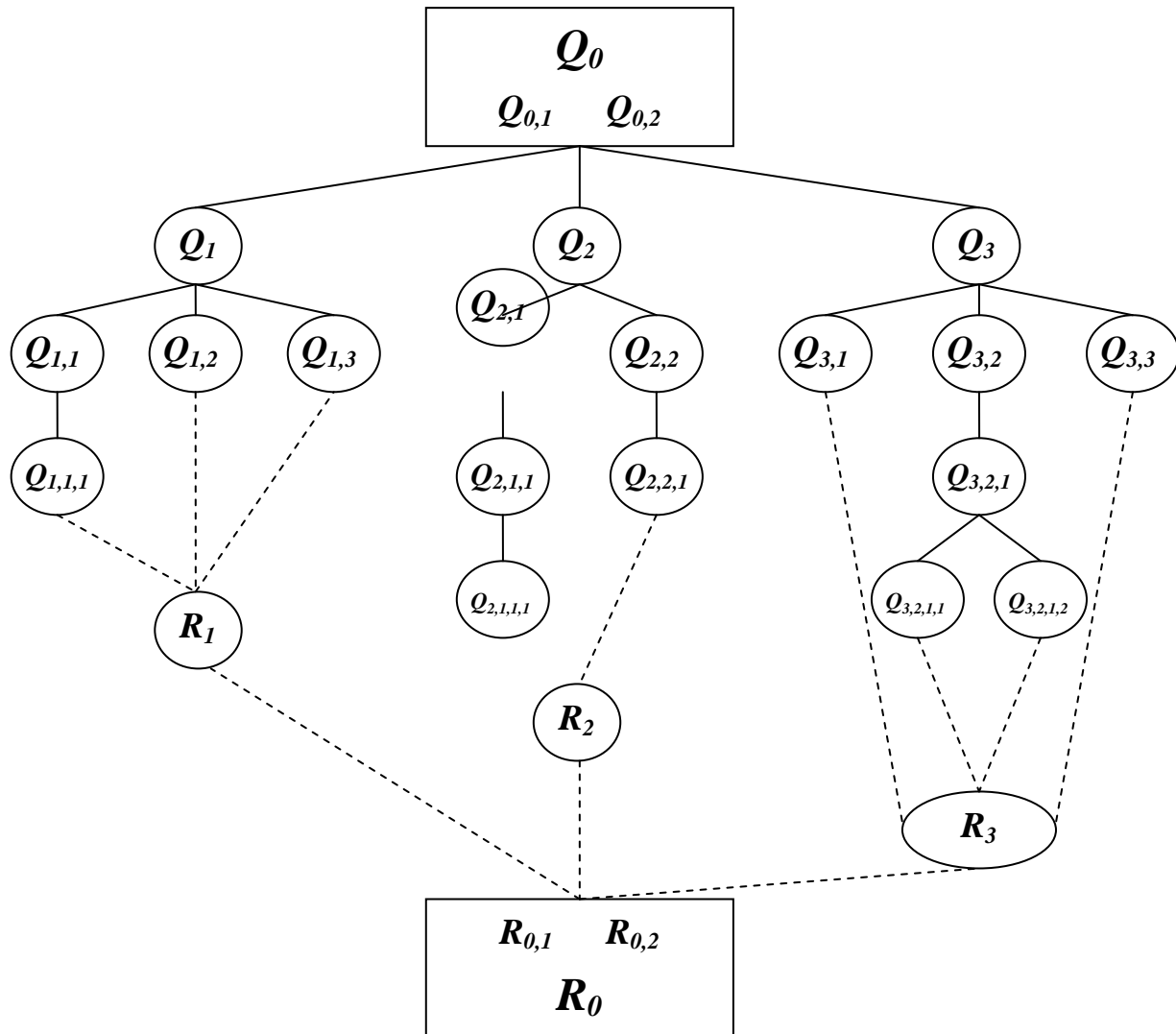
$Q_{3,2,1,2}$ : What is the inclination of the stairs?

$Q_{3,3}$ : My [one of the students] mother lives on the fifth floor – I wonder how long it takes walking up the stairs to her place?

The relationships between these (sub-)questions are illustrated in Figure 3. Note that the dotted lines in the tree-diagram display the dependence of the answers  $R_1$ ,  $R_2$ , and  $R_3$  respectively with respect to previously answers to questions in the tree. However, due to space limitations, and the fact that the focus of this paper is on the derived questions, these details are omitted here to be discussed elsewhere.

All branches except  $Q_{2,1,...}$  represent questions which answers contributed to the solving of the ESB-problem. The branch  $Q_{2,1,...}$  is about the classical mechanics concept of work, which the students briefly discuss as one possible strategy to get an estimate for the velocity of the elevators in the ESB.

After having spent about 15 minutes on the problem the group starts working on details concerning their suggested model on how to take the physical exertion into consideration in the  $Q_{0,2}$  question. They continue to do this in approximately 4 minutes, before the writing of the letter instructed in the problem formulation begins.



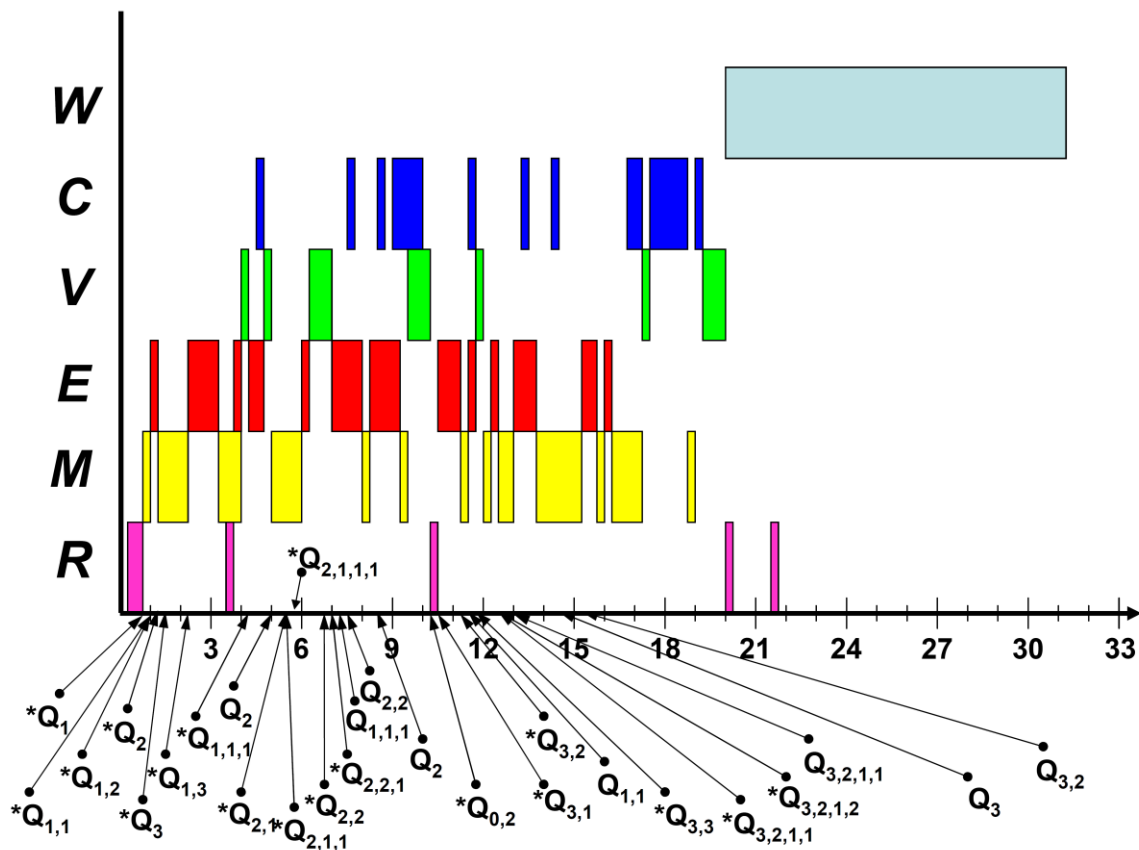
**Figure 3: A tree-diagram illustrating the relationship between the questions addressed by the students solving the ESB-problem**

Figure 4 illustrates the dynamic aspects of the addressed questions added to the MAD representation of the students' problem solving process. The first time a specific question is explicitly addressed it is preceded by an asterisk (\*).

## CONCLUSION AND DISCUSSION

One can notice that the actual modelling in terms of discussing, structuring and determining central variables and relationships important for solving the problem is something that is made implicitly and silently throughout the problem solving session. The praxeologies developed to address the questions (*tasks*)  $Q_{0,1}$  and  $Q_{0,2}$ , in fact all three groups in Ärlebäck (2009c) used the mathematical model  $t=s/v$  ( $t$  being the time,  $s$  the distance, and  $v$  the (average) velocity) as the basic *technique* to approach the questions. However, the decision to use this model was not explicitly uttered, or in any other way directly communicated, within the groups; it seems that all the participating students took it for granted that this was the model to use to

# 31



**Figure 4: The Modelling Activity Diagram (Ärlebäck, 2009c, s. 346) extended with the order and dynamics of the derived questions in the RSC.**

A majority of the praxeologies the students developed made use of estimation as the *technique* to resolve the *tasks* originating from all (but  $Q_{2,1,1,1}$  and  $Q_3$ ) of the derived questions the students engaged in. All these praxeologies have underdeveloped logos and the *technologies* and *theories* invoked to justify and verify the estimates are based on personal and often anecdotal experiences. This is most probable due to the feature of realistic Fermi problem to not provide the students with any explicit numbers to work with. It should be noted that one of the *technologies* applied and made use of to validate the estimate in some of these praxeologies are the same mathematical model as used as the *technique* in addressing  $Q_{0,1}$  and  $Q_{0,2}$ ;  $v=s/t$ .

The result suggests that there are some often used mathematical models, here exemplified by  $v=s/t$ , which are taken for granted used without second thought and reflection on underlying assumptions, limitations or alternatives. An explanation might be found in the different institutional conditions and constraints where these models are taught, learnt, practiced and applied. In particular, it would be interesting to study the didactical transposition of the notions and use of mathematical models

and modelling to see where these conditions and constraints arise. Though it has proven productive and useful to use realistic Fermi problems for the introduction of mathematical modelling at the upper secondary level, the challenge for the future is to design generative questions in the RSC so that also more advanced mathematical praxeologies are invoked and developed. The RSC ‘allows’ for the teacher to intervene, comment and make suggestions during the course of study, and this presents a possibility to achieve more, and perhaps specific, advanced mathematical content

## NOTES

1. For a critical discussion of this interpretation and its possible consequences see Ärleback (2009b).
2. For a discussion on the relations between realistic Fermi problems and other characterisations of problems (such as modeling eliciting activities, numerosity problems etc.) see Ärleback (2009c).

## REFERENCES

- Ärleback, J. B. (2009a). *Matematisk modellering i svenska gymnasieskolans kursplaner i matematik 1965-2000*. LiTH-MAT-R-2009-8. Linköpings universitet.
- Ärleback, J. B. (2009b). *Mathematical modelling in upper secondary mathematics education in Sweden. A curricula and design study*. Linköping: Department of Mathematics, Linköping University.
- Ärleback, J. B. (2009c). On the use of realistic Fermi problems for introducing mathematical modelling in school. *The Montana Mathematics Enthusiast*, 6(3), 331-364.
- Ärleback, J. B. (2010). Towards understanding teacher's beliefs and affects about mathematical modelling. *Proceedings of the CERME6*. Lyon, France.
- Ärleback, J. B., & Bergsten, C. (2010). On the use of realistic Fermi problems in introducing mathematical modelling in upper secondary mathematics. In R. A. Lesh, P. L. Galbraith, W. Blum & A. Hurford (Eds.), *Modeling Students' Mathematical Modeling Competencies. ICTMA 13* (pp. 597-609). NY: Springer.
- Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59(1), 235-268.
- Berry, J. (2002). Developing mathematical modelling skills: The role of CAS. *ZDM*, 34(5), 212-220.
- Blomhøj, M., & Højgaard Jensen, T. (2003). Developing mathematical modelling competence: conceptual clarification and educational planning. *Teaching Mathematics and its Applications*, 22(3), 123-139.
- Blomhøj, M., & Højgaard Jensen, T. (2007). What's all the fuss about competencies? In W. Blum, P. L. Galbraith, H. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study*. (pp. 45-56). NY: Springer.



- Blum, W., Galbraith, P. L., Henn, H., & Niss, M. (Eds.). (2007). *Modelling and applications in mathematics education. The 14th ICMI study*. NY: Springer.
- Blum, W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum & S. Khan (Eds.), *Mathematical modelling (ICTMA 12): Education, Engineering and Economics* (pp. 222-231). Chichester: Horwood.
- Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. *ZDM*, 38(2), 86-95.
- Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées De Didactique Comparée. Lyon*.
- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceeding of the CERME 4* (pp. 22-30). Barcelona: Universitat Ramon Llull Editions.
- Frejd, P., & Ärleback, J. B. (accepted). First results from a study investigating Swedish upper secondary students' mathematical modelling competency. *Proceedings of the ICTMA14 conference*. Hamburg.
- García, F., Gascón, J., Higuera, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM*, 38(3), 226-246.
- Hansen, B., & Winsløw, C. (2010). Research and study course diagrams as an analytic tool: the case of bidisciplinary projects combining mathematics and history. *To appear in proceedings of CATD-3*,
- Lesh, R. A., Galbraith, P. L., Haines, C. R., & Hurford, A. (Eds.). (2010). *Modeling Students' Mathematical Modeling Competencies*. NY: Springer.
- Maaß, K. (2006). What are modelling competencies? *ZDM*, 38(2), 113-142.
- Palm, T., Bergqvist, E., Eriksson, I., Hellström, T., & Häggström, C. (2004). *En tolkning av målen med den svenska gymnasie matematiken och tolkningens konsekvenser för uppgiftskonstruktion* (Pm nr 199). Umeå: Enheten för pedagogiska mätningar, Umeå universitet.
- Rodríguez, E., Bosch, M., & Gascón, J. (2008). A networking method to compare theories: metacognition in problem solving reformulated within the Anthropological Theory of the Didactic. *ZDM*, 40(2), 287-301.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.
- Skolverket. (2000). *Upper secondary school, Syllabuses, Mathematics*. Retrieved 09/15, 2008, from <http://www.skolverket.se/sb/d/190>
- Skolverket. (2003). *Lusten att lära - med fokus på matematik* (No. 221). Stockholm: Statens Skolverk.

# **HYPOTHESES AND ASSUMPTIONS BY MODELLING**

## **– A CASE STUDY –**

Roxana Grigoraş

University of Bremen, Germany

*By going through the mathematical modelling process, students often generate hypotheses and assumptions, in order to follow their strategies and to come to a solution. This generation of hypotheses is investigated by using the Scientific Discovery and Dual Search (SDDS) model. The design of the “Africa task” for a case study aims at stimulating students’ work with hypotheses in an a-priori non-mathematical setting. The students’ mathematical works are investigated with a modification of the SDDS model. This somewhat untypical task leads to assumptions and hypotheses of quite different nature than those one might be used to thinking of, that is, in more traditional tasks within mathematical modelling.*

### **INTRODUCTION**

During the mathematical modelling process, students go through distinct phases in the modelling cycle. However, some of these phases interfere with the others (Crouch and Haines, 2004) and some of them are even missing. In this study, we are interested in the process of mathematisation, and particularly in the means by which students express their thoughts and ideas in order to accomplish a mathematical modelling task. Mathematising, as part of mathematical modelling, is the process addressing the transition from a real model to mathematics. The modelling process students go through can be regarded as a path scattered with generation of hypotheses and assumptions, while following certain problem solving strategies.

The emerging hypotheses in students’ work are essentially of different nature than hypotheses in scientific research; like in physics, for example, where stated hypotheses are followed by experiments, and afterwards evaluated, therefore sustained or rejected. Here the students do not check many of these hypotheses, but simply state them and take them as granted. They are either led by intuition or the use of their background knowledge. It could also be that the means for proving does not stay at the students’ disposal, but the agreement in the group and intuition encourage them to go further with their stream of thoughts and ideas. Assumptions represent on the other hand another main component in modelling (Galbraith and Stillman, 2001; Ikahata, 2007), and according to the specificity of the task environment, in this study, their character is more or less mathematical. The model used as framework is applied here on a mathematical modelling task, and the procedure of testing hypotheses, as well as experimental space have atypical behaviour and form, closely dependent on the contextual situation, not following standardised hypotheses checking practices.

Since the current investigation concerns assignments where mathematics is somewhat hidden and no direct calculations are to be performed, the emphasis falls on elaborating strategies for pursuing with the task. Under these circumstances, collaborative working in groups appears fruitful, and students must proceed by loudly expressing their ideas. Assumptions are sometimes not easy to differentiate from hypotheses, thus working definitions for both are formulated in advance.

## **THEORETICAL FRAMEWORK**

On the one hand, the scientific acquisition of knowledge is seen as a complex form of problem-solving, where the problem-solving search takes place in two spaces: in the hypothesis-space and in the experiment-space (Simon, 1981). The SDDS model (Klahr, 2000) supports a structured description of hypotheses in the two spaces, as well as the evaluating evidence that comes in addition to these. On the other hand, the process of mathematical knowledge construction can be described with the help of the epistemic actions (Hershkowitz, Dreyfus & Schwarz, 2001).

### **SDDS**

The Scientific Discovery and Dual Search (SDDS) model is widely used in the area of scientific education; as a “general framework within which to interpret human behaviour in any scientific reasoning task” (Klahr, 2000). A special concern of the SDDS model is the evidence-based description of all aspects in the scientific acquisition of knowledge by the initial hypothesis statement, through the later acquisition of experimental evidences in order to check a hypothesis, and then ends up with a decision as to whether there is enough evidence for checking the hypothesis (Hammann, 2007). The scientific acquisition of knowledge begins with the searching in the hypothesis space, which also assumptions can be part of. Building hypotheses is described as a form of problem-solving, where initially domain-specific knowledge about a phenomenon or process following to be described, are provided. The final product is a tested hypothesis, explained with certain plausibility. Testing hypothesis has as output the description of evidence which sustains, respectively contradicts the hypothesis.

When predicting the finding of some research based on a theory or a logical common sense, a theoretical hypothesis is drawn. When a temporary conclusion is drawn, which will be validated with more data, it is handled with an empirical hypothesis. If an objective reasoning is important, in order to avoid bias, and if data is computed, a statistical hypothesis is formulated (Latief, 2009). More precisely, a theoretical and logical hypothesis can be defined as

[...] a tentative proposition suggested as a solution to a problem or as an explanation of some phenomena. It presents in simple form a statement of the researcher’s expectation relative to a relationship between variables within the problem (Ary, Jacobs and Razavieh 1979, p. 72).

Accommodated for our purposes, context and circumstances, the following working definitions are the basis for our identifications. We denote an *assumption* as being a proposition by means of which a certain thinking path is enabled. Reasons behind the generation of assumptions could be various; the need for pushing forward a stream of thoughts, reducing complexity of a situation, insufficient mathematical literacy or methods and algorithms near at hand, imposing conditions while smoothing the course of some solving. We denote a *hypothesis* as being any statement which is meant to answer a complex question, given a sophisticated situation and whose argumentation requires going through an unpredictable number of steps. Hypotheses differ from assumptions basically in their coherent and self-explanatory action containing it, meaningful by its own, with or without a succeeding proof.

### **RBC model**

The epistemic actions model of abstraction in context (AiC) aims at providing a framework for a micro-level description and analysis of processes of abstraction (Dreyfus, Hershkowitz & Schwarz, 2001). Empirically, abstraction can often be seen in the following “epistemic actions”: *recognising* (R), *building-with* (B) and *constructing* (C) (in the sense of vertical reorganisation of knowledge). These are mental actions by means of which knowledge is reconstructed. Conforming to the RBC, recognising takes place when students detect that a certain piece of knowledge is relevant for the context (s)he is dealing with. Building-with is the action where recognised constructs are combined in order to achieve a specific goal. Constructions are those activities indicating epistemic actions towards mathematical abstraction. The RBC model is used here to verify to which extent knowledge construction was realised by the students. The found constructions were finally embedded in the hypothesis space, that is, some of the hypotheses were identified as constructions.

### **EMPIRICAL SETTING**

A task on a predator-prey situation was designed with the purpose of encouraging students to make statements during argumentation and generate hypotheses and assumptions. Slightly different versions of this task have been developed over time, and the latest version of this Africa task (Grigoraş and Hoede, 2008) is:

*In Africa there are territories consisting of three areas. On one side of a river there is an area with grass and trees. On the other side there are two areas, one only with grass, and one only with trees. These two areas are separated by mountains.*

*There are seven species of animals: antelopes (A), crocodiles (C), Gnus (G), elephants (E), lions (L), monkeys (M) and panthers (P).*

*Grass and trees grow depending on the rain fall, grass serves as food for antelopes and Gnus, trees give food to monkeys and elephants. Lions feed on antelopes and Gnus, panthers feed on monkeys, and crocodiles feed on Gnus whenever these pass the river.*

*Gnus pass the river if there is not enough grass.*

*Elephants can also pass the river if they want to get to another area with trees, without being threatened by crocodiles.*

1. *In which areas do you think will the species be living? Make a drawing of the situation and indicate by letters where grass, trees, as well as each species is to be found.*
2. *Group the animals according to properties.*
3. *Suppose that there is no rain in the area where there grass and trees are. By this, the amount of grass and the number of trees are going down.*
  - a. *Investigate what is happening with all the species in the three areas.*
  - b. *What happens if the rain stops falling in the area where only grass grows?*
  - c. *What happens if the rain stops falling in the area where only trees grow?*
4. *In case the grass is disappearing in the area where there is only grass (case 3.b), Gnus will cross the river and start eating the grass on the other side.*
5. *Now focus upon grass, antelopes, Gnus and lions: describe your expectation of what will happen with the amount of grass and each of these three species.*

Students – aged 13-16 – of a secondary school worked on the given task, while being organised in groups, in a classroom environment. 7<sup>th</sup> graders were mainly chosen for the analysis, with some 9<sup>th</sup> graders also involved, in order to check whether significant differences in their approaches can be observed. Teams of three students were video-taped during working and the teacher had no intervention on the content, unless students had to clarify issues concerning task formulation.

### **Data analysis**

Given the complexity of the task, a fragmentation in smaller pieces of data contained in the formulation or inferred from students' reasoning was realised. The following categories were developed, which distinguish sources of knowledge as follows:

**Text information** belongs to the real world, and is extracted from the task's text. For reasons of space, just several pieces of the text information are given here, the others can be found in the assignment statement, presented in the previous section. For an easier manipulation in our analysis, the task was fragmented in nineteen pieces (T's):

*T<sub>1</sub>: In Africa there are territories consisting of three areas.*

*T<sub>2</sub>: On one side of a river there is an area with grass and trees, and so on.*

**Knowledge of biology/geography/etc.** (B's) belongs to real world and represents information showing relationships between concepts used in the task, being mainly background knowledge. It is also possible that some B's overlap or sometimes even contradict with some T's given in the formulation task. Some examples of B's are:

*B<sub>13</sub>: When the grass is gone, then the grass feeding animals are dying, and herewith other animals feeding from the grass feeding animals.*

*B<sub>15</sub>: When it stops raining in the area where only grass grows, then all animals die, except monkeys, elephants and panthers.*

*B<sub>16</sub>: If it stops raining in the area where trees and grass grow, then after a while they will all get dry.*

*B<sub>17</sub>: If it stops raining in the area where only grass grows, then antelopes will also cross the river to the area where grass grows, like the Gnus, and therefore lions will go to that area, too.*

We remark at this point that the difference between text information (T's) and biology information (B's) is basically that the text information may trace certain changes, by means of some actions, so that connections between different parts can make sense to be set up. The distinction between some T's and B's (which might be very similar) is done by the context in which B's were expressed.

**Logical clauses** (L's) belong to the mathematical world and act as kind of 'operators', by means of which logical inferences are achieved:

*L<sub>1</sub>:  $A \rightarrow B$*

*L<sub>2</sub>: Negation*

*L<sub>3</sub>: If " $A \rightarrow B$ " and " $B \rightarrow C$ ", then " $A \rightarrow C$ "*

*L<sub>4</sub>:  $x \rightarrow y \rightarrow x$  (if  $x$  influences  $y$ , then  $y$  will influence, in turn,  $x$ )*

A statement counting as construction in our analysis does not have to necessarily look or sound abstract in the sense of being intangible or imaginary difficult, but realised on a meta-level, so that further cognitive processes are enabled through it.

**Constructions:** (for reasons of space, just few of the ten C's are enumerated here)

*C<sub>5</sub>: When there are fewer lions, then there are more Gnus, but then there are again more lions.*

*C<sub>6</sub>: Less rain causes less grass, and therefore also fewer herbivores, namely antelopes and Gnus.*

*C<sub>7</sub>: When grass gets dry off, then Gnus and antelopes do not have food anymore and die off. Then lions and crocodiles die off, too,*

The list of suggested knowledge (B's and L's) gets longer with the number of subjects participating in the investigation. The present B's and C's resulted from gatherings of five teams (7<sup>th</sup> and 9<sup>th</sup> graders). An increased number of students, more advanced in background knowledge might easily produce considerably more items in number and quality, than the B's and C's delivered in the existing analysis.

There is no space to give all the hypotheses and assumptions found in students' results. Nevertheless, their dialogues abound in hypotheses, while assumptions are significantly less. 'Typical' hypotheses and assumptions produced by students are:

*Hypotheses:*

Student 1: The lion is by grass and trees, because the antelopes go on the other side, when not having grass there.

Student 2: I would say, by grass – only antelopes, and by grass and trees – Gnus are also coming, because the lions can eat sometimes...

- Student 3: I would rather write... here... eat grass, eat things from trees... eat Gnus and eat antelopes and make then cross... it is much practical.
- Student 4: The grass gets dry, and then Gnus and antelopes do not have food anymore and die off. When G and A are gone, then L and C die off.
- Student 5: When it does not rain, the trees are getting fewer, and thus fewer trees feeding animals.

*Assumptions:*

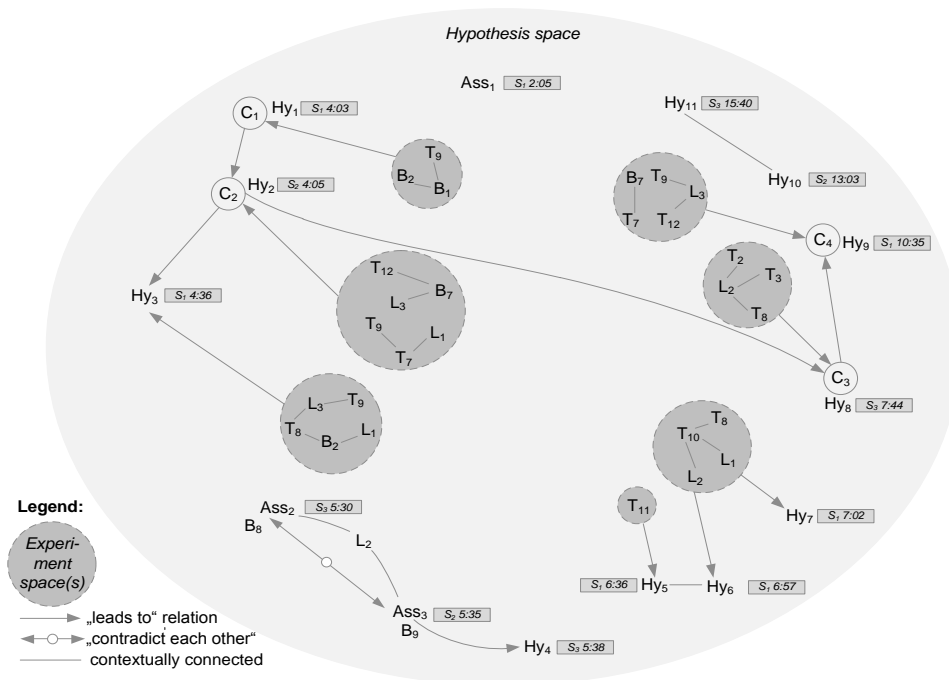
- Student 1: It must be so, that some... to be free from those which do not eat them... this is the feeding dynamic chain.
- Student 2: No, we have to describe that somehow... first by x... remains nothing more to eat, then by y no more food, then by z no more eating...
- Student 3: Yes, but we should restrict only to A, G and L. We should not do with the crocodiles, but only with antelopes, Gnus and lions.

**Findings**

In the following, we present a scheme to depict the hypotheses and assumptions generated by the students in form of figures, corresponding to every team under investigation. The entire dialogue is part of the hypothesis space, which contains, in turn, experiment spaces as subsets. Combinations of T's, B's and L's, seen as elements of the experiment spaces, lead to hypotheses, some of which are linked to each other, while assumptions act sometimes as junctions. When a certain direction can be defined, either in meaning or time, arrows are also coming along. A possible way to start reading the scheme is, for example, by taking time as backbone (the sense is here inverse clock-wise). The size of the experiment spaces spheres differs just in that they contain more or less dense clustered T's, B's, respectively L's.

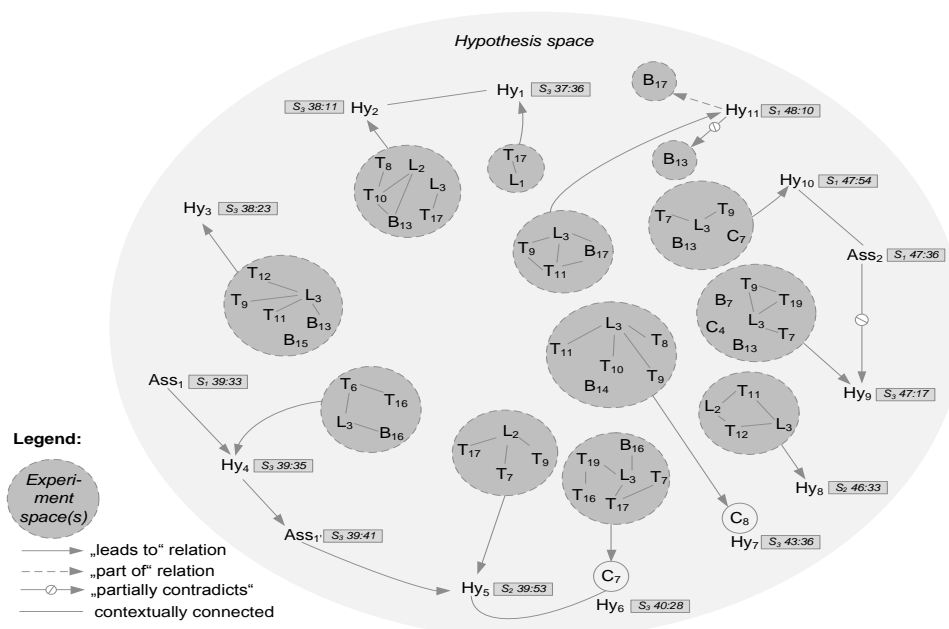
Figure 1 displays a rather typical view of an overall picture for the work performed by the young students for the Africa task. The four hypotheses ( $Hy_1, \dots, Hy_4$ ), which in this case happen to be constructions ( $C_1, \dots, C_4$ ), are connected in meaning, as it is illustrated in the diagram. Where the main subject of the statements expressed by students gets changed, no edge in the figure appears. If a certain assumption is related to a hypothesis stated previously or afterwards, then this can also be seen in the drawing. But it must not necessary be the case; there could be assumptions whose statements do not come in the logical reasoning of some previous hypothesis, and also do not directly yield the emergence of other hypothesis. A characteristic of this team is the weird way of communicating between students, sometimes the topic changing spontaneously, or some students ignoring the opinion of other colleagues. This is reflected in the 'solitary' hypotheses/assumptions, having no edges with others. It could also be that other peers just manifest the wish of immediately sharing a new idea, and thus a previous discussion gets abandoned. This can be seen in the

diagram, namely where even more isolated tiny systems lie. This particularity is independent of the quality of students' reasoning or the correctitude of their answers.



**Fig. 1: Hypotheses and assumptions in the 9<sup>th</sup> graders' team work**

As it can be seen in Figure 2, a considerable number of hypotheses are stated, having quite intensive discussions behind (many B's, L's, T's and dense connections between them in most of the spheres).



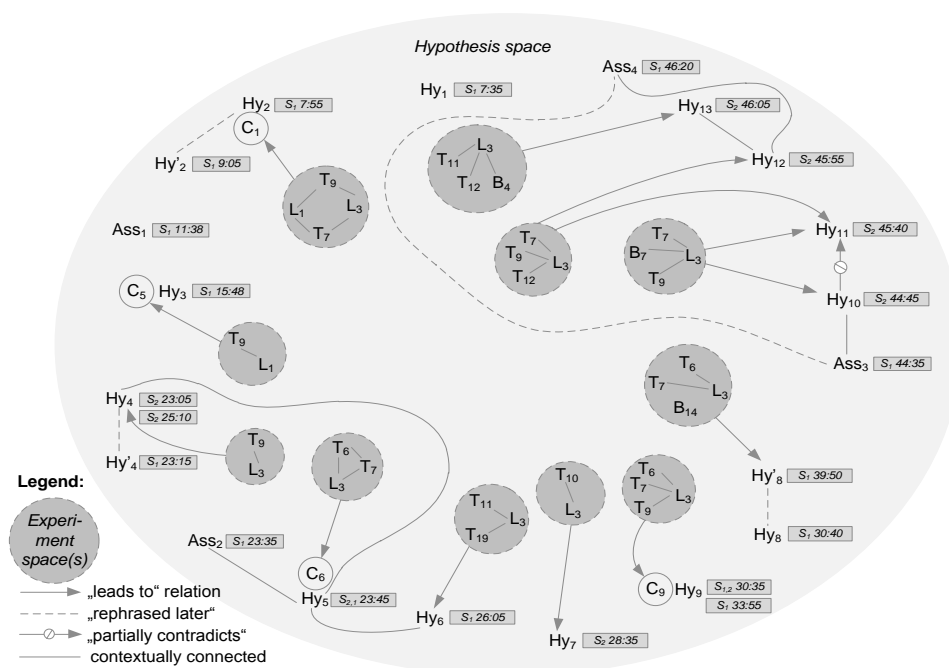
**Fig. 2: Hypotheses and assumptions in the 7<sup>th</sup> graders' first team**

The communication between students is well-established and efficiently maintained. Though sometimes certain misunderstandings are occurring in incipient phases of



some key-points of the conversation, they are thoroughly worked out and better ideas are emitted, approved and successfully implemented. The manner in which assumptions are produced, how they relate, sometimes contradicting to each other, as well as the way of leading to hypotheses are clear indications for students working carefully and likely to have previously experienced mathematical challenges. The constructions occurring here do not necessary seem to be decisive for the quality of the team, though some beliefs could speak about a direct proportionality in this regard. The overall work was performed in a rather mature manner and quite some complexity in elaborating strategies emerged, as far as the task allows.

The third team carries, as others as well, features of its quality peers communications. The work was often taken as a game, at least by the leading person. This influenced the results by preventing students from treating the task more thoroughly, and eventually to perform better. Four hypotheses and one assumption needed to be reformulated or rephrased, sometimes more than twice, and this indicates on the one hand uncertainty, but on the other hand consistency and the wish to make things better.



**Fig. 3: Hypotheses and assumptions in the 7<sup>th</sup> graders' second team**

The discussions between students are quite intensive; many hypotheses are stated, meaning that ideas are generated, but reasoning is not realised by means of an extensive number of assumptions. It is also the nature of this particular task, which does not necessary ask for imposing special conditions, in order that the reasoning process follows smoothly, and the answers to the posing questions to be found. The emitted hypotheses are tested through repetition, then expecting feedback, respectively confirmation from the other colleagues of the team. Quite particular in the working of this group of students is the way how they approached the last

question of the problem, and namely how they understood to discuss what happens “only with the three species of animals”. The specification given in the formulation, that just the three species (antelopes, Gnus and lions) are the object of investigation, created some confusion and efforts of handling the problem, because the requirement could not be achieved without involving other aspects, namely that crocodiles influence what happens with the Gnus. On the one hand, the impossibility of answering the question without considering other aspects was continuously sustained by one student. On the other hand, another colleague argued loudly for the denial of even mentioning other pieces of information than the required ones. These two opposing ideas met in a conflict, highly voiced, but nevertheless students maintained their collegial relationship. It was in fact controversy content versus form, and the noisiest ‘advocate’ finally won, namely the supporter of the second idea.

## DISCUSSION

Since hypotheses are sometimes hardly distinguished from assumptions, the SDDS model does help pointing this distinction while following the groups’ dynamics. When trying to map the empirical results to a SDDS diagram, some conclusions can be drawn, specific to the particular task it is worked with. The entire path covered while coming to a solution is a continuous hypotheses launching, which would fit to a searching hypotheses space. This seeking is done concomitantly with experimenting, i.e., testing hypotheses by continuously checking given information, as well as information obtained from these (T’s and B’s). The experimental space is formed by the logical block (mainly having at least one L), where the generation of hypotheses allows going deeper in the logical world, and staying in the same space. When nothing more in the direction of a hypothesis can be done by the students, then new interpretations take place and new logical constructs are attempted. These logical considerations are not always concluded as constructions (in the sense of the RBC theory), thus not all the hypotheses are regarded as constructions.

The assumptions are being conceived when changes in the model space are implicitly required, meaning that discussions get to such a point where no decision seems to be possible or agreed on. Sometimes assumptions opposing to each other are pronounced, then one of them is adopted, and afterwards a next hypothesis follows. Many hypotheses are tightly linked, like students’ thoughts rowed in a chain. The search hypotheses space cannot be separated from the space where testing hypotheses is acquired and hopefully achieved. The first space is containing the other space, since the components which help finding the hypotheses are the same by means of which hypotheses are tested, namely T’s, B’s and L’s. In fact, testing takes place through posing rhetorical questions, a proper combination of the background knowledge, which permits drawing satisfying (from a students’ point of view) conclusions. Evaluating evidence seems to be part of the testing hypotheses, and in

case agreement on certain hypotheses is performed, then no later contradictions, as result of evaluating evidence, are met.

Since operating with hypotheses is basically part of researchers' scientific work, one can ideally say that within every student lies a possible future scientist.<sup>7<sup>th</sup></sup> graders produced more elaborated results than the 9<sup>th</sup> graders, collaboration within the team being the main reason for it. They manipulate their own statements in no predefined and much more elaborate manner, but mainly on intuitive basis. Nevertheless, surprises in this regard are not excluded; novices are handling empirical and theoretical hypotheses (according to Latief's typology), and if properly challenged, it might be surprising how mature and constructive students' approach could become.

## REFERENCES

- Ary, D., Jacobs, L.C., & Razavieh, A. (1979). *Introduction to Research in Education*, Second edition. New York: Holt, Rinehart and Winston.
- Crouch, R., & Haines, C. (2004). Mathematical Modelling: transitions between the real world and the mathematical model. *International Journal of Mathematical Education in Science and Technology*. 35(2), 197-206.
- Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in context: the case of peer interaction. *Cognitive Science Quarterly*, 1(3), 307-368.
- Galbraith, P.L., and Stillman, G. (2001). Assumptions and context: Pursuing their role in modelling activity. In J.F. Matos, W. Blum, K. Houston and S.P. Carreira (Eds.), *Modelling and Mathematics Education: ICTMA 9: Applications in Science and Technology* (pp. 300-310). Chichester, UK: Horwood.
- Grigoraş, R. and Hoede, C., (2008). Reasoning Indicators – A Case Study. In: Eva Vásárhelyi (Ed.), *Beiträge zum Mathematikunterricht*, (pp. 417-420). Verlag Franzbecker, Hildesheim.
- Hammann, M. (2007). Das Scientific Discovery as Dual Search – Modell, In: D. Krüger, H. Vogt (Eds.), *Theorien in der biologiedidaktischen Forschung*. Springer, Berlin, 187-196.
- Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in Context: Epistemic Actions. *Journal for Research in Math. Education*, 32(2), 195-222.
- Ikahata, S. (2007). How Do Novice Students in Mathematical Modelling Estimate Assumptions? Presented at ICTMA 13, on July, 25<sup>th</sup>, accessed online on 09-09-2010 at <http://site.educ.indiana.edu/Portals/161/Public/Ikahata.pdf>
- Klahr, D. (2000). *Exploring Science: The Cognition and Development of Discovery Processes*. MIT, Cambridge Mass, London.
- Latief, M.A. (2009). *Assumption and Hypothesis in Language Learning Research*, accessed online on 05-19-2010 at <http://karya-ilmiah.um.ac.id/index.php/karya-dosen-fs/article/viewFile/4019/2128>.
- Simon, H. A. (1981). Wissenschaftliche Entdeckung und die Psychologie des Problemlösens. In H. Neber (Ed.), *Entdeckendes Lernen*, Beltz PVU, Weinheim, 104-125.

# **INTRODUCTION TO THE PAPERS OF WG 7: MATHEMATICAL POTENTIAL, CREATIVITY AND TALENT**

Leaders:     Roza Leikin (Israel) rozal@construct.haifa.ac.il  
              Demetra Pitta-Pantazi (Cyprus),  
              Florence Mihaela Singer (Romania),  
              Andreas Ulovec (Austria)

## **SCOPE AND FOCUS OF THE WORKING GROUP**

The purpose of the WG was to raise the attention of the mathematics education community to the field of mathematical potential, creativity and talent, and to promote empirical research that will contribute to the development of our understanding of the field.

The WG aimed at international exchange of ideas related to the didactics of teaching highly able students as well as the promotion of creativity in all students. Part of the discussion was focused on the ways that lead students to discover and realize their mathematical talents and develop their mathematical creativity. Special attention was devoted to activities which are challenging, fundamentally free of routine, inquiry-based, and rich in authentic mathematical problem solving.

The WG addressed the ways of identifying mathematically gifted and creative students as well as the relationship between exceptional mathematical abilities, motivation and mathematical creativity. The WG participants paid special attention to teacher education aimed at mathematics teaching that encourages creativity and fosters mathematical talent.

Finally WG members discussed interculturally- and culturally- related aspects of creativity and giftedness.

Twenty-five participants from 14 countries took part in the Working Group.

In what follows we describe the main topics for discussion that were suggested by group leaders and some outcomes of these discussions.

### **Topic 1: Theoretical foundations of mathematical talent and creativity**

Discussion on *theoretical foundations of mathematical talent and creativity* attended definitions and origins of creativity and of mathematical talent. Additionally, this session was devoted to questions of how to identify mathematically creative and talented students.

The study by Kontoyianni et al. served as a basis for the discussion of modelling mathematical giftedness through the design of a theoretical model. This model, which was based on previous theoretical works, was put into practice using large-scale empirical research. The paper by Leikin and Kloss offered a starting point for discussing theoretical models of creativity and their implications for empirical

research. The choice of a research paradigm for empirical investigation was found to be a basic element in the performance of empirical research. Brandl's model of mathematical giftedness was designed based on a variety of educational theories and paradigms. This work led to discussion of whether each theoretical model should be tested empirically and to what extent empirical investigation performed by a researcher should be analysed by means of the theoretical model he/she designed.

### **Discussion outcomes**

There was general agreement among group participants that the definitions of creativity usually depend on the context. Giftedness and creativity are expressed in extraordinary performance in problem solving. At the same time, problem posing can be a powerful tool for identifying mathematical creativity and giftedness. Moreover, problem posing can be considered as an essence of mathematics and, in contrast to problem solving, it facilitates better evaluation of elaborative acts.

The participants suggested that a characterisation of mathematically gifted students can be devised by asking expert mathematics teachers to distinguish between children who are good in math and those who are absolutely amazing. The teachers can be asked:

- What is the essence in this child that a teacher remembers for years?
- What is it that makes one certain student so special?

The group agreed that mathematical talent can rarely be identified using one (relatively short) specific test. Our assumption was that a combined test (e.g., combining achievements, personal traits and creativity) would have to be employed. We assumed that good questions and unique ideas raised by the students are probably indicators of their giftedness. However, the following question remains open:

- What are the predictors of mathematical giftedness and talent?

### **Topic 2: In-depth empirical studies on creativity and its development**

Three papers served as starting points for the discussion of empirical studies on creativity, its development and investigation of giftedness. This session ended up with more questions than answers.

The paper of Kattou et al. presented a complementary part of the study presented by Kontoyianni. This study presented empirical testing of a theoretical model in which mathematical creativity is considered as a predictor of mathematical ability. The results of testing 359 elementary school students in Cyprus using two instruments showed that mathematical ability may be predicted by mathematical creativity.

Levenson introduced the concept of collective creativity that was devised through empirical investigation of problem-solving directed at the development of creativity in elementary school students. We asked:

- How can this construct be used by other researchers?

- Does it characterise the work of professional mathematicians, or is it useful just on the level of elementary school mathematics?

We assumed that collective creativity does not necessarily lead to construction of shared mathematical meaning by a group of students since only few students can be participating at one time in this collaborative process.

Based on the paper by Levav-Waynberg et al. participants discussed the possibility of developing creativity in groups of students with varying abilities. The studies in Levav-Waynberg et al. demonstrated that development of mathematical creativity is significant not only in groups of high-achievers but also in groups of mid-achievers. Based on this study the following question was raised:

- What is a reasonable balance between qualitative and quantitative studies in the field of creativity and giftedness that can lead to a better understanding of these phenomena?

Other questions emerged from the discussions:

- What is the relationship between high ability and creativity?
- How can creativity be developed in each and every student?

### **Topic 3: Mathematical activities aimed at developing creativity and promoting mathematical giftedness**

Two sessions of the WG were devoted to the mathematical activities that are aimed at Development of Creativity and Promoting Mathematical Giftedness. We discussed papers by Aizikovitsh-Udi and Amit; Kyriakides; Maj; Novotná and Sarrazy; Singer, Pelczer and Voica; and Sophokleous and Pitta-Pantazi. We also learned about the NRICH project that was presented by Hewson and McClure.

A variety of creative ideas about developing mathematical creativity and the ways of working with gifted students were presented during two sessions of the WG. The discussion focused on ways of realizing students' mathematical potential including:

- teaching high-achieving students,
- promoting and fostering creativity in the mathematics classrooms,
- presenting mathematically challenging activities: non-routine, inquiry-based, and authentic mathematical, problem posing; and
- enhancing teacher education.

When discussing didactical principles of working with highly able students the participants made distinctions between the affective, social and cognitive domains. Among others, the following issues were found extremely important in discussion of the education of students with high mathematical abilities:

- When designing programs for highly able students, mathematics educators should be concerned with the social consequences of treating able children

differently. If these children are offered the same curriculum as everyone else, there is no point in placing them in separate groups. Ability grouping requires offering a qualitatively different curriculum.

- The interdisciplinary and multidisciplinary nature of the curriculum for mathematically gifted students is an important principle in the education of the gifted, since spending a majority of the time on mathematics could inhibit them from doing well in other subjects which they might otherwise excel at.
- The culture of the classroom/school for highly able students must have very well trained teachers and a culture in which ability is celebrated.
- Acceleration and in-depth programs have to be suited to individual differences among students, especially concerning the affective dimension.

Still, several questions were asked which require further discussion:

- Should the teacher of gifted students be gifted, or is it enough to be highly competent?
- What kind of didactical contract can best fit the needs of high ability students in mathematics?
- What kind of didactical contract can best promote creativity in all students?

**Topic 4: Culturally dependent and intercultural aspects related to the development of creativity in school mathematics and supporting mathematical giftedness:**

Following the paper by El Yacobi “Mathematical Creativity: Impediments and Challenges for Africa”, the group devoted one of its sessions to discussions concerning ways for developing creativity and fostering mathematical giftedness in schools in different countries. Participants developed a system of criteria for analysis and comparison of national school policy, the role of textbooks, management of special frameworks, and teacher education. We hope that this system of criteria will serve as a starting point for the international study in which the group participants take part.

# INTEGRATING THEORIES IN THE PROMOTION OF CRITICAL THINKING IN MATHEMATICS CLASSROOMS

Einav Aizikovitsh-Udi, Miriam Amit

Ben Gurion University, Beer-Sheva, Israel

*The present paper addresses the question of whether critical thinking can be most successfully taught in specialized courses (known as general skills approach) or by way of integration in the curriculum of regular courses (known as the infusion approach). Arguing in favour of the infusion approach, the paper discusses the possibility of integrating this approach with taxonomies of critical thinking. The qualitative data analyzed in the paper were obtained in a two-year experimental teaching of the learning unit "Probability in Daily Life." The unit was specially modified for teaching critical thinking in the infusion approach, while using as well as actively teaching Ennis' taxonomies.*

**Key words:** teaching critical thinking, critical thinking taxonomy, infusion approach, teaching probability

## INTRODUCTION

The field of education has recognized for decades the need to concentrate on the promotion of critical thinking (CT) skills. The question is how this can be best accomplished. Some educators feel that the best path is to design specific courses aimed at teaching CT, which is called the *general skills approach*. By contrast, integrating the teaching of these skills in regular courses in the curriculum is known as the *infusion approach*. The question at the heart of the argument is, whether CT skills are general or depend on content and on the system of concepts specific to that particular content. According to Swartz (1992), the infusion approach aims at teaching specific CT skills along with different study subjects, and instilling CT skills through teaching the set of instructional material. Swartz also emphasizes that the students should not only employ CT skills in class, but also be able to activate them in real-life situations and to recognize situations when these skills should be used. For this, an appropriate motivation should be fostered; otherwise these skills will remain passive.

The study on which this paper is based was conducted according to the infusion approach. We have combined the mathematical content of an existing learning unit "Probability in Daily Life" (Lieberman & Tversky, 2001) with CT skills according to Ennis' taxonomy, and restructured the curriculum. Further we tested different learning units and evaluated the participants' CT skills, to examine whether the modified learning unit "Probability in Daily Life," taught in the infusion approach, does indeed develop CT (Aizikovitsh & Amit, 2009).



## **The Defining Components of the Theoretical Framework**

This section presents the three fundamental components that this research is based on: Ennis' taxonomy, the modified "Probability in Daily Life" learning unit, and the infusion approach.

### *The Infusion Approach*

In light of the evidence that has accumulated in the field of teaching thinking, the question arises whether thinking skills are general or content-dependent (Perkins & Salomon, 1989). Out of this question there developed four major approaches: the general approach, the infusion approach, the immersion approach, and the mixed approach. The general approach teaches thinking skills as a range of general skills detached from other study subjects, as a separate course in the curriculum. In the infusion approach the skills are taught in the framework of a specific study subject, and thinking turns into an integral part of teaching specific materials, while general principles and terminology of thinking are explicitly emphasized. In the immersion approach, the study material is taught in a thought-provoking way and the students are "immersed" in the topic of study, without explicit reference to the principles of thinking. The mixed approach combines the general and the infusion approaches

### *Ennis' Taxonomy*

Ennis (1987, 1991) claims that CT is a reflective and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT: practical, reflective, moderate, belief and action. In accordance with the categories this definition employs, Ennis developed a taxonomy of CT skills that include both an intellectual and a behavioural aspect. In addition to skills, Ennis' taxonomy also includes dispositions and abilities. In this study, we focus on students' abilities rather than their dispositions. We have chosen to use Ennis' definition and taxonomy of CT because it distinguishes between abilities and dispositions, and because teaching thinking skills according to a taxonomy suits the hierarchical structure of our learning unit in probability studies.

### *The Learning Unit "Probability in Daily Life"*

This unit in probability studies is part of the formal high school curriculum of the Israeli Ministry of Education. It was chosen because its rationale is to make the students to "study issues relevant to everyday life, which include elements of critical thinking" (Lieberman & Tversky 2001, Introduction p.3). In this unit, students must analyze problems using statistical instruments, as well as raising questions and thinking critically about the data, its collection, and its results. Students learn to examine data qualitatively as well as quantitatively. They must also use their intuitions to estimate probabilities and examine the logical premises of these intuitions, along with misjudgements of their application. The unit is unique because it explores probability in relation to everyday problems. This involves CT elements

such as tangible examples from everyday life, evaluating reliability of information, accepting or dismissing generalizations, rechecking data, doubting, and comparing new knowledge with the existing knowledge. This unit is characterized by questions such as “Define the term ‘critical thinking’,” “Give examples of a problem while using a controlled experiment,” “Give examples of failures and misleading commercials,” and “Give examples of a scientific truth that was dismissed.” While studying the subject, the connection is checked between statistical judgment and intuitive judgment, and intuitive mechanisms that produce wrong judgments are explored (Aizikovitsh & Amit 2009). While studying the unit, students are expected to acquire the tools for CT. In the beginning, students learn the mathematical tools necessary for performing calculations, and later on they use the probability part: causal connection, and mechanisms of intuitive judgment, which are considered more of a psychological projection.

### **METHOD**

The instructional model consisted of a learning unit (30 hours) that focused primarily on statistics in everyday life situations. This unit was first implemented in an interactive and supportive environment for a class of mathematically gifted youth participating in the *Kidumatica* project at the Ben Gurion University of the Negev.

Seventy-one students between the ages of fifteen and sixteen participated in the first round of the experiment, an extra-curricular program aimed at enhancing the CT skills of students from different cultural backgrounds and socio-economical levels. Probability lessons were combined with CT skills, taught by several teachers, with most teaching done by the presenting author. Among the topics taught were bi-dimensional charts, Bayes’ formula and conditional probability. CT skills, such as raising questions, searching for alternatives and doubting, were evaluated quantitatively, using Cornell tests (Ennis, 1996, 2005) and the CCTDI test (Facione, 1992), as well as qualitatively. The present paper discusses the qualitative findings.

As sources of qualitative data, we used students’ products (papers, homework, exams etc.), pre and post questionnaires, personal interviews, and class transcriptions. Five randomly chosen students were interviewed individually at the end of each lesson and again one week after the final class of the course. The personal interviews were conducted in order to identify any changes in the students’ attitudes throughout the academic year. All lessons were video-recorded and all public dialogue was transcribed. The teacher also kept a journal (log) on every lesson. Data was processed by means of qualitative methods intended to follow the students’ patterns of thinking and interpretation with regard to the material taught in different contexts.

### **QUALITATIVE FINDINGS**

Twenty-seven interviews related to critical thinking were conducted with the students towards the end of our course, in order to closely examine their personal

attitude towards mathematics, critical thinking and the development of thinking, and to reveal the students' thinking patterns in their interaction with "Probability in Daily Life" and mathematics. The interviews allowed to create a direct, open and flexible dialog with the students, which provided an additional source of information for evaluating their critical thinking abilities. An additional body of findings is derived from the group discussions aroused by the learning unit, which shows the centrality of critical thinking in everyday life. With this set of findings, as with the others, the purpose of the analysis was to examine the students' patterns of critical thinking in the mathematical, social and cultural contexts. In the course of teaching the unit, we have interviewed a number of students and asked them a number of questions concerning critical thinking. During the interviews we have identified a number of recurrent elements presented below. The interviews were of two kinds: closed/structured interviews, where questions were composed in advance, and open/semi-structured interviews, where questions were also composed in advance but selected and/or modified according to the interviewee's answers. In all of the interviews, three main elements recurred throughout the students' answers: the usefulness of critical thinking as an instrument for life and learning; the importance of critical thinking as a more empowered attitude towards authoritative sources of information and opinion; and finally, the role of critical thinking in promoting the students' general understanding of the world.

### **The Findings of the Structured Interviews**

To the question, "*What is critical thinking?*" or the prompt "*Critical thinking is...*," the students gave the following answers, which define critical thinking in three main dimensions: as a tool they can use in life and studies, as an attitude towards authority and sources of information, and as a way to improve their general understanding of the world. The answers that define critical thinking as a useful tool would say, for example:

It's something for which you need to use your brain properly. Something about critique. For instance: an ad in a newspaper that is not true. [B536]

To know how to check findings, opinions, reliability; to research, to doubt. [R505]

Not to trust everything [you hear], to check before you decide. Not to believe any odd survey [right away]. To think about every thing. [A847]

The latter definition brings forward a strong aspect of critical attitude towards authoritative sources of information, as does the following one: "In my opinion, the importance of critical thinking is that you don't take everything they tell you for granted, but check whether it's true and whether it's possible that the person who is explaining is wrong, and you accept mistakes."

Yet another extensive definition focuses on the critical attitude towards information sources, but also makes a strong emphasis on the role of critical thinking in learning to understand the world better:

Often I used to see only the external aspect of things and wouldn't really see what they are about. All of a sudden things become explicit, something lights on me, and it has to do with understanding. When I understand something, it also helps me to understand myself better. I have a greater power. When we studied investigation, I felt that my voice was becoming strong. [I could ask,] Who is doing the research, how many people, what are the purposes? I got power out of understanding, to understand more things better. [E886]

The aspect of empowerment acquired by mastering critical thinking should be noted here as well.

Also the following definition, "Every time I study, I discover new things, things are becoming clearer to me" [A427], focuses on the aspect of improved general understanding that critical thinking provides. Another student says, "If we didn't have critical thinking we wouldn't be able to understand well" [Y318]. Finally, one student defined critical thinking as "A way of life" [E886],

While the students' answers elude going into a detailed analysis, they do capture the all-encompassing influence of acquisition of critical thinking on the students' lives and perception of the world.

To sum up, *the main elements in the students' definitions of **critical thinking*** are as follows:

- (i) Openness to a variety of opinions and ideas;
- (ii) Serious consideration of other points of view;
- (iii) Suspension of judgment when evidence and arguments are insufficient;
- (iv) Consolidating or changing an opinion when evidence supports doing so;
- (v) Looking for precision in information, searching for reasons and arguments, examining all the possibilities.

Answers to the question "*Who is a critical thinker?*" are closely related to the definitions of critical thinking itself, but also add an important dimension of personal wisdom and intelligence as traits closely associated with critical thinking, as, e.g., in the following answers:

A critical thinker knows how to examine things, put things into question, to go deep and think about what s/he sees. [R505]

A critical thinker for me is an intelligent person, with a lot of world knowledge and life wisdom, which they can draw on when they are thinking critically about what they read or what they get. They also need mathematical thinking. [S210]

To sum up, *the main elements of the students' definition of the **critical thinker*** are as follows:

- (i) Someone who tends to shape, correct and change their beliefs in light of convincing arguments.
- (ii) Someone who is capable of understanding at least two opposing, well-defined points of view on the same subject while maintaining one's own standpoint regarding the subject.

In answering the question “*In what ways can critical thinking be developed?*” the students emphasized learning from other people, reading, and the importance of patience and perseverance. One student said it can be “learned from reading books, criticisms, articles, listening to other people's opinions. In researches they discuss methods that one can examine” [R505]. Another student emphasized the challenge that learning and practicing critical thinking poses, and the importance of insistently pursuing it:

What's interesting about critical thinking is that at first everything is very difficult and complicated, and then, when you peel off leaf after leaf, you discover some little treasure; at first it seems very complex, so you need to remember that all the time you need to keep exploring, because as long as we go on it becomes more and more beautiful. [E886]

In their answers to the question about the ways of developing critical thinking, the students named three main abilities that need to be developed:

- (i) The ability to distinguish between opinion and fact: the difficulty of distinguishing between utterances expressing the position of the speaker/writer on a certain reality, and the expressions of facts/events comprising this reality.
- (ii) The ability to identify information intended to influence the reader emotionally, such as using emotional manipulation as a means for presenting an argument and persuading the reader.
- (iii) The ability to recognize stereotypes and avoid using them: it is difficult to identify overgeneralization that leads to stereotyping and is likely to create bias and acceptance of a stereotype as a scientific fact.

## DISCUSSION

Acquisition and construction of higher-order thinking skills by students in general and mathematics students in particular has become one of the main targets of the education system widely accepted by educators around the world. The acquisition of these skills will enable the student to function as an active and productive citizen, and the challenge at present is to find ways of teaching and developing this approach not only in the excellent students but in the total population of students in schools.

Higher-order thinking involves applying many different criteria that frequently contradict each other, as well as self-regulation of thinking processes (independence of others at every stage of thinking). With the qualitative methodology we have chosen for this research, it was possible to examine these different aspects from several perspectives that enabled observation and interpretation of the educational reality in each group and conducting direct dialogue with the research participants (Tsabar Ben Yehoshua, 2000). Thus it became possible to point out several tendencies that became apparent during the research.

Analyzing the findings, we have arrived at the following insights regarding the process of critical thinking skills construction and teaching:

- (i) It seems that critical thinking skills do not develop spontaneously and that even good students acquire them by means of explicit instruction. This finding is in direct opposition to Zohar's claim (2000) that learning skills and learning strategies develop in the student spontaneously, without direct instruction. The findings suggest that being familiar with the taxonomy of critical thinking skills has a potential to improve the students' success at acquiring and perfecting them.
- (ii) To a large extent, the construction and teaching of critical thinking skills are determined by specific contents and tasks the teacher uses. This finding corresponds with other researches, such as Bransford (2000). In this research, Ennis' taxonomy of skills enabled the researchers to choose skills most suitable to be taught at each stage of the course, with respect to the contents and the increasing difficulty level of the learning unit.
- (iii) It is possible to significantly influence and change the mathematical discourse in the classroom and the students' language of critical thinking, by providing appropriate conditions and terminology as well as using appropriate instruction methods. In the literature, this finding applies not only to older and/or more successful students, but also to younger and/or underachieving ones (Weinberger, 1992).
- (iv) Excellent students (the *Kidumatica* group) were capable of operating a greater number of skills automatically, quickly, utilizing a minimal degree of conscious effort. However, this automatic application of thinking has only been acquired after much practice and exposure to different learning contexts. What is more, even expert learners are likely to return to a much slower and more conscious way of learning when confronted with unfamiliar tasks or connections.

However, it has to be recognized that providing a systematic terminology for describing thinking skills is far from exhausting the problematics of teaching critical thinking. For instance, Resnick (1987) expresses a different view, namely, that it is difficult to define higher-order thinking skills, but easy to recognize them when used by someone. She believes that higher-order thinking is not algorithmic, and that thinking patterns are not clearly defined in advance. This type of thinking often

concludes with multiple solutions, each of which has its advantages and disadvantages, but does not yield a single clear solution. High-level thinking has to do with skills in solving problems, asking questions, thinking critically, making decisions and assuming responsibility (Ben-Chaim, Ron & Zoller, 2000; Zoller, 2001).

Decision is an essential part solving a problem that involves a gap between an initial situation and a final goal and there is no easy, well-known way of finding a solution. Nonetheless, based on the findings of this research, it seems that conscious learning and explicit discussion of thinking skills in the context of mathematics enables the students to more efficiently develop these skills by way of solving probability problems. This type of learning emphasizes the development of specific skills in the process of solving mathematical problems. The small-scale research described here constitutes a small step in the direction of developing additional learning units designed to develop critical thinking skills within the traditional curriculum.

### CONCLUDING REMARKS

This paper has focused on the qualitative findings of our research, which are supported by quantitative findings presented elsewhere (Aizikovitsh & Amit 2008, 2010). The analysis of CT skills and dispositions tests and math tests the students took revealed the following improvements

- (i) In all three iterations of this research design, a moderate improvement has been detected in the critical thinking dispositions of all experimental groups. This improvement may be attributed to maturation and accumulating life experience as well as learning proper. All of these are significant factors affecting the development of the students' critical thinking, particularly within the framework of probability.
- (ii) Throughout these iterations, a moderate improvement was also detected in the students' critical thinking abilities. As in the case of dispositions, this improvement can also be ascribed to maturation, accumulating life experience, knowledge in other mathematical fields (e.g. geometry contributes to the development of deductive skills), and learning proper.

Thus, the findings of this research suggest that when teachers consistently and explicitly emphasize specific critical thinking skills, the students are more likely to succeed in developing them.

### REFERENCES

- Aizikovitsh-Udi, E., & Amit, M. (2010). Is it possible to improve the students' critical thinking dispositions through teaching a course in probability? Brief research report. In P. Brosnan, P., Erchick, D. B., and Flevaris, L. (eds.). *Proceedings of the 32nd Annual Meeting of the North American Chapter of*

*the International Group for the Psychology of Mathematics Education*. Columbus, OH: The Ohio State University, 6, 31-36.

- Aizikovitsh, E., & Amit, M. (2008). Developing critical thinking in probability lesson. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepúlveda (Eds.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (Vol.2, pp. 9-13). Morelia, México: Universidad Michoacana de San Nicolás de Hidalgo, Centro de Investigación y de Estudios Avanzados del IPN.
- Aizikovitsh, E., & Amit, M. (2009). Promoting critical thinking abilities via probability instruction. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp.17-24). Thessaloniki, Greece: Aristotle University.
- Ben-Chaim, D., Ron, S., & Zoller, U. (2000). The disposition of eleventh-grade science students toward critical thinking. *Journal of Science Education and Technology*, 9, 149–159.
- Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (2000). *How people learn: Brain, mind, experience, and school*. Washington, DC: National Academy Press.
- Ennis, R. H. (1987). A taxonomy of critical thinking dispositions and abilities. In J. B. Baron & R. J. Sternberg (Eds.), *Teaching thinking skills: Theory and practice* (pp. 9-26). New York: Freeman.
- Ennis, R. H. (1991). Critical thinking: A streamlined conception. *Teaching Philosophy*, 14(1), 5-25.
- Ennis, R. H. (1996). Critical thinking dispositions: Their nature and assessability. *Informal Logic*, 18, 165-182.
- Ennis, R. H., & Millman, J. (2005). *Cornell critical thinking test, level X* (5<sup>th</sup> ed.). Seaside, CA: The Critical Thinking Company.
- Facione, P. A., & Facione, N. C. (1992). *The California critical thinking disposition inventory*. Millbrae, CA: California Academic Press.
- Lieberman, V., & Tversky, A. (1996). *Critical thinking: Statistical considerations and intuitive judgments*. Tel Aviv: Open University. (In Hebrew)
- Perkins, D. N., & Salomon, G. (1989). Are cognitive skills context-bound? *Educational Researcher*, 18(1), 16-25.
- Perkins, D. N., & Salomon, G. (1988). Teaching for transfer. *Educational Leadership*, 46(1), 22-32.
- Resnick, L. B. (1987). *Education and learning to think*. Washington DC: National Academy Press.



- Sabar Ben-Yehoshua, N. (2000). *Qualitative research*. Givatayim: Massada. (In Hebrew)
- Swartz, R. J. (1992). Critical thinking, the curriculum, and the problem of transfer. In D. N. Perkins, J. Lochhead, & J. Bishop (Eds.), *Thinking: The second international conference* (pp. 261-284). Hillsdale, NJ: Erlbaum.
- Swartz, R. J., & Parks, S. (1994). *Infusing the teaching of critical and creative thinking into content instruction*. Pacific Grove, CA: Critical Thinking & Software.
- Weinberger, Y. (1992). *Fostering critical thinking in biology instruction*. Unpublished master's thesis, Tel-Aviv University, Tel-Aviv, Israel. (In Hebrew)
- Zohar, A. (2000). Meta-cognitive knowledge of teachers teaching higher-order thinking skills. *Dapim*, 41-52. (In Hebrew)
- Zoller, U. (2001). Alternative assessment as (critical) means of facilitating HOCS-promoting teaching and learning in chemistry education. *Chemical Education: Research and Practice in Europe*, 2(1), 9–17. Retrieved from [http://www.uoi.gr/cerp/2001\\_February/pdf/04Zoller.pdf](http://www.uoi.gr/cerp/2001_February/pdf/04Zoller.pdf)
- Zoller, U., Ben-Chaim, D., Ron, S., Pentimalli, R., & Borsese, A. (2000). The disposition toward critical thinking of high school and university science students: An interintra Israeli-Italian study. *International Journal of Science Education*, 22, 571–582. doi: 10.1080/095006900289679

# HIGH ATTAINING VERSUS (HIGHLY) GIFTED PUPILS IN MATHEMATICS: A THEORETICAL CONCEPT AND AN EMPIRICAL SURVEY

Matthias Brandl

Chair for the Didactics of Mathematics Faculty of Philosophy University of Erlangen-Nuremberg

*There is a big difference between high attaining and (highly) gifted pupils in mathematics: first, being gifted in mathematics does not necessarily lead to high assessment in this subject; second, being high attaining in mathematics does not necessarily mean being mathematically gifted – depending on the context in which the notion of mathematics and the concept of mathematical giftedness is socially constructed. Based on a conceptual approach from systems theory, a model for the causality of giftedness and assessment is presented and supported by an empirical survey among teachers at a higher secondary German boarding school for high attaining pupils.*

**Keywords:** mathematical giftedness, high attaining, systems theory, teacher interview

## INTRODUCTION

For the purpose of fostering gifted pupils, the identification of those comes first. There has been done plenty of research emphasizing the quantitative aspect of different kinds of tests that historically range from a solely IQ-test over bringing in school marks up to modern multidimensional settings (Heller et al., 2000; Pfeiffer, 2008). However, in day-to-day school life there is one dominant (non-) identifier and fosterer of giftedness, for example, in mathematics: the teacher – and because of the viability of the construct, the teacher's idea of mathematical giftedness (MG) may differ from that of parents or professional mathematicians.

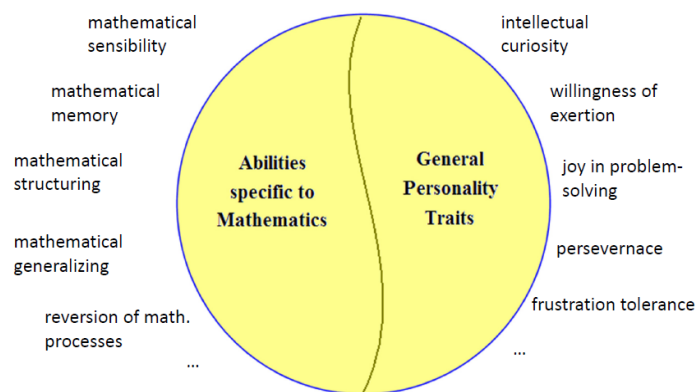
In order to make up a custom-made material-based program that suits the predominant concept of mathematical giftedness at a specific German school for high attaining pupils, the manifestation of the construct MG at this specific school and its difference to high assessment in mathematics were investigated by qualitative teacher interviews. Furthermore, a suitable theoretical model of mathematical giftedness is provided. Within that, MG is constructed by an ambivalent environment. As this environment on the one hand consists of (the viable notion of) mathematics and on the other hand, amongst others, of the educational system with its specific schools and mathematics teachers, the degree of being gifted in mathematics depends strongly on the maths teachers notion of mathematics.

Both exceeding and specifying the research question of identifying mathematically gifted pupils hitherto, is another question addressed here, too: How can the really

mathematically gifted ones be identified in a group of just and indeed mathematically high attaining pupils? Are there any characteristics that – again, in the teacher’s eye – distinguishes the gifted from the (just) high attaining ones? Those aspects that can be found out by qualitative single case interviews (as it was done here), may deliver aspects for the development of more quantitative orientated tests for samples already containing only high attaining test persons.

## MATHEMATICAL GIFTEDNESS

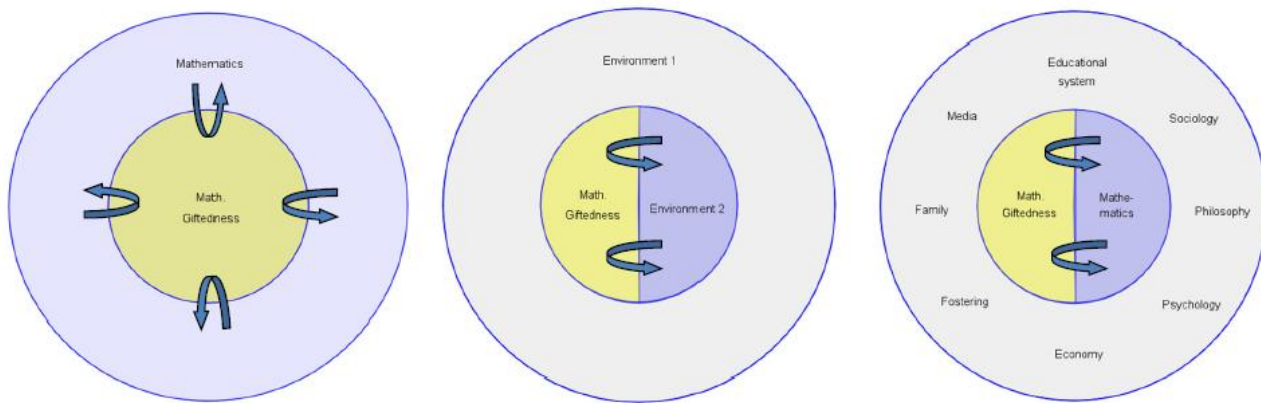
There are several concepts on giftedness. Mathematical giftedness (MG) is even more controversial as on the one side there are authors objecting to the existence of a specific MG (a. o. Fölsch, 1977; Treumann, 1974) and on the other side other authors testify this specific disposition based on empirical evidence (a. o. Greenes, 1981; Kruteskii, 1976; Kießwetter, 1992; Kämpnick, 1998). We stick to the latter ones and see MG as a synergetic potential that consists of two different aspects: a) abilities specific to mathematics and b) general personality traits. An incomplete list of items is given in figure 1.



**Figure 1: Mathematical Giftedness: abilities and traits.**

### An approach via Systems Theory

Because of the apparently vagueness of the concept MG, a conceptual approach by terms of systems theory was given in Brandl (2010) that is based on the “Law of cultural differentiation” (Irvine & Berry, 1988) and the so-called anthropological approach (Sternberg, 1996). Therein the system MG is seen as an open mental disposition construct surrounded by the environment mathematics which features necessary structural couplings between the system and its environment that describe the influence of predominant philosophical notions about mathematics on a conception of mathematical giftedness. Mathematics represents “environment 2” that “is the world that creates sense for the system.” (Krieger, 1996, p. 81). The further environmental factors that affect the system MG in a competitive way, too, but which do not necessarily endow the system with meaning or sense, are collected in „environment 1“ (figure 2).



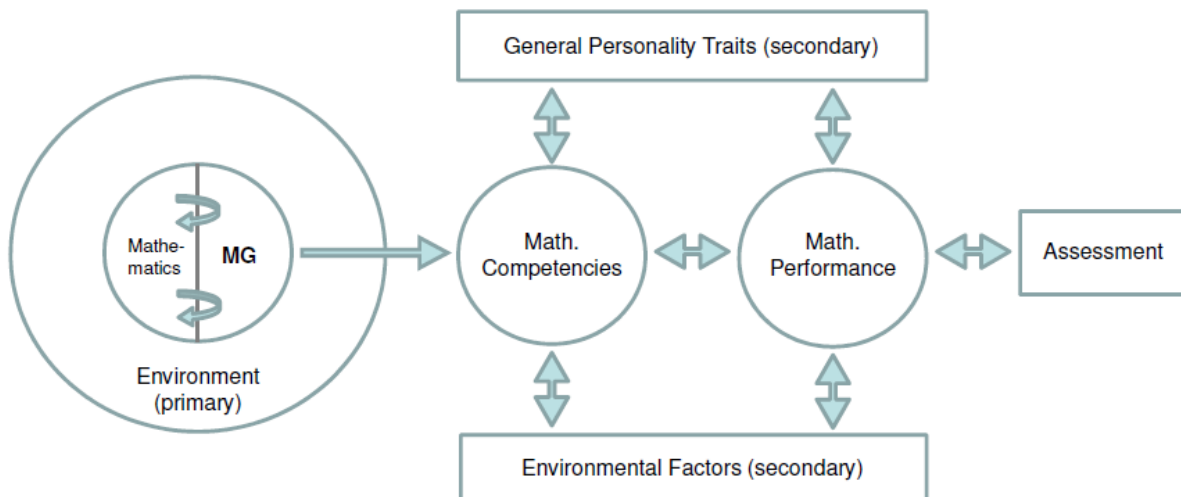
**Figure 2: System MG and environment(s) interacting by structural couplings.**

So MG manifests in different kinds and shapes depending on the notion about mathematics and the current state of the discourse concerning the authorities in environment 1. These items can only be detected by entering the field, like a certain school, and investigating the predominant notions and characteristics by qualitative and quantitative methods. Hence, the question “Is this person mathematically gifted?” should be altered to the question “Is this person mathematically gifted *within this specific environment(s)*?”.

### **Causality of Giftedness and Assessment**

In general, the system MG – as illustrated in Figure 1 and 2 – is seen as an individual potential for excellent assessment (Heller et al., 2002; Ulm, 2010). According to Ziegler (2009) a high attaining pupil is a person that fulfils a fixed criterion of achievement. So, in the first place, excellent assessment of a pupil in mathematics is measured by the teacher via recognizing very good marks in mathematics, for example.

A further developed model coming from the models in Heller et al. (2002) or Ulm (2010), respectively, for the causality between pupils’ MG and assessment which includes the systemic character, is given in Figure 3. The open mental construct MG that already contains some primary general personality traits (figure 1) depends on a predominant notion of mathematics in a certain situation which is located in a primary environment, the so-called environment 1. A person that possesses that kind of MG can develop mathematical competencies which may lead to mathematical performance and assessment. The actual formation of an observable performance as well as competencies is influenced by some secondary general personality traits (such as learning and working strategies, learning and working motivation, ability to concentrate, stress coping skills, ...) and secondary environmental factors (such as familial learning environment, teaching quality in class, atmosphere in school and class, peer group, critical personal experiences, ...) which operate as moderator variables.



Fig

**Figure 3: Causality between MG, competencies, performance and assessment.**

As indicated in figure 3 there is no equivalence between performance and MG: the arrow from MG to mathematical competencies only points in one direction. However, a well-known example for the fact that all dependences must not be seen as absolute but rather as possible is the (gifted) underachiever: he or she is MG, but his or her secondary moderators let him or her show no performance. On the other hand a high attaining pupil does not need to be (highly) gifted. The environment in his or her school can allow for very good grades in exams and the A-levels, for example, but does that mean that the pupil is a gifted mathematician? Perhaps the tasks the pupil had to solve only aimed at computational skills and did not stress on any kind of creative approaches at all. Of course, this implication depends on how MG is constructed in that specific environment. Therefore in the following paragraph an example from an empirical survey at a German school for high attaining pupils is sketched that gives insight to

- a) the manifestation of MG at this specific school and
- b) the different notions of being high attaining or highly gifted in mathematics from the view of teachers at this specific school.

## AN EMPIRICAL SURVEY

Description of field and sample: The school is a German boarding school for higher secondary education. So the pupils at this school are about between 15 and 18 years old. The classes are small and consist of 16 pupils with equally distributed gender. All pupils that want to apply for the school must have at least the mark “good” in the main subjects German, Mathematics, one foreign language and one natural science. Furthermore the average mark in the last two school reports has to be at least “good” as well. Pupils that fulfil the requirements with respect to marks have to pass a further selection process that consists of the intelligence structure test I-S-T 2000 R (Liepmann et al., 2007) and a two-day assessment centre concerning the social skills.

In the intelligence structure test pupils must have a total score that makes sure that the pupils finally selected are gifted in the sense of achieving very high IQ levels<sup>1</sup>.

All (eight) mathematics teachers were interviewed in an individual, problem-centred and episodic way with focus on their beliefs concerning their notion of mathematics, mathematical giftedness and the pupils at their school.

## RESULTS AND CONCLUSIONS

Transcript excerpts show the predominant concept of MG and different characteristics of high attaining versus (highly) gifted pupils in mathematics.

### Quotations from the individual interviews

Within this paper we limit ourselves to the presentation of quotations from two male mathematics teachers and denominate them as A and B. Teacher A is 47 and teacher B 59 years old. Both have worked at the school for six years. Teacher A earned a PhD in a non-math subject and had a several years experience of working with gifted pupils before entering the current school. Teacher B had worked as a teacher abroad. Both of them had worked at different secondary schools before entering the current school. The interviews lasted 55 and 26 minutes, respectively. Quotations from teacher A and B were chosen out of the eight interviewed ones for presentation in this paper because of their relative similarity, openness and extensiveness. But as there is no right or wrong but only *the* specific and viable construct of MG this is no restriction according to the theoretical background.

### Teacher A: notion of mathematics

When it comes to their notion of mathematics, teacher A addresses several aspects. His answers can be separated according to different topics. So, the themes are listed, followed up by some quotes stemming from the interviews<sup>2</sup>:

- leisure: “[...] thinking it over relaxed and quietly, this and then concentrating on it, that gives a better return”
- philosophical thoughts: “[...] it must make sense. I mean that philosophically, too. Because you said what pupils should answer to what is mathematics. Yes. (.) That is a matter of taste either way, but that it makes sense somehow.”
- communicative process: “Mathematics is a communicative process.”
- logically clearly defined and also communicative intellectual game: “I hope they don’t see mathematics as deterrent but as (.) a logically clear defined (.) and also (.) communicative intellectual game.”
- intellectual appealing world of thought: “[...] as an appealing, intellectually appealing, but that’s also a matter of taste, intellectual appealing mhh (..) world of thoughts”

- also applications: “That is a matter of taste either way, but that it makes sense somehow. ((hm)) (..) Either just cognitively or by application“
- intellectual joy, happiness: “The goals? ((yes)) Well, that the pupils have intellectual fun [...] Joy, actually that’s the highest goal.“

### Teacher A: manifestation of MG

When teacher A is asked for his idea of mathematical giftedness, the interview again delivers some main points:

- intuition; sensing the overall train of thoughts: “Well, I see that in fact that they actually sense the point of it all without that I would have spent many words about the problem. [...] a kind of intuition, aha, so now there is this and that and this. That they sense the overall train of thoughts where the method will lead us”
- interest in alternative definitions and the corresponding consequences: “[...] it is the permanent attempt (..) can’t we play differently? ((yes, yes)) And if yes, where would this lead us to? If no, why doesn’t it work out?”
- aesthetical sensation and joy: “Those have more fun than usual. [...] Over the fact that this is a beautiful construction of the theory ((ah, yes, certainly)) Or: what a wonderful exactness! ((mhm)) So this joy over a good construction [...] they really feel it.”

### Teacher A: looking at the structural coupling

Teacher A has a very high-level and sophisticated notion of mathematics. So, for example teacher A emphasises leisure, the communicative process, exactness and intellectual joy. Because of the structural couplings between the system MG and its environment of mathematics this leads to strong correlations with teacher A’s idea of MG. These correspondences are pointed out by colours and arrows in table 1.

<i>Notion of Mathematics</i>	<i>MG</i>
<ul style="list-style-type: none"> <li>• leisure</li> <li>• philosophical thoughts</li> <li>• communicative process</li> <li>• logically clearly defined and also communicative intellectual game</li> <li>• intellectual appealing world of thought</li> <li>• intellectual joy, happiness</li> <li>• also applications</li> </ul>	<ul style="list-style-type: none"> <li>• intuition: sensing the overall train of thoughts</li> <li>• interest in alternative definitions and the corresponding consequences</li> <li>• aesthetical sensation and joy about that</li> </ul>

**Table 1: Notion of mathematics and MG of teacher A; coloured text parts correspond to each other as also indicated by arrows.**



### Teacher B: notion of mathematics and manifestation of MG

For the sake of completeness the next summarized quotations give insight to the notion and manifestation of mathematics (first quotation) and MG (second quotation) of teacher B; the main topics are underlined:

- Maths: “[...] mathematics brings in a certain kind of systemization. [...] Mathematics: to search for, to see and to analyse structures. [...] They should (.) see that mathematics (.) is the art of looking at the complicated in an easy way. [...] Well, that knowledge to have, or are able to gain, an instrument by which we understand how the modern world operates. [...] But, of course, this is a bit of disenchantment. [...] this balancing act between the two different approaches to mathematics, on the one hand (.) I don’t need illustration at all ((mhm)) in order to design mathematics as just a glass bead game [...] and the other hint that I want to give the pupils: mathematics always was the answer to concrete questions. [...] I think either is important to get across. [...] Yes, it is inspiration and a bit of chaos, of course. Yes ((yes, yes)). And, of course, it is also the possibility, the frustration tolerance, to begin and to throw away ideas, to have ideas. ((yes)) (.) And from this, to see structures suddenly coming out of this confusion, so you can again throw away things and arrange them in a new way ((yes, yes)) Of course, there is a certain kind of joy, too. In this respect then there is something like mathematical beauty, too, of course.”
- MG: “[...] those really were satisfied with this (.) highly abstract chapter about vector spaces ((ok)) they took part in and saw some good connections [...] I don’t know if somebody is mathematically gifted. Well, of course, creativity, curiosity (.) plays a role, the ability to think out of the box or to see connections. Hence, the opposite to being uninspired/dutiful [...] what happens really often is that they find solutions to problems that I didn’t expect. [...]”

### Teacher A: high attaining pupils

When teacher A speaks of the pupils at his school he gives a good profile of these high attaining ones and their difference to the mathematically gifted. Again, the answers are sorted by main topics followed up by short quotations:

- want mathematics lessons to be done in detail and precisely: “The pupils often say (.) fully convinced: when we do it very accurately then we understand it best. ((mhm, yes, yes.)) This is a very clear statement which I get over and over again every year ((ok)) In all grades. That means (.) mathematical precision for these pupils leads to a better and not a worse understanding”
- want to be active: “In case of doubt they want to be active ((ok)) not hanging around, they are motivated.”



## Working Group 7

- are motivated (mainly intrinsic): “Actually, I think it is mainly intrinsic motivation ((hm)) just want to be intellectually active. That’s for sure.”
- are able to work in teams: “Able to work in teams, social cooperation ((yes)) in a positive sense”
- impose an introjected pressure to perform on themselves: “[..] it is a somehow introjected pressure to perform which they impose on themselves, too.”
- have extreme requirements on themselves: “[...] often it is a very high requirement on themselves [...] There are pupils weeping, because they only have the grade “good” [...] That means it is an internal pressure to perform on oneself ((hm)) I wouldn’t say that this is the classical grade pressure ((mhm)) but that these are pupils that partly have extreme requirements on themselves.”
- don’t dare to work independently: “I offer the pupils (.) that when they say they don’t want to take part in class to work independently about this or that mathematical aspect in the library (.) that’s what I offer them, and then do a short talk about it ((yes)) in order to differentiate. I offer that, also in other subjects, I offer that and who doesn’t make use of it? The pupils!”
- don’t want to isolate themselves: “They don’t want to isolate themselves”
- don’t want to miss anything in class: “[...] and, of course, they don’t want to miss anything in class. [...] They somehow don’t dare.”
- do well in all subjects: “The good and very good pupils from advanced math courses at normal schools would do well here especially in maths ((hm)) However, the pupils here do well in all subjects ((ok)) more or less.”
- are less non-conformistic: “Well, the pupils here are not more non-conformistic than at other schools. Not for sure. [...] It’s just the opposite.”
- are interested: “They are interested, they are motivated ((ok)) Let it be whatever it is, just make sure that it is interesting ((ok, hm)) It has to make sense and it has to be interesting”
- are very polite, respectful, sensitive and thus reserved: “On the interpersonal level they are (.) very polite, very respectful, very sensitive ((mhm)) and thus very reserved [...] just because they are so nice and respectful ((yes)) they don’t rebel so straightforward”
- are distinctively stubborn on an emotional an cognitive level: “[...] they are distinctively stubborn [...] in order to convince them to do something in different way or that something is better, you really need telling arguments. They have to be absolutely convincing [...] on the emotional level concerning something, but, of course, just as well when it comes to cognitive things ((ok)) [...] And the same with motivational questions in class. A problem is not interesting as a pseudo problem, it really has to be interesting [...] In this respect: a global stubbornness”

## Working Group 7

- are less qualified to be subordinates: “Actually they are less non-conformistic than at other schools ((ok)) but they are less qualified for being subordinates. ((mhm)) Yes. That’s the point”

Teacher B gives some more aspects:

- are teacher-orientated: “I experience the pupils here as interested, teacher orientated, committed [...] The pupils’ expectation is more significant than elsewhere that the teacher is the one, who knows the things and (.) has to find an appropriate way to teach it to the pupils. What I want to say is that they more often are less independent than I would actually wish. [...] They have the potential to work independently, but they don’t dare. [...] On the evaluation sheets that I hand out in every course, I always ask on which content we should have had more concentrated on or what else they imagine. And then surprisingly often there is the answer that I would never have heard at other schools: you will know it best, or, that’s what the curriculum says after all. [...] Well, that objects my experiences which I made at other schools.”
- are very dutiful: “We have many dutiful ones at this school. ((ah, ok, aha)) Probably too many dutiful [...] Many (...) do mathematics, because they know that one needs it and also has to at this school ”
- are resistant to psychological strain: “ [...] and they are (..) resistant to the psychological strain. They take on very much ((ok)) don’t show it as much as at other schools ((ok)) how much they suffer”
- are eager for reliability: “They want the teacher to take the lead, because he knows what’s going on. ((yes)) Especially with ours, who are eager for reliability”
- want to write good exams: “They want to write good exams. [...] they definitely want to do good A-levels.”

### **Teachers A and B: high attaining versus (highly) gifted in mathematics**

We chronologically list the important items that refer to the high attaining pupils at this specific school in table 2 and contrast them to the mentioned (not necessarily additional) characteristics of MG seen by these two teachers.

<i>High attaining pupils</i>	<i>MG pupils</i>
<ul style="list-style-type: none"> <li>• want mathematics lessons to be done in detail and precisely</li> <li>• want to be active</li> <li>• are motivated (mainly intrinsic)</li> <li>• are able to work in teams</li> <li>• impose an introjected pressure to perform on themselves</li> <li>• have extreme requirements on themselves</li> </ul>	<ul style="list-style-type: none"> <li>• have mathematical intuition: sensing the overall train of thoughts</li> <li>• are interested in alternative definitions and the corresponding consequences</li> <li>• feel aesthetical sensation and joy about that more than others</li> <li>• are satisfied with highly abstract</li> </ul>

## Working Group 7

<ul style="list-style-type: none"> <li>• don't dare to work independently</li> <li>• don't want to isolate themselves</li> <li>• don't want to miss anything in class</li> <li>• do well in all subjects</li> <li>• are less non-conformistic</li> <li>• are interested</li> <li>• are very polite, respectful, sensitive and thus reserved</li> <li>• are distinctively stubborn on an emotional and cognitive level</li> <li>• are less qualified to be subordinates</li> <li>• are teacher-orientated</li> <li>• are very dutiful</li> <li>• are resistant to psychological strain</li> <li>• are eager for reliability</li> <li>• want to write good exams</li> </ul>	<ul style="list-style-type: none"> <li>• objects</li> <li>• are creative</li> <li>• show curiosity</li> <li>• are able to think out of the box</li> <li>• "see" inner-mathematical connections</li> <li>• find unexpected solutions to problems</li> <li>• are the opposite of being uninspired/dutiful</li> </ul>
---	--

**Table 2: Characteristics of high attaining and gifted pupils in mathematics as described by teacher A and B opposed to each other.**

### Conclusion

The comparison of the two columns in table 2 shows well the difference between a high attaining and a (highly) gifted pupil in mathematics the way it is seen by those two exemplary teachers. Most of the aspects listed in the MG column correspond to the items listed in figure 1 both for abilities specific to mathematics and general personality traits. But especially when it comes to the latter ones, there is an overall theme that separates the gifted from the high attaining ones. In the eyes of both teachers, a pupil who is a brilliant mathematician features a specific characteristic that opposes the description of the typical high-attaining pupil: where a less non-conformistic, reserved, stubborn, dutiful and teacher-orientated attitude is typical for high attaining pupils, mathematical excellence results from creativity, curiosity, out-of-the-box-thinking, flexibility and the "opposite of being uninspired/dutiful". This stands in line with the following quotation from the teacher Carol Fertig at Prufrock's Gifted Information Blog: "Flexible thinkers go beyond the bounds of orthodox thinking and look for alternatives others fail to see. While rules are used as guidelines, they are not used as straightjackets that curb thinking. Flexible thinkers are those who are creative problem solvers. Flexibility requires that people escape from ruts and try new things. These thinkers are able to shift gears easily. They look for new ideas everywhere. They are not afraid of change." (Fertig 2007)

## Connection to the theoretical background

By the exemplarily examination of two teacher interviews at this specific school for high attaining pupils we could give an answer to the actual manifestation of MG in this special context: both teachers have a very high-level and sophisticated notion of mathematics. Furthermore it was illustrated in the case of teacher A, how the notion of mathematics influences the notion of MG. In system theoretical termini this fact shows the structural coupling between the two systems. Concerning the difference between gifted and high attaining pupils, the interviews clearly showed that there are more and/or other aspects that stand for being gifted in mathematics. Just looking at a very good performance in the subject does not mean that the pupil looked at is a gifted mathematician: the left horizontal arrow in Figure 3 is not reversible! Those aspects may be considered in tests to identify mathematically gifted pupils.

## NOTES

1. The fact that a psychologically defined numerical IQ exceeds a certain value does not necessarily imply that this person is mathematically gifted.
2. All quotations from the interviews and discussions have been translated from German into English by the interviewer and author of this paper.

## REFERENCES

- Brandl, M. (2010, August 3). *A Constructive Approach to the Concept of Mathematical Giftedness based on Systems Theory*. Paper presented at the 6th International Conference on "Creativity in Mathematics Education and the Education of Gifted Students", Riga, Latvia. Submitted for publication.
- Fertig, C. (2007, November 16). Creative Flexibility: Bending Gifted Minds. Retrieved from <http://resources.prufrock.com/GiftedChildInformationBlog/tabid/57/articleType/ArticleView/articleId/224/Creative-Flexibility-Bending-Gifted-Minds.aspx>.
- Fölsch, G. (1977). Die Stellung der begabten Schüler im Mathematikunterricht. *ZDM* 9, 1, 13-20.
- Greenes, C. (1981). Identifying the Gifted Student in Mathematics. *Arith. Teacher* 28, 6, 14-17.
- Heller, K., Reimann, R. & Rindermann, H. (2002). *Theoretische und methodische Grundlagen der Evaluationsstudie*. In K. Heller (Ed.), *Begabtenförderung am Gymnasium – Ergebnisse einer zehnjährigen Längsschnittstudie* (pp. 53-80). Opladen: Leske+Budrich.
- Heller, K., Mönks, J., Sternberg, R., Subotnik, R. (2000). *International Handbook of Giftedness and Talent* (2. Ed.). Oxford: Elsevier Science.

- Irvine, S. H. & Berry, J. W. (1988). The abilities of mankind: A revaluation. In S. H. Irvine & J. W. Berry (Eds.), *Human abilities in cultural context* (pp. 3-59). Cambridge University Press.
- Käpnick, F. (1998). *Mathematisch begabte Kinder. Modelle, empirische Studien und Förderungsprojekte für das Grundschulalter*. Frankfurt am Main.
- Kießwetter, K. (1992). „Mathematische Begabung“ – Über die Komplexität der Phänomene und die Unzulänglichkeiten von Punktbewertungen. *MU* 38 (1), S. 5-10.
- Kruteskii, V.A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. Chicago, London.
- Krieger, D.J. (1996). *Einführung in die allgemeine Systemtheorie*. München: W. Fink Verlag.
- Liepmann, D., Beauducel, A., Brocke, B. & Amthauer, R. (2007). *Intelligenz-Struktur-Test 2000 R (I-S-T 2000 R). Manual (2. ed.)*. Göttingen: Hogrefe.
- Pfeiffer, S. (2008). *Handbook of Giftedness in Children. Psychoeducational Theory, Research, and Best Practices*. New York: Springer.
- Sternberg, R. J. (1996). What is Mathematical Thinking? In: R. J. Sternberg & T. Ben-Zeev (Eds.). *The nature of mathematical thinking*. Mahwah, NJ: Lawrence Erlbaum Ass. Publishers, 303 – 318.
- Treumann, K. (1974). *Leistungsdimensionen im Mathematikunterricht*. Hamburg: Klett.
- Ulm, V. (2010). *Mathematisches Denken und mathematische Begabung*. In V. Ulm (Ed.). *Mathematische Begabungen fördern* (pp. 3-7). Berlin: Cornelsen Scriptor.
- Ziegler, A. (2008). *Hochbegabung*. München: Ernst Reinhardt Verlag.

# DOES MATHEMATICAL CREATIVITY DIFFERENTIATE MATHEMATICAL ABILITY?

Maria Kattou, Katerina Kontoyianni, Demetra Pitta-Pantazi, Constantinos Christou

Department of Education, University of Cyprus

*This paper focuses on the development of a theoretical model in which mathematical creativity constitutes a predictor of mathematical ability. Furthermore, we examine the existence of groups of students that differ across mathematical ability and investigate whether these groups present differences in their mathematical creativity. The study was conducted among 359 elementary school students in Cyprus, using two instruments. The results revealed that mathematical ability may be predicted by mathematical creativity. Moreover, the sample can be grouped in three distinct categories according to mathematical ability. The categories of students also present statistically significant differences in their mathematical creativity, suggesting that the level of mathematical ability depends on the level of mathematical creativity.*

**Keywords:** creativity, cognitive abilities, mathematics.

## INTRODUCTION

Creativity has been proposed as one of the major components to be included in the education of the 21st century (Mann, 2005). Therefore, contemporary curricula emphasize the development of students' creative thinking (Lamon, 2003). However, the conceptualization of creative learning varies due to the diversity of the proposed definitions of creativity. In particular, several definitions have been proposed for the concept of creativity; some of the definitions focus on process, while others emphasize the creative product (Haylock, 1987). For instance, creativity has been defined as a continuum of actions, as a process of bringing something new into being (Best & Thomas, 2007). In regard to the definitions that refer to the creative product, fluency, flexibility and originality are some of the commonly accepted characteristics that describe the outcome of the creative action (Torrance, 1995). In contrast to general creative ability, as has been defined previously, Gardner (1993) proposed that every individual has creative potential in a specific domain. Due to this fact, research literature makes a distinction between general and specific creativity (Piiro, 1999), whereas specific creativity "is expressed in clear and distinct ability to create in one area, for example mathematics" (Leikin, 2008).

## THEORETICAL FRAMEWORK

### Defining mathematical creativity

The lack of a uniform definition of general creativity consequently leads to the dearth of a commonly accepted conceptualization of mathematical creativity (Haylock, 1997). In some cases, researchers move away from mathematics to search ideas associated with creativity in general and then to select those which are the most

relevant to mathematics (Haylock, 1987). For example, the definitions given by Krutetskii (1976), Ervynck (1991), Silver (1997), Gil, Ben-Zvi and Apel (2007) are based on the concepts of fluency, flexibility and originality (Torrance, 1995) in the content of mathematics. In this case, fluency refers to the ability of producing many ideas, flexibility refers to the number of approaches that are observed in a solution and originality refers to the possibility of holding extraordinary, new and unique ideas (Gil, Ben-Zvi & Apel, 2007).

Moreover, several attempts have been made to generate definitions of mathematical creativity. These definitions include the abilities to solve problems and to develop thinking in structures (Ervynck, 1991), to observe patterns (Laylock, 1970), to abstract and generalize mathematical content (Krutetskii, 1976), to discern or choose (Hadamard, 1945), to understand which patterns are acceptable (Birkhoff, 1969) and to make connections between unrelated ideas (Haylock, 1987).

### **Mathematical creativity and mathematical ability**

As Silver (1997) suggested, “creativity is closely related to deep, flexible knowledge in content domains” (p. 750). However, there are conflicting results that describe the relationship between mathematical creativity and mathematical ability: could creativity affect an individual’s mathematical ability or mathematical knowledge may enhance mathematical creativity? According to Meissner (2000), solid mathematical knowledge is essential for the development of mathematical creativity. One important reason for this necessity is the fact that excellent knowledge of the content helps individuals to make connections between different concepts and types of information (Sheffield, 2009). Therefore, students who are characterized by mathematical accuracy and fluency are more able to think creatively in new mathematical tasks by providing original and meaningful solutions (Binder, 1996). Moreover, “creative work involves a certain amount of pre-existing domain knowledge and its transformation into new knowledge” (Nakakoji, Yamamoto, & Ohira, 1999). More specific, prior knowledge constitutes the backbone on which new information will be organised and determines the extent to which these information will be explored (Sheffield, 2009). On the contrary, other researchers proposed that creative potential contribute to the improvement of mathematical knowledge. In Starko’s words (1994) “students who use content in creative ways learn the content well”. The ability to solve a problem with several strategies or the ability to reach different answers in a specific task are valuable evidences of the development of mathematical reasoning (NCTM, 2000). In other words, mathematical creativity “is an essential aspect in the development of mathematical talent” (Mann, 2005, p. 29). The importance of mathematical creativity to mathematical ability is also proposed by Hong and Aquí (2004). In this research, Hong and Aquí (2004) studied the differences between high academically and high creative students in mathematics. The results verified that creative students in mathematics were more cognitively resourceful than their peers who achieved high

grades in school mathematics. On the basis of these results, Sternberg (1999) acknowledged that the essence of mathematics is to apply knowledge creatively in specific circumstances.

## PURPOSE OF THE STUDY

Despite the fact that the definitions referring to mathematical creativity suggest important abilities that may consist this construct, there is a lack of corresponding research regarding to the relationship between mathematical creativity and mathematical abilities. Therefore, the purpose of the present study is threefold: (1) to examine the relationship between mathematical creativity and mathematical ability, as they are projected in a theoretically driven model; (2) to trace groups of students that differ across the components of mathematical ability; (3) to investigate differences in creativity between the groups of students that vary in mathematical ability.

## METHODOLOGY

### Sample



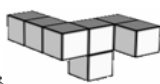
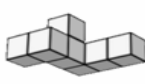
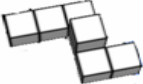


The sample for this study consisted of 359 Grade 4, Grade 5 and Grade 6 students from eight elementary schools in Cyprus. One hundred and fourty three students attended 4<sup>th</sup> grade, while 118 and 98 students attended 5<sup>th</sup> and 6<sup>th</sup> grade, respectively.

### Instruments

Each student completed two instruments: the mathematical abilities instrument and the mathematical creativity instrument (examples of tasks are presented in Figure 1). The mathematical abilities instrument consisted of 29 tasks measuring the following abilities: manipulation of quantities (quantitative ability), causal relationships (causal ability), visualization and spatial reasoning (spatial ability), processing of similarities and differences (qualitative ability), inductive/deductive reasoning (inductive/deductive ability). Students' answers were assessed as right or wrong. The mathematical creativity instrument included five tasks in which students were required to provide: (a) multiple solutions, (b) solutions that were different between them, and (c) solutions that none of his/her peers could provide. The assessment of students' creativity was based on the distinction of students' fluency, flexibility and originality (Torrance, 1995). A task from the mathematical creativity instrument is presented in Figure 1. In this example, fluency referred to the number of correct responses that students presented. For flexibility, the different types of responses were measured (e.g. whether students used additive or multiplicative structure). Originality was calculated by comparing a student's solutions with the solutions provided by all students and the rarest correct solutions received the higher score. Every component of the mathematical creativity instrument was converted to a score ranging from 0 to 1, with 1 being the highest score.



## Working Group 7

<b>Spatial Task</b>	<p>Observe carefully the following solid.</p>  <p>If you rotate the solid shown above, which of the following solids could be the result?</p> <div style="display: flex; justify-content: space-around; align-items: flex-end;"> <div style="text-align: center;">  A         </div> <div style="text-align: center;">  B         </div> <div style="text-align: center;">  C         </div> <div style="text-align: center;">  D         </div> <div style="text-align: center;">  E         </div> </div> <p>A. A      B. B      C. C      D. D      E. None of the previous</p>
	<p>Twenty passengers travel to Larnaca Airport by bus. Twelve of them carry a travel bag, eleven a computer bag and six people both types of suitcase. How many passengers carry only a computer bag?</p> <p>A. 5      B. 6      C. 8      D. 11      E. 17</p>
<b>Quantitative Task</b>	<p>George's marbles are three times as many as Spiro's marbles. Andreas has 20 marbles less than George. If Spiros has <math>S</math> marbles, how many marbles does Andreas have?</p> <p>A. <math>\frac{S}{3}-20</math>      B. <math>3S+20</math>      C. <math>S+50</math>      D. <math>S-50</math>      E. <math>3S-20</math></p>
<b>Inductive/ Deductive Reasoning Task</b>	<p>If the suitcase has stripes, then it belongs to Emmanuel. If the bag is yellow, then it does not belong to Emmanuel. The suitcase is yellow. Therefore,</p> <p>A) The suitcase has stripes. B) The suitcase does not have any stripes. C) The suitcase belongs to Emmanuel. D) I cannot decide.</p>
<b>Causal Task</b>	<p>Inside a vase there are four balls: one yellow, one red, one green and one blue. If I want to pull out of the vase only three balls, by pulling out one at a time, how many different possible combinations do I have? Write down all the possible combinations. (For example I can pull out a red, a yellow and then a green ball or a green, a yellow and then a red ball. These are only two possible combinations, but there are more).</p> <p>A. 6      B. 12      C. 18      D. 24      E. None of the above</p>
<b>Creativity Task</b>	<p>Look at the following number pyramid. All the cells must contain only one number. Each number in the pyramid can be computed by performing always the same operation with the two numbers that appear underneath it. Fill in the pyramid, by keeping on the top the number 35. Try to find as much solutions as possible.</p> 

**Figure 1: Examples of tasks from the mathematical abilities and mathematical creativity instruments.**

The two instruments were given in electronic form and students worked individually in the laboratory of their school in order to complete them. The administration time ranged from 40 to 80 minutes.

### Data Analysis

The objectives of the analysis were first to verify a theoretical model regarding to the relationship between mathematical creativity and mathematical ability, secondly to trace groups of students that differ across the components of mathematical ability

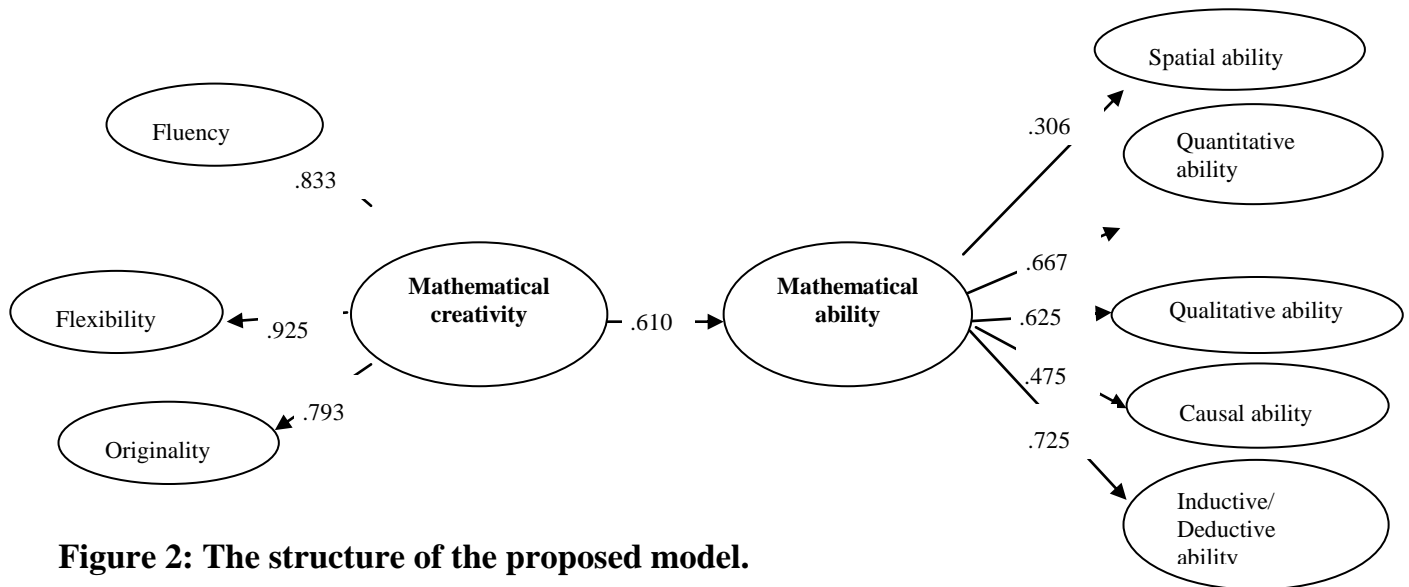
and finally to examine differences between the groups of students. In regard to the first objective, confirmatory factor analysis (CFA) was conducted in order to assess the fit of the theoretically driven model to the data of the present study. More specific, CFA was used to test whether several observed variables (e.g. fluency, flexibility, originality) may define a latent construct (e.g. creativity). For the purposes of this analysis, the statistical modeling program MPLUS (Muthen & Muthen, 2007) was used. The evaluation of model fit was based on three fit indices: The comparative fit index (CFI), the ratio of chi-square to its degree of freedom ( $\chi^2/df$ ) and the root mean-square error of approximation (RMSEA). According to Marcoulides and Schumacker (1996), for the model to be confirmed, the values for CFI should be higher than 0.90, the observed values for  $\chi^2/df$  should be less than 2 and the RMSEA values should be close to or lower than 0.08. For the accomplishment of the second objective, latent class analysis was used to explore whether there were different categories of students in our sample whose achievement could vary according to mathematical ability. Once the latent class model was estimated, subjects classified to their most likely class by mean of recruitment probabilities. Afterwards, analysis of variance (ANOVA) was conducted, in an effort to investigate differences between groups of students on mathematical abilities (spatial ability, quantitative ability, qualitative ability, causal ability, inductive/deductive reasoning ability) due to their different degree of mathematical creativity (fluency, flexibility, originality).

## RESULTS

### The Validation of the Model

A-priori we hypothesized that mathematical creativity, composed by fluency, flexibility and originality, could predict mathematical ability. Regarding to mathematical ability, we assumed that it was composed of five components: spatial ability, quantitative ability, qualitative ability, causal ability, inductive/ deductive reasoning ability. Figure 2 presents the structural equation model with the latent variables (mathematical creativity and mathematical ability) and their indicators.

The results of the analysis revealed that the theoretical model matched the data set of the present study and determined the “goodness of fit” of the factor model (CFI=0.990,  $\chi^2=29.269$ ,  $df=19$ ,  $\chi^2/df= 1.540$ , RMSEA=0.039). The analysis revealed that the statistically significant loadings of fluency ( $r=.833$ ,  $p<.05$ ), flexibility ( $r=.925$ ,  $p<.05$ ) and originality ( $r=.793$ ,  $p<.05$ ) consist a first order factor, that of mathematical creativity. Moreover, the five cognitive abilities, namely spatial ability ( $r=.306$ ,  $p<.05$ ), quantitative ability ( $r=.667$ ,  $p<.05$ ), qualitative ability ( $r=.625$ ,  $p<.05$ ), causal ability ( $r=.475$ ,  $p<.05$ ) and inductive/deductive reasoning ability ( $r=.725$ ,  $p<.05$ ) can model the performance of students in mathematics. Mathematical creativity contributes to the prediction of mathematical ability ( $r=.610$ ,  $p<.05$ ).



**Figure 2: The structure of the proposed model.**

### Categories of students and differences on mathematical abilities and creativity

The second aim of the study concerns the extent to which students in the sample vary according to their mathematical ability. To this end, we examined whether variation on mathematical ability leads to discrepancy on mathematical creativity components. The latent class analysis (LCA) used a stepwise method—that is, the model was tested under the assumption that there are two, three, and four groups of subjects. The best fitting model with the smallest AIC (6366.99) and BIC (6467.96) and the largest entropy (0.733) indices (Muthén & Muthén, 1998) was the one with three groups.

Taking into consideration the average group probabilities as shown in Table 1, we may conclude that categories are quite distinct, indicating that each class has its own characteristics. The means and standard deviations of the three categories of students on the specific mathematical abilities are presented in Table 2.

	Category 1	Category 2	Category 3
Category 1 (N=111)	.878	.122	.000
Category 2 (N=189)	.102	.861	.037
Category 3 (N=59)	.000	.083	.916

**Table 1: Average Latent Class Probabilities.**

Table 2 reveals that students in Category 3 outperformed students in Category 1 and Category 2 across all mathematical abilities. Students in Category 2 outperformed their counterparts in Category 1. It is important to note that across the three categories of students, there are statistically significant differences ( $p < 0.05$ ) among all mathematical abilities.

Due to the differences on mathematical abilities across the three categories of students, it can be deduced that our sample can be grouped in three distinct levels of

abilities; Category 1 (N=111) consists of low mathematical ability students, Category 2 (n=189) consists of average mathematical ability students and Category 3 (n=59) consists of high mathematical ability students.

	Category 1 $\bar{X}$ (SD)	Category 2 $\bar{X}$ (SD)	Category 3 $\bar{X}$ (SD)	Total $\bar{X}$ (SD)
Spatial ability	1.02 (0.88)	1.32 (1.09)	2.07 (0.96)	1.35 (1.07)
Quantitative ability	0.80 (0.81)	1.54 (1.04)	3.34 (1.03)	1.61 (1.28)
Qualitative ability	0.65 (0.82)	1.25 (0.94)	3.14 (0.96)	1.37 (1.23)
Causal ability	1.02 (0.76)	1.74 (0.83)	2.34 (0.71)	1.61 (0.91)
Inductive/Deductive ability	1.11 (0.59)	2.57 (0.67)	3.56 (0.65)	2.28 (1.07)

**Table 2: Means and Standard Deviations of the three Categories of Students in mathematical abilities.**

Regarding to mathematical creativity, Table 3 presents the differences between the three categories of students across the three components of mathematical creativity, namely fluency, flexibility and originality.

	Category 1 $\bar{X}$ (SD)	Category 2 $\bar{X}$ (SD)	Category 3 $\bar{X}$ (SD)	Total $\bar{X}$ (SD)
Fluency	0.93 (0.44)	1.46 (0.57)	2.04 (0.63)	1.39 (0.66)
Flexibility	1.37 (0.42)	1.83 (0.46)	2.21 (0.38)	1.75 (0.52)
Originality	1.67 (0.66)	2.20 (0.79)	2.76 (0.73)	2.13 (0.82)

**Table 3: Means and Standard Deviations of the three Categories of Students in mathematical creativity.**

In particular, students with high mathematical ability (Category 3) are the high creative students as well. Likewise, average mathematical ability students (Category 2) have an average performance across fluency, flexibility and originality, while low ability students (Category 1) have low creative potential in mathematics. The differences on fluency, flexibility and originality are statistically significant across the three groups of students ( $p < 0.05$ ).

## DISCUSSION

Creativity is currently discussed as an essential component of the aim “mathematics for all” (Pehkonen, 1997). Given the importance of creativity in school mathematics, several researchers investigated the relationship between mathematical creativity and

school mathematics (e.g. Mann, 2005), but few of them examined the impact of the former to the latter. Hence, the first goal of this study was to articulate and empirically test a theoretical model, in which abilities in mathematics may be predicted by creative potential. In the theoretical model, spatial conception (spatial ability), arithmetic and operations (quantitative ability), proper use of logical methods (inductive/deductive reasoning), formulation of hypotheses concerning cause and effect (causal ability) and the ability to think analogically (qualitative ability) constitute mathematical ability, as has been proposed by Krutetskii (1976). With respect to mathematical creativity, fluency, flexibility and originality were its three components as recommended by Torrance (1995) and adopted by researchers on mathematics education (e.g. Silver, 1997). The model extended the literature in a way that mathematical creativity is a predictor of mathematical ability.

Therefore, it appears that the assessment of mathematical creativity can provide useful information in regard to students' profile and more specifically to their mathematical performance. Unfortunately, mathematical tests which are used in schools value mainly speed and accuracy and neglect creative thinking abilities (Mann, 2005). For this reason, creative tasks should be included in the assessment methods of mathematics, in order to capture not only students who do well in school mathematics and are computationally fluent but also students who have the potential but have not manifested their abilities yet.

The second aim of the study concerned the extent to which students in the sample vary according to their mathematical ability. The analysis illustrated that three different categories of students can be identified. Category 1 students had low mathematical ability, Category 2 students had average mathematical ability and Category 3 students had high mathematical ability. This distinction between the three categories of students appeared across the five mathematical abilities. With respect to the third aim of the study, an investigation of differences in mathematical creativity components, due to the differences in mathematical ability took place. The results of the present study verified that the three categories of students varying in mathematical ability, reflect three categories of students also varying in mathematical creativity. To this end, it can be assumed that creativity is one of the components that contribute to the development of mathematical abilities. These results are in accordance with Hong's and Aqui's study (2004).

To sum up, creative applications of mathematics in the exploration of problems and in the teaching of mathematical content are essential (Pehkonen, 1997). Encouragement of mathematical creativity in combination with computational accuracy is important for students to further develop their mathematical ability and understanding (Mann, 2005).

**Acknowledgements:** This work was funded by the Cyprus Research Promotion Foundation (Grant: ANTHROPISTIKES/PAIDI/0308(BE)/13).

## REFERENCES

- Best, B. & Thomas, W. (2007). *The Creative Teaching and Learning Toolkit*. New York: Continuum International Publishing Group.
- Binder, C. (1996). Behavioral fluency: Evolution of a new paradigm. *The Behavior Analyst*, 19, 163-197. Retrieved from <http://www.abainternational.org/TBA.asp>
- Birkhoff, G.D. (1969). Mathematics and psychology. *Society for Industrial and Applied mathematics review*, 11, 429-469. Retrieved from <http://www.siam.org/journals/sirev.php>
- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42-53). Dordrecht: Kluwer.
- Gardner, H. (1993). *Creating minds*. New York: Basic Books.
- Gil, E., Ben-Zvi, D., & Apel, N. (2007). What is hidden beyond the data? Helping young students to reason and argue about some wider universe. In D. Pratt & J. Ainley (Eds.), *Proceedings of the Fifth International Research Forum on Statistical Reasoning, Thinking and Literacy: Reasoning about Statistical Inference: Innovative Ways of Connecting Chance and Data* (pp. 1-26). UK: University of Warwick. Retrieved from <http://srtl.stat.auckland.ac.nz/srtl5/presentations>
- Hadamard, J.W. (1945). *Essay on the psychology of invention in the mathematical field*. Princeton: NJ: Princeton University Press.
- Haylock, D. (1987). A framework for assessing mathematical creativity in school children, *Educational Studies in Mathematics*, 18(1), 59–74. doi: 10.1007/BF00367914
- Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren, *ZDM*, 29(3), 68-74. doi: 10.1007/s11858-997-0002-y
- Hong, E., & Aquilino, Y. (2004). Cognitive and motivational characteristics of adolescents gifted in mathematics: Comparisons among students with different types of giftedness. *Gifted Child Quarterly*, 48, 191-201. doi: 10.1177/001698620404800304
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago Press.
- Lamon, S. (2003). Beyond constructivism: An improved fitness metaphor for the acquisition of mathematical knowledge. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching* (pp.435-448). Mahwah, NJ: Lawrence Erlbaum and Associates.
- Laycock, M. (1970). Creative mathematics at Nueva, *Arithmetic Teacher*, 17(4), 325–328. Retrieved from <http://www.nctm.org/publications/content.aspx?id=8590>

- Leikin, R. (2008). Teaching mathematics with and for creativity: An intercultural perspective. In P. Ernest, B. Greer, & B. Sriraman (Eds.), *Critical Issues in Mathematics Education* (pp. 39-43). USA: Information Age Publishing Inc. & The Montana Council of Teachers of Mathematics.
- Mann, E. (2005). *Mathematical Creativity and School Mathematics: Indicators of Mathematical Creativity in Middle School Students* (Doctoral dissertation). Retrieved from [www.gifted.uconn.edu/siegle/Dissertations/Eric%20Mann.pdf](http://www.gifted.uconn.edu/siegle/Dissertations/Eric%20Mann.pdf)
- Marcoulides, G. A., & Schumacker, R. E. (1996). *Advanced Capacity equation modelling: Issues and techniques*. NJ: Lawrence Erlbaum Associates.
- Meissner, H. (2000, August). *Creativity in Mathematics Education*. Paper presented at the meeting of the International Congress on Mathematics Education, Tokyo, Japan.
- Muthén, L. K., & Muthén, B. O. (1998). *Mplus user's guide*. Los Angeles, CA: Muthén & Muthén.
- Nakakoji, K., Yamamoto, Y., & Ohira, M. (1999). A Framework that Supports Collective Creativity in Design using Visual Images. In E. Edmonds & L. Candy (Eds.), *Proceedings of the 3rd conference on Creativity & Cognition*, pp. 166 – 173. New York: ACM Press. Retrieved from <http://www.informatik.uni-trier.de/~ley/db/conf/candc/candc1999.html>
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Pehkonen, E. (1997). The state-of-art in mathematical creativity, *ZDM*, 29(3), 63-67. doi: 10.1007/s11858-997-0001-z
- Piirto, J. (1999). *Talented children and adult: Their development and education*. New Jersey: Prentice-Hall Inc.
- Sheffield, L. (2009). Developing Mathematical Creativity-Questions may be the answer. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in Mathematics and the Education of Gifted Students* (pp. 87-100). Rotterdam: Sense Publishers.
- Silver, E. A. (1997). Fostering creativity through instruction rich mathematical problem solving and problem posing. *International Reviews on Mathematical Education*, 29(3), 75-80. doi: 10.1007/s11858-997-0003-x
- Starko, J. A. (1994). *Creativity in the classroom*. New York: Longman.
- Sternberg, R. J. (1999). *Handbook of creativity*. New York: Cambridge University Press.
- Torrance, E. P. (1995). *The beyonders' in why fly? A philosophy of creativity*. Norwood, NJ: Ablex.

# UNRAVELING MATHEMATICAL GIFTEDNESS

Katerina Kontoyianni, Maria Kattou, Demetra Pitta-Pantazi, Constantinos Christou

University of Cyprus

*This study purports to develop a multiple criteria identification process for mathematically gifted students, in an effort to clarify the construct of mathematical giftedness. The study was conducted among 359 4th, 5th and 6th grade elementary school students in Cyprus, using four instruments measuring mathematical ability, mathematical creativity, self-perceptions of mathematical behaviour and fluid intelligence. The results revealed that mathematical giftedness can be described in terms of mathematical ability and mathematical creativity. Moreover, the analysis illustrated that although self-perceptions and fluid intelligence do not consist mathematical giftedness, they could predict it. Implications for researchers and teachers are discussed.*

**Keywords:** giftedness, mathematical ability, creativity, self-perceptions, intelligence.

## INTRODUCTION

To meet the challenges of the new millennium, the field of giftedness research has to expand its conceptions with revised models and approaches (Ziegler, 2009). Despite the number of publications that introduced several conceptions of giftedness, only a small number of them have been empirically examined (Stoeger, 2009). Thus, the need for more empirical studies in the future is evident. The field-dependent character of giftedness was pointed out by Csikszentmihalyi (2000). However, prior research in the field of giftedness focused on the examination of general giftedness rather than domain-specific giftedness. As a result, there is limited focus on theoretical models of mathematical giftedness as well as specially designed procedures and instruments for students' identification.

This study attempts to complement the lack of empirical studies in giftedness and the lack of studies, models and identification processes focused on mathematical giftedness. Specifically, the present study aims to investigate the construct of mathematical giftedness in students aged 10-12 years old (4<sup>th</sup>-6<sup>th</sup> grades) and to develop an identification process based on a multiple criteria approach.

## THEORETICAL FRAMEWORK

### Conceptualizations of giftedness

Although IQ was for decades considered as the only and predominant index of giftedness, a major shift was later noted in the research field. Namely, environmental influences were acknowledged resulting to the decrease of the influence of intelligence (Hartas, Lindsay & Muijs, 2008). At the same end, there were studies which have questioned the validity and liability to cultural and social bias of



standardized measures such as IQ (Black, 2001). Despite the criticism, Silverman (2009) claims that instruments with the richest loadings on general intelligence, such as the Wechsler scales, are the most useful for identification for giftedness. Turning away from intelligence as an indicator of giftedness, among the conceptualizations of giftedness that have been proposed over the years, a widely accepted definition was proposed by Renzulli (1978), emphasizing above-average ability and creativity as characteristics of gifted individuals. In the area of mathematics, the relationship between mathematical giftedness and creativity has also been documented by a number of researchers (e.g., Sriraman, 2005).

### **Identification of giftedness**

Due to the lack of conceptual clarity as to the nature of giftedness, identification processes have varied widely. Identifying gifted individuals raises important issues regarding the types of evidence of giftedness and the validity of assessment processes, since gifted children will be provided with opportunities not accessible to others.

For gifted children to be identified, researchers should decide upon their discriminating characteristics and assess their ability in the specific domain, in our case mathematics. In addition to above average mathematical ability and mathematical creativity, researchers suggested that gifted students can also be identified by examining students' learning pace, depth of understanding and interests (Maker, 1982). According to other researchers (Hartas, Lindsay & Muijs, 2008), information about certain students' personality characteristics (e.g., persistence, perseverance, resilience), motivation and interests should be collected. Equally prevalent is the desire to understand students' perceptions with respect to these characteristics and behaviours.

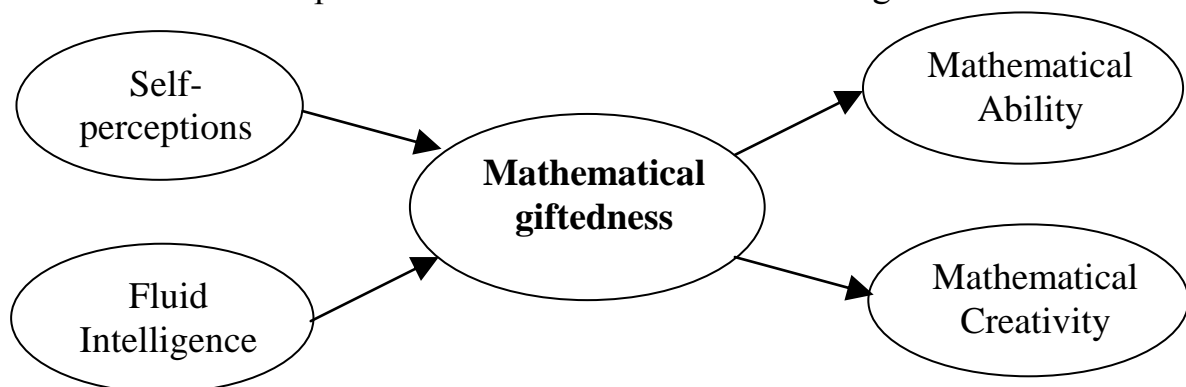
Particularly in the case of mathematically gifted students, researchers should pay attention to mathematical abilities of highly able students and characteristics related to mathematical reasoning. For example, Krutetskii's work (1976) revealed a number of characteristics and abilities that mathematically able children possess: ability for logical thought with respect to quantitative and spatial relationships, number and letter symbols; the ability for rapid and broad generalization of mathematical relations and operations, flexibility of mental processes and mathematical memory. Moreover, a number of characteristics of mathematical giftedness have been proposed by several researchers (Benbow & Minor, 1990; Feldhusen, Hoover, & Sayler, 1991; House, 1987; NCTM, 2000; Olszewski-Kubiliyus, Kulieke, Shaw, Wilhus, & Krasney, 1990; Sriraman, 2005; Stanley, 1993; Wiczerkowski & Prado, 1993), such as high spatial ability, the ability to develop unique relations, produce original, insightful solutions/methods for solutions or formulate imaginative questions and the ability to organise data in such a way to consider patterns or relationships.

Following, this wide variety of characteristics of gifted individuals calls upon the use of a multiple-criteria approach during their identification, employing a combination of valid and reliable tools and multiple sources of evidence (Hoeflinger, 1998). Among other instruments, tests, self-report questionnaires, teacher rating scales, checklists and inventories have been introduced as measures of the identification of giftedness. To sum up, both evaluation of academic performance and cognitive abilities are used (Naglieri & Ford, 2003), despite their conceptual differences in combination with evidence of students' perceptions.

## PURPOSE OF THE STUDY

Having in mind the abovementioned considerations, the purpose of this paper is twofold. Firstly, the study attempts to investigate the construct of mathematical giftedness which comprises of mathematical ability and mathematical creativity. To clarify this concept, the relationship between students' self-perceptions, fluid intelligence and mathematical giftedness is investigated. Secondly, the study purports to develop a valid and reliable identification process for identifying mathematically gifted students based on multiple measures. These measures assess mathematical ability, mathematical creativity, students' self-perceptions with respect to mathematical behaviour and fluid intelligence.

To fulfil the purpose of the study, a theoretical model was a-priori created (see Figure 1). In this model, we hypothesized that mathematical giftedness consists of mathematical ability and creativity. Furthermore, we assumed that self-perceptions regarding students' behavioural characteristics in mathematics and fluid intelligence would contribute to the prediction of students' mathematical giftedness.



**Figure 1: The proposed model.**

## METHODOLOGY

### Sample and instruments

To fulfil the aims of the study, four instruments were administered to 359 students ranging from 9 to 12 years of age; the mathematical abilities instrument, the mathematical creativity instrument, the self-report questionnaire and the fluid intelligence instrument. The mathematical instrument comprised of 29 mathematical

items measuring spatial, quantitative, qualitative, causal and inductive/deductive abilities. The creativity instrument included five open-ended mathematical tasks. The self-report questionnaire consisted of 20 statements describing behaviours with special focus on mathematics. Students responded on a 5-point Likert scale regarding the frequency of each behaviour observed. To measure fluid intelligence, we used the subtest Matrix Reasoning Scale from the Wechsler Abbreviated Scale of Intelligence (WASI) (Wechsler, 1999). The WASI Matrix Reasoning Scale provides a measure of nonverbal fluid abilities using 32 tasks for students of 9 to 11 years old and 35 tasks for students older than 11 years. All instruments were group administered in electronic form except from the WASI Matrix Reasoning Scale which was completed in a hard-copy form.

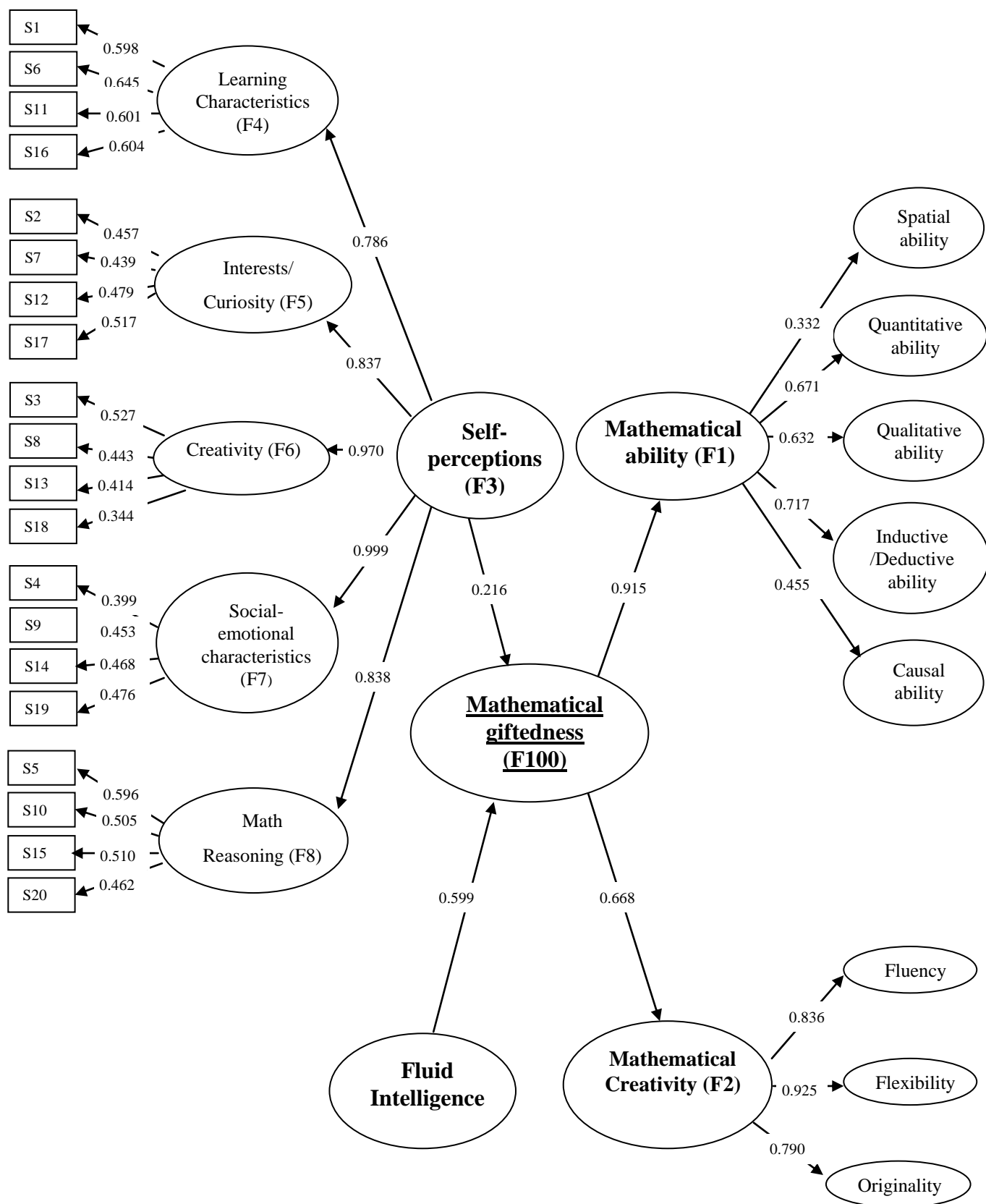
### **Data analysis**

For the analysis of the data confirmatory analysis was employed using the statistical package MPLUS. In this study, confirmatory factor analysis (CFA) was used to investigate whether the proposed model for the identification of mathematically gifted students fits our data. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio ( $\chi^2/df$ ), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). For the model to be confirmed, the values for CFI should be higher than 0.9, the observed values of  $\chi^2/df$  should be less than 2 and the RMSEA values should be close to zero.

### **RESULTS**

For the construct validity of the model to be evaluated, a confirmatory factor analysis (CFA) was employed. CFA showed that all tasks and statements of the four instruments loaded adequately (i.e., they were statistically significant, because the  $z$  values were greater than 1.96) on each factor (see Figure 2). Figure 2, presents the structural equation model with the latent and observed variables and their indicators. CFA also showed that the observed (students' responses to each task and statement) and theoretical factor structures (the components of the theoretical model) matched the data set of the present study and determined the "goodness of fit" of the factor model (CFI=0.923,  $\chi^2=566.627$ ,  $df=366$ ,  $\chi^2/df= 1.626$ , RMSEA=0.039). Therefore, the analysis suggested a model representing distinct components that should be considered during the identification of mathematical giftedness. Thus, (a) spatial abilities, (b) quantitative abilities, (c) qualitative abilities, (d) verbal abilities and (e) causal abilities constitute mathematical abilities, while fluency, flexibility and originality comprise mathematical creativity. Both mathematical abilities and mathematical creativity constitute mathematical giftedness. In addition, the analysis revealed that fluid intelligence and self-perceptions of mathematical behaviour could predict mathematical giftedness. Moreover, with regard to students' self-perceptions about mathematical characteristics the data illustrated that they can be organised

across five dimensions and should be considered during the identification of mathematically gifted students.



**Figure 2: The structure of the proposed model.**

In particular, students' descriptions in regard to their (a) learning characteristics, (b) interests/curiosity, (c) creativity, (d) social-emotional characteristics and (e) mathematical reasoning are important for the identification of mathematically gifted students.

Specifically, the analysis revealed that the five types of mathematical abilities measured by the mathematical instrument constitute one general factor, that of mathematical ability (F1). In particular, the statistically significant loadings of spatial abilities ( $r=.332$ ,  $p<.05$ ), quantitative abilities ( $r=.671$ ,  $p<.05$ ), qualitative abilities ( $r=.632$ ,  $p<.05$ ), inductive/deductive abilities ( $r=.717$ ,  $p<.05$ ), and causal abilities ( $r=.455$ ,  $p<.05$ ) verify that these abilities constitute general mathematical ability. The data suggest that for this age group the quantitative, qualitative and inductive/deductive abilities contribute more than the causal and spatial abilities to mathematical abilities. Likewise, the loadings of fluency ( $r=.836$ ,  $p<.05$ ), flexibility ( $r=.925$ ,  $p<.05$ ) and originality ( $r=.790$ ,  $p<.05$ ) suggest that these three first order factors constitute the second order factor of mathematical creativity (F2). Furthermore, one general factor, that of mathematical giftedness (F100), was generated from mathematical ability ( $r=.915$ ,  $p<.05$ ) and mathematical creativity ( $r=.668$ ,  $p<.05$ ) as shown by their statistically significant loadings.

In addition, students' self-perceptions of their characteristics with regard to mathematics comprise of five factors (F4-F8) with statistically significant loadings; learning characteristics ( $r=.786$ ,  $p<.05$ ), interests/curiosity ( $r=.837$ ,  $p<.05$ ), creativity ( $r=.970$ ,  $p<.05$ ), social-emotional characteristics ( $r=.999$ ,  $p<.05$ ) mathematical reasoning ( $r=.838$ ,  $p<.05$ ). The data suggest that according to students' responses for this age group, characteristics describing social-emotional characteristics and characteristics describing creative behaviours contribute more than learning characteristics, interests/curiosity and mathematical reasoning to their self-perceptions.

The structure of the proposed model also addresses that students' perceptions and fluid intelligence are able to significantly predict students' mathematical giftedness ( $r=.216$ ,  $p<.05$  and  $r=.599$ ,  $p<.05$ , respectively).

## DISCUSSION

Given the controversy prevailing in the field of giftedness, reflected both in the variety of concepts and identification processes proposed, the identification of mathematically gifted students is considered to be extremely challenging (Hoeflinger, 1998). Hence, this study comes to complement the lack of empirical studies in giftedness as well as the lack of studies, models and identification processes focused on mathematical giftedness. Firstly, the study attempts to investigate the construct of mathematical giftedness. Namely, it was expected that the relationship between students' self-perceptions, fluid intelligence and mathematical giftedness, composed by mathematical ability and creativity, would be

clarified. Secondly, the study aims to develop a valid and reliable identification process based on multiple criteria and instruments. This process is focused on the identification of mathematically gifted students at the upper grades of the elementary school.

To fulfil the purpose of the study, a theoretical model was conceived and it was later empirically tested and confirmed. With respect to the first objective, data analysis revealed that mathematical ability can be defined in terms of five abilities; spatial conception (spatial ability), number relationships (quantitative ability), the ability of analogical thought (qualitative ability), experimentation skills (causal ability) and logical reasoning (inductive/deductive abilities), confirming the abilities suggested by Krutetskii (1976) as characteristics of mathematically able children. Mathematical creativity can also be described in terms of fluency, flexibility and originality, as proposed by Torrance (1974). Moreover, the findings showed that mathematical ability and creativity constitute a more general factor, that of mathematical giftedness.

The model extended the literature to include two specific measures that may predict mathematical giftedness, complementing the other two measures, the mathematical ability and the mathematical creativity instruments. Specifically, it was shown that although fluid intelligence and self-perceptions of mathematical behaviour are not components of mathematical giftedness, they could predict it. The finding for the predictive power of self-perceptions about mathematics is in accord with other researchers who claim that self-efficacy of gifted students contribute to the prediction of math performance (Pajares, 1996). Moreover, data analysis revealed that students' responses to the self-report questionnaire can be organized across five distinct factors; (a) learning characteristics, (b) interests/curiosity, (c) creativity, (d) social-emotional characteristics and (e) mathematical reasoning. These characteristics have been mentioned by researchers listing the traits of gifted students (e.g., Hartas, Lindsay & Muijs, 2008; Maker, 1982).

The findings imply that the identification process of mathematically gifted students should include multiple measures in order to capture the variety of characteristics that these students present. Since we aim to identify mathematical and not general giftedness, instruments measuring mathematical ability and mathematical creativity are fundamental. Furthermore, the use of self-report questionnaires such as the one reported in this study and a measure of fluid intelligence could be used as supplementary means to collect information about mathematical giftedness, since they have been found to predict it. In particular, the self-report questionnaire addresses a variety of characteristics focusing in mathematics. Therefore, it may complement the lack of domain-specific identification instruments for gifted students, in this area. At the same time, the fluid intelligence instrument sheds light on the relationship between intelligence and giftedness.

From the theoretical point, the model may inform the controversial concepts of intelligence, giftedness and domain-specific abilities. This model suggests valid evaluation of mathematical giftedness could be made both with processes where general and domain-specific abilities and processes are measured. In practice, the model is expected to facilitate the identification process of mathematically gifted students and to afterwards promote the nourishment of students' mathematical talent. Namely, teachers could identify gifted students through the identification instruments which encompass the components of the theoretical model proposed. Certainly, further research is required in order to introduce and examine additional instruments that might contribute to the identification of mathematical giftedness.

**Acknowledgements:** This work was funded by the Cyprus Research Promotion Foundation (Grant: ANTHROPISTIKES/PAIDI/0308(BE)/13).

## REFERENCES

- Benbow, C. P., & Minor, L. L. (1990). Cognitive profiles of verbally and mathematically precocious students: Implications for identification of the gifted. *Gifted Child Quarterly*, 34(1), 21-26. doi: 10.1177/001698629003400105
- Black, P. (2001) Dreams, strategies and systems. Portraits of assessment past, present and future. *Assessment in Education*, 8(1), 65-85. doi: 10.1080/09695940120033261
- Csikszentmihalyi, M. (2000). *Becoming adult: How teenagers prepare for the world of work.* New York: Basic Books.
- Feldhusen, J.F., Hoover, S.M., & Saylor, M.F. (1991). Identifying and educating gifted students at the secondary level: Purdue Academic Rating Scales. *Sydney: Hawker Brownlow.*
- Hartas, D., Lindsay, G., & Muijs, R. D (2008). Identifying and selecting able students for the NAGTY summer school: emerging issues and future considerations. *High Ability Studies*, 19(1), 5-18. doi: 10.1080/13598130801980265
- Hoeflinger, M. (1998). *Developing mathematically promising students.* Roeper Review, 20(4), 244-247. doi: 10.1080/02783199809553900
- House, P.A. (1987). *Providing Opportunities for mathematically gifted, K-12.* Reston, Virginia: National Council of Teachers of Mathematics.
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren.* Chicago: University of Chicago Press.
- Maker, J. (1982). *Curriculum development for the gifted.* Rockville, MD: Aspen Systems Corporation.
- Marcoulides, G. A., & Schumacker, R. E. (1996). *Advanced Capacity equation modelling: Issues and techniques.* NJ: Lawrence Erlbaum Associates.

## Working Group 7

- Naglieri, J. and Ford, D. (2003) *Addressing underrepresentation of gifted minority children using the Naglieri Nonverbal Ability Test (NNAT)*. *Gifted Child Quarterly* 47(2) , 155-160. doi: 10.1177/001698620304700206
- National Council of Teachers of Mathematics (2000). *Principles and standards for mathematics*. Reston, Virginia: National Council of Teachers of Mathematics.
- Olszewski-Kubiliyus, P., Shaw, B., Kulieke, M.J., Willis, G.B., & Krasney, N. (1990). *Predictors of achievement in mathematics for gifted males and females*. *Gifted Child Quarterly*, 34(2), 64-71. doi: 10.1177/001698629003400203
- Pajares, F. (1996). Self-efficacy beliefs and mathematical problem-solving of gifted students. *Contemporary Educational Psychology*, 21, 325–344. doi: 10.1006/ceps.1996.0025
- Renzulli, J. S. (1978). *What Makes Giftedness? Reexamining a Definition*. *Phi Delta Kappan*, 60(3), 180-184. Retrieved from <http://www.pdkintl.org/kappan/index.htm>
- Silverman, L.K. (1993) A developmental model for counseling the gifted. In L. K. Silverman (Ed.), *Counseling the gifted & talented* (pp. 51–78). Denver: Love..
- Silverman, L.K. (2009). The measurement of giftedness. In L. V. Shavinina (Ed.), *International handbook on giftedness* (pp.947-970). Amsterdam: Springer Science and Business Media.
- Sriraman, B. (2005). *Are giftedness & creativity synonyms in mathematics? An analysis of constructs within the professional and school realms*. *The journal of secondary gifted education*, 17, 20–36. doi: 10.4219/jsge-2005-389
- Stanley, J. C. (1993). Boys and girls who reason well mathematically. In G. Bock & K. Ackrill (Eds.), *The origins and development of high ability* (pp. 119–138). New York: Wiley.
- Stoeger, H. (2009). The history of giftedness research. . In L. V. Shavinina (Ed.), *International handbook on giftedness* (pp.17-38). Amsterdam: Springer Science and Business Media.
- Torrance, E.P. (1974). *Torrance Tests of Creative Thinking*. Bensenville, IL: Scolasting Testing Service.
- Wechsler, D. (1999). *Wechsler Abbreviated Scale of Intelligence*. San Antonio, TX: Psychological Corporation.
- Wiczerkowski, W., & Prado, T.M. (1993). *Programs and strategies for nurturing talents/gifts in mathematics*. In K. Heller, F. Monks & A.H. Passow (Eds.), *International Handbook of Research and Development of Giftedness and Talent* (pp.443-451). Oxford: Pergamon.
- Ziegler, A. (2009). Research on giftedness in the 21<sup>st</sup> century. In L. V. Shavinina (Ed.), *International handbook on giftedness* (pp.1509-1524). Amsterdam: Springer Science and Business Media.



# QUESTIONING ASSUMPTIONS THAT LIMIT THE LEARNING OF FRACTIONS: THE STORY OF TWO FIFTH GRADERS

Andreas O. Kyriakides

European University Cyprus

Dedicated to the memory of my colleague Georgia Kyriakidou

*While standard procedures associated with learning to compute fractions could be quickly and easily memorized, many students appear highly vulnerable in cases they need to explain aspects of knowledge they had not discerned before. In this article, the author provides a window on his own teaching practice and explores the story of two fifth-grade students as they try to work out how to perform fraction subtraction using diagrams. Analysis of data suggests that familiarity with a limited class of examples of computations could be responsible for the constraint of learners' perception of the possible values a fractional difference might take. Implications for the role diagrams could play as tools to occasion students to discover their mathematical potential are discussed in light of these findings.*

**Key words:** Fractions, elementary school mathematics, computations, diagrams

## INTRODUCTION

Learning of fractions has traditionally been one of the most problematic areas in school mathematics. The most comprehensive analyses of fractional meanings stem from the work of Kieren (1988, 1993) and the Rational Number Project (Behr, Harel, Post, & Lesh, 1992; Behr, Harel, Post, & Lesh, 1993; Lesh, Behr, & Post, 1987). This body of research suggests that transition from fractions as operators to fractions as numbers is a major hurdle for young learners. Many have no idea what they are doing when they are combining fractions and, thus, appear highly vulnerable when they forget or misremember an algorithm they had learned to apply but not to re-construct for themselves.

In this paper, I provide a window on my own teaching practice to highlight some of the themes which have struck me as a teacher and researcher of a group of twenty-two fifth-grade Cypriots (10-11 yr.). What is to be recounted is part of an ongoing work studying the complexities of learning to compute fractions as revealed from the use of diagrams. Knowing about diagram use and being able to use that knowledge appropriately is a component of visual literacy, which, in the words of Hortin (1994), is “the ability to understand [read] and use [write] and to think and learn in terms of images” (p. 25).

The story zooms in on two students (Marvin and Mina) as they try to work out how to perform fraction subtraction using rectangular areas. This research is particularly important because is based in a cultural context in which students have poor individual experiences in using diagrams to compute fractions.

## THEORETICAL BACKGROUND

The key theory that has guided and shaped the direction of this work is variation theory. It has developed from and is grounded in phenomenographic research, which accounts for how the same thing or the same situation could be seen, experienced or understood in a limited number of qualitatively different ways (Marton, 1981; Marton & Booth, 1997; Bowden & Marton, 1998). Central to this theory is the object of learning, that is, there is no learning without something being learned. How this something is experienced is constituted by the simultaneous discernment of certain aspects or features of what is experienced. Those aspects of an object that come to the forefront of our awareness are called dimensions of variation.

While variation seems to be an important condition for learning, Runesson (2006) stresses that it is variation in the critical aspects of the object of learning that is significant and not any variation. For students to appreciate or see a mathematics concept in a certain way, it is essential to be at least subliminally aware of what is exemplary about an example: what aspects, what dimensions can change and still the example remains an example of the concept (Mason & Watson, 2005; Watson & Mason, 2005). This hints at the kernel of the notion of *dimensions of possible variation* which according to Johnston-Wilder and Mason (2005) underpins all conceptual development. The adjective possible is added because different people may be aware of different possibilities. Teachers, for instance, are usually aware of aspects which can be varied but have not yet occurred to learners. Johnston-Wilder and Mason (2005) also note that the variation permitted in each dimension might be referred to as *range of permissible change* because learners may think that the range of change is more constrained than is actually the case. Understanding grows when they become aware that the range of permissible change is greater than previously thought.

My sense is that the power of variation theory lies in exemplifying what it is possible to learn, in terms of what could be discerned, and in locating which conditions/aspects are necessary to be present in the learning environment. Worthy underlining is that none of these necessary conditions for learning something should be perceived as absolute since these “are relative to the individuals and the situation. Nor is there a guarantee that the students will learn if the necessary conditions exist” (Runesson, 2005, p. 84).

## METHOD AND METHODOLOGY

Two methodological traditions have contributed to the design of this study: phenomenography and action research. Because the ultimate aim of phenomenographic research was made clear through the analysis of variation theory, I will now elaborate on how action research has also contributed to the development of the methods I used.

Mason (2002) notes that action research has become a label for a form of research with many different interpretations in practice: it “has an enormous literature and a wide range of detailed methods of implementation, whether for socio-political critique or for effecting change in some complex situation” (p. 199). For the purpose of this work, I consider only one of its variants: teacher research.

My personal understanding of teacher research is built upon Ainley’s (1999) perspective. As she explains, assuming the role of the teacher or the role of the researcher simply refers to ways of behaving, perceptions and expectations other people have of that behaviour (Ainley, 1999). The notion of roles is, thus, used “as a deliberate device to recognize and label choices, and to allow me to re-enter experiences imaginatively in order to explore other choices which I could have made” (Ainley, 1999, p. 45).

During the time my teacher research took place, though I followed the fraction-related content outlined in the syllabus, I digressed from the way this was presented in pupil’s textbooks because I did not want standard algorithms to dictate my practice. My desire to avoid a direct exposure of my students to procedural rules along with the low status of diagrams in Cypriot curriculum (Kyriakides, 2009, 2010), led me to implement a visual component. This was ensured by asking learners to produce and partition diagrams as a response to the tasks or by inviting them to comment on given diagrams. I intentionally chose the rectangular area instead of the circular one as a diagrammatic form because drawing equal parts inside a circle is technically difficult. In the words of Ball (1993),

this difficulty makes it harder to determine whether a child intends to divide the circle equally –and just does not know how- or whether the child is even considering the importance of equal parts (p. 180).

Another limitation of the circular model is that it might provide

a source of simple examples for adding and subtracting, maybe even for other numerical operations with fractions, but becomes unwieldy for denominators that are not closely related multiplicatively (Watson & Mason, 2005, p. 95).

Considering the suggestion made by Adams and Sharp (2006) that “classroom research requires an intricate question precisely because the ebb and flow of the classroom is taken into account” (p. 16), I set out to explore the subsequent research question: What are the complexities (if any) of learning fractions as revealed by students’ use of diagrams? The foregoing was an important issue to study because

though in Cypriot culture school mathematics textbooks introduce the concept of fraction with images of partitioned rectangles and circles, they make little or no use of diagrams when they show students the way to compute (Kyriakides, 2010, p. 1003).

Thus, the use of diagrams in the process of coming to understand and learning to calculate with fractions promised to be fruitful in revealing complexities beneath the surface of students' facility, and lack of it, in calculating with fractions.

During the analysis conducted on transcripts of single tasks and classroom discussions, I became aware that there were some students, whose voices were heard more than others across the entire collection of tapes, suggesting that these learners could serve as subjects of individual case studies. I have deliberately chosen Marvin and Mina to be the focus of the current paper because Marvin's sophisticated reasoning, which seemed to rise above others with an evident persistence, in the long run made it possible for Mina to discover and realize her mathematical potential.

Using as analytic frames the variation theory as well as personal reflections on pedagogy, my goal in the current paper is not only to assess critically two fifth graders' growth of understanding but to demonstrate, through their story, how it is possible for any student to question early constraints and experience the extension of personal "example spaces" (Watson & Mason, 2005, p.60). This likelihood is important because it could lead to a shift in perspectives and change of individual's thought with regard to the learning of fractions. As Watson and Mason (2005) explain, "no one has access to all possible elements and features of a potential example space associated with a specific topic. ...example spaces prevalent at a given time may be extended or altered in the future" (p. 60).

## CLASSROOM FINDINGS

In this section, I have chosen to present part of a classroom discussion on subtraction of fractions. The quoted transcript has been intentionally split into Episode A (lines 318-333) and Episode B (lines 413-453). This division is absolutely artificial and it does not imply any lack of succession in terms of time or place. Rather, it is meant simply to organize structurally the data and facilitate the development of commentary later on. The discussion starts with a reference to the example of four sevenths minus one half and then shifts to the example of three sixths minus one half. What counts in both episodes is not only the content and structure of the mathematics itself but also the ways in which it is talked about, perceived and assimilated by the involved learners.

### Episode A

318 Teacher: Now, we will do something different...I would like you all to think what will happen if we have four sevenths minus one half. Who would like to come on the board and show us?

[Bob comes to the board]

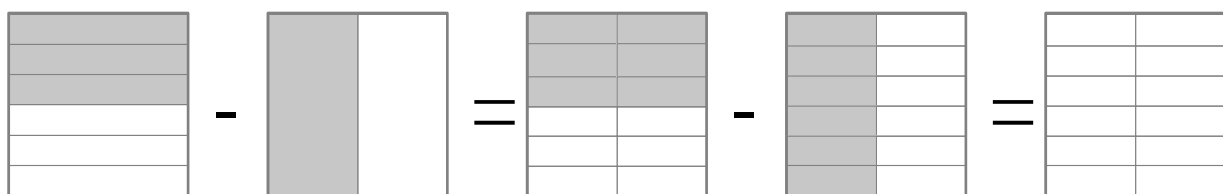
319 Bob: I did one area model and I divided it into seven vertical equal parts and I took the four. Then I drew another area model and I divided it into two vertical parts and took one... [Then Bob erases his second drawing]

## Working Group 7

- 320 Teacher: Why did you erase it?
- 321 Bob: It should have been divided horizontally not vertically because here it is vertical and here it should be horizontal.
- 322 Teacher: Why?
- 323 Bob: To be able to add?
- 324 Teacher: To add?
- 325 Bob: No, to subtract...we should make it the same...you know, to transfer these rows to the columns here and vice versa because if they were the same all columns, for example, I wouldn't be able to see it when I would exchange.
- 326 Marvin: Can I say something sir?
- 327 Teacher: Marvin would like to add something, let's hear.
- 328 Marvin: No sir I don't want to add anything I just want to tell you something in person.
- [I approach Marvin's seat and he whispers to me]
- 329 Marvin: To confuse them sir I have an idea: how about showing them three sixths minus one half?
- 330 Teacher: And why do you think this will confuse the class?
- 331 Marvin: Well, maybe there will be someone who doesn't know.
- 332 Teacher: Will he not know?
- 333 Marvin: That one half for instance is half.

### Episode B

- 413 Teacher: Now, who would like to come up to the board and show us how we can find three sixths minus one half by using rectangular areas?
- [Mina is coming up to the board and developing the first rectangle of Figure<sup>1</sup> 1]
- 425 Mina: I will draw another area model for one half but instead of a horizontal line I will draw a vertical line to divide the area into two equal parts.
- 426 Teacher: Mina could you show us what you are talking about?
- [Mina is developing the rest of Figure 1]



**Figure 1: Mina's diagrams for  $3/6 - 1/2$**

- 427 Mina: I draw a second area model and divide it into two vertical columns and shade one.
- 428 Teacher: Yes and then?
- 429 Mina: Then I take the rows of the first area and shift them to the second and the columns of the second I'm shifting them to the first one.

## Working Group 7

- 430 Teacher: So Mina has exchanged the rows with columns and vice versa. What do we have now Mina?
- 431 Mina: We have the same denominators, 12ths. We have  $\frac{6}{12}$  in the first area model and  $\frac{6}{12}$  in the second.
- 432 Teacher: Can this happen?
- 433 Richard: No.
- 434 Teacher:  $\frac{6}{12}$  minus  $\frac{6}{12}$ , what's the result Mina?
- 435 Marvin: It was my idea, I caused you difficulty.
- [Mina is smiling]
- 436 Mina: Hm...nothing.
- 437 Teacher: That is?
- 438 Mina:  $\frac{0}{12}$ , zero?
- 439 Teacher: So the answer is zero. Well, class I have to say that Marvin asked me to work on this example.
- 449 Mina: Sir in the beginning I felt a bit strange. I thought that the answer was  $\frac{0}{12}$  but I found it strange to be true.
- 450 Teacher: Could you tell us why you found it strange?
- 451 Mina: Hm...because sir I haven't come across such a fraction before.
- 452 Teacher: So what did you learn today?
- 453 Mina: That it happens...it's possible to find zero.

## DISCUSSION

What Marvin proposed confidentially to me in line 329 might have sounded like a jump to his classmates -if they listened to what he had said- or as something disconnected to the ongoing discussion. However, a deliberate attempt to explore the line of thinking that stands behind Marvin's words could suggest that when he saw one half drawn by Bob, Marvin possibly noticed connections between alternative images of one half, predicted how these might be achieved and, in turn, constructed a class-like mental object, including all equivalent fractions occupying half of a whole. Worthy of consideration here is that in Greek language the fraction  $\frac{1}{2}$  is not read as one half but as one second. This cultural clarification is meant to elucidate any triviality or vagueness likely assigned to Marvin's statement that one half is half (line 333).

The first part of the discussion (lines 318-333) explicitly shows that Marvin does not see learning as an absolute event but prefers, instead, to engage himself in the "eventing" (Mason, 2002, p. 228), by increasing the possibilities and potential he recognizes through the structure of his own attention, as he participates in the moment by moment flow of unfolding events. The boy does not only integrate what he hears, but also consciously searches to extend his own and peers' "example space" (Watson & Mason, 2005, p. 60) of the difference of two fractions.

When I posed Marvin's suggestion to the whole class, insightful verbal exchanges (lines 413-453) were brought to the fore. As it stood, the case of zero was not an ordinary difference of two fractions for the rest of his peers. The case of Mina suggests an example. The girl appears competent in drawing and manipulating diagrams to show the fractions three sixths and one half (see Figure 1). Evidence also exists with regard to her fluency in explaining her actions to secure common denominators and thus subtract (lines 413, 425-431). The problematicity lies in the interpretation aspect since Mina is observed (face, gestures, tone of voice) to be somehow hesitant (lines 434-438) to acknowledge the validity of zero as the difference of three sixths minus one half. Her reference to "nothing" (line 436) could suggest, on the one hand, an image of "emptiness" for the concept of zero which "is, quite reasonably but probably unconsciously, influenced by the common language meaning of nothing" (Pirie & Martin, 2000, p. 136). This, in conjunction with her expressed confession (lines 449-453) of what she experienced while standing in front of the board, could serve as evidence of an internal struggle, which in turn gave rise to an awareness that the "range-of-permissible-change" (Watson & Mason, 2005) of the difference of two proper fractions is greater than previously thought (line 453).

An associated implication is that learners make sense of fractional computations from what is available to them. This is frankly implied in Mina's words: "I found it strange to be true (line 449)... because sir I haven't come across such a fraction before (line 451)". In terms of variation theory (Johnston-Wilder & Mason, 2005; Runesson, 2006), familiarity with a limited class of examples of fractional computations could hence be responsible for the constraint of learners' perception of the possible values a fractional sum, difference or product might take. Lakatos (1976) notes that mathematics develops as people question the implicit assumptions that limit the examples offered or that come to mind. The story of Marvin and Mina comes to suggest that placing learners in the situation of board presentation might have its ebbs and flows but it could also occasion individuals who, on their own, may lack the persistence or courage to try something new (Movshovitz-Hadar, 2008), to think creatively.

What has been described here is a new kind of practice of teaching and learning for the participants. Rather than attempt to isolate or emphasize diagrams artificially, I came more and more during this paper to note how diagrams could function naturally with words, numbers and other symbols used on board, or perhaps in children's heads. I found that my students moved among these multiple representations in an interwoven, fluid process to embody and communicate their growing sense of the concept of fractions and the operation of subtraction. This, of course, should neither be regarded as an unproblematic, linear task nor as something that can be certain (Kieren, 1988, 1993). Watching the gradual change in Mina's notion of zero revealed complexities but also allowed me to capture a useful description of resolving challenges as well as celebrating advantages of my students' encounter with

diagrams. The contribution of this paper lies, therefore, in exemplifying the relationship between diagrams and the process of coming to understand fractions, as well as, the roles diagrams could play in promoting individual mathematical creativity and supporting students to question the status of an example.

## NOTES

1. The initial form of Figure 1 was drawn on the classroom's board. Soon after it was made, the teacher-researcher hastily copied it (by hand) on a piece of paper. Afterwards, based on his handwritten rough sketches he constructed a clearer and more accurate copy with the use of technology.

## REFERENCES

- Adams, B., & Sharp, J. (2006). The impact of classroom research on student and teacher learning: Division of fractions. In C. W. Langrall (Ed.), *Teachers engaged in research: Inquiry into mathematics classrooms in grades 3-5* (pp. 13-32). Greenwich, Connecticut: NCTM.
- Ainley, J. (1999). Who are you today? Complementary and conflicting roles in school-based research. *For the Learning of Mathematics*, 19(1), 39-47.
- Ball, D. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-195). Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio and proportion. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.
- Behr, M., Harel, G., Post, T., Lesh, R. (1993). Rational numbers: Toward a semantic analysis - Emphasis on the operator construct. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 13-47). Hillsdale, NJ: Erlbaum.
- Bowden, J., & Marton, F. (1998). *The university of learning. Beyond quality and competence in higher education*. London: Cogan Page.
- Hortin, J. A. (1994). Theoretical foundations of visual literacy. In D. M. Moore & F. M. Dwyer (Eds.), *Visual literacy* (pp. 5-29). Englewood Cliffs, NJ: Educational Technology Publications.
- Johnston-Wilder, S., Mason, J. (2005). *Developing thinking in geometry*. London: Paul Chapman.
- Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Reston, VA: National Council of Teachers of Mathematics.



## Working Group 7

- Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Erlbaum.
- Kyriakides, A. O. (2009). Learning to add fractions: A progression of experiences or an experience of the progression? In J. Houssart & J. Mason (Eds.), *Listening counts: Listening to young learners of mathematics* (pp. 85-100). Stoke on Trent, UK: Trentham Books.
- Kyriakides, A. O. (2010). Engaging everyday language to enhance comprehension of fraction multiplication. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the CERME 6* (pp. 1003-1012). Lyon, France: CERME.
- Lakatos, I. (1976). *Proofs and refutations*. Cambridge, England: Cambridge University Press.
- Lesh, R., Behr, M., & Post, T. (1987). Rational number relations and proportions. In C. Janvier (Ed.), *Problems of representation in teaching and learning of mathematics* (pp. 41-57). Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- Marton, F. (1981). Phenomenography-describing conceptions of the world around us. *Instructional Science*, 10, 177-200.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, NJ: Erlbaum.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.
- Mason, J., & Watson, A. (2005). Mathematical exercises: What is exercised, what is attended to, and how does the structure of the exercises influence these? Paper presented at the 11th Biennial European conference for research on learning and instruction, August 22-27, Nicosia, Cyprus.
- Movshovitz-Hadar, N. (2008). Intellectual courage and mathematical creativity. In R. Leikin (Ed.), *Proceedings of the 5th international conference on creativity in mathematics and the education of gifted students* (pp. 173-185). Tel Aviv, Israel: CET - The center for educational technology.
- Pirie, S. E. B., & Martin, L. (2000). The role of collecting in the growth of mathematical understanding. *Mathematics Education Research Journal*, 12(2), 127-146.
- Runesson, U. (2005). Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, 35(1), 69-87.
- Runesson, U. (2006). What is it possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research*, 50(4), 397-410.
- Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, New Jersey: Lawrence Erlbaum Associates.

# MATHEMATICAL CREATIVITY OF 8<sup>TH</sup> AND 10<sup>TH</sup> GRADE STUDENTS

Roza Leikin and Yona Kloss

University of Haifa, Israel

*One hundred and fifty-eight 8th graders and one hundred and eight 10th graders were asked to solve four multiple solution tasks (MSTs). Creativity is evaluated using Leikin's (2009) model that employs MSTs. Problem solving performance of 8<sup>th</sup> and 10<sup>th</sup> graders is compared with respect to correctness of the solutions, and the three components of creativity: fluency, flexibility and originality. While success in problem solving is highly correlated with fluency and flexibility, originality is shown to be a special mental quality. Consistent with other studies, we demonstrate that originality determines creativity stronger than fluency and flexibility do. This study once again demonstrates the validity of the model for evaluation of mathematical creativity by means of MSTs.*

**Key words:** Mathematical creativity, relative creativity, problem solving, multiple solution tasks.

## EVALUATION OF MATHEMATICAL CREATIVITY IN THIS STUDY

The study presented in this paper is one of a group of studies that evaluate relative mathematical creativity in different groups of school students using Multiple Solution Tasks. A *multiple-solution task (MST)* is an assignment in which a student is explicitly required to solve a mathematical problem in different ways (see the definition and various examples of MSTs in Leikin, 2006, 2009, Leikin & Levav-Waynberg, 2008). The model utilises the notion of *solution spaces* (Leikin, 2007). *Expert solution spaces* include the most complete set of solutions known for a problem at a particular time. They can be conceived as a set of solutions that expert mathematicians can suggest to the problem. These spaces include both *individual solution spaces*, which are collections of solutions produced by an individual to a particular problem, and *collective solution spaces*, which are a combination of the solutions produced by a group of individuals. Solution spaces are used here as a tool for exploring the students' mathematical creativity.

There is no single, authoritative perspective or definition of creativity (Mann, 2006; Sriraman, 2005; Leikin, 2009). There is a diversity of views on creativity and they keep changing over time. Based on research literature, Mann (2006) argues that there are more than 100 contemporary definitions of creativity.

We view creativity as personal creativity that can be developed in schoolchildren. This view requires drawing a distinction between *relative* and *absolute* creativity (Leikin, 2009, p. 151).

*Absolute creativity* is associated with “great historical works” (in Vygotsky’s terms, 1982, 1984), with discoveries at a global level. Examples of absolute creativity may be seen in the inventions of Fermat, Hilbert, Riemann, and other prominent mathematicians (Sriraman, 2005). *Relative creativity* refers to discoveries by a specific person within a specific reference group, to human imagination that creates something new (Vygotsky, 1982, 1984).

Mathematical creativity in school mathematics is usually connected with problem solving or problem posing (e.g., Silver, 1997). Kwon, Park, and Park (2006) proposed two major criteria for mathematical creativity: the creation of new knowledge and flexible problem-solving abilities. Chiu (2009) connected mathematical creativity with the students’ ability to solve routine and non-routine problems and to approach ill-structured problems.

The current study represents uses of a multidimensional creativity scoring scheme that takes into account the relative nature of creativity and links creativity with problem solving. It draws on the views of Ervynck (1991), Krutetskii (1976), Polya (1973), and Silver (1997) that solving mathematical problems in multiple ways is closely related to personal mathematical creativity, and suggests evaluating mathematical creativity by means of Multiple Solution Tasks (MSTs).

The model presented herein (Leikin, 2009) makes possible not only an evaluation of the students’ personal mathematical creativity but also of the efficiency of MSTs in evaluating such creativity (see Figure 2). The model contains operational definitions and a corresponding scoring scheme to evaluate creativity based on three dimensions (originality, fluency, and flexibility), as suggested by Torrance (1974). To evaluate originality it uses Ervynck’s insight-related levels of creativity in combination with the conventionality of the solutions. *Conventional solutions* are usually determined by the curriculum, displayed in textbooks, and usually taught by the teachers, while *unconventional solutions* are based on strategies usually not prescribed by the school curriculum. Conventionality of the solutions refers also to the individual student’s educational history.

*Fluency* (Fl) is detected by the number of solutions in the individual solution space. To evaluate *flexibility* (Flx), we established groups of solutions for the MSTs. Two solutions belong to separate groups if they employ solution strategies based on different representations, properties (theorems, definitions, or auxiliary constructions), or branches of mathematics. With respect to the corresponding solution spaces, we evaluated flexibility as follows (see Figures 1 and 2):  $Flx_1 = 10$  for the first appropriate solution (see an explanation of this scoring in the section on scoring creativity). For each consecutive solution  $Flx_i = 10$  if it belongs to a group of solutions different from the solution(s) performed previously;  $Flx_i = 1$  if the solution belongs to one of the previously used groups but has a perceptible minor distinction;  $Flx_i = 0.1$  if the solution is almost identical with a previous solution. A student’s

*total flexibility score* on a problem is the sum of the student's flexibility on the solutions in his/her individual solution space. We evaluate *originality* ( $Or$ ) in written settings as follows: If  $P$  is the percentage of students in the group that produces a particular solution, then (*relative evaluation*):  $Or_i = 10$ , when  $P < 15\%$  for an insight-based or unconventional solution;  $Or_i = 1$ , when  $15\% \leq P < 40\%$  or for a model-based or partly-unconventional solution;  $Or_i = 0.1$ , when  $P \geq 40\%$ . A student's *total originality score* on a problem is the sum of the student's originality on the solutions in his/her individual solution space. (For the explanation of decimal basis we use in the scoring and the decisions about 15% and 40%, see Leikin, 2009).

The *creativity* ( $Cr$ ) of a particular solution is the product of the solution's originality and flexibility:  $Cr_i = Flx_i \times Or_i$ . Producing flexibly non-original solutions and producing repeatedly original solutions is less creative than flexibly producing (different) original solutions. The total *creativity score* on an MST is the sum of the creativity scores on each solution in the individual solution space of a problem:

$$Cr = \sum_{i=1}^n Flx_i \times Or_i.$$

	Flu-ency	Flexibility	Originality	Creativity
Scores per solution	1	$Flx_1 = 10$ - for the first solution $Flx_i = 10$ - solutions from a different group of strategies $Flx_i = 1$ - similar strategy but a different representation $Flx_i = 0.1$ - the same strategy, the same representation	$Or_i = 10$ $P < 15\%$ or for insight / unconventional solution $Or_i = 1$ $15\% \leq P < 40\%$ or for model-based / partly unconventional solution $Or_i = 0.1$ $P \geq 40\%$ or for algorithm-based / conventional solution	$Flx_i \times Or_i$
Total score	$n$	$Flx = \sum_{i=1}^n Flx_i$	$Or = \sum_{i=1}^n Or_i$	$\sum_{i=1}^n Flx_i \times Or_i$

$n$  is the total number of appropriate solutions

$P = (m_j / n) \cdot 100\%$  where  $m_j$  is the number of students who used strategy  $j$

**Figure 1. Scoring scheme for the evaluation of creativity**

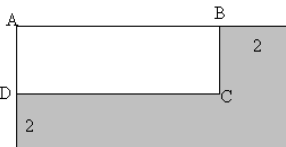
## THE STUDY

### Population

The target population in this study consisted of 108 tenth-grade students and 158 eighth grade students. The students were from families with a similar socio-economic level and similar ability levels. The tenth-grade participants, when they were eighth-graders, learned according to textbooks identical to those with which the eighth-grade participants of the study were learning.

## Tasks in the experiment

The tenth-grade and eighth-grade students were asked to perform identical tests. There were 4 different variants of the tests distributed in different classes. The variants differed in the order of problems and in slight changes in the problems themselves (e.g., different numbers in the system of equations – see below, or a different context for the word problem). Figure 2 depicts tasks in one of the variants.

1. System of equations	$\begin{cases} 3x + 4y = 14 \\ 4y + 3x = 14 \end{cases}$	4. Algebraic expressions	Given: $a + b = 1$ , $a > b$ , What is bigger $a^2 + b$ or $b^2 + a$ ? From Maison (2010)
2. Jam problem	Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. Once, her workers distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute the jam from these jars equally among other jars. She realized that she had added exactly $\frac{1}{4}$ of the previous amount to each one of the jars. How many jars did her workers prepare in the beginning?		
3. Rectangle problem		When the sides of rectangle ABCD were enlarged by 2 cm, its area enlarged by $24 \text{ cm}^2$ . Find the perimeter of rectangle ABCD.	

**Figure 2: The test**

The test was combined from the problems that 8<sup>th</sup> graders had studied before the test was administered. Thus, for 10<sup>th</sup> grades these problems were definitely solvable. The problems were taken from different mathematical topics: Task 1- Systems of linear equations (SE), Task 2 and 3 - word problems: Task 2 – Jam Problem (JP) according to the curriculum could be solved using equations with fractions, Task 3 – Rectangle Problem (RP) could be solved using linear equation, Task 4 – algebraic expression (AE) – required manipulations with algebraic equations using reduced multiplication formulas. Clearly, all the tasks had additional solutions (see Figures 3a, 3b for JP and AE).

The test design also included preliminary evaluation of the creativity embedded in the tasks. In our previous research (Leikin & Lev, 2007; Leikin, 2009) we concluded that the tasks with higher "embedded creativity" are better tools for the comparison of creativity in different groups of students. Thus we searched for those problems in which embedded creativity was higher than 200. Figures 3a and 3b display calculations of the embedded creativity for JP and AE tasks.

Figures 3a and 3b demonstrate that an expert problem solver can receive a higher creativity score on the AE task than on the JP task. However, when school students solve MSTs they rarely receive maximal flexibility and originality scores. In the next section of the paper we describe the problem-solving performance of the participants in our study.

## Working Group 7


<b>Jam problem:</b> Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. Once her workers distributed equally 80 liters of jam among a number of jars. She decided to save 4 jars and to distribute jam from these 4 jars equally among other jars. She realized that she added exactly $\frac{1}{4}$ of the previous amount to each one of the jars. How many jars did her workers prepare in the beginning?					
		Fluency	Flexibility $Flx_i$	Originality $Or_i$	Creativity $Cr_i = Flx_i \times Or_i$
J1.	System of equations in two variables	1	10	0.1	1
J1a	Another system or variations in solving the system	1	0.1	0.1	0.01
J2	Equation in 1 variable	1	1	0.1	0.1
J2.a	Another equation	1	0.1	0.1	0.01
J3	Fractions	1	10	1	10
J3.a	Percents	1	1	1	10
J4	Diagram 	1	10	10	100
J5	Insight Solution	1	10	10	100
	Maximum Total	8	42.2	13.3	221.12

Figure 3a. Scoring creativity embedded in a JP task

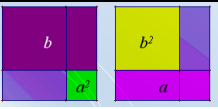
<b>Algebraic expressions:</b> Given: $a + b = 1$ , $a > b$ , Which is bigger $a^2 + b$ or $b^2 + a$ ?					
		Fluency	Flexibility $Flx_i$	Originality $Or_i$	Creativity $Cr_i = Flx_i \times Or_i$
A1	Algebraic manipulation – straightforward $a = 1 - b$	1	10	0.1	1
A3	Diagram 	1	10	10	100
A4	The value of the quadratic expression in several points	1	10	1	10
A2	Algebraic manipulation – reduced multiplication $(a + b)(a - b) = a - b$	1	10	10	100
A5	Symmetry considerations	1	10	10	100
	Maximum Total	5	50	31.1	311

Figure 3b. Scoring creativity embedded in an AE task

## FINDINGS

### Differences in the problem solving performance of 8<sup>th</sup> and 10<sup>th</sup> graders

Tenth-grade students achieved significantly higher scores in both correctness and fluency, on all four tasks in the test, than eighth-grade students (see Tables 1). This finding was not expected since the topics from which the tasks were borrowed are

learned in eighth grade. Tenth-grade students were significantly more flexible than eighth-grade students on three of four tasks (JP, RP and AE tasks). At the same time eighth-grade students were significantly more flexible than tenth-graders when solving the system of equations. Eighth-grade students produced significantly more original and creative solutions than tenth-grade students when solving the JP task whereas tenth-graders were significantly more original and creative than eighth-graders when solving the AE task (see Table 1). No significant differences were found with respect to students' originality and creativity on two of the tasks (SE and RP). However, there were non-significant differences in students' originality and creativity on the RP task, which were similar to significant differences found for another word problem in the test (i.e., JP task).

**Table 1. Problem-solving performance in two groups of students**

Task	N <sub>8</sub> =158 N <sub>10</sub> =108	Correctness		Fluency		Flexibility		Originality		Creativity	
		Range	Mean (SD)	Range	Mean (SD)	Range	Mean (SD)	Range	Mean (SD)	Range	Mean (SD)
1 Sys	8 <sup>th</sup> grade	0-25	23.48 (4.54)	0-5	1.98 (0.87)	0-31	14.4 (5.8)	0-20.1	1.75 (3.5)	0-201	11.20 (29.11)
	10 <sup>th</sup> grade	10-25	24.68 (1.84)	1-5	2.31 (1.0)	10-31	11.99 (3.5)	0.1-21.1	1.28 (3.13)	1-202	10.65 (31.07)
	F	6.843**		7.89**		15.11***		1.25		0.02	
2 JP	8 <sup>th</sup> grade	0-25	11.72 (9.27)	0-3	0.82 (0.54)	0-21	7.39 (4.62)	0-10.2	1.59 (3.41)	0-101	15.89 (34.12)
	10 <sup>th</sup> grade	0-25	16.47 (9.31)	0-3	1.05 (.58)	0-20	8.73 (4.03)	0-10.1	0.59 (1.65)	0-101	5.71 (16.53)
	F	17.06***		10.09**		5.92*		7.99**		8.27**	
3 RP	8 <sup>th</sup> grade	0-25	8.83 (8.45)	0-2	0.59 (0.5)	0-20	5.2 (5.14)	0-11	0.87 (2.44)	0-110	8.61 (24.4)
	10 <sup>th</sup> grade	0-25	19.06 (8.45)	0-2	1.02 (0.45)	0-20	9.15 (3.25)	0-11	0.44 (1.1)	0-110	4.11 (11.0)
	F	74.49***		50.30***		50.09***		2.87		3.23	
4 AE	8 <sup>th</sup> grade	0-25	9.13 (9.27)	0-2	0.7 (0.47)	0-10	5.82 (4.94)	0-10	0.66 (2.18)	0-100	6.62 (21.84)
	10 <sup>th</sup> grade	0-25	17.44 (10.11)	0-3	1.11 (0.68)	0-20	8.88 (5.09)	0-11	1.51 (2.83)	0-110	13.82 (26.98)
	F	50.52***		33.86***		23.88***		7.61***		5.76*	

\*p<0.05; \*\*p<0.01; \*\*\*p<0.001

Whereas tenth-grade students were significantly more successful and fluent when solving all the tasks, the differences between tenth-graders' and eighth-grader' flexibility, originality and creativity appear to be *task dependent* (see Tables 1 and 2). Tenth-grade participants performed significantly better on all the examined criteria when solving AE task. Eighty-three percent of tenth-grade students solved this task successfully; 31% of these students found more than one solution to this problem. At the same time only 69% of eighth grades managed to solve this task, and only 3 of 158 (2%) students managed to solve this problem in 2 ways. We found that, first, the algebraic manipulations that the task required appeared to be difficult for the eighth-graders and, second, they did not connect this problem to the topic of

reduced multiplication formulas. Tenth-graders performed rarer (more original) solutions and used more different strategies (i.e., were more flexible) when solving the task.

Our findings pertaining to students' problem-solving performance on the SE task are different. As on the other tasks tenth-grade students were more successful: 100% tenth-graders vs. 98% eighth-graders produced appropriate solutions to the system of equation. Though students from both groups were more successful when solving this task, the differences between the students from tenth and eighth grades were statistically significant (taking into account the correctness of the solutions). As on all other tasks tenth-graders were more fluent when solving the SE. However, tenth-grade students produced more "repeating" solutions when solving the SE, while eighth-grade students were more flexible: by presenting both algebraic and graphical solutions (e.g, 60 of 158 (38%) eighth-graders solved the system graphically vs. only 10 of 108 (9%) tenth-graders). Higher flexibility of eighth-grade students on the SE task was quite surprising since tenth-grade students have more available tools for solving the system. A symmetry-based solution was produced by 3 (2.7%) tenth-grade students and by 6 (4%) students from 8<sup>th</sup> grade. These solutions contributed to the differences in originality of the solutions which appeared to be non-significantly better for eighth grade students.

### Summary

Although the differences between eighth- and tenth-graders appeared to be significant on many criteria for different problems in the test, the ranges of the scores on all the criteria were similarly low for all the participants. The highest scores received by eighth- and tenth-grade students were equal on all the tasks and all the criteria except fluency and flexibility scores for SE and AE tasks. Only tenth-graders produced 4 and 5 solutions to SE while the solutions varied mainly within the space of algebraic solution strategies. Only 2 tenth-graders produced 3 solutions for the AE task, of which 2 were based on different solution strategies. Moreover, flexibility mean-scores for all the tasks except SE task are lower than 10. This finding reflects the fact that only a small number of students produced more than 1 solution to a particular problem. Additionally, when the students produced several solutions most of them were repetitive.

Based on these findings we argue that Israeli school mathematics teaching is not directed towards the development of mathematical creativity. Both for eighth- and tenth-graders we expected better results with respect to all the criteria presented here. Moreover, while students in tenth grade are more successful and fluent in solving the problems than eighth grade students, this success and fluency can be attributed to the additional tools that students have attained during 2 more years of school. We cannot say that students from tenth grades are consistently more flexible than eighth grade students.



## CORRELATIONS BETWEEN THE DIFFERENT COMPONENTS IN THE MODEL

One of the most interesting findings in this study is related to the examination of correlations (by Pearson correlation coefficient) between different criteria in the scoring scheme on each one of the problems. Table 2 demonstrates correlations received for the four tasks in the test.

**Table 2: Pearson correlation coefficient for correctness and different components of creativity**

N=266		Fluency	Flexibility	Originality	Creativity
SE	Correctness				
	Fluency				
	Flexibility				
	Originality				
JP	Correctness				
	Fluency				
	Flexibility				
	Originality				
RP	Correctness				
	Fluency				
	Flexibility				
	Originality				
AE	Correctness				
	Fluency				
	Flexibility				
	Originality				

\* $p < 0.05$ ; \*\* $p < 0.01$ ; \*\*\* $p < 0.001$

First we find that correlations among the three components of creativity, as presented in the model, are significant and, thus, we consider the model reliable. Second, while correlations between correctness, fluency and flexibility are significant for all the tasks, correlations between originality and correctness are not necessarily significant. Moreover, even when significant, correlations between correctness and originality are low (see Table 2). That is, students who solve mathematical problems in multiple ways more successfully than other students are not necessarily more original. This observation is consistent with the findings that follow a comparison between problem solving performance of eighth and tenth grade students on 3 of 4 tasks: SE, JP and RP.

Correlations between fluency and flexibility are always significant at different levels. For the tasks in this study, the Pearson correlation between fluency and flexibility varies from 0.53 to 0.85. This finding allows us to argue that fluency and flexibility are strongly related to each other, and very reasonably, students who produce more solutions also have a chance to produce more solutions that are different. However,

on some problems fluency does not necessarily lead to flexibility. Both the values of Pearson correlation for SE task (Table 2) and the comparison of tenth-graders' and eighth-graders' problem-solving performance on the SE task support this observation (Table 1).

Interestingly, correlations between originality and flexibility and between creativity and flexibility are significant but low. At the same time, originality and creativity are highly correlated (Table 2). While our model suggests an identical scale for scoring flexibility and originality, and the calculation of creativity includes flexibility and originality symmetrically (Figure 1), originality (in our model) appears to be the main criterion for determining creativity. This finding is consistent with the common view of creativity as production of novel ideas (both on relative and absolute levels), and thus it proves the validity of the model presented in this paper.

## REFERENCES

- Chiu, M.-S. (2009). Approaches to the teaching of creative and non-creative mathematical problems. *International Journal of Science and Mathematics Education*, 7, 55–79.
- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 42–53). Dordrecht, Netherlands: Kluwer.
- Krutetskii, V. A. (1976). The psychology of mathematical abilities in schoolchildren. (J. Teller, Trans., J. Kilpatrick & I. Wirszup, Eds.). Chicago: The University of Chicago Press.
- Kwon, O.-N., Park, J.-S., & Park, J.-H. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Education Review*, 7, 51–61.
- Leikin, R. & Levav-Waynberg, A. (2008). Solution spaces of multiple-solution connecting tasks as a mirror of the development of mathematics teachers' knowledge. *Canadian Journal of Science, Mathematics and Technology Education*, 8(3), 233–251.
- Leikin, R. (2006). About four types of mathematical connections and solving problems in different ways. *Aleh - The (Israeli) High School Mathematics Journal*, 36, 8–14. (In Hebrew)
- Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education - CERME-5* (pp. 2330–2339). (CD-ROM and On-line). Retrieved from <http://ermeweb.free.fr/Cerme5.pdf>

- Leikin, R. (2009). Multiple proof tasks: Teacher practice and teacher education. In the *Proceedings of ICMI Study-19: Proofs and Proving*.
- Leikin, R., & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31<sup>st</sup> International Conference for the Psychology of Mathematics Education* (Vol. 3, pp. 161–168). Korea: The Korea Society of Educational Studies in Mathematics.
- Levav-Waynberg, A., & Leikin, R. (2009). Multiple solutions to a problem: A tool for assessment of mathematical thinking in geometry. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Conference of the European Society for Research in Mathematics Education - CERME-6*. 6 (pp. 776-785). Lyon, France. Available on-line [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)
- Levav-Waynberg, A., & Leikin, R. (under review). Developing creativity and knowledge in geometry through employing multiple solution tasks.
- Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, 30, 236–262.
- Pólya, G. (1973). *How to solve it*. Princeton, NJ: Princeton University.
- Silver, E. A. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM*, 3, 75–80.
- Torrance, E. P. (1974). *Torrance tests of creative thinking*. Bensenville, IL: Scholastic Testing Service.
- Vygotsky, L. S. (1982). Imagination and its development in childhood. In V. V. Davydov (Ed.), *Vol. 2: General Problems of Psychology. The collected works of L. S. Vygotsky* (pp. 438–454). Moscow, SSSR: Pedagogika (In Russian).
- Vygotsky, L. S. (1984). Imagination and creativity in adolescence. In D. B. Elkonin (Ed.), *Vol. 4: Child Psychology. The collected works of L. S. Vygotsky* (pp. 199–219). Moscow, SSSR: Pedagogika (In Russian).

# EMPLOYING MULTIPLE SOLUTION TASKS FOR THE DEVELOPMENT OF MATHEMATICAL CREATIVITY: TWO COMPARATIVE STUDIES

Roza Leikin<sup>(1)</sup>, Anat Levav-Waynberg<sup>(1)</sup>, Raisa Guberman<sup>(1,2)</sup>

<sup>(1)</sup> University of Haifa, Israel

<sup>(2)</sup> Achva Academic College of Education, Israel

*The paper presents two empirical studies that examine employment of multiple solution tasks (MSTs) for the development of creativity. Based on the findings of the two studies we compare development of fluency, flexibility and originality in two different learning environments: a problem-solving workshop for pre-service mathematics teachers and a geometry course for high school students. The two studies compare development of creativity in groups of participants with different attainment levels. Similarities in the findings of these studies led us to several theoretical hypotheses. Students' flexibility and fluency significantly increased in both studies. Students' originality decreased non-significantly. We demonstrate that originality appears to be the strongest component in determining creativity.*

**Key Words:** mathematical creativity, multiple solution tasks, developing creativity

## THEORETICAL BACKGROUND

The studies presented in this paper consider creativity as a dynamic characteristic that can be developed if appropriate learning opportunities are provided to learners (Vygotsky, 1930/1984; Sheffield, 2009; Subotnik et al., 2009; and Yerushalmy, 2009). According to Leikin (2009) we distinguish between absolute and relative creativity as follows: *Absolute creativity* is associated with “great historical works” (in Vygotsky’s 1930/1984 terms), with discoveries at a global level. *Relative creativity* refers to discoveries by a specific person within a specific reference group, to human imagination that creates something new (Vygotsky, 1930/1984). Whereas absolute creativity is quite easy to identify, we lack tools for the identification of relative mathematical creativity.

Our conception of creativity utilizes the views of Ervynck (1991), Krutetskii (1976), Polya (1973) and Torrance (1974). Following the perspective suggested by Torrance (1974) we consider three categories: *Fluency* relates to the continuity of ideas, flow of associations, and use of basic and universal knowledge. If time is unlimited it refers to the pace of solving a procedure and switches between different solutions. If time is limited it refers to the number of solutions generated by a solver. *Flexibility* is associated with changing ideas in producing a variety of solutions and refers to the number of different solutions generated by a solver. *Originality* is characterized by a unique way of thinking and unique products of a mental activity. Additionally, we accept Ervynck's (1991) position that an insight-based solution is an indication of

creativity. Based on these definitions Leikin (2009) proposed a model that evaluates mathematical creativity by means of multiple solution tasks (see also Leikin & Kloss, in the proceedings of this conference, Levav-Waynberg & Leikin, 2009). A multiple-solution task (MST) is an assignment in which a student is explicitly required to solve a mathematical problem in different ways. We use the notion of solution spaces (Leikin, 2007) which enables us to examine the various aspects of problem-solving performance using MSTs. *Expert solution spaces* include the most complete set of solutions to a problem known at a particular time. Expert solution spaces include *conventional* and *unconventional* solution spaces. *Individual solution spaces*, which include all the solutions displayed by an individual, are used here as a tool for exploring the students' mathematical creativity and for assessment of the potential of a task to evaluate mathematical creativity. *Collective solution spaces* include all the solutions produced by a group of students.

We will shortly explain the model and demonstrate its use in the two studies.

## DESCRIPTION OF THE RESEARCH INSTRUMENT

To analyze students' performance on a certain MST, the first step is to construct its expert solution space, which is the collection of solutions proposed by researchers and students. The scoring scheme for the evaluation of creativity of an individual solution space is borrowed from Leikin (2009). The number of all the appropriate solutions in one's individual solution space indicates one's *fluency*. *Flexibility* is measured by the differences between appropriate solutions in one's individual solution space. To carry out the analysis of flexibility, solutions of a specific MST in an expert solution space are divided into groups according to the level of differences between the solutions. The first solution presented in a certain solution group is scored 10 even if it is the only solution in the individual solution space. Additional solutions from the same, already represented, group receive a score of 1 or 0.1, depending on the degree of difference between the two solutions.

The originality of a student's specific solution is measured by how rare its solution group is in the mathematics class to which the student belongs. Solutions from a solution group that recur in more than 40% of the individual solution spaces of a certain class are scored 0.1. Solutions from less frequent solution groups (15%-40%) are scored 1. Solutions are considered to be most original if they belong to groups with a frequency lower than 15%. The originality of these solutions is scored 10. The originality score for the same solution group may vary in different classes as a result of its different rate of recurrence in those classes. The border values of 15% and 40% for different levels of originality were established based on experimentation (Leikin, 2009).

The decimal basis of the flexibility/originality scores makes it easy to interpret the total flexibility/originality score. For example, a total flexibility score of 21.3 means that the individual solution space contains two solutions belonging to different

solution groups, one solution that uses a solution strategy similar to one of the previously used ones but different in some essential characteristics (e.g., representation), and three solutions that are repetitive. If the total originality score is 21.3 for a solution space, this means that the solution space includes two original solutions, one solution that is partly original, and three conventional (non-original) solutions.

The data analysis of the two studies described in this paper employed the model, and we compared development of fluency, flexibility and originality in two different learning environments: a problem-solving course for pre-service mathematics teachers (PMTs) and a geometry course for high school students. Similarities in the findings of these studies have led us to several theoretical hypotheses which will be presented in this paper.

## **STUDY 1: EMPLOYING MSTs IN A PROFESSIONAL DEVELOPMENT COURSE**

This study examines development of problem-solving expertise in PMTs through employment of MSTs at a professional development course (Leikin & Guberman, in preparation). Problem-solving expertise in this study was determined by correctness of PMTs' solutions and their mathematical creativity (as defined above).

### **Setting**

Twenty-seven pre-service mathematics teachers (PMTs) who were studying towards their B.Ed. in elementary school mathematics participated in the research. They took part in the Problem-Solving course during the 3<sup>rd</sup> or 4<sup>th</sup> year of academic studies. One of the researchers conducted a teaching experiment focusing on MSTs.

When coping with MSTs during the course, PMTs were encouraged to compare their individual solution spaces and discuss the quality and aesthetics of different solutions in the collective solution spaces. We analyzed changes in teachers' mathematical creativity and their success in solving MSTs. The pre-test was given to PMTs during the 2<sup>nd</sup> lesson of the 56-hour courses and the post-test during the last lesson of the course. Each test lasted approximately 90 minutes.

We compared the development of problem solving expertise in two groups of PMTs who participated in the course: HA (high achievers) and LA (low achievers). For this purpose we examined PMTs' achievements at three elementary mathematics courses (Arithmetic, Algebra and Euclidian Geometry) which were central to the professional development program in which participants of this study took part. The division of the PMTs into two groups -- HA and LA -- was based on the mean score ( $M_m$ ) they received in the three courses. The HA group included participants with  $M_m$  between 85 and 97, whereas the LA group included participants with  $M_m$  between 70 and 84. This division generated an approximately equal number of PMTs within the groups (13 LA and 14 HA).

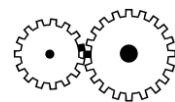
## Mathematical tasks in the study

The problems used in the study were chosen based on the following considerations:

- The tasks are non-conventional with respect to the standard Israeli textbooks while their solutions require knowledge from elementary school mathematics curriculum only. Thus we assumed that study participants were able to approach the problems in at least one way.
- The problems belong to different mathematical topics and require knowledge of different mathematical concepts.
- Each of the problems allows performing at least 2 different solutions.

**Task: Solve the following problem in as many ways as possible**

Two gears, one with 15 teeth and the other one with 20 teeth, fit together as shown in the figure. Each gear has a marked tooth as indicated in the figure. After how many rotations of the gears will the marked tooth be together again for the first time?



**Figure 1: Example of a task used in this study**

## Data analysis

Each individual solution space in the pre- and post-test was scored for correctness, fluency, flexibility, originality and creativity. The tool reliability of the model was confirmed by the significant high correlations between all the criteria. We compared scores for the participants' problem-solving expertise on the pre-test and post-test for all the criteria for each of the two groups – HA and LA. We also compared problem solving performance of HA and LA on the pre-test and post-test. The comparison was performed using T-tests and Repeated Measures ANOVA.

## Findings

**Table 1: Changes in success and creativity – Study 1**

	Success			Fluency			Flexibility			Originality			Creativity		
	HA Mean (SD)	LA Mean (SD)	t	HA Mean (SD)	LA Mean (SD)	t	HA Mean (SD)	LA Mean (SD)	t	HA Mean (SD)	LA Mean (SD)	t	HA Mean (SD)	LA Mean (SD)	t
Pre-test	66.2 (20.2)	62.9 (26.9)	0.35	2.25 (.8)	2.1 (.9)	.57	21.7 (8.3)	21.3 (9.9)	.09	3.3 (4.5)	3.1 (4.7)	1.27	30.6 (45)	38.9 (62.3)	.386
Post-test	100 (0)	86.6 (22.8)	2.26*	6.8 (1.4)	4.7 (1.4)	2.81**	51.1 (12.3)	35.1 (14.7)	3.0**	1.2 (.8)	.5 (.4)	2.70*	48.5 (36)	15.8 (18.3)	3.06**
t	5.8***	3.13**		9.47***	4.2***		5.63***	3.28***		1.53	2.12		.92	1.4	
$\Delta T$	33.8	23.7	F= 1.03	4.55	2.6	F= 5.93*	29.4	13.8	F= 5.54*	-2.1	-2.6	F= .005	17.9	-23.1	F= 2.63

During the course we observed meaningful changes in PMTs' problem-solving expertise. Contrary to pre-test at which PMTs' solutions were non-systematic, mainly based on trial and error strategy, at post-test, when solving the problems, PMTs used

systematic mathematical concepts and rules. Only at post-test PMTs employed a variety of representations, e.g., algebraic expressions, tables, number line, diagrams.

HA and LA began their participation in the course with similarly low results on all the examined criteria: correctness, fluency, flexibility, originality. The differences between these two groups on all the criteria were not significant.

Both HA and LA significantly improved their *correctness, fluency and flexibility* through their participation in the course. Changes in fluency manifested in the number of solutions produced by PMTs to each one of the problems: from 0, 1 or 2 solutions at pre-test to 2 to 8 solutions at post test. The flexibility changed from one type of solution to each of one of the problems on the pre-test to four types of different solutions on the post-test. At the same time, the improvement of correctness, fluency and flexibility in HA was higher than in LA. The differences in the increase of fluency and flexibility between HA and LA were significant. As a result, on the post-test, correctness, fluency and flexibility in HA became significantly higher than in LA. These differences [probably] explain teachers' [mistaken] belief that MSTs are appropriate for implementation with high achievers only. However, as noticed above, our study clearly demonstrates that MSTs are an effective tool for the development of PMTs' problem-solving expertise both in HA and LA.

Contrary to the findings related to fluency and flexibility, originality in both groups of participants decreased non-significantly from pre-test to post-test. We assume that the increase of flexibility caused the decrease in the numbers of original solutions that PMTs were able to produce. The decrease in originality was [non-significantly] lower for HA and, as a result, originality in HA was significantly higher than originality in LA.

Correlation between originality and creativity was higher than correlations between fluency and creativity and between flexibility and creativity. Additionally, the changes in creativity were similar to changes in originality and different from changes in fluency and flexibility. Thus, we hypothesize that originality is the major component that determines creativity.

## **STUDY 2: MSTs IN SCHOOL GEOMETRY**

This study explores systematic implementation of Multiple Solution (Proof) Tasks (MST) in school geometry (Levav-Waynberg, in progress; Levav-Waynberg & Leikin, 2009, submitted). In addition to the development of a mathematical creativity compound of fluency, flexibility and originality, in this study we examine development of the connectedness of students' geometry knowledge as reflected in the multiple proofs that they produce. We compare the changes in geometric creativity and knowledge on high level (HL) students versus regular level (RL) students.

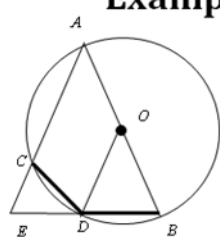


## Setting

Three hundred and three tenth-grade students from 14 instructional groups participated in this study. The classes were divided between experimental and control groups HL and RL. In this paper we address the study involving the experimental group only.

Students in the experimental group were consistently encouraged to perform and discuss various proofs to geometrical statements during a whole school year. The students were given a pre-test (at the beginning of the school year) and a post-test (at the end of the school year). Each test included 2 proof problems and lasted 60 minutes. Students were asked explicitly to perform as many proofs as they could for each one of the problems.

**Example:** **AB-diameter, O-center of the circle**  
 **$OD \parallel AC$ , Prove:  $DB=CD$**



**$OD$  is a midline of the triangle  $EAB$ :  $ED=DB$ . We need to prove  $CD=DB$**

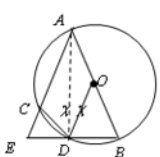
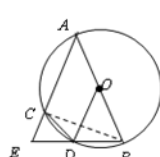
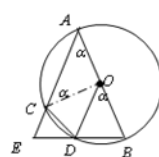

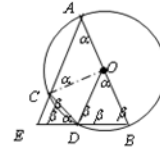
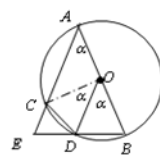
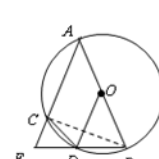
<p><u><b>Proof 2.1</b></u>            Construct – <math>AD</math> – median,  <math>\angle ADB = 90^\circ \Rightarrow</math>  <math>\triangle EAB</math> – isosceles  <math>AD</math> bisect angle <math>EAB</math>  <math>\Rightarrow CD=DB</math></p> 	<p><u><b>Proof 2.2</b></u>            Construct – <math>CB</math>  <math>OD</math> midline in <math>\triangle ACB</math>  <math>\Rightarrow OD</math> bisects <math>CB</math>  <math>\Rightarrow CD=DB</math></p> 	<p><u><b>Proof 2.3</b></u>            Construct – <math>CB</math>  <math>\angle ACB = 90^\circ</math>  <math>OD \parallel AC \Rightarrow OD \perp CB</math>  <math>\Rightarrow CD=DB</math></p> 	<p><u><b>Proof 2.2</b></u>            Construct – <math>OC</math>  <math>\angle DOB = \angle EAO = \alpha</math>  <math>\Rightarrow \angle ACO = \alpha \Rightarrow \angle COB = 2\alpha</math>  <math>\Rightarrow \angle COD = \alpha</math>  <math>\Rightarrow CD=DB</math></p> 
<p><u><b>Proof 2.3</b></u>            As in proof 2.1  <math>EAB</math> – isosceles triangle            As in 2.2 – calculation of the angles <math>\Rightarrow</math>  <math>\triangle CDE</math> is an isosceles triangle</p> 	<p><u><b>Proof 2.7</b></u>            Construct – <math>OC</math>  <math>\angle DOB = \angle EAO = \alpha</math>  <math>\Rightarrow \angle COB = 2\alpha</math> (central angle)  <math>\Rightarrow \angle COD = \alpha \Rightarrow CD=DB</math></p> 	<p><u><b>Proof 2.6</b></u>            Construct – <math>CB</math>  <math>\angle ACB = 90^\circ</math> rests on the diameter  <math>\Rightarrow \angle ECB = 90^\circ</math>; <math>ED=DB</math>  <math>CD</math> – median in the right angle triangle <math>ECB</math>  <math>\Rightarrow CD=DB=ED</math></p> 	

Figure 2: Example of a MST in Study 2

## Data analysis

Each individual solution space was scored for correctness, connectedness, fluency, flexibility and originality. Connectedness was scored according to the number of concepts and theorems in the individual solution space relative to the total number of concepts and theorems in the corresponding expert solution space. Fluency, flexibility, originality and creativity were scored according to the scoring scheme described earlier in this paper. Repeated measures MANOVA was performed in order to compare the development of the different criteria (for the experimental and

control groups of HL and RL). The tool reliability was confirmed by significant correlations between all the criteria. High correlations were found between correctness, connectedness, fluency and flexibility, on the one hand, and between originality and creativity, on the other hand. Originality appeared to be the strongest component in determining creativity

## Findings

**Table 2: Changes in success and creativity – Study 2**

	Success			Fluency			Flexibility			Originality			Creativity		
	HL Mean (SD)	RL Mean (SD)	F(1,165)	HL Mean (SD)	RL Mean (SD)	F(1,165)	HL Mean (SD)	RL Mean (SD)	F(1,165)	HL Mean (SD)	RL Mean (SD)	F(1,165)	HL Mean (SD)	RL Mean (SD)	F(1,165)
Pre-test	83 (23)	67 (30)	45.3 *** (Time)	3 (1.5)	2 (1)	88.1 ***	23 (9)	18 (8)	45.4 ***	2.4 (4.5)	1.2 (2.9)	0.4	55 (129)	19 (52)	0.002
Post-test	96 (14)	88 (21)		4.8 (2.3)	3.1 (1.5)		30 (10)	23 (9)		1.6 (3.5)	1.5 (3.7)		48 (164)	24 (62)	
(Time× group level) F (1,165)	2.8			4.6*			1.1			1.5			0.16		

Students of HL and RL improved significantly the *correctness* of their solutions, without significant differences between the participants of RL and HL. *Connectedness* of students' knowledge improved non-significantly in groups of both HL and RL, but in contrast to the changes in correctness, changes in connectedness were significantly higher in HL than in LA. All the groups improved their *fluency* and *flexibility* over time. These findings also show that fluency and flexibility are strongly based on knowledge. Students of HL demonstrated a significantly higher change in fluency in comparison with the students from RL group.

There was no significant effect of the experimental intervention on originality. For HL students the mean originality score decreased while there was a slight increase in originality for RL students (although this difference between HL and RL was not significant). We explain this phenomenon by the relative nature of the evaluation of originality in our study: when students become more fluent they have less chance to be original.

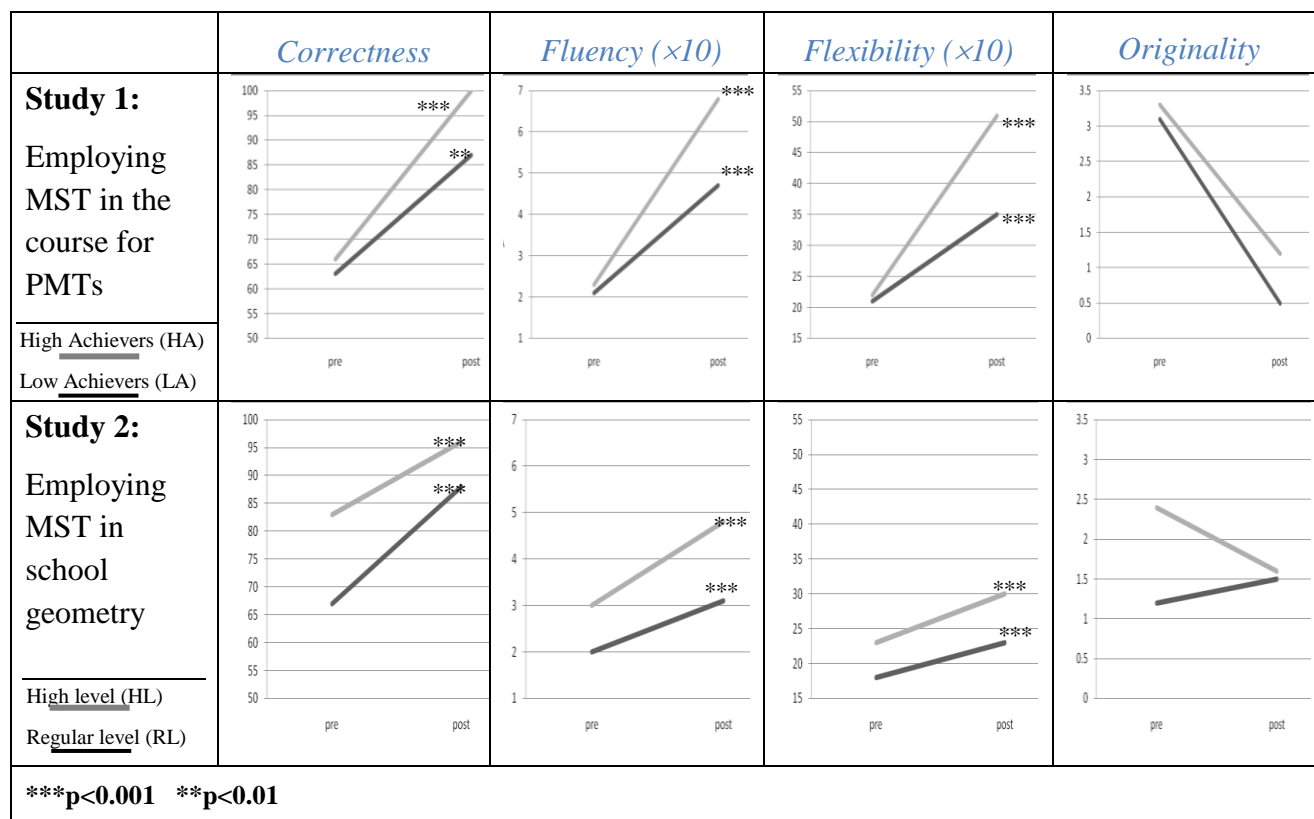
When focusing on the most original solutions produced by the students in this study we found that these solutions were produced by HL students only.

## DISCUSSION

The two studies presented in this paper examined employment of MSTs in two different educational settings. In Study 1, we used MSTs in the problem-solving course for pre-service mathematics teachers (PMTs), comparing development of PMTs' creativity in high achievers (HA) and low achievers (LA). In Study 2 we

implemented MSTs in school geometry; we compared development of geometry knowledge and creativity in high level (HL) and regular level (RL) students. These two studies demonstrate several common phenomena (see Figure 3).

The graphs in Figure 3 show the changes in correctness, fluency, flexibility and originality found in the two studies for the different groups of participants (LA versus HA groups for Study 1 and RL versus HL in Study 2).



**Figure 3: Developing problem-solving expertise by employing MSTs: Two studies**

We found that in both studies:

- Participants at both levels significantly improved their problem solving correctness, fluency and flexibility.
- The improvement in all the criteria was shown for participants of both levels: high level (HA/HL) and lower level (LA/RL).
- The improvement of fluency (in both studies) and flexibility (Study 1) was significantly greater for the high level participants than for participants of lower level. (In Figure 2 this is depicted by a bigger slope for HA/HL than for LA/RL).
- Originality of the solutions decreased non-significantly in high level participants (in Study 1 also in low level participants).

- e. Correlations between the three combined components of creativity (fluency, flexibility and originality) and the creativity outcome are significant.
- f. Correlations between fluency and flexibility as well as between originality and creativity are high.
- g. Correlations between originality and creativity are higher than 0.9

Figure 3 summarizes the changes in all the examined components in the two studies.

The consistency of the findings concerning the increase of fluency and flexibility along with the decrease in originality validates the model. The two studies demonstrate that MSTs are an effective didactical tool. The comparison between control and experimental groups in Study 2, which was not the focus of this paper, strengthened this assertion by the findings that experimental HL group, which systematically employed MSTs, improved significantly higher on its connectedness, fluency and flexibility than the control group.

We hypothesize that in the fluency-flexibility-originality triad, **fluency and flexibility** are of a **dynamic nature**, whereas **originality is a "gift"**. We demonstrate that **originality** appeared to be the strongest component in determining **creativity**.

The strength of the relationship between creativity and originality can be considered as validating our model, being consistent with the view of creativity as an invention of new products or procedures. At the same time, our studies demonstrate that this view is true for both absolute and relative creativity.

Based on the research findings, we hypothesize that, one of the ways of identifying mathematically gifted students is by means of originality of their ideas and solutions. Systematic research should be performed to examine our hypotheses.

Finally, we assume that developing mental flexibility and fluency in and of themselves is of great importance. Probably other types of activities, such as problem posing (which includes elaboration and generalization) and mathematical explorations, are also effective in developing originality, but this assumption requires a different empirical investigation. The model presented in this paper can be useful for such an investigation.

## REFERENCES:

- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42–53). Dordrecht, Netherlands: Kluwer.
- Krutetskii, V. A. (1976). *The psychology of mathematical abilities in schoolchildren*. (Translation from Russian: J. Teller, Trans., J. Kilpatrick & I. Wirszup, Eds.). Chicago: The University of Chicago Press.

- Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. *The Fifth Conference of the European Society for Research in Mathematics Education - CERME-5*. (pp. 2330-2339) (CD-ROM and On-line). Available: <http://ermeweb.free.fr/Cerme5.pdf>
- Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*. (pp. 129-145). Rotterdam, Netherlands: Sense Publishers.
- Leikin & Guberman (in prep). Developing creativity in pre-service mathematics teachers using multiple solution tasks.
- Levav-Waynberg, A. (in prep). Employing multiple solution tasks in school geometry. PhD dissertation, University of Haifa, Israel.
- Levav-Waynberg, A., & Leikin, R. (2009). Multiple solutions to a problem: A tool for assessment of mathematical thinking in geometry. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Conference of the European Society for Research in Mathematics Education - CERME-6*. 6 (pp. 776-785). Lyon, France. Available on-line [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)
- Levav-Waynberg, A., & Leikin, R. (under review). Developing creativity and knowledge in geometry through employing multiple solution tasks.
- Pólya, G. (1973). *How to solve it*. Princeton, NJ: Princeton University.
- Sheffield, L. J. (2009). Developing mathematical creativity – Questions may be the answer. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*. (pp. 87–100). Rotterdam, Netherlands: Sense Publishers.
- Subotnik, R. F., Pillmeier, E. and Jarvin, L. (2009). The psychosocial dimensions of creativity in mathematics: Implications for gifted education policy. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 165-179). Rotterdam, Netherlands: Sense Publishers.
- Torrance, E. P. (1974). *Torrance tests of creative thinking*. Bensenville, IL: Scholastic Testing Service.
- Vygotsky, L. S. (1930/1984). Imagination and creativity in adolescent. In D. B. Elkonin (Ed.), *Vol. 4: Child psychology. The collected works of L. S. Vygotsky* (pp. 199–219). Moscow, SSSR: Pedagogika (In Russian).
- Yerushalmy, M. (2009). Educational Technology and Curricular Design: Promoting Mathematical Creativity for All Students. In R. Leikin, A. Berman & B. Koichu (Eds.), *Mathematical creativity and the education of gifted students* (pp. 101-113). Rotterdam, Netherlands: Sense Publisher.

# MATHEMATICAL CREATIVITY IN ELEMENTARY SCHOOL: IS IT INDIVIDUAL OR COLLECTIVE?

Esther Levenson

Tel Aviv University

*This study combines theories related to collective learning and theories related to mathematical creativity to investigate the notion of collective mathematical creativity. Collective learning takes place when mathematical ideas and actions, initially stemming from an individual, are built upon and reworked, producing a solution which is the product of the collective. Referring to characteristics of individual mathematical creativity, such as fluency, flexibility, and originality, this paper examines the possibility that collective mathematical creativity may be similarly characterized. The paper also explores the possible relationship between individual and collective mathematical creativity.*

**Key words:** mathematical creativity, elementary school mathematics, individual creativity, collective creativity.

Many studies have investigated ways of characterizing, identifying, and promoting mathematical creativity. Haylock (1997), for example, and more recently, Kwon, Park, and Park (2006) assessed students' mathematical creativity by employing open-ended problems and measuring divergent thinking skills. Leikin (2009) explored the use of multiple solution tasks in evaluating a student's mathematical creativity. What many of these studies had in common was that they focused on the mathematical creativity of individuals. This study focuses on the collective, not as the aggregation of a few individuals, but as a unit of study. Although some of the studies mentioned above acknowledged the effect of classroom culture on the development of mathematical creativity, and others considered the creative range of a group of students, these studies did not necessarily investigate mathematical creativity as a collective process or as the product of a collective endeavour.

This study combines theories related to collective learning and theories related to mathematical creativity to investigate the notion of *collective mathematical creativity*. The notion of collective creativity has been used to investigate creativity in several contexts including the work place (Hargadon & Bechky, 2006) and the global community (Family, 2003). In those cases, collective creativity was considered to occur when the social interactions between individuals yielded new interpretations that the individuals involved, thinking alone, could not have generated. Can the notion of collective creativity also be applied to mathematics education?

## THEORETICAL BACKGROUND

Two main issues are at the heart of this study, the nature of mathematical creativity and the collective nature of mathematical learning. This section begins by describing

three key components of mathematical creativity: fluency, flexibility, and originality. These components are mentioned in several studies which view creativity as “an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population” (Silver, 1997, p. 75). The section then describes the collective nature of mathematical learning.

### **Fluency, flexibility, and originality**

Silver (1997) related fluency to “the number of ideas generated in response to a prompt” (p. 76). He claimed that the use of ill-structured and open-ended problems in instruction may encourage students to generate multiple solutions developing fluency. Leikin (2009) referred to fluency as the pace at which solving proceeds and measured the number of solutions produced by a student.

Flexibility, according to Silver (1997) refers to “apparent shifts in approaches taken when generating responses to a prompt” (p. 76). Leikin (2009) evaluated flexibility by establishing if different solutions employ strategies based on different representations, properties, or branches of mathematics. At times, it helps to think of flexibility in relation to its counterpart, fixation. In problem solving, fixation is related to mental rigidity (Haylock, 1997). Flexibility is then shown by overcoming fixation or breaking away from stereotypes. Haylock further differentiated between content-universe fixation and algorithmic fixation. Overcoming the first type of fixation requires the thinker to consider a wider set of possibilities than at first is obvious and extend the range of elements appropriate for application. The second type of fixation relates to when an individual adheres to an initially successful algorithm even when it is no longer appropriate. Originality is related to creating new ideas. For example, the systems model of creativity suggests that when an individual employs the rules and practices of a domain to produce a novel variation within the domain content, then that individual is being creative (Sriraman, 2009). With regard to mathematics classrooms, this aspect of creativity may manifest itself when a student examines many solutions to a problem, methods or answers, and then generates another that is different (Silver, 1997). Leikin (2009) measured the originality of a solution based on its level of insight and conventionality according to the learning history of the participants.

### **The collective nature of mathematical learning**

Different studies have taken into account the social context of mathematical learning. For example, Yackel and Cobb (1996) introduced the notion of sociomathematical norms to describe “normative aspects of mathematical discussions that are specific to students' mathematical activity.” (p. 458). They were interested in how normative aspects of mathematics discussion are developed, such as what counts as mathematically different, mathematically efficient, and mathematically elegant. While these norms are important for guiding interactions between individuals, they are not necessarily indicative of the individuals acting as a collective.

According to Martin, Towers, and Pirie (2006) collective mathematical understanding emerges from coactions. Coactions describe particular mathematical actions carried out by an individual but which are “dependent and contingent upon the actions of the others in the group” (p. 156). Collective understanding does not necessarily occur whenever two or more people collaborate or interact. Instead, “coacting is a process through which mathematical ideas... initially stemming from an individual learner, become taken up, built on, developed, reworked and elaborated by others...” (p. 156).

### **Research aims**

According to Haylock (1997) there are two main approaches to recognizing creativity. The first is to consider the cognitive process which is indicative of creative thinking. Overcoming fixation is one such process. The second approach is to consider the product which indicates creative thinking has taken place. An original solution would be one such product. The first aim of this study is to explore both the process involved in collective mathematical creativity as well as the product of collective mathematical creativity.

The three key components of creativity discussed above, fluency, flexibility, and originality, are sometimes related to the process (Shriki, 2010) and sometimes related to the product (Haylock, 1997). The second aim of this study is to examine if and how the notions of fluency, flexibility, and originality used when describing individual mathematical creativity may also be used to describe collective mathematical creativity. That is, is it possible to speak of collective fluency, collective flexibility, and collective originality?

Finally, although the aim of this study is to explore the emergence of collective mathematical creativity and describe its nature in elementary school mathematics classes, it also acknowledges the significance of individual mathematical creativity. Thus the third aim of this study is to explore the interrelationship between individual mathematical creativity and collective mathematical creativity.

### **METHODOLOGY**

One fifth grade and one sixth grade class participated in this study. The sixth grade teacher had 14 years experience teaching fifth and six grades. The fifth grade teacher had eight years experience teaching fifth and sixth grades. Both teachers taught according to the mandatory mathematics curriculum using state approved textbooks. The teachers did not collaborate with the researcher and were not explicitly implementing a program aimed at promoting creativity. During the school year, each class was observed approximately ten times. During classroom observations, the focus was on students’ interactions with materials, other students, and teachers and the ways in which “ideas are picked up, worked with, and developed by the group”



(Martin, Towers, & Pirie, 2006, p. 152). All lessons were video recorded and transcribed by the researcher who also took field notes during the observations.

## RESULTS

In this section I describe three classroom episodes that illustrate different aspects of mathematical creativity. The first episode was taught by the first teacher. The second and third episodes were taught by the second teacher. For each episode I describe the main aim of the lesson, the classroom interactions, and the mathematical creativity, individual and collective, observed.

### Episode 1: Collective fluency and collective flexibility?

This episode was taken from a sixth grade class, in the middle of the school year, where the main topic of the lesson was multiplication of decimal fractions. The class had already been introduced to this topic and had already practiced the procedure for multiplying decimal fractions during previous lessons. The teacher put the following problem on the board,  $\_ \times \_ = 0.18$ , and asked the class, “What could the missing numbers possibly be?” Many children raise their hands while the teacher comments, “There are many possibilities.” She then calls on one at a time:

Gil: 0.9 times 0.2.

Teacher: Another way. There are many ways.

Lolly: 0.6 times 0.3.

Teacher: More.

Tammy: 0.90 times 0.20.

Teacher: Would you agree with me that 0.2 and 0.9 is the same [as 0.90 and 0.20]? I want different.

Miri: I’m not sure. 9 times 0.02.

Teacher: Nice. Can someone explain what she did?

(The teacher and students then review the rules for multiplying decimal fractions.)

First, we note that although Gil and Lolly gave different answers both answers may be considered similar in that they consisted of two numbers with one digit after the decimal point. The teacher does not acknowledge their similarity. However, when Tammy attempts to break the mould, the teacher does not accept the answer because 0.9 is equal to 0.90. This relates to sociomathematical norms and the establishment of what it means for solutions to be different. In this case, the teacher is establishing that merely expanding a number from 0.9 to 0.90 does not qualify for difference. She then requests a different solution, which is supplied by Miri.

Tom: What about 0.18 times 0.1?

Tad: No.

## Working Group 7

Gad: 0.18 times 1.

Toby: 18 times 0.1?

Many students: 18 times 0.01.

Teacher: Let's move to another problem

This episode illustrates how one child may have the germ of an idea but another child may develop it. In the first vignette, Gil, factoring 18 into 9 and 2, comes up with one solution. Tammy attempts to use the same factors as Gil, but is not successful. Miri then follows up on the idea, producing an additional solution. The same scenario occurs in the second vignette. Tom has the idea of factoring 18 into 1 and 18 but comes up with an incorrect solution as witnessed by Tad. Gad follows up on the idea and comes up with a correct solution. Toby also attempts to find a solution with the same basic factors which is then corrected by other students in the class.

In this episode we also see an illustration of collective fluency which seems to have been promoted by the teacher. The teacher has the class working together to produce many different solutions to the same problem. Up until this point in the lesson, the class came up with five different correct solutions. Perhaps, if more time was available, the class may have produced more solutions.

Regarding flexibility, we note that the second solution,  $0.6 \times 0.3$  followed more or less the same strategy as the first solution  $0.9 \times 0.2$ . The last three solutions,  $9 \times 0.02$ ,  $0.18 \times 1$ , and  $18 \times 0.01$ , may also be considered similar to each other. Each example consists of one factor which is a whole number and a second factor which is a decimal fraction with two digits after the decimal point. Thus, it seems as though this problem promoted collective fluency but not necessarily collective flexibility. Only one student, Miri, was successful in employing a different strategy, a sign of flexibility. And yet, Miri's solution came after Tammy's suggestion. Although the teacher did not accept Tammy's solution as being different from the first, it was Tammy who attempted a solution with two digits after the decimal point. Perhaps, the flexibility manifested in Miri's correct solution was the result of working on Tammy's suggestion. Perhaps, Miri understood what was meant by a solution being different and had the courage to think flexibly after Tammy paved the way. In other words, it is possible that in this case collective flexibility refers to a collective process and not necessarily that the group, as a whole, produced solutions based on different strategies.

### **Episode 2: Individual or collective originality?**

This episode took place in the fifth grade classroom with the second teacher. The students had previously been introduced to decimal fractions, had learned to convert back and forth between decimal fractions and simple fractions, and had recently learned to add and subtract decimal fractions. The main topic of the current lesson

## Working Group 7

was reviewing addition and subtraction of decimal fractions. The following problem, taken from the classroom textbook, was given as a homework assignment, and, at the request of one of the students, was reviewed in class.

Complete the following sequence:  $\frac{5}{100}, \frac{30}{100}, \frac{\quad}{100}, \frac{\quad}{100}, \frac{\quad}{100}$

The following discussion ensues:

Teacher: After 30/100, mmm hundredths, and again, mmm hundredths, and again. They want a sequence. What is a sequence?

Oren: It continues with jumps.

Teacher: Equal jumps. The jumps must be equal. What types of jumps are there?

(A few students say out loud different numbers: 25 and 25/100.)

Teacher: That's the size of the jump. You mean to add 25/100.

Oren: Can the jumps be in multiplication?

Teacher: Wonderful. That's exactly what I mean. If I jump by adding 25/100 then the next will be 30/100. Now, you mentioned another type of jump. We didn't learn that yet...it's part of next year's syllabus. But, there are also multiplication jumps. Who said that going from 5/100 to 30/100 means that I added 25 [hundredths]. I can also multiply...

Sam: By 6.

Sarah: 180.

Teacher: 180 hundredths.

At this point it is worthwhile to note that although multiplication of simple fractions and decimal fractions had not yet been introduced, Oren came up with this idea by himself. Furthermore, although it is technically part of next year's curriculum, the teacher does not dismiss this idea. Finally, two more children contribute to the idea by carrying out the actual multiplication

Oren: That's what I did at first. But I thought it was a mistake.

Teacher: Is that allowed?

Tina: I thought that it would be a mistake.

Noam: But then you get big numbers.

Teacher: So, you can use a calculator.

Note that at least two children thought of multiplying but ended up dismissing the possibility for various reasons. Noam, who claims that he would end up with "big numbers", shows signs of content fixation. This may have been brought on by previous textbook examples which refrained from using 'big' numbers. The teacher

## Working Group 7

is quick to negate this excuse, keeping the door open for additional possibilities. Noam then takes Oren's idea one step further, by considering division.

Noam: Then you can also divide.

Teacher: You can divide, but not here (referring to the jump from  $5/100$  to  $30/100$ ).

Noam: You can multiply by 6 and then divide by 3.

Teacher: Ok. That's also a type of sequence. Multiply by 6, divide by 3, and then again multiply by 6 and divide by 3.

Tina: But, that's not good. You need equal numbers.

Teacher: This is a different type but it is certainly acceptable. Let's try it.

Tina: But, it won't come out. You need equal numbers.

Teacher: Let's just say that when the textbook requests a sequence, they generally don't mean this type. They usually mean jumps that are the same each time. But, this is definitely a sequence.

Tina: But, they are not all equal.

Aaron: You can also have more than two types of jumps. Multiply, divide, and add.

Noam expands on the idea of multiplication and division jumps by including both in the same sequence. This is a novel idea, which the teacher accepts. Yet, the teacher does not succeed in convincing Tina that two types of jumps may also be considered a sequence. While conceding that this type of sequence may not be the norm, the teacher attempts to legitimize thinking that may be outside the norm. At the end of this vignette, Aaron, who has previously remained silent, but apparently has been listening, joins in and adds yet another novel idea.

Looking back, it is apparent that several children displayed original ideas. First, we have Oren. The children had not yet learned multiplication of fractions and yet, he considers the possibility. Then there is Noam who suggested employing both multiplication and division in the same sequence and raises the possibility that the jumps do not have to be equal. Recall that the teacher had previously said that a sequence consists of equal jumps. Finally, Aaron suggests employing three mathematical operations at once in the same sequence. Perhaps, employing multiplication and division may be allowed because they are essentially inverse operations. But to consider addition in the same sequence is indeed novel. So, who displayed original thought? Certainly, Oren, Noam, and Aaron suggested novel ideas. Yet, looking at the sequence of events, it also seems that each student built on the previous student's idea. So, was Oren the only student to display original thinking? Perhaps, taken together, we can say that this is an illustration of collective originality.

### Episode 3: Individual or collective flexibility?

## Working Group 7

This episode is actually a continuation of the previous episode. The teacher presents another problem from the book.

Teacher: Let's look at another problem. Build a sequence that has in it the numbers 0.2 and 1.1.

(Four children raise their hands.)

Aaron: Add 0.9.

Teacher: You're saying to place them next to each other and then the difference is 0.9. Then what would be the next number?

Judy: 2.

Teacher: And then?

Mark: 2.9.

Teacher: But who says that the two numbers have to be next to each other?

Tali: You can do 0.3.

Teacher: Jumps of 0.3. (The teacher writes on the board 0.2, 0.5, 0.8.)

Aaron: You can put the sequence in backwards order and do subtraction.

Teacher: Ok. You can start with 1.1.

In this episode, the class is working on an open-ended or ill-defined task. Unlike the previous task in the previous episode where the first two numbers in the sequence were given, in this task, two numbers are given but are not placed in any specific order. The teacher takes advantage of the situation in order to promote flexibility. In other words, she seems to be less interested in promoting fluency and more interested in trying to encourage the students to think of various ways of placing the numbers. She then raises another suggestion, moving in an entirely different direction than the ones suggested by the students.

Teacher: I have another idea. You can expand the numbers. (The teacher writes on the board 0.20, leaves a lot of space, and then writes 1.10.)

Tomer: 0.9.

Shay: Nine and a half.

Teacher: 0.90 so the expansion is by 10 and then I can do jumps of 0.45. Is that allowed?

Tomer: Yes.

This last part is interesting because it seems that the teacher is also displaying flexible thinking and joining the collective effort to come up with various ways of placing the two numbers in a sequence. If we look at the different solutions to this problem, we may count four solutions where each solution stems from a very

different way of combining the numbers into a sequence. Perhaps, we can call this collective flexibility.

## DISCUSSION

When reviewing the episodes presented in this paper, it is possible to discern both the process and product of collective mathematical creativity. Regarding the collective process, we focus on the interplay of ideas put forth by individuals which is then woven together. When describing the product of collective mathematical creativity, we look not only towards the individual solutions but towards the end product of the collective process. For example, in the first episode, one student put forth an idea, another expanded it, another assessed the correctness of a solution, and the teacher clarified boundaries. The end product consisted of five solutions brought forth by the collective. On the one hand, this is similar to the notion of a collective solution space which is “a combination of the solutions produced by a group of individuals” (Leikin, 2009, p. 134). On the other hand, rather than looking at each solution as the product of one person’s creativity and then merely collecting them together, we look at the five solutions as the product of the collective.

Throughout this paper fluency, flexibility, and originality were used to describe mathematical creativity related to the collective. In the first episode, we reflected on collective fluency as a product of the collective effort and pondered the possibility of discussing collective flexibility. Can we talk about a collective being flexible? Perhaps collective flexibility may refer to the dynamic process by which one student leads the group in one direction which then reminds another student about the possibility of another direction. Together, the group tries out various strategies and possibly produces solutions based on different mathematical properties or different representations. Thus collective flexibility may be used to describe a process as well as a product. Regarding originality, it seems almost far fetched to talk about collective originality. Originality presumes a unique or novel idea. If the idea is unique, is it not individual? Leikin (2009) suggested that originality is relative, that a solution’s novelty is relative to the solutions produced by the group. And yet, although a solution may be unique, it may be the product of a collective process of creativity. Which brings us to the question, what is the relationship between individual and collective mathematical creativity?

Although the focus of this study was on collective mathematical creativity, one cannot help but notice the many instances of individual mathematical creativity. Raising the idea that the jumps of a sequence do not have to be equal, illustrates how one student was able to break away from a stereotype. By focusing on collective mathematical creativity this study does not claim that individual mathematical creativity is not important. On the contrary. As noted by Martin, Towers, & Pirie (2006) a musical performance, such as a jazz improvisation, is highly dependent on the creativity of each individual player. Yet, together, the musicians create a

performance that is more than the sum of its parts. So too, individuals, working on a mathematics problem, may each contribute insights, ideas, and directions building eventually to a collective idea. At the end of the second episode, one student came up with the original idea of creating a sequence that employed three mathematical operations. This idea was not put forth in a vacuum. Before him, there was a student who introduced the idea that a sequence may employ two operations and before him a student who introduced the idea that the jumps in a sequence do not have to be equal. In other words, the students built on each other's ideas coming up with many different solutions as well as some very novel solutions. So, can we say that the final original idea was only that of the one student? Or, might we say that the final original idea was the product of the collective?

In considering the relationship between individual mathematical creativity and collective mathematical creativity, we might also consider affective issues. On one's own, a student may lack the persistence sometimes necessary for creative mathematical thinking or the courage to try something new (Movshovitz-Hadar, 2008). Working as a collective may actually encourage students to keep at it and try new ideas. In other words, by promoting collective mathematical creativity we may also be promoting individual mathematical creativity.

This paper combined two fields of study, collective learning and mathematical creativity, in an initial investigation into the notion of collective mathematical creativity in elementary school classrooms. Many questions remain. What is the teacher's role in this endeavour? Will the same types of tasks used to promote individual mathematical creativity promote collective mathematics creativity as well? What are the ramifications of the age of the students? Additional research is necessary in order to answer these as well as other questions.

## REFERENCES

- Family, G. (2003). Collective creativity: A complex solution for the complex problem of the state of our planet. *Creativity Research Journal*, 15(1), 83-90.
- Hargadon, A. & Bechky, B. (2006). When collections of creatives become creative collectives: A field study of problem solving at work. *Organization Science*, 17(4), 484-500.
- Haylock, D. (1997). Recognizing mathematical creativity in schoolchildren. *ZDM Mathematics Education*, 27(2), 68-74.
- Kwon, O. N., Park, J. S., & Park, J. H. (2006). Cultivating divergent thinking in mathematics through an open-ended approach. *Asia Pacific Education Review*, 7(1), 51-61.
- Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.) *Creativity in mathematics and the education of gifted students* (pp. 129-135), Sense Publishers.

## Working Group 7

- Martin, L., Towers, J., & Pirie, S. (2006). Collective mathematical understanding as improvisation. *Mathematical Thinking and Learning*, 8(2), 149-183.
- Movshovitz-Hadar, N. (2008). Intellectual courage and mathematical creativity. In R. Leikin (Ed.) *Proceedings of the 5<sup>th</sup> International Conference on Creativity in Mathematics and the Education of Gifted Students* (pp. 173-185). Tel Aviv, Israel: CET – The Center for Educational Technology.
- Shriki, A. (2010). Working like real mathematicians: developing prospective teachers' awareness of mathematical creativity through generating new concept. *Educational Studies in Mathematics*, 73, 159-179.
- Silver, E. (1997). Fostering creativity through instruction rich in mathematical problem solving and problem posing. *ZDM Mathematics Education*, 3, 75-80.
- Sriraman, B. (2009). The characteristics of mathematical creativity. *ZDM Mathematics Education*, 41, 13-27.
- Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 22, 390-408.



# DEVELOPING CREATIVE MATHEMATICAL ACTIVITIES: METHOD TRANSFER AND HYPOTHESES' FORMULATION

Bozena Maj

University of Rzeszow

*This paper describes a series of workshops conducted among mathematics teachers. The aim of the workshops was to develop creative mathematical activities among the teachers. The analysis of the transcripts is focused on two kinds of such activities: transfer of a method and hypotheses' formulation and verification. The results show a considerable improvement in the teachers' ability and their attitude towards creative mathematical activities.*

**Keywords:** creativity, hypotheses' formulation and verification, transfer of a method

## INTRODUCTION

Mathematics education should harmonize with the socio-economical need of preparing people to everyday life. In many professions creativity, ingeniousness and a creative attitude to problems are required even from a young person who just entered the field. At the same time, that creative side of education is almost absent at school. Mathematics teaching very often has an imitative and reproductive character. It is focused on elementary activities and skills leading the students to learn schematic behaviours. This is because the teachers are not sufficiently prepared to promote creativity in mathematics among their students. They do not have sufficient knowledge, skills, experience and didactical tools to develop creative mathematical activities among their students (Klakla, 2008; Maj, 2006). The essential condition for the development of the skills needed for different kinds of creative mathematical activities among students is the deep understanding of those issues by the mathematics teachers. Only then the teachers can effectively form and develop these activities in their work with the students. According to Nęcka (2005) "a creative teacher will educate creative pupils, a not much creative teacher will rather discourage pupils from unconventional thinking" (p. 201).

## THEORETICAL BACKGROUND

According to Ervynck (1991) mathematical creativity is the ability of problem solving and/or developing structural thinking by taking into consideration the logical and deductive nature of mathematics. He gives some examples of creativity in mathematics: the ability to formulate a valuable definition by using the concepts which ensure the usefulness of the defined object in a later theory; the mathematization of basic ideas taken from a real context which were initially the base of a mathematical problem. Therefore, according to him mathematical creativity is both the ability of creating mathematical objects and discovering mutual relations between them. Ernest (2008) sees creative mathematical activity as solving

problems, using mathematical methods or performing investigative work; this usually includes the formulation of more complex tasks. It requires a novelty and an insight into the choice of transformations and the choice of the elements of a task that could be used in order to create a sequence of such transformations. For Lakatos (2005) mathematics is not predictable, you cannot create it step by step in a specific direction. You can rather compare it with discovering a new territory during which the ‘trip’ could be pointless. It is similar to mathematical thinking, which contrary to ready-made knowledge is a creative activity which allows the possibility of mistake.

From the above we can conclude that mathematical creativity is related to different kinds of creative mathematical activities. A detailed description of such activities has been worked out by Klakla (2002) who distinguishes seven kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians. These are: (1) hypotheses’ formulation and verification; (2) transfer of a method; (3) creative receiving, processing and using mathematical information; (4) discipline of thinking and critical thinking; (5) problems’ generation in the process of the method transfer; (6) problems’ prolonging; (7) placing the problems in open situations. These activities can be formed through multistage tasks which consist of series of tasks, problems and didactic situations which have a specific structure. They are based on problematic situations and connect different kinds of creative mathematical activities with each other in complex and rich mathematical-didactic situations.

From the previous activities, the transfer of a method, which is described in literature also as ‘analogical transfer’, is for some researchers the main (or even the only) means to solve novel problems (Reeves & Weisberg, 1994; Polya, 1957). In order to undertake the transfer of a method, firstly you have to familiarize with a problem’s solution based on an idea. The analysis of the solution makes you realize the essence of that idea by abstracting any unimportant data or context. This leads to the awareness of the class of tasks for which the idea of the solution will function. For this class that idea becomes an effective method of reasoning. Sometimes the trials of a method’s transfer can be made beyond the class of the tasks, i.e. in similar, analogous or general situations. Then the method requires some modifications and the trials can be successful or not (Klakla, 2002). It is very important to put students in a situation in which the transfer of a method fails.

Another important aspect of creative work in mathematics is formulating and verifying hypotheses. The creative mathematician puts hypotheses in situations when s/he has some premises but does not have a sufficient proof. Such situations may occur in the work of both a mathematician and a student. The difference lies in the cautiousness of the mathematician which is based on previous experiences. By working on some mathematical situations one can perform various empirical investigations: calculations in some concrete cases, experimentations with drawings or reasoning based on analogies. However, accepting that the discoveries are not

enough to state that they are true is not easy for the students. There is a need for formal and conscious evaluating of all stages of reasoning (Klakla, 2002).

## METHODOLOGY

In this paper we present a part of a wider research (Maj, 2009) carried among a group of mathematics teachers. We mainly analyse fragments of the workshops, which show the work of the teachers in the direction of the transfer of a method and hypotheses' formulation and verification.

A group of seven teachers of mathematics (of gymnasium and high schools) took part in a series of workshops from March to September 2006. The workshops were organised as part of the Professional Development of Teachers Researchers (PDTR) project, during the mathematics course. The main content of that course was solving different kind of mathematical problems which were supposed to be challenging for the teachers. The workshops were organised around three multistage tasks. They consisted of solving some chosen tasks – open-ended problems and discussing the possibilities of introducing the students to the particular problems. The main aims of the workshops were:

- developing the skills of undertaking creative mathematical activities among mathematics teachers,
- raising the mathematics teachers' awareness of the need to develop creative mathematical activities among students.

We will present an analysis of four fragments of the workshops; these fragments are related to the first and the second multistage task. The analysis was conducted in the direction of answering the following questions:

- Did the teachers gladly engage in working on multistage tasks (emotional aspect)?
- Did the teachers use the transfer of a method or the transfer of a method with modifications?
- Did they formulate some problems, put questions, hypotheses and did they verify them? Were the formulated problems non-trivial?
- Were discipline of thinking and critical thinking evident in the way of verifying the hypotheses?
- What was the role of the instructor during working on multistage tasks?

The data collected comprised of an audio recording of the workshops (24 lesson hours) and notes. After this, a full transcription of the audio recording and analysis of this transcription was made.

## Example of the multistage tasks used at the workshops

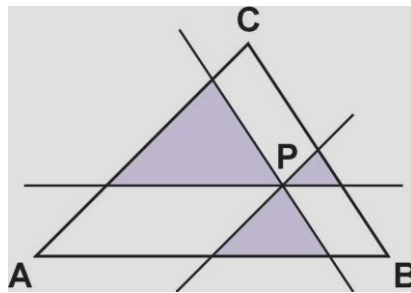
### Task 1 – ‘Lengthening the sides of a triangle’

An acute triangle  $ABC$  is given. Construct the triangle  $A'B'C'$  by lengthening each side of the given triangle  $ABC$  by its own length in the same sense of circulation. Compare the area of the triangles  $A'B'C'$  and  $ABC$ . Give the ratio of their areas.

- What happens with the ratio of the areas when you make 2-times, 3-times,  $n$ -times ( $n \in \mathbb{N}_+$ ) longer the sides of the triangle  $ABC$ ?
- Consider analogous situation for a convex quadrilateral  $ABCD$ .

### Task 2 – ‘Butterfly’

An acute triangle  $ABC$  is given. Through any point  $P$  which belongs to the inside of the triangle, three lines parallel to the every side of the triangle are drawn. The lines divide the triangle  $ABC$  of the area  $S$  into six parts. Three of them are triangles of areas  $S_1, S_2, S_3$ . The figure which is comprised of these three triangles with the common vertex  $P$  is called ‘butterfly’. The three triangles are called the ‘wings’ of the ‘butterfly’ (Figure 1).



**Figure 1. The initial situation of the task ‘Butterfly’.**

Without any suggestions on the direction of enquiry the teachers were asked by the instructor: “What questions would you like to ask to that situation?”.

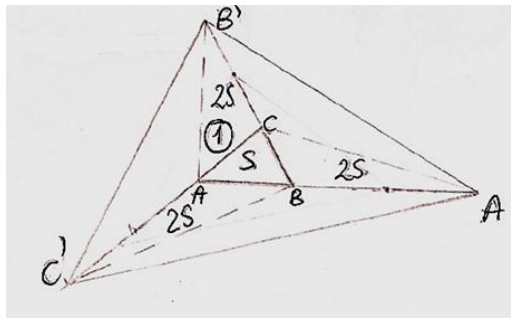
## RESULTS

Four fragments of the workshop will be analysed. The dialogs were realised either between the instructor (I) conducting the workshops and the teachers (T) or between the teachers.

The ‘Lengthening the sides of a triangle’ task was the first to be considered at the workshops. During the work on the task three methods of solving were used: proper ‘cuttings’, use of a particular formula of the triangle’s area and use of triangles of the same areas. Then the teachers had the task to prolong the initial situation by changing the number of lengthening into 2-times, 3-times and  $n$ -times:

- 1 I: So we have the task: we are lengthening 2-times the triangle’s sides...
- 2 T2: Lengthening 2-times ... (on the blackboard):

## Working Group 7



The height is the same ( $ACB'$  and  $ABC$ ) and the base is 3-times longer (the triangle  $ABB'$ ).

- 3 T1: 2-times.
- 4 T4: But it depends on which triangle we are talking about...
- 5 T2: Exactly, maybe we're talking about this one (he is showing the triangle  $ACB'$ ) and here the base is 2-times longer.
- 6 I: So write the area.
- 7 T1:  $2S$ .
- 8 I: And now what?
- 9 T2: Now...
- 10 T3: Where else is  $2S$ ?
- 11 T2: And now here... Here it will be the same (the triangle  $ABC'$ ), won't it be?
- 12 T1: Yes.
- 13 T2: Because there is the base  $AC$  and the height here (he is showing). And the third connection (he is drawing a triangle  $CBA'$ ), here is again the same, I mean  $2S$ . Now, what about these ones? (the other triangles:  $CA'B'$ ,  $AB'C'$ ,  $BC'A'$ ).
- 14 I: Yes, what about it?
- 15 T1, T3: It is 2-times bigger (the triangle  $AB'C'$ ).
- 16 I: When I told you about this student, what I said about this and this? ( $ACB'$  and  $AB'C'$ ) [the story of a 12-year old student who solved that problem]
- 17 T2: I must have missed it... what is happening here? (he is thinking)
- 18 T1: I also... I must have missed it too.
- 19 I: He compared this one with this one ( $ABC$  and  $ACB'$ ) and then this with this one ( $ACB'$  and  $AB'C'$ ).
- 20 T4: And then he turned this situation and there he had the same heights ( $ACB'$  and  $AB'C'$ ).
- 21 T2: So he compared this with this, the heights are the same, so it has to be  $4S$ .
- 22 T1: Once again!
- 23 T2: It will be  $2 \cdot 2S$  – because of that (he is showing on the drawing).
- 24 I: He is now comparing this triangle and this triangle (she is showing).
- 25 T2: The height is the same (showing) and the base is 2-times longer, so the area is 2-times bigger than this one ( $2S$ ), so  $4S$ .
- 26 T1: I see now!

## Working Group 7

27 T2: The same is here 4S and 4S (he is writing the areas for the triangles which are left).

28 I: So the previous ratio was  $1/7$ , what is now?

29 T2: Now we have 3 times 4 and 3 times 2, so 18 plus one, so  $1/19$ .

Comment. In spite of the fact that the third solution ('triangles of the same areas') was introduced by the instructor (by giving some clues) as an alternative solution, the teachers liked that way the most. Probably it was because of its elegance and simplicity. And they chose that method for solving the next problems. Although that method seemed to not require advanced knowledge, when the teachers wanted to transfer it on the analogous task they had difficulties.

Analysis. The example of that task shows that by solving an analogous problem you can develop an important creative mathematical activity – transfer of the method (Klakla, 2002). Although the method was quite easy, using it in the next problem was challenging for the teachers. They started asking questions by themselves, explaining to each other and after such discussions we can say that they really familiarised themselves with this method and understood it. The important fact was that group work gave them the opportunity for such common discussions.

The other prolonging of the task contained a quadrangle instead of a triangle:

30 I: Let's work on the next task: a quadrangle.

The teachers are working silently; they are drawing, cutting the figure into triangles. Then they are discussing in pairs (or triads) trying to use the method of triangles of the same areas:

31 T7: These ones are for sure equal, these also, but I have too many of these Ps...

32 T4 and T1 (they obtained the result  $1/5$  and are happy like children): I have it, hurrey!!!

33 T3 asks T4: How to calculate this?

34 T4: You have to only notice that  $S_2 + S_4 = S_1 + S_3$ .

35 T3: How nicely you wrote it!

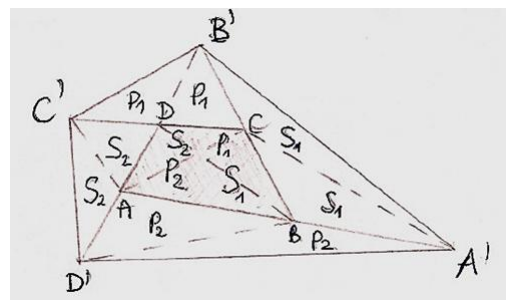
36 T2 (drawing):

37 T7: It can't be in 2, you have to do it in 4 (divide the quadrangle in triangles).

38 T5: You should do it in 4 from the beginning.

39 T1: In 4? Aha! I also did it in 2.

40 T2:  $2P_1 + 2P_2 + 2S_1 + 2S_2 = 2P + 2P$  plus P – that initial one, that is 5P.



Comment. This time the teachers did not have any problems with the transfer of the assumptions of the task. They started working in small groups. Their strategy was to divide the initial quadrangle into triangles and this was done in two ways. In both cases the teachers were searching for triangles of the same areas with these which arose as the result of the division of a triangle. Therefore they modified the previous

method of solving and they transferred it into the new problem. But not all managed to do it so easily (e.g. T7 in [31] or T3 in [33]). After T2 presented his solution on the blackboard a discussion on how many triangles should a quadrangle be divided into started.

Analysis. In this fragment the teachers undertook a creative mathematical activity – transfer of the method of solution on an analogous task (Klakla, 2002). They did it by themselves without any suggestions. They tried to transfer the method which they used in the case of triangles. When it showed that it cannot be transferred directly they modified it in such a way to use it also in the case of a quadrangle. It is worth noting the joy that the solution of this task gave them [32]. It is also important to underline the special role of the instructor who only initiated the problem and then backed out and let the teachers decide by themselves how to solve it.

The next two fragments are related to the second multistage task called ‘Butterflies’:

- 41 I: What more can we ask about that situation?
- 42 T4: Maybe what is the ratio of the painted areas to the non-painted?
- 43 T3: No, if they are similar triangles, we can ask...
- 44 I: Similar triangles... So the wings of the butterfly are similar triangles to ...?
- 45 Everybody: To the initial one.
- 46 I: To the initial one... Are they?
- 47 Everybody: the property (of similar triangles) Angle – Angle – Angle  
(corresponding angles are the same), because they are lines parallel to  
the sides.
- 48 T3: Are they similar to each other?
- 49 I: Exactly, are they?
- 50 T1: If all of them are similar to the initial one, then they are similar to each other.
- 51 I: Yes, and on what base we can conclude it?
- 52 T1 and T3: On transitivity.
- 53 I: Of the relation?
- 54 T1 and T3: Relation of similarity.
- 55 I: Ok. In what scale are they similar to each other? The same scale, different scales?
- 56 T4: Probably it can be done to be the same scale.
- 57 T3: It depends on the point P.
- 58 T6: It depends on the position of the point P.
- 59 I: On the position of the point P... Listen, where would have to be the point P in  
order for those scales to be the same? Is it possible to answer that  
question?

Teachers are discussing...

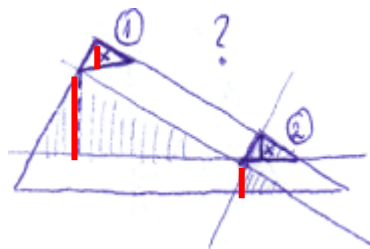
- 60 T3: The centroid of the triangle?
- 61 T4: Maybe in the equilateral triangle?

Comment. After formulating and solving some problems proposed by the teachers, the instructor asked another question [41] that made the teachers put other questions and hypotheses. The first one [42] could be connected with the first multistage task, but the next question of T3 [43] is related to the property noticed on the drawing. Actually, most questions of the teachers were not specified that is why the instructor's interventions were needed. This resulted in discipline of thinking and critical thinking of the teachers (Klakla, 2002).

Analysis. In this fragment the teachers showed an ability of putting hypotheses and the need of verifying them. They had the ideas on what directions can be taken during considering that task. They formulated non-trivial problems, they were engaged in investigation (Ernest, 2008). Their language was characterized by a lack of precision, thus their talk had to be complemented by the instructor. But eventually the teachers started supplementing each other's talk.

In the next fragment the teachers continued the work on the task "Butterfly" by searching another relation in the initial situation:

- 62 T2: T3 has something.  
 63 I: T3, show us what you have.  
 64 T3: But I'm not sure if it is true, it's a question for now...  
 65 T2: Put a hypothesis!  
 66 Everybody: Put a hypothesis!  
 67 T3: But it's not a theorem!  
 68 T2: A bold hypothesis and a good hypothesis is precious by itself.  
 69 T4: I'm not sure if it can be related to the point P – that the sum of the heights of the small triangles equals the height of the big one.  
 70 I: T3, show us what you have.  
 71 T3: I don't know if it is so. For now it is only a conjecture (she draws on the blackboard):



For sure those heights (red lines) give us the height of the triangle ABC. The question is if it is  $x$  – if that height and the height of that triangle (1 and 2) are equal!

Comment. In this fragment we can observe how the workshop participants forced one of them to put a hypothesis. The reaction of the teachers showed us on the one hand the willingness of searching a hypothesis and on the other hand the joy of putting it [68]. It also proved that the participants of the workshop became aware of undertaking mathematical activities and their significance. Putting a hypothesis



resulted immediately in a trial of its verification. T3 knew that the sum of the heights (in red) is equal to the height of the triangle ABC, but she had doubts if the heights of the triangles 1 and 2 had the same length. Although the drawing suggested that it is so, the teacher was careful in formulating the theorem. Her reasoning was characterized by discipline of thinking (Klakla, 2002).

Analysis. The described fragment showed us that the teachers noticed that the work on the task makes sense when you put other hypotheses and try to verify them. The teachers had a need of prolonging the task and this explains the fact that when one of them made an observation, the others forced her to present it and then everybody was working on its verification (in the later part of the workshop). When they finished working on a previous problem they started searching for new relations and new ways of prolonging the task by trying to use the whole potential of the given situation. In the short presented fragment many kinds of creative mathematical activities appeared: putting and verifying hypotheses, prolonging the task and discipline of thinking (Klakla, 2002). Moreover, the work was accompanied by intense emotions.

## CONCLUSIONS

The presented fragments reveal the character of the teachers' and the instructor's work. The first multistage task 'Lengthening the sides of a triangle' had the aim to introduce the teachers to creative mathematical activities. Already during solving that task the first tentative attempts of independent thinking and undertaking different kind of creative mathematical activities appeared. The teachers tried to transfer the method of 'triangles of the same areas' into an analogous problem. This made them aware that firstly you have to familiarize with a problem's solution in order to realize the essence of the idea. Only then you can use that method to solve another problem even if it needs some modification (Klakla, 2002). That leads to noticing the whole potential of a task and developing mathematical methods.

Later, we observed a greater freedom in putting questions, hypotheses' formulation and verification. The teachers decided by themselves what they want to work on, what they want to discover and what relation they want to investigate, which all fit in the frame of mathematical creativity according to Ervynck (1991). The formulated hypotheses were non-trivial; the teachers made a progress in expressing their thoughts, in being more careful during verifying their hypotheses and in transferring their reasoning into analogous problems. Additionally, we could observe the satisfaction that teachers felt when they succeeded in solving the tasks. The role of the instructor was minimalized to the only necessary actions: delicate suggestions of the directions of searching, small numbers of clues, evaluating discipline of thinking and critical thinking of the teachers.

Creative mathematical activities do not develop spontaneously. The conscious attitude of the teacher can be formed through the personal experience of different

forms of such activities. Creativity requires conscious didactical methods and tools on every educational level (Maj, 2009). The multistage tasks used during the workshops support the development of activities such as transfer of a method and hypotheses' formulation and verification (Klakla, 2002; Maj, 2009). In order to transfer a method of solution or reasoning, the teachers have to have the opportunity to work with such series of tasks, try to use the worked out methods, experience the situations in which they cannot be used or the situations in which in order to use them, they have to modified them. Open-ended problems and rich mathematical situations favour the formulation of hypotheses. The actions of the instructor should make the teachers aware of the need for verifying them. At the same time, the participants of the workshops have to feel that they can stand on their own feet and they do not need the help of the instructor any more. That confidence can improve their professional development.

## REFERENCES

- Ernest, P. (2008). Towards a semiotics of mathematical text (Part 1). *For the Learning of Mathematics*, 28 (1), 2-8.
- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 42-53). Dordrecht: Kluwer Academic Publishers.
- Klakla, M. (2002). Kształcenie aktywności matematycznej o charakterze twórczym na poziomie szkoły średniej. In J. Żabowski (Ed.), *Materiały do studiowania dydaktyki matematyki, Tom III. Prace dr Macieja Klakli* (pp. 263-273). Płock: Wydawnictwo Naukowe Novum.
- Klakla, M. (2008). Universal Basic Mathematical Education and Education through Mathematics. *Acta Didactica (University in Nitra)*, 3, 9-30.
- Lakatos, I. (2005). *Dowody i refutacje. Logika odkrycia matematycznego*. Warszawa: Tikkun.
- Maj, B. (2006). Teachers' Beliefs and Competencies of Creative Mathematical Activities. *Proceedings of PME 30* (Vol. 1, p. 292). Prague: PME.
- Maj, B. (2009). *Zadania wieloetapowe w rozwijaniu twórczej aktywności nauczycieli matematyki*. Unpublished doctoral dissertation. Kraków: Pedagogical Academy.
- Nęcka, E. (2005). *Psychologia twórczości*. Gdańsk: Gdańskie Wydawnictwo Psychologiczne.
- Polya, G. (1957). *How to solve it*. Princeton, NJ: Princeton University Press.
- Reeves, L. M., & Weisberg, R. W. (1994). The Role of Content and Abstract Information in Analogical Transfer. *Psychological Bulletin*, 3 (115), 381-400.

# DIDACTICAL VS. MATHEMATICAL MODELLING OF THE NOTION COMPETENCE IN MATHEMATICS EDUCATION: CASE OF 9-10-YEAR OLD PUPILS' PROBLEM SOLVING

Bernard Sarrazy, Jarmila Novotná

Université Victor Segalen Bordeaux 2, Charles University in Prague

*In the paper, the authors investigate experimentally the notion of “competence” applied to mathematical education, more precisely the idea by which teaching cognitive and metacognitive strategies could allow pupils to learn mathematics better. They first identify the psychological model of this perspective and then underline the insufficiency of that approach, showing that pupils’ solving strategies are not independent of situations and solving contexts.*

**Key words:** competence, cognitive strategies, meta-cognitive strategies, didactical situation.

Key competences for lifelong learning are a combination of knowledge, skills and attitudes appropriate to the context. They are particularly necessary for personal fulfilment and development, social inclusion, active citizenship and employment. [...]Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations, with the emphasis being placed on process, activity and knowledge.

Recommendation of EU from December 18, 2006 <sup>22</sup>

## INTRODUCTION

The notion of competence is often used for explaining differences in the levels in mathematics; but the origins of these differences are never explained. Nowadays, the competence is used in relationship with talent for explaining the differences in creativity.

The recent notion of “competence” reintroduces the ancient idea that teaching cognitive and metacognitive strategies will allow pupils to solve problems in a better way and hence to learn mathematics more efficiently. As a result of evolution of this perspective, a didactical model based on a psychological model with origin in information processing was developed. This model focuses on the development of pupil’s mathematical thinking (Novotná & Sarrazy, 2009).

The model supposes that for being efficient, the taught strategies have to be independent of situations and contexts of solving. The aim of this paper is to challenge this hypothesis.

---

<sup>22</sup> [http://europa.eu/legislation\\_summaries/education\\_training\\_youth/lifelong\\_learning/c11090\\_en.htm](http://europa.eu/legislation_summaries/education_training_youth/lifelong_learning/c11090_en.htm)

## THEORETICAL FRAMEWORK AND QUESTIONS

The psychological model referred to in this paper is a theoretical follow-up of works of W. Kintsch, J.G. Greeno (1985), M.-C. Escarabajal (1986) (among other) built on the theory of scheme developed in artificial intelligence: either the scheme is identified and thus allows interpretation of information contained in the assignment (processing from the scheme to its components), or the subject transfers elements from the task to the scheme which allows integrating pieces of information. In this perspective, solving a problem consists of application of a solving scheme. To understand a problem means to make it calculable. In reality, the pupil has to identify useful elements in the wording of the assignment and find a sequence of relationships from the surface characteristics in order to end in a numerical operation. In this point of view, a mistake is considered as selecting an inappropriate scheme or as a “shift of meaning”.

Three invariants structuring psychological models can be found. Our observations suggest they inspire teachers to use various lesson plans (see e.g. Roiné, 2009):

1. *Apply a scheme*: In order to understand a problem the pupil must construct a representation of the problem from the pieces of information that he/she selects and that he/she interprets in a conceptual scheme. He/she must refer to knowledge that allows him/her to apply the solving scheme corresponding with the problem to be solved.
2. *Develop heuristics*: The pupil should be able to check validity of his/her interpretation. When looking for a suitable solution, he/she must develop heuristics: select the operation, control the validity of his/her result, e.g. by using the so called “analogical transfer” of a solving procedure already known to be efficient for a problem with whose solution the pupil is familiar, to the solution of a new problem which he/she is supposed to solve.
3. *Make the pupil become aware of the used procedures*: The teacher must make the pupils realize what procedures of checking have been used. The pupils should also be able to adapt these procedures to use in other contexts.

Even if the principles allow *description* of what “experts” (good pupils) do, nothing guarantees that these pupils will use the procedures efficiently. The idea of teaching competences lies in the implicit principle by which teaching cognitive strategies allows to prefer the transfer of knowledge from one solving context to another. This is the principle that we will call the *principle of transversality*. This principle will be investigated in the following text in the framework of the Theory of Didactical Situations (Brousseau, 1997).

## METHODOLOGY<sup>23</sup>

The experimental protocol is relatively simple: The same task is assigned to 9-10-year old pupils in four different situations. What makes this task unusual/special is the fact that it involves an unusual use of multiplication, not common in school practice. In fact, the pupil has to use his/her knowledge of multiplication to discover that the assigned task does not involve this operation. We call such problems “pseudo-multiplicative”; in the text they are labeled as “PPM”.

The problem can be formulated simply: If the pupil is able to distinguish between PPM and usual problems in a given context, then, according to the principle of transversality (inherent in the psychological model), he/she should manifest it also in other similar contexts. If not, we can conclude that this principle is not valid and therefore it is wrong to expect any improvement if cognitive and metacognitive strategies are taught.

Four PPM were presented in four different school situations:

1. *“Evaluation by the experimenter”*  
If we know that one car covers the distance between the town A and B in 6 hours, how long will I take before 3 cars cover the same distance?
2. *“Mathematical competition”*  
A ship covers the distance between Marseille and New York in 6 days. How long will it take 3 ships to cover the same distance?
3. *“Evaluation by the teacher”*  
A snail is at the bottom of a well. He decides to leave the well. We know that it will take him 6 days to get out of the well. How long will it take three snails to get out of the well?
4. *“Warning”*  
If we know that one biker covers the distance between the towns A and B in 6 hours, how long will it take 3 bikers to cover the same distance if they set off together?

Each of these four target problems was assigned to pupils from seven classes in four different situations ( $N = 155$ ). All these situations are situations of evaluation because in each of them the pupil must demonstrate that he/she is able to carry out what is requested. These situations differ in the degree of analogy to common situations in three aspects: their level of formality, the types of input linked to them (individual vs. collective; the result of evaluation taken into consideration by the teacher or not) and the position of the proposer of the assignment.

However, each of the situations may be described in many more details:

---

<sup>23</sup> Experiments for this research were carried out in France by B. Sarrazy.

- *Situation 1* (“a researcher’s practice”): It is the experimenter who proposes the assignment; the pupils were informed in advance that the test would not be marked; intentionally other information about the test was not disclosed (Will the teacher, the parents be informed about the results? Will the results be communicated publically in the classroom to other pupils? ...)
- *Situation 2* (explicit frame “a mathematical competition between classes”): This test is presented to pupils as a competition between classes in which each class chooses which of the posed problems will be assigned to the other classes. The test was divided into two phases: in the first phases the rules of the competition were presented and then each pupil posed a problem and then submitted it to the experimenter. The test itself was carried out in the second phase, a few days later, and the researcher posed his own problems. Here, the nature is not individual as in the other situation but collective.
- *Situation 3* (explicit frame “summative evaluation at the end of term”): Each teacher was asked to carry out such evaluation that they would normally use in the end of the term. This evaluation should in addition include the target problem. Here, the nature of the activity is clear to pupils: individual and marked by their teacher.
- *Situation 4* called “warning” (frame “experience of a researcher”): The aim of this situation is to verify that pupils are able to correct a “defective” problem assignment. That is why the pupils were informed of presence of both non-calculable and calculable (classical) problems in the given set of problems. Below is an example of the two forms:

Non-calculable form: If we know that one biker covers the distance between the towns A and B in 6 hours, how long will it take 3 bikers to cover the same distance if they set off together?

Calculable form: A decorator paints one wall of a building in 6 hours. How much time will 3 decorators need for painting the same wall if they decide to work together?

Note: In case of all four situations, the target problems were included in a longer set of problems with both unusual problems (e.g. incomplete assignments ...) and classical problems.

## RESULTS AND ANALYSES

Table 1 and Figure 1 present detailed results of the types of answers to PPM (in percentages) for each school level in mathematics in the four situations.

	Good pupils				Average pupils				Weak pupils			
	S1	S2	S3	S4	S1	S2	S3	S4	S1	S2	S3	S4
Exact	20,5	32,6	43,5	91,1	14,3	31,1	33,3	73,8	10,0	20,8	14,6	71,1
No answer	0,0	0,0	8,7	8,9	1,8	1,6	8,3	14,8	2,5	4,2	10,4	13,3
Calculation	79,5	67,4	47,8	0,0	83,9	67,2	58,3	11,5	87,5	75,0	75,0	15,6
N	39	46	46	45	56	61	60	61	40	48	48	45

**Table 1: Frequency (in %) of types of answers to PPM**

- “Exact”: The student provides the answer included in the assignment without doing any calculation
- “No answer”: No answer provided.
- “Calculation”: The pupil multiplies (6x3)

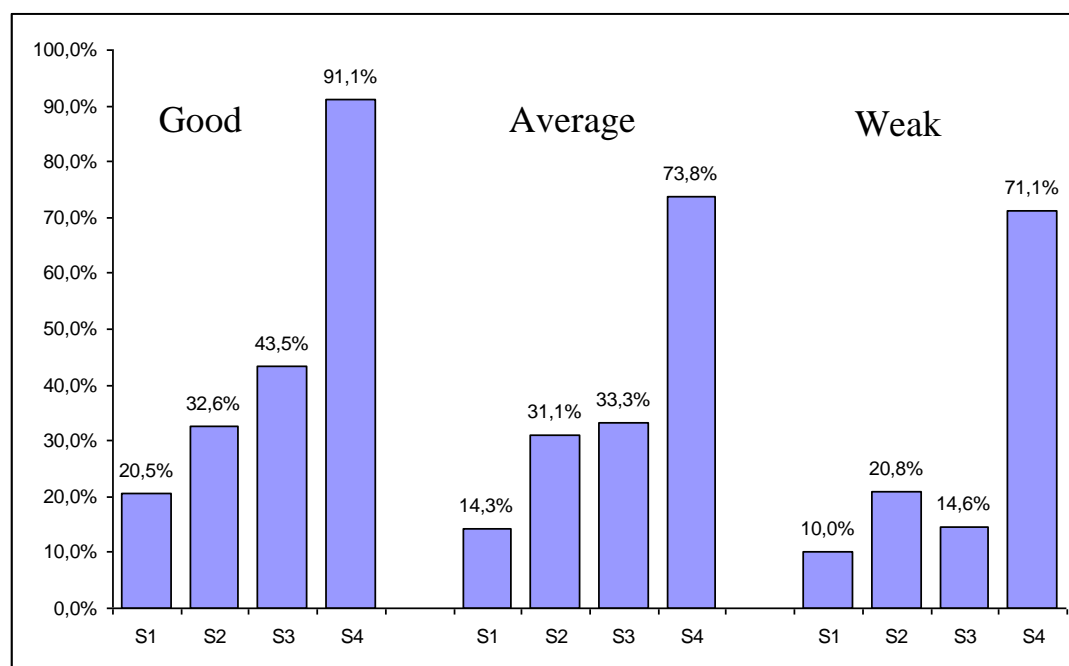


Figure 1: Histogram of success in PPM for each situation and with respect to pupils' age levels

**1) It is clear that in all four situation pupils find PPM difficult. These difficulties on the one hand reveal insufficient mastering of multiplicative structures, but they can also be interpreted as the effect of *didactical contract* (for this notion**

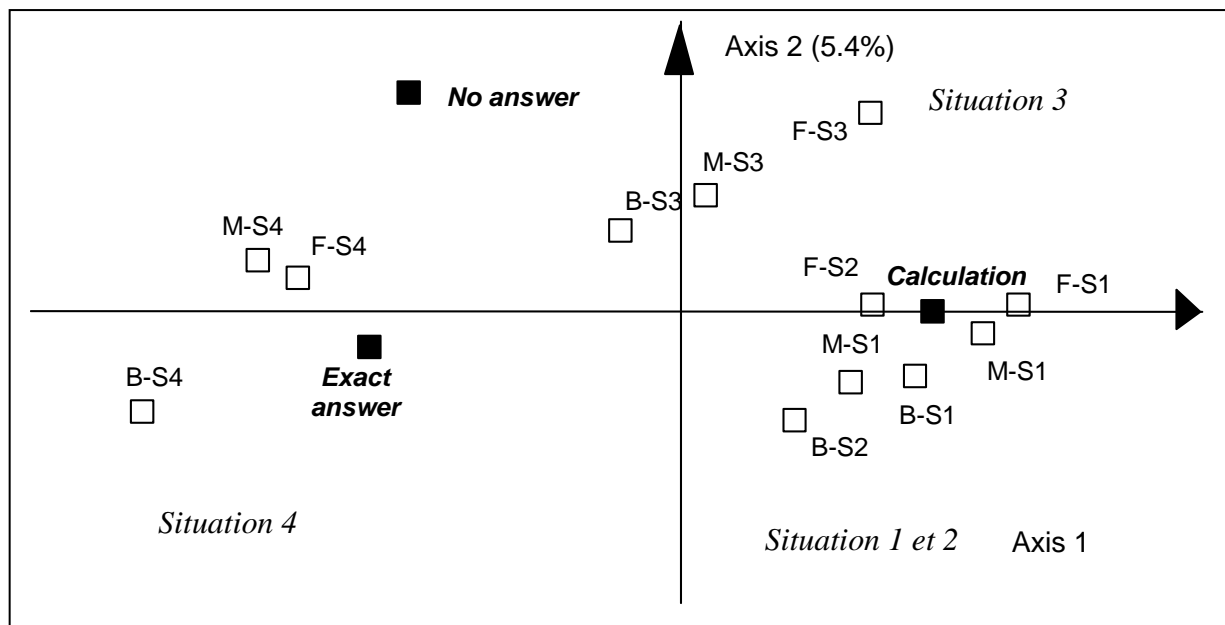
see Brousseau, 1997; Sarrazy & Novotná, 2005) connected to the proximity of classical multiplicative problems; it is highly probable that the problems are caused by both of the factors.

2) The results for situation 4 clearly show that pupils are able to distinguish between “non-calculable” and “calculable” problem ( $\epsilon$ ; s.;  $p < .001$ ).

3) Frequency of success increases with the pupils’ school level ( $\chi^2 = 7,89$ ; s.;  $p < .02$ ); This phenomenon is more evident when the degree of formality increases and when the evaluative nature is strong (as in case of situation 3).

4) Finally, a similarity of profiles of results can be observed in case of all three school levels in all four situations; one can conclude that the difference is a difference in degree but not in situational competences (because the profiles are similar).

These two last results (3 and 4) seem to confirm our hypothesis but in a reverse direction, good pupils get better results regardless of the situation. Factor analysis of correspondences allows us to assess the role the two factors (level of competence and type of situation) play in handling of the task and to examine what the most valid statement is; see Figure 2.



**Figure 2: Simple factor analysis of correspondence. Types of answers to PPM, types of situations and pupils’ school level**

The results of factor analysis of correspondence clearly show that production of answers to the same problem is, regardless of the pupils’ school level, more strongly linked to the situation than to mathematical abilities: in the set of pupils  $\chi^2 = 88,01$ ; s.;  $p < .001$ .



Situation 4 is not typical (because it is a situation where pupils are warned about the possible presence of untypical problems). The analysis verifies that the result holds if we do not consider the three first situations  $\chi^2 = 8.19$ ; s.;  $p. < .02$ .

This result allows to contradict the initial hypothesis that production of an answer to the same task is independent on the solving context. Pupils' decisions are more determined by the type of situation than by their competences in mathematics. In other words, the notion "task", central in cognitive psychology, fails to provide meaningful feedback on the pupil's work. The principle of transversality is not valid here and consequently cannot be classified as one of the constituents of the base for cognitive improvement enabling development of pupils' mathematical thinking.

## CONCLUSION

In specific situations, a considerable proportion of pupils do not find themselves capable of providing an answer that does not require any numerical calculation if they are not warned of this possibility before they start work. Thus, although it was the same assignment, pupils from S1, S2, S3 and S4 did not solve the same problem. Our results show that the reasons are not to be looked for in the pupil's psychology but in the situation that, by the limitations that define it, imposes certain forms of attitudes. The "official" relationship, aimed at this type of the task, is not present in majority of primary school pupils. However, it cannot be concluded that this relationship cannot be established as the results obtained in situation 4 confirm.

This research also highlights the didactical weaknesses of those approaches that come out from a limited conception of the situation (the situation of the system "subject-task"). This definition does not correspond to the observation reported in this paper: the observed variations from one situation to the other – that must not be regarded as simple artefacts – are not comprehensible if the situational dimension of pupils' activities is overlooked.

As one of the authors has already demonstrated (Novotná, 2009), models guided by psychological modelling of the activity often result in the situation when pupils regard problems as mere tools of evaluation, not as an occasion to learn mathematics. Consequently, problems tend to be conceived as tools that enable division of pupils between those who succeed and who fail.

What we propose is to proceed in the opposite direction than described in the previous paragraph by re-centering teacher's and pupils' activities on problems themselves (see Novotná, 2009, Sarrazy, 2002). We need tools that would allow the pupils to develop a living culture of problems and that would change pupils to "experts in problems". Only then will they regard problems as tools necessary for their learning and as an opportunity for mutual visits to some regions of mathematics.

This means we must develop a less didactically narrow relationship towards mathematical problems in pupils, guide them to regard problems as work, and, let us stress this again, not merely as an instrument used by the teacher for instruction and for evaluation. This form of problem knowledge could develop in pupils some kind of mathematical culture that would trigger, among other, a more homogeneous behaviour of classes.

## REFERENCES

- Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*. Kluwer Academic Publishers. 304 p.
- Brousseau, G., & Novotná, J. (2008). La culture scolaire des problèmes de mathématiques. In B. Sarrazy (Ed.), *Les didactiques et leurs rapports à l'enseignement et à la formation. Quel statut épistémologique de leurs modèles et de leurs résultats ?* Bordeaux : AFIRSE, IUFM d'Aquitaine –Université Montesquieu Bordeaux IV, LACES – Université Victor Segalen Bordeaux 2. [CD ROM]
- Escarabajal, M.-Cl. (1986). Utilisation de la notion de schéma dans un modèle de résolution de problèmes additifs. In C. Bonnet, J.-M., Hoc, G. Tiberghien (Eds.), *Psychologie, intelligence artificielle et automatique*. Pierre MARDAGA, Belgique.
- Kintsch, W., & Greeno, J.G. (1985). Understanding and Solving Word Arithmetic Problems. *Psychological Review*, 92, n° 1, 109-129.
- Novotná, J. (2009). Contribution à l'étude de la culture scolaire. Cas de la résolution de problèmes dans l'enseignement des mathématiques. In *Proceedings CIEAEM 61*. Montreal, Canada. In print.
- Novotná, J., & Sarrazy, B. (2009). Mathematical and psychological modelling in mathematics education. In J. Novotná, & H. Moraová (Eds.), *Proceedings of International Symposium Elementary Mathematics Teaching SEMT 09* (pp. 190-197). Praha, UK-PedF.
- Roiné, C. (2009). Effets de cécité didactique des discours noosphériens dans les pratiques d'enseignement en S.E.G.P.A.: contribution à la question des inégalités. Doctoral thesis s/d Pr B. Sarrazy. Université Victor Segalen Bordeaux 2.
- Sarrazy, B. (1995). Le contrat didactique. *Revue Française de Pédagogie*, 112, 85-118.
- Sarrazy, B. (2002). Effects of variability on responsiveness to the didactic contract in problem-solving among pupils of 9-10 years. *European Journal of Psychology of Education*, vol. XVII. n° 4, 321-341.
- Sarrazy, B., & Novotná, J. (2005). Didactical contract: Theoretical frame for the analysis of phenomena of teaching mathematics. In J. Novotná (Ed.), *Proceedings SEMT 05* (pp. 33-24.) Prague: Charles University, Faculty of Education.

**Acknowledgement:** This research was supported by the project 2-09-04 in the programme Partenariat Hubert-Curien (PHC) Barrande 2010.

# PROBLEM POSING AND MODIFICATION AS A CRITERION OF MATHEMATICAL CREATIVITY

Florence Mihaela Singer

University of Ploiesti  
Romania

Ildikó Pelczer

École Polytechnique  
Canada

Cristian Voica

University of Bucharest  
Romania

*We explore different types of behavior during the problem posing process by looking at the ways students value the problem data in solving and extending their own posed problems. Based on the outcomes of these analyses we explain the differences in students' success and failure in the problem posing approaches in relation to the level of understanding the solution of a problem and the novelty of the posed problems. We notice that the more the student advances in the abstract dimension of the problem and its context, the more mathematically relevant are the newly obtained versions. The abstraction level of the solution process determines the novelty of the newly posed problems and it seems to be a good predictor of the child's creative potential.*

**Key-words:** problem posing, problem reformulation, novelty, creativity

## INTRODUCTION

Mathematical creativity raises special interest due to its links with mathematical giftedness and advanced mathematical thinking. Once considered a valid research subject only in relation with professional mathematicians, the focus on mathematical creativity has shift in the last decade towards the creativity of schoolchildren in classroom settings. However, there is an ongoing controversy about how to define it and what tasks allow identifying creative behaviors. A source of such tasks is problem posing (PP). In this study, we focus on the key elements students use in the problem posing process. More specific, we explore how students value the problem data in solving and extending their own posed problems. We compare students' behaviors during the process of modification of a posed problem in order to identify the differences among them. Based on the outcomes of these analyses, we try to explain the differences in students' success and failure in PP approaches in relation to their potential for mathematical creativity.

There are different terms that are used in reference to problem posing, such as problem finding, problem sensing, problem formulating, creative problem-discovering, problematizing, problem creating, and problem envisaging (Dillon, 1982; Jay & Perkins, 1997). In the present study, we adopt Silver's (1994) position, in accordance to which „problem posing refers to both the generation of new problems and the re-formulation, of given problems”.

The literature on PP shows that this activity is important from various perspectives and emphasizes connections between PP and creativity. For example, Jensen (1973) considers that for being creative in mathematics students should be able to pose mathematical questions that extend and deepen the original problem as well as solve the problems in a variety of ways; therefore, exhibiting PP capacities is a condition of mathematical creativity. From another perspective, Silver (1997) argued that inquiry-oriented mathematics instruction which includes problem-solving and problem-posing tasks and activities can assist students to develop more creative approaches to mathematics.

The question still remains: To what extent the production of problems validate or not students' creative capacities? Some studies are reserved in this respect. For example, Yuan & Sriraman (2010) conclude that "there might not be consistent correlations between creativity and mathematical problem-posing abilities or at least that the correlations between creativity and mathematical problem-posing abilities are complex". Silver's (1997) statement suggests that any relationships between creativity and problem posing might be the product of previous instructional patterns. Other studies, for instance Haylock (1997) and Leung (1997), who did not agree that there was correlation between creativity and problem posing in mathematics, did not take instruction into consideration. To answer the question, thus involving us into the existent controversy in the literature, we have analyzed the problems devised by students who voluntarily responded to a PP task.

Before entering the details of our study, we have to face another, related question: how do we identify creative behavior in students, or, a more tackled one: how do we define creativity? In a large sense, creativity is defined as «the ability to make or otherwise bring into existence something new, whether a new solution to a problem, a new method or device, or a new artistic object or form» (Encyclopædia Britannica). The definition of creativity benefits of a large spectrum of studies, going from broad characterizations to particular aspects about problem identification, problem posing and problem solving (e.g. Csikszentmihalyi, 1998). The topic of mathematical creativity also received much attention from researchers who focused on defining, characterizing it, or establishing criteria for its assessment (see for example, Ervynck, 1991; Freiman & Sriraman, 2007; Silver, 1997; Sriraman, 2004).

In this paper, we will step out of adopting a certain definition of mathematical creativity. We prefer to specify some aspects of creativity by exemplifying situations in which student's behavior is certainly non-creative. Through comparison, behaviors that are situated at distance from these examples will shape certain features of creativity.

## METHODOLOGY

Usually, in the Romanian schools, students are not asked to pose problems. They are trained in problem solving, and once the problem is solved, it is not even followed by a heuristic stage that could lead to modifications of the initial text. Consequently, because there is not a PP practice in the common Romanian classes, we can undoubtedly consider that PP is a task with creative potential.

The participants in the study were 120 students from grades 3 to 6 who voluntarily answered to a call for problems, from a total of 220 students who participated to a mathematical summer camp. The students in the camp were selected from 89,872 students via a two-round national competition; therefore their experience in problem solving was high. During the camp, the students were asked to create two problems – one easy and one difficult, and to deliver their proposals, including the problems solutions, after a few days.

Subsequently, we have chosen and interviewed 40 respondents. The choice was made depending on the nature of the problems posed by the students and their proposed solutions.

We structured the protocol interviews around questions such as: Can the given data of your problem be changed? (And, if yes, can you devise new data that fit your problem?) Are there redundant/ insufficient data in your proposal? Can you define a more general situation? Is there any interesting particular case? What happens if you change a smaller/larger part of the problem?, etc.

During the interview session, the students had about 15 minutes to re-read their initial proposals and to think to the addressed questions. After this reflection time, the discussion started individually and followed children's ideas until the interviewer had a clear image of the student's approach. Along each interview, other questions have been addressed, suggested by the students' answers.

## DATA ANALYSIS

In the next sections, we present some of the students' reactions that are relevant for the different contexts identified in the gathered data. From the database of 240 of problems posed by the children of our sample, we first selected the problems that were likely to be far away from common, usual and conventional school problems. The term conventional has been used in the literature in relation with solution spaces (Leikin, 2007), to express that a solution respects the recommendations made by curriculum and it can be found in textbooks. Here we use it in the same spirit, as an expression of what it is expected, usual and habitual at a given grade, in the context of a given school curriculum.

In order to better explain the selection strategy, we start the presentation with a standard conventional problem, proposed by a 5<sup>th</sup> grader, Sorin:

What are the values of  $x$  if:

(a)  $7+x \leq 40-5x+30$

(b)  $x-4+7 \geq 30+5-20:4$

The problem posed by Sorin is a classic exercise in each textbook or workbook. In addition, the text is elliptic: we realized from his solving that Sorin's presumption was to solve the inequations in the set of natural numbers (He gives for (a) the solution  $x \in \{0,1,2,3,\dots,37,38\}$ , and for (b):  $x \in \{27,28,29, \dots \infty\}$ ). For these reasons, this problem was framed in the category conventional, even if one can say that it is "too algebraic" for the student's age (12 years old).

Other students, however, posed problems that, although they might be recognized as conventional, they were put in a context that made them interesting. This is the case, for example, for one of the problems proposed by Patricia (grade 6):

Martha and Helen invented a game: "maximal sums". Martha tells Helen:

"- We have two numbers  $a$  and  $b$ . Knowing that  $[a, b] \cdot (a,b) = 34$ , find the maximal sum!"

Help Helen by solving this problem!

Patricia's problem resorts to a conventional situation that evokes the formula  $[a, b] \cdot (a, b) = a \cdot b$ . However, some features of the problem: game style, characters that have a conversation, the case-analysis necessary for solving, and the invitation addressed to the reader to get involved in the solving are mostly unusual in the common teaching strategy. Therefore, we take these features as testimonies in favor of Patricia's originality in posing this problem.

While we use the term originality in relation with a person's behavior, we shall use the term novelty as a quality of the outcome of the process (for example, as quality of the problems resulting from the posing process). Novelty might be defined as surprising connections between the concepts or elements involved within a problem. As in the case of the term "conventional", we define the habitual connections between concepts as the ones promoted by the school curriculum and by the problems found in the textbooks. Surprising connections are not habitual. From this perspective, the smallest common multiple is strongly connected to product of numbers, factors and prime numbers. The vast majority of the textbook problems related to this concept will involve in some way the product between the numbers. Taking into account the students' age, we generally considered as a sign of novelty the cases when symbols and mathematics formula were not presented in the problem text, although they contributed to the solving.

Going back to Patricia's problem, we can say that her proposal displays novelty, since she asks a question about the sum of the numbers, while we would be expecting something about their product.

## Working Group 7

As far the relationship between originality and novelty of the results is concerned, we observe that all combinations are possible (maybe except the one no originality – novel result). Nevertheless, we had to distinguish between originality and training. To clarify this aspect, we analyze the solution given by Claudia (grade 6) to the problem she proposed:

I am 13 years old and my sister is 3 years younger than me. What will be the difference of ages between us 5 years later?

The problem can be considered a mathematical charade (the difference between the ages remains constant!). However, the solution given by Claudia shows that she was not aware of this fact:

“Let’s note by  $x$  the difference of ages between the two sisters.

$$x=13+5-(13-3+5)$$

$$x=18-(10+5)$$

$$x=18-15$$

$$x=3 \text{ (years)}”$$

In Claudia’s case, the problem text moves away from the conventional style. However, is it the result of a creative capacity or just the consequence of intensive training? Silver (1994) connects problem re-formulation to problem solving: „when solving a nontrivial problem a solver engages in this form of problem posing by recreating a given problem in some ways to make more accessible for solution”. From Claudia’s solution, we can see that she does not understand an essential fact in the logic of a problem: a problem should not be formulated only for the sake of the algebraic game, but its data should have consistency. The algebraic mechanism mobilized by Claudia in solving the problem is not consistent; it is just a way to encrypt what one can deduce through a brief analysis of the text.

This example is an argument in favor of the assumption that the degree of conventionality of a problem is not enough to decide whether the student acts or not creatively when devising a problem. At this stage of argumentation, some explanations are needed. Because the sample is composed of students who participated at a very selective contest, they are familiar with the problems vehiculated during previous competitions. This is why, although some students’ proposals look non-standard, these problems are, in fact, similar to problems belonging to the sets used for training. From this point of view, the respective problems are considered already known, conventional.

It was obvious in our sample that, in achieving the problem posing task, many of the students were influenced by „problem models” they knew, and they actually proposed re-formulations of these models. For this reason, in order to refine the

analysis, we looked closer at the “distance” between the posed problem and the possible model from which the student seemed to start.

We further analyze students’ proposals from this perspective.

Consider, for example, one of the Teona’s (grade 6) problems:

On a 20x20 square table colored in 2 alternative colors (like a chess table) one must position rectangular tiles that cover exactly 2 squares. The tiles can be positioned vertically or horizontally. Can we fill in the table with 199 tiles, if the squares from the ends of one of the diagonals were cut?

In its „classical” shape, the problem was about a chess table (8x8), from which two squares from diametrically opposite corners were cut. Teona gave a more general case of this problem, keeping the same context, but varying the dimensions of the table. The starting model is quite obvious since she even refers to 2 color-squares, albeit this information is not needed in the text of the problem (but, it is an important reference point for the solving).

Another example is the one of Stefan (grade 6):

Andrei and Bogdan play the following game. Each of them has a candy box. They alternatively take a number of candies from 1 to 6 inclusively, and put them into an urn. The one who first puts the 100<sup>th</sup> candy wins. Andrei is starting. How many candies should he put into the urn for the first time, in order to secure his victory?

Stefan’s proposal is closely connected to a known game (called „Who tells first 100?”), where two children successively say numbers from 1 to 9, which they add to the previous sum. The winner is the one who first arrives at 100. By changing the data and the context (which is no more purely mathematical but related to daily life), Stefan manifests flexibility in thinking. Stefan, as well as Teona, has built analogies of already known problems.

In general, what we noticed is that the students reacted to the problem posing task by resorting to analogies that reflect a near transfer (Salomon & Perkins, 1989) even if they had complete freedom in processing the task, including the non-compulsory reporting of the task.

However, we also found some situations of what we considered to be far transfer.

For example, Paul (grade 6) proposed the following problem:

Andrei goes to the shop and buys 10 chocolates. For each chocolate he gets a ticket and with two tickets he can buy a new chocolate [and, of course, a new ticket – a.n.]. How many chocolates does Andrei get?

This problem seems not to start from a previous model (fact confirmed during the interview). We account that this problem moves away from both conventional and training problems because the student combines the idea of equivalent exchange with successive iterations.



A general conclusion of the previous paragraphs is that when facing a problem posing task, the students proceed through analogy. They start from a known problem and vary one or more parameters, obtaining a re-formulation of the starting model.

Thus they resort to a transfer, usually near transfer, most frequently based on analogy. However, a more restrained category of students tries to create novel contexts within the problem posing task.

### A MORE FOCUSED ANALYSIS

In the second part of the study, we wanted to see how deeply the students entered the philosophy (and the mathematical mechanisms) of the problems they proposed. More precisely, because the re-formulation can be done on various tiers of understanding of the solving, we have analyzed the relationship between the deep understanding of the solution and the quality of the final problem. We thus realized that the degree of novelty of a problem is not enough to conclude that the student is mathematically creative.

We first discuss below some answers given by Teona (see the problem above) during her interview. In order to have an insight into the mechanisms used by Teona when she moved from the “model problem” to her proposal (mainly characterized by new dimensions of the table), we asked:

- a) What would happen for a  $13 \times 13$  table?
- b) What would happen for a  $20 \times 20$  table? Is it possible to cover such a table (from which 2 squares have been cut) with  $1 \times 3$  tiles?

Although Teona found relatively easy the argument for  $13 \times 13$  (the answer is **no** because of the odd number of squares), she failed to find the answer for the  $1 \times 3$  tiles (which is based on divisibility and which, also, constitutes the deeper argument for the **no** answer for the odd number of squares). Thus, she unnecessarily restricted the range of numbers she could use to make up a new problem. The term *unnecessarily* expresses the fact that those restrictions originate from the student, internally, and not from the situation described in the problem’s text. In the terms of Haylock (1997), she remained “prisoner” of a content-universe fixation.

Nevertheless, this was not the case for Stefan (see his problem above). The questions in his interview have been focused on the relevance of the numbers 6 and 100 for his problem. He not only was able to explain accurately his solution, but he was also able to give the following generalization:

”Let’s say that each child can take away between 1 and  $n$  candies and the winner is the one who puts the candy number  $m$  into the urn. Then the first player wins (no matter how the second proceeds) if  $n+1$  does not divide  $m$ , and the second may win (no matter how the first plays) in the opposite case.”

This generalization witnesses deep understanding of the way in which the variation of the parameters can influence the result.

Paul's case (see above the text of his problem) is even more interesting. Although his proposed problem is novel, the interview showed us that Paul remains at a superficial, concrete and numerical level of understanding. For example, Paul was not able to propose any new situation in which after all the possible exchanges are made, the child in the story has no ticket at all.

We have now a list of examples to point out to the question: How does the transfer made by some students in problem posing process relate to creativity?

The deep understanding of a problem determines flexibility in devising new problems, because deep understanding allows exploring the relationships between the elements of the problem at different levels. Thus, the level of understanding of the solution is decisive for the variety of the new problems developed by the students.

In order to clarify this aspect we resort to a complexity-abstraction scale, in which a gradual progression from operating with concrete objects to operating with symbols and symbol systems is seen as a spiral development with different tiers of abstraction. Constructing abstraction implies reorganizing previous knowledge by incorporating it within new systems, which are hierarchically structured (Singer, 2007).

At an immediate, superficial level of understanding, we will find students that keep the problem at a concrete tier, just changing, for instance, the numerical data. For example, on such scale, Teona remains anchored of the chess table and she is not able to climb up on the spiral to a more abstract level, where she would use divisibility without physical support. This means a minimal understanding of the connections among the elements from the problem text (numerical data, parameters, constraints – Singer & Voica, 2008).

In order to better illustrate the way in which this scale functions, we return to the problem posed by Sorin. Although his problem looks abstract, he is at a low level on the scale because he cannot satisfactorily operate with that abstraction.

At a higher level are situated the students that modify large parts of the problem model and advance in the direction of abstraction-generalization. In this last case, the newly posed problems are similar to the “models” only at an abstract tier – that of the mathematical structures that describe the problem. For example, Stefan makes a clear jump in abstraction, because he gives a strong algebraic generalization.

## CONCLUSIONS AND FURTHER RESEARCH

We included in this paper examples of problems proposed by students of 11-13 years old. On this gamut of ages, we found that students of our sample are at different

levels of abstraction capabilities, although all of them were selected via complex mathematical tests. We noticed that the more the student advances in the abstract dimension of the problem and its context, the more mathematically relevant are the newly obtained versions. Supplementary evidence in sustaining this claim is offered by Paul's case: his problem is surprising, original, but because he does not master the abstracting possibilities beyond his proposal, he cannot make variations. We see this – namely, the role of abstraction ability in creative output – as a potential avenue for further research about the differences between mathematical creativity and creativity in general.

In conclusion, if the student is able to construct coherent and novel variations in problem posing activities by changing some parameters of his/her posed problem and he/she understands the consequences of these changes, then he/she proves capability for deep transfer creative approaches.

By assessing the distance between the conventional model of a problem – frequently met by the students in their solving activity – and the final problem emerged along the interviews, we noticed that in some cases there is a considerable leap. This advancement can indicate disposition toward mathematical creativity. In all the cases we analyzed, it is connected to student's capacity to effectively operate with abstract information. Therefore, this capacity might be a feature of mathematical creativity.

## ACKNOWLEDGMENTS

This research was partially supported by the involvement of the first author in the project POSDRU/19/1.3/G/14373.

## REFERENCES

- \*Encyclopædia Britannica, [www.britannica.com/EBchecked/topic/142249/creativity](http://www.britannica.com/EBchecked/topic/142249/creativity).
- Csikszentmihalyi, M. (1998). Motivation and creativity: Toward a synthesis of structural and energistic approaches to cognition. *New Ideas in Psychology*, 6(2), 159-176.
- Dillon, J. T. (1982). Problem finding and solving. *Journal of Creative Behavior*, 16, 97-111.
- Ervynck, G. (1991). Mathematical Creativity. In D. Tall (Ed.), *Advanced Mathematical Thinking*. 42-53. Dordrecht, the Netherlands: Kluwer Academic Publishers.
- Freiman, V. & Sriraman, B. (2007). Does mathematics gifted education need a working philosophy of creativity? *Mediterranean Journal for Research in Mathematics Education*, 6(1-2), pp. 23-46.
- Haylock, D. (1997). Recognising mathematical creativity in schoolchildren. *ZDM*, 29(3), 68-74.

- Jay, E. S., & Perkins, D. N., (1997). Problem finding: The search for mechanism. In M. Runco (Ed.), *The creativity research handbook*, pp. 257-293. New Jersey: Hampton Press.
- Jensen, L. R. (1973). *The relationships among mathematical creativity, numerical aptitude, and mathematical achievement*. Unpublished Dissertation. The University of Texas at Austin, Austin, TX.
- Leikin, R. (2007). Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education - CERME-5*, pp. 2330–2339. (CD-ROM and On-line). Retrieved from <http://ermeweb.free.fr/Cerme5.pdf>
- Leung, S. S. (1997). On the role of creative thinking in problem posing. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 97(2), 48-52.
- Salomon, G., & Perkins, D. N. (1989). Rocky roads to transfer: Rethinking mechanisms of a neglected phenomenon. *Educational Psychologist*, 24(2), 113-142.
- Silver, E.A. (1994). On mathematical problem posing. *For the learning of mathematics*, 14(1), 19-28.
- Silver, E. A. (1997). *Fostering creativity through instruction rich in mathematical problem solving and problem posing*. Retrieved, July 29, 2007, from <http://www.emis.de/journals/ZDM/zdm973a3.pdf>
- Singer, F. M. (2007). Modelling both complexity and abstraction: a paradox? In W. Blum, P. Galbraith, H. W. Henn and N. Mogens (Eds.), *Applications and Modelling in Mathematics Education*, New York: Springer, pp. 233-240.
- Singer, M. & Voica, C. (2008). Operational categories for assessing problem solving (in Romanian). *ROMAI Educational Journal* 3, 36-41
- Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator*, 14(1), 19–34.
- Yuan, X., & Sriraman, B. (2010). An exploratory study of relationships between students' creativity and mathematical problem posing abilities. In B. Sriraman, K. Lee (eds.), *The Elements of Creativity and Giftedness in Mathematics*, Rotterdam, The Netherlands: Sense Publishers.

# CREATIVITY IN THREE-DIMENSIONAL GEOMETRY: HOW CAN AN INTERACTIVE 3D-GEOMETRY SOFTWARE ENVIRONMENT ENHANCE IT?

Paraskevi Sophocleous & Demetra Pitta-Pantazi

Department of Education, University of Cyprus

*This paper reports the outcomes of two empirical studies undertaken to investigate the creative abilities of sixth grade students in three-dimensional (3D) geometry and to analyse the way in which an interactive 3D geometry software environment enhanced that abilities of two students. The analysis indicated that students' creative abilities in terms of fluency, flexibility and originality were very low in both studies. But the interactive 3D geometry software improved the creative abilities of students who worked in this environment. The interactive geometry environment offered opportunities to students to imagine, synthesize and elaborate. It appears that enhancing students' ability to imagine, synthesize and elaborate may be a way to enhance students' creative abilities.*

**Key-words:** 3D geometry, interactive environment, creativity, imagining processes, synthesizing skills, elaborating skills

## INTRODUCTION

Creative performance is an essential part of doing mathematics (Pehkonen, 1997). This recognisable value of creativity in mathematical thinking led a number of researchers to define, assess and promote creativity in various mathematical concepts such as problem solving, problem posing and two dimensional geometry (e.g., Pehkonen, 1997; Leikin & Lev, 2007; Levav-Waynberg & Leikin, 2009; Pitta-Pantazi & Christou, 2009). Technology was proposed by a number of researchers as a tool that promotes students' mathematical creative abilities (Mevarech & Kramarski, 1992; Dunham & Dick, 1994; Clements, 1995; Subhi, 1999). However, although substantial work has been done in this area, little attention has been given to the ways in which technology can enhance mathematical creative abilities and processes. In this paper, we tried to examine the impact of an interactive 3D geometry software environment on students' creative abilities and processes in 3D geometry.

## THEORETICAL BACKGROUND

### Creativity and mathematics

Creative thinking is an essential aspect of the Integrated Thinking Model and involves "using and going beyond the accepted and reorganised knowledge to generate new knowledge" (Iowa Department of Education, 1989, p. 7). In the same line, Ervynck (1991) argued that creativity is an important factor of advanced

mathematical thinking and refers to it as process of creating new knowledge, in other words “making a step forward in new direction” (p. 42), based on previous knowledge.

The creation of new knowledge is not simply a process, but it is a multicomponent process. More specifically, the new knowledge is created by imagining, synthesizing and elaborating processes with accepted or reorganised knowledge (Iowa Department of Education, 1989). In other words, creative thinking involves imagining processes which require original ideas through intuition, visualisation, prediction and fluency. Creative thinking also involves synthesizing skills which depend on the abilities to combine parts to form a new whole using analogies, summarizing main ideas in one word, hypothesizing and planning a process. Elaborating skills refer to the abilities to develop an idea fully by expansion, extension and modification (Iowa Department of Education, 1989). In the same line, Levav-Waynberg and Leikin (2009) describe mathematical creativity as the act of “integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution” (p. 778).

A number of researchers characterised and evaluated creative responses in mathematics mainly by fluency (the number of acceptable responses), flexibility (the number of different ideas or categories of responses used) and originality/novelty (the relative infrequency of the responses) (Torrance, 1974; Leikin & Lev, 2007).

### **Creativity, mathematics and technology integrated environment**

In the literature we come across two conflicting views regarding the relationship between creativity and technology (Clements, 1995). One view is that technology enhances only uncreative, mechanistic thinking. The second view is that technology is a valuable tool of creative production (Clements, 1995). This is in line with the argument of the National Advisory Committee on Creative and Cultural Education (1999) on the role of technology who suggests that technology enables students to find new modes of creativity. The results of empirical studies showed that technological environments enhance students’ creative abilities too. More specifically, Mevarech and Kramarski (1992) found that students who participated in problem solving activities using the Logo environment had higher creative scores in specific parts of the Torrance Test of Creative Thinking (TTCT) than students who participated in Guided Logo environment. Subhi’s (1999) research extended these results and indicated that problem solving via the Logo environment can enhance creativity in all figural and verbal domains of TTCT. Furthermore, in a study by Dunham and Dick (1994) students who used graphing calculators appeared to be more flexible problem solvers.

We could argue that most of the studies conducted until now on the impact of technology in students’ creative abilities are “results oriented”. They concentrate only on whether a specific software environment may enhance or not students’

creative abilities. In this study, we try to examine the ways in which 3D geometry environment enhances students' creative abilities and processes while being engrossed in the task.

## **THE PRESENT STUDY**

### **Purpose**

The purpose of this paper was to investigate students' creative abilities in 3D geometry and to examine the impact of an interactive 3D geometry environment on these abilities. More specifically, we address the following questions:

- (a) What were students' creative abilities in 3D geometry in terms of fluency, flexibility and originality?
- (b) Can an interactive 3D geometry environment enhance students' creative abilities?
- (c) How did an interactive 3D geometry environment enhance students' creative abilities and processes in 3D geometry?

### **Participants and procedure**

To answer the first research question, we conducted two empirical studies on two kinds of 3D geometrical abilities that young students are expected to perform according to the Cypriot mathematics curriculum: 3D rectangular arrays of cubes (Study 1) and nets (Study 2).

In Study 1, a hundred and twenty one 6<sup>th</sup> grade primary school students (54 males and 67 females), ranging from 11 to 11.5 years of age completed two 3D rectangular arrays of cubes tasks. The first task required students to create as many constructions with nine cubes as possible (9-cubes constructions). The second task called students to create as many constructions as possible with four cubes and surface area equal with 18 square units (4-cubes constructions). In Study 2, a hundred and twenty eight 6<sup>th</sup> grade primary school students (59 males and 69 females) completed two nets tasks. These tasks required from students to create as many cube nets and cuboid nets as possible. It needs to be stressed that all tasks used in both studies asked students to find multiple solutions, a characteristic of creative mathematical activity (Leikin & Lev, 2007). To evaluate students' mathematical creative abilities we measured: fluency (number of correct constructions), flexibility (number of different types of constructions or categories of constructions) and originality (extraordinary, new and unique constructions) (Torrance, 1974). Every response in the two 3D arrays of cubes tasks and in the two nets tasks was given a score from 0 to 4 for each one of these three dimensions (fluency, flexibility and originality). These scores are relative and based on the categories raised from all of the students' answers.

To answer the second and third research questions, we used a case study approach because it was the most appropriate to respond to the "how" research questions according to Yin (2003). In addition to this, this approach offers in details the

description of a specific situation and provides an example of “real people in real situations” (Cohen, Manion & Morrison, 2000, p. 181). We selected two sixth grade students who participated in the two studies with creative tasks. These two students worked for two sessions, one hour each, on four 3D geometry activities, which were the same with those that they worked in paper. They used two applications of an interactive 3D software environment, DALEST; a powerful tool in the teaching of 3D geometry which provides conditions of observation and exploration (Christou et al., 2007). The first application, Cubix Editor, can be used by students to “create 3D structures built of unit-sized cubes” (Christou et al., 2007, p. 4). The second application, Origami Nets, can be used by students to create different nets of various 3D geometric figures. The two students were asked to read again the task instructions that they solved earlier on paper and this time solve them with the use of the software. The students’ work on DALEST applications was videotaped and at the same time the researcher was taking notes. The researcher was recording the students’ comments during the time that they were working on the software, their strategies and constructions on each task.

This second part of the paper was conducted in the frame of the European project InnoMathEd (this project is funded with the support of the Lifelong Learning Programme of the European Union).

### **Data analysis**

To investigate students’ creative abilities in 3D geometry, we conducted descriptive analysis on both studies. To examine the ways in which DALEST applications environment enhances students’ creative abilities and processes in 3D geometry, we analysed students’ strategies in the creativity tasks which they solved with the software and tried to underline the different creative thinking skills which arose from students’ solutions. The presentation and discussion of students’ solutions to the four creativity tasks are organised around three phases: (a) the phase before students express creative abilities (before students give numerous, different and unusual responses), (b) the creative phase and (c) the expansion phase.

## **RESULTS**

### **Creative abilities in 3D geometry in paper**

Table 1 presents the mean and the standard deviation of students’ performance in the creative tasks in both studies in terms of fluency, flexibility and originality. The means of students’ performance shown in Table 1 are all smaller or equal to one, since scores of students in fluency, flexibility and originality were divided by four (the maximum score).



CREATIVE TASKS	FLUENCY $\bar{X}$ (SD)	FLEXIBILITY $\bar{X}$ (SD)	ORIGINALITY $\bar{X}$ (SD)	TOTAL $\bar{X}$ (SD)
<b>STUDY 1 (N=121)</b>				
9-cubes constructions	0.49 (0.30)	0.48 (0.30)	0.45 (0.30)	0.47 (0.29)
4-cubes constructions	0.25 (0.22)	0.23 (0.20)	0.23 (0.20)	0.24 (0.20)
<b>STUDY 2 (N=128)</b>				
Cube nets	0.48 (0.29)	0.44 (0.29)	0.44 (0.27)	0.45 (0.27)
Cuboid nets	0.22 (0.19)	0.19 (0.15)	0.19 (0.14)	0.20 (0.15)

**Table 1: The means and standard deviations of students' performance in creative tasks on study 1 and 2 in terms of fluency, flexibility and originality**

According to Table 1, the total mean performance of students in creative tasks in both studies was below 0.5. These tasks appear to be very difficult and complicate to solve on paper and students probably did not have the opportunity to solve similar tasks in their mathematical textbooks in the past. However, it seems that students' creative abilities are better in some tasks than others. More specifically, the total mean of creative performance of sixth grade students in Study 1 in the task with 9-cubes constructions ( $\bar{X}=0.47$ ) was double the mean of performance in the task with 4-cubes constructions ( $\bar{X}=0.24$ ). The same pattern appears in Study 2 where students' mean performance in cube nets task ( $\bar{X}=0.45$ ) was double the mean performance in cuboid nets task ( $\bar{X}=0.20$ ). We hypothesise that students provided a larger variety and more unique solutions in tasks which are given to them without any "limitations" than in tasks with certain "limitations". Specifically, in the 9-cubes constructions task students were free to create constructions without any limitations in regard to the size of the surface area. On the other hand, in the 4-cubes constructions task, students were asked to create constructions with a specific size of surface area (18 square units). Similarly in Study 2, students were more creative while working with the cube nets, since all the faces were equal, instead of working with the cuboid nets where not all faces were equal.

Moreover, students in both studies had a higher mean score in fluency rather than in flexibility and originality. Therefore, it appears that students tended to be more fluent than innovative. They provided a number of answers but not always were these of a great variety or unique. More specifically, almost all students of Study 1 made at least one 9-cubes construction. This was usually a nine cubes tower or a cuboid 3X3X1. Many of them provided one to four 9-cubes constructions and only twenty students made more than five constructions with nine cubes. Therefore, the mean score in fluency was 0.49. In addition to this, many of the students of Study 1 created one to four different types of 9-cubes constructions and only fourteen students produced more than five different types of 9-cubes constructions. Thus, their total mean score in flexibility was 0.48. Students' originality score was also under 0.50 ( $\bar{X}=0.20$ ), since only a small number of students created 'irregular' constructions

(with small base or to have a specific shape). In the task where students asked to create 4-cubes constructions with surface area 18 square units, sixty students could not create such as construction. Almost all students who created a correct 4-cubes construction had as one of their constructions a tower (an alignment of four cubes). Only eight students created more than five 4-cubes constructions and the rest of them created up to four constructions (fluency). In addition to this, only three students build four different types of 4-cubes constructions and the rest of the students, who responded correctly, gave up to three different types of constructions (flexibility) which were not very unique (originality). Thus, students' fluency ( $\bar{X}=0.25$ ), flexibility ( $\bar{X}=0.23$ ) and originality scores ( $\bar{X}=0.23$ ) were very low.

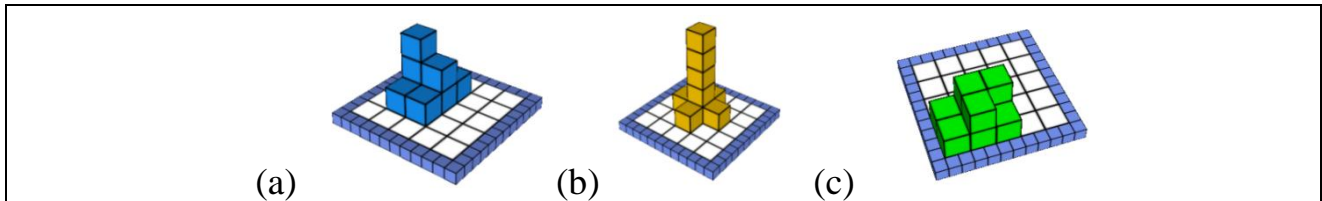
From table 1, it can be deduced that students of Study 2 were more able to construct many, different and unique cube nets than cuboid nets. All students that completed these creativity tasks in Study 2, produced at least one cube net and this was the cross-net, but only twenty four students produced more than five cube nets. This is why students fluency score was under 0.50 ( $\bar{X}=0.48$ ). Many of the students created up to four different types of cube nets, but only thirteen students constructed more than five different types of cube nets (flexibility), which however were not unique (originality). Thus, students' flexibility ( $\bar{X}=0.44$ ) and originality scores ( $\bar{X}=0.44$ ) were under 0.50. In the cuboid nets tasks, more than half of the students did not create any net and only five students produced more than five cuboid nets. The rest of the students created at least one cross cuboid net and this is why their fluency score was very low ( $\bar{X}=0.22$ ). Students who produced cuboid nets were able to construct at the most three different types of cuboid nets. Only five students created more than four different types of cuboid nets (flexibility). In addition to this, students' cuboid nets were not innovative, since many of them were either the cross-net or T-net (originality). This is why the flexibility ( $\bar{X}=0.19$ ) and originality scores ( $\bar{X}=0.19$ ) were very low.

### **Creative abilities in 3D arrays of cubes in an interactive 3D geometry environment**

*The phase before creativity.* In this phase, students explored the functions of the software by creating the constructions they already drew on paper. More specifically, for the task with the 9-cubes constructions both students created a tower of nine cubes and a cuboid 3X3X1. These were the most “popular” constructions in students' paper solutions. For the task with 4-cubes constructions, both students constructed only a tower of four cubes. These explorations allowed students to *visualise* the constructions that they had already created on paper. They could *see* their constructions from different perspectives by rotating them and *check* their solutions with the use of the “statistics window”. The “statistics window” shows the number of cubes used and the surface area of the constructions.

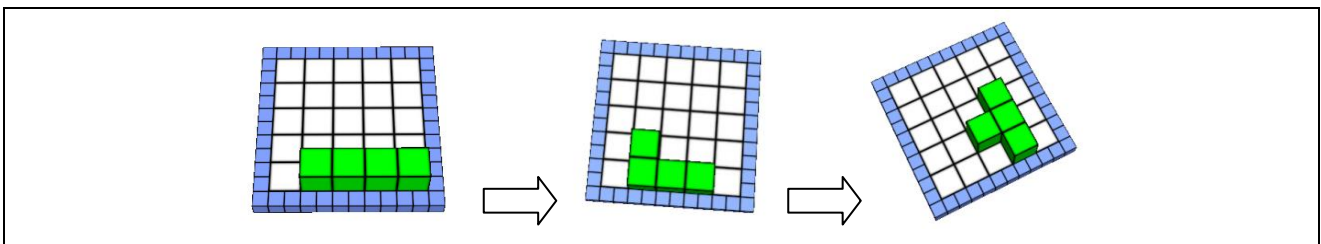
*The creative phase.* Through the visualisation processes and the checking potential that the software environment offers, both students provided six different types of 9-

cubes constructions. It was notable that students built these constructions by *expressing their ideas fluently* and without any assistance from the researcher. Sample of students constructions are given in Figure 1. These 9-cubes constructions appear to be difficult to be drawn on paper. In addition to this, students selected to build construction with “big base” (the number of cubes at the base is more than the number of cubes of other layers of the constructions).



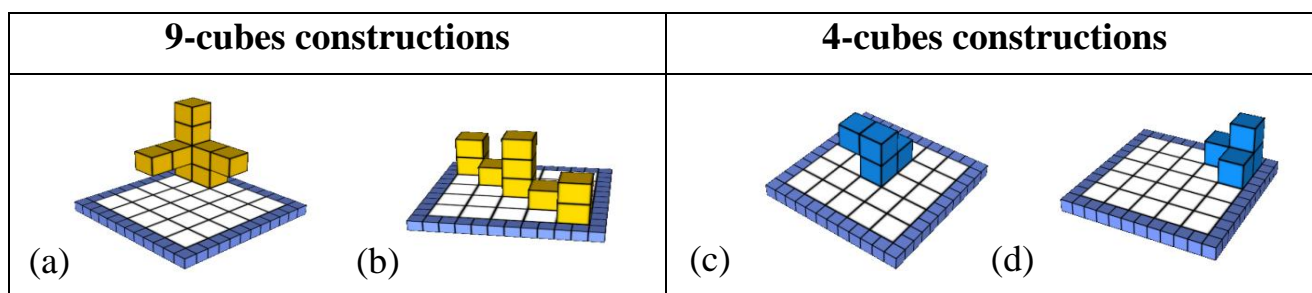
**Figure 1: Students’ 9-cubes constructions**

In the task where students were asked to create 4-cubes constructions with 18 square units surface area, both students started making *hypothesis* about which cube to move from the original tower (their first construction) to transform it to a different 4-cubes construction and *checked* simultaneously their solutions using the “statistics window”. They repeated this *procedure* (see Figure 2) until they provided a number of constructions. Student A provided nine 4-cubes constructions and Student B made five 4-cubes constructions. They stopped this step-by-step procedure when they realised that they could not find any other solution.



**Figure 2: A step-by-step procedure for providing 4-cubes constructions**

*The expansion phase.* In this phase, students appeared to use their *intuition* and *imagination* to extend their constructions. They tried to provide unique constructions employing a *modified* version of their previous constructions. More specifically, Student A constructed an additional 9-cubes construction by *removing* some cubes from the base of his previous construction and *adding* them on other layers above (see Figure 3(a)). Student B constructed another 9-cubes construction by *removing* cubes from the top of his previous construction and extending it horizontally (see Figure 3(b)). Similarly, in the 4-cubes constructions task both students created two further constructions without depending on the step-by-step procedure (see Figure 3(c) and Figure 3(d)). The step-by-step procedure, which characterised their way of processing during the creative phase, was replaced in the extension phase by their intuition and imagination.

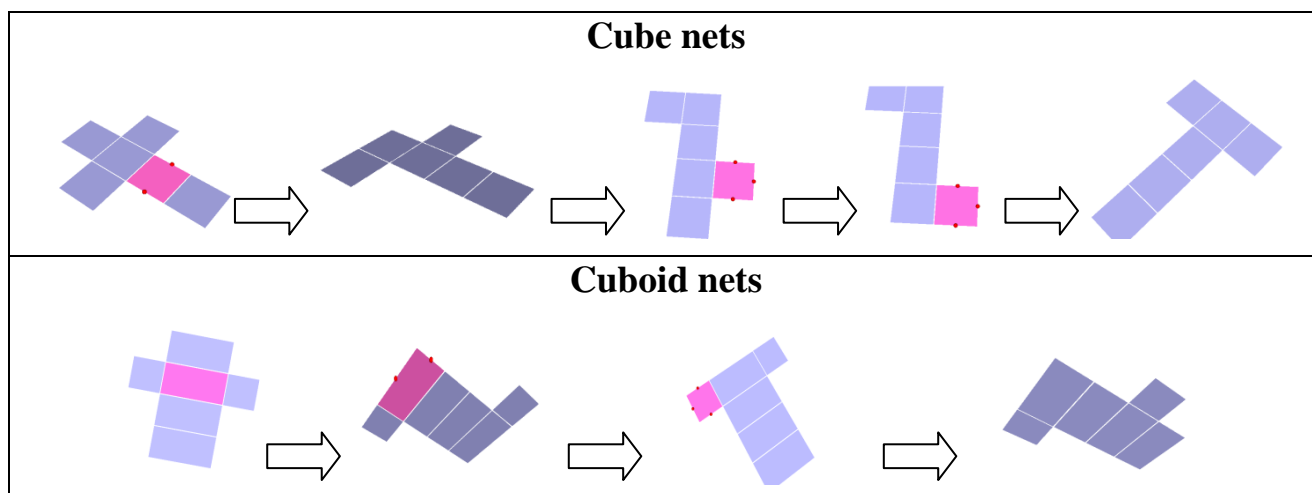


**Figure 3: Students' constructions in the expansion phase**

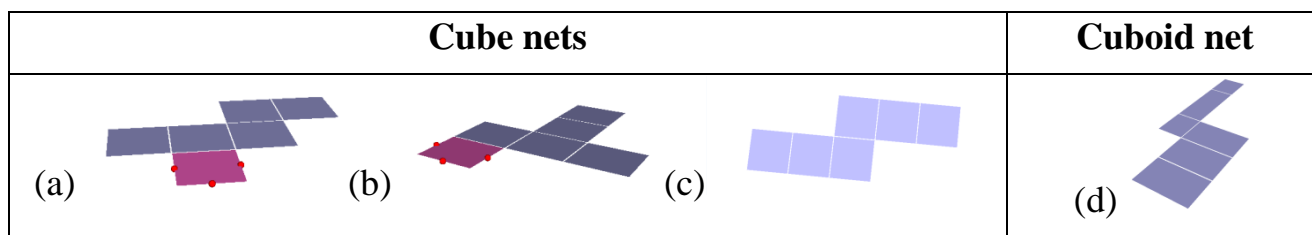
### Creative abilities in nets in an interactive 3D geometry environment

*The phase before creativity.* The two students when solved the nets tasks on paper produced only the cross-net for cube and cuboid. This cross-net, according to Stylianou, Leikin and Silver (1999) is considered as the easiest since it can be straightforwardly “opened up” whereas other nets need more transformations. Students claimed that they remembered it from their mathematical textbooks. In this phase, students *created* a cross cube net and a cross cuboid net in the software application, Origami Nets and *checked* whether these nets could be folded.

*The creative phase.* In this phase, both students *observed* and *explored* unfolded cross-net of cube and cuboid and tried to *plan a process*. In both tasks, students kept four squares or rectangles attached together in a row and moved the other two squares around that row (see Figure 4). Similar process was used by students in the study by Stylianou et al. (1999) and was characterised as systematic. With the above procedure, both students managed to construct 6 cube nets and 4 cuboid nets.



strategy and succeeded in “extending” their answers by providing three extra solutions for the cube nets and one extra solution for the cuboid nets. Students’ “extended” constructions of cube nets and cuboid nets are given in the Figure 5.



**Figure 5: Students’ cube nets and cuboid net in the expansion phase**

It is important to note that during the creative phase as well as in the expansion phase, the two students justified their solutions and at the same time they worked in the interactive environment. Their approach was not at all mechanistic (i.e. give an answer by chance, check and correct it).

Overall, the two students were able to construct many, different and unique solutions in the four creativity tasks in the interactive 3D geometry environment. They expressed a more creative performance in this interactive 3D geometry environment, by visualising their ideas and using their intuition and imagination (imagining skills), by planning a step-by-step procedure and hypothesizing (synthesizing skills) and finally by extending and modifying their previous solutions (elaborating skills).

## DISCUSSION

The current paper examined sixth grade students’ creative abilities in 3D geometry and the ways in which an interactive software environment enhanced their creative abilities and processes.

Overall, the results of the two studies indicated that students’ creative abilities in terms of fluency, flexibility and originality were very low. But the interactive 3D geometry software environment enhanced the creative abilities of students who worked in that environment, by facilitating them to provide more, different and unique solutions. This finding confirms previous studies results about the value of technology as a tool for creative production (Clements, 1995; Subhi, 1999). Moreover, we found through the two case studies that the interactive 3D geometry environment forced and also enhanced students’ imagining, synthesizing and elaborating thinking skills. More specifically, it seems that during the phase before creativity, students used mainly imagining processes, while during the creative phase they used a combination of imagining and synthesizing skills. During the expansion phase, they used a combination of imagining, synthesizing and elaborating skills. These processes appear to have empowered students’ creative performance.

## REFERENCES

- Christou, C., Pittalis, M., Mousoulides, N., Pitta, D., Jones, K., Sendova, E., & Boytchev, P. (2007). Developing an active learning environment for the learning of stereometry. In E. Milková (Ed.), *Proceedings of the 8th International Conference on Technology in Mathematics Teaching (ICTMT8)*. Czech Republic: University of Hradec Králové. Retrieved from <http://eprints.soton.ac.uk/47577/>
- Clements, D. H. (1995). Teaching creativity with computers. *Educational Psychology Review*, 7(2), 141–161. doi:10.1007/BF02212491
- Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education*. London: Routledge Palmer.
- Dunham, P., & Dick, T. (1994). Research on graphing calculators. *Mathematics Teacher*, 87, 440–445. Retrieved from [http://www.nctm.org/eresources/journal\\_home.asp?journal\\_id=2](http://www.nctm.org/eresources/journal_home.asp?journal_id=2)
- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42–53). Dordrecht: Kluwer Academic Publishers.
- Iowa Department of Education (1989). *A guide to developing higher order thinking across the curriculum*. Des Moines, IA: Department of Education. Retrieved from ERIC database (ED 306 550).
- Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam, The Netherlands: Sense Publishers.
- Leikin, R., & Lev, M. (2007). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 161–168). Seoul: PME.
- Levav-Waynberg, A., & Leikin, R. (2009). Multiple solutions for a problem: A tool for evaluation of mathematical thinking in geometry. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of Sixth Conference of European Research in Mathematics Education* (pp. 776–785). Lyon, France: Institut National de Recherche Pédagogique. Retrieved from <http://www.inrp.fr/editions/editions-electroniques/cerme6/>
- Mevarech, Z. R., & Kramarski, B. (1992). How and how much can cooperative logo environments enhance creativity and social relationships? *Learning and Instruction*, 2(3), 259–274. doi:10.1016/0959-4752(92)90012-B
- National Advisory Committee on Creative and Cultural Education (NACCCE). (1999). *All our futures: Creativity, culture and education*. London, UK: DfES.

- Pehkonen, E. (1997). The state-of-art in mathematical creativity. *ZDM*, 29(3), 63–67. doi:10.1007/s11858-997-0001-z
- Pitta-Pantazi, D., & Christou, C. (2009). Mathematical creativity and cognitive styles. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 377–384). Thessaloniki, Greece: PME.
- Stylianou, D., Leikin, R., & Silver, E. A. (1999). Exploring students' solutions strategies in solving a spatial visualization problem involving nets. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 241–248). Haifa, Israel: Technion - Israel Institute of Technology.
- Subhi, T. (1999). The impact of logo on gifted children's achievement and creativity. *Journal of Computer Assisted Learning*, 15(2), 98–108. doi:10.1046/j.1365-2729.1999.152082.x
- Torrance, E. P. (1974). *The Torrance tests of creative thinking*. Bensenville, IL: Scholastic Testing Service, INC.
- Yin, R. (2003). *Case study research design and methods*. Thousand Oaks, CA: Sage Publications.

# MATHEMATICAL CHALLENGING TASKS IN ELEMENTARY GRADES

Isabel Vale & Teresa Pimentel

School of Education of the Polytechnic Institute of Viana do Castelo, Portugal

*Mathematics teaching and learning should include much more than routine tasks, which appeal to memory and drill, being complemented by others, mathematically more rich and challenging, as problem solving and investigations, that push for reasoning, creativity and connections. So teachers must have the ability to choose, formulate and adjust tasks according to their intended objective. In this paper we present some mathematical pattern tasks to use in elementary mathematics classes that, intending to be challenging, can develop students higher order thinking and creativity.*

**Key words:** Challenging tasks, investigations, teacher education, elementary teachers

## INTRODUCTION

The new Portuguese curriculum proposes a different perspective about the learning and teaching of school mathematics, with great challenges for both teachers and students. Despite of being well written and transparent about its aims, pathways for implementing these changes in real classroom settings are not fully defined and can be very complex and difficult to accomplish by the teachers when they haven't innovative instructional materials, and in fact only a few are available yet. More than ever we only can have students engaged in knowledge and critical citizenship if school and teachers promote creativity in their own classroom - being creative both in developing curriculum to get higher levels of students understandings through adequate tasks and in promoting creativity in students work. In this setting we are designing a two-year project to develop a research-based professional development curriculum focused on mathematics tasks, which will assist teachers in their practices and teachers in their initial preparation of grades k-6. Our attention is directed not only to mathematical themes of the curriculum but also to cross-mathematics processes - communication, problem solving and reasoning. Thus we intend to draw sequences of tasks, materials, expectations for each, and methodological notes about their use, and in particular selecting illustrative episodes of students' creative solutions of the tasks and teachers' creative ways of exploring them.

## THEORETICAL FRAMEWORK

### Teacher education and school mathematics

Literature identifies the major obstacles to reforms as teachers' lack of familiarity with innovative instructional practices and tools, teachers' lack of understanding of



mathematics they teach and their inability to communicate mathematics with students in ways other than direct instruction; and teachers' reluctance to conform to new methods of teaching due to their beliefs about what students need to know (Heibert et al., 2007). Several researchers (Raymond, 1997; Remillard & Bryans, 2004; Schoenfeld, 2008) suggest that teachers' knowledge, beliefs and attitudes influence their actions in the classroom and their interactions with students. Towards a mathematics preparation, Ma (1999) claims that teachers must have a profound understanding of fundamental mathematics to provide teaching and learning processes. In this pathway reflection plays an important role as the reconstruction of the teachers' experience and knowledge (Hodgen, 2003). Actually, teachers with more explicit and organized knowledge tend to use with their students more conceptual connections, appropriate representations and active student discourse (Warren, 2006).

Nowadays, widespread tendencies in mathematical education suggest that effective learning requires that students be active and reflexive when they are involved in significant and diversified activities. This idea follows a way of thinking where higher order and critical thinking skills are privileged, where lectures are substituted by dialogue and discovery methods. According to Boaler (2002), different teaching methods are not just vehicles to produce more or less knowledge, they shape the nature of knowledge production and define the identity of students toward mathematics through the practices in which they involve. As such, teachers must have an in-depth understanding of the mathematical thinking of their students. In doing so, they can support the development of their mathematical competence (Franke et al., 2007). Further, they need an understanding of how to mobilize this knowledge for their students' learning. We believe that students' mathematical thinking must support teachers' practice so teachers must construct or adapt good mathematical tasks to use in the classroom. They should have the capacity to be creative for themselves in the tasks they propose but be also mathematically competent to analyse their students solutions.

Research shows that learning heavily depends on teachers. They must make a set of decisions during the instructional process that depend on various factors affecting their actions, including how to interpret the curriculum and select curricular materials and strategies to use in the classroom. Within this context we as teacher educators have to design *good* mathematical tasks that should be used to achieve a variety of goals, in particular further mathematical understanding and creative thinking in order to motivate students to learn. So we have the dual responsibility of preparing mathematics teachers, both mathematically and didactically, and discuss the way tasks can be designed and refined for the purposes of promoting mathematical understanding.

## Mathematics tasks and the development of mathematical knowledge

Having an explicit understanding of how and why mathematical tasks are used in teacher education gives us insight into what qualities a *good* task must have. However, it does not automatically provide us with the ability to design *good* tasks. The designing of *good* tasks requires an interface between the theoretical and the practical, between the intended and the actual, between the task and the student. The process of designing mathematical tasks is a recursive one that applies to the creation of entirely new tasks as well as it does to the adaptation (or refinement) of already existing tasks (Liljedahl et al., 2007). Serpinska (2003) regards the design, analysis and empirical testing of mathematical tasks, whether for the purposes of research or teaching, as one of the most important responsibilities of mathematics education.

What students learn is largely influenced by the tasks given to them (Stein & Smith, 1998). In fact, the tasks used in the classroom provide the starting point for the mathematical activity of students (Doyle, 1988) and the way of implementing a task determines its cognitive level. Its nature significantly influences the type of work that is done in math class; they should be diverse in nature and in context, giving rise to a variety of representations, using different resources and promoting discussion. Discussion that focuses on cognitively challenging mathematical tasks, namely those promoting flexible thinking, reasoning and problem solving, is a primary mechanism for promoting conceptual understanding of mathematics (Smith et al., 2009). Such discussions give students opportunities to share ideas and clarify understandings, develop convincing arguments, develop a language for expressing mathematical ideas and learn to see things from other perspectives (NCTM, 2000).

Although discussions about higher-level tasks provide important opportunities for students to learn and to promote creativity in their solutions, they also present challenges to the teacher who must determine how to organize discussion built from a diverse set of responses. The teacher must decide what aspects of a task to highlight, how to organize the work of students, what questions to ask to challenge those with different levels of expertise and how to support students without taking over the process of thinking for them and thus eliminating the challenge (NCTM, 2000). Giving students too much or too little support or too much direction can result in a decline in the cognition demands of the tasks (Stein et al., 1998). On the other hand, mathematical challenging tasks are not just difficult tasks or with a higher level of mathematization (Holton et al., 2009; Stillman et al., 2009) but much of the challenge may be provided by the teacher.

Patterning tasks are a specific kind that allows a depth and variety of connections with all topics of mathematics leading to consider patterns as cutting across all of mathematics education, both to prepare students for further learning and to develop skills of problem solving and communication (NCTM, 2000; Orton, 1999; Polya, 1945; Vale et al., 2009). Thus we will give special attention to this kind of tasks,

mainly for representations they raise (very different and usually taking the shape of analogies, drawings, manipulative or tables). Research shows that the use of multiple representations is beneficial in the teaching and learning of mathematics (Tripathi, 2008). Our previous research analyzed the impact of an intervention centred on the study of patterns in the learning of mathematics concepts (Vale et al., 2009; Barbosa, 2009). Since patterns are powerful in the mathematics classroom and can suggest numerical, visual and mixed approaches (Orton, 1999) and exploring growing patterns in elementary levels lays in the foundation for the algebraic reasoning (e.g. Usiskin, 1999; Rivera & Becker, 2008) we designed a didactical experience grounded on figural patterns as a suitable context to get expression of generalization and contribute to approach algebraic thinking.

The research has also shown that pattern tasks are a fruitful focus to support teacher inquiry and students learning towards the implementation of the current curriculum orientations. Moreover, patterning tasks challenge for different representations and to look for creative ways to reach the solutions; for instance, if students look for different ways of counting a collection of elements in figures, or a general rule of a growing figurative pattern. We identify the five different interconnected representations proposed by Clement (2004) for the mathematical ideas: pictures, manipulative, written symbols, spoken language and relevant situations. They must be used in mathematics lessons in order to provide a lens for making sense of students' solutions and responses and can be a guide for teachers to plan their lessons. We privilege problem tasks that require manipulative because, in doing so, specially young students seem to create a more significant and long-lasting experience, becoming involved in their own learning (NCTM, 2000; Weiss, 2006; Vale, 2003).

#### **Challenge and creativity in the mathematics classrooms**

We can read in ICMI Study 16 (Barbeau & Taylor, 2005) that mathematics is engaging, useful, and creative. The sentence itself was a challenge for us in the way that it conducted us to wonder what can we do to make it accessible to our students. In the last decades we developed our work and research around problem solving which led us to believe that it can be a fruitful context to engage both students and teachers to perceive those mathematics characteristics. A problem solving approach can reflect the creative nature of mathematics and give students opportunities both to learn mathematics and to feel the way in which mathematicians develop mathematics.

Learning mathematics is much more than facts, memorizing and mastering rules, techniques and computational algorithms, despite their importance and role. It entails incorporating experiences and conceptual understanding to solve different tasks like problems, investigations, games and puzzles, that promote mathematical knowledge in a reflective way, and developing creative processes to get solutions.

We only can be creative if we are attracted and challenged by the task. Challenging situations provide an opportunity to think mathematically. Holton et al. (2009) defend the importance of challenge in mathematics classroom when they state: “Students can become unmotivated and bored very easily in “routine” classroom unless they are challenged and yet it is common to hold back our brightest students” (p. 208). A mathematical challenge occurs when the individual is not aware of procedural or algorithmic tools that are critical to solve the problem and seems to have no standard method of solution. So he/she is required to engage in some kind of reflection and analysis of the situation, possibly putting together several factors, therefore having to build or invent mathematical actions to get the solution. Those challenges must respond to the situation with flexibility and imagination (Barbeau & Taylor, 2005; Powell et al., 2009).

Challenging tasks usually require creative thinking. Creativity begins with curiosity and involves students in exploration and experimentation drawing upon their imagination and originality (DFES, 2000). Creativity is typically used to refer to the ability to produce new ideas, approaches or actions and manifest them from thought into reality. The process involves original thinking. According to the *Wikipedia* creative thinking is a mental process involving creative problem solving and the discovery of new ideas or concepts, or new associations of the existing ideas or concepts, fuelled by the process of either conscious or unconscious insight. For Meissner (2000) there are several descriptions for creative thinking having not a standardized answer. But this author claims that creative thinking may develop as a powerful ability to interact between reflective and spontaneous *internal representations*. An examination of the research that has attempted to define mathematical creativity found the lack of a consensual definition. According to several researchers (Leikin, 2009; Polya, 1981), we also believe that creativity can be developed if we provide students with tasks that, allowing autonomous approaches, can generate new insights in underlying mathematical ideas. As we said before we are timely in the new national curriculum, so we designed some mathematics tasks fitting 1-6 grades, admitting multiple approaches and providing the development of creative processes of solution. We hope that we can motivate students to involve in class and challenge them for mathematics activity.

## **EXAMPLES OF CHALLENGING MATHEMATICAL TASKS FOR THE CLASSROOM**

In a constructivist perspective the exploratory tasks in several contexts increase the development of students’ knowledge and mathematical skills. In this setting, while mediator between students and mathematical knowledge, the teacher must offer students diversified tasks that allow them to access mathematical content as well as to highlight and develop mathematical processes such as to experiment, conjecture, investigate, communicate and create, contributing to a more effective learning of

mathematics. On the other hand, good tasks must call for mobilization, integration and application of different knowledge. According to NCTM (2000) a task is a good one when it deals or serves as an introduction to fundamental mathematical ideas that constitutes an intellectual challenge to students and allows different approaches.

We present four examples of tasks included in the research plan, of different nature and designed to different grades. Some of them intend to develop number sense while others stress algebraic thinking or geometric and spatial reasoning. However, they have a common objective: to develop new approaches and creative ideas. That is, the tasks must provide multiple solutions in order to raise the student flow of mathematical ideas, flexibility of thought and originality in the responses. According to students' age, the teacher may encourage the use of manipulative materials so that children can have an useful involvement in the task.

We briefly discuss possible ways of exploration of the tasks.

### Task 1. Visual Counting - The shells



*The sea girl organized this way the shells she caught yesterday.*

*Can you find a quick process to count them?*

**Fig 1: Shells**

We claim that a previous work with counting tasks in figurative settings can be a particularly good way to develop skills of *seeing* (identification, decomposition, rearrangement) to facilitate similar processes in growing pattern tasks. In fact, in the exploration of growing patterns in figurative sequences it is crucial for students to see the relations among successive terms in order to translate visual patterns into numerical expressions conducting to the generalization process - the heart of algebraic thinking.

Numerical expressions translating students' thinking in seeing the collection of shells arranged in this manner must be explained. For instance

$$40 = 4 \times 4 + 4 \times 6 = 3 \times (4 \times 4) - 2 \times (2 \times 2) = 2 \times (4 \times 4 + 2) + 4 = \dots$$

“Four rows of four shells and four rows of six shells” or “Three 4x4 squares minus two overlapping 2x2 squares” or “A 4x4 square plus two above, its reflected image and 4 shells more in the reflection axis”.

The horizontal representation stresses the equivalence of the numerical expressions and allows a new meaning of the equal sign. Our expectations of students' creativity in this task lay in the different original ways of seeing/counting the number of shells.

## Task 2. Figurative growing patterns - Trucks

Observe the pattern.

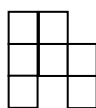


Fig. 1

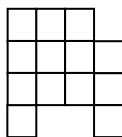


Fig. 2

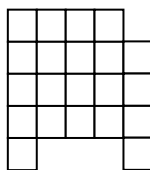


Fig. 3

### Fig 2: Trucks growing pattern

1. Sketch the next figure.
2. What is the area of each figure if the little square is the unit? Write a numerical expression translating a way to calculate that number.
3. Describe with a written explanation how you could construct the figure 25.
4. Describe with words how you could determine the area of any figure of the sequence.
5. Explain, to a fellow that doesn't believe in your rule, why does it work.

We intend that students look for a pattern, describe it, and produce arguments to validate it using different representations. The previous work with visual counting may help to see a visual arrangement that changes in a predictable form and write numerical expressions translating the way of seeing, in order to make it possible the generalization to distant terms.

Students must be encouraged to observe and see the figures in different ways and to register those several modes in a table looking for a functional relation (Table 1) using more or less formal representation. Creativity can be revealed in the search of different ways of seeing the arrange, in order to choose the best way to get an expression of far generalization.

Number of the figure	Area
1	$2 \times 3 - 1 + 2 = 3 + 2 \times 2 = \dots$
2	$3 \times 4 - 1 + 2 = 4 + 3 \times 3 = \dots$
3	$4 \times 5 - 1 + 2 = 5 + 4 \times 4 = \dots$
...	...
25	$26 \times 27 - 1 + 2 = 27 + 26 \times 26 = \dots$
...	...
n	$(n + 1) \times (n + 2) - 1 + 2 = (n + 2) + (n + 1) \times (n + 1)$

**Table1: Registration of modes of seeing**

## Task 3. The Euclid's Game

Number of players:2

Material: - a hundred chart

- overhead projector pens or translucent marks

## Working Group 7

*Development:*

*Toss who is player 1. The 1<sup>st</sup> player chooses a 1-100 number and marks this number on the chart. The 2<sup>nd</sup> player chooses and marks any other number.*

*In turn, the player subtracts any two marked numbers in order to find a difference that has not yet been selected. The players play alternately until they cannot choose a chart number. The player who can mark the last number wins.*

*Play the game several times. Try to discover a winning strategy.*

Throughout games students can develop a greater motivation for mathematics work. The links between the game and mathematics are sometimes surprising and unexpected: the fact that a game has a simple mathematical explanation and its knowledge may entail the possibility of gain provides a good way to enjoy mathematics, its beauty and power. The Euclid's game is a numerical one in which, with a few persistence, students will be able to discover patterns in the numerical structure and relations in a flexible manner in order to reach a winning strategy. The fluency of basic knowledge to relate the numeric data collected must give insight to the underlying mathematical concepts in the task and associate them to produce the bright idea of the solution.

The hundred chart may be replaced by a table of 6x6, for example, to facilitate and not becoming boring to perform several calculations.

The general conclusion about the game is complex and may be presented as follows: Let  $n$  and  $s$  the numbers chosen, respectively, by the 1<sup>st</sup> and the 2<sup>nd</sup> player. Let  $d = \text{gcd}(n, s)$ ; Let  $m = \text{Max}\{n, s\}$ . Then the game ends after  $m/d$  steps.

It is the evenness of  $m/d$  that determines the winner. So if the 2<sup>nd</sup> player knows how to play, the 1<sup>st</sup> hasn't any possibility of winning: if the 1<sup>st</sup> chooses an even number, it is enough for the 2<sup>nd</sup> to choose its half to take immediate winning; if the 1<sup>st</sup> chooses an odd number, it is enough for the 2<sup>nd</sup> to choose its double, to take immediate winning, or then the successive even number to warrant victory after a number of steps equal to the chosen number.

However, students can take partial conclusions based on particularization with various numbers resulting from several games. For this purpose it is useful to ask questions such as:

*Compare the patterns of numbers marked in each game. Can you explain why some games have so few numbers marked and others so many?*

*If the first player chooses the number 26, what number should I choose to be sure of winning?*

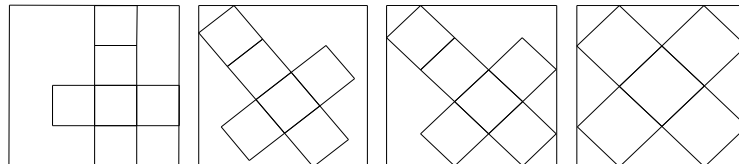
*If I start the game by choosing an odd number can I win? How?*

### **Task 4. The Cube Problem**

*From a square sheet of paper, draw the net of a cube with the largest possible volume. Then build the cube by folding.*

This problem involves geometric and spatial reasoning.

Many nets can be drawn in a square sheet of paper but there is only one that fits the condition. Fig 3 shows some different attempts students may do to get the solution. This one – the last drawing in Fig 3 – may be achieved either by drawing or by folding. It is necessary insight and divergent thinking to admit one face of the cube divided into four triangles. This requests a novel idea. However, the evolution in the consideration of the different nets as suggested by Fig 3, as well as the intuitive notion of balance and symmetry, may provide this good idea.



**Fig 3: Different attempts to optimize**

Another promising exploration for elder students is the relation between maximum volume/area. The construction of those successive figures with dynamic geometry software may be a good tool to discover the solution and to verify that it is indeed the optimal one.

## TO CONCLUDE

Mathematics tasks are not the only feature in promoting mathematical challenge. In this endeavour, the teacher has a critical role in the process of fostering mathematical learning of students. To provide students with challenging situations is in itself a challenge for teachers. They have to choose adequate tasks and help students to carry on with appropriate assistance. This implies a wide and deep knowledge of the mathematics they teach and also a didactical one, in order to interpret students discourse and support students who are working on in new situations.

Creativity is new for us as a research field. We expect some results about the impact of teaching and learning elementary mathematics students with the support of these tasks and about if they are real opportunities to obtain creative solutions that contribute to the learning process. To check these goals we privilege classroom communication including questioning, oral presentations, written work and discussions, as well as focusing on the analysis and comprehension by the teachers of their students' mathematical thinking.



## REFERENCES

- Barbeau, E. & Taylor, P. (2005) (Eds.). ICMI Study 16: Challenging mathematics in and beyond the classroom: discussion document. *Educational Studies in Mathematics*. New York: Springer.
- Boaler, J. (2002). Open and closed mathematics: Student experiences and understandings. In J. Sowder, & B. Schappelle (Eds.), *Lessons learned from research* (pp. 135-42). Reston, VA: NCTM.
- Clement, L. (2004). A model for understanding, using and connecting representations. *Teaching Children Mathematics*, 11, 97-102.
- DFES (2000). *The Curriculum Guidance for the Foundation Stage*. London: QCA.
- Franke, M.L., Kazemi, E. & Battey, D. (2007). Mathematics teaching and classroom practice. In Frank Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, Vol.1 (pp. 225-256). Reston: NCTM.
- Holton, D., Cheung, K., Kesianye, S., Losada, M., Leikin, R., Makrides, G., Meissner, H., Sheffield, L. & Yeap, B. (2009). Teacher development and mathematical challenge. In Edward J. Barbeau and Peter J. Taylor (Eds.), *Challenging Mathematics In and Beyond the Classroom. The 16th ICMI Study* (pp. 205-242). New York: Springer.
- Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman and B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students*. (pp. 129-145). Rotterdam, Netherlands: Sense Publishers.
- Liljedahl, P., Chernoff, E. & Zaskis, R. (2007). Interweaving mathematics and pedagogy in task design: a tale of one task. *Mathematics Teacher Education*, 10, 239-249.
- Ma, L. (1999). *Knowing and teaching mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
- Meissner, H. (2000, July-August). *Creativity in mathematics education*. Paper presented at the meeting of the International Congress on Mathematical Education, Tokyo.
- National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston: NCTM.
- Orton, A. (1999). *Pattern in Teaching and Learning of Mathematics*. London: Cassel.
- Polya, G. (1981). *Mathematical discovery*. New York: John Wiley & Sons.

- Powell, A., Borge, I., Fioriti, G., Kondratieva, M., Koublanova, E. & Sukthankar, N. (2009). Challenging Tasks and Mathematical Learning. In Edward J. Barbeau and Peter J. Taylor (Eds.), *Challenging Mathematics In and Beyond the Classroom. The 16th ICMI Study* (pp. 133-170). New York: Springer.
- Remillard, J. & Bryans, M. (2004). Orientation Toward Mathematics Curriculum Materials. *Journal for Research in Mathematics Education*, 35, 352-388.
- Rivera, F. & Becker, J. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM Mathematics Education*, 40 (1), 65-82.
- Schoenfeld, A. (2008). On Modeling Teachers' In-the-Moment Decision Making. *Journal for Research in Mathematics Education. A Study of Teaching – Multiple Lenses, Multiple Views. Monograph No 14*, 45-96. Reston: NCTM.
- Sierpinska, A. (2003). *Research in mathematics education – through a keyhole*. Plenary Address CMESG 2003 Conference.
- Smith, M., Hughes, E., Engle, R. & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School*, 14(9), 548-556.
- Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3(4), 268-275.
- Stillman, G., Cheung, K., Mason, R., Sheffield, L., Sriraman, B. and Ueno, K. (2009). Challenging Mathematics: Classroom Practices. In Edward J. Barbeau and Peter J. Taylor (Eds.), *Challenging Mathematics In and Beyond the Classroom. The 16th ICMI Study* (pp. 243-284). New York: Springer.
- Usiskin, Z. (1999). Doing Algebra in Grades K-4. In Barbara Moses (Ed.), *Algebraic Thinking, grades K-12*. Reston: NCTM.
- Vale, I. (2003). Didactic materials in initial elementary mathematics teacher education: the use of manipulative in geometry. In Pawel Wlodkowic (Ed.), *Proceedings of CIEAEM 55*. University Plock, Poland.
- Vale, I., Barbosa, A., Borralho, A., Barbosa, E., Cabrita, I., Fonseca, L. & Pimentel, T. (2009). *Padrões no ensino e aprendizagem da matemática – propostas curriculares para o ensino básico*. Viana do Castelo: ESEVC – Projecto Padrões.
- Warren, E. (2006). Supporting learning in early algebra: a model of professional learning. *Proceedings of MERGA 29, Mathematics Education Research Group of Australasia* (pp.535-542), Canberra, Australia.
- Weiss, D. (2006). The Rationale for Using Manipulatives in the Middle Grades. *Mathematics Teaching in the Middle School*, 11( 5), 238-242.

## **INTRODUCTION TO THE PAPERS OF WG 8**

### **AFFECT AND MATHEMATICAL THINKING**

Marilena Pantziara<sup>1,2</sup> (Chair) <sup>1</sup>University of Nicosia, <sup>2</sup>Pedagogical institute, Cyprus

Pietro Di Martino, University of Pisa, Italy

Kjersti Wæge, Norwegian University of Science and Technology, Norway

Wolfgang Schloeglmann, Universität Linz, Austria

## **INTRODUCTION**

Affect has been a focus of increased interest in mathematics education for the last fifteen years. At Cerme 7, Working Group 8 was as stimulating and interesting as it had been in previous years with once again an atmosphere of collaboration between the 20 participants of the group. 13 research papers were submitted to the group and 12 were accepted for presentation after the review process. In the proceedings, 11 papers and two poster presentations were included, each of which investigated the multifaceted world of Affect.

The coordinators scheduled the seven conference sessions so that each participants paper could be presented to and discussed within the group. Fruitful discussion was encouraged by presenters also taking the role of responder to another paper, allowing the interactions between the papers to be highlighted. The program was scheduled in such a way as to enable the discussion to go beyond the papers, to a reflection on the structure and dynamics of the affective domain in an attempt to develop and expand the field in new directions. To this end, the first five sessions, based on the presented papers, were organized around five thematic topics: 1) Attitudes, 2) Beliefs, 3) Affect in mathematics thinking and learning, 4) Beliefs and Attitudes-Change, and 5) Other issues on Affect. In the two last sessions the work presented in the group was further discussed and possible new directions in the field were highlighted. Specifically the group discussed how to get a deeper insight into the affective constructs based on the questions: (a) How are the different constructs related to each other? What are the similarities and differences? (b) The different role of qualitative and quantitative approaches in the development of the Affective Domain. (c) The affective domain and other areas in the sphere of mathematics education. (d) Ideas for further studies.

### **The structure of the affective domain**

Research into affect has always been accompanied by a discussion of the various definitions of the affective concepts (e.g. Di Martino & Zan, 2010). Therefore the goals of this working group on affect remained the clarification of the definitions used, the introduction of new concepts in order to broaden the field and the elucidation of the relation between the affective constructs. A starting point for this discussion was McLeod's (1992) first categorization of the affective constructs. McLeod (1992) introduced the categories of beliefs, attitudes and emotions, each

## Working Group 8

distinguished by certain characteristics. Beliefs were seen as the one end of a continuum, more cognitive, more stable and less intense and emotions to be at the other end, affective, less stable and more intense. The context of the discussion of the group took a socio-constructivist perspective on learning continuing from the previous two working groups. This theoretical framework is characterized by its focus on the situatedness of learning and problem solving and by the close interaction between (meta)cognitive, motivational and affective factors in students' learning (Op't Eynde et al., 2006).

The concept of attitudes was presented to the group (Pepin; Pezzia & Di Martino). The authors moved beyond the simple definition that is attitude as the positive or negative degree of affect in relation to mathematics, to the model proposed by Di Martino and Zan (2010). The two studies investigated students' and teachers' attitudes shifting from the normative approach, measuring students' and teachers' attitudes towards mathematics, to an interpretive approach investigating causes that may shape students' and teachers' attitudes towards mathematics. An issue raised in the discussion was the nominalization in comparative studies (Pepin), the same statement may be used with different meaning in different context. The poster presentation by Vankúš was also related to students' attitudes towards mathematics.

Other concepts discussed in the group referred to self related beliefs, particularly self-concept and self-efficacy beliefs (Kleanthous; Panaoura et al.; Tuohilampi). In an attempt to clarify the concepts used and establish the relation between them, the discussion concluded that both self-concept and self-efficacy beliefs include individuals' evaluation of their ability in a domain, their confidence and how they see themselves in comparison to others. Kleanthous investigated students' self-efficacy beliefs and their dispositions to study mathematically demanding courses. Panaoura investigated differences in primary and secondary students' self-concept, self-efficacy beliefs and actual performance. Tuohilampi drawing from self beliefs and motivational theories (achievement goals) investigated the discrepancy between real and ideal self, a new direction in the field of affect. In the same realm and in the context of achievement motivation (Elliot & Church, 1997) the fear of failure was introduced as a self-evaluative framework (Pantziara & Philippou). In this study the authors investigated the causes of students' fear of failure by considering students' characteristics, family background and teachers' practices, once again revealing the complex world of affect.

New concepts were introduced and discussed in the group such as Personal Meaning. Vollstedt investigated the impact of context and culture on the different kinds of students' personal meaning in two different countries. Another new issue in the group was Resilience. Lee and Johnston-Wilder investigated ways to develop a mathematically resilient community of learners, who were confident enough to recruit other pupils to their way of thinking about effective ways to learn mathematics and to communicate those ways to teachers. Identity (Heyd-

Metzuyanım & Anna Sfard) was also introduced to the group as a collection of narratives that are told by or about individuals, narratives that are *reifying, endorsable, and significant*.

In an attempt to interpret the nature of affect in the social context of mathematics classrooms, new theoretical frameworks were presented. Lewis introduced the Reversal Theory. The foundation of this theory is the structure of the motivational landscape, and its eight constituent motivational states. Students experience the world as they move or reverse, between opposite states. Commognitive theory (Heyd-Metzuyanım & Anna Sfard) was also presented. This framework recognizes the centrality of communication in all our activities, including uniquely human forms of learning and thus mathematical learning. In the same vein, Edwards' poster presentation referred to the impact of friendship groups on mathematical reasoning.

Intentionally changing affect was an important issue in this group's discussion as in the previous two affect groups. The studies by Pezzia and Di Martino and by Stylianides and Stylianides described intervention programs specifically designed to change teachers' and student teachers' affect.

### **Research methods for investigating affect**

Various research methods were presented designed to investigate the multifaceted and complex affective domain. Some of the studies introduced qualitative approaches, such as observations and interviews (e.g. Heyd-Metzuyanım & Anna Sfard; Vollstedt) and others used quantitative approaches (questionnaires) (e.g. Tuohilampi). The trend in the group was for their research methods to combine qualitative and advanced quantitative approaches (e.g. SEM, Chic, Rasch) (Kleanthous & Williams; Lee & Johnston-Wilder; Panaoura et al.; Pantziara & Philippou) and also to use both teachers' and students' narratives (Peppin; Pezzia & Di Martino; Stylianides and Styliades).

### **Discussion and further considerations**

The discussion of the results focused on the development and clarification of the concepts and instruments in the domain. The contribution of a combination of qualitative and quantitative methods to this development was stressed. Clarity of the concepts within the domain was further enhanced by an increasing unity in the language used. The new concepts introduced may develop and refine existing concepts but the definitions of these new concepts must combine with or grow out of the existing ones. A second focus was the relation between different constructs in the affective domain and their connection to other areas in the realm of mathematics education such as problem solving and students' achievement in geometry, algebra etc. Several different relations have been revealed, such as the relation between students' negative/positive affect and achievement. The complex nature of these relations was also stressed, revealing that students characterized by the same affective constructs may behave differently in a mathematical learning context.

## Working Group 8

A third focus was the change in teachers' and students' affective constructs, considering how change can happen and how to instigate lasting change. Through the discussion different approaches to the stability of affective constructs emerged. The difference between state and trait was discussed and also the issue of the resistance to change in teachers' and students' affect. In addition, the distinct characteristics of intervention programs that changed affect in teachers were revealed. A last issue was whether the change in teachers' affect is also mirrored in their practices.

A last focus was the need to deepen our knowledge of the structure and dynamics in the affective domain. To this aim we followed Hannula's new theoretical framework on affect (Hannula, 2011). The framework is based on the integration of emotions with cognition and motivation, providing a dynamic approach to affect by using different angles. The broad distinction between affective states and traits is approached through an angle of motivation, cognition and emotion and at the same time through a psychological, psychological and social angle emphasising the dynamics of the different constructs involved.

The concluding remarks of the group included the necessity of continuing the research in the affect field using multi-method approaches; qualitative analysis of structures could describe patterns in students' engagement in the learning process while quantitative analysis of structures could focus on affective traits. Collaborative research between the members of the group was discussed. The work will go on.

## NOTES

1. We would like to acknowledge the contribution of Clare S. Lee in proof-reading the final version of the introduction.
2. Special thanks to the members of our group, Clare S. Lee, Gareth Lewis, Julie-Ann Edwards, Karen Skilling and Sue Johnston-Wilder for proof-reading the papers of the group.

## REFERENCES

- Di Martino, P., & Zan, (2010). "Me and maths": towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education*, 13(1), 27-48.
- Elliot, A., & Church, M. (1997). A hierarchical model of approach and avoidance achievement motivation. *Journal of Personality and Social Psychology*, 72, 218-232.
- Hannula, M. (2011, February). *Structure and dynamics of affect in mathematical thinking and learning*. Plenary presentation presented at the CERME7n, Rzeszów, Poland. Retrieved from <http://helsinki.academia.edu/MarkkuHannula>
- McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Learning and Teaching* (pp. 575-596). New York: MacMillan.
- Op't Eynde, P., De Corte, E., & Verschaffel, L. (2006). Accepting emotional Complexity. A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Education Studies in Mathematics*, 63, 193-207.

# **SELF – BELIEFS ABOUT USING REPRESENTATIONS WHILE SOLVING GEOMETRICAL PROBLEMS**

**Areti Panaoura\*, Eleni Deliyianni\*\*, Athanasios Gagatsis\*\*, Iliada Elia\*\***

\* Department of Primary Education, Frederick University, Cyprus

\*\* Department of Education, University of Cyprus, Cyprus

*The purpose of the present study was to investigate students' self-beliefs about using representations as a useful tool for understanding geometrical concepts and for solving geometrical tasks. The interest was concentrated on finding the differences between primary and secondary education in respect of students' self-concept beliefs, their self-efficacy beliefs about using representations and their real performance in geometry. Results indicated that, as was expected, secondary school students' performance is higher in all the dimensions of geometrical figure understanding. There is a coherent model of self-concept dimensions about the use of representations for understanding the geometrical concepts, which becomes more stable across the educational levels.*

Keywords: representations, self-beliefs, self-efficacy, self-concept, geometry

## **INTRODUCTION**

Students experience a wide range of representations from their early childhood years onward. Mathematics textbooks use a variety of representations in order to enable students to better understand mathematical concepts. In geometry, the understanding of mathematics requires that there is not any confusion between mathematical objects and the respective representation (Duval, 1999). In education it is important to investigate how pupils use and react to each teaching tool or procedure and what beliefs or conceptions develop. The present work correlates the students' beliefs about the use of representations in the learning of geometry with their respective performance on using them. Researchers embrace the belief that the development of geometrical concepts is multifaceted (Walcott, Mohr, & Kastberg, 2009) and agree that concept formation in geometry is potentially different from concept formation in other mathematics disciplines. We consider the present study to be a contribution to the extension of theoretical approaches of the cognitive and affective processes that underlie understanding in the learning of geometry.

## **Representations and the teaching of geometry in mathematics education**

The importance of studying and teaching geometry is well established in the literature and is stressed in contemporary mathematics curricula not only as an autonomous mathematics field, but also as a means to develop other mathematical concepts. Through the study of geometry, students are expected to learn about geometric shapes and structures and how to analyse their characteristics and relationships (NCTM, 2000), building understanding from informal to more formal thinking and passing from recognizing different geometric shapes to geometry

reasoning and geometry problem solving. Geometry is typically regarded as a difficult branch of mathematics for many students. As a mathematical domain, geometry is to a large extent concerned with specific mental entities, the geometrical figures. At a mathematical level, geometrical figures are mental entities, which exist only based on their definitions and their properties. But, a distance is identified between the geometrical-mathematical meaning of these specific concepts and students' personal meanings of geometrical figures, since, in students' minds, they are often related to real objects or are dealt with as pictures.

During the past twenty years, several mathematics educators have investigated students' geometrical reasoning based on different theoretical frames. For example, van Hiele (1986) developed a model referring to levels of geometric thinking, Fischbein (1993) introduced the theory of figural concepts and Duval (1999) reported the cognitive analysis of geometrical thinking. Duval distinguishes four apprehensions for a geometrical figure: perceptual, sequential, discursive and operative. Each has its specific laws of organization and processing of the visual stimulus array. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three dimensions. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, perceptual apprehension refers to the recognition of a shape on a plane or in space. It indicates the ability to recognize, in the perceived figure, several sub-figures. Sequential apprehension is required whenever one must construct a figure or describe its construction. Discursive apprehension is related to the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehensions.

In geometry, three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because, in a figure, all the relations of an object with other objects are depicted. However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. According to Mesquita (1998), an external representation has the nature of an object when the geometrical relationships utilized in the construction of the representation can be reutilised. In this case, it is possible to infer geometrical relationships from the construction of the figure; these geometrical relationships, such as parallelism, right angles etc. may be used in geometrical reasoning and proof. In the case when the representation has the nature of an object, the visual perception of the figure is consistent with the verbal statements of the problem. On the contrary,



when the external representation has the nature of an illustration, it is then impossible to directly extract a geometrical relationship from the construction of the figure and in this case the figure seems to “mislead”.

### **Self-concept and self-efficacy beliefs in mathematics**

Cognitive development of any concept is related with affective development. Affective domain in mathematics education is an area to which considerable research attention continues to be directed (Leder & Grootenboer, 2005). The relationship between affective factors and learning in mathematics is not simple, linear or unidirectional; rather it is complex and convoluted (Grootenboer & Hemmings, 2007). The relationship between cognition and affect has attracted increased interest on the part of mathematics educators, particularly in the search for causal relationships between affect and achievement in mathematics (Panaoura, Gagatsis, Deliyianni & Elia, 2009). Marsh and Craven (1997) maintain “enhancing a child’s academic self-concept is not only a desirable goal but is likely to result in improved academic achievement as well” (p. 155). The anticipated improvement of student performance is based on the existence of a reciprocal relationship between self-concept and academic achievement (Marsh, Trautwein, Ludtke, Koller & Baumert, 2005).

The literature suggests that there is an influential connection between affective mathematical views and performance in mathematics (Ai, 2002; Schreiber, 2002). One’s behavior and choices, when confronted with a task, are determined by one’s beliefs and personal theories, rather than one’s knowledge of the specifics of the task. Beliefs is a multifaceted construct, which can be described as one’s subjective “understandings, premises, or propositions about the world” (Philipp, 2007, p. 259). In the present study, attention is concentrated on the self-beliefs and their relation to the performance.

According to Pajares (2008) “self-efficacy should not be confused with self-concept, which, as a broader evaluation of one’s self, is often accompanied by the judgments of worth or esteem that typically chaperone such self-views” (p. 114). Self-efficacy beliefs refer to matters related to one’s capability and revolve around questions of “can”, whereas self-concept beliefs refer to matters related to being and reflect questions of “feel”. Academic self-concept is referred to as self-perceptions of ability, which affects students’ effort, persistence, anxiety (Pajares, 1996), and indirectly their performance. Self-concept includes beliefs of self-worth associated with one’s perceived competence (Pajares & Miller, 1994). Besides an individual impression, students could develop their academic self-concept externally through a comparison with their classmates (Wang, 2007). People who believe that they are capable of performing academic tasks use more cognitive and metacognitive strategies, and, regardless of previous achievement or ability, they work harder, persist longer, and persevere in the face of adversity. People with a strong sense of efficacy approach difficult tasks as challenges to be mastered rather than as threats to

be avoided. They have greater intrinsic interest in activities and they set themselves challenging goals and maintain a strong commitment to them (Pajares, 2008).

Based on the above, the purpose of the present study was to investigate students' self-beliefs about the use of representations for solving geometrical problems in relation to their self-efficacy beliefs, their self-concept beliefs and their real mathematical performance in primary and secondary education. The main purpose of the study was twofold: (1) to identify the statistically significant differences between primary and secondary school students concerning their understanding in the various geometrical figure dimensions and (2) to propose and validate a framework which describes the components of students' self-beliefs to solve tasks involving different geometrical figures and mainly the interrelations between those different types of beliefs.

## **METHOD**

The study was conducted among 1086 students, aged 10 to 14, of primary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus (250 in Grade 5, 278 in Grade 6, 230 in Grade 7, 328 in Grade 8). A test with 12 tasks (Deliyianni et al., 2009) was constructed and a questionnaire with 23 items was developed. The test consisted of three groups of tasks: (1) the first group of tasks includes two tasks concerning students' geometrical figure perceptual ability and their recognition ability, (2) the second group of tasks includes area and perimeter measurement tasks, which examine students' operative apprehension of a geometrical figure. Three other tasks require a reconfiguration of a given figure, while a task demands to synthesize two given figures in a new one, (3) the third group of tasks includes verbal problems that correspond to discursive figure apprehension.

A questionnaire (Table 2) was developed for measuring students' beliefs about the use of different types of representations for understanding geometrical concepts such as the area and the perimeter. Items were created with responses using a 5-point Likert scale ranging from 1=strongly disagree, to 5=strongly agree. The reliability of the whole questionnaire was high (Cronbach's alpha was 0.807 and 0.856 for primary and secondary education respectively).

## **RESULTS**

In respect of the first objective of the study, we present the results of the multivariate analysis of variance in order to identify the significant differences between primary and secondary school students concerning their performance in the different aspects of geometrical figure understanding. Then the collected data of the questionnaire about self-concept beliefs were first subjected to exploratory factor analysis in order to examine whether the factors that guided the construction of the questionnaire were presented in participants' responses. We present the statistically significant differences for self-beliefs factors in respect of students' grades. Finally we present the results of the confirmatory factor analysis, indicating the structure of the specific

affective factors, their main interrelations and the differences between primary and secondary education.

In order to determine whether there are significant differences between primary and secondary school students concerning their performance in the different aspects of geometrical figure understanding, a multivariate analysis of variance (MANOVA) is performed. Overall, the effects of students' educational level (primary or secondary) are significant ( $F_{6,1079}=34.43$ ,  $p<0.001$ ). The presentation of the results in Table 1 is indicative.

		$\bar{X}$	SD	$\bar{X}$	SD
Geometrical figure perceptual ability	$F_{1,1079}=79.51$ , $p<0.001$	0.62	0.38	0.45	0.41
Students' recognition ability	$F_{1,1079}=38.81$ , $p<0.001$	0.62	0.26	0.72	0.27
Operative apprehension of a geometrical figure	$F_{1,1079}=74.34$ , $p<0.001$	0.49	0.35	0.32	0.31
Place value modification tasks	$F_{1,1079}=36.03$ , $p<0.001$	0.45	0.42	0.31	0.38
Verbal problems (7,8)	$F_{1,1079}=105.38$ , $p<0.001$	0.63	0.40	0.38	0.4
Verbal problems (9,10,11)	$F_{1,1079}=0.03$ , $p=0.85$	0.25	0.31	0.24	0.21

**Table 1: Means and standard deviations of students' (primary and secondary education) performance on specific types of geometrical tasks**

The principal component analysis of students' responses to the items of the questionnaire revealed five factors ( $KMO=0.887$ ,  $p<0.001$ ) with eigenvalues greater than 1. Varimax rotation was used and as a consequence uncorrelated factors were revealed (Costello & Osborne, 2005). The eigenvalues, percentages of variances explained by factors and the factor loadings of the items are presented in Table 2.

The first factor corresponded to students' beliefs about the use of representations and materials for the better understanding of mathematical concepts (F1). The items at the second factor expressed students' self-efficacy beliefs in using representations for the understanding of geometrical concepts (F2). The third factor represented their self-efficacy beliefs about solving problems of area (F3), while the fourth factor represented their self-efficacy beliefs about solving problems of perimeter (F4). The fifth factor corresponded to students' self-concept beliefs about the use of diagrams, figures and representations in understanding mathematical concepts and their beliefs in using them as useful tools for explaining their geometrical thinking.

Analysis of variance (ANOVA) indicated that there were statistically significant differences in respect of grades for the factors F1, F2, F3 and F5. Those results depended on post-hoc analyses by using Scheffe. Specifically in the case of F1, there were differences at the means ( $F_{3,1060}=4.943$ ,  $p<0.001$ ) of the Grade 5 with the Grade 7 and the Grade 8 ( $\bar{X}_5=3.62$ ,  $\bar{X}_6=3.73$ ,  $\bar{X}_7=3.92$ ,  $\bar{X}_8=3.88$ ). In the case of the students' self-efficacy beliefs about using representations (F2), the statistically significant differences ( $F_{3,999}=7.349$ ,  $p<0.001$ ) were found between the Grade 5 with Grades 7 and 8 ( $\bar{X}_5=3.66$ ,  $\bar{X}_6=3.84$ ,  $\bar{X}_7=3.95$ ,  $\bar{X}_8=3.92$ ). The results were

## Working Group 8

different in the case of students' self-efficacy beliefs about solving geometrical problems of area ( $F_{3,1057}=4.237$ ,  $p<0.005$ ). The differences were between Grade 5 and Grade 8 with Grade 6 and Grade 7 ( $\bar{X}_5=3.84$ ,  $\bar{X}_6=3.76$ ,  $\bar{X}_7=3.72$ ,  $\bar{X}_8=3.59$ ). In the case of self-concept beliefs about using representations in order to solve or explain problems ( $F_{3,1034}=4.377$ ,  $p<0.005$ ), the differences were between Grade 7 with all the other Grades ( $\bar{X}_5=3.60$ ,  $\bar{X}_6=3.71$ ,  $\bar{X}_7=3.83$ ,  $\bar{X}_8=3.64$ ).

Item	F1	F2	F3	F4	F5
The diagrams (e.g. circle area, rectangle area, number line) are useful tools for the problem solving in mathematics	.53				
The use of materials (e.g. fraction circles, dienes cubes) is important for the primary school students.	.62				
The construction of a figure or a diagram is useful for the problem solving in mathematics.	.67				
I prefer solving problems which present the data at a schema.	.70				
I can easily solve area problems.	.78				
I can easily explain the solution of a perimeter problem verbally.	.79				
I can easily solve problems which present the data at a schema.	.70				
I can easily solve problems which present the data verbally.	.56				
If someone asks me to explain the solution of an area problem, I prefer to do it verbally.	.69				
I can easily indicate the data of a perimeter problem at a geometrical schema.	.59				
I prefer solving problems perimeter problems which present the data only verbally.	.73				
I can easily solve the problems of area which need the construction of a schema.			.68		
I am very good in solving area problems.			.69		
The schemata help me in solving area problems.			.64		
I can easily solve the problems of perimeter.				.55	
I can easily explain to my classmate the solution of a problem of a perimeter by using a schema.				.65	
I can easily find the perimeter of a figure.				.59	
I can easily use formulas for solving problems of perimeter (e.g the perimeter of a rectangle is $2X(\text{length} + \text{width})$ ).					.65
The good student in mathematics can present his/her solution and explain it by using many different ways.					.48
For the correct solution of a mathematical problem, the use of an equation is necessary.					.59
I find it easy to explain at my classmate how I have solved an area problem by using a formula (e.g. rectangle area=widthXlength).					.67
When I solve an area problem I construct a schema.					.47
If someone asks me to explain the solution of a perimeter problem I prefer to do it by using a schema.					.51
Eigenvalues	7.87	5.28	2.92	1.85	1.45
Percentage of variance explained	30.1	16.7	8.77	6.01	4.20
Cumulative percentage of explained variance	30.1	46.8	55.5	61.5	65.7

**Table 2: Factor loadings of the factors against the items associated with participants' beliefs**

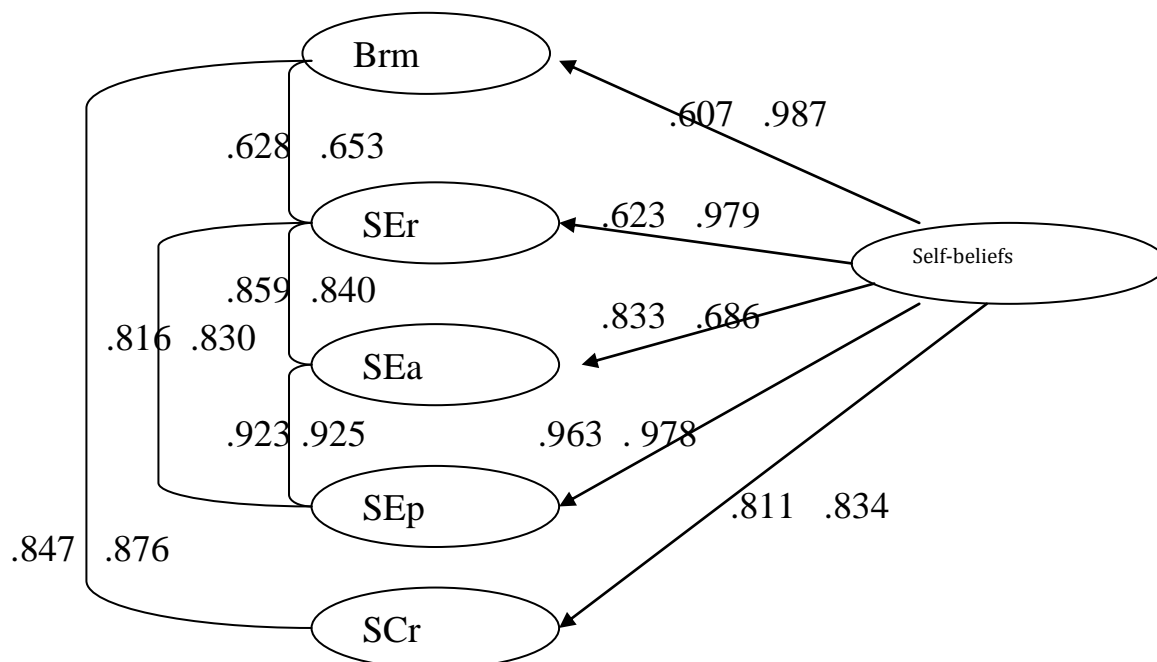
In order to confirm the structure of students' self-concept beliefs in respect of the use of geometrical representations, a CFA (Confirmatory Factor Analysis) model was constructed by using the Bentler's (1995) EQS programme. The tenability of a model can be determined by using the following measures of goodness of fit:  $\chi^2/df < 1.95$ , CFI > 0.9 and RMSEA < 0.06. Firstly, a first-order model was examined within the structural equation modeling framework. This model involved only one first-order factor, which associated all the items involved. This model did not have a good fit to the data and therefore, could not be considered appropriate for explaining students' behavior.

Figure 1 presents the results of the model that fits the data reasonably well for both the levels of education (primary education:  $\chi^2/df = 1.14$ ,  $df = 208$ , CFI = 0.972 and RMSEA = 0.020, secondary education:  $\chi^2/df = 1.45$ ,  $df = 209$ , CFI = 0.949 and RMSEA = 0.03). The second-order model, which is considered appropriate for interpreting students' self-representation beliefs, involves the 5 first-order factors, which were the results of the above exploratory factor analysis and one second-order factor. The first order factors regressed on a second order factor explaining the students' self-beliefs about using geometrical representations for solving geometrical tasks and understanding geometrical concepts, indicating that those factors are not independent.

The loadings of the whole model are higher in the case of secondary education almost in all cases. It is an integrated model of self-beliefs factors concerning the use of representations for solving geometrical tasks which becomes more stable across the educational levels, as a result of the continuous experiences in the teaching procedure and the more precise self-representation about the cognitive and affective performance. Students realize which tools and external or internal procedures help them to understand better the geometrical concepts.

There is a high statistically significant interrelation between the students' beliefs about using representations and their self-concept beliefs about using them for solving or explaining the solution of geometrical tasks [primary school (0.847), secondary school (0.876)], indicating that students who believe that representations are useful tools for understanding geometrical concepts tend to use representations in order to solve tasks and in order to explain to someone else the solution of a problem. At the same time students who use representations have already positive beliefs about the usage of this learning material. As was expected, the highest statistically significant interrelation is between students' self-efficacy beliefs for solving tasks concerning the concept of area with their self-efficacy beliefs for solving tasks concerning the concept of perimeter [primary school (0.923), secondary school (0.925)]. Students develop similar self-efficacy beliefs for both the concepts because in primary education they solve many problems, which ask them at the same time to find the area and the perimeter of a geometrical figure. Students with high self-efficacy beliefs about their ability to use representations, express positive beliefs about the use of representations on teaching and learning. The relation is higher in

secondary education (0.853) than in primary education (0.728) where students have more experiences with Euclidean geometry, they have more positive beliefs about the usefulness of representations and they have more positive self-efficacy beliefs.



Notes: 1. Brm= beliefs about the use of representations and materials, SEr= Self-efficacy beliefs about using representations, SEa= Self-efficacy beliefs about using representations for the concept of area, SEp= Self-efficacy beliefs about the using representations for the concept of perimeter, SCr= Self-concept beliefs about using and explaining geometrical representations, 2. The first and second coefficients of each factor stand for the application on the model of primary and secondary education.

**Figure 1: A confirmed model of students' self-beliefs about using representations in solving geometrical tasks**

## DISCUSSION

The first objective was to identify the statistically significant differences between primary and secondary school students concerning their understanding of the various geometrical figure dimensions and the second was to propose and validate a framework, which describes the components of students' affective performance to solve tasks involving different geometrical figures and mainly the interrelations between those factors. In respect of the first objective, differences existed in the geometrical figure understanding performance of primary and secondary school students. Particularly, secondary school students' performance was higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance. The performance improvement can be attributed to the general cognitive development and learning taking place during secondary school. In fact, the secondary school curriculum in Cyprus involves many concepts already known and mastered during primary school. This repetition of concepts leads to higher performance even though primary and secondary school instructional practices differ.

Students' self-efficacy beliefs in using representations are lower in primary education. The results were different in the case of the self-efficacy beliefs about solving geometrical problems with the concept of area where the means are lower in secondary education. Students at Grade 7 have high self-concept beliefs about using representations in order to explain solutions of geometrical tasks. Actually there is an increase up to this age and then there is a decrease. It seems that students start having a more precise self-image about their abilities and do not overestimate their abilities.

Confirmatory factor analysis indicated that there is a stable coherent model of affective dimensions about the use of representations for understanding the geometrical concepts. It becomes more stable across the educational levels as a result of the continuous experiences. General beliefs about the use of representations are related to the self-concept beliefs about using them as a tool to explain geometrical tasks problem solving. Students seem to connect the concept of area with the concept of perimeter and for this reason they have similar self-efficacy beliefs. It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis.

## REFERENCES

- Ai, X. (2002). Gender differences in growth in mathematics achievement: Three-level longitudinal and multilevel analyses of individual, home and school influences. *Mathematical Thinking and Learning*, 4 (1), 1-22.
- Bosnyak, A., & Nagy-Kondor, R. (2008). The spatial ability and spatial geometrical knowledge of university students majored in mathematics. *Acta Didactica Universitatis Comenianae*, 8 , 1-24.
- Costello, A., & Osborne, J. (2005). Best practices in exploratory factor analysis: Four recommendations for getting the most from your analysis. *Practical Assessment Research and Evaluation*, 10 (7), 1-9.
- Deliyianni, E., Elia, I., Gagatsis, A., Monoyiou A., & Panaoura, A. (2009). *A theoretical model of students' geometrical figure understanding*. In V. Durand-Guerrier, S. Soury-Lavergen & F. Arzarello (Eds.), *Proceedings of the CERME 6* (pp. 695-705). <http://www.inrp.fr/editions/editions-electroniques/cerme6>. Lyon.
- Duval, R. (1999). Representation, vision and visualizations: Cognitive functions in mathematical thinking. In F. Hitt & M. Santos (Eds.) *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3-26). Mexico.
- Fischbein, E. (1993). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. W. Scholz, R. Strasser, B. Winkelmann (Eds.). *Didactics of mathematics as a scientific discipline*. (pp. 231- 245) Netherlands, Dordrecht. Kluwer.
- Grootenboer, P., & Hemmings, B. (2007). Mathematics performance and the role played by affective and background factors. *Mathematics Education Research Journal*, 19 (3), 3-20.

## Working Group 8

- Leder, G., & Grootenboer, P. (2005). Affect and Mathematics Education. *Mathematics Education Research Journal*, 17 (2), 1-8.
- Marsh, H. W., & Craven, R. (1997). Academic self-concept: Beyond the dustbowl. In G. Phye (Ed.), *Handbook of classroom assessment: Learning, achievement, and adjustment* (pp. 131-198). Orlando, FL: Academic Press.
- Marsh, H. W., Trautwein, U., Ludtke, O., Koller, O., & Baumert, J.(2005). Academic selfconcept, interest, grades, and standardized test scores: Reciprocal effects models of causal ordering. *Child Development*, 76(2), 397-416.
- National Council of Teachers of Mathematics, (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- Pajares, F. (1996). *Review of Educational Research*, 66(4), 543-578.
- Pajares, F. (2008). Motivational role of self-efficacy beliefs in self-regulated learning. In D. Schunk & B. Zimmerman (Eds). *Motivation and Self-regulated learning. Theory, Research and Applications* (pp. 111-140). New York: Routledge.
- Pajares, F., & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: a path analysis. *Journal of Educational Psychology*, 86, 193–203.
- Panaoura, A., Gagatsis, A., Deliyianni, E., & Elia, I. (2009) Students' beliefs about the use of representations in the learning of fractions. *Educational Psychology*, 29 (6), 713-728.
- Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing.
- Schreiber, J.B. (2002). Institutional and student factors and their influence on advanced mathematics achievement. *The Journal of Educational Research*, 95 (5), 274-286.
- van Hiele, P. (1986). *Structure and Insight: A Theory of Mathematics Education*. Orlando, FL: Academic Press.
- Walcott, C., Mohr, D., & Kastberg, S.(2009). Making sense of shape: an analysis of children's written responses. *The Journal of Mathematical Behavior*, 28, 30-40.
- Wang, J. (2007). A trend study of self-concept and mathematics achievement in a cross-cultural context. *Mathematics Education Research Journal*, 19(3), 33-47.

**ACKNOWLEDGE:** This paper draws from the medium research project MED19 that investigated the role of multiple representations in mathematics learning during the transition within and between primary and secondary school, which is supported by the University of Cyprus.



# **A COMPARATIVE STUDY OF NORWEGIAN AND ENGLISH SECONDARY STUDENTS' ATTITUDE TOWARDS MATHEMATICS**

Birgit Pepin

Sør-Trøndelag University College, Trondheim, Norway

*This presentation reports on a comparative study of English and Norwegian pupils' attitude toward mathematics, using a mainly qualitative questionnaire to develop a deeper understanding of the factors that may shape and influence pupil attitude. Albeit based on a small statistical sample, in both countries students' attitudes has very similar trends between grades 7/8 and 10/11. However, pupils' qualitative comments showed that the most influential factors were due to pupils' experiences in their respective mathematics classrooms and larger school environments. It is argued that by comparing it was possible to deepen our understanding of pupil attitude as a socio-cultural construct, and as a 'lived' construct influenced by the meanings that are made in different contexts.*

Keywords: pupil attitude towards mathematics; comparative study; situated-ness of pupil attitude.

## **INTRODUCTION**

Much research has been conducted in the area of 'affect in mathematics education': in terms of the role of emotions in mathematical thinking in general; in terms of the role of affect in learning; and of the role of affect in social contexts such as the classroom (Hannula, Evans, Philippou & Zan, 2004). Particular attention has been given to the influence of socio-cultural context on the formation of beliefs, and it is generally accepted (e.g. Op 't Eynde, De Corte & Verschaffel, 2006) that students' mathematics-related beliefs are "more complex, personal and context-specific ... than is generally thought." (p. 68) They argue that students' previous experiences in instructional environments were closely associated with beliefs.

Whilst the definitions of, and relationships between, constructs such as 'belief', 'attitude', 'emotions' and 'values' have been an area of great concern (Zan, Brown, Evans & Hannula, 2006), it is also unclear how to measure these, what influences them and how they are comparable across countries or cultures. Comparisons of how these constructs are 'lived' in different contexts and cultures (or countries), can not only identify similarities and differences, but are likely to deepen our understandings of what we mean by 'affect in mathematics education'.

## **THEORETICAL FRAMEWORK**

Without making judgement on any particular constructs and frameworks, for our comparative study where culture is likely to play a significant role, the socio-constructivist view of learning and emotions appears most appropriate, as it stresses

the situatedness of learning activities and connects the ‘cognitive, conative and affective factors in students’ learning’ (Op ‘t Eynde et al., 2006).

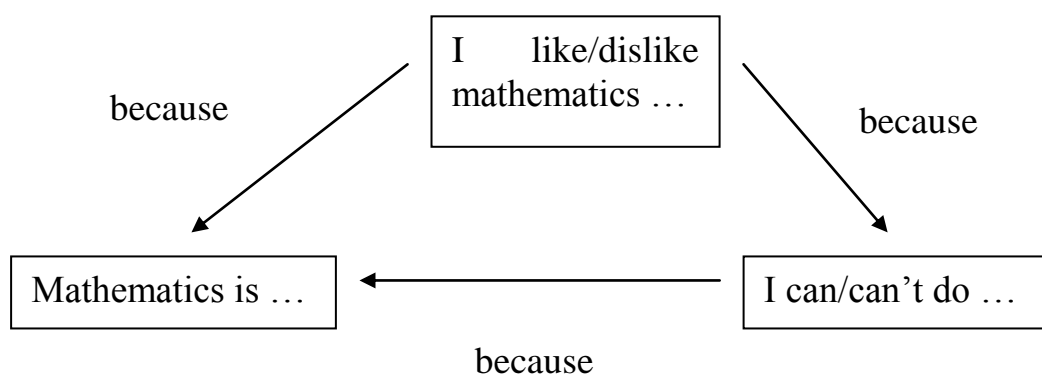
The theoretical framework of ‘attitude’ used in this study builds on Zan and Di Martino’s work (2007) who had investigated this construct for many years in an Italian project. Students are asked three ‘questions’ (see fig 1) and given the opportunity to “tell their own story with mathematics through an autobiographical essay” (p.163). Analyses of their data identified three core themes:

Emotional disposition towards mathematics expressed with “I like/dislike mathematics”;

Perception of being/not being able to be successful in mathematics, expressed with “I can/cannot do mathematics”;

Vision of mathematics expressed with “mathematics is ...”

(Zan & Di Martino, 2007, p.163)



**Fig. 1: Triad of Zan and Martino (2007)**

The rationale for choosing this scheme is that the scheme provides a strong link to practice and the educational experiences of pupils. Moreover, students are given the opportunity to narrate their story (Bruner, 1990), and it is not important whether the story is ‘likely’ or ‘true’, but students’ perceptions of what they have done, the underlying reasons, and the types of situations they encountered are at the core of this frame. This is of interest. According to Bruner (1990) the narrative is concerned with the explication of the person’s intentions ‘in the context of action’ and through the telling of stories, one is likely to be able to make sense of events, or work out meanings of actions and processes.

## THE STUDY

In previous studies the author developed an understanding of identity and ‘culturally figured worlds’ (Holland, Lachicotte, Skinner, & Cain, 1998) in mathematics classrooms in England (and other countries) and it emerged that national educational traditions are a large determinant and influence on teacher pedagogic practice and

classroom culture. More recently, she investigated Norwegian and English classrooms, in particular with respect to ‘attitude towards mathematics’. The collected data (for this study) consisted of qualitative questionnaires and selected lesson observations. The qualitative questionnaire was based on three questions/statements (‘I like/dislike mathematics because ...’; ‘I can/cannot do mathematics because ...’; ‘Mathematics is ...’) where pupils could add their comments, describe their experiences and generally tell their story. The questionnaires were administered to 307 Norwegian pupils in grades 6 to 11, altogether to 13 groups/classes (distributed over several schools), and the same (translated) questionnaires to 194 English pupils in grades 7 to 11, altogether to nine groups/classes.

It is acknowledged that the data sets are too small and are not likely to be representative for England or Norway. However, in terms of catchment areas and situation of schools, they are comparable: the English data were taken mainly in one inner city comprehensive school in a large city in the Mid-North of England; and the Norwegian data were taken mainly in one ‘inner city’ school of the largest city in the Mid-North of Norway. Considering that the population of the whole of Norway is only half of the population of London, it may not be appropriate to talk about ‘inner city schools’ in Norway; but by Norwegian standards the schools was situated in a large city environment.

Moreover, and in terms of the validity of the data, the data were analysed on the basis of the author’s understandings of the construct of ‘attitude toward mathematics’ and its potential influences. The main questions addressed are:

- (1) How do students perceive their learning of mathematics, and what are the aspects of their attitude towards mathematics?
- (2) What are the main influences which appear to shape pupil attitude towards mathematics?
- (3) What are the similarities and differences in the different settings, and how does that influence our understandings of the ‘attitude’ construct?

In terms of analysis the main emphasis was on discovery rather than testing of theory, and the analysis involved category generation and saturation based on *constant comparison* as advocated by Glaser and Strauss (1967). Concepts emerged from the data, they were checked and re-checked against further data, compared with other material, strengthened and refined, similar to a procedure described by Woods (1996). Di Martino and Zan (2010) used a related way of analysing their data.

Moreover, the author tried at one level to maintain the coherence of the groups’ responses in schools, by analysing the responses with respect to selected observations. At another level s/he analysed across cases with each country, using the conceptual framework of ‘attitude’ and testing the hypotheses offered by the literature, and building explanations and theorisations grounded in the data. On a

third level, she looked for similarities and differences of pupil attitude towards mathematics across the two countries and contexts. However, due to the additional cross-cultural dimension, it was important to address the potential difficulties with cross-national research, in particular issues related to conceptual equivalence, equivalence of measurement, and linguistic equivalence (Warwick and Osherson, 1973). Particularly important were the validity checks with respect to the ‘questionnaires’, and considerable time was spent amongst researcher colleagues to ensure ‘equivalent’ meanings and constructs. In this respect, it was important to locate and understand mathematics classroom practices and the classroom cultures in England and Norway, and it was useful to draw on knowledge gained from previous research conducted by the author, which highlighted the complex nature of classroom practices and environments in the two countries. In addition, national curriculum documents and guidelines (including textbooks) were analysed in order to study the contextual background of mathematics classrooms and the potential influences of these texts in each country.

## THE FINDINGS

Albeit based on a small sample, it is interesting to compare the two data sets quantitatively: interestingly more students like mathematics in Year 8 in Norway than in Years 7 or 9 in England, and especially the English Year 9 data (which are comparable in terms of age) ‘stick out’ there. At school leaving age (Year 10 for Norway, Year 11 for England) slightly more pupils like mathematics in the Norwegian schools than in English ones. It is also interesting to note that the same trend that Hodgen, Küchemann, Brown and Coe (2009) claimed, namely the drop of interest in mathematics from age 12 to 14, can also be claimed in both the Norwegian and the English data. However, after age 14 it seems that the interest increases again, and this is evident in both countries’ data sets: in Norway the positive attitude toward mathematics in Year 10 even overtakes the Year 8 ‘marks’. Thus, it is suggested that Hodgen et al. (2009) may have found a similar increase in interest, had they surveyed older children. Perhaps the most interesting finding here is that the ‘trends’ in attitude, going from Years 7 to 11, are similar in both countries’ data sets. The pupil comments appear to suggest that this was the case because students had accepted that, given the right conditions, they were ready to work at this subject in order to get good grades in their (compulsory) school leaving examinations, which in turn were likely to give them more opportunities in life. Thus, the ‘exchange value’ of mathematics, and this is linked to the assessment system, appears to influence pupils’ attitude towards mathematics.

## Working Group 8

<b>England</b>			
(194 pupils)	Like mathematics	Indifferent	Dislike mathematics
Y7 (age 11/12) 74 pupils- 3 classes	(31/74=) 42%	9 %	49%
Y9 (age 13/14) 57 pupils- 3 classes	(22/57=) 39%	16%	46%
Y11 (age 15/16) 63 pupils- 3 classes	(26/63=) 41 %	22 %	37%
<b>Norway</b>			
(278 pupils)	Like mathematics	Indifferent	Dislike mathematics
Y8 (age 13/14) 41 pupils- 2 classes	(18/41=) 44%	22%	34%
Y9 (age 14/15) 158 pupils- 6 classes	(51/158=) 32%	23%	45%
Y10 (age 15/16) 79 pupils- 3 classes	(36/79=) 46%	16%	38%

**Table 1: Pupil attitude (%) per year group and country**

Analysing the qualitative comments, this paper argues in support of ‘pupil attitude toward mathematics’ as a socio-cultural construct that connects the cognitive, motivational and affective factors in students’ learning of mathematics. Leaning on the work of Op ‘t Eynde et al. (2006) the author contends that pupil attitude is embedded and shaped by the context in which it develops. Looking across pupils’ comments (on the questionnaires) there were seven themes that emerged from the data:

- (1) Mathematics for jobs and ‘later life’;
- (2) Mathematics is interesting, but hard and challenging for some, and boring and frustrating for others;
- (3) Repetitive nature of mathematics in classroom lessons;
- (4) Importance of working in groups (also for thinking) and support of friends;

- (5) The role of the teacher;
- (6) The support of the family and primary school for being able to do mathematics;
- (7) Examinations play an important part, both in terms of individual success as well as what doing mathematics means.

Taking the example of Theme 1 (Mathematics for jobs and ‘later life’), and looking across pupils’ comments, it appears that in terms of liking/disliking mathematics most students stated that ‘mathematics is necessary for life’. In Norway, this is supported by curricular guidelines. Large scale international comparative studies in mathematics and science (TIMSS; PISA) have shown that Norwegian pupils perform relatively poor and significantly lower than the mean of other countries (e.g. Grønmo & Onstad, 2004), and there has also been noted a decrease in recruitment to science related studies (Schreiner, 2008). This, together with the low performance on international achievement tests, is likely to have influenced the Government in terms of increasing the emphasis on sciences and mathematics education.

Many pupils in the study had dreams of particular professions and needed mathematics to fulfil those dreams.

”You get to learn new skills and it will help you later on in life ... because algebra etc can help you in jobs such as an accountant...” (EY7-SO)

“... Maths can be tiring, but it opens opportunity doors.” (EY11- U8)

“... I struggle with any work ... my dream is to become a police officer and you need to have a GCSE in maths.” (EY9- JF)

However, this emphasis on the utilitarian aspect of mathematics, in the sense of using it as a means to achieve their own particular ‘distant’ goal, was more evident in English pupils’ comments than in Norwegian answers (from the questionnaires). This can be understood in the light of the different contexts in which pupils were working and living. In particular, the education system in England expects pupils to direct their studies very early, and pupils have to decide latest at the GCSE level (age 16) what kind of line they want to pursue at A level, which in turn has implications for further study (e.g. Higher Education) and hence for job opportunities. This also means that depending on their A level choices, they may not have any mathematics instruction after GCSE. In Norway every child has the right for upper secondary education regardless of their grades at the end of compulsory schooling (grade 10), and they have to continue mathematical studies for at least another two years after age 16. More than 95% of students in Norway continue with upper secondary education, either in ‘professional’ (e.g for apprenticeships) or ‘theoretical’ (e.g. traditional upper secondary mathematics and science) tiers.

Theme 2 (Mathematics is interesting, but hard and challenging for some, and boring and frustrating for others) and theme 3 (Repetitive nature of mathematics in

classroom lessons) are interrelated. Approximately half of the pupils, in particular those who seemed to succeed in mathematics (according to their comments), talked about mathematics as a ‘challenging’, ‘interesting’ but ‘hard’ subject to learn. Others characterised it as ‘boring’, ‘non-creative’ and ‘confusing’. These perceptions went right through all classes, ages, and achievement groups.

“It is fun and easy and very interesting subject ... I like being made to think in maths ... I also like to be made to write and do many activities in maths. I love maths so much that I make sure I don’t forget a calculation ...” (EY7- TO)

“... it is boring and it does not allow you to be creative. ... mathematics is also very complicated as you have to use many different formulas... it is stressful..” (EY11-U12)

Pupils’ perceptions were often linked to particular classroom practices and atmospheres. For example, many Norwegian pupils commented that it was a very theoretical subject where topics build on and connect to one another, and where very little practical work was done.

“I don’t like mathematics because it is such a theoretical subject. If you miss one theme the class works on, you don’t get the next theme. Mathematics builds upwards. And it is some calculations I don’t get so I have given up, so now I’m lost in mathematics classes. In addition, I have fear of failure in this subject, so I don’t give a crap about anything called mathematics.” (NY9- GN1)

English pupils perceived the nature of mathematics more rigidly as ‘getting the right answer’, and where little creativity is encouraged.

“... I find it easy to understand and because it is the same wherever you go. Numbers are always the same in every country, so anyone can do it. With maths, the answer is either right or wrong, whereas with other subjects like English and History, there are many different answers.” (EY11- DB)

“There is always a key book so there is always just one correct answer.” (NY9- GN5)

“... it is boring and it does not allow you to be creative. ... mathematics is also very complicated as you have to use many different formulas....” (EY11- U12)

In terms of classroom practices, and particularly in Norway, pupils commented on the repetitive nature of lessons, with textbook work and teacher presentation playing the main role.

“ ... mathematics could be funny if it was varied and not just writing ... I don’t like mathematics because it is too much of the same thing.” (NY9- GN2)

“... it is not an interesting subject. There is nothing fun towards this subject and a lot of it just feels like you’re doing the same thing all the time. I like the class I am in and how we really understand things ...” (EY11- U13)

## Working Group 8

“.. once you learn how to work out an equation or formula, you keep repeating the same method and there is always one correct answer. ...” (EY11- U14)

However, pupils wanted to work ‘differently’, in a problem solving way and with more open questions (than provided by textbooks), in order to develop a better understanding of the mathematics.

“I like problem solving. I think it is fun. I like to think and reflect long, because I then have a better chance of understanding. I become really happy when I manage to solve the problem. With Pascal’s triangle it was quite fun and I got the system, but when we went back to calculating arithmetic problems again I became bored. ... there should be more variation in the teaching. All we do is star down in to a book all the time. I find it a total waste of useful time.” (NY6- GN10)

Whilst not being able (for limited space) to explain all themes, the seventh theme (Examinations play an important part, both in terms of individual success as well as what doing mathematics means) provided evidence of examinations influencing pupils’ attitudes.

“I’m good at it and I like it because it’s good trying to work it out. But sometimes I don’t like it, because of exams ...” (EY7- DB)

“.. good because when you leave school you will have good exam results and you will be brainy. Also if you go to university you will get really good grades, and if you get good exam results in university then you will get a really good job.” (EY7- GE)

It seemed that the assessment system played a crucial role in pupils’ perceptions of what mathematics is and how to become a ‘proficient’ mathematics learner. For example, in both countries, but more in England, pupils practice on examination questions several months before the examination. This means that nearly all curriculum teaching is suspended, and pupils and teachers go over past examination papers- ‘teaching to the test’. Examinations appear to define whether a pupil is ‘good at maths’ or not.

## CONCLUSION

This article argues in support of ‘pupil attitude towards mathematics’ as a socio-cultural construct that connects the cognitive, motivational and affective factors in students’ learning of mathematics. Leaning on the work of Op ‘t Eynde et al. (2006) the author argues that pupil attitude is embedded and shaped by the context in which it develops. Furthermore, and looking at the qualitative data, one can conclude that the main dimensions identified by Op ‘t Enyde et al. (ibid) are represented: beliefs about self; beliefs about social context; and beliefs about social norms in pupils' own classes.

Interestingly, whilst there are differences which can be seen to be accounted for by differently ‘figured’ environments, there are also many similarities. For example, it



was interesting to see that, albeit based on a small sample, in both countries students had a positive attitude towards mathematics in year 7/8, which dropped in Year 9, and increased again in years 10/11. The pupil comments appear to suggest that this was the case because students had accepted that, given the right conditions, they were ready to work at this subject in order to get good grades, which in turn were likely to give them more opportunities in life. Thus, the ‘exchange value’ of mathematics, and this is linked to the assessment system, appears to influence pupils’ attitude towards mathematics.

In terms of differences it was interesting to see how pupils ‘lived’ their mathematics worlds, in school and ‘at home’. Pupil attitude appeared to be influenced by several factors which in turn were influenced by the different contexts in which pupils (and teachers) were working. For example, most pupils in both countries perceived mathematics as ‘non-creative’ and ‘theoretical’. However, the perceptions that underpinned these notions were different. In Norway ‘theoretical’ was explained as connected and topics logically building on each other. In England, however, the connotation was that it was theoretical if one could ‘understand the workings out and how numbers and statistics work’, in short a more formulaic approach - different from making connections in mathematics. This is likely to be influenced by how mathematics is perceived, also by teachers.

In theoretical terms, it is argued that the seven before mentioned themes influence and shape pupil attitude towards mathematics. Whilst not being entirely new constructs, the identified influences emphasise the situatedness of ‘pupil attitude towards mathematics’ within a ‘meaningful’ environment. Furthermore, it has been shown how these constructs are intrinsically interwoven, and come together in ‘attitude towards mathematics’, when pupils write about their experiences. Thus, it is claimed that it is not enough to identify the factors that may shape and influence pupil attitude, but more importantly to study how these are ‘lived’ by pupils, what meanings are made in classrooms and in different contexts, and how the factors interrelate and can be understood.

## REFERENCES

- Bruner, J. (1990). *Acts of meaning*. Cambridge: Harvard University Press.
- Di Martino P., & Zan R. (2010). 'Me and maths': Toward a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education*, 13(1), 27-48.
- Glaser, B.G., & Strauss, A.L. (1967). *The discovery of grounded theory. Strategies for qualitative research*. Chicago: Aldine.
- Grønmo, L.S., & Onstad, T.S. (Eds). (2009). *TIMMS 2007: Tegn til bedring?* Oslo: Universitetet i Oslo.

## Working Group 8

- Hannula, M., Evans, J., Philippou, G., & Zan, R. (2004). Affect in mathematics education- exploring theoretical frameworks. In M.J. Høines & A.B. Fuglestad (Eds.), *Proceedings of the 28<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education: Vol. 1*(pp. 107- 136). Bergen: PME.
- Hodgen, J., Küchemann, D. Brown, M., & Coe, R. (2009). Lower secondary school students' attitudes to mathematics: evidence from a large-scale survey in England. In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics: Vol. 29*(3) (pp. 49-54). Loughborough: BSRLM.
- Holland, D., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge, Mass. and London, UK: Harvard University Press.
- Op 't Eynde, P., De Corte, E., & Verschaffel, L. (2006). "Accepting emotional complexity": a socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, 63, 193-207.
- Schreiner, C. (2008). Noen som passer for meg? Ungdoms valg av utdanning og yrke. *Kimen* (1). Oslo: Naturfagsenteret.
- Warwick, D., & Osherson, S. (Eds.). (1973). *Comparative research methods: An Overview*. Englewood Cliffs: Prentice-Hall.
- Woods, P. (1996). *Inside schools: Ethnography in educational research*. London: Routledge & Kegan Paul.
- Zan, R., Brown, L., Evans, J., & Hannula, M.S. (2006) Affect in mathematics education: an introduction. *Educational Studies in Mathematics*, 63, 113- 121.
- Zan, R., & Di Martino, P. (2007) Attitude toward Mathematics: overcoming the positive/negative dichotomy. *The Montana Mathematics Enthusiast*, Monograph 3, 157- 168.

# THE PUPILS' VOICE IN CREATING A MATHEMATICALLY RESILIENT COMMUNITY OF LEARNERS

Clare Lee and Sue Johnston-Wilder

Open University and University of Warwick

*We report on our work with one school to develop a mathematically resilient learning community. Pupils acted as Ambassadors to take the ways of working developed together into their mathematics classes and to bring the voice of all the pupils to our meetings. We describe our work within the school and what the pupils told us about the way that they learned mathematics and the way that they felt their learning could be improved. The knowledge we gained was shared with the teachers in the school. Some staff welcomed the ideas; others felt threatened by the notion of working in different ways. We argue that in order for the school to develop into a mathematically resilient learning community a change in thinking is needed by many mathematics teachers.*

Keywords: Mathematical Resilience, community of learners, pupil voice

## INTRODUCTION

In this paper, we report on a single cycle within an ongoing action research project undertaken in one school to facilitate formation of a mathematical learning community, where pupils learned from teachers and from each other, and teachers sought to learn from each other and the pupils. The intention behind the development of this community was to start to work towards increasing the overall attainment of the school's pupils in mathematics examinations taken at age 16. Schools in England are increasingly concerned about their pupils' attainment in mathematics as they can be judged to be 'failing to meet the needs of their pupils' if examination results in mathematics fall below an arbitrary measure. There is also an agenda, driven by government agencies, to increase the number of students studying STEM (science, technology, engineering, and mathematics) subjects at University (Roberts, 2002).

## THE CONTEXT

We work regularly alongside teachers of mathematics, and recently began an action research project focussed on the notion of mathematical resilience (Johnston-Wilder & Lee, 2010a), the first cycle of which was reported in Johnston-Wilder and Lee (2010b). The first cycle indicated that, in addition to raising mathematical resilience, working in this way also raised attainment of the pupils in a difficult situation.

The cycle of action research presented here took place in an all-girls school in an urban setting in the West Midlands region of England. The school is considered a 'high attaining school', as it is always towards the top in school league tables. However, the managers saw a problem; the overall results in English examinations were always higher than in mathematics. As a result of our previous work, the

Advanced Skills Teacher in maths invited us to work with the mathematics department in order to see whether we could help to narrow this gap. The cycle reported in this paper focused on introducing pupils and selected teachers to strategies for engaging and empowering pupils in learning mathematics, building their understanding of mathematical resilience and using them as conduits for change. We worked with pupils in Year 8 (aged 12-13 years) to change their stance towards learning mathematics and thereby raise their attainment (Dweck, 2000).

Our particular role was to enable the pupils to have an informed mathematical voice and to allow that voice to be heard. We were concerned that, when we asked the pupils how their mathematical learning could be improved, most pupils would give stereotypical or naive answers because they had not experienced different ways of learning mathematics. We knew from the theoretical considerations discussed below that, for example, increasing the pupils' ability and opportunity to take part in mathematical discussions, and encouraging them to work collaboratively, would enable them to surmount some of the barriers that learning mathematics often presents and to become more mathematically resilient. The results clearly showed that many of the pupils already knew this.

Typically for an English school, the school had grouped its pupils into classes for mathematics according to their attainment in internal examinations. The teacher of each of these 12 classes or 'sets', chose two girls to take part in the project as 'Ambassadors'. The girls were chosen because of their ability to speak out and take a lead within their own group. We had, in the community of Ambassadors, a mixture of girls in terms of mathematical attainment and mathematical confidence.

### **THEORETICAL BASIS**

The way that we worked in the school was based on the ideas of building mathematical resilience that we have published elsewhere (Johnson-Wilder & Lee, 2010a, 2010b). Our underlying intention was to encourage the teachers to act to make the classroom a more positive place to be and one where barriers to learning mathematics could be overcome. If mathematics is difficult to master, and we see that it often is, then pupils need to develop a positive adaptive stance towards mathematics which will allow them to continue learning despite barriers and difficulties. This positive adaptive stance towards mathematics we have named as ***mathematical resilience*** (Johnston-Wilder & Lee, 2010a). Characteristics of mathematical resilience as we have described it include perseverance when faced with mathematical difficulties, working collaboratively with peers, having the language skills needed to express mathematical understandings and having a growth theory of mathematical learning (Dweck, 2000). Any learning may require resilience at times and can be actively promoted (Newman, 2004); however, we argue that pupils particularly require resilience in order to learn mathematics because of various factors that include: the type of teaching often used (Nardi & Steward, 2003; Ofsted,

2008), the nature of mathematics itself (Mason, 1988; Jaworski, 2010) and pervasive beliefs about mathematical ability being ‘fixed’ (Dweck, 2000, Lee, 2006).

It has been established that emotions have an important role in mathematical thinking generally (McLeod, 1992) and that powerful affective structures are a key factor for effective mathematical learning (Goldin, 2002). We see resilience as an important positive affective construct. Resilience enables pupils to make positive use of their affective domain and is built when teaching takes account of the four aspects of affect: emotions, attitude, beliefs and values (Hannula, Evans, Philippou, & Zan, 2004). We consider that building mathematical resilience offers a way to counter the well-known global affective structures that impede mathematical learning, commonly called “maths anxiety” (Richardson & Suinn, 1972). We see teaching for resilience as facilitating a positive self-belief or self-efficacy in pupils learning mathematics, which have been shown to be influential factors determining the interpretation and appraisal processes constituting their affective responses and emotions (McLeod, 1992).

In this study, we wanted to develop a mathematically resilient community of learners, who were confident enough to recruit other pupils to their way of thinking about effective ways to learn mathematics and to communicate those ways to teachers, thereby including both teachers and pupils in the purpose of improving mathematical learning in the school. Communicative aspects of resilience were particularly important to us in this project. We aimed to promote dialogic interaction, thereby enabling intra-mental ideas to subsequently become inter-mental (Vygotsky, 1981). The strategies that we asked pupils to use arose from work such as Vygotsky (ibid) and Lee (2006), which show that thinking and communicating are intricately intertwined. For example, we invited the pupils to try making mathematical videos and peer teaching. We have previously demonstrated (Lee and Johnston-Wilder, 2010) that video-making is a device that can be used in mathematics departments to increase pupil articulation and autonomy, so we incorporated video-making into this cycle. Peer teaching has also been demonstrated (Lee, 2006) to increase articulation, change pupils’ mathematical identities and increase agency.

### ***what we Did***

Our work with the school consisted of three days over half a term spent with the mathematics Ambassadors and additional time emailing and meeting three times with representatives of the teachers in order to plan and review the days. We were joined by a drama teacher from Creative Partnerships ([www.creative-partnerships.com](http://www.creative-partnerships.com)), whose role was to inform about and model the use of drama to support mathematical learning. Drama can be seen as enabling dialogic communication and therefore this expert practitioner added to our expertise in building a mathematically resilient community. We used the three days in two distinct ways. Firstly, we introduced the pupils to different ways of learning about mathematics. Secondly, we used the days

to enable the pupils to collect and analyse data about the ways that Year 8 pupils in the school felt would be effective in enabling them to learn mathematics.

On the first day, we introduced a questionnaire to examine how pupils felt about the way school encouraged them to learn maths. The questions we asked derived from the work of Dweck (2000) on fixed and incremental theories of learning and the work of Fennema and Sherman (1976) on assessing attitudes to mathematics. We asked the Ambassadors to examine the questionnaire and to suggest changes to make questions more accessible to their peers and any extra questions that may be needed. The original questionnaire was changed in the light of the Ambassador's suggestions and they administered the questionnaire to their mathematics groups. We collated pupils' responses before our next day in school, when we asked the Ambassadors to analyse the response data and identify points that seemed important to them. They reported finding this both challenging and interesting. We also asked Ambassadors to keep journals describing their feelings about, and reactions to, the mathematics they were learning and the way that they were learning it, both in lessons and during our days with them. The Ambassadors were asked to focus in their journals on their own and their peers' feelings and reactions and that their journals were not to critique teachers or record any negative personal reactions other than to learning mathematics. These journals were brought to the second and third day workshops and each pupil drew attention to important points in their journal.

Therefore our data on the way that the mathematics learning community developed in this school consisted of: field notes about our plans for the days and the reasons for revisions of the plans, field notes about our discussions with the teachers in the school and the drama teacher, evaluations from the pupils from the day workshops and notes from our discussions with them, the results from the questionnaire on attitudes to mathematical learning and the Ambassadors' journal entries along with the pupils own analyses of both the questionnaire results and the journal entries.

### ***What we found out***

#### **The Questionnaire Results**

The Ambassadors distributed the questionnaire to all the pupils in their mathematics classes, collected the completed questionnaires and posted them to us for analysis. Therefore the responses we have represent the views of all the Year 8 girls in 2010. We entered the responses into a spreadsheet, created pie charts and asked the Ambassadors to tell us what the results indicated to them. Thus our analysis of the questionnaire results is informed by the Ambassadors. The results showed that many of the attitudes to learning displayed by the Year 8 girls in this high-achieving school corresponded to those that we would see as resilient. For example, 78% said that they worked hard in mathematics lessons and 80% agreed with the idea that 'I can get smarter at maths if I work hard'. However, 16–20% of the girls were rather more

disaffected; it appears that the school will need to work with this cohort and engage them in mathematics in order to raise the school's attainment in mathematics.

We were struck by how resilient in general our respondents appeared to be; 94% reported being sure that they would be able to learn new work in all subjects. This level of confidence dropped by 6% when pupils were asked specifically about mathematics; nevertheless, 88% of the girls were confident in their ability to learn more mathematics. The resilient stance of the majority extended to their willingness to undertake tasks even if they knew that they might not 'do well' at the task; only 17% said that they would not engage with such tasks.

This resilient stance was not so evident when we asked if they "... sometimes would rather get good marks than understand the work." Only 40% agreed and a further 33% were not sure, leaving 27% valuing understanding over good marks. A further 78% said that they preferred getting a good mark to being challenged. Such results may indicate that the majority of pupils in the school are currently motivated by the idea of 'good marks' rather than understanding mathematics. This attitude is further emphasised by the fact that 53% disagreed with the statement "In addition to getting a right answer in maths, it is important to understand why the answer is correct." and 58% did not disagree with the statement "It does not really matter whether you understand a mathematics problem if you can get the right answer."

The answers also showed that for many of the pupils mathematics is 'a chameleon' (Johnston-Wilder & Lee, 2010a). 24% of the girls were not sure that studying mathematics would help them earn a living and 23% thought studying mathematics might be a waste of time; for such girls, mathematics lessons did not help them to distinguish how mathematics appears or is useful in the world outside school.

The data showed that 28% of Year 8 girls enjoy mathematics all the time, 55% some of the time but 17% not at all. It is to be expected that the girls did not all indicate that they enjoyed mathematics all the time; however, that 17% of the girls do not enjoy mathematics at all in the early years of their secondary school careers is rather worrying and, we suggest, underlies the relatively lower attainment in mathematics.

### **The Workshop Days**

The tasks that we used during the workshop days involved learning with ICT using Grid Algebra (Hewitt, n.d.) and Autograph software, making videos (see Johnston-Wilder and Lee (2010b) for a description of this in another context) to explore where mathematics can be found in the real world, some drama activities as well as data analysis. Our notes from the days, and the pupil's own evaluations, show that all 24 girls enjoyed making the videos: the particular elements that they mentioned about the days were the team work, being able to go outside, using ICT in the form of video recorders, the boost that the activities gave to their confidence and the fact that the activities were more interesting and fun than they had expected. One of the pupils who worked with Grid Algebra wrote in her evaluation of the day: '*something like*

*nth term is usually boring but we understood it*'. After being asked to show the class their work on Grid Algebra, one girl wrote *'I enjoyed making the presentation as I learned more about algebra. I would like to do something like this in my lessons as we could perform to each other and learn more'*.

Pupils particularly valued finding out some mathematics as part of a group and they enjoyed both working with friends and working with people that they had not worked with before. A pupil wrote: *'we brought our confidence out, writing and really being creative'*. The elements of choice they were offered were also important, as were using visual aids and sharing work. Another pupil wrote: *'all the projects were interesting and my thoughts about maths have really changed'*. The pupils also mentioned enjoying the more active way of learning that they were offered and that they would like to do such activities more often in their mathematics lessons.

### **The Journals**

The pupils gave us permission to take their journals away for analysis. We noticed ideas in the data that informed us about the pupils' learning; we collected like ideas together and reflected on the import of what they were saying. From this we devised 'stable categories' (Cohen, Manion & Morrison, 2007) that reflected the key ideas discussed by the pupils. A draft letter to the teachers was constructed from this information. This letter was discussed with the Ambassadors during the third day workshop and the quotes given below either come directly from the pupils' journals, were recorded during the discussions of the letter or are from the wording of the letter agreed with the Ambassadors.

We discovered that these pupils intuitively knew ideas about effective teaching and learning of mathematics that were supported by research literature. For example, they either knew or had discovered that, in the best lessons, teachers talk less and consequently pupils talk more. Many of the journals mentioned that mathematics teachers talk too much. *"When we are not involved enough we lose focus so we would like less teacher talk, more pupil work and more expectation of effort."*

It is important that pupils feel able to ask their teachers when they do not understand and that the teachers *"explain and help if we are stuck"*. Peers are also important to learning; the Ambassadors said that classmates should be allowed to help one another and recognised that pupils learn best when they can support each other and *'have a laugh occasionally'*. Contrary to the questionnaire results, the Ambassadors' descriptions of lessons valued understanding and reported that pupils liked lessons where all are given a chance to understand the essential elements. The timing of the lessons was important; according to the pupils, lessons should be well timed and not involve *"sitting still for too long and being bored"*. The pupils are aware that they do not do well in an environment where the work is *"boring and repetitive"*.

The journals also made clear that the pupils *"would like teachers to have higher expectations of us"*. They wanted more challenge and felt they would be more



## Working Group 8

engaged when challenged. *“We don’t mind hard work. We are not afraid to work hard.”* They liked teachers to be sufficiently strict in lessons to ensure that pupils learn, but not so severe that they discourage questions and problems. Pupils stressed that they like teachers who set high expectations and expect them to do well. Pupils enjoy working on difficult questions *“that will help us in the long run.”*

These pupils would like more variety in mathematics lessons, less book work, a diversity of tasks and more group work. They do not believe they learn or remember when working solely from books. They like to be active, interacting with other people, giving presentations, working independently and completing projects or extended work. They said such activities boosted their confidence and helped them learn to work independently. They enjoyed it when a pupil prepared and taught part of a lesson; they said it is useful when pupils work at the board for the rest of the class to consider and question. They see value in complex tasks, using a range of skills in a lesson. Similarly, they would like projects and extended work.

*“We would like more interactivity, more games and interesting activities, more practical work and creative tasks, like making and testing helicopters as some did this term. We like more fun activities and we like adventures ... We like maths we can recognise in the real world”.*

The pupils like to learn using computers and said computers are not just for playing games. They felt that work such as making presentations using ICT helped them to learn and those who had opportunity to use Grid Algebra recognised its value. They also suggested that they should be given optional ICT tasks for homework.

Pupils would like to support each other more; they asked for more group activities, team work and collaborative work. *“For example, one day this term, we did a GCSE problem and had to work as a group. It went well and everyone enjoyed it and began to work as a team.”* Sometimes, teachers could split the class into *“those who can do it and those who can’t.”* The pupils pleaded: *“Make sure all pupils understand the topic”.* They recognised that they needed to be proficient in using the mathematics register if they were to fully understand and become confident in using mathematics. *“We would like teachers to give us more help on the meaning of words.”*

The working environment was important; the pupils liked the room to be not too hot so that they could concentrate well and they liked interesting wall displays. The ethos of the class was crucial; pupils needed a relaxed environment where they could feel trusted and allowed to talk to one another whilst working. They do not enjoy working in silence: *“we don’t like the atmosphere of silence and it makes us feel locked in. We like it when people are talking, getting on with interesting work and able to ask questions with a helpful teacher.”* They also told us that they did not like to be asked if they did not know – it made them feel *“dumb.”*

## CONCLUSION

We were struck by the extent that the pupils' ideas resonated with research about learning mathematics and our own research about how pupils become more mathematically resilient. Pupils understood the importance of collaborative learning (Wiliam, 2008; Mercer & Littleton, 2007) and how important it is for the pupils to use the language of mathematics (Vygotsky, 1981; Lee, 2006). They also understood the importance of variety in keeping them motivated and interested.

There was a contradiction in the data concerning the importance of understanding. In the questionnaire, only 27% unequivocally valued understanding over getting good marks, whereas the Ambassadors' journals clearly plead for the teachers to "*Make sure all pupils understand the topic*". This could be a false dichotomy; it is likely that the pupils appreciate that, when they understand their work, they get good marks in examinations. It seems more likely from the questionnaire results that the dialogue in school values marks over understanding and that the pupils' plea about understanding should be listened to, if attainment in mathematics is to be improved.

The majority of pupils were willing to form a learning community in order to learn more about succeeding in mathematics. Most of them willingly presented their views about what they thought best helped them to learn mathematics. Those few who were less willing told us that they did not expect to be listened to and hence considered the process a waste of time. They became more willing as it became clear that the data they provided was recorded and considered. However, their conviction that teachers would not listen was partly supported. The teachers themselves, who had all been willing participants at the start of the process, became divided along a continuum by the second workshop. There were some who dropped in on the pupil days to see what was going on, discussed ideas with us and invited their pupils to demonstrate new ideas. However, others were deeply suspicious; they came to share lunch with us but were reluctant to talk and did not allow pupils write in their journals during lessons. Other mathematics teachers varied between these two extremes.

Why the divergence of attitudes on behalf of the teachers who had agreed to take part in the project in the first place? We know, from teachers who have discussed the ideas with us, and from the completed journals, that the reluctant teachers have a view of teaching that conforms with the stereotypical teaching described by Nardi and Stewart (2003). Therefore, the changes that were being promulgated may have appeared to deskill these teachers, possibly making them feel incompetent.

The data collected from the Year 8 ambassadors conformed to the way research defines 'effective' teaching: active, reflective, collaborative and grounded in the real world (Askew & Wiliam, 1995). Many of the pupils said firmly that they were not afraid of 'hard work'; they enjoyed being challenged, and working on complex, tractable problems. This view is far from the atomised practice of mathematics teaching prevalent in many schools in England. However, from our data, based in the

pupils' experience, we argue that part of enabling the school to develop into a mathematically resilient learning community will be a change in identity (Holland, Skinner, Lachicotte, & Cain, 1998) on the part of some of the mathematics teachers; moving from practices that tend to be repetitive, and focussed on techniques, to working with the pupils to develop mathematical understanding; moving from teacher as deliverer of knowledge, to pupil as an active, resilient participant in the learning process.

The pupil data clearly confirms our previous research that, if pupils are to become sufficiently mathematically resilient to overcome the barriers presented in learning mathematics, they must be more involved at all stages in the learning process. They must feel that they have the ability, opportunity, time and confidence to work to overcome any obstacles that are presented in learning mathematics. Therefore, they must feel supported by teachers and their fellow pupils, be challenged by the activities used and have time to fully engage in and succeed with their mathematical learning. Their questions must be fully answered and understanding should be valued over everything else. Above all pupils must not be made to feel "dumb".

## REFERENCES

- Askew, M., & Wiliam, D. (1995). *Recent Research in Mathematics Education*, London: HMSO.
- Autograph (3.3) [Computer software]. Oundle: Eastmond Publishing Ltd. Available from <http://www.autograph-maths.com/>
- Cohen, L. Manion, L., & Morrison, K. (2007). *Research Methods in Education* (6<sup>th</sup> Edition). London: Routledge/Falmer.
- Dweck, C. (2000). *Self Theories: Their Role in Motivation, Personality and Development*. Lillington NC: Psychology Press, Taylor & Francis.
- Fennema, E., & Sherman, J.A. (1976). Fennema-Sherman Mathematics Attitudes Scales: Instruments designed to measure attitudes toward the learning of mathematics by males and females. *Catalog of Selected Documents in Psychology*, 6(1), 31. [doi:10.2307/748467](https://doi.org/10.2307/748467)
- Hannula, M., Evans, J., Philippou, G., & Zan, R. (2004). Affect in mathematics education – exploring theoretical frameworks. In M. J. Høines, & A. B. Fuglestad, (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education: Vol. 1* (pp. 107-136). Norway, Bergen: PME.
- Hewitt, D., Grid Algebra [Computer Software]. Derby: ATM. Available from: <http://www.atm.org.uk/shop/products/sof071.html>
- Holland, D., Skinner, D., Lachicotte, W., & Cain C. (1998). *Identity and agency in Cultural Worlds*. Cambridge, Mass.: Harvard University Press.

- Goldin, G.A. (2002). Affect, meta-affect, and mathematical belief structures. In G.C. Leder, E. Pehkonen, & G. Törner, (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (59–72). Dordrecht: Kluwer.
- Jaworski, B. (2010). The Nature of Mathematics. In S. Johnston-Wilder & C. Lee (Eds.), *Leading Practice and Managing Change in the Mathematics Department*, (15–32). St Albans: Tarquin Books.
- Johnston-Wilder, S., & Lee, C. (2010a). Mathematical Resilience. *Mathematics Teaching*, 218, 38–41.
- Johnston-Wilder, S., & Lee, C. (2010b). *Developing Mathematical Resilience*. Paper presented at the BERA annual conference at Warwick University.
- Lee, C. (2006). *Language for Learning Mathematics- assessment for learning in practice*. Buckingham: Open University Press.
- Lee, C., & Johnston-Wilder, S. (2010) Children overheard, mathematical resilience *Mathematics Teaching MT218i*.
- Mason, J. (1988). *Learning and Doing Mathematics*. London: Macmillan.
- McLeod, D.B. (1992). Research on affect in mathematics education: A reconceptualization. In D.A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (575–596). New York: Macmillan.
- Mercer, N., & Littleton, K. (2007). *Dialogue and the development of children's thinking: a socio-cultural approach*. London: Routledge.
- Nardi, E., & Steward, S. (2003). Is Mathematics T.I.R.E.D.? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal*, 29(3), 345–366. [doi:10.1080/01411920301852](https://doi.org/10.1080/01411920301852)
- Newman, T. (2004). *What works in building resilience?* Ilford: Barnardo's
- Ofsted (2008). *Understanding the Score*. London: Ofsted.
- Richardson, F.C., & Suinn, R.M. (1972) 'The Mathematics Anxiety Rating Scale' *Journal of Counseling Psychology*, 19, 551–554.
- Roberts, G. (2002). *SET for Success: the Supply of People with Science, Technology, Engineering and Mathematics Skills*. Retrieved from <http://www.employment-studies.co.uk/pubs/report.php?id=1440robert>.
- Vygotsky, L.S. (1981). The genesis of higher mental functions. In J.V. Wertsch (Ed.) *The Concept of Activity in Soviet Psychology* 144–188). Armonk, NY: Sharpe.
- William, D. (2008). Changing Classroom Practice. *Educational Leadership*, 65(4), 36–42.

# **"YOU UNDERSTAND HIM, YET YOU DON'T UNDERSTAND ME?!" - ON LEARNING MATHEMATICS AS AN INTERPLAY OF MATHEMATIZING AND IDENTIFYING**

Einat Heyd-Metzuyanin and Anna Sfard

Haifa University

*Leaning on commognitive theory, we attempt to demonstrate how affective, social and cognitive aspects of learning can be treated with the same set of theoretical concepts and analytical tools. This will be done by developing the basic idea of learning mathematics as interplay between activities of mathematizing (talking about mathematics) and subjectifying (talking about the participants of the discourse). We shall exemplify our proposed tools of analysis on a group of usually successful 7<sup>th</sup> grade mathematics students whose ability to benefit from significant opportunities for learning is shown to be highly sensitive to how they identify themselves and their peers.*

**Keywords:** identity, social interactions, affect, commognitive theory, discourse analysis.

Affective aspects of learning mathematics such as attitudes and beliefs have been studied extensively both for discovering differences between groups and for finding relations between those aspects and mathematics achievement (i.e. McLeod, 1992). However, emotional reactions during mathematics learning have received less attention (Leder & Forgasz, 2006), and, in particular – their interaction with the cognitive processes of mathematics learning have seldom been researched. Social aspects such as status and positioning in class has received growing attention in recent years (Lerman, 2000), yet less work has been done about the links between social interactions and the emotional and cognitive processes going on within the individual student in class.

Most probably, the lack of studies dealing with the interaction between 'affective' reactions, cognitive processes and social interactions stems from the fact that very specific, dissimilar conceptual frameworks and research tools have been used for studying each of these three arenas. Therefore, the long term goal of our research is to contribute to the effort of creating a unified framework, where cognitive and affective, as well as individual and social aspects of learning would all be seen as members of the same ontological category, to be studied with one integrated system of tools, grounded in a single set of foundational assumptions. In what follows, we report on the conceptual tools we have developed, while trying to address this challenge. We also exemplify the usage of these tools in an interesting episode of social, emotional and cognitive conflict in class.

The proposed conceptual framework leans on the 'commognitive' theory, which recognizes the centrality of communication in all our activities, including the

uniquely human forms of learning. This theory, stemming from participationist (Lave & Wenger, 1991) and discursive theories (Harré & Gillette, 1994), maintains that thinking can be viewed as an interpersonal form of communication and mathematics, being a particular way of thinking, turns out to be a special form of discourse (Sfard 2008).

Learning mathematics can be seen as interplay between two concomitant activities: that of *mathematizing* – communicating about mathematical objects; and that of *subjectifying*, that is, communicating about participants of mathematical discourse. Of all subjectifying activities, the most consequential for learning seems to be that of identifying – the activity of talking about the properties of persons rather than about what the persons do. Scrutinizing the activity of mathematizing is the commognitive counterpart of cognitive analysis, whereas studying the activity of identifying means attending to all those phenomena that other researchers label with adjectives such as affective, interpersonal or social. The empirical material presented in this article demonstrates how the activity of identifying may interfere with the activity of mathematizing, and thus with the learning of mathematics.

### *The Study*

The study was held over a period of 5 months, in an extra-curricular program, where one of us taught three groups of 7<sup>th</sup> grade students: students with very high scores in mathematics, with moderate-to-high scores, and with low scores. In each group there were 4 students, 12 students overall (7 boys and 5 girls). The episode discussed here occurred during the 11th lesson with the moderate-to-high achievers group. This group included two boys - Ziv and Dan, and two girls - Edna and Idit. Generally speaking, Ziv and Dan had a history of high achievement in mathematics, Idit was generally successful though in some areas she encountered problems, and Edna had usually moderate scores.

The lessons were video filmed by 3 stationary cameras, directed at the student's fronts. Additionally, all written material performed during the lesson was collected. The recordings were transcribed in Hebrew, and the examples given here were translated into English by the authors.

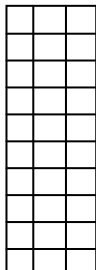
The episode was chosen for deep and thorough analysis because of the very high occurrence of subjectifying utterances that accompanied what appeared to us as an inexplicable block in the advancement of the group towards a solution of a given mathematical problem. The students were presented with a worksheet containing a problem called the Chocolate Factory Problem (Figure 1). Its purpose was to help students to advance toward a discourse on fractions (though fractions were not explicitly mentioned).

What mystified us was the fact that the participants seemed impervious to one student's (Ziv) seemingly lucid and cogent explanations and that the eventual eye-

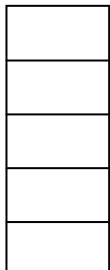
opening effect came from what looked to us as a much less coherent and quite opaque argumentation of another student (Dan). In what follows we show that much more than scrutinizing the flow of the "mathematical content" of the conversation is necessary in order to understand what blocked the learning process.

**The Chocolate Factory**

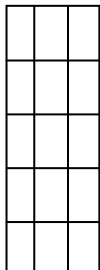
The chocolate factory produces four different types of chocolate bars, all of the same size, as shown in the drawing.




A



B



C



D

When bars get damaged, the factory can't sell the chocolate in its original package. The factory donates the damaged bars after repacking them in bags. Each bag contains the amount of chocolate equal to one type-D piece and two type-B pieces.

1. In how many different ways can one pack such a bag?
2. What are these ways?
3. How can you be sure you have found all the possible ways?

**Fig. 1: The Chocolate Factory problem**

## Mathematizing

While participating in mathematical discourse, interlocutors combine mathematical keywords and mediators into mathematical objects. Indeed, to act as a competent participant in mathematical discourse, one has to realize (translate) words such as *numbers*, *functions* or *sets* with the help of other mathematical words and mediators. A *mathematical object* is a mathematical signifier together with its *realization tree*, a hierarchically organized set of all the realizations of the given signifier, together with the realizations of these realizations and so forth.

One way to analyze the activity of mathematizing is to follow the flow of mathematical objects, that is, to try to identify the ways in which the participants realize the focal signifiers at different points in the process of solving the problem. One important question that can be answered on the basis of this information is that of the effectiveness of communication: We can decide in a systematic, testable manner whether two interlocutors are speaking of the same object while using the same words.

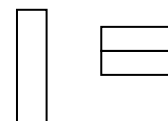
## Working Group 8

Let's take a close look at how two of the students, Ziv and Dan, realized the signifier *a bag[containing] the amount of chocolate equal to one type D piece and two type B pieces* (which we shall shortly term 'the required bag', or simply 'bag') while trying to solve the Chocolate Factory problem.

### *Dan's first try*

As evidence by numerous utterances during the first minutes of the episode, Dan and the two girls had considerable difficulty trying to understand the question. Apparently, they were unable to unpack the complex 'bag' signifier so as to be able to start thinking about any realization procedure. After grappling with the question for three minutes, a breakthrough occurred and Dan offered an idea:

Dan: so it's actually they say that they mean this.. that one like this,  
each piece of one type, of type D, equals two of type B  
[pieces] right? (lines 233-235)

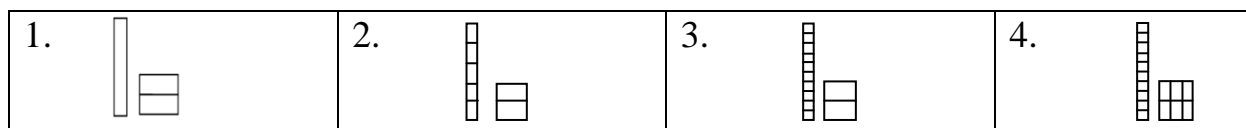


Dan had arrived at the (wrong) impression that the word *equal* in the description of the task signify the relation between the amount of chocolate in a type-D piece and the amount of chocolate in two type-B pieces (in other words,  $1/3 = 2/5$ ). Not surprisingly, this interpretation stymied his further attempts at solving the problem. Indeed, from here, it was not clear what the words "number of different ways to pack such bag" could possibly refer to. His idea was questioned by the teacher, who encouraged Ziv to offer his alternative solution.

### *Ziv's try*

Ziv remained silent throughout the conversation between Dan, the girls and the teacher. During the first 10 minutes of the episode he worked on his own solution. When he eventually spoke up, his proposals flowed one after another, contributing to the gradual emergence of a coherent realization tree for the requested 'bag'.

Ziv: O.K. (..) the first way (.) is the way they showed us, which is one stripe from here (*points to Bar D*) and two from here (*Bar B*, see fig 2. No. 1), that's the first way<sup>24</sup>. The second way is to take all these (*Bar C*) and two from here (*Bar B*, see No. 2). A third way is to take a=ll<sup>25</sup> these (*Bar A*), two from here (*Bar B*, see No. 3). Now let's move here (*points to Bar C*). A third way is to take a=ll these (*Bar A*), and all this part (*Bar C*, see No. 4), all the six-, all the-, these



<sup>24</sup> The visual realizations of the students' words are *our* interpretations, based on the way the students pointed on the worksheet

<sup>25</sup> The transcription signs are as follows: = marks - a prolongation of the syllable; (.) marks a very short halt.



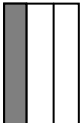

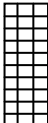
**Fig 2. Ziv's realizations of the requested 'bag'.**

In spite of the ostensive clarity of Ziv's explanations (supported by suggestive pointing), his classmates claimed they were unable to understand what he was saying. This left space for Dan's new take on the problem.

### *Dan's second try*

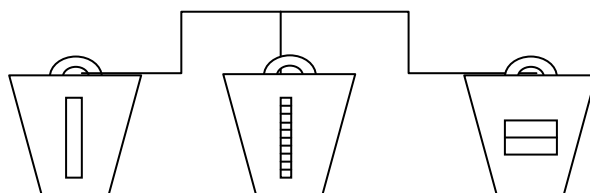
After his  $2B=D$  proposal was rejected by the teacher, Dan went back to coping solitarily with the problem, hardly paying attention to Ziv's solution. He then volunteered to explain his own solution to the girls and set out into a long and somewhat muddled explanation.

Dan: Uh- the chocolate factory did, um- things-, ah like, did a defected [pack], and did that the defected chocolates pack, will be this size, O.K? (*Pointing up and down a D rectangle, see fig. 3-1*) .. and in this size (*fig 3-2*). Right? Wait. (*Turning the worksheet towards himself*) If it's actually in this size, then why? Ah, I got it. 'nyway he did in this size, O.K? (*pointing to D, fig 3-1*) And in this size (*pointing to B, fig 3-2*). Two different packs. Now, they have a few types of packs, and they wanted actually, each, each pack of the defected [packs] they wanted actually to make it into such a pack (*pointing to D, fig 3-1*). They only have two types of packs, that they can get, and they can't change the size of the packs, so they have to insert [it]. Now, here (*pointing to bar A, fig 3-3*) there are many small ones, so they can actually divide them so they be put into such [a pack] or such [a pack] (*fig 3-1 and 3-2*). That's why they asked you here, if you can pack such a bag with squares of type A alone, so yes... So actually you can divide them so they'll be in this (*pointing to bar D, fig 3-1*), or like this (*bar B, fig 3-2*) because they are bi-, they're small. (Lines 536-554).

1. D piece		2. 2B pieces		3. type A pack	
------------	---	--------------	---	----------------	---

**Fig 3 – The diagrams Dan was pointing to (the pointed parts are shaded gray)**

Apparently Dan had arrived at a realization of a 'bag' he was quite content with (though it was wrong, again). He figured there are two 'legitimate' types of chocolate bars: one of the size of a single D piece, and one of the size (shape) of two B pieces. Thus, his realization tree for the requested 'bag' was as depicted in fig 4.

**Fig 4 – Dan's realization tree for the signifier 'bag containing the amount of chocolate equal to 1 type D piece and 2 type B pieces'**

Surprisingly enough, the girls, who earlier insisted they 'don't understand a thing', now showed signs of comprehension. Idit claimed she understood Dan, and Edna offered: "and then it's like ten cubes" (line 557).

What was happening here? These students seemed to be learning from the participant who himself was struggling for understanding, and who offered a realization which was much too blurred and ambiguous to be helpful. All this happened after they let the obvious opportunity for learning slip away – after remaining unimpressed by a solution which, according to the observers, was not only correct, but also presented quite clearly and convincingly. Although nobody seemed to doubt the correctness of Ziv's solution, no visible effort was made to find out what his proposal was all about. We found no evidence for the other students being even interested in Ziv's explanation.

In order to understand what was happening, it was necessary to look into what we claim is a process that always runs in parallel to 'mathematizing', the process of 'subjectifying' and, in particular, 'identifying'.

## SUBJECTIFYING

Subjectifying could be about oneself (1<sup>st</sup> person subjectifying), about others (3<sup>rd</sup> person subjectifying) or stated explicitly toward the subject of the communication (2<sup>nd</sup> person subjectifying). Following Sfard & Prusak (2005) we shall denote every subjectifying utterance with a specific notation that will clarify who the informer, the subject and the receiver of the communication are. Thus the notation  $_{\text{Edna}}\text{Ziv}_{\text{Teacher}}$  will signify an utterance about Ziv, made by Edna and directed at the Teacher.

Subjectifying acts can be verbal or non-verbal (such as in gestures, facial expressions or meaningful intonation of the voice). They can also be direct or indirect. *Direct* subjectifying utterance refers directly to the subject (such as Dan (to Ziv): "no one can understand you"). *Indirect* subjectifying acts are acts that can be interpreted as being about one of the participants, even if the object of the act is not stated explicitly (such as when Ziv says about the other student's failure to comprehend "so they should try harder", indirectly stating he himself is superior to the others in his mathematical performance: in other words, this utterance may be marked  $_{\text{Ziv}}\text{Ziv}_{\text{B}}$ ).

### *Moving from subjectifying to identifying*

By using the term 'identity', we refer to Sfard & Prusak's definition of *identity as a narrative*. More specifically,

"identities may be defined as collections of stories about persons or, more specifically, as those narratives about individuals that are *reifying*, *endorsable*, and *significant*" (Sfard & Prusak, 2005, p. 16)

Sfard & Prusak made the distinction between narratives of *current identity* which consist of stories about the actual state of affairs, and *designated identity* "consisting of narratives presenting a state of affairs which, for one reason or another, is

expected to be the case, if not now then in the future." (p. 18). This distinction will serve us when we look at different designated identities the participants are attributing to one another during this episode, and how such designated identities might conflict with the current 1<sup>st</sup> P identities the participants hold.

One of the most problematic obstacles for studying the mechanisms of identity building is the inaccessibility of some of the 'identity narratives' during a non-intrusive observation of students in class. Our first task, therefore, was to prepare an operational means for extracting 'identifying processes' (processes of identity building) from the natural discourse of the students in class and, in particular, for deciding which subjectifying utterances could count as identifying. We did this by assessing how general the subjectifying message is. At the lowest level of generalization we placed utterances in which an interlocutor evaluates what she just did or is about to do (e.g., Idit: "I didn't understand a thing"). A remark about a general characteristic of the speaker's or other person's participation (such as Edna: "When Ziv speaks I never understand him") was classified as representing a higher level of generalization. We decided that the highest level of reification occurs when rather than assessing the participation (what people do), the speaker evaluates properties of a person (what the person *is* or *has*) or of this person's memberships (with whom the person belongs), e.g., Dan (to Ziv): "You'll never be a teacher". Such reifying utterances are *identifying* by definition. However, *general participation* evaluation and even *specific participation evaluation* utterances can sometimes add up to form a coherent identifying narrative, provided they are *recurring* and *consistent*.

Most researchers interested in students' identities rely in their analysis exclusively on students' verbal 1<sup>st</sup> person identification, such as those obtained in interviews. And yet students of this age seldom talk extensively about themselves, and thus these are the non-verbal and indirect subjectifying actions which may often be a major, sometimes the only, 'window' to students' first-person identities. In our analysis, we found that when examined in the context of verbal direct identity narratives, *repetitive*, *consistent non-verbal* and *verbal indirect* subjectifying actions may often provide the most valuable information about how the students view themselves and others.

Additionally, emotional declarations and gestures are key signs of significance, and thus indicate which narratives and activities are important for the interlocutors. It can be reasonably assumed that whatever an emotionally loaded statement is, it has something to do with the student's identity – with how the student sees herself in the longer run. For instance, when Ziv solemnly hides his face behind his worksheet, avoiding eye contact and sitting ubiquitously quiet after remarking that Edna has 'understood' Dan and not him, we may assume he's emotionally hurt and that 'being understood', and 'explaining better than Dan' is important for his 1<sup>st</sup> P identity.

To summarize, we will define identifying utterances (which may be either verbal or non-verbal) as *those subjectifying acts that signal that the identifier considers a given feature of the identified person as permanent and significant.*

## *Technique of analyzing identifying actions*

In order to understand the identifying processes going on in this episode, we extracted all the subjectifying verbal and non-verbal acts made by the participants. These utterances were inserted into four tables (one for each student), each table containing all the group member's references toward the subject of the table. Looking at the tables, it appeared immediately that most of the subjectifying actions were made about Ziv. His table was the richest in subjectifying utterances in general, and in 2<sup>nd</sup> and 3<sup>rd</sup> P utterances in particular. In other words, not only was Ziv the participant who talked directly about himself the most, he was also the one who got the most attention from the other participants. As we shall shortly see, this attention was far from being positive, at least on the part of the other students.

In the following table<sup>26</sup>, we bring examples of the most telling subjectifying acts from Ziv's table, those which can be classified as 'identifying' according to the definition made above.

Looking at this table, and from the analysis of all the subjectifying acts in the transcript, a clear picture of the identifying processes going on in this episode emerged. Let's first look at Ziv's 1st P identifying. It included a few main themes: First, Ziv made it clear that he thought himself superior to the others (especially to Idit and Edna). Second, he was very competitive with Dan (for instance by trying to 'beat' Dan and insulting him when he attempted to catch up). Finally, he seemed to get very hurt when the girls failed (or refused) to understand him conveying that 'being a leader of the mathematical discussion' was an important part of his identity.

The Teacher's 3rd P subjectifying of Ziv was very consistent, verbal and direct, and thus it could safely be claimed she was identifying Ziv as the most competent participant (mathematically speaking) in the group, and as one who could, and should, help the others understand the task.

The girls (both Edna and Idit) consistently identified Ziv as an "Incapable teacher" who may be smart but does not know how to explain. He was identified directly by Edna as condescending and admittedly complied with this identity, for instance by rolling his eyes at Edna's disability to comprehend him (though all the students had long histories together, thus it might be that the interaction seen here was rooted in the long past of their acquaintance).

---

<sup>26</sup> Out of space restrictions, the structure of this table is different from the one used for our analysis. The analysis table used a column for each participant, where all his/her utterances about Ziv were documented.

**Table 1 – Ziv's subjectifying acts**

<b>Speaker, object and addressee</b>	<b>What is said (<i>and how</i>)</b>	<b>Subjectifying category</b>
TeacherZIV <sub>Ziv</sub>	Teacher: [explain your solution to the others] because you're the only one who understood [the question] (line 99)	Verbal 2nd P <b>specific participation</b> evaluation
DanZIV <sub>Ziv</sub>	Dan ( <i>quietly, while Ziv starts explaining again</i> ): Enough, Ziv, you won't be a teacher. (line 678)	Verbal 2nd P <b>designated identifying</b>
EdnaZIV <sub>Ziv+Teacher</sub>	Edna ( <i>to the Teacher, in an annoyed voice</i> ): He just- he talks to me like I'm his [little] girl! ( <i>to Ziv, mummifying angrily a 'teaching voice'</i> ) Ziv, you understand that if it's ^so and so^ than it's this and that? Yeah?! ( <i>Turning back to the teacher</i> ) That's how he talks to me! (line 704)	<b>Verbal and Non-verbal identifying</b> of a complex type: Edna speaking of how she thinks Ziv is identifying her.
IditZIV <sub>Teacher+Others</sub>	Idit: No [I didn't understand Ziv]. Ziv talks to himself and he thinks everyone is listening to him so he ( <i>mummifying a 'babbling' voice</i> ) >pa pa pa pa pa, wa wa wa wa wa< (line 443)	Verbal 3 <sup>rd</sup> P <b>General participation evaluation</b> and <b>non-verbal identifying</b>
ZivZIV+DAN <sub>Edna</sub>	Ziv ( <i>to Edna, in an annoyed voice</i> ) Ah! Like you understand him [Dan], and me you don't? ( <i>sits back and looks hurt: covers part of his face with the worksheet, avoids eye contact and seems to 'close up'</i> ) (line 555)	<b>Verbal + Emotional-Non-Verbal</b> 2nd P specific participation evaluation, highlighting the rivalry between Ziv and Dan.

It can be reasonable to conclude that the combination of Ziv's arrogant behaviour and 'superior' 1st P identity as enacted by him during the lesson - combined with the "he's the competent leader" 3rd P identity afforded to him by the Teacher - created a clear resistance from the girls to learn anything from him. Look at Edna, for instance, at the point when Ziv was trying to explain his solution to her (the 3<sup>rd</sup> time around) and she aggressively rejected it (see table 1, line 3). From her point of view, at this particular moment, she was offered a very undesirable identity by Ziv, and indirectly by the teacher - that of the 'little, inferior', perhaps even 'incapable' student. If she had cooperated by 'understanding' Ziv, she would have simply approved this identity, something she probably very much wanted to avoid.

Dan, on the other hand, being confused himself and showing no condescending behaviour, was much less threatening. In other words, his acts offered no endangerment for the girls designated identities as 'intelligent learners'.

To conclude, what we see here is a *struggle of identities*. As a result of this struggle, something very basic was missing from the students' discourse which would enable a useful learning process: a *teaching-learning agreement*. Sfard (2008, p. 283) coined this term for the implicit understanding, formed between two or more interlocutors, that one of them is the 'teacher' who has the authority to determine what is 'true', and the others are the 'students' who 'learn'. In our case, it is clear that the teacher believed such an agreement should exist, and as Ziv was making signs he had solved the problem, it was natural from her point of view, that he would take the role of the

'teacher'. However, as this agreement would endanger the designated identities of the girls, they strongly rejected it, even at the price of blocking their advancement in solving the presented problem.

## CONCLUSIONS

This case shows how powerful processes of identifying may be in hindering mathematical discourse. The way the student chooses to participate in the mathematical discourse is affected not only, and perhaps even not mainly, by her mathematical competence, but also by her 1<sup>st</sup> P identity. Of principal importance here are her designated identities constructed by herself and by other participants.

It should also be noted that it is not always the 'weak' or peripheral students who pay the social price during participation in mathematical discourse. As Ziv's behaviour shows, it is the strong student who would sometimes be hurt.

Finally, teachers' actions, even if well-intended and performed in the attempt to advance the mathematical discourse in the class, can fuel and shape counterproductive identifying interactions among the students. As we have shown, such identifying interactions may hinder the learning process, achieving an effect exactly the opposite to the one intended by the teacher.

## REFERENCES

- Harre, R., & Gillett, G. (1994). *The Discursive Mind*. Thousand Oaks, CA: Sage Publications.
- Lave, J., & Wenger, E. (1991). *Situated learning: legitimate peripheral participation*. Cambridge: Cambridge University Press.
- Leder, G., & Forgasz, H. J. (2006). Affect and mathematics education. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on Mathematics Education*, (pp. 403-428). Rotterdam: Sense Publishers.
- Lerman, S. (2000). The social turn in mathematics education. In J. Boaler (Ed.), *Multiple perspective on mathematics education* (pp. 19-44). Westport, CT: Ablex.
- McLeod, D. B. (1992). Research on affect in mathematics education: a reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575-596). New York: Macmillan Publishing Company.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge, UK: Cambridge University Press.
- Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14-22.

# AN INTERVENTION ON STUDENTS' PROBLEM-SOLVING BELIEFS

Gabriel J. Stylianides

University of Oxford, UK

Andreas J. Stylianides

University of Cambridge, UK

*Many students have certain beliefs about problem solving that tend to impact negatively on their engagement with problem solving. Previous interventions that achieved a positive impact on some of these student beliefs lasted over extended periods of time, thereby producing research knowledge that cannot be easily replicated in other settings. Findings from a 4-year design experiment in an undergraduate mathematics course suggest that it is possible to achieve a notable positive impact on 4 common and counter-productive student beliefs about problem solving with an intervention of duration as short as 75 minutes.*

Keywords: beliefs, problem solving, classroom interventions, design experiments

## INTRODUCTION

Students' ability or willingness to engage with the important mathematical activity of *problem solving* is shaped by students' *beliefs*, "the set of understandings about mathematics that establish the psychological context within which individuals do mathematics" (Schoenfeld, 1985, p. 5). According to Schoenfeld (1985), "[o]ne's beliefs about mathematics can determine how one chooses to approach a problem [...] and how long and how hard one will work on it" (p. 45).

Many students believe that those who understand the content can solve assigned problems in five minutes or less (Schoenfeld, 1992). Students with this belief tend to "give up working on a problem after a few minutes of unsuccessful attempts even though they might have solved it had they persevered" (Schoenfeld, 1992, p. 359). Also, many students tend to believe that "[t]here are always numbers in formulations of math problems" (Callejo & Vila, 2009, p. 116) or other clearly identifiable mathematical referents such as formulas. Students with this belief tend to give up working on problems that do not fit in with their expectations, because they cannot see how to make progress on these problems (see, e.g., Callejo & Vila, 2009).

These student beliefs not only interfere with students' ability to engage productively with problem solving, but also often lead to low success rates and negative reactions to mathematics more broadly (e.g., Furinghetti & Morselli, 2009). Previous studies (e.g., Perrenet & Taconis, 2009; Philippou & Christou, 1998; Swars et al., 2009) developed interventions that impacted positively on students' problem-solving beliefs, and identified specific features of the respective interventions that seemed to have played major role in their success (e.g., an emphasis on the history of mathematics). Despite the significance of these interventions, their success cannot be easily replicated in other settings due to their long duration and the associated large number of variables that were not explicit parts of the interventions but

possibly influenced their outcomes (the interventions lasted from 10 weeks to several years). Would it be possible to develop interventions of short duration in mathematics classrooms that would impact positively on students' problem-solving beliefs? Such interventions would be more easily replicated in other settings than long-term interventions, for they would allow more control over confounding variables. Evidence from a 4-year design experiment we conducted in an undergraduate mathematics course suggests that a notable positive impact on students' problem-solving beliefs can be achieved with an instructional intervention of duration as short as 75 minutes. With this intervention we targeted specific student problem-solving beliefs that tend to impact negatively on students' problem-solving engagement. Specifically, we aimed to promote the following 4 learning goals. Help students:

1. Recognize that some problems they perceive to be “unsolvable” can actually be solvable and within their capabilities (Goal 1);
2. Realize that effective problem solving requires perseverance (Goal 2);
3. See that the formulation of a mathematical problem can include more than just clearly identifiable mathematical referents (numbers or formulas) (Goal 3); and
4. Appreciate that problem solving can be a satisfying or enjoyable activity (Goal 4).

### **THEORETICAL FRAMEWORK**

“[B]elief systems are composed mainly of ‘episodically’-stored material” and episodic memory “is organized in terms of personal experiences, episodes or events” (Nespor, 1987, p. 320). Thus, beliefs can often be traced back to vivid memories of highly influential episodes that shape people's interactions with subsequent events. Following Nespor's work, we aimed to engineer a dramatic and positive episodic memory for our students that would overpower their earlier episodic memories. This new episodic memory would be engineered by means of purposeful implementation of a carefully-designed problem. Below, we discuss design and implementation features of such a problem that can be used to impact positively on students' beliefs, with a focus on the 4 learning goals. Our theoretical framework has 2 components, which emerged both from prior research and knowledge from our design experiment.

#### **Component A: Problem design features**

The problem should have a memorable name (feature A1) so that students can more easily store in and retrieve from their episodic memory their engagement with the problem. The remaining design features relate explicitly to the 4 learning goals.

To promote Goal 1, the problem should initially appear to be unsolvable to students and, of course, should be presented to the students in a way that leaves open the possibility that it can actually be unsolvable (feature A2). At the same time, however, the problem should be within students' capabilities (feature A3a): if the students persevere, they should be able to solve the problem within reasonable time



by working in small groups and by receiving (perhaps) limited scaffolding from the instructor. If under these conditions students solve the problem, Goals 2 and 4 are also promoted: students are expected to see that their perseverance paid off (Goal 2) and feel satisfaction/accomplishment for solving an “unsolvable” problem (Goal 4). To further promote Goals 1, 2, and 4, the problem’s solution should involve achievement of multiple milestones (feature A3b). If it involved only one milestone, then, once a student in a small group had an insight into the solution, the other students would be deprived of opportunities to: (a) offer something substantial to the solution of a problem they originally perceived unsolvable (cf. Goal 1), (b) see their own perseverance as a contributive factor to the solution (cf. Goal 2), or (c) experience personal satisfaction from their engagement with the problem (cf. Goal 4). To promote Goal 3, the problem should include few clearly identifiable mathematical referents, which should offer by themselves insufficient data for the problem’s solution (feature A4). The fact that students will need to consider other kinds of referents can help expand their view about what counts as a legitimate referent.

### **Component B: Problem implementation features**

The problem implementation should involve a set of conceptual awareness pillars (CAPs; Stylianides & Stylianides, 2009) that aim to help students become more aware of specific beliefs that are to be challenged or formed during students’ engagement with the problem (feature B1). CAPs is a “construct that ... describe[s] instructional activities that aim to direct students’ attention to their conceptions [understandings, beliefs, etc.] about a particular mathematical topic [such as problem solving]” (ibid, p. 322). CAPs at the beginning of the intervention can help students become more aware of their original beliefs, while CAPs at the end can help them reconsider these beliefs in relation to new beliefs provoked by the intervention.

The assumption underlying feature B1 is that the more aware students are of their existing beliefs, the more likely they will be to problematize and potentially change these beliefs when they engage in a problem-solving situation that challenges their existing beliefs and encourages the formation of alternative beliefs. This assumption finds support in the literature. For example, Nespor (1987) noted the following:

[Trying to change or shape teachers’ beliefs] would mean helping teachers and prospective teachers become reflexive and self-conscious of their beliefs and, as Fenstermacher [1979] suggests, presenting objective data on the adequacy or validity of these beliefs. However, this can result in transformations of teachers’ beliefs and practices only if alternative or new beliefs are available to replace the old. (p. 326)

The features we discussed in Component A, along with the experience of solving the problem, are expected to provide students with objective data that they can use to judge the validity of their original beliefs and begin to develop new ones. Feature B2 concerns the instructor’s critical role in the implementation of the problem (Stylianides & Stylianides, 2009). We view the instructor as the representative of the

mathematical community in the classroom and thus as the guarantor that the development of new knowledge will be consistent with conventional understandings (feature B2a). Also, the instructor organizes and facilitates social interactions among the classroom participants so that new knowledge in the classroom (including solutions to mathematical problems) is interactively constituted (feature B2b). Related to the latter aspect of the instructor's role is the implementation by the instructor of scaffolding strategies and CAPs.

## METHOD

The findings we report in this paper were derived from the last research cycle of a 4-year *design experiment* (e.g., Cobb et al., 2003) we conducted in a semester-long mathematics course for prospective elementary teachers in the United States. The design experiment included 5 research cycles of implementation, analysis, and refinement of a set of instructional interventions that aimed to promote students' mathematical knowledge (including beliefs). The students pursued different majors and tended to have weak mathematical backgrounds. Our overarching objective was to develop effective instructional interventions to promote hard-to-achieve goals and also theoretical frameworks that would help explain how the interventions achieved their goals. Only one of the interventions explicitly aimed to promote the 4 goals we focus on here. This intervention was implemented toward the middle of the course.

### The problem used in the intervention: a solution outline and design features

Below is the problem we used in the focal intervention, called the Blonde Hair Problem (BHP). This is a slightly modified version of a problem we found in Philippou and Chiristou (1995, p. 132). Next we outline a solution to the problem that our students develop by the end of their engagement with the problem.

#### The Blonde Hair Problem

- 1 After having many years to see each other, two friends who really loved math, Hypatia and
- 2 Pythagoras, meet again. They have the following conversation:
- 3
- 4 *Pythagoras:* Are you married? Do you have any children? How many? How old are they?
- 5 *Hypatia:* Yes, I am married! I have three children and the product of their ages is 36.
- 6 *Pythagoras:* (After doing some thinking.) I cannot figure out their ages. I don't have
- 7 enough clues.
- 8 *Hypatia:* Right! What if I told you that the sum of their ages is the same as the number
- 9 of your address?
- 10 *Pythagoras:* (After doing some thinking again.) I still can't figure out their ages. I need
- 11 another hint.
- 12 *Hypatia:* Well done! I also tell you that the oldest has blond hair.
- 13 *Pythagoras:* Aha! Now I can, without any doubt, figure out the ages of your children.
- 14
- 15 What are the ages of Hypatia's children? (their ages can only be natural numbers)

The first hint (line 5), together with the fact that the ages are natural numbers (line 15), reveals that there are 8 possibilities for the children's ages. With regard to the second hint (lines 8-9), although we are not told the number of Pythagoras' address, we can infer that Pythagoras knows this number since he is trying to see how this hint could be useful to him (line 10). Eventually, Pythagoras concludes that he is still unable to figure out the children's ages (lines 10-11). This suggests that there are at least 2 possibilities in which the sum of the ages is equal to the number of

Pythagoras' address; the only two possibilities in which this happens are 1, 6, 6 and 2, 2, 9. In the final hint (line 12), the hair color is a distraction; the relevant information is that there *is* an oldest child. So the ages of Hypatia's children are 2, 2, and 9. (Given that the ages can only be natural numbers, for a child to be recognized as older than another child their ages should correspond to different natural numbers.) The BHP fulfills features A1-A4 of our theoretical framework. Regarding feature A1, it has a memorable name, which includes a color referent that is not only atypical for names of mathematical problems but also a distraction to the problem's solution. The BHP provides only 2 numbers (3 and 36), which are the only clearly identifiable mathematical referents in the problem and by themselves offer insufficient data for a solution (feature A4). Also, it includes some distractive information (e.g., the hair color) and some "mystifying" information, which however turn out to be crucial for its solution (e.g., that Hypatia had an oldest child). Thus, the problem may initially appear to be unsolvable (feature A2). Features A2 and A4 are further satisfied by the fact that the BHP cannot be solved by application of a formula and does not fit in any traditional domain (algebra, geometry, etc.). Regarding feature A3, the BHP can be solved using only basic concepts and systematic listing/elimination of cases, and so it can be accessible to students of varied backgrounds (feature A3a). Also, its solution involves 3 milestones, each corresponding to one of the hints (feature A3b).

### **Implementation plan for the problem in research cycle 5**

The problem's implementation was done according to a highly refined lesson plan (outlined below) that we developed during our design experiment. The lesson plan derived from: critical comparison between different versions of the plan over the 5 cycles of the design experiment and the relative success of these versions in promoting Goals 1-4, relevant literature, and our emerging theoretical framework.

1. The instructor presents a PowerPoint slide entitled "Blonde Hair Problem" that includes the problem statement, without commenting on whether the problem is solvable or not. The students read the BHP and think about it individually. After a few minutes students are expected to start laughing or wonder whether the instructor is joking with them by presenting to them such a "weird" problem.
2. When the students do that, the instructor maintains a serious tone and asks the students to respond individually and in writing to the following prompts (CAPs).

*CAP 1:* Describe your initial reactions to the BHP.

*CAP 2:* Does this problem differ in any way from most of the other problems you encountered in mathematics classes before? If so, how?

The instructor is careful not to say anything that could influence students' responses to the CAPs (e.g., that the problem is solvable). CAPs 1 and 2 aim to help make students more aware of their original thoughts about the BHP, including their current beliefs about problem solving (cf. feature B1).

## Working Group 8

3. The instructor collects the students' responses to CAPs 1 and 2. At this point, he assures the students that the BHP can be solved mathematically, encourages them to think carefully about the information that each hint may give them, and asks them to continue their work on the problem (first individually and then in their small groups). Telling the students at this point that the BHP is solvable is important because, otherwise, many students would give up on the problem. This instructional move is consistent with our view of the instructor as the representative of the mathematical community (cf. feature B2a) and the fact that students need some scaffolding (in this case encouragement/reassurance) in order to overcome their original conviction that the BHP is unsolvable (cf. feature B2b).
4. The small groups work on the BHP independently until all of them come up with a solution. It is important that all groups solve the BHP prior to a whole group discussion, because, otherwise, the groups that would not solve the problem would not benefit from the experience in regard to Goals 1, 2, and 4. The instructor provides necessary scaffolding only to those groups that really need it.
5. The whole class discussion focuses on the mathematics of the BHP, not on students' experience with solving the problem. One of the small groups is asked to present first its solution for the consideration of the rest of the class. Other groups present alternative solutions as appropriate. Based on our experience, most groups will come up with all 8 possibilities for the first hint, but few will list these possibilities in a systematic way to show that these are all the possibilities. The issue of systematic enumeration is brought up for discussion by the instructor.
6. The instructor provides some historical remarks on the 2 characters in the BHP and asks the students to respond individually and in writing to a new CAP.

*CAP 3:* Describe your experience with working on the BHP. CAP 3 is intended to help students reconsider their original reactions to the BHP in relation to their subsequent experience of working on and solving the problem.
7. The instructor collects the students' responses to CAP 3 and the class period ends (time elapsed: about 60 minutes). At the beginning of the next period, the instructor returns students' responses to CAPs 1-3 and poses a new CAP:

*CAP 4:* (A) Read your responses and compare what you wrote at the beginning of your work on the BHP [cf. CAPs 1 and 2] and at the end [cf. CAP 3]. (B) Share your observations with the other members of your small group.

Activity A aims to help the students reflect on the anticipated sharp contrast between their reactions to the BHP at the 2 different stages of the problem's implementation. In communicating these observations to the other members of their small group in Activity B, the students are expected to become more aware of possible changes in their beliefs, thereby making it more likely that students' engagement with the BHP will become an influential episodic memory for them.

## Data and analytic method

The data come from the implementation of the focal intervention in research cycle 5, which was conducted in 2 sections of the course taught by the first author and attended by a total of 39 students. The implementation in each section lasted approximately 75 minutes. We used videos of the focal seminars and field notes of students' small-group work in order to create an account of how the BHP was implemented and examine whether it matched the lesson plan we outlined earlier. Also, we coded written data from all 39 students to assess their learning related to the 4 goals. Students' responses to CAPs 1-3 offered data for examining the short-term impact of the intervention. Their responses to a question in the last homework assignment of the course (2 months after the implementation of the BHP) offered data for examining its possible long-term impact. In that homework question, we asked the students to individually consider all the activities done in the course (more than 40 activities), identify 3 activities they felt contributed the most to their learning, and write a paragraph for each activity explaining what they found useful about it. A notable methodological consideration is the open phrasing of the questions. By not directly eliciting student comments related to the 4 goals in CAPs 1-3 and the homework question, we aimed to obtain valid data about the possible effectiveness of the intervention. We coded independently all responses, compared codes, and discussed disagreements until we reached consensus. We did not code students' responses to CAP 4, for these were based on their responses to CAPs 1-3.

### *Coding students' responses to CAPs 1-3*

Regarding Goal 1, the relevant data came from the combined consideration of students' responses to CAPs 1 and 3. We coded students' responses to CAP 1 using 2 codes: (a) "the problem is unsolvable" and (b) "do not know how to solve the problem." Although it was possible for a student to produce a response that fitted neither code, all responses ended up receiving one of the 2 codes. A precondition for us to say there was evidence a student made progress in Goal 1 was that the student's response to CAP 1 received code a. No claim could be made for responses that received code b. For the subset of students whose responses received code a, we also examined their responses to CAP 3 to see whether there was evidence to suggest they got any insights related to Goal 1. Such evidence included comments that a problem they originally perceived to be unsolvable was actually within their capabilities. Regarding Goals 2 and 4, we examined students' responses to CAP 3. Depending on whether there was evidence for one or both goals, each response to CAP 3 could receive a code for one or both goals. Regarding Goal 3, the relevant data came from the combined consideration of students' responses to CAPs 2 and 3. Given that no student explicitly wrote in response to CAP 2 that the BHP was a legitimate or a solvable problem, students' responses to CAP 2 could not provide by themselves sufficient support for learning related to Goal 3. There needed to be also some evidence in their responses to CAP 3 to indicate an expansion in their conceptions of

the nature of referents a (legitimate) mathematical problem can have. A student's response to CAP 3 could potentially provide by itself sufficient evidence for Goal 3.

*Coding students' responses to the homework question at the end of the course*

We analyzed students' responses to the homework question at 2 levels. At the first level we identified all the students who included the BHP in their list of 3 activities and we examined whether there was evidence in their responses to suggest understanding related to each goal. A response could receive up to 4 codes (one corresponding to each of the 4 goals). At the second level we focused on the students we identified in the first level, and we examined whether a student's response to the last homework assignment got any of the same codes that the student got for his/her responses to CAPs 1-3. Repeated codes were considered as evidence for sustained (long-term) student learning related to the goals that corresponded to those codes.

## RESULTS

The implementation of the intervention in the 2 sections of the course played out in almost identical ways, and was faithful to the lesson plan. So below we discuss together our findings based on data from all 39 students in the two sections.

### **Short-term impact of the intervention: Responses to CAPs 1-3**

For each goal, students' responses to CAPs 1-3 offered evidence that more than 35% of them developed understanding related to that goal: 36% for each of Goals 3 and 4, 54% for Goal 1, and 77% for Goal 2. Regarding Goal 1, out of the 26 students who had originally deemed the BHP unsolvable (thereby setting 67% to be the maximum possible percentage for Goal 1), 21 of them ended up recognizing that a problem that appears to be unsolvable can actually be solvable and within their capabilities.

### **Long-term impact of the intervention: Responses to the last homework question**

Given that the students could choose from more than 40 activities in the last homework question, we can say that the number of students who included the BHP in their list of 3 activities was a lower bound of the actual number of students who found the BHP important for their learning: for some students, the BHP could come, for example, 4th or 5th on their lists, but they could only list 3 activities (the data do not allow us to investigate this issue further). Also the fact that the question was asking students to identify and comment on the influence of individual activities rather than on collections of activities that had a specific effect on their learning helped address the issue of the possible contribution activities other than the BHP had on students' problem-solving beliefs. The lower bound we identified for the long-term impact of the intervention was markedly high: 13 students. The responses of 77% of these students offered evidence of understanding related to each of Goals 1 and 2. The corresponding percentages for Goals 3 and 4 were 54% and 46%.

To explore sustained learning, we checked whether a student's response to the last homework assignment got any of the same codes the student got for his/her responses to CAPs 1-3. The percentages of repeated codes for each of Goals 1-4 were 88%, 82%, 57%, and 100%, respectively. Thus, we may say that the intervention had a rather long-lasting effect on students' learning. In their explanations about why they chose the BHP in their list of 3 activities, students articulated broader implications of their engagement with the BHP for their problem-solving beliefs, which is a positive outcome of the intervention. For example, Aleara (all names are pseudonyms) noted that her engagement with the BHP changed her "whole outlook on how to solve problems" and learnt that one does not "need specific numbers, or even variables, when solving a problem." ira said she used to get angry and give up when she was given a problem she did not think she could solve; she "either didn't do it or waited for someone else to solve it and copied their answer." Her experience of independently solving the BHP was fulfilling and helped challenge her prior way of thinking about problem solving. Finally, Evans noted that the BHP taught him the importance of perseverance.

## DISCUSSION

The combined evidence for the short- and long-term impact of the intervention was more robust for Goals 1 and 2. The similar outcomes in relation to these 2 goals might be explained in terms of their possible relationship. Students who hasten to conclude that certain problems are "unsolvable" are likely to lack perseverance with working on such problems. The converse can also be true. Although the impact of the intervention in Goals 3 and 4 was not as powerful as in Goals 1 and 2, the positive results are encouraging given the short duration of the intervention and the difficulties encountered by research and practice in promoting Goals 3 and 4. A possible explanation for the lower impact in Goal 3 is that changes in students' beliefs about the legitimacy of different referents in a problem would require students to carefully examine the referents in the BHP. For methodological reasons, we avoided channeling students' activity to any particular direction (cf. the open phrasing of CAPs), and so we did not guide students to conduct such an examination. Regarding Goal 4, the lower impact of the intervention might be attributed to the challenges involved in getting university students who tend to have negative attitudes towards mathematics to explicitly say that problem solving can be enjoyable. The contribution of the paper is not limited, however, to showing that the focal intervention "does" or "can" promote the 4 learning goals. Consistent with the 2 primary intents of design experiments, our design experiment had, in addition to the pragmatic concern to promote the 4 goals, a theoretical concern to explain how the intervention supported the goals. Due to the lack of a randomized controlled experiment, we could not establish the specific contribution of each design and implementation feature in promoting each goal. However, we identified a "package" of design and implementation features whose collective function can yield the

desirable outcomes. The multiple research cycles of our design experiment enabled us to compare different versions of the intervention, thereby allowing us to identify weaknesses of the early versions of the package and generate possible explanations for what provoked these weaknesses that informed subsequent refinements. Regarding the issue of replicating the intervention in other settings, the following aspects of the intervention suggest its potential to be successfully incorporated into different educational programs at both the school and university levels: (a) it has a short duration and a stand-alone nature; (b) it involves ideas that are accessible to a wide range of student populations; (c) it targets beliefs that prevail in a wide range of student populations; and (d) its design and theoretical underpinning were not contingent upon special characteristics of the research participants in our study.

### AUTHOR NOTE

The two authors contributed equally to the preparation of this paper. The research was supported by funds from the Spencer Foundation (Grant Numbers: 200700100, 200800104). The opinions expressed in the paper are those of the authors.

### REFERENCES

- Callejo, M.L., & Vila, A. (2009). Approach to mathematical problem-solving and students' belief systems. *Educational Studies in Mathematics*, 72, 111-126.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.
- Furinghetti, F., & Morselli, F. (2009). Every successful problem solver is unsuccessful in his or her own way: affective and cognitive factors in proving. *Educational Studies in Mathematics*, 70, 71-90.
- Nespor, J. (1987). The role of beliefs in the practice of teaching. *Journal of Curriculum Studies*, 19, 317-328.
- Perrenet, J., & Taconis, R. (2009). Mathematical enculturation from the students' perspective: shifts in problem-solving behaviour during the bachelor programme. *Educational Studies in Mathematics*, 71, 181-198.
- Philippou, G., & Christou, K. (1995). *Teaching mathematics*. Athens: Dardanos.
- Philippou, G., & Christou, C. (1998). The effects of a preparatory mathematics program in changing prospective teachers' attitudes towards mathematics. *Educational Studies in Mathematics*, 35, 189-206.
- Schoenfeld, A.H. (1985). *Mathematical problem solving*. Orlando: Academic Press.
- Schoenfeld, A.H. (1992). Learning to think mathematically. In D. A. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.
- Stylianides, G.J., & Stylianides, A.J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, 40, 314-52.
- Swars, S.L., Smith, S.Z., Smith, M.E. et al. (2009). A longitudinal study of effects of a developmental teacher preparation program on elementary prospective teachers' mathematics beliefs. *Journal of mathematics Teacher Education*, 12, 47-66.
- Törner, G., Schoenfeld, A.H., & Reiss, K.M. (Eds.) (2007). Problem solving around the world. *Special Issue in ZDM Mathematics Education*, 39(5-6).



# **A REVERSAL THEORY PERSPECTIVE ON DISAFFECTION USING TWO EXAMPLES**

Gareth Lewis

School of Education, University of Leicester

*In this paper arguments are presented for a qualitative approach to researching affect in order to illuminate a richer, more complex and more dynamic motivational and emotional landscape in relation to children's experience of school mathematics. Reversal Theory is introduced as an approach to motivation and emotion which can provide a richer description of the phenomena than has been done hitherto. Data from the literature is re-analysed using Reversal Theory as a lens to provide insights. Conclusions are drawn about the implications for future research.*

**Keywords:** Disaffection; Motivation; Attitude to School Mathematics; Reversal Theory

## **INTRODUCTION**

### **Disaffection**

Disaffection with school mathematics education is a major problem. It is not just an educational problem, but an individual tragedy as well as having social and economic consequences. In the last 30 years there has been a great deal of research on attitudes to mathematics, much of it documenting and quantifying the worryingly high level of negative attitudes to mathematics amongst young people (Underwood, 2009).

A recent report into mathematics education by the Royal Society notes the widespread nature of current concern, 'no decade since the 1970's has seen so much being written about the disaffection young people appear to have for science and mathematics' (The Royal Society, 2008, p.171). The report points out that there has not been enough quality research into this area, and cites only three studies in relation to mathematics.

### **Affect and Attitude**

In order to understand disaffect, it is important to understand affect. However, this is not easy to accomplish as the field is fragmented and confusing (Hannula, 2006; Zan & Di Martino, 2007). Theorisation of affect has been strongly influenced by the framework suggested by McLeod (1987), and this has informed the CERME working model of affect (Hannula et al., 2010). However, research to date on affect has been dominated by attention to attitude. Although the study of attitude has helped to identify and document disaffection with school mathematics, and has provided useful comparative data across countries, gender and social class, it has not provided explanatory evidence for achievement (Furnham, 2009; Ma & Kishor,

1997). Thus there is strong evidence of a consensus that real insight and explanation will only come with a widening of the study of affect.

Schorr and Goldin (2008) argue :

‘We share with other researchers the need to study affect more deeply than the study of attitude permits....But it is increasingly clear that the functioning of affect is far more complex than is suggested by considerations of positive versus negative emotions and attitudes.’(p.132)

## **WIDENING THE STUDY OF AFFECT : MOTIVATION AND EMOTION**

To investigate disaffection more fully, I propose that a focus on motivation and emotion, and a widening of the methodologies used will provide a deeper, richer and more dynamic picture of the landscape, and thus create new insights. More recently, there have been studies giving a more fine-grained picture of motivation and affect in educational settings (Op't Eynde, De Corte, & Verschaffel, 2006; Zan & Di Martino, 2007).

It has been argued that classroom contexts, pedagogy and teacher attitude and behaviour are critical to engaging the motivation of students (Boaler, 2010; Pintrich, 2003; Zembylas, 2005). But there is a need to understand more fully the motivational factors that influence how students engage or disengage in mathematics. One such study was conducted by Nardi & Steward (2003). They used the acronym TIRED to describe the factors identified.<sup>27</sup> Yet underneath these broad factors there are motivational mechanisms and needs involved that need further investigation.

Much of the literature on motivation has focused on achievement goals, but it has been argued that a broader research focus is required to capture the richness of motivational impact (Weiner, 1990). To do this, it is necessary to delve more deeply into aspects of motivation and emotion to find exactly what is going wrong for disaffected individuals. Vygotsky (1986) argued that cognition and affect are indivisible, and that emotion, motivation, attitude and beliefs all interrelate in a seamless web. He advocates a more holistic approach, as does Zembylas (2005), arguing that we can understand the inter-relatedness of motivation, emotion, values, goals and beliefs (in addition to cognition) - what Vygotsky called the ‘fullness of life’ – via the use of ethnographic methodologies which he believes create space for the voices of those studied.

Aside from achievement goals, there are other formulations of motivational needs. One such influential formulation is that the basic needs are for competence, autonomy and relatedness (Deci & Ryan, 1985). However, as research has begun to develop a more detailed, qualitative picture of motivation and emotion, a good deal of complexity has been encountered. At the same time, a range of motivational

---

<sup>27</sup> TIRED stands for : Tedium; Isolation; Rote learning; Elitist; Depersonalisa

factors and motivational and emotional phenomena that do not fit neatly into current theorisations have emerged, as the following makes clear.

### **Evidence of Motivations :**

Schorr and Goldin (2008) point out that the complexity of affect in the social contexts of mathematics classrooms is exceptionally difficult to characterise for the purposes of research. The variety of emotions they report was substantial, including: curiosity, anticipation, frustration, anger, fear, excitement, pride, pleasure, elation, satisfaction. They note a number of important aspects of the motivational climate that encourages students to engage with the mathematics. These include an emotionally safe environment; the central importance of social interactions; relationships based on dignity and respect. Important as these are, it is not clear how they relate to motivational needs.

Swain (2005) studied the complex and multiple motivations of adult underachievers adopting mathematics education later in life, and found that ‘to help my children; to prove that I can do it; for understanding, engagement, enjoyment’ were more frequent and important motivators than utility. Swain also reports evidence of the excitement and pleasure of these adults in being able to do mathematics, *“It’s exciting.” “I enjoy doing it...” “It gives you a buzz....it’s exciting.”* Again we come across evidence that mathematics can be a source of pleasure and satisfaction for its own sake. Such evidence is also reported in Buxton (1981, p.132).

Theorising about emotions in the literature has been dominated by attention to anxiety or a limited set of negative emotions, and yet a whole range of other emotions has been reported, as referenced above. It has proved difficult to provide any theoretical or explanatory framework for many of these emotions, and they are rarely related to other aspects of motivation or affect. For instance, how does anger occur? How does excitement relate to motivation? These are important questions that need to be explored in a research setting. The CERME working group has set the agenda:

‘One apparent main focus for research and practice in this domain has been to develop richer theoretical frameworks using aspects and develop better concepts and instruments, preferably combining qualitative and quantitative methods. The frameworks should recognise the close relation between beliefs, motivation and competence.’ (Hannula et al., 2010, p.32)

One such framework that addresses these challenges is Reversal Theory. In the rest of this paper I set out some key aspects of the theory, and an analysis of data sources from the literature that evidence the application of the theory in the context of mathematics education.

## REVERSAL THEORY

Reversal Theory was developed over 30 years ago as a way of trying to explain and illuminate problematic behaviour of young people. This led to a focus on the subjective experience and the meanings ascribed by children to their own behaviour. The foundation of the theory is the structure of the motivational landscape, and its constituent eight motivational states. Time and space will not allow a detailed exposition of the theory, but a full description can be found in the literature, and is perhaps summarised best in Apter (2001). A motivational state is defined as a distinctive orientation to the world based on a fundamental psychological value. Methodologically, Reversal Theory is a structural phenomenological approach, in that it begins with subjective experience, but seeks to find structure in the complexity of that experience. Our experience is ordered into four fundamental domains:

- means-ends, about directionality or purpose;
- rules, including expectations, conventions, norms, customs, and the constraints put on us by social contexts of various sorts;
- transactions, which is those people or things we interact with; and
- orientation, which is a focus on self or identification with external entities – ‘the other’.

One of the key insights of Reversal Theory is that we can experience each of these four domains in two, entirely opposite ways. So, for instance, with the means-ends domain, we can experience it as focusing on the goal or outcome, or we can experience it as focusing on the journey rather than the destination. The former is defined as the telic motivational state (or *serious* in everyday language). The latter is defined as the paratelic or *playful* state. In the case of the former, we are interested in goals or outcomes, and progress towards them. The latter involves doing things for their own sake – in effect, for the pleasure and excitement for its own sake.

This aspect of our motivational make-up has strong resonance in many aspects of human experience, but it is not well represented in current theorisations of motivation. Specifically in the educational context, there is significant evidence that fun and excitement (both associated with the paratelic state) are important. The evidence from (Buxton, 1981; Swain, 2005) and others, above confirms this. One of the advantages of Reversal Theory as a framework for studying motivation in an educational context is that it does account for phenomena that are evidenced in research, but which resist classification in terms of current theorising. Examples include :

- the notion that high arousal can be experienced in a pleasant way (as excitement or thrill). This runs counter to the notion, often adopted in mathematics education research, that high arousal is always experienced negatively (as fear or anxiety).

- That motivationally, goals do not always drive our experience and behaviour.
- Reversal Theory proposes motivational needs that are poorly explained or absent from current motivational research in education. These include: rebelliousness and anger; altruism – the need to help others. It will be instructive to see if there is evidence of these in mathematics education contexts, and to widen the theoretical base to include them.
- The theory gives an account of primary emotions and their relationship to motivational states.

What Reversal Theory states is that one will always (at any one moment) experience the world as either serious (telic) or as playful (paratelic), but never both at the same time. What is very different about this treatment of motivational states (as opposed, for instance to trait descriptions), is that we frequently reverse between states.

The other domains and states describe aspects of our motivational experience. The domain of rules can be experienced as conformity or negativism. The domain of transactions can be experienced as mastery or sympathy, and relationships can be experienced as self-oriented (autic) or other-oriented (alloic). When we are in a state, it colours every aspect of our felt experience – from what we are paying attention to, to what we value, to how we view events, and the emotions we experience. More specifically, each motivational state has associated a value, a feeling, a way of experiencing, and associated emotions. People move between states (and combinations of states) frequently. And since these states are opposites, not only are we bi-stable, but we are multi-stable. That is, we can literally, be different people at different times. In this way, Reversal Theory captures the dynamic and changeable (and even contradictory) aspects of our personality, emotions and motivation.

In summary, we can see that our motivation shifts around (by reversals and by changes of focus) in a dynamic flow. There is strong evidence to show that motivational efficacy and psychological health requires that we have available and experience all eight states in the daily course of our lives, and psychological dysfunction caused by inability to do this is well documented (Apter, 2001). It is very interesting to speculate how, and to what degree these eight states are available to students (or not) in typical mathematics lessons.

## **REVERSAL THEORY IN MATHEMATICS EDUCATION: TWO EXAMPLES**

Reversal Theory has been applied successfully as an explanatory framework in many areas of human experience, such as child guidance, psychotherapy, drug addiction, anti-social behaviour, smoking cessation, sporting performance (Apter, 2001). In educational terms it has been used much less. In the UK the Learning and Skills Council evaluated a psychometric instrument based on the theory very positively, and stated: ‘There is an impressive amount of empirical evidence which supports

Reversal Theory ...which has major implications for how we think about learning styles.’ (Coffield, Moseley, Hall, & Ecclestone, 2004, p.54)

The first step in my own research has been to identify data and evidence for the constructs set out In Reversal Theory. Two examples will be briefly described here.

### **Reversal Theory perspective on the case of Frank**

The case study of Frank (Op't Eynde & Hannula, 2006) was addressed by the contributors to the issue of Educational Studies in Mathematics, devoted to affect. An alternative analysis is presented here.

Perhaps the most interesting data about Frank is the map of his dynamically changing emotions. Using the ideas from Reversal Theory we can use the sequence of his emotions to infer the motivational states. Frank is reported as being a little nervous about the task, and this suggests that he is in the serious state. In fact, in subtask 1 we see that he is worried, which suggests serious/high arousal. This is likely to be caused by the problems he encountered in his progress. However, he then finds a solution, arousal decreases, and he becomes relieved. Reversal theory predicts that in the serious state the emotion will move from anxiety to relaxation/relief as arousal lowers (as the solution is found).

In subtask 2 he again encounters a problem and gets stuck. This time he panics. I interpret this to be a stronger form of anxiety, due presumably to his perception of the depth of his ‘stuckness’. That is, he can’t see a way to progress. The panic causes him to reach for the calculator, which is seen as a strategy to make progress. It is possible that the panic is amplified by the fact that using the calculator brings him into conflict with other aspects of his motivational system (e.g. possibly a mastery-based self belief that if you are clever, you don’t need it). Without the data, we can only speculate. However, he does invoke a sound learned strategy for dealing with problems (and unpleasant high arousal) in mathematics contexts – stop and think. This seems to work, as he says ‘I know again what I have to do’. The emotions map shows that at some stage the panic changes to frustration/anger. This indicates that a reversal has taken place. His inability to make progress in the serious-conforming state has triggered a reversal from conformist to rebellious state, and the anxiety has now changed to anger. He now seeks to move outside of his own norms – to break the rules (whatever they are perceived to be). Is this where he impulsively reaches for the calculator?

When he finally reaches the end the emotion reported is pride. This suggests a shift of focus from the serious (which appears to have been the operative state so far) to that of self-mastery with the felt transactional outcome being experienced as winning. This analysis enables us to evidence the dynamic flow through motivational states within the problem-solving process, and more specifically we are able to relate states to the associated emotions. We can see also evidence of reversals, and change of focus of motivational state.

## Quiet disaffection revisited

The qualitative research by Nardi and Steward (2003) is one of the most influential analyses of disaffection with school mathematics. Below is a summary of the analysis of one of the interviews conducted with two 14- year old girls. Clearly, the interviews were not conducted with the intention of examining the complete motivational and emotional landscape for each participating individual. Nonetheless, it is possible to determine, even from this short extract, some key aspects of individual motivational factors. I identified in the interview transcript any motivationally or emotionally significant statements or segments. There were approximately thirty such statements. For each one the implied motivational inferences were tabulated.

Examples include :

“I don’t like being shouted at”	“I need things explaining”
“I like it when it’s fun”	“I remember it better when it’s fun”
“I need to know why it’s important	“I need to know the rules”
“Not knowing the answer makes... ..me feel stupid”	“I enjoy it more when I understand it”
“Maths makes me panic. (then)... ..I don’t know what to do”	“I need to do it my own way”
“Talking to friends helps me to understand”	

If I use Reversal Theory as a framework or as a lens to refract the data through, I encounter these points:

**(Self)Mastery.** The need to understand is a very strong focus for the interviewees. There are many statements here reflecting that. Having something explained to us is an important component of being able to understand. When I understand it I can do it, and when I can do it I enjoy it more. Not understanding can leave us feeling powerless (*‘oh my god I can’t do it’*), or humiliated. There are a numerous comments about feeling stupid. There is also evidence here of the issue of agency. For student J, there is clearly a power tussle in which she obviously needs to assert her own way of doing things.

**Seriousness.** Many of the comments here are about the lack of a sense of purpose. They demonstrate that both girls need to know why they are doing this. Comments like *‘what’s the point in it?’* and *‘I don’t see what it would help us with’* can be viewed as cries for a sense of purpose and direction.

**Playfulness.** Playfulness is not about play in the everyday sense of the word, but about enjoyment in the moment. It is related to fun, but also to excitement, intrigue, curiosity. It is arousal-seeking, and so is also often associated with risk-taking.

However, when arousal is low, playfulness will be experienced as boredom or sullenness, and there is good evidence of that here. It is interesting to note that ‘fun’ is mentioned a number of times by both girls. They make the association with being more interested, paying more attention and remembering better. Activities that were ‘different’ (i.e. not following the book) sparked playfulness, as did practical activities like looking through catalogues.

**Conformity and rebelliousness.** Conformity is about fitting in – complying with the norms, expectations and rules of the socio-cultural environment. There are quite specific and distinct rules and expectations that operate in mathematics classrooms (even though they may not be explicitly codified), and students will want to know what they are. Comments like *“I don’t get most of maths. It’s really weird.”* *“I do everything differently to Mrs R because I don’t understand how most of the time”*, give away the discomfort caused by the students not understanding the ‘rules of the game’. And even when they do, it is sometimes not sufficient to create mastery (*“you remember how to write it out but not how to do it.”*) This is particularly interesting, because there is often a received notion that all pupils need to do in school mathematics is to follow the rules and the correct answer will follow. This evidence shows that even disaffected students (or maybe especially disaffected students) know that this is not the case. In contradiction to this, all healthy individuals will also spend time in the rebellious state, and will need to express this in some way. This is rarely legitimised in a school setting, and perhaps in mathematics classrooms least of all. Unfortunately, this is an opportunity lost, as constructive cognitive and behavioural rebelliousness can be extremely creative, and is a requirement for mathematicians at a higher level.

In these sequences, we do see evidence of rebelliousness operating. For student C, boredom or sullenness creates the need to raise arousal levels, and when associated with the rebellious state, involves doing something ‘naughty’ (*“winding up my little brother”*) to raise arousal. Playfulness + rebelliousness + low arousal can cause problems when expressed in a classroom (as every teacher knows!). What is interesting is that C knowingly chooses to express this outside of the classroom, which demonstrates an element of mature self-regulatory behaviour. An element of rebelliousness is also expressed later by C in her rejection of the teacher’s way of doing things, and her assertion of her need to do it her own way.

**Sympathy and other.** There is some evidence here of the importance of relationships, and the need for affinity. Many comments refer to the need to have friends around. But friends also play a quite specific role in terms of being able to discuss, and the need to ask questions and receive explanations.

There is also evidence of interesting motivational sequences reported here. One is: ‘Maths makes me panic. When I panic I don’t know what to do. When I don’t know what to do I feel stupid.’ Motivation here shifts from serious to self-mastery (losing). Another sequence goes: ‘I need to understand. When I understand, I can do it. When



I can do it, I enjoy it more.’ The sequence moves from self-mastery (winning) to paratelic enjoyment. It will be interesting to find evidence of other such sequences in further research.

So we can see from this interview evidence of all states being operative in the mathematical experience of these two girls. Of course, because we are in a state, does not mean we experience it positively or gain the satisfaction from the state, and that is often the case here.

## CONCLUSION

Recognising the agenda of the CERME working group on affect, ideas have been presented from a wider point of view than the study of attitude and beliefs, and I have presented a theoretical framework that offers one answer to the call for a more holistic approach that relates motivation to emotion and other aspects of affect.

Reversal Theory has significant potential to provide an explanatory framework for mapping the motivational and emotional landscape of students in mathematics classrooms, and to provide a coherent basis for integrating theory about different aspects and constructs in the affective field.

The next stage of my research is to apply a multi-method approach to investigating how these phenomena manifest in the experience of young people in mathematics classrooms.

## REFERENCES

- Apter, M. J. (Ed.). (2001). *Motivational styles in everyday life: A guide to reversal theory*. Washington, DC: American Psychological Association.
- Boaler, J. (2010). *The elephant in the classroom*. London: Souvenir Press.
- Buxton, L. (1981). *Do you panic about maths: Coping with maths anxiety*. London: Heinemann Educational.
- Coffield, F., Moseley, D., Hall, E., & Ecclestone, K. (2004). *Learning styles and pedagogy in post-16 learning*. London: Learning and Skills Research Centre.
- Deci, E. L., & Ryan, R. M. (1985). *Intrinsic motivation and self-determination in human behavior*. New York: Plenum.
- Furnham, A. (2009). The use of psychometrics to predict GCSE outcomes. *British Journal of Educational Psychology*, 79(4).
- Hannula, M., Pantziara, M., Waage, K., & Schloglmann, W. (2010). Introduction: Multimethod approaches to the multidimensional affect in mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *European Research in Mathematics Education VI: Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 28–33). Lyon: Institut National de Recherche Pédagogique.
- Hannula, M., S. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics*, 63(2), 165-178.

- Ma, X., & Kishor, N. (1997). *Assessing the relationship between attitude towards mathematics and achievement in mathematics, a meta-analysis. Journal for Research in Mathematics Education*, 28(1), 26-47.
- McLeod, D. (1987). A constructivist approach to research on attitude towards mathematics. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of the 11th Conference of the International Group for the Psychology of Mathematics Education: Vol. 1* (pp. 33-140). Montreal: PME.
- Nardi, E., & Steward, S. (2003). Is mathematics T.I.R.E.D.? A profile of quiet disaffection in the secondary mathematics classroom. *British Educational Research Journal*, 29(3), 345-367.
- Op't Eynde, P., De Corte, E., & Verschaffel, L. (2006). Accepting emotional complexity: A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, 63, 193-207.
- Op't Eynde, P., & Hannula, M. (2006). The case study of Frank. *Educational Studies in Mathematics*, 63, 123-129.
- Pintrich, P. (2003). A motivational science perspective on the role of student motivation in learning and teaching contexts. *Journal of Educational Psychology*, 95(4), 667-686.
- Schorr, R., Y., & Goldin, G., A. (2008). Students' expression of affect in an inner-city simcalc classroom. *Educational Studies in Mathematics*, 68, 131-148.
- Swain, J. (2005). Beyond the daily application: Motivations for adults attending numeracy classes. *Research in Post-Compulsory Education*, 10(3), 305-324.
- The Royal Society. (2008). *State of the nation: Science and mathematics education, 14-19*. London: The Royal Society.
- Underwood, J. (2009). *Narrowing the gap: Literature review*. London: Becta.
- Vygotsky, L., S. (1986). *Thought and language* MIT Press.
- Weiner, B. (1990). History of motivational research in education. *Journal of Educational Psychology*, 82(4), 616-622.
- Zan, R., & Di Martino, P. (2007). Attitude towards mathematics: Overcoming the positive/negative dichotomy. *The Montana Mathematics Enthusiast*, 3, 157-168.
- Zembylas, M. (2005). Three perspectives on linking the cognitive and the emotional in science learning: Conceptual change, socio-constructivism and poststructuralism. *Studies in Science Education*, 41(1), 91-115.

# **STUDENTS' DISPOSITIONS TO STUDY FURTHER MATHEMATICS IN HIGHER EDUCATION: THE EFFECT OF STUDENTS' MATHEMATICS SELF-EFFICACY**

Irene Kleanthous and Julian Williams

University of Manchester

*The present study aims to investigate what factors affect students' dispositions to study further mathematics in Higher Education (HE) and the role of students' maths self-efficacy in particular. A questionnaire was designed for the purposes of this study and was distributed to 563 adolescent students in Cyprus. The validity of the scales designed to measure students' dispositions and maths self-efficacy was investigated using the Rasch model. It was found that students' maths self-efficacy had a statistically significant effect on students' dispositions to study further mathematics but when students' mathematics course was included in the model maths self-efficacy was not statistically significant anymore. Possible reasons for the non-significant effects are discussed drawing on some qualitative data from this study.*

Keywords: dispositions, further mathematics, HE studies, mathematics self-efficacy

## **LITERATURE REVIEW**

The problem of students dropping out of mathematics, especially advanced mathematics has become one of the major contemporary concerns of educators, parents and politicians (Ma, 2001). Students' dispositions towards mathematics influence their decisions to choose advanced mathematics at school and to pursue further studies in mathematically demanding courses in Higher Education (HE). A considerable number of studies have investigated the role of students' mathematics self-efficacy in students' decision making for choosing a major course (e.g. Betz & Hackett, 1983; Hackett & Betz, 1989; Pajares & Miller, 1994). Most of these studies support Hackett's argument (1985) which highlights the critical role self-efficacy beliefs play in the academic and career choices of students.

The importance of considering students' self-efficacy beliefs, in addition to test scores is stressed in recent research findings, mainly because of its positive effect on students' academic choices (e.g. Pajares & Miller, 1997; Chen & Zimmerman, 2007). Chen and Zimmerman (2007) argue that perceived academic self-efficacy positively influences students' academic choices, academic performance, effort and persistence as well as choices in careers related to mathematics and science. Nevertheless, the relationship between self-efficacy and outcome expectations is not always consistent. According to Usher and Pajares (2009) "a student reasonably confident in her mathematics capabilities may choose not to take an advanced statistics course" (p.89).

Bandura (1986) initially defined the notion of self-efficacy as "people's judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (p.391). With regards to mathematics, Hackett and Betz (1989) defined mathematics self-efficacy as "a situational or problem-specific assessment of an individual's confidence in her or his ability to successfully perform or accomplish a particular [mathematics] task or problem" (p. 262). Mathematics self-efficacy has been assessed broadly in terms of individuals' judgments of their capabilities to solve specific mathematics problems, perform math-related tasks, and succeed in math-related courses (Betz & Hackett, 1983).

Bandura (1986) cautioned that, because judgments of self-efficacy are task and domain-specific, "ill-defined global measures of perceived self-efficacy or defective assessments of performance will yield discordances" (p. 397). As Pajares and Miller (1995) point out measures of self-efficacy should be specifically tailored to the criterial task being assessed. They argue that mismatch between self-efficacy and criterial task assessment is a recurring theme in educational research, often producing confounded relationships and ambiguous findings.

Pajares and Miller (1995) also remarked that the confidence assessment should consist of students' judgments of their confidence to solve specific problems rather than of global confidence statements infused with personal judgments of self-worth. In addition, they argued that such global statements decontextualise efficacy beliefs and transform the construct of self-efficacy into a generalized personality trait rather than the context-specific judgment Bandura (1997) suggested it should be.

In a recent study in England, called "*Keeping open the door to mathematically demanding courses in Further and Higher Education*", Wake and Pampaka (2007) reported the use of contextualised questions with mathematics problems to measure students' mathematics self-efficacy and to investigate its effect on students' dispositions to study further mathematics. They argue that students' mathematics self-efficacy is a multidimensional construct which could be measured separately for applied and pure mathematics. Furthermore, Pampaka, Kleanthous, Hutcheson and Wake (in press) found a positive effect of students' mathematics self-efficacy on students' dispositions to study further mathematics in HE, and they noted the effect of gender and mathematical attainment on students' mathematics self-efficacy.

The present study aims to investigate the effect of students' mathematics self-efficacy on their dispositions to study further mathematics in HE in a different cultural context, namely Cyprus. We build on previous research by Williams et al. (2008) for the ESRC-TLRP project "*Keeping open the door to mathematically demanding courses in Further and Higher Education*" which investigated students' dispositions to study mathematically demanding courses in HE and the effects of different socio-cultural backgrounds. We hypothesise that students' mathematics self-efficacy will have a positive effect on students' dispositions to study further mathematics as the literature in different educational contexts suggests.

## METHODOLOGY

For the purposes of this study a questionnaire was designed and distributed to 563 students aged 16-17 years old, attending four different upper secondary schools (lyceums) in Cyprus (boys N=266, girls N=297). Additional semi-structured interviews were conducted with a subset of 22 students, who were selected according to the year group and mathematics course they were attending. We drew on previously validated instruments from the ESRC-TLRP project for designing the questionnaire used in this study. We used the same scale designed for the ESRC-TLRP project, to measure students' dispositions to study further mathematics in HE (DISP.MATHS). The DISP.MATHS scale consists of 5 items aiming to capture information on students' dispositions to studying mathematically demanding subjects in the future in HE. This scale was translated into Greek and revalidated using the Rasch model. Cronbach  $\alpha$  for this scale is  $\alpha=0.93$ . An indicative item of the DISP.MATHS scale is the following:

My preferred options for any future study will include:


- ☐ a lot of mathematics
- ☐ quite a lot of mathematics
- ☐ a moderate amount of mathematics
- ☐ as little mathematics as possible
- ☐ no mathematics
- ☐ Don't know

In order to measure students' mathematics self-efficacy we used some items from the mathematics self-efficacy (MSE) scale designed for the ESRC-TLRP project, which consists of contextualised questions with mathematics problems (Wake & Pampaka, 2007). Additional items were designed according to the mathematics curriculum for each year group. These were again contextualised questions with mathematics problems drawn from the national textbooks of mathematics for lyceum. The items for the MSE scale were presented by the question: "*How confident are you in your ability to solve each maths problem?*". Students were asked to rate their confidence in solving sixteen mathematics problems ranging from 1= Not confident at all to 4= Very confident, without actually solving the problems.

A total of thirty-one items were developed with twenty items drawn from the national mathematics textbook for lyceum and a further eight items drawn from the MSE scale used in the ESRC-TLRP project. Three additional items were included in the scale in "pure" mathematics areas because most MSE items drawn from the ESRC-TLRP project dealt with modeling and applying mathematics (Wake & Pampaka, 2007). The thirty one self-efficacy items (sixteen on each questionnaire) were organised into four different versions of the questionnaire, which were

## Working Group 8

appropriately adjusted according to the year group and mathematics course the students were attending (Advanced or Core mathematics). Cronbach  $\alpha$  for the MSE scale was  $\alpha = 0.91$ . Examples of a ‘pure’ and an ‘applied’ MSE item are given in Figure 1.

<b>D6. Solving equations in algebra with square roots, such as:</b>	<b>I am ...</b>	
$\sqrt{3x^2 - 9x + 7} = 2x - 3$	1	Not confident at all
	2	Not very confident
	3	Fairly Confident
	4	Very confident
<b>D14. Solving practical problems involving quadratic equations, such as:</b>	<b>I am...</b>	
<div style="display: flex; align-items: center;">  <div> <p>A golfer hits a ball so that its height, <math>h</math> metres, above horizontal ground is given by <math>h = 20t - 5t^2</math>. Find when the ball is 5 metres above the ground by solving <math>5 = 20t - 5t^2</math>.</p> </div> </div>	1	Not confident at all
	2	Not very confident
	3	Fairly Confident
	4	Very confident

**Figure 1: An ‘applied’ (bottom) and a ‘pure’ (top) item from the MSE scale**

In summary, therefore, our MSE instrument meets Pajares and Miller’s (1995) urge for specificity in assessment when measuring students’ mathematics self-efficacy, due to its contextualised questions for eliciting students’ self-efficacy in solving specific mathematics problems.

## FINDINGS

The validity of the scales was investigated separately using Rasch analysis. According to Bond and Fox (2007) Rasch analysis provides indicators of how well each item fits with the underlying construct. Model fit statistics and item analysis was carried out for each scale using Winsteps, a software application for carrying out Rasch analysis (Bond & Fox, 2007). Tests of fit showed acceptable fit suggesting that our instruments could be used to measure the desired constructs, since there were no statistically significant misfitting items in any of the scales.

Once the Rasch analysis was conducted to check scalability, a step-wise model selection procedure was adopted to build generalized linear models (GLM) in the statistical software ‘R’. Students’ mathematics self-efficacy (MSE) was used as an explanatory variable to model students’ dispositions to study further mathematics (DISP.MATHS). Other variables used in these models were background variables

## Working Group 8

such as gender, socio-economic status (SES) measured in terms of their parents' occupation, and students' mathematics course (Advanced or Core mathematics).

The model in table 1 shows the effect of MSE on students' dispositions to study further mathematics in HE (DISP.MATHS). Clearly the F-value ( $F=9.886$ ,  $p=0.00$ ) shows the model below is statistically significant although it can only explain 6% of the variance of DISP.MATHS ( $R^2=0.061$ ). In this model MSE and gender were statistically significant for predicting students' dispositions to study further mathematics but students' SES was not.

$$\text{DISP.MATHS} \sim \text{MSE} + \text{SES} + \text{GENDER}$$

Coefficients				
	Estimate	Std. Error	t-value	p-value
(Intercept)	-0.76231	0.26335	-2.895	0.00395 **
MSE	0.30183	0.05924	5.095	4.82e-07 ***
SES[T.low]	-0.42433	0.27809	-1.526	0.12762
SES[T.medium]	-0.25212	0.26469	-0.952	0.34127
Gender[T.Male]	0.49288	0.16640	2.962	0.00319 **

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

**Table 1: Generalised linear model (GLM1) for predicting DISP.MATHS.**

In table 2 a second generalised linear model (GLM2) is presented with mathematics course (MATHS.COURSE) included in the model as an explanatory variable. When MATHS.COURSE was introduced in the model MSE and GENDER were no longer statistically significant. In the model below the F-value ( $F=54.5$ ,  $p=0.00$ ) shows the model is statistically significant and it explains 33% of the variance of DISP.MATHS ( $R^2=0.3284$ ).

Surprisingly, when mathematics course was introduced in the model MSE was no longer statistically significant. In all the models built, students' mathematics course (MATHS.COURSE) was a statistically significant predictor for DISP.MATHS which might explain why the relationship between MSE and DISP.MATHS is not statistically significant. Apparently, students who are more self-efficacious in mathematics tend to choose Advanced mathematics at school so the effect of MSE is 'hidden' in students' choice of mathematics course.

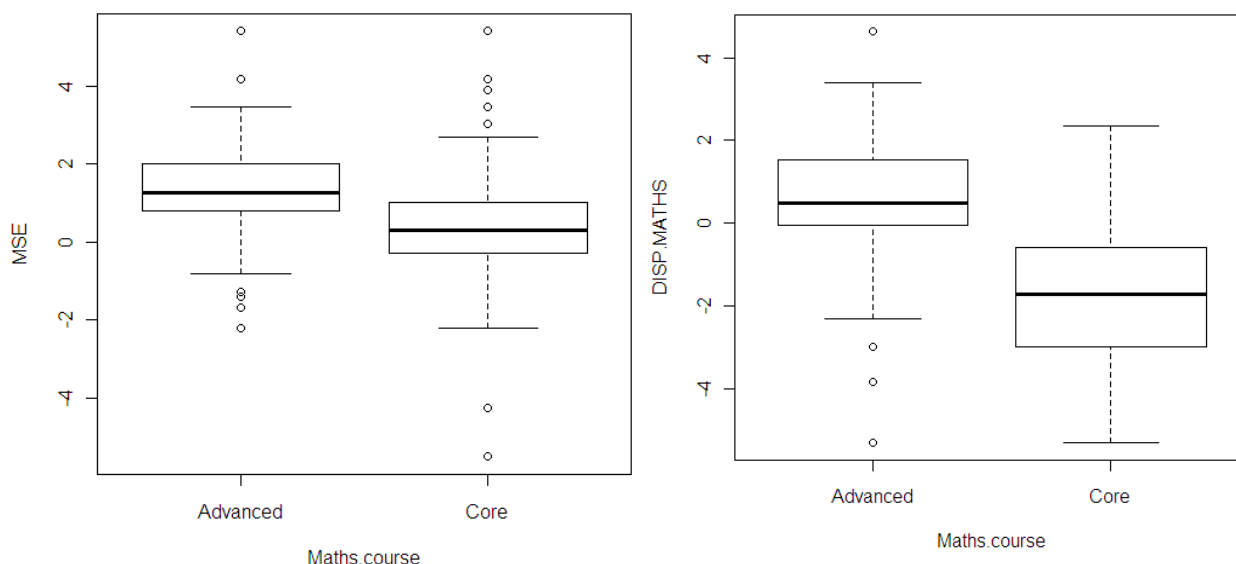
$$\text{DISP.MATHS} \sim \text{MSE} + \text{SES} + \text{GENDER} + \text{MATHS.COURSE}$$

Coefficients				
	Estimate	Std. Error	t-value	p-value
(Intercept)	0.58807	0.24083	2.442	0.0149 *
MSE	0.04402	0.05306	0.830	0.4072
SES[T.low]	-0.24675	0.23549	-1.048	0.2952
SES[T.medium]	-0.07073	0.22419	-0.315	0.7525
Gender[T.Male]	0.20595	0.14206	1.450	0.1477
Maths.course[T.Core]	-2.20956	0.14992	-14.738	<2e-16 ***

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

**Table 2: Generalised linear model (GLM2) for predicting DISP.MATHS.**

Further analysis showed that students who attend Core mathematics have lower mathematics self-efficacy (MSE) and are negatively disposed towards studying further mathematics in HE (DISP.MATHS). On the contrary, students who attend Advanced mathematics are more self-efficacious in mathematics and are positively disposed to studying mathematically demanding courses in HE. These differences are illustrated by the boxplots in figure 2.



**Figure 2. Students' mathematics self-efficacy and dispositions by mathematics course**



### **Insights from the qualitative data of the study**

In order to interpret the non-significant effect of students' mathematics self-efficacy on their dispositions to study further mathematics in HE, we draw on the qualitative data of this study. We chose to present here two case studies of students who were interviewed twice over a period of one year, because these two students were considered to be representative cases of students with high and low mathematics self-efficacy. They are both taking Advanced mathematics at school: a male student (Charalambos) and a female student (Chryso). They both aspire to study mathematically demanding courses in HE; Charalambos aspires to study Civil Engineering and Chryso wishes to study Computer Science. Although they are both attending Advanced mathematics at school their self-efficacy in mathematics differs tremendously.

When Charalambos was asked where he will rank mathematics in terms of difficulty comparing to his other subjects he seemed quite self-efficacious about mathematics:

Interviewer: Where will you rank mathematics?

Charalambos: For me? For me it's the easiest.

Similarly, when he was asked about the contextualised questions with mathematics problems for the MSE scale he responded quite confidently:

Interviewer: Is there something you saw on the questionnaire that you would like to comment on?

Charalambos: I think it is important if we can think about general exercises, like say there were some exercises which weren't part of the school curriculum but we can solve them. This is important because based on our knowledge regardless if we had never seen them before we could, if we thought about them, solve them.

On the contrary, Chryso seems less confident about mathematics and she uses the word 'phobia' when she refers to mathematics:

Interviewer: I wanted to ask you about your relationship with mathematics in general, what do you think about maths? If you want to you can go back to primary school or gymnasium. What experiences have you had with mathematics?

Chryso: In previous years, in primary school I was very good. But then when I went to gymnasium, I don't know I had a phobia, I didn't do so well. At gymnasium I had B but sometimes I had C. So it was a bit harder for me. I didn't like it much either but ok now I think it is useful and it's worth the effort to try.

The reason for this 'phobia' were not her mathematics teachers she argues, but her low self-efficacy in mathematics made her feel stressed and worried about mathematics.

## Working Group 8

Interviewer: What was the reason maths gave you a hard time at gymnasium? Do you think it had something to do with your teachers or with the curriculum that is being taught?

Chryso: No the teachers, all right they played a role but ok sometimes... They were good, the teachers I had, but I don't know, I was a bit stressed and scared.

Interviewer: Only in mathematics or in other subjects as well?

Chryso: Yeah basically mathematics because I didn't like it much and I don't know sometimes I was afraid I won't do well and things like that.

During their interview both students seem to have chosen Advanced mathematics to enable them to study mathematically demanding courses in HE. Chryso explains:

Interviewer: You said you have chosen advanced mathematics although you are afraid of maths?

Chryso: Yeah.

Interviewer: What led you to this decision?

Chryso: Because I liked computer science and I wanted to pursue it, I knew it required [of you] to know maths and physics, you needed them. OK I can't say I didn't like mathematics at all, I liked it but not as much as some other subjects.

Students were also aware that they would have to face challenging mathematics during their studies. Charalambos seems to be quite confident about the mathematics he will have to do at university:

Interviewer: Do you think that Civil Engineering will have lots of maths for the degree?

Charalambos: Yes.

Interviewer: Are you prepared for that? To face lots of mathematics during your studies?

Charalambos: Ok (laughs). I feel I can do it, but I don't know what I have to face exactly. I saw more or less what subjects are being taught. Maths and Physics but I don't know exactly.

On the other hand, Chryso's low mathematics self-efficacy almost made her change her mind about which studies to follow in HE. This indicates the effect students' mathematics self-efficacy can have on their decision making for future studies in HE.

Chryso: Last year when we were about to make our choices I was thinking not to choose computer science because the other subjects would be difficult for me despite the fact that I was good at computer science. I was thinking that maybe the other subjects will be difficult for me physics, mathematics...

## DISCUSSION AND CONCLUSION

In this study we sought to explore students' dispositions to study further mathematics in Higher Education and the effect of students' mathematics self-efficacy. Our

statistical analysis showed that students' mathematics self-efficacy is statistically significant for predicting students' dispositions to study further mathematics, which aligns with previous research findings (Betz & Hackett, 1983; Hackett & Betz, 1989; Pajares & Miller, 1994). Nevertheless, when students' mathematics course was introduced in the model students' mathematics self-efficacy was no longer statistically significant. This is probably because the mathematics course 'masks' the effect of MSE; more self-efficacious students in mathematics tend to choose Advanced mathematics at school.

Moreover, we argue that students' mathematics self-efficacy is an important factor influencing some students' decision making for future studies in HE and we have illustrated this with some qualitative data from this study. We also found that students' mathematics self-efficacy varies across the spectrum of confidence in mathematics to math-phobia.

A possible explanation for the non-statistically significant effect of students' mathematics self-efficacy on their dispositions to study further mathematics could be students' inaccurate ratings of their self-efficacy in mathematics. This aligns with Chen and Zimmerman (2007) who argue that students are generally not well calibrated or accurate in prejudging their capabilities to solve mathematical problems. Chen and Zimmerman (2007) also argue that researchers should examine the difficulty of the tasks that elicit students' judgments. Although our instrument was validated using Rasch analysis students might still have overestimated or underestimated their self-efficacy in mathematics.

Another plausible explanation why students' mathematics self-efficacy was not statistically significant in some models could be the multicollinearity between the explanatory variables of the model. This serves as a warning to studies that adopt correlational designs in this complex, multi-collinear field. It should be noted that previous research in the area of mathematics self-efficacy is mostly quantitative in nature. Apparently, students' mathematics self-efficacy needs to be further investigated by integrating quantitative and qualitative research methods.

## REFERENCES

- Bandura, A. (1986). *Social foundations of thought theory*. Englewood Cliffs, NJ: Prentice – Hall.
- Bandura, A. (1997). *Self-efficacy: the exercise of control*. New York: Freeman.
- Betz, N.E., & Hackett, G. (1983). The relationship of mathematics self-efficacy expectations to the selection of science-based college majors. *Journal of Vocational Behaviour*, 23(3), 329-345.
- Bond, T.B., & Fox, C.M. (2007). *Applying the Rasch model: Fundamental Measurement in the Human Sciences* (2nd ed.). Mahwah: New Jersey Lawrence Erlbaum Associates.

- Chen, P., & Zimmerman, B. (2007). A cross-national comparison study on the accuracy of self-efficacy beliefs of middle-school mathematics students. *Journal of Experimental Education*, 75(3), 221-244.
- Hackett, G. (1985). Self-Efficacy in Career Choice and Development. In Bandura, A. (Ed), *Self-Efficacy in Changing Societies*. New York: Cambridge University Press.
- Hackett, G., & Betz, N.E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education*, 20(3), 261-273.
- Ma, X. (2001). Participation in Advanced Mathematics: Do expectation and influence of students, peers, teachers and parents matter? *Contemporary Educational Psychology*, 26, 132-146.
- Pajares, F., & Miller, M.D. (1994). The role of self-efficacy and self concept beliefs in mathematical problem solving: A path analysis. *Journal of Educational Psychology*, 86, 193-203.
- Pajares, F., & Miller, M.D. (1995). Mathematics self-efficacy and mathematics performances: The need for specificity of assessment. *Journal of Counselling Psychology*, 42, 190-198.
- Pajares, F., & Miller, M.D. (1997). Mathematics self-efficacy and mathematical problem solving: Implications of using different forms of assessment. *Journal of Experimental Education*, 65(3), 213-228.
- Pampaka, M., Kleanthous, I., Hutchenson, G., & Wake, G. (in press). Measuring mathematics self-efficacy as a learning outcome. *Research in Mathematics Education*.
- Usher, E. L., & Pajares, F. (2009). Sources of self-efficacy in mathematics: A validation study. *Contemporary Educational Psychology*, 34(1), 89-101.
- Wake, G., & Pampaka, M. (2007). Measuring Perceived Self-Efficacy in Applying Mathematics. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Conference of the European Society for Research in Mathematics Education - CERME-5* (pp. 2210–2219). Retrieved from <http://ermeweb.free.fr/Cerme5.pdf>
- Williams, J., Black, L., Davis, P., Hutcheson, G., Nicholson, S., & Wake, G. (2008). Keeping open the door to mathematically demanding programmes in further and higher education. *Research Report RES-139-25-0241*. Swindon: ESRC.

# AN EXAMINATION OF THE CONNECTIONS BETWEEN SELF DISCREPANCIES' AND EFFORT, ENJOYMENT AND GRADES IN MATHEMATICS

Laura Tuohilampi

Helsinki University

*Research studies show that there is a connection between individuals' behavior and the discrepancies between their perceived, self-related skills and ideals: for example, high expectancies without success may lead to mathematics anxiety. In this study a hypothesis of a quadratic connection between self-discrepancies and desired learning action is formulated: an ideal level of discrepancy will associate with desirable action, whereas too large or too small a discrepancy will associate with undesirable action. The connections between examined variables were found to be non-linear, statistically significant associations; some evidence in support of the hypothesis was found.*

Keywords: self discrepancy, effort, enjoyment of mathematics, self concept, self beliefs.

## INTRODUCTION

Research in mathematics-related attitudes has put emphasis on studying linear relationships between different dimensions of affect and achievement. In this article, some theoretical elaboration and empirical results about a more complex model will be presented. This examination is a continuation of the investigation by Tuohilampi (2011), in which the threats and possibilities of internal discrepancies in one's self-concept were presented.

### **self-concept**

Self-concept seems to be a compendious and essential, but intricate, construct. It has been divided into *the material self*, *the social self*, *the spiritual self* and *the pure Ego* by James (1890). Later on, it is said to contain certain essential ingredients: self-related beliefs, which may or may not be valid (self image), the emotional and evaluative connotations around those beliefs (self-esteem), and a consequent likelihood of responding in a particular way (behavior) (Burns, 1982). Self can be viewed from two very different angles: self as a subject (I) and self as an object (me) (Harter, 1999). According to Bandura (1986), self-concept is a composite view of oneself that has formed through direct experience and evaluations adopted from significant others.

Burns (1982) has identified three roles of self-concept: to maintain inner consistency, to interpret experiences, and to determine a set of expectations. Self-concept is powerful and resilient, which is mostly a consequence of maintaining inner consistency. The strength of one's self-concept affects the interpretations of the

experiences of a person. Further, the self-image determines the expectations of a person's experiences and the expectations of his / her success. A person with poor self-image does not expect any success, which leads to either behavior or the adoption of an interpretation of circumstances that fulfills the prediction in order to maintain inner consistency. A person not only has certain beliefs within him/herself which need to be fulfilled, but also the perceived beliefs of other people in their social environment.

### **self-perceptions and behavior**

Perceptions of one's self-concept inspire, or cause, an individual's behavior (Burns, 1982). The connection between self-perceptions and behavior has been confirmed by empirical research. For example, mathematics anxiety is determined by outcome expectancy and outcome value (Kyttälä & Björn, 2010). In view of the effects of self on behavior, achievement and self-esteem, what perception of self should we be striving to instil in ourselves and others?

According to Bandura (1986), competent functioning requires both *skills* and *self-beliefs of efficacy* to work out effectively, and further, perceived self-efficacy is a judgment of one's capability to accomplish a certain level of performance, whereas an outcome expectation is a judgment of the likely consequence such behavior will produce. A circle of production can thus be described: good achievement implies better self-image (Marsh, Byrne & Shavelson, 1988), better self-image implies better performance (Bandura, 1986; Korpinen, 1990), which again leads to better self-image.

According to Harter (1999), self-concept has three types of functions. Firstly, there are *organizational functions* to provide expectations and to give meaning to life. Secondly, there are *protective functions* to maintain and maximize the pleasure. Thirdly, there are *motivational functions* to "energize the individual to pursue selected goals", and to "identify standards that allow one to achieve ideals in the service of self-improvement" (Harter, 1999, p. 10).

### **real self and ideal self**

An individual not only has perceptions of the *real self*, but also of the *ideal self*. An initial idea of the relationship between real self and ideal self was presented as early as 1890 by James' formula of self-esteem as a ratio of success and pretensions. The idea is that self-esteem can be enhanced by increasing the nominator (success, which can be seen as real self) or reducing the denominator (pretensions, which can be seen as ideal self) (James, 1890). The difference between real and ideal selves is then something to work on, as unrealistic pretensions can be risky: it is important for an individual not to fail when pursuing the ideal self; failure will result in negative outcomes, such as anxiety and depression. Further, the discrepancies between the *success* of pursuing an ideal and the *importance* of that ideal is a determinant of one's level of self-esteem (Harter, 1999).

Other discrepancies emerge from the *disparity* between real and ideal selves. The idea of the magnitude of the disparity between ideal and current self was first explored in the work of Rogers and his colleagues (Rogers & Dymond, 1954). In Roger's view, this disparity was the main cause of an individual's maladjustment. The idea of inequalities was operationalized, for example Butler and Haigh (1954) designed Q-sort task to measure the difference; subjects were instructed to sort cards to describe themselves on the day (self-sort) and to describe their ideal person (ideal sort).<sup>28</sup>

The disparity between real and ideal selves seemed to have some negative effects, such as poor self esteem (Harter, 1999). However, Harter writes that Rosales and Zigler have argued that the disparity could also be considered as an inspirer of desired action. Thus, the function of the disparity would be positive. In addition, Rosales and Zigler have presented that at some point the significance of the disparity might start to decrease. This suggests a quadratic function: diminutive difference would not provide any positive effect onto motivation, nor does large difference, whereas a suitable difference would generate a motive for the desired behavior. From that point of view, the increased disparity developed with age is not necessarily detrimental; it can also be seen as a more effective way of encouraging motivation.

Quite apart from considerations of the magnitude of the disparity, the same discrepancy, when associated with a negative sense of one's real self, may produce more distress than if it is associated with a more positive evaluation (Harter, 1999). As Bandura (1986) wrote, self theories have had difficulty explaining how the same self-concept can give rise to diverse types of behavior. By examining the disparity, its magnitude, its sign (whether the value of the discrepancy is negative or positive), and its starting point (the degree of the basis of the discrepancy, i.e. the degree of the real self), some improvements to the self theories might follow.

### **Aims and hypotheses**

As Tuohilampi (2011) has pointed out, the idea that the connection between self-discrepancy (disparity between the real self and the ideal self) and desired action can be quadratic needs to be examined. The association between these variables is quadratic, if both large and small discrepancies connect with minor action, whereas moderate discrepancy connects with good action. If the connection does appear quadratic, one needs to determine the best conditions, i.e. the optimum amount of discrepancy to ensure good action. This requires the direction of the connection to be from discrepancy to action; otherwise one has to study the possible benefits of connection in the opposite direction, i.e. from action to discrepancy.

---

<sup>28</sup> Q-sort (or Q-method, Q-technique), developed by William Stephenson, in which the items are put in an order of representativeness or significance for the individual (Block, 1961).

Based on the previous studies, the hypothesis is that the connection is from discrepancy to action, and that if the discrepancy is too large or too small, the motivation to bridge that gap is depressed. In the case of the discrepancy being too large, this is because the goal appears unattainable, and this leads to academic frustration; in the case of the distance being too small, it is because of the lack of a perceived challenge. If the distance between the real self and the ideal self is a negative number, i.e. the value of the ideal self is smaller than the value of the real self, no hypotheses as to how this may affect action are available from previous studies. However, an educated guess is that in that case the motivation to act as desired is potentially destroyed.

The study presented in this paper concentrates on the type of the connection when discrepancy is considered to be independent variable, and *effort*, *enjoyment of mathematics* and *last grade in mathematics* are considered to be dependent variables. The direction of the connections, however, cannot be revealed in this study, as this requires longitudinal analysis.

Specific research questions are: 1. Are there any associations between discrepancy and a) effort, b) enjoyment of mathematics (referred to as: EoM), and c) the last grade awarded in mathematics (referred to as: grade)? 2. If the association exists, what sort of is it regarding a), b), and c)? 3. What numerical value of discrepancy is connected to best values of a), b), and c)?

### Method

The present analysis is based upon data collected from December 2009 to January 2010 in Finland. The participants were aged 13 to 15 years. The data was collected in four local authority areas (*kunnat*; singular: *kunta*), and the numbers of participants in each *kunta* was 120 (598), 100 (472), 186 (540), and 0 (373); the sample size for each *kunta* is given in parentheses, giving a response rate of 20 %. The total number of participants was 406.

Data was collected as part of a research project, the aim of which is to investigate the affects of a learning environment called *Opit*. The *kunnat* differ in their application of *Opit*: some have used *Opit* for a long time; some started to use *Opit* at the time of the first data collection (three collections were carried out altogether); some have not used *Opit* at all. *Opit* is a Finnish learning environment that consists of a variety of tools, such as basic applications, customer service, technical support and teacher training.

In the questionnaire that was used to collect data, there were questions from several existing instruments. In this study, the following instruments were applied: *Students' view of learning mathematics*, developed by Rösken, Hannula, Pehkonen, Kaasila and Laine (2009), with sections on competence, effort, enjoyment of mathematics, difficulty of mathematics, and confidence; *Patterns of Adaptive Learning Scales* (PALS), developed by Midgley & al. (2000), with sections on mastery goal



orientation, performance-approach goal orientation, performance-avoidance goal orientation, avoiding novelty, classroom mastery goal structure, classroom performance-avoidance structure, parent mastery goal, and parent performance goal; and *Self-Regulation*, developed by Schwarzer, Diehl and Schmidz (as cited in Diehl, Semegon, & Schwarzer 2006).

In this study, the data was used to find out exploratory results concerning the hypothesis. In the data, there were items that measured perceptions of present skills and abilities (students' competence, confidence and self-regulation). These items were used to formulate mean variable of perceived self, representing students' *real self*. Items for mastery goal orientations (MGO) represent students' *ideal self*, as they measure how students wish to act. Discrepancy is then defined as the distance between perceived self and their ideal goals (MGO minus mean variable of perceived skills). The justification for choosing these items comes from previous theory: mastery goal orientations can be seen as pretensions (James, 1890); or as ideal self (Burns, 1982); the sum of competence, confidence and self-regulation can be seen as perceived skills, i.e. real self (e.g. Rogers & Dymond, 1954; Harter, 1999; Burns, 1982).

## Results

The analysis started by looking at the distributions of 'discrepancy', 'effort', 'EoM' and 'grade'. All distributions, except the 'grade', could be interpreted as normal according to their histograms and absolute values of skewness and kurtosis (all absolute values < 1) albeit the tests gave an alternative result (e.g. Nummenmaa, 2009). The parameters were: discrepancy ~ N (0.18 ; 1.05<sup>2</sup>), range [-3 ; 3.5]; effort ~ N (3.25 ; 0.84<sup>2</sup>), range [1 , 5]; and EoM ~ N (2.93 ; 0.96<sup>2</sup>), range [1 , 5]. Grade had a near normal distribution, but the absolute value of the kurtosis was slightly over 1. Mean of the 'grade' was 7.88, variance was 1.36<sup>2</sup>, and range was [4 , 10].

Reliability of all the factors were examined by Cronbach's alpha, though discrepancy had to be disassembled into perceived skills (confidence + competence + self-regulation) and into mastery goal orientation. The values of the alphas were: skills=.91, MGO=.84, effort=.77, EoM=.87. None of the reliabilities could be improved significantly by removing items.

The second phase of the analysis was to examine the scatter plots. When taking 'discrepancy' as the independent variable and other variables as dependent, none of the connections (discrepancy-effort, discrepancy-EoM, discrepancy-grade) seemed linear. On the contrary, when allowing the "fit line" to be loose ("loess" in SPSS-program), the connection seemed rather quadratic in nature, as expected by the hypothesis. This was the case for all the connections, except the connection between discrepancy and grade. However, the connections were not very clear.

How to examine the quadratic connection? According to Kendrick (2005), one way to better see whether two variables are quadratically connected is to view their

contingency table. This is what was done next. The variables were categorized, the contingency tables formulated and the distributions of percentages were interpreted (Kendrick, 2005). This time one could clearly see that the connection was either quadratic or at least curvilinear in two of the cases, discrepancy-effort, discrepancy-EoM (see tables 1, 2, and 3).

The significance of this result can be determined statistically by using the  $\chi^2$ -independence test for a null hypothesis “There is no association between independent variable ‘discrepancy’ and dependent variables ‘effort’, ‘EoM’, ‘grade’”. To satisfy the conditions of the test, the number of categories of variables was decreased: only about 20% of the cells may have frequency of less than 5, while the frequencies of all cells ought to be more than zero (e.g. Nummenmaa, 2009). At this point, the variables ‘effort’, and ‘EoM’ were condensed into three categories: -1=not at all or not very much, 0=neither a lot nor a little, 1=quite a lot or very much; ‘grade’ was also distributed into three categories: 1=poor grade, 2= moderate grade, 3=good grade; and ‘discrepancy’ into 5 categories: -2=orientations much less than interpreted skills, -1= orientations less than interpreted skills, 0=orientations in line with interpreted skills, 1=orientations higher than interpreted skills, 2=orientations much higher than interpreted skills.

For all the variable pairs (discrepancy-effort, discrepancy-EoM, discrepancy-grade), the null hypothesis “there is no association between” was rejected. The exact results were: discrepancy-effort  $\chi^2(8)=19.99$ ,  $p=.01$ ; discrepancy-EoM  $\chi^2(8)=14.92$ ,  $p=.06$ ; discrepancy-grade  $\chi^2(8)=107.28$ ,  $p=.00$ . Although the p-values were not all under .05, the trend was similar in all cases. Whether the null hypothesis got rejected incorrectly in any of the cases seems unlikely because of this trend as well as the theory behind what has been tested.

To view the connections, see the tables below. To make interpretation easier, the highest percentages have been bolded if the value is very high or remarkable: these values bend the connections from clearly linear.

Finally, a comparison was made between the linear and quadratic models linking the variable pairs. When discrepancy was an independent variable and effort was dependent variable, the results were:  $p=.35$  in linear model,  $p=.07$  in quadratic model,  $R^2_{\text{linear}}=.00$ ,  $R^2_{\text{quadratic}}=.01$ . When enjoyment was dependent variable, discrepancy again independent, the results were:  $p=.02$  in linear model,  $p=.00$  in quadratic model,  $R^2_{\text{linear}}=.01$ ,  $R^2_{\text{quadratic}}=.047$ . When grade was dependent variable, and discrepancy independent, the results were:  $p=.00$  in linear model,  $p=.00$  in quadratic model,  $R^2_{\text{linear}}=.16$ ,  $R^2_{\text{quadratic}}=.18$ . The differences were not very obvious; neither of the models seemed very suitable in any of the connections. However, the coefficient of determination ( $R^2$ ) is not a reliable measure if the connection is not monotonic, i.e. for example, quadratic (e.g. Nummenmaa, 2009).

## Working Group 8

		Discrepanc					Total
		-2	-1	0	1	2	
Effort	-1	2 15,4%	11 14,1%	27 13,4%	10 10,4%	8 <b>47,1%</b>	58 14,3%
	0	3 23,1%	36 <b>46,2%</b>	94 <b>46,5%</b>	44 <b>45,8%</b>	3 17,6%	180 44,3%
	1	8 <b>61,5%</b>	31 39,7%	81 40,1%	42 <b>43,8%</b>	6 35,3%	168 41,4%
Total		13 100%	78 100%	202 100%	96 100%	17 100%	406 100%

**Table 1: Contingency table ‘discrepancy’-‘effort’**

		Discrepanc					Total
		-2	-1	0	1	2	
EoM	-1	5 <b>38,5%</b>	20 25,6%	55 27,2%	31 32,3%	10 <b>58,8%</b>	121 29,8%
	0	5 38,5%	30 38,5%	96 <b>47,5%</b>	40 41,7%	7 41,2%	178 43,8%
	1	3 23,1%	28 <b>35,9%</b>	51 25,2%	25 26,0%	0 ,0%	107 26,4%
Total		13 100%	78 100%	202 100%	96 100%	17 100%	406 100%

**Table 2: Contingency table ‘discrepancy’-‘enjoyment of mathematics’**

		Discrepanc					Total
		-2	-1	0	1	2	
Grade	Poor	0 ,0%	2 2,6%	23 11,4%	26 27,4%	10 <b>58,8%</b>	61 15,1%
	Moderate	4 <b>33,3%</b>	21 26,9%	107 <b>53,2%</b>	60 <b>63,2%</b>	5 29,4%	197 48,9%
	Good	8 <b>66,7%</b>	55 <b>70,5%</b>	71 35,3%	9 9,5%	2 11,8%	145 36,0%
Total		12 100%	78 100%	201 100%	95 100%	17 100%	403 100%

**Table 3: Contingency table ‘discrepancy’-‘grade’**

### Interpretation

As far as the first connection, discrepancy-effort, is concerned, we can clearly see that the discrepancy should be negligible or negative rather than positive. However, the interplay is complicated to explain. If discrepancy causes effort, it is because

those whose skills are perceived as poor or who have high aspirations start to put in more effort. If effort causes discrepancy, it is either because those putting in a lot of effort without reaching their goals may revise the interpretation of their skills downwards, or those who like working hard set themselves too high aspirations as a realistic vision. The causations may even be circular. To be exact, those having large positive discrepancy are more likely to be connected with no effort, those having moderate positive or negative discrepancy or large negative discrepancy are more likely to be connected with moderate or strong effort. Counter to the hypothesis, negative discrepancy does not connect to putting in nonexistent effort. The result in the case of large positive discrepancy, however, supports the hypothesis that aspirations which are too high in relation to one's perceived skills do not result in increased effort.

In the second table, the connection seems to be curvilinear, to the point of being quadratic, but the cause-result relationships are hardly clear: the high frequency in cell<sub>1,1</sub> can be explained by claiming either that dislike of mathematics has led to lowered orientations or that lowered orientations has led to the dislike, whereas the high frequency in cell<sub>1,5</sub> may be explained by claiming that the non-fulfillment of high orientations has led to dislike but it can hardly be claimed that dislike has led to high orientations. Again, those having moderate positive or negative discrepancy are more likely to be connected with moderate or strong enjoyment of mathematics, but, this time large negative discrepancy is more likely to be associated with no enjoyment of mathematics, which is supported by the hypothesis.

The third connection seems to be somewhat similar. Moderate discrepancy connects with good or moderate grades, whereas large positive discrepancy clearly connects with poor grades. However, this time the connection between large negative discrepancy and poor grades is not obvious. The causation is again complicated: either poor achievement has led to high orientations (the wish to do better) or too much discrepancy has led to poor achievement.

The results printed above were subjected to further comparison by carrying out checks on the connections of all the independent variables (effort, EoM and grade) with 1) mean variable of competence, confidence and self-regulation (referred to as: skills), and 2) MGO. All the connections were clearly linear, and most of the correlations were quite strong and positive:  $R_{\text{skills} - \text{effort}} = .61$ ,  $R_{\text{skills} - \text{EoM}} = .66$ ,  $R_{\text{skills} - \text{grade}} = .62$ ,  $R_{\text{MGO} - \text{effort}} = .59$ ,  $R_{\text{MGO} - \text{EoM}} = .57$ ,  $R_{\text{MGO} - \text{grade}} = .23$ , wherein  $R$ =Pearson correlation. This means, that although high orientations correlate positively with examined variables *and* perceived skills correlate positively with examined variables, the effect changes if *the distance* between the two variables (perceived skills and orientations) is taken into account. Thus, there is strong evidence for the importance of this kind of perspective.

However, the analysis did not give a good model for the connections. This may be due to the fact that the number of subjects examined turned out to be on the low side.

Consequently, there was only a small number of cases showing large positive or large negative discrepancy (i.e. extreme cases), which limited the possibilities for statistical analysis. In future studies, this has to be taken into account: one has to pay attention to the number of participants.

## Discussion

Previous studies of self discrepancies show that there are connections between perceptions of the self and behavior (e.g. Burns, 1982; Kyttälä & Björn, 2010). The results of this study confirm this connection: perceptions of self can be separated from aspirations, and the disparity between the two gives an interesting perspective to examine the self. However, even a small discrepancy was connected with poor behavior, which was against the suggestion that some degree of the disparity would be necessary to emerge motivation, brought out by Harter (1999).

The results in this study linked the best connection to either small negative discrepancy or to no discrepancy at all. This means that those students who have high feelings of their skills, as well as those students who do not feel so competent but can set their goals very moderately, can enjoy mathematics, put effort on it and are able to get better grades. This suggests that it is good for a student to either be able to trust on own abilities, or to have realistic view on own skills and proceed step by step rather than trying to take huge leaps.

What are the causations of the connections? Bandura (1986) argues that competent functioning requires skills and self-beliefs of efficacy to work out effectively, but how can these efficacy feelings be improved? Perhaps those students, who tend to set their goals realistically, experience feelings of success more often, which again strengthens their ability feelings. If so, is the large discrepancy one cause for losing ability feelings? Do some students protect their *inner consistency* (Burns, 1982) by setting unrealistic aspirations to make sure they are not able to achieve them? If this is the case, students' *protective functions* (Harter, 1999) seem to overrun *motivational* and *organizational functions*. This suggests that the discrepancy is rather a protection of self than an initiation of desired behavior.

This study gives some preliminary results in the field of self-discrepancy's connections in mathematics, and builds the basis for further hypotheses.

## REFERENCES

- Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Prentice Hall NJ: Englewood Cliffs.
- Block, J. (1961). *The Q-sort method in personality assessment and psychiatric research*. Springfield, Illinois: Charles C Thomas publisher.
- Burns, R. (1982). *Self-Concept Development and Education*. London: Holt, Rinehart and Winston.

- Butler, J. M., & Haigh, G. V. (1954). Changes in the relation between self-concepts and ideal concepts consequent upon client-centered counseling. In C. R. Rogers & R. F. Dymond (Eds.), *Psychotherapy and personality change*. Chicago: The university of Chicago press.
- Diehl, M., Semegon, A. B. & Schwarzer, R. (2006). Assessing Attention Control in Goal Pursuit: A Component of Dispositional Self-Regulation. *Journal of personality assessment*, 86(3), 306–317.
- Harter, S. (1999). *The construction of the self. A developmental perspective*. London: The Guildford Press.
- James, W. (1890). *The principles of psychology*. New York: Dover Publications, Inc.
- Kendrick, Jr., J. R., (2005). *Social statistics. An introduction using SPSS for Windows*. Boston: Pearson Education Inc.
- Korpinen, E. (1990). *Peruskoululaisen minäkäsitys*. Kasvatustieteiden tutkimuslaitos.
- Kyttälä, M., & Björn, P. M. (2010). Prior mathematics achievement, cognitive appraisals and anxiety as predictors of Finnish' students' later mathematics performance and career orientation. *Educational psychology*, 30(4), 431-448.
- Marsh, H. W., Byrne, B. M., & Shavelson, R. (1988). A multifaceted academic self-concept: Its hierarchical structure and its relation to academic achievement. *Journal of educational psychology*, 80(3), 366-380.
- Midgley, C., Maehr, M. L., Hruda, L. Z., Anderman, E., Anderman, L., Freeman, ...Urdu, T. (2000). *Manual for the Patterns of Adaptive Learning Scales*. Retrieved from <http://www.umich.edu/~pals/manuals.html>.
- Nummenmaa, L., (2009). *Käyttäytymistieteiden tilastolliset menetelmät*. Tammi.
- Rogers, C. R., & Dymond, R. F. (1954). *Psychotherapy and personality change*. Chicago: The university of Chicago press.
- Rösken, B., Hannula, M., Pehkonen, E., Kaasila, R., & Laine, A. (2009): Structuring students' view of learning mathematics. In: L. Häggblom, L. Burman & A-S. Røj-Lindberg (Eds.), *Kunskapens och lärandets villkor*. Åbo Akademi, Pedagogiska fakulteten.
- Tuohilampi, L. (2011). Examining real / ideal self discrepancies as a source for motivation and goal-directed behavior. In: K. Kislenko (Ed.), *Current state of research on mathematical beliefs XVI. Proceedings of MAVI-16 conference*. Tallinn university, Institute of mathematics and natural sciences.

# THE IMPACT OF CONTEXT AND CULTURE ON THE CONSTRUCTION OF PERSONAL MEANING

Maike Vollstedt

Leibniz-Institute for Science and Mathematics Education at the University of Kiel,  
Germany

*What impact do context and culture have on the kinds of personal meaning constructed by students of the lower secondary level in the context of learning mathematics at school? To answer this question, this paper takes the perspective of students from Germany and Hong Kong as examples for cultures from the West and the Confucian Heritage Culture in East Asia. The concept of personal meaning will be developed denoting the personal relevance a person assigns to an object or action. Personal meaning is put in a concept framework of didactics and educational psychology and the relation to context and culture is shown. Finally, exemplary results from a qualitative study enriched with exploratory statistical analyses suggest the cultural impact on the construction of different kinds of personal meaning.*

Keywords: personal meaning, culture, context, learner's perspective, East/West

## INTRODUCTION

Humans have a “need for meaning” and “[a]lthough meaningful learning and meaningful life seem to be different concepts, it is quite possible that they have the same origin” (Vinner, 2007, p. 6). If meaningful learning is a special case of “man's search for meaning” (ibid), this specific human attitude does not disappear before entering the classroom. Meaning is also sought inside the classroom when students engage in learning and dealing with subject contents. Thus, the demand for meaning in education has been detected for many years, and meaningful learning has been identified as one of the major goals of education (ibid, p. 10).

Meaning seems an important topic when considering learning processes in school in general, and in learning mathematics in particular. But what exactly do we mean when we talk about *meaning*? Do students and educators intend the same thing when using this term? And taking the students' perspective, which meaning do students construct when dealing with mathematics in a school context? Does culture play a significant role for the construction of (personal) meaning? The last two questions are answered in the qualitative study reported on in this paper.

The study is included in the context of research on Educational Experience and Learner Development, which focuses on the learners' perspective of their own educational process. The paper's main focus is the development of the theory of personal meaning and hence, first presents different interpretations of the term *meaning*. The concept of *personal meaning* is then developed with regard to the perspectives of German and Hong Kong students. The two countries represent

cultures from the West and the Confucian heritage in East Asia. The theoretical part of the paper concludes with the relation of personal meaning with context and culture. The theory then is illustrated with exemplary results of the study presenting the influence of culture on the construction of meaning.

## FROM MEANING TO PERSONAL MEANING

A review of the relevant literature shows that the term *meaning* is used in very different contexts denoting different concepts. In education, we think about meaningful learning, and, even within the field of mathematics education, the term *meaning* can be used in different ways. The absence of a commonly accepted interpretation of *meaning* has resulted in a diversity of interpretations. This is due to a mixture of philosophical and non-philosophical interpretations as Kilpatrick, Hoyles, and Skovsmose (2005a, p. 2) point out: “on the one hand, we may claim that an activity has meaning as part of the curriculum, while students might feel that the same activity is totally devoid of meaning”. Different kinds of meaning can also be reconstructed when we turn to the students’ perspective (Kilpatrick, Hoyles, & Skovsmose, 2005b, p. 9):

Some students find it pointless to do their mathematics homework; some like to do trigonometry, or enjoy discussions about mathematics in their classrooms; [...] other students are told that because of their weakness in mathematics they cannot join the academic stream.

Although many interpretations of the term *meaning* relate to very different aspects, Howson (2005, p. 18) distinguishes two different aspects of *meaning*:

those relating to relevance and personal significance (e.g., ‘What is the point of this for me?’) and those referring to the objective sense intended (i.e., signification and referents). These two aspects are distinct and must be treated as such.

In this paper the terms *personal meaning* denotes the personal relevance of an object or action, and *objective meaning* a collectively shared meaning of an object or action (Vollstedt, 2010) [1].

To maintain the productive range of interpretations, the criterion of *personal relevance* here defines the term *personal meaning* of an object or action (Vollstedt & Vorhölter, 2008, pp. 29–31). It comprises the following aspects: objective meaning, usefulness, goal, purpose and value. Therefore, depending on the context, personal meaning can have an intentional as well as functional character. In addition to these aspects, I assume that personal meaning is subjective and individual, that the construction of personal meaning is context bound and that personal meanings can, but do not have to be reflected on (Vollstedt, 2010).

## RELATIONAL FRAMEWORK OF DIDACTICS AND EDUCATIONAL PSYCHOLOGY



The concept of personal meaning can be found in a relational framework of different concepts from mathematics didactics, the didactics of Educational Experience and Learner Development, and educational psychology. These concepts are understood to have an impact on the construction of personal meaning as they denote aspects which are personally relevant for the students. Let us assume we have a context in which an individual is dealing with a certain situation. In school, this might be a learning context in which a student deals with a mathematical problem. This student, let us call him Toby, judges the situation solely with respect to his personal attributes and goals. Toby automatically answers the question whether dealing with the situation makes sense to him, i.e. whether it is personally relevant, or not. He also considers in which way a possible following action might affect his personal goals.

Appraisal of the situation requires different concepts from mathematics didactics, the didactics of Educational Experience and Learner Development, and educational psychology. Toby might judge differently depending e.g. on his mathematical beliefs (Op ‘t Eynde, Corte, & Verschaffel, 2002). He might also be influenced by his interpretation of the developmental tasks he is dealing with at that point in time (Havighurst, 1972), a concept prominently discussed in the research context of Educational Experience and Learner Development. In addition, judgement is influenced by different aspects of learning motivation; a concept from educational psychology (Wild, Hofer, & Pekrun, 2001). The academic self concept, i.e. Toby’s judgement of his abilities in mathematics and how he perceives them (Möller & Köller, 2004), is relevant for the process of constructing personal meaning. Also, the three basic needs for autonomy, competence, and relatedness (Ryan & Deci, 2002) are believed to play a decisive role.

### **PERSONAL MEANING IN RELATION TO CONTEXT AND CULTURE**

Personal meaning cannot be constructed in a vacuum but is related to context. In this study, the term *context* covers both situational context (i.e. the context of the learning situation in terms of topic and the situation in the classroom) and personal context. The personal context may consist of the student’s personal traits (i.e. aspects which concern the student’s self like his/her self-concept, motivation, or beliefs) and his/her personal background (i.e. aspects regarding the world around the student like his/her socio-economic status, migration background, or learning and cultural environment) (Vollstedt & Vorhölter, 2008). Mercer (1993, pp. 31–32) takes the student’s perspective when he describes context in the following way:

What counts as context for learners [...] is *whatever they consider relevant*. Pupils accomplish educational activities by using what they know to make sense of what they are asked to do. As best they can, they create a meaningful context for an activity, and the context they create consists of whatever knowledge they invoke to make sense of the task situation. (Italics in original, MV)

According to Mercer, the students decide which information and experiences are relevant to deal with the posed task.

In my study, Mercer's description is interpreted in a broad way. In a learning situation not only knowledge but also beliefs, goals or other kinds of personal traits or background may be relevant for the students. According to Leung, Graf, & Lopez-Real (2006), these are subject to cultural influence as culture has a strong impact on how learning takes place in any learning situation. This is consistent with Mercer (1993, p. 43), who states that learning in the classroom depends both on culture and context as it is:

(a) culturally saturated in both its content and structure; and (b) accomplished through dialogue which is heavily dependent on an implicit context constructed by participants from current and past shared experience.

Both culture and context of a learning situation differ greatly in the Confucian Heritage Culture of East Asia and the Western traditions, as they are based on Chinese/ Confucian and Greek/Latin/Christian traditions respectively (Leung, 2001). As students from East Asia outperformed Western students in large scale comparative studies, it is interesting to see whether cultural differences can provide an explanation for these significant differences. Therefore, Leung (2006) examined a number of different characteristics of the Chinese/Confucian culture. He shows that:

there are indeed different cultural values pertinent to education that may explain the differences. This is of course no proof that differences in student achievement are caused by cultural differences. But in the absence of clues from variables at other levels, it is probable that culture does matters [sic]. (Leung, 2006, p. 44)

Therefore, one can stress that "the impact of cultural tradition is highly relevant to mathematics learning" (Leung, Graf, & Lopez-Real, 2006, p. 7) and therefore, also for personal meaning constructed in the context of learning mathematics at school.

Leung (2001) describes six features relevant for mathematics education distinct for East Asian and Western cultures. These features are not understood as dichotomies but as continua on which the relative position of countries from East Asia or the West differ. They are product (content) vs. process, rote learning vs. meaningful learning, studying hard vs. pleasurable learning, extrinsic vs. intrinsic motivations, whole class teaching vs. individualised learning, and pertaining to the competence of teachers: subject matter vs. pedagogy. All these aspects relate to complex underlying cultural differences and not just different kinds of established practice (Leung, 2001; for detailed discussion of the differences cf. Vollstedt, 2011). As the aspects given above are relevant for the learning context of mathematics, they are supposed to influence the students' processes of constructing personal meaning. Accordingly, the aim of this study is to contrast different types of personal meaning constructed in Germany and Hong Kong.

## THE STUDY

The study is based on 34 guided interviews conducted in Germany and Hong Kong with students of the lower secondary level (about 15-16 years old). Seventeen students participated from each country; all attended the highest school type in the respective educational system. In Hong Kong, I collaborated with schools using English as medium of instruction so as to conduct the interviews in English.

The guided interviews lasted for about 35 to 45 minutes and began with a sequence of stimulated recall (Gass & Mackey, 2000). For this, the students were shown a five to ten minutes video abstract of the last lesson they attended. Their task was to verbalize and reflect on the thoughts they had during the lesson. The interviews then tackled various topics inspired by the relational framework of personal meaning (see above) to come as close as possible to the aspects related to learning mathematics which are personally relevant for the students in a school context. Students were for instance asked about their associations of the words *mathematics* and *mathematics lesson*, they were interrogated about their beliefs with relation to mathematics, mathematics lessons and their learning of mathematics as well as about their feelings, their learning strategies, their goals and so on.

The data was coded in the style of Grounded Theory (Strauss & Corbin, 1996) by developing concepts directly from the interview material as well as theory-governed, taking the theoretical framework of the study as sensitising concepts into account. The coding was done partly in teamwork, partly independently but together with a team member so that the results could be discussed afterwards, and – after having received consistent results – partly on my own. By comparison and by using a coding paradigm, relations between concepts were disclosed so that core categories were developed denoting 17 different kinds of personal meaning. They vary in their orientation towards mathematics and the self so that a broad range is covered from the fulfilment of duty and the wish for cognitive challenge when dealing with mathematics to the experience of social relatedness. They can be grouped in seven different types of personal meaning.

Strictly speaking, the model of personal meaning presented above is also a result from the study as the concept and the relational framework were specified and developed further throughout the research process. To assure validity, the results as well as aspects which lacked clarity from my Western perspective were later discussed with Hong Kong professors of mathematics education as well as Hong Kong mathematics teachers.

After reconstructing the kinds and types of personal meaning, they were analysed with respect to the students' cultural background. Bound to the reconstructive framework, I opted not to generate hypotheses concerning cultural differences or similarities on a theoretical basis with relation to the relevant literature. Instead, I conducted exploratory statistical analyses using the software SPSS (version 15) to generate culture-specific hypotheses from the data. Hence, the unspecific and undirected hypotheses used in the *t*-tests of my study are strictly speaking not 'real'

hypotheses. On the contrary, I only checked whether differences exist between the two places to generate hypotheses about similarities and differences between the two places from empirical data. Statistics therefore acted as a means to conduct exploratory analyses instead of making general statements about Germany and Hong Kong.

To obtain a measure of the students' personal preferences, the relative frequency of every kind of personal meaning was coded for every student. In order to achieve this, the codings of every kind of personal meaning were counted. Then, the percentage of codings of one kind of personal meaning was calculated from all codings of personal meaning of this student. These relative frequencies formed the basis for the calculation of *t*-tests. From these results, empirically grounded hypotheses were generated for the relation between different kinds of personal meaning and the students' cultural background. The results are rather tentative as each sample only includes 17 students. Therefore, to support these results with a non-parametric test, the Mann-Whitney-*U*-test was calculated. This paper only includes the results of the *t*-tests, because the results from the two tests were comparable (for further information cf. Vollstedt, 2011).

## DISCUSSION OF EXEMPLARY RESULTS

As described above, seven different types of personal meaning could be generated. Here, example discussions include the type *fulfilment of societal demands*. The characteristics for this type of personal meaning are that it is personally relevant for the students to deal with mathematics and to learn mathematics to gain appreciation by other people due to their achievement, as well as to fulfil certain requirements for their intended future profession. The type consists of four different kinds of personal meaning. In *examinations*, it is important for the students to prepare themselves for examinations as passing the exams may result in opportunities in their personal future or their education. In addition, the students intend to create a *positive impression* and impress others through their achievement in mathematics. With respect to their future vocation, students must fulfil *vocational prerequisites* to ensure admission e.g. to their chosen field of study. Regarding *duty*, it is personally relevant for the students to meet the demands perceived as unavoidable as well as to deal with performance pressure.

When looking at *fulfilment of societal demands*, no significant differences could be found between Germany ( $M_{\text{Ger}} = 12.91$ ,  $SE_{\text{Ger}} = 5.21$ ) and Hong Kong ( $M_{\text{HK}} = 12.70$ ,  $SE_{\text{HK}} = 4.99$ ),  $t(32) = .12$ ,  $p = .905$ , Cohen's  $d = .04$ . This, however, does not automatically imply that underlying kinds of personal relevance do not show significant differences as opposite results may cancel each other out. Similar as with the type itself, no significant differences could be found for *duty* and *vocational prerequisite*. The differences between results from Hong Kong and Germany for *positive impression* and *examination* turned out to be significant as will be shown in more detail below.

As no significant differences could be found with the two kinds of personal meaning *duty* ( $M_{\text{Ger}} = 4.17$ ,  $SE_{\text{Ger}} = 2.33$ ,  $M_{\text{HK}} = 4.08$ ,  $SE_{\text{HK}} = 2.44$ ;  $t(32) = .10$ ,  $p = .918$ ,  $d = .04$ ) and *vocational prerequisite* ( $M_{\text{Ger}} = 4.94$ ,  $SE_{\text{Ger}} = 2.73$ ,  $M_{\text{HK}} = 5.08$ ,  $SE_{\text{HK}} = 3.06$ ;  $t(32) = -.14$ ,  $p = .891$ ,  $d = -.05$ ), it seems personally relevant for Hong Kong as well as German students to learn mathematics either because it is a school subject or due to other's expectations of them. In both countries students also consider some mathematical competencies relevant for their desired future vocation. Therefore, learning mathematics appears meaningful in preparation for future vocational demands.

When looking at *positive impression* and *examinations*, the results of the two countries are quite different. On the one hand, creating a *positive impression* of oneself in others seems more important for German students ( $M_{\text{ger}} = 3.57$ ,  $SE_{\text{Ger}} = 2.54$ ) than for those from Hong Kong ( $M_{\text{HK}} = 0.78$ ,  $SE_{\text{HK}} = 1.80$ ),  $t(32) = 3.70$ ,  $p < .01$ ,  $d = 1.27$ , as the value of the effect size Cohen's  $d$  is positive and bigger than 0.8. At first glance, this result seems surprising, as keeping face, acting according to others' expectations and worrying about others' opinions are aspects of social orientation and, therefore, part of the Confucian Heritage Culture (Leung, 1998).

The differences between Germany and Hong Kong can be traced back to the category of appreciation by other people with the appreciation of the teacher being the focal aspect for German students. A possible interpretation of these results with respect to (lesson) context and culture might be found in the different importance of oral participation in class, and examinations respectively. Oral participation is highly important in German lessons as it comprises a great share in the overall mark students get for a subject. In addition, performance measurement is continuous and dependent on the teachers' judgement rather than on examination boards (Kaiser, 1999). Therefore, it is of great importance for German students to continually create a positive impression of themselves for their teachers. In Hong Kong the focus lies on whole class teaching instead of individualised learning (Leung, 2001). Hence, oral participation in lessons is not as important. Accordingly, in the Hong Kong educational system, great importance is placed on results of examinations – an aspect which may historically be related to the prevalent belief in the Confucian heritage culture that practice makes perfect (Li, 2006). This goes along with the historic role of examinations in China. As early as 597 AD examinations were designed to choose savants for government positions. Therefore, it was possible to further social advancement by taking examinations (Leung, 2008). This attitude is still prevalent today. Today particularly the *Hong Kong Certificate of Education Examination* (HKCEE, similar to GCSE in the United Kingdom) is of great personal relevance for students. At the time of interview, students were going to take it in one to two years time, but already this early on it was governing their thoughts. Therefore, studying hard, practising a lot (rote learning as a necessary step towards deep understanding of the contents), and extrinsic motivation resulting from the importance of the results

from HKCEE are crucial aspects for Hong Kong students (Leung, 2001; Vollstedt, 2010). This, then may be an explanatory approach for the differences detected with *examinations*: examinations seem to play a more important role in Hong Kong ( $M_{HK} = 3.71$ ,  $SE_{HK} = 3.16$ ) than in Germany ( $M_{Ger} = 0.47$ ,  $SE_{Ger} = 0.90$ ),  $t(18.6) = -4.06$ ,  $p < .01$ ,  $d = -1.39$  as the value of Cohen's  $d$  is negative and  $|d| > .8$ . A similar situation did not arise in Germany with relation to the *Abitur* (University Entrance Examination) or other kinds of regular examinations. Two possible explanations are suggested here: On the one hand, as stated above, oral participation has a stronger weight for marks in Germany than regular examinations have. On the other hand, the *Abitur* exams are taken one year later than the HKCEE so that they may not have been as acute for the students during the interviews as for the Hong Kong students.

## CONCLUSION

When considering meaningful learning in a school context, it makes sense to take the perspective of the students as they are the people focused upon. Therefore, one must ask what is personally relevant for them when learning and dealing with mathematics. Characterizing personal meaning with the help of the criterion of personal relevance encompasses the range of interpretations denoted by the blurred concept of meaning. Theoretical coding of interview data from Germany and Hong Kong led to 17 different kinds of personal meaning covering aspects such as fulfilment of duty, cognitive challenge, and social relatedness. All these are personally relevant for students to deal with mathematics or learn the contents. Varying in degrees of relation to mathematics and the self, the 17 kinds of personal meaning could be grouped into seven types.

The paper discusses exemplary results including the four underlying kinds of personal meaning of the type *fulfilment of societal demands*. As shown above, no sound statements can be made on type level, as the results of the underlying kinds of personal meaning may be oppositional and cancel each other out. Therefore, only the results from the kind level were discussed in more detail. Here, no differences could be found for *duty* and *vocational prerequisite*, whereas differences became apparent for *examination* and *positive impression*. The students' cultural backgrounds were considered as a possible explanation for the differences detected in the last two kinds of personal meaning. Some instances of lesson culture as well as cultural beliefs were related to the results suggesting possible explanations: the cultural belief that practice makes perfect and the historic role of central examinations is prevalent in Hong Kong whereas in Germany oral participation in lessons plays a greater role. Therefore, the results of the qualitative study reported on in this paper suggest that a connection may be drawn between some personal meanings and the context, as well as the culture in which they had been constructed. In the East Asian and Western culture other kinds of personal meaning seem universal. This of course does not prove that culture is the reason for these differences. However, it may provide a

partial explanation. To provide more conclusive evidence, the hypotheses developed in this study will be tested on a broader field of data.

## NOTES

1. The German term for *personal meaning* we use in our research is *Sinnkonstruktion*. *Objective* or *collective meaning*, on the other hand are equivalent to *Bedeutung*.

## REFERENCES

- Gass, S. M., & Mackey, A. (2000). *Stimulated Recall Methodology in Second Language Research. Second language acquisition research*. Mahwah, NJ: Lawrence Erlbaum.
- Havighurst, R. J. (1972). *Developmental Tasks and Education* (3rd ed.). New York: Longman.
- Howson, A. G. (2005). "Meaning" and School Mathematics. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 17–38). New York, NY: Springer.
- Kaiser, G. (1999). *Unterrichtswirklichkeit in England und Deutschland: Vergleichende Untersuchungen am Beispiel des Mathematikunterrichts*. Weinheim: Deutscher Studien Verlag.
- Kilpatrick, J., Hoyles, C., & Skovsmose, O. (2005a). Introduction. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 1–8). New York, NY: Springer.
- Kilpatrick, J., Hoyles, C., & Skovsmose, O. (2005b). Meanings of 'Meaning of Mathematics'. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in Mathematics Education* (pp. 9–16). New York, NY: Springer.
- Leung, F. K. S. (1998). Implications of Confucianism for education today. *Journal of thought*, 33(2), 25-36.
- Leung, F. K. S. (2001). In Search of an East Asian Identity in Mathematics Education. *Educational Studies in Mathematics*, 47(1), 35–51.
- Leung, F. K. S. (2006). Mathematics Education in East Asia and the West: Does Culture Matter? In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics Education in Different Cultural Traditions. A Comparative Study of East Asia and the West. The 13th ICMI Study* (pp. 21–46). New York: Springer.
- Leung, F. K. S. (2008). In the books there are golden houses: Mathematics assessment in East Asia. *ZDM – The international journal of mathematics education*, 40(6), 983-992.
- Leung, F. K. S., Graf, K.-D., & Lopez-Real, F. J. (2006). Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West. In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics Education in Different Cultural Traditions. A Comparative Study of East Asia and the West. The 13th ICMI Study* (pp. 1–20). New York: Springer.

## Working Group 8

- Li, S. (2006). Practice makes perfect: A key belief in China. In F. K. S. Leung, K.-D. Graf, & F. J. Lopez-Real (Eds.), *Mathematics Education in Different Cultural Traditions. A Comparative Study of East Asia and the West. The 13th ICMI Study* (pp. 129-138). New York: Springer.
- Mercer, N. (1993). Culture, Context and the Construction of Knowledge in the Classroom. In P. Light & G. Butterworth (Eds.), *Context and Cognition. Ways of Learning and Knowing* (pp. 28–46). Hillsdale N.J.: Lawrence Erlbaum.
- Möller, J., & Köller, O. (2004). Die Genese akademischer Selbstkonzepte: Effekte dimensionaler und sozialer Vergleiche. *Psychologische Rundschau*, 55(1), 19–27.
- Op 't Eynde, P., Corte, E. de, & Verschaffel, L. (2002). Framing student's mathematics related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Mathematics education library: Vol. 31. Beliefs. A hidden variable in mathematics education?* (pp. 13–37). Dordrecht Boston: Kluwer Academic Publishers.
- Ryan, R. M., & Deci, E. L. (2002). Overview of self-determination theory: An organismic dialectical perspective. In E. L. Deci & R. M. Ryan (Eds.), *Handbook of self-determination research* (pp. 3–33). Rochester, NY: University of Rochester Press.
- Strauss, A. L., & Corbin, J. (1996). *Grounded theory: Grundlagen qualitativer Sozialforschung*. Weinheim: Beltz.
- Vinner, S. (2007). Mathematics education: Procedures, rituals and man's search for meaning. *Journal of Mathematical Behaviour*, 26(1), 1–10.
- Vollstedt, M. (2010). “After I do more exercise I won't feel scared anymore”: An example of personal meaning from Hong Kong. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *European Research in Mathematics Education VI: Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 131–140). Lyon: Institut National de Recherche Pédagogique.
- Vollstedt, M. (2011). *Sinnkonstruktion und Mathematiklernen in Deutschland und Hongkong: Eine rekonstruktiv-empirische Studie. Perspektiven der Mathematikdidaktik: Vol. 2*. Wiesbaden: Vieweg+Teubner.
- Vollstedt, M., & Vorhölter, K. (2008). Zum Konzept der Sinnkonstruktion am Beispiel von Mathematiklernen. In H.-C. Koller (Ed.), *Studien zur Bildungsgangforschung: Vol. 24. Sinnkonstruktion und Bildungsgang* (pp. 25–46). Opladen: Barbara Budrich.
- Wild, E., Hofer, M., & Pekrun, R. (2001). Psychologie des Lerner. In A. Krapp & B. Weidenmann (Eds.), *Pädagogische Psychologie. Ein Lehrbuch* (pp. 207–270). Weinheim: Beltz Psychologie Verlags Union.



# THE EFFECT OF A TEACHER EDUCATION PROGRAM ON AFFECT: THE CASE OF TERESA AND PFCM

Maria Pezzia\* & Pietro Di Martino\*\*

\* University of Naples Federico II – Italy

\*\* University of Pisa – Italy

*Affective factors are strictly intertwined and have a strong influence on teachers' professional practices. Literature underlines that primary teachers often have a negative attitude towards mathematics. For these reasons it is fundamental that professional development programs in primary mathematics teaching consider affective issues. In this study we analyze – describing the case study of Teresa, an experienced primary teacher with negative attitude towards mathematics – the effect of a Portuguese in-service program in mathematics and its teaching for primary teachers that includes, among its goals, the development of a positive attitude towards mathematics and its teaching.*

Keywords: attitudes towards mathematics, primary teachers' education, beliefs, emotions, teachers' professional development.

## INTRODUCTION AND THEORETICAL BACKGROUND

Many researchers have underlined the influence of several factors on teachers' effectiveness: Shulman (1986) highlights the role of subject matter knowledge, pedagogical knowledge and pedagogical content knowledge, others scholars (for example Ernest, 1989) underline the role of affective factors. As Tsamir and Tirosh (2009) claim, mathematical subject knowledge, pedagogical knowledge and affective factors are strictly intertwined and have a strong influence on teachers' professional practice. Therefore a professional development program in mathematics teaching limited to cognitive and pedagogical issues is “doomed to failure unless placed within an affective frame in which teachers have space to question mathematics and mathematics teaching” (Hodgen & Askew, 2006, p.41).

Literature shows that the common traits shared by many primary teachers (they are non specialist in mathematics and have often had negative experiences with mathematics as students) could generate uncertainty, low perceived self-efficacy as a teacher (Tschannen Moran et al., 1998), negative emotions such as shame (Bibby, 2002) and anxiety (Hannula et al., 2007) and produce “deep seated beliefs [that] often run counter to contemporary research on what constitutes good practice” (Liljedahl, 2007, p.320).

Adapting the model proposed by Di Martino & Zan (2010) for *attitude towards mathematics*, we characterize *attitude towards mathematics and its teaching* by three strictly interconnected dimensions: emotional disposition towards mathematics and its teaching, vision of mathematics and its teaching, and perceived competence in teaching mathematics. According to this theoretical framework, primary teachers'

common traits could generate a negative attitude towards mathematics and its teaching.

For these reasons it appears to be very important, both for the teacher as a person and for the quality of mathematics teaching and learning, that education programs for primary teachers focus on the several aspects that characterize the affective side. Nevertheless there are few professional development programs focalised on affective aspects (above all for what concerns in-service primary teachers) and “there has been little theoretical or empirical research exploring teachers’ emotional relationship with mathematics” (Hodgen & Askew, 2007, p.470).

Reporting some major trends in teacher education research, Krainer and Goffree (1999) stress the increasing importance of “teaching stories”, seeing case studies as:

an outcome of teachers’ efforts to investigate into their own teaching (...) for them as an additional circle of reflection, for other colleagues and researchers as an insight into teachers’ challenges and change. (Krainer & Goffree, 1999, p.230)

The issue of our study is related to the theme of understanding, by means of small scale qualitative research, teachers’ opportunities to change their attitude towards mathematics.

We focus on PFCM (Programa de Formação Continua em Matemática): a Portuguese in-service primary teacher education program that includes among its goals the need to develop a positive attitude towards mathematics and its teaching in order to improve mathematical learning in primary school. Some studies on PFCM and its effects on teachers have been conducted by Portuguese researchers, above all teacher educators directly involved in the program (e.g. Menezes, 2008; Canavarro & Rocha, 2010). These studies underline the good results of the program together with some obstacles (usually organizational difficulties).

Our study is different from the others mentioned above, mainly for two fundamental aspects: first, it is focused on teachers’ change related with the affective domain of mathematics education; second, it offers an external point of view (the authors are not Portuguese and they are not involved in the PFCM program). This aspect can be of some relevance: most research on mathematics teacher education is conducted by teacher educators studying the teachers with whom they are working and “*we do need more external research*” (Adler et al., 2005, p. 371).

In this paper we will briefly describe the PFCM’s structure and, through the analysis of one case (Teresa), we will evaluate what elements of PFCM either hinder or promote change in an experienced teacher with a deeply seated negative attitude towards mathematics.

### ***methodology***

This paper is based on the analysis carried out for the first author’s doctoral thesis. Four teachers have been involved in the research, chosen according to two common

features: their declared negative attitude towards mathematics and their feeling that, consequently to their involvement in PFCM, *something* changed in their attitude towards mathematics and its teaching. In this paper - due to the limited space - we only describe the case of Teresa. It seems to us particularly interesting because we recognized in Teresa all the common and critical traits highlighted by the literature on primary teachers; moreover her declared attitude towards mathematics and previous experiences with mathematics as a student are particularly negative. These elements, in our opinion, make of Teresa a real challenge for PFCM's efficacy.

We used a qualitative and interpretive approach. This methodological choice follows our attention to the processes below the teacher's change and not only to the certification of the possible change. We believe that qualitative analysis can highlight these processes. As Bruner (1986) states, narrative is the primary way in which we organize our own experience, trying to give it a meaning. Furthermore, the meanings given by teachers to their own professional development experience has an influence on their practice. For these reasons the main focus of the analysis has been on Teresa's narratives.

Data collection was realized during the second year of Teresa's participation in the program. It included two observations of two-hours school lessons (two hours each) and a semi-structured interview a week after each observation. Each interview went on for about two hours. The first observation was in the first term, while the second observation took place in the third term. Moreover our first author could see Teresa's first-year portfolio: it is a 45-pages document including an introduction, her reflections upon two activities carried out in the first year and her conclusions.

### **Context**

PFCM (Programa de Formação Continua em Matemática) is a national in-service program started in 2005, as a response by the Education Ministry to the Portuguese students' worrying results in mathematics emerging from PISA 2003 survey. The main aim of the program is to improve the quality of mathematics teaching in 1<sup>st</sup>-6<sup>th</sup> grade through teachers' professional development in mathematics and didactics.

From 2009/2010 PFCM has also had the goal of helping teachers to implement the new Mathematics Curriculum (Ponte et al., 2007). PFCM and the new Curriculum are oriented by a socio-constructivist model of teaching and learning processes, where mathematical knowledge is built in the classroom mainly through problem solving and "investigation" activities. As Ponte (2001, p.1) claims "there is a parallel between the activity of the research mathematician and the activity of the pupil in the classroom". The author describes investigation activity in the classroom as follows:

"A mathematical investigation stresses mathematical processes such as searching regularities, formulating, testing, justifying and proving conjectures, reflecting, and generalizing. When one starts working on an investigation, the question and the conditions are usually not completely clear and making them more precise is the first part

## Working Group 8

of the work. That is, investigations involve an essential phase of problem posing by the pupil—something that in problem solving is usually done by the teacher. However, investigations go much beyond simple problem posing and involve testing conjectures, proving, and generalizing.” (Ponte, 2001, p.3)

The new Mathematics Curriculum and the PFCM are proposing a neat break with “traditional methods” of Portuguese primary schools. These were mainly based on knowledge transmission, with a central focus on drilling exercises and learning by heart definitions or procedures. In order to make this *revolution*, teachers need to acquire new pedagogical and mathematical competences. As the national coordinators of the program recognize<sup>29</sup>, primary teachers’ professional development needs to be based on the improvement of pedagogical content knowledge, reflective attitude about professional practice and, in many cases, on a change of “attitude” towards mathematics and its teaching.

PFCM’s goal is to promote professional development starting from teachers’ reflection on their own practice. Indeed, the main features of PFCM are its close relationship with teacher practice and school context and the long duration. The training program lasts two school years and it is composed by two typologies of sessions: (i) Group sessions (two per month, three hours each) involving 8-10 teachers and held in the school after the curricular school-time. In these sessions, teachers and tutor discuss the mathematics curriculum, wondering about the content and pedagogical knowledge needed to plan lessons and possible obstacles in their classes. They also discuss questions arising from supervision; (ii) Individual supervision sessions of classroom work (about ten hours per year), where the teacher, with the trainer’s help, implements selected tasks explored in the previous whole-group working sessions.

The development of the teacher’s reflective attitude is also promoted through the editing of a portfolio.

### *Teresa’s story*

In the first meeting, Teresa (a very experienced teacher) introduces herself saying “I have never liked maths”. Her story with mathematics is a story of difficulties. In the first interview she tells that, when she was a pupil, she was terrified by multiplication tables and counting:

Teresa: I did not like tables and I was afraid...I was afraid of counting money. I was terrified of counting money. I got stuck on it, I got stuck! I am not able to count money!

Maria: According to you why did you get stuck on counting?

---

<sup>29</sup> Comissão de Acompanhamento, 2005. Programa de Formação Contínua em Matemática para professores do 1.º Ciclo, unpublished document.

## Working Group 8

Teresa: My father had a grocery shop...At that time there were many coins. He knew that I did not like to count the coins and he obliged me to count and I failed! Then he beat me (she laughs).

The emotional charge of this story is clear. It is interesting and surprising, at the same time that, in Teresa's view, this experience has not had any impact on her difficulties in mathematics. She strongly rejects Maria's hypothesis about the relationship between her experiences at home and her attitude towards mathematics. She explains that, in her family, two brothers out of four have their father's *maths genes*. Describing her father, she says that he "adored" maths and that he had an extraordinary intelligence in this field. She seems to suggest that if she were as gifted as her brothers she would not have been beaten.

Teresa tells about her aversion towards memorization (and towards all topics that she links to memorization like tables, numbers...) and, vice versa, her appreciation for visualization in mathematics. Her very few positive marks in mathematics as student (in grade 8) are linked to particular topics like geometry:

Teresa: When I got those marks, I do not know the reason. But I remember that I was able to see everything quickly! I discovered all. I don't know...I adore geometry!

The discovery of visual strategies in mathematics represented a turning point for Teresa: in her experience as a student they were accepted only in few topics and contexts. Starting from the new approach to mathematics met in PFCM (based on the use of multiple problem solving strategies including visual ones, not limited to geometry) Teresa thinks back to her experience as a student. This reflection, together with the experience in PFCM, convinces her about the need, for students, to use different strategies in mathematics and to seek their preferred ones and, for teachers, to give prominence to all the different strategies used by the students.

In grade 10 Teresa chose a humanistic course of study and "turned away from mathematics". Three years later, she enrolled in a university course necessary for becoming primary teachers in Portugal and during the first two years she took mathematics courses. She describes this experience with mathematics as very difficult but positive. Teresa was enthusiastic of her teacher:

Teresa: I had a wonderful mathematics teacher. Even now I love him, he is my life! (... ) I had again good marks with him and I did not like maths at all!

In spite of this last positive experience as a student, Teresa seems to be still confined in the role of someone who does not like maths. In her long teaching experience (almost thirty years) Teresa had never been involved in a significant mathematics development program before PFCM.

### ***TERESA and the PFCM***

In the first interview, Maria asks Teresa why she decided to enrol in PFCM.

## Working Group 8

Maria: Why did you decide to enrol in PFCM last year?

Teresa: First of all we are obliged<sup>30</sup> to enrol in some teacher development program. Then I thought that I had to follow something new related with maths...I haven't done mathematics for too long, I do not like mathematics and I think I have to learn something more because we have to start with the new Curriculum. So I thought that I had to enrol. And I like it very much.

Maria: Did you enjoy it?

Teresa: I enjoyed it. I liked my trainer very much, I think that he has been wonderful, he taught much. He has been excellent!

From this excerpt it emerges that, although Teresa decided to enrol in a teacher development program for an extrinsic reason, it was nevertheless her need to be ready for starting with the new Curriculum that addressed her towards the PFCM. Moreover, Teresa underlines the role of the trainer in triggering her enthusiasm for the first year of the project. Afterwards she gives more details about what she had learned from her trainer, also observing him to put his suggestions into practice.

Teresa: F. taught us...taught me...if I noticed that then the others noticed too...simple and practical words: “look, look again, try to explain with your words, pay attention, reflect, how did you do this thing? Are you able to explain how you did that? Ah explain! Try to write it down”. Because many students are not motivated to write and explain what they know, what they have done (...) F. taught me this way to ask students to explain how they have done something. We used to order them: “not like this! Not like this!!” But this is not the right way to teach this thing! (...)

F.'s suggestions about the way to interact with students during maths lessons (“the simple and practical words”) and his behaviour in the classroom are related with a view of mathematics in which error is not the focus. Arguing skills and comparison between different strategies become the main points. This new way of viewing mathematics seems to cause a radical change not only in Teresa's beliefs towards mathematics, but also in her emotions and attitude. It seems that now mathematics “is totally different” for her:

Teresa: I think that [mathematics] is difficult, I keep thinking that it is difficult, anyway I like it much more. I think that...it is totally different.

Maria: Why do you like it now and before you disliked it?

Teresa: Sure. Because...how can I explain that? Why do I like it much more now? Because there is not a unique way to do things!

---

<sup>30</sup> Actually it is not compulsory, but it is needed to have career progression.

## Working Group 8

At the beginning of the second year of the PFCM two significant events happened: Teresa was assigned another tutor and the new Curriculum started in its first implementation. As the first interview shows, Teresa lives *dramatically* the change of tutor. She feels that the significant path started with F. is definitely interrupted:

Teresa: I need to learn much more about mathematics. Do you understand? Much more! It is what F. did with us: he taught maths.

Maria: Then do you need to continue this path?

Teresa: Exactly! Because, after all, we have done all on the surface. Do you understand? On the top! Now we need to go in depth!

She attributes that to the inexperience of J. (the new tutor): unlike F., J. is not a school teacher and Teresa thinks she cannot deeply understand teachers' needs and difficulties. Moreover, the overlap of commitments linked to the start of the new Curriculum caused the lack of time for reading and studying in depth mathematical or teaching-related issues. According to Teresa, the previous year she positively faced the changes in her practice, because it was a gradual change and teachers had the time to both share and manage this change. Moreover, she recognizes some aspects of disorganization in the start of the new Curriculum, for instance there were no texts or handbooks ready for the new Curriculum. The work load, the uncertainties and the lack of an adequate support cause negative emotional reactions in Teresa and, according to her, in her colleagues. She claims: "we cry"!

The second interview is carried out in May, at the end of the second year of PFCM. The stress period has gone, Teresa is surely more relaxed and she shows a renewed enthusiasm towards the experience in PFCM. She is conscious that she changed her practice from the traditional way towards an inquiry oriented mathematics teaching. This change gave such unexpected results with students, that Teresa tells Maria with emphasis, underlying her happiness:

Teresa: Students from a school on the mountains getting A mark! I was admired! I was so happy! (...) I think that the few things made the last year have opened their minds and perspectives and they got A in maths!!!

She has not a definitive idea about the validity of her new method to teach maths:

Teresa: Perhaps this way to do maths is easier. Or rather it teaches to work better and then the students do everything more easily. I am not able to discover that! I need time to understand. Perhaps they understand more and if they understand they do better. Perhaps this new way makes it easier to understand.

She keeps attributing this success principally to the action of F.:

Teresa: He gave me a different perspective about mathematics, completely different. More practical, I do this in my classroom now. It is all a discovery! All, all, all! And I don't know how I can be so ahead with mathematics now.

## Working Group 8

The enthusiasm of Teresa is not only justified by the good marks of her students but also by the changes in pupils' attitude towards the mathematical activity:

Teresa: It is very nice that they [students] realize that one answers in one way, another in a different way, another in a further way...but all the ways are right! "teacher, I have done it different from you!" I find it funny!

Due to these results Teresa's perceived self-efficacy as mathematics teacher grew:

Teresa: Now I think that I would be able to orient a group of teacher, do you understand? Help other colleagues! Now I feel that I have the capacities to do that! Now I know how to explain the new Curriculum.

She reports an episode in which other primary teachers recognized her as an expert in mathematics and she is aware that her beliefs about mathematics have changed:

Teresa: Some days ago a colleague told me: <Teresa, you that are learned in maths, explain that thing to me>. He said that and I thought I have never been learned in maths, they wouldn't believe that I didn't like maths! But it is true! Now I think that I view maths in another way and in another form.

This professional change makes Teresa more confident about possible results with pupils in the future. She is now convinced about her ability to obtain the expected goals of the new curriculum. Teresa concludes the second interview showing her awareness that with PFCM and the implementation of the new curriculum she has started a path towards autonomy in teaching maths and she wants to continue it:

Teresa: This year has been an experience. Now we have to begin to get organized.

In this growth, Teresa changes also her consideration of the textbook: from the feeling of being discouraged without it, to the idea that the textbook is only the starting point to develop teaching ideas individually.

### **Conclusions**

Teresa is a difficult case: she has a personal history with mathematics full of difficulties and her emotions towards mathematics are strongly negative; moreover, she is a very experienced teacher with very deep-seated beliefs about mathematics and its teaching. The analysis of her case shows that, despite these difficulties, at the end of the two years of PFCM Teresa, her emotions and beliefs have changed radically. It is true that the main reason to enrol in PFCM was extrinsic (the need for career progression), but this decision gives Teresa the chance to engage in an unexpectedly significant developmental path:

Teresa: Sometimes we leave for a journey and we don't know why! But when we discover new things during the journey, we find new perspectives on life!

Teresa's change happens as a consequence of the encounter with a "*different mathematics*", introduced by the tutor, that Teresa implements both in the group sessions and in the classroom (the tutor in this phase represents a model and also a



facilitator). The change in Teresa's view of mathematics is relevant also because it elicits an emotional change: Teresa begins to appreciate mathematics (*"now I like it because I see that there are many ways to do the same thing"*). In PFCM Teresa has had the possibility of developing a new idea of mathematics, to learn aspects of mathematics that she ignored and to appreciate a more open method to teach mathematics (related and consistent with her new view of mathematics). This possibility has originated a virtuous process in which Teresa develops the pleasure to do and teach maths, increasing her perceived self-efficacy towards mathematics and its teaching. In this process Teresa changes her attitude towards mathematics and she is rewarded and also comforted by students' reactions (and results) to her new way of teaching mathematics.

It is a process and, like all processes, on the one hand it is full of crossroads and on the other hand it needs time (we cannot expect to get radical changes without time expense). The structuring of PFCM (through the group sessions where the teacher can explain his/her doubts and the continuous support of the tutors) provided a great help to Teresa in overcoming the crossroads she met; concerning the time variable, in our analyses we could notice how the second year was crucial to realize the change in Teresa's attitudes and practices (it is important to underline that PFCM is one of the few in-service teachers programs lasting for two years).

We are aware that our analysis is limited to a single case, nevertheless we think that this case has the strength of an *existence theorem* in mathematics: a radical change of in-service teachers' attitude towards mathematics and its teaching is possible also when it appears to be very difficult to be realized.

Furthermore, as emerges from our analysis, the change we observed is due to PFCM's specific features, that provided the necessary conditions to overcome the teacher's difficulties, by paying attention to content, to teachers' practice and to the affective side of teaching at the same time. We think that Teresa's case sheds light on the great potential of this teacher education model for the professional development of in-service teachers.

### REFERENCES

- Adler, J., Ball, D., Krainer, K., Lin, F., & Novotna, J. (2005) Reflections on an emerging field: researching mathematics teacher education. *ESM*, 60, pp. 359-381.
- Bruner, J. (1986). *Actual minds, possible worlds*. Cambridge, MA: Harvard University Press.
- Bibby, T. (2002). Shame: an emotional response to doing mathematics as an adult and a teacher. *British Educational Research Journal*, 26(5), pp.705-721
- Canavarro, A.P., & Rocha, I. (2008). Professional development of mathematics teachers: Challenges from a national in-service teacher education program in Portugal. Published at <http://dg.icme11.org/document/get/147>

- Di Martino, P., & Zan, R. (2010) 'Me and maths': towards a definition of attitude grounded on students' narrative. *JMTE*, 13(1), pp.27-48.
- Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: a model. *Journal of Education for Teaching*, 15(1), pp.13-33.
- Hannula, M., Liljedahl, P., Kaasila, R., & Rösken, B. (2007). Researching relief of mathematics anxiety among pre-service elementary school teachers. In Woo, JH, Lew, HC, Park, KS & Seo, DY (Eds.), *Proceedings of 31st PME*, 1, pp.153-156.
- Hodgen, J., & Askew, M. (2006). Relationship with/in primary mathematics: identity, emotion and professional development. In D. Hewitt (ed.), *Proceedings of the British Society for Research into Learning Mathematics*, 26(2), pp. 37-42.
- Hodgen, J., & Askew, M. (2007). Emotion, identity and teacher learning: becoming a primary mathematics teacher. *Oxford Review of Education*, 33(4), pp.469-487.
- Krainer, K., & Goffrey, F. (1999). Investigations into teacher education: trends, future research, and collaboration. In K. Krainer, F. Goffrey (eds), *From a study of teaching practices to issues in teacher education* (pp.223-241). Forschungsinstitut fuer mathematikdidaktik, Osnabruck.
- Liljedahl, P., Rolka, K., & Rösken, B. (2007). Affecting affect: The reeducation of preservice teachers' beliefs about mathematics and mathematics teaching and learning. In W. G. Martin, M. E. Strutchens, & P. C. Elliott (Eds.), *The learning of mathematics*, pp. 319–330. Reston, VA: NCTM.
- Menezes, L. (2008). Desenvolvimento profissional de professores em matemática num contexto de formação contínua. In A. Domingos, C. Carvalho, J.M. Matos & P. Texeira (Eds.), *Actas SIEM 2007*, Lisboa: APM (CD ROM edition).
- Ponte, J. P. (2001). Investigating in mathematics and in learning to teach mathematics. In F. L. Lin and T. J. Cooney (Eds.), *Making sense of mathematics teacher education* (pp. 53-72). Dordrecht: Kluwer.
- Ponte, J.P., Serrazina, L., Guimarães, H.M., Breda, A., Guimarães, F., Sousa, H., Graça Martins, M.E., & Oliveira, P. A. (2007). *Programa de matemática do ensino básico*. Available at <http://www.dgidc.minedu.pt/matematica/Documents/ProgramaMatematica.pdf>
- Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, 15(2), pp.4-14.
- Tsamir, P., & Tirosh, D. (2009). Affect, subject matter knowledge and pedagogical content knowledge: the case of a kindergarten teacher. In J. Maaß and W. Schlöglmann (Eds.), *Beliefs and attitudes in mathematics education: New Research Results* (pp.19-32). Rotterdam, the Netherlands: Sense Publishers.
- Tschannen-Moran, M., Woolfolk Hoy, A., & Hoy, W.K. (1998). Teacher efficacy: ist meaning and measure. *Review of Educational Research*, 68, pp. 202-248.

# **FEAR OF FAILURE IN MATHEMATICS. WHAT ARE THE SOURCES?**

Marilena Pantziara<sup>1,2</sup> and George Philippou<sup>2</sup>

Cyprus Pedagogical Institute<sup>1</sup> –University of Nicosia<sup>2</sup>

*This paper presents some results of a larger study that concern the causes of students' fear of failure in mathematics. Data was collected from 321 sixth grade students through a questionnaire comprising five-point Likert-type scales measuring among other constructs students' fear of failure and self-efficacy beliefs: An observation protocol was developed to identify teachers' practices fostering students' fear of failure. Findings revealed that fear of failure is a complicated affective construct based on several sources such as family context, students' characteristics and teachers' practices. The implications of these findings for understanding and improving students' behaviour in the mathematics classroom are discussed.*

Keywords: primary students, fear of failure, self efficacy, teachers' practices.

## **INTRODUCTION**

Research on achievement motivation provides empirical data about the nature and consequences of fear of failure (FF) (Conroy & Elliot, 2004; Macgregor & Elliot, 2005). Particularly, in Educational Psychology, achievement motivation theory emphasizes FF as a determinant of students' behaviour and performance (e.g. Elliot & Church, 1997), however there is little research on reasons why individuals are fearful of and motivated to avoid failure in mathematics. In recent studies the origins of FF were found in students' family context (parental socialization) and in students' associated feelings of shame (Macgregor & Elliot, 2005). In mathematics education, negative feelings have been reported by researchers such as fear, anxiety and frustration and their relation to students' mathematics performance (Ho et al., 2000) has been established. However, there is a need to investigate causes of students' FF in the mathematics classroom.

In this respect the present study investigates variations in students' internal and family characteristics and focuses on teachers' practices in the mathematics classroom that may cause students' FF. By raising awareness of the negative consequences of students' FF on their mathematical performance and on their stance towards mathematics, we believe that the results of the study will shed some light on factors that may lead to the development of FF in mathematics and inform teachers about desirable and undesirable practices in the classroom.

## **THEORETICAL BACKGROUND AND AIMS**

### **Fear of Failure in Educational Psychology**

Research on motivation identified the motive to avoid failure or more commonly fear of failure, as an energizing means for human behaviour (Conroy & Elliot, 2004). McGregor and Elliot (2005, p.219) state that FF is a self-evaluative framework that influences how the individual defines, orientates to, and experiences failure in achievement situations. More explicitly, an individual with a high FF perceptually and cognitively orientates to failure-relevant information and thus encounters anxiety prior to, and during, task engagement. He/she seeks to avoid failure by avoiding the situation, by quitting or withdrawing effort, or by trying hard to succeed and thus avoid failure. The core emotion of FF is most likely shame, a devastating emotion that entails a sense of one's global incompetence. Other origins underlying students' FF mentioned by these researchers were students' relationship with their parents and social background. Specifically, students' high in FF had mothers who punished failure but reacted neutrally when the students were successful or they had mothers who set high standards for achievement but believed that their children could not reach them. Among other causes of students' FF identified by researchers is the experience of embarrassment, the devaluation of one's self-esteem and also that important others may become upset, Conroy, Poczwardowski and Henschen, 2001 (in Conroy & Elliot, 2004). In the context of the hierarchical achievement goal framework proposed by Elliot and Church (1997), motive-based and goal-based variables appeared to be an integral part. Elliot and Church (1997) assert that FF negatively predicts adaptive behaviour. Particularly, FF is found to negatively predict mathematics performance and interest through achievement goals (e.g. Zusho, Pintrich & Cortina, 2005) and it is negatively correlated to self-efficacy beliefs (Pantziara & Philippou, 2006).

### **Fear of failure in Mathematics Education**

In mathematics education, a term that is close in meaning to Fear of Failure is mathematics anxiety (Ho, et al., 2000). Mathematics anxiety has been investigated as a two dimensional construct, in a two factor model comprised of affective and cognitive dimensions. Affective anxiety refers to an emotional component of anxiety such as fear, feelings of nervousness, tensions etc. Cognitive anxiety refers to the worry component of anxiety which is often displayed through negative expectations, preoccupation with failure and self-deprecatory thoughts (Ho et al., 2000, p.2). The conceptual nature of fear of failure as developed in the realm of Educational Psychology we believe is closer to the cognitive anxiety as described above.

Similar results to the ones in Educational Psychology have been reported in mathematics education with mathematics anxiety correlating negatively to students' mathematics performance and behaviour. The indirect effects of mathematics anxiety are worth noticing, even in cases when the negative correlation with performance is poor, such as students' negative attitudes to mathematics, avoidance of mathematics classes and elementary school teachers spending less time in teaching mathematics. (Ho et al., 2000; Pantziara & Philippou, 2006).

We limited our attempt to investigate multiple sources underlying FF in mathematics to the context of a socio-constructivist perspective on learning (Op't Eynde, De Corte & Verschaffel, 2006). More specifically Op't Eynde et al. (2006) recognised the close relation between (meta) cognitive, motivational and affective factors in students' learning and problem solving. They believed that students' understanding of, and behaviour in the mathematics classroom is a function of the interaction between who they are (their identity), and the specific classroom context. Students' identity, their values and what matters to them and in what way is revealed to them through their emotions (Op't Eynde et al., 2006). In this respect, students' attitudes toward mathematics are the outcomes of a conscious or unconscious personal evaluation of mathematics, the students' self and their mathematics learning situations.

Based on this theoretical framework and in an attempt to inform educators as to the factors creating students' FF, we investigated variables regarding students' mathematical performance and self-efficacy beliefs, mothers' and fathers' educational background, as well as variables referring to the context of learning mathematics (teachers' practices) that may influence students' FF for mathematics.

### **Self-efficacy beliefs**

Friedel, Cortina, Turner, and Midgley (2007) refer to academic self-efficacy as children's confidence in their ability to master new skills and tasks, often in a specific academic domain such as mathematics. In this study we consider self-efficacy beliefs in relation to broader types of tasks (mathematics tasks) and not to specific ones (e.g. fraction tasks) to attain broader results; but not seeking to establish a general competence construct.

Numerous studies have found that students with high self-efficacy beliefs are more devoted, show intense interest, work harder, persist longer and have fewer adverse emotional reactions when they come across difficulties, than students who doubt their capabilities (Zimmerman, 2000). Also self-efficacy beliefs were found to be related to mathematical performance (Zimmerman, 2000; Pantziara & Philippou, 2007).

Students' self-efficacy beliefs which manage the demands of academic tasks were found to have an emotional influence by decreasing their stress, anxiety and depression, Bandura, 1997 (in Zimmerman, 2000). Moreover self-efficacy beliefs were found to be more predictive of mathematical performance than students' mathematics anxiety, Pajares and Kranzler, 1995 (in Zimmerman, 2000). These results suggests that educators should focus more on fostering positive characteristics in students, like self-efficacy rather than merely diminishing negative characteristics like anxiety and FF.

### **Instructional practices**

Elliot and Church (1997) draw attention to the role of teachers' practices in the classroom; they note that if the setting is strong enough, it alone can establish situation-specific characteristics that lead to different motivational constructs, either in the absence of a priori propensities or by overwhelming such propensities. Earlier studies in the context of achievement motivation and mathematics education specified various classroom instructional practices that contribute to the development of different patterns of motivation and achievement outcomes (e.g. Ames, 1992; Patrick, Anderman, Ryan, Edelin & Midgley, 2001; Stipek et al., 1998).

Achievement motivation theorists, relying on a large literature on classroom environments, proposed six sources that contribute to the classroom motivational environment represented by the acronym TARGET (Task, Authority, Recognition, Grouping, Evaluation and Time). All these sources have been examined with regard to teachers' specific practices. Several studies in classroom environments have shown that teachers' different practices in each of these sources ended in different levels of students' motivation in the classroom. In the mathematics education domain, Stipek et al.'s (1998) study, referring to instructional practices and their effect on learning and motivation, found that the affective climate was a powerful predictor of students' motivation and mastery orientation.

The various and vital consequences of students' FF in the mathematics classroom together with the absence of studies investigating the sources of their FF obliged us to identify the origins of this construct by investigating students internal and contextual characteristics. In this respect, the purpose of this study was:

- To test the validity of the measures of the factors, fear of failure and self-efficacy, in a specific social context.
- To identify students' characteristics (mathematical performance, self-efficacy beliefs, mothers' and fathers' educational background) which affect the level of their fear of failure.
- To identify teachers' practices that trigger students' fear of failure, using an observational protocol that includes convergent variables referring to instructional practices in the classroom.

## **METHOD**

The participants were 321 sixth grade students (136 males and 185 females) from 15 intact classes and their 15 teachers. All student-participants completed a questionnaire reflecting among other motivational constructs (achievement goals, interest), fear of failure and self-efficacy beliefs.

We further collected information about the students' parents, including their educational background. We measured students' mathematics performance through a specially constructed mathematics test on fractions. Most of the tasks comprising the test were adapted from published research and specifically concerned students'

understanding of fraction as part of a whole, as measurement, equivalent fractions, fraction comparison and addition of fractions with common and non common denominators (Lamon, 1999).

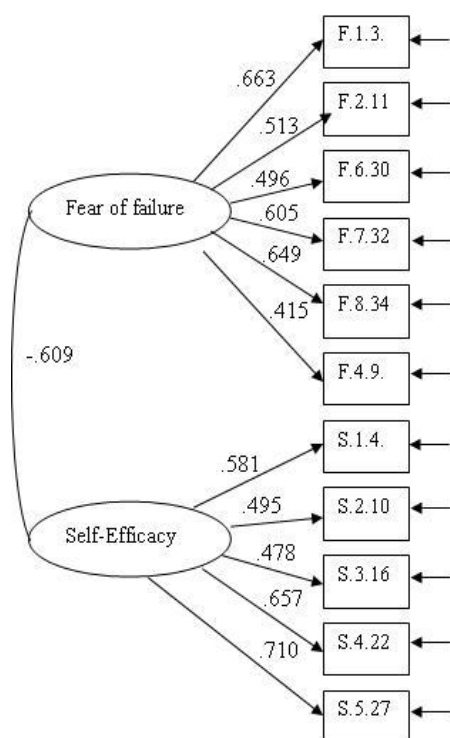
Herman's Fear of Failure scale (Elliot & Church, 1997) was used to measure students' FF; Herman's 27-item Fear of Failure scale was revised by Elliot and Church (1997) who tested its reliability (Cronbach's  $\alpha=0.88$ ) and construct validity. A specimen item from the nine items we used in the study was "I often avoid a task because I am afraid that I will make mistakes". Students' self-efficacy beliefs were measured using the five scale measure of the Patterns of Adaptive Learning Scales (PALS) (Midgley et al., 2000). The items measured students' perception of their competence to do their work in the classroom. A specimen item was "I'm certain I can master the skills taught in mathematics this year" which the researchers reported had a reliability of  $\alpha=0.78$ . We adjusted the items in the scale to measure students' perception of competence in the mathematics classroom.

For the analysis of teachers' instructional practices we developed a protocol for the observation of teachers' practices in mathematics in the 15 classes. The observational protocol was based on the convergence between instructional practices described by Achievement Goal Theory and the Mathematics education reform literature. Specifically, we developed an inventory of codes around six constructs, based on previous literature (Ames, 1992; Patrick et al., 2001; Stipek et al., 1998), which were found to influence students' motivation and achievement. These six constructs were: task, instructional aides, practices concerning the task, affective sensitivity, messages to students, and recognition.

The construct *task* included algorithms, problem solving, teaching self-regulation strategies, open-ended questions, closed questions, constructing the new concept on an acquired one, generalizing and conjecturing. We also checked whether teachers made use of instructional aides during their lesson. *Practices concerning the task* included the teacher giving direct instructions to students, asking for justification, asking multiple ways regarding the solution of problems, pressing for understanding by asking questions, dealing with students' misconceptions, or seeking only for the correct response, helping students and rewording the question posed. Behaviour referred to *affective sensitivity* included teachers' possible anger, using sarcasm, being sensible to students, having high expectations of the students, teachers' interest towards mathematics or teachers' fear for mathematics. *Messages to students* included learning as students' active engagement, reference to the interest and value of the mathematics tasks, students' mistakes being part of the learning process or being forbidden, and learning as receiving information and following directions. Finally, *recognition* referred to the reward for students' achievement, effort, behavior and the use of external rewards by the teachers. During two 40 minute classroom observations for each teacher, we were able to identify the occurrence of each code in each structure.

## RESULTS

Since we have conducted an exploratory factor analysis involving 302 students concerning the same scales (Pantziara & Philippou, 2006), in the present study we have proceeded with Confirmatory Factor Analysis using structural equation modelling and the program EQS (Hu & Bentler, 1999) in order to identify the factors corresponding to fear of failure and self-efficacy beliefs. To this end, we followed a process including the reduction of raw scores to a limited number of representative scores, an approach suggested by proponents of Structural Equation Modelling (Hu & Bentler, 1999). Particularly, regarding FF, some items were deleted because their loadings on the factor were very low and some items were grouped together because they had high correlation with each other. The reliability for the factor FF was Cronbach's  $\alpha=.726$  and for the factor Self-efficacy was Cronbach's  $\alpha=.710$ . The correlation between the factors was  $-.609$ .



**Fig 1: Two factor measurement model**

Figure 1 shows ( $\chi^2=68.908$ ,  $df=43$ ,  $p<0.000$ ; CFI=0.961 and RMSEA=0.044). In order to investigate the second aim of the study, regression analysis was performed to determine which of the antecedent variables (self-efficacy beliefs, students' mathematics performance) predicted students' FF. Multiple regression analysis revealed that self-efficacy beliefs and students' performance were negative predictors  $\beta=4.346$ ,  $F(36.413)$ ,  $p<0.001$  of students' FF. Specifically, the regression equation was:

Students' FF =  $4.346 - 2.18 \times \text{self-efficacy} - 3.52 \times \text{mathematics performance}$ .



One-way ANOVA (GLM1) indicated a statistical significant difference between the FF of students whose fathers had different educational background,  $F(5, 290) = 2.569$ ,  $p < 0.05$ . Hochberg's GT2 post-hoc test revealed that students whose father had low educational background (gymnasium) reported a higher fear of failure ( $M=2.584$ ) than students whose father had higher educational background (postgraduate studies), ( $M=1.996$ ). No statistical significant difference was found between students' FF whose mothers had different educational background.

Investigating the third aim of the study, we used one-way ANOVA (GLM1) to identify possible significant differences between students' FF placed in different classes. The analysis showed significant differences between classrooms in students' FF,  $F(14, 300) = 2.545$ ,  $p < 0.05$ . Gabriel post-hoc test identified that students in class 11 and in class 13 had non-significant means. Specifically students in class 11 declared the highest FF in mathematics (Mean=2.93) and students in class 13 the lowest FF in mathematics (Mean=2.06). Worth noticing is that students in class 11 performed better in mathematics (Mean=10.20) than students in class 13 (Mean=9.11).

#### *Analysis of the teachers' observations*

To assess teachers' practices we calculated the mean score of each code for the two observations using the SPSS and creating a matrix display of all the frequencies of the coded data from each classroom. Each cell of data corresponded to a coding structure. Being aware that FF constitutes a complicated construct, a first analysis of the observational data involved isolating the two classes at the highest and lowest extremes of specified motivational construct and comparing the means of each code in the six structures to identify commonalities and differences in teacher behaviours and instructional practices in the two classes. This approach is similar to the one used by Patrick et al. (2001).

T11 (the teacher in class 11 where the highest FF appeared) had 15 years of experience, a strong background in mathematics and a master's degree in mathematics education. T13 (the teacher in class 13, where the lowest FF appeared) had 29 years of experience, and lower qualifications in mathematics. As far as it concerns the *task*, T11 used more problem solving activities than T13, while T13 used more routine activities than T11. Teacher 11 used less open-ended questions and more closed questions than T13. During the observations teacher T11 taught problem solving strategies, while T13 did not. T11 did not connect new mathematical knowledge with the students' existing knowledge and he/she did not make connections between the mathematical ideas while T13 did so. T11 did not give students the opportunity to generalize or conjecture while T13 did so. T11 made use of visual aids in the mathematics lesson, while T13 did not. Relating to *practices concerning the task*, T13 was observed giving direct instructions to students while T11 did not. T11 asked the students to provide reason for their choices and solution plans more than T13 while T13 was observed asking for multiple solutions for a

problem and pressed students for understanding more than T11. Dealing with misconceptions was observed in T11's class while in many cases T13 accepted the correct responses from students without any explanation. While both teachers gave individual help to their students, T11 gave more individual help than T13. As far as practices of *affective sensitivity* were concerned T11 was observed in some instances showing anger and using sarcasm while T13 did not. None of the other affective variables were observed. With respect to *messages* sent to students, both teachers informed students that erroneous answers were part of the lesson with T11 doing so with greater frequency than T13. Concerning *recognition*, both teachers rewarded students for their mathematical performance, while T11 also verbally rewarded the students' for their efforts. T11 gave rewards publicly to the students.

## CONCLUSION

Regarding the first aim of the study, data revealed that factors referred to the two motivational constructs (FF and self-efficacy beliefs) were confirmed as present in a different social context. The data referred to the second aim of the study revealed that a source of FF could be traced to students' family context (Macgregor & Elliot, 2005). The explanatory factors for these findings were the fathers' job, social status and the consequences of these factors (e.g. earnings) or the help students receive whilst engaged in their work at home. In addition, students' mathematics performance and their self-efficacy beliefs were found to predict negatively their FF. Numerous studies (e.g. Ho et al., 2000; Pantziara & Philippou, 2006; 2007) revealed that students' mathematics performance negatively predicts students' FF. Naturally students with low mathematics performance may more often experience feelings of shame and incompetence in the mathematics classroom, both feelings were found to be implicated in students' FF (e.g. McGregor & Elliot, 2005). Moreover, other studies (Pantziara & Philippou, 2007) revealed the negative relation between students' self-efficacy beliefs and their FF. Students' positive self-efficacy beliefs concerning their ability to manage academic task demands, influence them emotionally and thus decrease their stress, their anxiety and depression. In addition the positive relation between students' self-efficacy beliefs and their mathematics performance may contribute to a limitation of their FF. Instructional and social influences are found to be the most influential source of students' self-efficacy (Zimmerman, 2000). Therefore this study indicates that educators who work in such a way that students' self-efficacy beliefs are raised will help the students confront unpleasant situations.

Regarding the third aim of the study, we uncovered that students' FF in mathematics is sensitive to the classroom context, finding statistically significant differences in students' FF from different classes. We found that teachers' practices contributed to students' different motivational constructs (e.g. FF) in line with other studies (Patrick et al., 2001; Stipek et al., 1998). Here we discuss these practices in parallel with the findings of other studies, being aware that the identification of teachers'

practices is not easy to attempt. Students in the class with the highest FF had higher average mathematics performance than students in the class with the lowest FF. This complexity may be due to the close interaction between teachers' practices and students' other motivational and cognitive factors (Opt' Eydne et al., 2006).

Our results indicate that affective *sensitivity* is the most predictive structure for students' FF, which is in line with the results of Stipek et al. (1998). Stipek et al. (1998) revealed that teachers' positive affect was the most predictive variable in students' positive emotions. Our conclusion was based on observations showing that in a class with high FF, the teacher had knowledge of and used practices that have been shown to raise students' positive affect (use of problem solving activities, giving help, errors part of the lesson). However even the traces of anger and sarcasm in the classroom might have proved to be stronger than these positive practices in affecting students' FF. Anger and sarcasm may bring shame and embarrassment to students, which are both found to be origins of students' FF. Similarly, Patrick et al. (2001) described classes characterized by negative motivation in which teachers insulted students. Another practice found to affect students negatively belongs in the category *recognition* and refers to making rewards to students public. Rewarding publicly may raise competition between students affecting them negatively (Ames, 1992). Practices from the category *task*, such as the use of open – ended questions, making conjectures, and connecting new knowledge to existing knowledge were observed in the class with low FF. These practices were found by researchers (Patrick et al., 2001; Stipek et al., 1998) to raise students' motivation and our results indicate also may be considered as diminishing students' FF. From category *procedures concerning the task*, discussing multiple solutions of a problem and pressing students for understanding were also observed in class with low FF. Lastly giving individual help to students was also a practice observed in the low FF class.

Fear of failure is found to be a multiphase and complex structure with various consequences in students' performance and behaviour. More research is needed to illuminate origins of students' FF especially in the educational setting. Such information will guide teachers so as to avoid certain practices that increase students' FF and therefore use practices that raise students' motivation to learn mathematics.

## REFERENCES

- Ames, C. (1992). Classrooms: Goals, structures, and student motivation. *Journal of Educational Psychology*, 84(3), 261–271.
- Conroy, D., Elliot, A. (2004). *Anxiety, Stress and Coping*, 17(3), 271-285.
- Elliot, A., Church, M. (1997). A hierarchical model of approach and avoidance achievement motivation. *Journal of Personality and Social Psychology*, 72, 218-232.
- Friedel, J., Cortina, K., Turner, J., Midgley, C. (2007). Achievement goals, efficacy beliefs and coping strategies in mathematics: The roles of perceived parent and teacher goals emphases. *Contemporary Educational Psychology* 32, 434-458.

## Working Group 8

- Ho, H., Senturk, D., Lam, A., Zimmer, J., Hong, S., Okamoto, Y., Wang, C. (2000). The affective and cognitive dimensions of math anxiety: A cross-national study. *Journal for Research in Mathematics Education*, 31(3), 362-380.
- Hu, L., Bentler, P.M. (1999). Cut off criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. Hu, L., Bentler, P.M. (1999).
- Lamon, S. (1999). *Teaching fractions and ratios for understanding. Essential content knowledge and instructional strategies for teachers*. London: Lawrence Erlbaum Associates.
- Mcgregor, H., Elliot, A. (2005). The Shame of Failure: Examining the Link Between Fear of Failure and Shame. *Personality and Social Psychology Bulletin*, 31(2), 218-231.
- Midgley, C., Maehr, M. L., Hruda, L. Z., Anderman, E., Anderman, L., Freeman, Urdan, T. (2000). *Manual for the Patterns of Adaptive Learning Scales*. Retrieved from <http://www.umich.edu/~pals/manuals.html>.
- Op't Eynde, P., De Corte, E., Verschaffel., L. (2006). Accepting emotional Complexity. A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Education Studies in Mathematics*, 63, 193-207.
- Pantziara, M., Philippou, G. (2006). Measuring and relating primary students' motives, goals and performance in mathematics. In J. Novotna, H. Moraonva, M. Kratka, N. Stehlikova (Eds.), *Proceedings of the PME30: Vol. 4* (pp. 321-328). Prague, Czech Republic: PME.
- Pantziara, M., Philippou, G. (2007). Students' Motivation and Achievement and Teachers' Practices in the Classroom. In J. Woo, H., Lew, K. Park, D. Seo. (Eds.), *Proceedings of the PME31: Vol. 4* (pp. 57-64). Seoul, Korea: PME.
- Patrick, H., Anderman, L., Ryan, A., Edelin K., Midgley, C. (2001). Teachers' communication of goal orientations in four fifth-grade classrooms. *Elementary School Journal* 101(1), 35-58.
- Stipek, D., Salmon, J., Givvin, K., Kazemi, E., Saxe G., MacGyvers, V. (1998). The value (and convergence) of practices suggested by motivation research and promoted by mathematics education reformers. *Journal of Research in Mathematics Education*, 29, 465-488.
- Zimmerman, B. (2000). Self-Efficacy. An essential motive to learn. *Contemporary Educational Psychology*, 25, 82-91.
- Zusho, A., Pintrich, P., Cortina, K. (2005). Motives, goals and adaptive patterns of performance in Asian American and Anglo American students. *Learning and Individual differences*, 15 (2), 141-158.

# **INTRODUCTION TO THE PAPERS OF WG 9: LANGUAGE AND MATHEMATICS**

Maria Luiza Cestari, University of Agder, Kristiansand, Norway

Alain Mercier, Université de Provence, Marseille, France

Pier Luigi Ferrari, Università del Piemonte Orientale, Italia

Konstantinos Tatsis, University of Western Macedonia, Greece

## **INTRODUCTION**

*Maria Luiza Cestari*

The presentations and discussions based on written papers submitted to Working Group 9 focusing on language and mathematics reveal a multiplicity of theoretical backgrounds and, at the same time, the use of methodologies from different areas of research, particularly from social sciences. The mathematical topics and the way in which they have been analysed demonstrate that this area of research in Mathematics Education continues to be in constant expansion. This fact reflects the intersection of disciplines from different areas of knowledge which contribute to illuminating the complexities observed in mathematical lessons.

In connection with the structure and group dynamics, the sections were divided into time slots of 40-45 minutes for each presentation and had as a basic schedule the following steps: 1. Authors presented the main ideas of their papers including research questions, theoretical framework, methods and main findings. Some have also showed a video or introduced transcriptions of the empirical material. 2. Clarifications requested by the audience; 3. First discussant commented on the paper and raised questions or issues for discussion; 4. Small group (2- 3 participants) discussed together in order to point out issues which caught their attention; 5. A whole-group discussion. 6. Second discussant made central remarks and gave feedback to the author. On many occasions, at this point, the author responded to the comments, and included new insights concerning the understanding of his work.

For the presentation of posters we organized a special conference section: the authors displayed their posters before the section began, participants had the opportunity to read them and during the section a short time slot was allocated to every author. Participants were invited to comment and discuss special issues and contribute new ideas.

In relation to the topics approached by the authors, as has been observed and pointed out in earlier Working Groups, the variety of mathematical concepts and ways to treat them in discourses has been remarkable. From a total of 10 papers, four have focused on mathematical concepts, four on ways in which mathematics has been treated in classrooms conversations and two related to teacher's identity and

bilingual students. It is important to mention that in the majority of studies, empirical material – in general from classrooms – is included. This fact has facilitated the group discussions particularly with respect to the focus on the object analysed and implications for theoretical development.

In the first group of papers the mathematical concept has been the focus, for example the one by Rønning analysing 8-year-old pupils engaged in activities related to different aspects of symmetry: reflection (design of symmetric pictures in various ways) and rotation (building three dimensional objects and being encouraged to talk about them using mathematical words). Coppola, Mollo and Pacelli also analyse episodes involving children from primary school. They show convincingly how “*actions* of shortening the language sentences and checking the equivalence between two sequences in two activity systems, allowed the children to *construct the concept of equivalence*”.

Morgan and Alshwaikh in a teacher experiment with students from a secondary school in year 8 (ages from 12 to 13 years) that includes the class teacher, the researchers and a student teacher are concerned with “the evolution and use of a system of gestures for communication about movement in three dimensional space”. They make distinctions between movements related to 2D and 3D space showing the contrast between everyday vs. specialised resources, using the concept of rotation.

Another study where the mathematical concept has been the focus is the one by Bardelle. From a pragmatic perspective, particularly the functional linguistics and a semiotic approach, she analyses how the concept of infinite and unbounded sets is used among seven second-year undergraduate mathematics students. The author pointed out that they do not recognize the importance of mathematical definitions. Her data showed also that different systems of representation evoke different meanings related to “bounded/unbounded” and “finite/infinite” sets.

Among the second group of papers, four of them focus on the social interaction in the classroom. The first one by Brandt and Höck analyse transcribed interactions by dyads using a framework provided by conversation analysis. Two collaborative processes are identified and compared illuminating chances and as well impediments in problem solving situations. They pointed out the possibilities of a collective cognitive convergence being productive and also provoking limitations to the moment when the problem is not anymore perceived as the focus of conversations. The second one by Gellert presents an exemplary episode between a teacher and two students discussing the decimal place value system in a class of fourth graders at a primary school. The concept of contention is used to locate a point which presents different possibilities for interpretation. One of the concluding remarks is related to the fact that “the engagement in the point of contention and the arguments seem to be difficult and unfamiliar” especially at these pupils’ age. The third study by Ingram, Briggs and Johnston-Wilder focuses on turn-taking in secondary mathematics lessons including pupils aged 12-13 years, also with a conversation analysis

approach. In all lessons it has been observed that turn-taking is controlled by the teacher. Three situations have been identified from the turn-taking analysis, all of them involving self-selection of a pupil as the next speaker. Ingram and colleagues locate the first when pupils ask their own questions; the second when they initiate or perform repairs related to the teacher's previous turn or a peer's turn and the third when they respond to indirect teacher questions. At the end of this study, the author elaborates on an open question related to the effectiveness of this kind of interaction compared to the more "traditional" interactions between teacher and pupils. We could say that this question has been open for a quite long time in this area of research. And the fourth study by Planas and Morera is related to two theoretical constructs: revoicing ("re-telling") and collective mathematical argumentation. Data from secondary mathematical classrooms are analysed by a group of mathematics teachers and researchers. Some of the data shows how students' use of revoicing can reinforce mutual understanding. The authors exemplify this phenomenon when "a student partially explains an argumentation, and another in the group uses revoicing to emphasize particular aspects of that argumentation..." This additional information is recognized as a facilitator for a more adequate mathematical understanding.

The third group of studies includes the works by Tatsis and Ní Ríordáin. Tatsis presents a study where language is conceived as a tool for shaping teachers' identity. Data consists of a collection of narratives registered during an in-service training course. Typical examples show how the notion of identity as narrative can be more operational than the beliefs and attitudes approach to teaching. Ní Ríordáin introduces a working model for improving mathematics teaching and learning for bilingual students. This model has been elaborated basically from data generated in the Irish context. The focus is on the transition from Irish language Gaeilge-medium mathematics education to English-medium mathematics education in Ireland. The description of the working model includes Mathematics Understanding, Pedagogy and Culture and the so-called quartet: Mathematics Understanding, Bilingual Factors, Conceptions of Mathematics and Language Use.

From the brief presentation of papers above we can identify three main research focuses: 1. the mathematics content and how it is conveyed in communication; 2. the interaction in the classroom; and 3. new topics emerging from recent theoretical and methodological developments, for example, mathematics teachers' identity. Even though the group of participants in this Working Group 9 has been changing from conference to conference, we still have a stable commitment to advance knowledge in this research area of mathematics education.

# INFINITE AND UNBOUNDED SETS: A PRAGMATIC PERSPECTIVE

Cristina Bardelle

University of Eastern Piedmont 'A. Avogadro', Alessandria, Italy

*This paper explores some of the ambiguities inherent in the notions of finite/infinite sets and bounded/unbounded sets for what concern Euclidean spaces. The study, carried out with seven mathematics students, shows that wrong attitude towards mathematical language is a major source of misconceptions about cardinality and boundedness. In particular, everyday use of mathematical terms combined with a lack of coordination of different representations proves to be a hindrance in the conceptualization of basic properties of subsets of Euclidean spaces.*

## INTRODUCTION

In the last years research in mathematics education has pointed out the central role of language in the learning of mathematics and has addressed this topic from a variety of perspectives.

The complexity of 'mathematical language' does not depend on the symbolic component only, but on all of the semiotic representations used in mathematics, such as symbolic notations, diagrams, figures, and verbal texts as well.

From the point of view of semiotics many studies stressed the importance of both treatment within the same semiotic system and conversion between different representations of mathematical objects as well (Janvier et al., 1987; Duval, 1993).

From a pragmatic perspective, and in particular within the functional linguistic framework some studies (e.g. Pimm, 1987; Morgan, 1998; Ferrari, 2004) provided ideas to interpret some language-related difficulties in the learning of mathematics. For example Ferrari (2004, p. 384) claimed "that students' competence in ordinary language and in the specific languages used in mathematics are other sources of troubles". In particular some studies pointed out difficulties arising from the overlapping of everyday language and mathematical language (see e.g. Tall, 1977; Cornu, 1981; Mason & Pimm, 1984; Ferrari, 2004; Kim, Sfard & Ferrini-Mundy, 2005; Bardelle, 2010). But not enough attention has been paid to the fact that most of mathematical terms are borrowed from everyday language and used with meanings different from everyday-life usage. The aim of this paper is to focus on this topic. In particular here we deal with the concept of boundedness and infiniteness of subsets of Euclidean spaces. The study of metric spaces and their properties is basic in the curricula of mathematics undergraduate students and the concept of boundedness is fundamental for the learning of other topological properties such as compactness.



## Research questions

This research is aimed at investigating some typical students' behaviours concerning the recognition of bounded or unbounded subsets of  $Y^n$ , endowed with the Euclidean distance. For example, some students seemingly justify that 'a set is unbounded because it has infinite elements'. In particular, the study explores the interplay of pragmatic aspects with the flexibility in switching between different representations of bounded/unbounded sets and finite/infinite sets. In particular the research questions are:

- How the everyday meanings of the words 'finite/infinite' and 'bounded/unbounded' influence students' behaviour in the resolution of problems involving such words?
- What is the role of the formal definition of 'bounded/unbounded set' and 'finite/infinite set'?
- What are the factors (problem formulation, system of representation of the set, context, etc.) that might influence the way words like 'bounded/unbounded set' and 'finite/infinite set' are interpreted and used by undergraduate students?

## THEORETICAL FRAMEWORK

### The functional linguistics approach

Functional linguistics is a theoretical perspective (in the frame of pragmatics) that studies language in relation to its functions rather than to its form. According to this framework, the main functions of language are the *ideational*, *interpersonal* and *textual* ones (Halliday, 1985). In particular, the interpersonal function involves social and cultural aspects of language and language use, including the processes of dynamic and negotiated meaning generation through interaction.

Here I adopt the pragmatic notion of *register*, which has been thoroughly discussed from a functional linguistics perspective by Leckie-Tarry (1995). A register denotes a linguistic variety based on use that is a conventional pattern or configuration of language that corresponds to a variety of situations or contexts. Ferrari (2004), following Leckie-Tarry (1995), distinguishes between 'colloquial registers' and 'literate registers'. The former refer to the linguistic resources adopted in spoken communications prevalently but also in informal written communication such as sms messages, e-mails, etc. whereas the latter refer to written-for-others texts mainly such as books but also to formal spoken communication such as in academic lectures. Ferrari (2004, p.387) argued "that the registers customarily adopted in advanced mathematics share a number of features with literate registers and may be regarded as extreme forms of them" and provided evidence to corroborate this claim.

## Semiotic approach

As mentioned before, registers used in mathematics are highly literate; in fact, the development of mathematics but also its teaching and learning requires the introduction of a variety of representations such as symbolic notations, verbal texts, geometrical figures, diagrams and so on. According to Duval (1993) *there cannot be no  sis without s  miosis*, where *s  miosis* denotes the production of a semiotic representation and *no  sis* denotes the conceptual learning of an object. Moreover Duval states that the cognitive functioning of human thought needs multiple semiotic systems and he applied his ideas to the learning of mathematics where very different representations occur. Duval makes a distinction between *treatment of a representation*, which is a transformation (manipulation) within the same semiotic system, and *conversion* (translation) between different semiotic systems<sup>31</sup>. For example, computing the sum of two fractions is an example of treatment, whereas translating a fraction into an equivalent decimal expansion is an example of conversion.

## THE EXPERIMENT

The research involved seven second year undergraduate mathematics students at the University of Eastern Piedmont in Italy. The students were attending a Geometry course focused on point set topology and on introductory algebraic topology. Students had already dealt with the concept of finite/infinite set and of bounded/unbounded set in  $\mathbb{Y}^n$ , endowed with the Euclidean metric, in their first year classes. In this second year course the definition of finite/infinite set is supposed as well known, whereas the definition of boundedness of a set is generalized to an arbitrary metric space, paying particular attention to  $\mathbb{Y}^n$  with Euclidean metric. This study focused on Euclidean spaces only. The data have been collected from individual interviews. The interviews were semi-structured and based on two different kinds of questions.

### Q1-questions

The first kind of questions were aimed to recognize if a given set in  $\mathbb{Y}^n$  with Euclidean metric is unbounded and infinite. The sets were given by their graphical representation or by symbolic notations and in this case, whenever it was useful and possible, students were encouraged to represent it in the Cartesian coordinate system. Some examples of given sets are  $\{1/n, n \in \mathbb{N} \setminus \{0\}\}$ ,  $\{(x,y) \in \mathbb{Y}^2 : x^2+y^2=1\}$ ,

$\{(x,y) \in \mathbb{Y}^2 : y=x\}$ ,  $\{(x,y) \in \mathbb{Y}^2 : xy=1\}$ ,  $\{(x,y,z) \in \mathbb{Y}^3 : x^2+y^2=1, z=3\}$ ,  $\{(x,y,z) \in \mathbb{Y}^3 : x^2+y^2=3\}$ ,  $\{(x,y) \in \mathbb{Y}^2 : y=\sin x\}$ , etc.

---

<sup>31</sup> Duval used the term ‘register’ referring to a semiotic system; here ‘register’ is used with its pragmatic meaning only.

## Q2-questions

The second kind of questions were aimed to understand the meaning of ‘finite/infinite set’ and ‘bounded/unbounded set’ used to answer to Q1-questions. Students were asked to say what is an infinite and unbounded set or to produce some examples.

The interviews were carried out in order to help students to achieve a proper understanding of the subjects as well as to identify their behaviours and the *concept images* (Tall & Vinner, 1981) they were adopting. The term ‘concept image’ is used “to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes” (Tall & Vinner, 1981, p. 152). The generation of the concept image of an individual can be influenced also by the *concept definition* that is “a form of words used to specify that concept” (Tall & Vinner, 1981, p. 152). The concept definition can be different from the *formal* concept definition, i.e. a definition accepted by the mathematical community at large. Moreover Tall and Vinner introduced the *evoked concept image* as the portion of the concept image which is activated at a particular time. The concept image may have conflicting aspects that may be evoked at different times. When such conflicting aspects are evoked *simultaneously* they cause a cognitive conflict. In particular, Q2-questions investigated the concept definition and Q1-questions were aimed to study the influence of the symbolic and graphical representation on the concept image. The interviews were conducted, when appropriate, in order to cause cognitive conflicts helping students to understand the concepts involved.

## SELECTED FINDINGS

Just two students (students A, E) out of seven showed no kind of problems dealing with the topic of this study. The remaining students presented difficulties that seem, as we shall see shortly, to be due to their lack of skills for the treatment of symbolic expressions, lack of coordination of representations of subsets of  $\mathbb{Y}^n$  ( $n=1,2,3$ ) and also to the improper adoption of colloquial registers.

In what follows only the responses about the sets  $\{(x,y) \in \mathbb{Y}^2 : x^2+y^2=1\}$  and  $\{1/n, n \in \mathbb{N} \setminus \{0\}\}$  are presented. This choice is due to the fact that the students were familiar with them and their conversion into the related graphical representation was within their reach and did not require a previous treatment of the symbolic representation, which would have been beyond the purpose of this study. Some results concerning more examples of sets, which were provided ad hoc to some of the students in order to better recognize their concept image, are also presented.

Table 1 summarizes the students’ answers concerning the boundedness of the set  $\{(x,y) \in \mathbb{Y}^2 : x^2+y^2=1\}$ .

### Working Group 9

Student	$\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$	Explanation
A	Bounded	" $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} \subseteq [-1,1] \times [-1,1]$ "
B	Unbounded	" $x^2+y^2=1$ has infinite solutions, therefore the set is unbounded"
C	Bounded	"It is a circumference, it is bounded"
D	Bounded	"It is a circumference and hence it is bounded"
E	Bounded	" $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\} \subseteq [-1,1] \times [-1,1]$ "
F	Bounded	"It is a circumference, it is bounded"
G	Bounded	"It is a circumference, it is bounded"

**Table 1: Answers to the question on the boundedness of  $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$**

The first thing to say is that all students, except B, recognized at a first glance that  $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$  is a circumference in the plane and were able to represent it graphically. Students A and E showed a literate use of mathematical language (literate register) and preferred the symbolic representation. The remaining students, except B, could not explain why the set is bounded. Therefore such students, among the others, were encouraged to say when a set is bounded/unbounded.

The set  $\{1/n, n \in \mathbb{N} \setminus \{0\}\}$  revealed itself to be more troublesome for students. Table 2 summarizes the answers about its boundedness.

Student	$\{1/n, n \in \mathbb{N} \setminus \{0\}\}$	Explanation
A	Bounded	" $\{1/n, n \in \mathbb{N} \setminus \{0\}\} \subseteq [0,1]$ "
B	Unbounded	"It is unbounded because $n$ varies in $\infty$ "
C	Unbounded	"It is unbounded on the right...the naturals are infinite and hence there is no end for this set"
D	Unbounded	"It is unbounded because there are infinite $n$ "
E	bounded	"It is contained in $[0,1]$ "
F	bounded	"The set is bounded from above and below"
G	Unbounded	"It is not bounded because you get infinite fractions"

**Table 2: Answers to the question on the boundedness of  $\{1/n, n \in \mathbb{N} \setminus \{0\}\}$**

Only three students (A, E and F) answered correctly. Since the explanation of student F, even if correct, seemed to be not so usual as a topologic argument, some investigations were conducted in his interview (see below). The remaining four students, among the others, were encouraged also after this question to give the definition of bounded/unbounded set and of finite/infinite set. In this case some students evoked a concept definition which was different from the one evoked after

### Working Group 9

the first question. Table 3 summarized the responses of students about their concept definition of the boundedness of a set.

Student	What is a bounded/unbounded set in $\Upsilon^n$ ?	
A	“A set $X$ is bounded if there exists a ball $B$ such that $X \subseteq B$ ”	
B	She was not able to give a definition but she gave the closed interval $[0,1]$ as an example of bounded set in $\Upsilon$ and the interval $]0,1]$ as an example of unbounded set in $\Upsilon$ . She could not come up with an example in $\Upsilon^2$	
C	“A set $X$ is bounded if there exists a ball $B$ such that $X \subseteq B$ ”	
D	She answered “An unbounded set is an infinite set” when she referred to sets that she could not visualize	She answered “A bounded set is a set placed in a limited space” when she referred to sets that she could visualize
E	“A set $X$ is bounded if there exists a ball $B$ such that $X \subseteq B$ ”	
F	“A set $X$ is bounded if there exists a ball $B$ such that $X \subseteq B$ or a product of intervals that contains it”	
G	She answered “An unbounded set is an infinite set” when she referred to sets that she could not visualize	She answered “A set is bounded when, from the drawing, it is in a narrow space” when she referred to sets that she could visualize

**Table 3: Answers to the question on the concept definition of bounded/unbounded set**

For what concern the concept of finite/infinite set all students grasped its correct meaning. All students did not give the formal definition but used a more colloquial (colloquial register) but effective argument as showed in Table 4.

Student	What is a finite/infinite set in $\Upsilon^n$ ?
all	All students answered that a finite set is a set with a finite number of points (or elements)

**Table 4: Answers to the question on the concept definition of finite/infinite set**

Data shows that students use an everyday meaning for the boundedness of a set even if their concept definition is correct and given with a literate register. Such a meaning is sufficient to give an answer to problems where the set is already drawn in the Cartesian coordinate system or students can sketch its graphical representation. In this case the definition of ‘bounded’ used in the everyday meaning in Italian language that is “something that has limits referring to space or time” clearly fits with the concept definition of students D and G. Moreover, it seems that also students C, F adopted this colloquial meaning since they are not able, at least

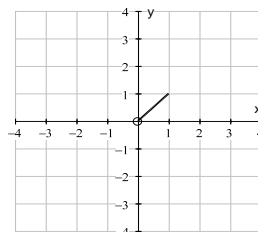
## Working Group 9

apparently, to provide an explanation to the boundedness of the circumference (Table 1) even if their concept definition coincide with the formal one (Table 3). One has to highlight that this everyday meaning is not more sufficient in order to answer to problem  $\{1/n, n \in \mathbb{N} \setminus \{0\}\}$  (Table 2). In this case all the students who gave an incorrect explanation (B, C, D, G) did not think of sketching the set, neither of working within the symbolic system to find an appropriate ball of  $\mathbb{Y}$  containing it. The symbolic aspect of the representation of this set seemed to evoke in these students the idea of infinite elements connected to the set of natural numbers, indeed a set that tends to infinity and hence with no upper bound. Here the matter is not that students did not understand the concept of ‘infinite set’. Rather, the first evoked meaning of ‘infinite’ in this context is colloquial, that is “something that never ends with reference to space and time” indeed something which is unbounded. The problem is not even that they could not represent it graphically. Indeed, the students were asked to sketch it and they did it correctly (sometimes with some help). After that they recognized that they were dealing with a bounded set. Notice that student B seemed to apply this reasoning also for the circumference since she could not sketch it (Table 1). During the interview students were asked to explain their apparently incoherent behaviour. Some examples of responses are:

- G: I used I used ‘infinite’ in order to justify that a set was unbounded because I imagined an infinite set as a set stretching to infinity.
- C: I think more to the idea of infinity in order to decide about the boundedness of a set.

The research proves also the everyday usage of the term ‘finite’ by student D, as one can see from the following transcript:

- 1 I: Is  $\{(x,y) \in \mathbb{Y} : x^2+y^2=1\}$  unbounded?
- 2 D: It is a circumference [she draws it] ... it is bounded
- 3 I: It is infinite?
- 4 D: It is closed, not infinite.
- 5 I: What does it mean that a set is finite?
- 6 D: It is limited in space.
- 7 I: Is the set [in its graphical representation]



- bounded?
- 8 D: Yes, it is
- 9 I: It is infinite/finite?
- 10 D: It is finite

### Working Group 9

- 11 I: Can you find points belonging to it?  
12 D: yes, for example  $(1/2, 1/2)$ ,  $(1/3, 1/3)$ , etcetera,  $(1/n, 1/n)$  .....no, then it's infinite!  
13 I: Why did you answer finite before?  
14 D: Because it seems finite from the drawing.

Also in this case the use of the colloquial register (colloquial definition of 'finite') prevailed on the literate one (mathematical definition of 'finite'). When she was asked to convert the set in the symbolic system she immediately could give the proper answer. This behaviour as the previous ones show that students are not aware of the role of mathematical language and of its functions.

Finally the research highlights another kind of behaviour due to a misleading interpretation of mathematical language. Student F showed no kind of problems with sets like in Table 1 and 2. Problems arose with sets like  $\{(x,y) \in Y^2 : y=\sin x\}$ ,  $\{(x,y) \in Y^2 : xy=1\}$ ,  $\{(x,y) \in Y^2 : y=\arctg x\}$ , etc. where functions are involved. In this case the recognition of well known functions (both graphical and symbolic representations) evoked a meaning of 'boundedness' related to functions rather than sets. The following transcript shows this fact:

- 1 I: Is  $\{(x,y) \in Y^2 : y=\sin x\}$  bounded or unbounded?  
2 F: Bounded because it can assume values between -1 and 1

After some explanation about his wrong answer he declared

- F: I think that the set  $\{(x,y) \in Y^2 : y=\sin x\}$  was in  $Y$ ..... if I see something simple like  $8^x = 8^y$  or  $y=\sin x$  I don't see the other particulars.

Finally, the formal concept definition of bounded/unbounded set was showed to the students (B, D, G) that did not manage to provide it and they were asked if they remembered it. Student B answered that she could not remember it and the other two that they knew it but they did not think to use it.

### DISCUSSION AND TEACHING IMPLICATIONS

This paper provides an example of a wrong use of mathematical language by some Italian mathematics undergraduate students. In particular, these students do not recognize the importance of mathematical definitions. Mathematical terms, such as 'bounded/unbounded' and 'finite/infinite', evoke different meanings at different times. This is a typical thinking habit that concerns the everyday-life language. In mathematical language, as well as scientific languages in general, terms are usually coined with one meaning only in order to avoid interpretative problems. This purpose should be shared with the students in order to prevent improper use of mathematical language. Moreover, since data showed that different systems of representation evoke different meanings of 'bounded/unbounded' and 'finite/infinite', another goal of mathematics education should be promoting flexibility in switching from one representation to another. By 'promoting flexibility'

I mean, helping students not just to learn *how* to shift from a representation to another, but also to *think* of doing so, that is to work at metacognitive level. Actually, the students involved in this experiment, when prompted, were able to switch between the symbolic representation of sets and their graphical representation. Finally, we have to add that most of the students involved have great difficulty in the manipulation of symbolic representation (or ‘treatment’ in the sense of Duval (1993)). This issue has not been investigated in this research but it might be interesting to explore causes and connections between weakness in the treatment within the symbolic system and lack of awareness of the role of language in mathematics.

In my opinion a good practice of teaching, in order to promote the learning of the concept of boundedness of sets, should include not only the formal definition, but also examples of sets satisfying the definition as well as examples of sets that do not verify it. The choice of examples has to be done in order to evoke possible cognitive conflicts, for example like those presented in this paper. Some examples related to other topics can be founded in Tall and Vinner (1981). Moreover, when possible, following the idea of Duval (1993), sets have to be presented in at least two kinds of semiotic representations. Finally, a good practice should include the assignment of exercises requiring students to make explicit their concept definitions in order to help them to focus on the proper meaning of concepts.

## REFERENCES

- Bardelle, C. (2010). Interpreting monotonicity of functions: a semiotic perspective. In M.F. Pinto, & T.F. Kawasaki (Eds.), *Proc. of the 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education: Vol. 2* (pp. 105-112). Belo Horizonte, Brazil: PME.
- Cornu, B. (1981). Apprentissage de la notion de limite: modèles spontanés et modèles propres. *Actes du Cinquième Colloque du Groupe Internationale PME* (pp. 322-326). Grenoble, France.
- Duval, R. (1993). Registres de représentations sémiotique et fonctionnement cognitif de la pensée. *Ann. de Didactique et de Sciences Cognitives*, ULP, IREM Strasbourg. 5, 37-65.
- Ferrari, P.L. (2004). Mathematical language and advanced mathematics learning. In M.J. Høines & A.B. Fuglestad (Eds.), *Proc. of the 28<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education: Vol. 2* (pp. 383-390). Bergen, Norway: PME.
- Halliday, M.A.K., & Hasan, R. (1985). *Language, context, and text: aspects of language in a social-semiotic perspective*. Oxford: Oxford University Press.
- Janvier, C. (Ed.). (1987). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, NJ: Lawrence Erlbaum.



### Working Group 9

- Kim D., Sfard A., & Ferrini-Mundy J. (2005). Students' colloquial and mathematical discourses on infinity and limit. In H.L. Chick, & J.L. Vincent (Eds.). *Proceedings of the 29<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education: Vol. 3* (pp. 201-208). Melbourne: PME.
- Leckie-Tarry, H. (1995). *Language & context. A functional linguistic theory of register*. London: Pinter Publishers.
- Mason, J., & Pimm, D. (1984). Generic examples: seeing the general in the particular, *Educational Studies in Mathematics*, 15, 277-289.
- Morgan, C. (1998). *Writing mathematically: the discourse of investigation*. London: Falmer Press.
- Pimm, D. (1987). *Speaking mathematically: communication in mathematics classrooms*. London: Routledge Kegan and Paul.
- Tall, D. (1977). *Cognitive conflict and the learning of mathematics*. Paper presented at the 1<sup>st</sup> Conf. of the Int. Group for the Psychology of Mathematics Education. Utrecht, Netherlands. Retrieved from [www.warwick.ac.uk/staff/David.Tall/pdfs/dot1977a-cog-confl-pme.pdf](http://www.warwick.ac.uk/staff/David.Tall/pdfs/dot1977a-cog-confl-pme.pdf).
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

# **MATHEMATICAL JOINT CONSTRUCTION AT ELEMENTARY GRADE - A RECONSTRUCTION OF COLLABORATIVE PROBLEM SOLVING IN DYADS**

Birgit Brandt and Gyde Höck

*Elementary dyads solving arithmetical problems have been the focus within this study. With regard to an interactional theory of learning mathematics, the exploratory focus are the students' acts and how these acts function on the emerging joint construction. Analysing the structure of individual participation brings light into joint construction mechanisms, described as different 'types' on the basis of Christine Howe's research project (Howe 2009). The micro-analytical analysis of transcribed discourses will highlight significant differences in the participation of one child interacting in two different dyads as well as in the emerging types of joint constructions.*

## **INTRODUCTION**

On the basis of social interaction as a fundamental element for individual learning, joint construction is giving a great chance of supporting each others ideas in the process of problem solving. Considering international studies aiming at the complex structure of interaction in peer group work, there are two main trends to mention. On the one hand, these studies focus on asymmetric peer group constellations, where there is one advanced group member taking the lead, to support the zone of proximal development (Vygotski 1978). On the other hand, they point out mainly symmetric dyads or groups, which are moving towards a new (mathematical) understanding in a more balanced way, by resolving socio-cognitive conflicts, emerging by different perspectives or strategies in peer interaction (e.g. Bearison, Magzamen, & Filardo 1986). Coordinating the different perspectives, the result of a joint constructed process can become more than an addition of different ideas or propositions.

## **TYPES OF CO-CONSTRUCTION IN PEER INTERACTION**

Howe (2009) has been looking into the processes of joint construction in the developed collaborative settings for 8 to 12-year-old pupils, exploring motion under different conditions. In many years of research, Howe has focused on collaborative learning processes, collecting data from various group settings, and their way of creating an idea for a reasonable solution. Throughout these years, Howe has been following the question of how the underlying processes can transform the individual knowledge of each group member. In an extensive re-analyses, she concentrates on finding certain structures in the problem solving process of groups, to gain some hints of how the learners can progress by taking part in collective problem solving processes. Howe's analyses lead to two main forms of co-constructive processes, named as Type 1 and Type 2 joint construction (p.217).

Type 1 indicates a discourse, where there are different group members involved in the process of problem solving, in a very balanced way. Everyone participates individually with their own ideas and, as a group, coordinate all contributions “into a relatively advanced whole” (p. 217). Given the fact that there are different ideas to consider, it needs a “coordination of competing perspectives” (p. 217) within the group. In the quoted study, Howe concentrates on “groups where members were different as regards ideas expressed at pre-test” (p. 223). Thus her examples are mainly considering the phenomenon of coordinating competing perspectives, although she does not restrict this Type 1 to situations where different opinions are discussed. The basic idea of this type is a relatively equal allocation of substantive contributions among the group members, which is consistent with a symmetric group constellation.

Type 2 describes a constellation where “only one idea has to be considered” (p. 218), as one group member is suggesting one idea which is accepted by the others. In this context, Howe underlines that “less advanced children progress when they work with more advanced peers” (p. 219). That does not mean that this type excludes constellations where the main idea is presented by a less advanced pupil, but “relatively advanced children are likely to be the source of appropriate ideas whose acceptance results in Type 2 joint construction” (p. 219). Thus, Type 2 can be characterized by unequal distributed substantive contributions of different members to the joint construction, which often goes along with asymmetric group constellations<sup>32</sup>

The types represented by Howe serve us as a basis for our own reconstructions, on the empirical data of elementary pupils, solving mathematical problems. During our research, we found Type 1 and Type 2 in different variations and mostly in the allocation to asymmetric and symmetric constellations, as assigned above. We will expose this allocation in two different dyadic constellations, where one child is involved in both. In addition, we encountered a phenomenon of joint construction which in its whole dimension could not be classified by the existing types.

### **THE EMPIRICAL STUDY: COLLECTIVE PROBLEM SOLVING**

As above mentioned, this paper is focused on the general idea of an interactional theory of learning mathematics and deals with everyday classroom interactions. Particularly, we analyse videotaped peer interactions, which originate from the project “Collective problem solving”<sup>33</sup> and has been based on the concept of learning by participating in collective argumentation processes (Krummheuer 2007). relying on the capability of the others leading to a result.

---

<sup>32</sup> A more or less active participation in a problem solving process is not to compare with a “freeriding-effect“, where there are members of the group not been focussed on the shared task and only

<sup>33</sup> The project was sponsored by the Zentrum für Lehrerbildung, Schul-und Unterrichtsforschung of the Goethe-Universität Frankfurt (Feb. 2009 – Jan. 2010).

## Methodological and analytical Background

In this study, collaborative learning will be discussed as a theoretical question, concerning a general understanding on how children learn. The research is design-based, which means that the object of investigation is the real-world-practice, which reverts to theoretically founded learning materials. In our study, the theoretically deduced collaborative learning settings were prepared by the involved teachers, for the application in their classrooms. Therefore, "...design-based research is concerned with using design in the service of developing broad models of how humans think, know, act, and learn." (Barab & Squire 2004, p. 5). We will not evaluate the settings, but we will explore the execution of them in the peer interaction. Given the fact that this empirical study occurs in every day classrooms, moments of collaborative learning environment can be accomplished, which render the possibility of collective problem solving.

We adopt qualitative methods for the analysis of transcribed interaction processes. The basic analysis method is a turn-by-turn reconstruction of the interaction processes that originate from ethno-methodological conversation analysis (basically pointed out in Sacks 1998). In addition, we reconstruct the participation structure by decomposing the everyday concept 'speaker' into more detailed analytical elements (cf. Krummheuer 2007). We included the production design in our analysis to trace the responsibility for the ideas emerging in the (joint) construction. Following Goffman's (1981) idea of decomposing, each utterance consists of three analytical aspects:

a) gestical or acoustical appearance, b) formulation and c) idea/motive. For instance, person A can introduce an idea to the discourse and person B can repeat the utterance more or less literally (e.g. as a kind of ratification), or can express the idea with their own words (e.g. to establish a common understanding). Taking into account the responsibility these aspects have, there are four ways of participation for the speaker of the actual utterance (+ gestical/acoustical appearance):

Role of the actual speaker	Idea	Formulation
author <sup>34</sup>	+	+
ghostee	+	-
spokesman	-	+
relayer	-	-

The last two forms (spokesman and relayer) ensure a stabilization of the negotiation of meaning, while the other two forms (author and ghostee) are challenging that negotiation, in the sense of enrichment as well as in the sense of threat to communication. Each interactive process is kept alive by balancing the two elements

---

<sup>34</sup> The denominations are adopted from Levinson (1988, p. 172).

– and the success of group work among pupils depends on the fact how the differing ideas can be associated with each other in the interactive process.

### Design of the study

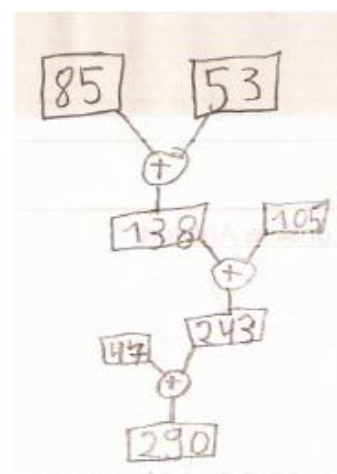
The research project took place in two elementary schools in Frankfurt a.M./Germany. At the beginning of the project, the two classes were joining the third grade, and after the summer holidays we continued with both classes during the first half of the fourth grade. The mathematic lessons we filmed were held by their mathematic teacher. Together with both teachers, we were developing four different main themes of arithmetical problem solving:

Theme	Form of cooperation	Start	Grade
written subtraction	think – pair – square	individual	3 <sup>rd</sup>
division with remainder	math-conference <sup>35</sup>		
number sequences	pair – double-pair	collaborative	4th
word problems	pair – cross-over-pair		

Over the whole period of the project, rules for cooperation were asked in classroom discussions and matched to the currently passed collaboration (cf. the “thinking together project”, e.g. Mercer & Littleton 2007). The above mentioned pairs were kept in the same combination together during the whole project (cf. Edwards 2007). It was interesting to follow the development of how two partners, who had to discuss considerably about the organisational way of co-operating, became dyads who were then able to better focus on the task itself, given the fact that there was less distraction from collaborating within the process of joint construction.

In this article we are concentrating on two transcripts of the last set of lessons, where two pupils are constantly working together on word problems. The first three lessons of this last theme were based on long-term constellations of dyads, trying to discover the arithmetical structure of a word problem and designing a diagram<sup>36</sup> of their strategy, as the following example is representing:

*Your class teacher has asked you to bring some lego bricks to school, because you have to build something with it. All the children of your class are putting their lego bricks on the table. On table one there are 85 bricks, on table two there are 53, on table three there are 105 and on the last table there are 47 bricks displayed.*



<sup>35</sup> In a small group of four pupils, each has to present their individual solution they worked out beforehand

<sup>36</sup> This diagram is called “Rechenbaum” in German.

In the last lesson of this subject, the dyads were split and arranged into new pairs, by changing the partners (cross-over method). The main focus of our analysis is now on a dialog of a long-term dyad in the last setting: *Josefine and Janina and the remaining money*. We will contrast their dialog with an excerpt of the subsequent cross-over phase: *Josefine and Belen and their cookie-task*. By contrasting these two dyads, we encountered significant differences between the work flow of a long-term dyad and the interactional approach of a complete new collaboration. Following Howe, the joint constructions of the partnership between Josefine and Belen could be clearly classified as Type 2, in an asymmetric constellation (whereas Josefine is the ‘better mathematician’). The collaboration between Josefine and Janina, instead, does show some interesting characteristics which could not be found clearly classified within the two main types described by Howe.

### **Collaboration Process A: Josefine and Janina and the remaining money**

Josefine and Janina are two female pupils which have been observed by the teacher as children with very similar performances in mathematics. Generally, both are able to resolve mathematical problems in the normal lessons, without further support. Their grading is usually quite high and they are seen as more competent in calculating and solving problems than most of the other children in their class. In the beginning phase of their team work they still had to discuss a lot about their way of organising the work flow. But during the lessons, the understanding towards each other was increasingly growing, to a point where many elements were resolved in a non- verbal way, by taking for granted that the other one understands and thinks similarly. For somebody supervising the two pupils, one got the impression of a dyad which collaborates on a very harmonic and equal basis, being productive and target-oriented, as you can see by the following extract of transcript:

86 M1 who is calculating/  
 87 Jo both  
 88 <<sup>37</sup> Ja [*pushes Josefine*] thirty-nine divided by >three  
 89 <Jo >three yes

One girl from another group, next to them, is asking who is responsible in their dyad for calculating the result. There are actually quite a number of teams, where the group members are dividing the tasks between themselves, such to have a clear responsibility of who is writing, reading, calculating or supervising the time. In this case, the spontaneous answer is “both”, and it goes to show how serious the participants take on the instructions from the teacher of really working together. This a typical sequence which characterizes their way of collaborating, and it will be highlighted by the following transcript.

60 Jo well once again\ *reading the word problem*. You have been saving your

---

<sup>37</sup> < signals simultaneously voiced utterances. In the utterances the overlapping is marked with >.

### Working Group 9

- 61 pocket money for quite a while to be able to buy some presents.  
62 Altogether you own 39 Euros. You want to keep one third. With the  
63 leftover (Rest)<sup>38</sup> you spent 8 Euros on your parents.  
65 Ja well then we have to calculate  
66 Jo well this divided by three somehow/ but then there is  
67 a remainder\ (Rest)  
68 Ja yes remainder (Rest) is (incomprehensible) no with the leftover (Rest)  
69 you spend eight Euros  
70 Jo yes  
71 Ja well thirty-nine Euros divided equals by divided<sup>39</sup> ...no one third divided  
72 thirty-nine Euros divided by three  
73 Jo yes

First of all, Josefine starts the problem-solving process by reading the task out loudly one more time. By reading the word problem out loud, she shares the information with her partner and provides some structure to a concrete beginning of the problem-solving process<60-63>. This starting point is implying the minimum which is required for a co-constructive process, as both are focused on the same problem. Janina is taking the next turn and makes the point of seeing the need to calculate <65>. She also demonstrates her interest to deal with the problem together with her partner by using the personal pronoun “we”. It is not clear if she would have completed her remark, but Josefine is offering the first idea how to translate the word problem into an arithmetic operation. Josefine’s idea is paraphrasing an important part of the word problem “one third”, into a concrete division by using “divided by three” <66>. Both seem to agree on this translation offered by Josefine, because there is no debate arising – this sequence is actually demonstrating a first mathematical idea shared between each other as a joint construction.

In the following lines, the word “leftover/remainder” (Rest) is shifting into the centre of attention as Josefine brings this point into consideration <67>, and Janina is at first assenting, but then feeding back the word “leftover/remainder” to the text, to underline the fact that with the “leftover” you spend eight Euros <68/69>. Josefine does agree with Janina’s point of view, by saying “yes” <70>. Her affirmation could be also interpreted as a signal for only listening and letting the partner keep the turn, but in both interpretations no contradiction emerges on the stage of interaction. The next turn is taken by Janina, with the attempt of translating their ideas into a first mathematical term. At first, it seems quite confused what Janina is trying to express in line<71>, but in the end she brings up the term  $39 : 3$  <72>, and Josefine does

---

<sup>38</sup> In the original transcript, the girls were using the German word „Rest“ which stands for both left over and remainder. This caused the problem of mixing the meaning of the ‘leftover’ in the word problem with the understanding of division with remainders.

<sup>39</sup> “Divided by“ is translated as “geteilt durch“ in German and sometimes just the word „durch“ is representing the meaning of division in an everyday language: <71-73-German> also 39 Euro geteilt ist durch...nein ein Drittel durch 39 Euro geteilt durch 3.

agree without having any problems of understanding <73>. In a similar way, they agree on having a remainder for this term, but after calculating the term, they assert:

95 Ja no there is no remainder (Rest)

96 Jo that's right, there is no remainder (Rest) ... this will be thirteen

After a short pause, Josefine proposes minus as a new operation to get a kind of "remainder/leftover" (Rest). Janina contributes with a new first calculation for the word problem and they finish the problem in the following way:

102 Ja thirty-nine minus eight

106<sup>40</sup> Jo minus eight equals thirty-one\ ..and now

109 Ja divided by three

113 wait a moment this will be ten but now you have one Euro leftover (übrig)

114 Jo yeah this will be ten remaining one

At the end, both are happy of having achieved a "remainder" (Rest) and they do not doubt their solution, even though they tie their result to the situation presented in the word problem <113, 114>. They close this word problem with the written answer: *She has got 10 remaining 1 for herself*. Thus, by starting with 39 minus 8, they jointly 'produce' a number (31) which ensures a remainder taking 3 as divisor.

The prompt way of supporting the input of the partner is characteristic of their joint constructions, and during the whole problem-solving process their way of communicating grows from this supportive answering to an overlapping conversation, where the last word or two are repeated by each other, before completing the sentence, up to a collective cognitive convergence<sup>41</sup> where the ideas and sentences are merging into each other. Thus, only by taking utterances of both partners together, a substantive contribution to the joint construction will occur. The detailed analyses of their participation leads to the perception of having two equal "authors" within this problem-solving process, as they are both carrying forward their partner's ideas and joining their input into one whole joint construction, where in the end they can sign their result to the word problem on a working sheet, agreeing that they both understand their approach to the solution.

### **Collaboration Process B: Josefine and Belen and their cookie-task**

In contradistinction to Josefine and Janina's highly symmetric joint construction, we are going to show another sequence of team work between Josefine and another female pupil of her class, named Belen. This dyad is a new constellation; these two

---

<sup>40</sup> Missing lines refer to a side dialogue.

<sup>41</sup> The phenomenon of collective cognitive convergence in groups of people, who are interacting frequently together, has been discussed by several studies. For different aspects of collective cognitive convergence in cooperative learning see Teasley et.al. (2008). In particular, van Dyke Parunak, is engaged in this development between group members who work together over quite a long period of time, losing the capability of questioning themselves and being aware of the advantage of differing and contrasting ideas (e.g. Parunak et.al 2009).



pupils did not work together on mathematic problems before this lesson. Belen is usually having some problems in solving mathematical tasks alone, which requires a lot of support by the teacher in the everyday lessons. Alike the dyad we have analysed above, this new team is taking the task as a problem which has to be resolved together – joint construction has also been built by these two pupils, but in a complete different approach than before.

Again, Josefine is setting up a joint focus by repeating the task as a spokesman of the teacher “well now we have to think up of a word problem” <45>. Belen throws in immediately an idea where Tom and Kim own respectively 55 and 50 clothes. Josefine’s reaction shows a sceptical attitude which is demonstrated by her way of questioning Belen’s input. After rejecting Belen’s first idea, Josefine phrases the following:

- 51 Jo .. well wait let’s do it rather let’s do it well mhmh on Sunday a  
 52 Christmas market takes place at the Opernplatz \ (square in front of the  
 53 opera) there\ there Belen sells Belen has– has got a  
 55 <Jo stall - has got hund red >eigh ty cookies  
 56 <Be >eighteen  
 57 Jo of these she sells ninetyfive\ so Josefine has got also a stall.. she has..  
 58 Be three. fourhundredfifty

Josefine induces her contribution as something new, not related to the former idea of Belen <51>. After rejecting Belen’s input, she places emphasis on the idea of working together: Within her new attempt of creating a word problem, she integrates Belen as a protagonist <53>. This could be seen as a kind of a face-saving act towards Belen after face-threatening her by rejecting Belen’s idea (cf. Brandt & Tatsis 2009). Furthermore, she reverts to the underlying structure of Belen’s suggestion, using two people (Belen and Josefine) and a certain amount of objects (cookies). But Josefine doesn’t leave it to the additional term Belen verbalised, as she is bringing in the subtraction by selling the objects <55>. At several stages, Belen contributes single numbers at the right moment <56-58>, such that Josefine can integrate them into their word problem while creating it. Later, Josefine prompts Belen to calculate:

- 110 Jo Belen now strain your head do you know what this plus that is  
 111 [*pointing at the work sheet*]  
 112 Be this is (incomprehensible)  
 113 Jo threehundredfive plus hundredninety  
 114 Be becomes than fourhundredninetyfive

Belen is also joining the process of developing the task, by calculating an addition problem which Josefine confers on her <113-114>. By analysing this new dyad, it highlights that Josefine, after taking on Belen’s first idea as a ghostee, is leading the team work process like a tutor, as she is the more capable one. She takes on the part of being the “author” of the word problem structure and of the final result. Belen

takes part as a “spokesman” and “relayer”. Thus, she is integrated in the joint construction of Josefina’s idea.

## COMPARING AND CONCLUSION

Collating the two dyads, there is a special form of joint construction to emphasize: In the interaction between Josefina and Janina there is not even one moment of misunderstanding and there is no antithesis to the suggestion of a single person (like the direct rejection of Belen’s idea). They collectively reject their first joint construction (39 : 3) without even questioning each others ideas. Another interesting phenomenon is how the turn-taking is accepted by both partners in all moments of their problem solving. There is never a dissent between them which leads to a conflict or even a contradictive discussion. They are both responsible for creating their solution in a complete balanced way. This means that the joint constructions in their word problem solving process could not be classified as Type 2 – there may be one idea considered through the collaborative process of Josefina and Janina, but it is not a “relatively advanced contribution of one child becom(ing) a group product” (Howe, 2009, p.217, 218). Instead the two partners show a complete equality of how they develop and coordinate their resolution. Considering Type 1 as a coordination of substantive contributions from more than one child into a relatively advanced whole (Howe, 2009, p.217), it also does not encompass the complete phenomenon of Josefina’s and Janina’s joint constructions process, although it may fall under the category of Type 1, being a very special subcategory. It is a conjointly creation of a common idea, where the single turns of both partners gear into each other – and both partners are aware of this common idea just from the first glow in the interaction. Thus, this subtype involves elements of collective cognitive convergence and can be described as follows: Joint construction of one united idea, in complete symmetry, as one idea has to be considered from both partners from the beginning.

Something new is constructed, a new idea unfolded without any dissent, but resolving a ‘joint problem’ – even if the solution is not responding ‘correctly’ to the problem. In our example, the ‘mistake’ Josefina and Janina do is underlining the singularity of their joint construction – there is actually a real ‘problem’ to solve together and they find a collaborative way to surmount the difficulty.

This example of Josefina’s und Janina’s collaboration is highlighting the chances and impediments of working in dyads over a longer period of time, as collective cognitive convergence can be very productive on the one hand, as there are less misunderstandings between the team members, but on the other hand it can lead to such closeness that problems are not perceived anymore (cf. Parunak et. al 2009). Using the cross-over method as well as pair-square to bring in new ‘experts’ with new ‘ideas’ from outside, can help to detect these ‘blind alleys’ and offer new incitements for working more productively.

## REFERENCES

- Barab, S. & Squire, K (2004): Design-based research: Putting a stake in the ground. *The Journal of Learning Sciences*, 13 (1), 1–14.
- Bearison, D. J., Magzamen, S. & Filardo, E. K. (1986): Socio-conflict and cognitive growth in young children. *Merill-Palmer Quarterly*, 32, 51–72.
- Brandt, B. & Tatsis, K. (2009): Using Goffman's concepts to explore collaborative interaction processes in elementary school mathematics. *Research in Mathematics Education* 11 (1), 39–55.
- Edwards, J. (2007): The language of friendship: Developing sociomathematical norms in the secondary school classroom. *Proceedings of CERME 5* (pp. 1190–1200). Lanarca, Cyprus: ERME. [<http://ermeweb.free.fr/CERME5b/WG8.pdf>] (14.09.2010)
- Goffman, E. (1981): Footing. In Goffman, E. (Ed.), *Forms of talk* (pp. 124–150). Oxford: Basil Blackwell.
- Howe, C. (2009): Collaborative group work in middle childhood – joint construction, unresolved contradiction and the growth of knowledge. *Human Development* 52, 215 – 239.
- Krummheuer, G. (2007): Argumentation and participation in the primary mathematics classroom. Two episodes and related theoretical abductions. *Journal of Mathematical Behavior* 26, 60 – 82.
- Levinson, S. (1988): Putting linguistic on proper footing. Explorations in Goffman's concepts of participation. In Drew, P. & Wootton, A. (Eds.), *Erving Goffman - Exploring the interaction Order* (pp. 161–227). Cambridge: Polity Press.
- Mercer, N. & Littleton, K. (2007): *Dialogue and the development of children's thinking: A sociocultural approach*. London: Routledge.
- Parunak, H., Belding, T., Hilscher, R. & Brueckner, S. (2009): Understanding collective cognitive convergence. In Nuno, D. & Sichman, J. (Eds.) *Multi-Agent Based Simulation IX: International Workshop, MAPS 2008, revised selected papers*. (pp. 46–59). Berlin, Heidelberg: Springer. [<http://www.newvectors.net/staff/parunakv/MABS08UC3.pdf>] (14.09.2010)
- Sacks, H. (1998): *Lectures on conversation*. Malden, Ma, Blackwell.
- Teasley, S., Fischer, F., Weinberger, A., Stegmann, K., Dillenbourg, P., Kapur, M. & Chi, M. (2008): Cognitive convergence in collaborative learning (Symposium). *International Conference on Learning Sciences* (pp. 360–367). Utrecht: International Society of the Learning Sciences.
- Vygotski, L. (1978): *Mind in society: Development of higher psychological processes*. Harvard University Press. (edited and translated by Cole, M. et.al.)

# THE CONCEPT OF EQUIVALENCE IN A SOCIALLY CONSTRUCTED LANGUAGE IN A PRIMARY SCHOOL CLASS

Cristina Coppola, Monica Mollo, Tiziana Pacelli

Università degli Studi di Salerno - Italy

*We analyze two episodes of a teaching experiment involving children of primary school. The children, handling a procedural language they previously constructed, single out equivalence rules. So they construct the concept of equivalence intuiting a semantic equivalence in the language. We adopt a qualitative approach mainly based on observational and ethnographic methods. We present a data analysis regarding how the children, in interaction, co-construct the solutions to the tasks, according to the activity theory and the epistemological triangle.*

Key words: procedural language, equivalence, interaction

## INTRODUCTION AND THEORETICAL BACKGROUND

The study we are presenting is part of a wider research, whose main aim is to analyse the relationship between language and the development of logical tools through linguistic-manipulative activities, in primary school classrooms. By *logical tools* we mean the skill to find and choose different strategies to solve problems regarding logical tasks. The activities regard both assertive and procedural aspects of the language. Both aspects play a fundamental role in the development of child's mathematical thinking (Gerla et al., 1990). In this paper we refer to a simple procedural language.

Our theoretical background refers on the one hand to the social-genetic constructivism (Perret-Clermont, 1979; Iannaccone, 2010) and to the contributions of the soviet cultural-historical school (Vygotskij, 1934; Leont'ev, 1978), on the other hand to formal logic [1] (Kneale, 1962). The latter can represent a basic point of reference for the study of the development of mathematical thinking provided it is seen as the explicit expression of the strong relationship among the language, the construction of mathematical objects and the development of mathematical concepts. Mathematical objects arise not only through abstraction processes from the direct experience, but also through the language objectification, through the singling out of manipulation rules of linguistic objects and through the intuition of the equivalence among different procedures. According to the Vygotskian theories, the use of the symbols deeply affects the cognitive functioning. Thinking cannot be traced back only to abstract processes of knowledge and reasoning, but it is feed with motor and sensory processes; the tools, intended as "mediation devices" between the individual and the social context, modify the whole flow and the structure of the mental functions, transforming and affecting the actions (Anolli, 2005). Symbols are intended as "tools" supporting the accomplishment of the actions required by the context and the activities which the individuals are engaged with (Radford, 2000).

Concerning signs in the *Activity Theory* Leont'ev agrees with Vygotskij in that tools mediate activity. Psychological processes are based on the *activity*. It represents the reference global unit, which is naturally social and so it implies necessarily the interaction among the different actors. Activities consist of goal-oriented conscious *actions*. The actions consist of automatic *operations* which are independent from the nature of the activities (Anolli, 2005). In the subsequent developments of the activity theory Leont'ev includes the rules and the division of the work inside the community (*activities' systems*), maintaining that an activity and its progress require a social coordination and a meanings' sharing (Ligorio, 2010). According to the socio-genetic constructivism, on some conditions the peers interaction represents the key for a cognitive progress and moreover the cognitive activity cannot be studied without considering the social and cultural contexts in which this activity is carried out (Perret-Clermont, 1979; Iannaccone, 2010). Thinking, and so mathematical thinking, cannot be considered as an individual information processing, but "thinking is a form of *social praxis*" (Wartofsky, 1979; Radford et al., 2005). The construction of new mathematical knowledge is based on the relations built in the classroom interactions between *signs/symbols* and concrete or abstract *objects/reference contexts*. Signs themselves do not have an isolated meaning, which has to be constructed by the learner. The meaning given to the not yet familiar systems of symbols is continuously enriched by the mediation with suitable reference contexts (Steinbring, 2000).

The use of the symbols in the culture of mathematics teaching is constituted in a specific way, giving social and communicative meaning to letters, signs and diagrams during the course of ritualized procedures of negotiation (Steinbring, 1997).

An important theoretical tool to describe and analyze the processes through which the children, in an activity system, construct new mathematical knowledge is the *epistemological triangle*.

[...]the epistemological triangle is used for modeling the nature of the (invisible) mathematical knowledge by means of representing relations and structures the learner constructs during the interaction (Steinbring, 2006).

The characteristic of the epistemological triangle consists in the mutual relations among the three vertices of the triangle, "sign/symbols", "object/reference context", "concept". These relations are not fixed a priori, they constitute a balancing system. As the knowledge develops in an interactive way, the interpretation of the systems of signs and the corresponding reference contexts modify (Steinbring, 2000; 2006).

## RESEARCH PURPOSES

Our research hypothesis is that activities based on the creation and the handling of simple procedural languages, in a context of social interaction, can spur in the children a reflection about the language functions and the use of symbols. The aim is to lead the children, by group activities, to a change in the representation and in the

use of the language, from being only a communication tool to being also an object to manipulate. We believe that such a change could be an important process in the development of mathematical thinking and that through manipulation the construction of new mathematical knowledge could be facilitated. With regard to the activities we describe in this paper, we expect that the children would carry out interactive processes of constructing the concept of equivalence.

## THE METHODOLOGY AND THE PARTICIPANTS

The participants were 24 children, aged 9-10 years, from a fourth grade class on a primary school, working in small groups in a period of about three months. We adopted a qualitative approach mainly based on observational and ethnographic methods. The activities referred to the “poor informatics paradigm” [2]. During the activities our role was to ask questions in order to stimulate thought processes without correcting any errors. We collected a corpus of data consisting of videotapes, audio recordings, “observable traces” (Vermersch, 1994) [3], initial and final tests and narrative interviews (ongoing and at the end of the activities). We analyzed: 1) how the children, in interaction, co-constructed the solutions to the tasks; 2) the cognitive representation of the tasks’ solution that the children, individually, developed during and after the activities. In this paper we present only some results of the data analysis regarding point 1), according to the activity theory (Leont’ev, 1978; Engeström, 2001) and the epistemological triangle (Steinbring, 2000; 2006).

## DESCRIPTION AND ANALYSIS OF THE ACTIVITIES

We identified different situated activity systems (Leont’ev, 1978; Engeström, 2001) by analyzing the video recordings and by transcribing the communicative interactions among the children in the activities progress. We transcribed the communicative exchanges using a lightened version of the Jeffersonian system (Jefferson, 1985). We analyzed the *systems* according to the three levels of the *activity theory* (Leont’ev, 1978; Engeström, 2001): the level of the goal-oriented *activity* (the given task); the level of the *actions* the children carried out to solve the task; the level of the *operations* underlying the performed actions.

In this work we focus on two activity systems that are part of a teaching experiment starting with a problematic situation in which there is a “child-robot” in a room. The other children have to “invent” the instructions in order to make the child-robot move. After several discussions and negotiations, the children create a language constituted by letters. The *interpretation* of each letter is a basic movement of the robot. Therefore, a word of the language is a “program”, since it is a sequence of elementary instructions. The invented language with its (translated) interpretation is: *A is Forward; I is Backward; N is Turn to the north; S is Turn to the south; E is Turn to the east; O is Turn to the west; L<sub>O</sub> is West side step; L<sub>S</sub> is South side step; L<sub>N</sub> is North side step; L<sub>E</sub> is East side step; F is No-moving* [4]. This first phase ends with

the institutionalization of the language: the children write it on a billboard after the negotiation and the sharing by all the groups. Furthermore, the classroom walls are labelled with the cardinal points.

The aim of the two activities we are going to present is to bring out some semantic equivalence rules in the language [5] in the first activity and to get children use these rules in the second activity.

### The construction of the equivalence rules

At the beginning of the first activity the groups have to represent on a squared sheet [6] two sequences of symbols of the language:  $EAISAAAL_E L_O$  and  $NAEEANNL_E L_E L_O$ . In Fig. 1 there is an example of the children's representation. Then, for any sequence, they have to answer the following questions: *Is the path you have drawn the shortest one for the robot to achieve the goal? Can you make it shorter by replacing some instructions with some others? Which instruction can be replaced? With which ones? How come?*

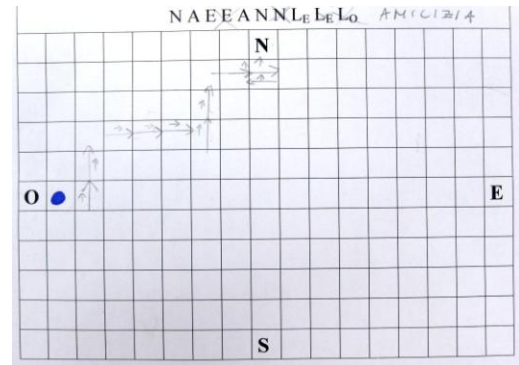


Figure 2

Below we report two episodes in which the *action* is *shortening the sequences*, previously represented. We identify the *operations* that the children carry out and the *tools* they use in order to construct the rules.

### Excerpt n.1

In the following excerpt [7] we can observe how a group, working on the first sequence, constructs the rule “ $AI = F$ ”:

- 1 S1: we can remove Backward in this way (*pointing and touching the sequence on the worksheet*)
- 2 S2: Backward and [...]
- 3 S3: let's remove this Forward here (.) and this Backward (*pointing AI in the sequence touching the worksheet*)

R1 encourages the children to observe pairs of adjacent instructions

- 4 R1: for example if I go forward and then backward (.) what happens?
- 5 S4 and S5: [[I come back on the same tile]]
- 6 R1: and so?
- 7 S5: I have to remove Forward and Backward

## Working Group 9

R1 draws the group's attention to the fact that the task is "to replace" instructions and not only "to remove" them

8 R1: what can we put instead of Forward and Backward? What do Forward and Backward correspond to?

9 S5: to nothing (0.5) (*looking at the billboard with the instructions*) to No-moving

In this example, the activity is built collaboratively by the children, which in turn (lines 1, 2, 3) try to find a solution to the questions asked by the researcher. The strategy pursued by the group is to remove the instructions. The operations are the specific modalities of execution by which the action is performed: in this example they correspond to the use of the sheet as a mediation device. For example, in lines 1 and 3 the children point at the sheet on which the instructions are written. In line 5, two children answer in choir to the researcher stating that "by forward and backward" it is possible to come back on the same tile. This verbal formulation is the result of the previous interaction with the other children. The researcher reminds the children of the task: it is about "replacing" and not only "removing". So a child (line 9) says that "Forward and Backward" can be replaced with "nothing". The operation "looking at the billboard" moves the attention of the group to the language. The children answer that "Forward and Backward" can be replaced with "No-moving". In the development of this first action it seems that the group is approaching to an insight of a semantic equivalence in the language (lines 5, 9). After the response of the child about the "No-moving" (line 9), the group goes on with the activity, interacting with the researchers, *making marks on the sheet, simulating the movements of the robot* and referring to the context in which they are working. Afterwards a child sums up trying to answer the questions. The group asserts that the initial sequence can be simplified.

10 S4: (*reading again the task*) which instructions can be replaced? Backward and Forward (.) we have replaced with No-moving (.) then (.) with what? With No-moving and we said it (.) How come?

11 S3: because (.) forward and backward is always the same thing so you might as well keep still (.)

The group receives the second sequence. In the following excerpts there is the construction of the rule " $L_E L_O = F$ ". After the negotiation of the task inside the group and with us, a child *crosses* by the pencil the instructions  $L_E L_O$  on the sequence. Another child asks for an explanation

12 S2: why did we remove them?

13 S4: because (.) no look at=look at! (*pointing the worksheet*) Wait! As before (.) see (.) we said orally (.) because at the end we always come back to the same point

14 S2: ok (0.5)

15 S4: so these [instructions] here can be removed



## Working Group 9

R2 asks for explanations about the abbreviations and about the marks on the sheet

16 S4: These as before (0.5) which actually get [us] reach always the same point

17 R2: So we can replace them (.) with what?

18 the group: [[nothing]]

19 R2: and which is the instruction for nothing?

20 the group: [[No-moving]] (*looking at the billboard with the instructions*)

Finally, as for the first sequence a child sums up in order to answer the questions

21. S4: which have to be replaced? (0.5) East side West side North East (0.5) (*looking at the sheet with the task*) with what? (.) with No-moving (.) and then (.) how come? Because (.) because you always come back to the same point

The other members of the group nod and confirm. S4 while giving us the sheet

22.S4: ‘cause then (.) at the end (.) if it is really how we did (.) the solution (.) we can do differently (.) that is on the contrary (.) in the way that (0.5) that we have to lengthen the path (.) as for example (.) like here (.) but we have to get it longer

23.R1: longer (.) but always reaching the same point?

24.S4: yes (.)

The previous excerpt shows that the children are constructing the concept of equivalence step by step. At every step every child spontaneously recovers what the other children have built. The children, after they have understood and internalized the rule, try alternately to solve the task. At the same time they reflect on the answers given by the others (lines 12 to 15). Furthermore, they construct new rules referring to how the other rules have been previously constructed by other children (line 13, lines 16 to 20). We cannot claim that in another situation or educational context the same interactional situation would arise. It is interesting to notice that the activity of rules' creation is a *socially constructed* culturally situated (lines 19, 20, 21) activity. Besides it is mediated by both materials and linguistic cultural artefacts.

### The use of the equivalence rules

The second activity we analyze follows the singling out of the equivalence rules, which have been institutionalized by writing them on the blackboard. The task is to verify if the two symbols sequences of the language  $SAAIAEEAFL_S L_N L_N AAANN$  and  $SAFAEAFL_N 3AN$  are equivalent, using only the equivalence rules. Only afterwards the groups have to verify the task by drawing the corresponding paths. We report an excerpt regarding the communicative exchanges of the resolution process of a different group from the one of the previous activity. We identify the *operations* and the *tools*. A child *reads the task aloud* and *reads the two sequences*. At first the group concentrates on the instructions.

### Excerpt n.2

## Working Group 9

- 1 S1: Turn to the south (.) then there is Forward (0.5) it is replaced with No-moving (0.5) because it stays in the same point (0.5) then AA then E (.) Turn to the east (0.5) then it always stays in the same point (*pointing EE in the sequence*) (0.5) Forward (.) Backward [...] they aren't equivalent (.) because there is South side and here there isn't (*moving from a sequence to the other one touching the sheet*) (0.5) but let's see now (.) we have to check it yet

Hurriedly the group states that the sequences are not equivalent and soon it passes to verify the answer *representing* the corresponding paths. Nevertheless, using the drawing the children get confused, they don't succeed in reaching a correct solution and there is not agreement.

- 2 S3: they always arrive at the same point (0.5) no (.) they don't always arrive at the same point
- 3 S4: yes (.) they are equivalent (0.5) look (.) it's the same tile
- 4 S2: it's not true look that this is the yellow tile (0.5) and this (.)
- 5 S1: ok let's revise the path

At this point the researcher suggests reading the task again. Then the group goes back to the symbols dropping the drawing. After a discussion the children conclude

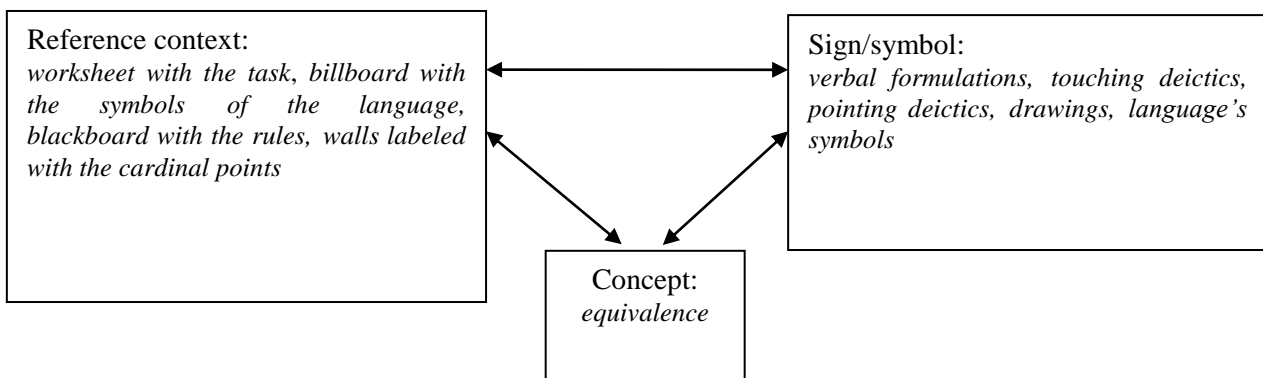
- 6 S2: (*looking at the two sequences in the first of which there is  $L_S L_N L_N$ , in the second one there is  $L_N$* ) the robot takes a step south side (*pointing by his hand the wall labeled with the south-card*) and then it takes a step north side (*pointing by his hand the wall labeled with the north-card*) if then you write North side (.) it's the same thing because here there is one instruction more (.) but they always are equivalent
- 7 S3: so are they equivalent?
- 8 S2: wait (.)
- 9 S1: N (0.5) South (.) Forward (.) it's ok (.) No-moving it's ok (0.5) it's ok
- 10 S3: is it ok?
- 11 S2: the instructions are ok

In the excerpt the group applies the equivalence rules, previously constructed, to the two sequences given in the task. At first (line 1) the children's action is oriented to the solution of the task in the shortest time. The "hurry" is due to the competition among the groups and it leads the children to mistake. The operation in this case is "reading the task" hurriedly and not carefully and "representing the paths" on the sheet. In lines 2-5 we can observe how the children, debating one another, realize that their solution does not fit the task. The discussion seems to activate a real *awareness* of the error. The researcher by a *guidance action* leads again the group to a deeper reflection about the task. The action of "reading again the task" changes the initial goal of the children from giving a solution in the shortest time to giving a

correct solution. This action involves new operations: a child represents the movements of the robot “pointing out the cardinal points” fixed on the walls (lines 6-10). Again the interaction in the group allows the co-construction of the solution and also the awareness of the initial error.

### The concept of equivalence

The process of constructing the concept of equivalence can be described by the epistemological triangle (Fig. 2). In the activity systems different kinds of signs are produced (Steinbring, 2006). There are *verbal formulations* (in both the activities): the children use everyday language expressions, which can represent preliminary forms of interactively produced signs useful to construct mathematical concepts; “*touching deictics*” (above all in the first activity): the children touch the small squares on their worksheet simulating by the fingers the robot’s movement; “*pointing deictics*” (above all in the second activity): the children point out by their hands the walls labeled with the cardinal points; *drawings*: the children use graphical representations as a support in the task resolution. Besides the children produce and handle the language’s symbols. Really the latter turn out to be the most useful to gain the goal of the activities. Indeed, especially in the second activity, when the children use the drawing they do not succeed in solving the task, while they are successful when they use and handle the symbols of their language. The meaning of the signs is produced by the children in a continuous reference to what they employ as *reference context*, that is, *the worksheet with the task*, *the billboard with the symbols of the language* (in the first activity), *the blackboard with the rules* (in the second activity) and *the walls labeled with the cardinal points*. Through the continuous mediation between signs and reference context they construct the concept of “equivalence”.



**Figure 2**

In both activities the concept of equivalence comes out in the repeated use of expressions like “is the same thing”. The use of expressions like “it remains always on the same point”, “it comes back to the same tile”, “you come back always to the same point” points out that the first step of the construction is the singling out of simple equivalence rules (as an example  $AI=F$ ). Showing that the composition of some instructions is “the same” than the composition of other ones, the children

“construct mathematical knowledge”, because in doing this they identify an equivalence relation. The produced signs are “mathematical signs” because they denote a semantic equivalence relation in the language. The understanding of the equivalence is not an end in itself, since it is basic for the construction of the “mathematical objects”, which arise just from unifying equivalent “words” [8].

In the collective discussion among the groups, taking place after the activities, it emerges how the children’s previous knowledge strengthens the mediation between signs and reference context and the whole construction process of the “new” concept. “The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge-one cannot understand these signs without any presuppositions” (Steinbring, 2006). Indeed the children refer to the already known notion of equivalence regarding weights and measures. Besides it is interesting the comparison they do with the “synonymous” words in the Italian language. In their “new language” two syntactically different words are two sequences with different symbols. These words are “equivalent” if, even though they are different, they have the same meaning where the meaning is the robot’s movement.

## CONCLUSIONS

According to our research hypothesis, the shared creation of a language and its manipulation allowed the children to acquire the awareness that symbols depend on the context of use and that they presume an agreement in the community of the receivers (Ferrari, 2002). Moreover the language, seen as a manipulation object as well as a communication device, was a basic tool in the knowledge construction, an “amplifier” of the intellectual capabilities (Bruner, 1996). Although the activities don’t deal with argument strictly linked to mathematical school topics, they aim to stimulate linguistic skills which are basic for the mathematical knowledge construction. By the analysis of the activity systems it comes out how in a particular educational situation the children, freely interacting, were able to co-construct a socially shared meaning of the concept of equivalence. The actions were negotiated and shared by the groups. So the underlying operations became common and understandable. The *actions* of shortening the language sequences and checking the equivalence between two sequences, in the two activity systems, allowed the children to *construct the concept of equivalence*. The *operations* generated by the actions spurred a continuous mediation between *reference context* and *signs/symbols* produced and handled by the children (Leont’ev, 1978; Steinbring, 2006). The repeated use of expressions like “is the same thing” highlights that the children *intuit* a semantic equivalence relation in the language. They arrive *gradually* to the concept of equivalence, through the *singling out* of simple equivalence rules.

## NOTES

1. Formal logic is seen as the expression of an historical path bringing to a change of the language’s role in mathematics. This path starts with the passage from the rhetorical algebra to the symbolic one and keeps up to modern logic.

## Working Group 9

2. According to this paradigm it is possible to develop abilities near to a data processing frame by means of simple and “poor” materials. The children become “inventors” (they code a language), “interpreters” (they decode a language), “manipulators” and users of “programs” in the language (Gerla et al., 1990).
3. According to Vermersch (1994) the traces are the material signs produced by the children during the activities, such as intermediate and final written notes and outcomes.
4. As an example, *AAIS* is a word corresponding to the temporal sequence of actions: take a step forward, take a step forward, take a step backward, turn to the south.
5. We call *equivalent* sequences representing paths which, starting from the same initial position, allow the robot to arrive at the same end position. An example of an “equivalence rule” is  $AI = F$  since both the words represent “no-moving”. So as an example, the sequence *AIA* is equivalent to the sequence *A*.
6. The squared sheet simulates the classroom’s tiled floor and N,O,S,E represent the cardinal points labelling the walls.
7. In the excerpts we denote by S1, S2,...the students and by R1, R2,...the researchers involved in the activities
8. This is the point of view of formal logic which shows how mathematical models can be constructed starting from the used language (closed terms) and from a suitable equivalence relation.

## REFERENCES

- Anolli, L. (2005). *Psicologia della cultura*. Bologna: Il Mulino.
- Bruner, J.S. (1996). *The culture of education*. Cambridge, MA.: Harvard Un.Press.
- Engeström, Y. (2001). Expansive learning at work: toward an activity theoretical reconceptualization. *Journal of Education and Work*, 14, I, 133-156.
- Ferrari, P.L. (2002). Developing language through communication and conversion of semiotic systems. In A.D. Cockburn & E. Nardi (Eds.), *Proc. of the 26<sup>th</sup> Conf. of the Intern. Group for the Psychology of Math. Ed.* Norwich (UK), 2, 353-360.
- Gerla, G., Sestito, L. & Vescia, S. (1990). Linguaggi algebrico-procedurali nella scuola elementare: un progetto di ricerca. *La matematica e la sua didattica*, 4, 39-48.
- Iannaccone, A. (2010). *Le condizioni sociali del pensiero. Contesti, attività e ricerca di senso*. Milano: Edizioni Unicopli.
- Jefferson, G. (1985). An exercise in the transcription and analysis of laughter. In T. Van Dijk (ed.). *Handbook of discourse analysis*, 3. London: Academic Press.
- Kneale, W. & Kneale M. (1962). *The development of logic*. London: Oxford Un. Press.
- Leont’ev, A.N. (1978). *Activity, consciousness, personality*. Englewood Cliffs: Prentice-Hall.
- Ligorio, M.B. (2010). La scuola come sistema di attività. In M.B. Ligorio & C. Pontecorvo. *La scuola come contesto:prospettive psicologico-culturali*. Roma: Carocci.

## Working Group 9

- Perret-Clermont, A.N. (1979). *La construction de l'intelligence dans l'interaction sociale*. Berne: Peter Lang, coll. Exploration.
- Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Ed. Studies in Mathematics*, 42, 237-268.
- Radford, L., Bardini, C., Sabena, C., Diallo, P. & Simbagoye, A. (2005). On embodiment, artifacts, and signs: a semiotic cultural perspective of mathematical thinking. In H.L. Chick & J.L. Vincent (Eds.). *Proc. of the 29<sup>th</sup> Conf. of the Intern. Group for the Psychology of Math. Education*, 4, 113-120. Melbourne: PME.
- Steinbring, H. (1997). Epistemological investigation of classroom interaction in elementary mathematics teaching. *Ed. Studies in Mathematics*, 32, 1, 49-92.
- Steinbring, H. (2000). Interaction Analysis of Mathematical Communication in Primary Teaching: The Epistemological Perspective. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 32, 138-148.
- Steinbring, H. (2006). What makes a sign a *mathematical sign*?-An epistemological perspective on mathematical interaction. *Ed. Studies in Mathematics*, 61, 133-162.
- Vygotskij, L.S. (1934). *Thought and language*. Moscow-Leningrad: Sozhegiz.
- Vermersch, P. (1994). *L'entretien d'explicitation*. Paris: ESF.
- Wartofsky, M. (1979). *Models*. Dordrecht: D. Reidel.

# CONTENTION IN MATHEMATICAL DISCOURSE IN SMALL GROUPS IN ELEMENTARY SCHOOL TEACHING

Andrea Gellert

University of Duisburg-Essen

*Abstract. The paper presented refers to the elaboration of the theoretical construct called “focusing teaching strategies” within the research project “Probing and Evaluating Focusing Instruction Strategies in Elementary Mathematics Teaching (ProFIT)”. By teachers’ interventions an attempt is made to create a learning situation, in which the joint attention is focused on a critical feature of a mathematical problem. Especially the following two perspectives are investigated: How can the teacher make a mathematical reasoning need more accessible for students and in which way can the needs be developed in the common discourse? And: How can pupils understand a mathematical reasoning need and how are they able to agree on it?*

**Keywords:** *interpretative discourse analysis, focusing teaching strategies*

## INTRODUCTION: CONTENTION IN MATHEMATICAL DISCOURSE

Mathematical classroom interaction is a complex process. In the research project “Probing and Evaluating Focusing Instruction Strategies in Elementary Mathematics Teaching (ProFIT)”, the interaction competence of mathematics teachers will be worked out by means of videotaped small group discussions with four pupils and one teacher. The five participating teachers are from one school so the video data consists of 40 videos of group discussions and the associated school lessons. Selected sequences, in which a contention or disagreement is evolved interactively, will be interpreted by theory-based analyses for further elaboration and differentiation of the theoretical construct “focusing instruction strategies”. The project is related to social, interactionist and epistemological theories, which are associated with discursive learning and the construction of new mathematical knowledge.

Furthermore, it will be analysed for how focusing teaching strategies affect concrete discourse situations and pupils’ learning processes with an aim of a deeper understanding of the nature of mathematical classroom discourses. This is also elaborated by means of interpretation of the students’ reaction in the selected video episodes. The special analysis refers to the interaction evolved by the contention to the apparent mathematical content and to the interplay between them.

## **THEORETICAL BACKGROUND**

### **Mathematical Discourse and discursive learning**

The particular mathematical discourse is part of the social teaching and learning activity in school. It especially attends to clarifying jointly identified problems and statements, which are related to the mathematical content of a task on which not all participants agree. From a sociologist perspective, discourses are subjected to certain rules determining who can say what in which way and under which circumstances. Nevertheless, these rules are not rigid. They can be changed and indeed are changed frequently (Miller 2006, 202f.). The teaching practice of mathematics in elementary schools often does not comply with these changes. For example in the pattern of the “thematic procedure of mathematization” elaborated by Voigt (1990, 305), the teacher tries to achieve a pretended uniqueness in the interpretation of ambiguous mathematical classroom objects. However, the ambiguity of these objects admits numerous initial points for classroom discussions (e.g. Voigt 1990; Steinbring 1994; Söbbeke 2005).

The communicative classroom exchange about these different views and the development of intersubjectivity through the participants’ negotiation processes are fundamental for the learning of mathematics in social interaction (Voigt 1994). So the pupils involved in these learning processes depend on communication with other people, e.g. classmates, parents, teachers or others, because fundamental learning demands constructing actively and negotiating dialogically new general relations between already existing segments of knowledge. Miller (1986) speaks about “basic theories” to acquire applied knowledge. Moreover, discourses are a necessary condition for them to emerge.

In his epistemological research, Steinbring speaks of “new mathematical knowledge”, which “has to exceed the old knowledge systematically” (Steinbring 2005, 61). Thus, new mathematical knowledge is understood as an extension of the old knowledge at the same time. Hence, the construction of new knowledge in mathematics lessons takes place in a tension between consistent base knowledge and new knowledge relations, which is in need of fundamental reorganisation and enhancements of knowledge systems. For “mathematical knowledge” being negotiated interactively, the subject “mathematics” cannot be understood as a predetermined matter but has to be interpreted in accordance with the epistemological conditions of their dynamic, interactive development (compare Steinbring 2005, 34f.).

### **Interaction patterns and routines in the mathematical classroom**

From an interaction-theoretical and epistemological point of view, teaching and learning processes are seen as dynamic and reflexive. In this process, teachers and students jointly constitute the classroom’s reality by interaction. So the situation is necessarily influenced by interaction and communication patterns (Mehan 1979;



Bauersfeld 1988; Wood 1994). They are mutually produced by the routines of teachers and students (Bauersfeld 1988). The teacher reverts to a functioning repertoire of procedures of acting and communication patterns, which are internalized by his/her own school and teaching experiences (Bauersfeld 1980) and from which he/she can hardly diverge. The students adapt their behaviour and over the time, teacher and students jointly and interactively produce particular regularities in their behaviour and their way of communicating. Finally, they form steady routines for the behaviour in different classroom situations. So future situations become predictable for the students, the focus of their attention can be conducted to unknown or problem oriented mathematical situations (Voigt 1984; 1990). Thus, routines are important and useful: they are responsible for classroom interaction being as trouble-free as possible.

However, if these routines become patterns, by which teacher and students are so well-attuned to each other that a deeper mathematical understanding is not possible, learning can be delayed. The tripartite exchange (teacher's question – student's response – evaluation by the teacher) as well as the often-discussed funnel pattern described by Bauersfeld (1988) show in their analyses only an unsatisfying mathematical understanding by the students. Often, it is sufficient to follow the signals given by the teacher to react to the teacher's question adequately. Thus, Steinbring demands:

The intended uniqueness of communicative, mutual understanding on the layer of interaction through the funnel pattern or the interactive negotiation of significance has to be absorbed, referring to the epistemological content of mathematical knowledge, by establishing theoretical ambiguity (Steinbring 1994, 205; Translation: AG).

If there are any created possibilities for producing theoretical ambiguity and if they are embedded in the interactive process, discourses are developed, in which mathematical meaning is negotiated and the understanding of mathematical strategies is supported.

### **Contention in mathematical discourse**

Discourses constitute a basic method to solve interpersonal problems of coordination (Miller 2006, 13). The first aim of discourse is that a controversial question, the *Quaestio*, is answered by the persons communicating together (Miller 1986, 143). According to Miller, these negotiations of contention are fundamental for learning. For Kamii (1986) too, conflicts (cognitive as well as social) are central, because social influences probably lead to conflicts. Thus, children have to try to approximate their divergent perspectives and by this learning processes can be initiated. In the presented project episodes from small group discussions are interpreted, in which a *res controversia*, a controversial question or statement, can be observed. The video episodes are selected by sequences in which there is a kind of irritation, initiated by a student or the teacher.

### **The focusing pattern**

In several aspects, the teacher's role is decisive. He has to apprentice, whereas teaching in this case is not meant as transporting subjects, but arranging favourable learning conditions (see also Cooney 1988; Mason 1987):

The teacher's role is seen as providing learning situations in which students have to contribute their own potential for actively reconstructing knowledge, for establishing a personal relationship towards his knowledge (Steinbring 1994, 91; Translation: AG).

Thus, it is in question how new mathematical knowledge can be developed in classroom discourse and how the teacher can make his/her contribution to accompany children on their way to autonomous learning. Bromme et al. (1990, 2f.) speak of an epistemological dilemma in the teacher's role and in his/her activities. The teacher moves steadily in an intermediate field between his/her own knowledge of mathematics science and the already existing students' knowledge. Thus, it is also in question, to what extent teachers are able to dissociate from their own ideas and intentions and to what extent they are open to the children's knowledge structures. However, this openness is equated neither with an unreflective orientation towards the students nor with the exploratory character used in clinical interviews. The teacher's openness is understood with regard to the interpretation of the children's constructions and reasoning attempts.

Wood (1998) characterized this teaching practice with the concept of the *focusing pattern*. She highlights the importance of involving the children in the teachers' process of interaction and structuring: The teacher gives the students the possibility to make the mathematical content as well as making the discourse accessible for them. He provokes them to explaining and reasoning, and to discussing their ideas with other students. Thus, the teacher's intention is different: focusing the students' attention on a critical aspect of the problem, raising a question that turns the discussion back to the students leaving him/her with the responsibility of clarifying the situation (Wood 1994).

For example, this is possible by demanding them to write down his/her statements or reasons. The notation can probably help to clarify his/her own reasoning, because supported by the possibility of pointing to something he/she is perhaps able to explain more decidedly. On the other hand, for the students listening it affords an opportunity to follow up the given explanations and reasons. Additionally, this course creates possibilities of connecting their informal explications with the formal, in their cultural setting. Students start to realize that the symbols have a mathematical meaning. The teacher has to anticipate which aspects are critical or rather controversial, and he/she has to focus on clarifying questions, for example. One aim is to draw the joint attention to important aspects, so that these aspects do not fade out nor that the topic is changed without having been clarified. The students should be aware of the teacher's expectation that they are responsible for understanding the

explanations and reasons given by their classmates and that they are responsible for asking upcoming questions, and thereby creating an opportunity to clarify a mathematical aspect (Wood 1994).

In his investigations of argumentation processes, Schwarzkopf (e.g. 2000) also pointed out that classroom discussions can be structured by the teacher through guiding the students' attention to a reasoning-needed and/or a discussion-worthy statement. Thus, a reasoning need must be demanded explicitly by the teacher. The objects, on which the attention is focused, are jointly produced in the interactive process in spite of different functional parts:

Thus, argumentations are not arbitrarily opened by the teacher, but they arise from the interaction between teacher and students (Schwarzkopf 2000, 588; Translation: AG).

As students should have the opportunity to hold their own point of view, they also should accept others'. In meetings with the teachers being involved in the project ProFIT, an attempt was made to create awareness for this kind of learning as well as to sensitize these teachers to their own routines.

## **RESEARCH QUESTIONS AND FIRST IDEAS OF AN ANALYSIS**

The preliminary analyzing aspects *interaction*, *mathematical content* and the *interface* between them arise from these theoretical considerations:

### **The interaction**

In a first analysis of the discursive context, three different types of context are distinguished: *Report Ways*, *Inquiry* and *Argument*. This distinction is based on a grid developed by Wood and Turner-Vorbeck (2001). They used this grid for structuring and planning discursive contexts on the basis of their own research, experiences, and analyses. The context type *Report Ways* is mainly related to the *comparing* and *contrasting* of mathematical students' solutions in classroom discussions. The *Inquiry*-Type implies *reasoning* and *questioning* based classroom discourse, while for the *Argument*-Type, *justifying* and *challenging* is central. A first pre-analysis within the project ProFIT showed that the grid is useful for a first pre-structuring of the episodes on an interaction level. Using the grid as an instrument for reconstruction, in particular the following questions arise: How does the teacher develop opportunities for different students' responses during the interaction? When does a firstly *reporting* context exceed to an *inquiring* or an *argumentative* context?

### **The mathematical content**

The tasks together with the mathematical learning environments are the indispensable base of autonomy in teaching, but the interface of the teacher's and students' activities are fundamental, too (Bromme et al. 1990, 2f.). As already mentioned in the theoretical part, the fundamental assumption from an interaction-theoretical perspective on teaching is that the objects of discourse should be

“ambiguous”, i.e. open to different interpretations by the participants (Voigt 1994, 77). As it provides numerous opportunities for discourses in classroom, this ambiguity can produce arguments. But this is only possible if the teacher is aware of that ambiguity and if he/she incorporated it as a constitutive element in the mathematical discourse. The richness of mathematical structures is a possible basis for explaining, reasoning and perhaps also for clarifying contentions. Thus, the following research questions are of interest: To what extent does the ambiguous mathematical content comprise something structural in its topic, its presentation, its visualization, etc., what is interpreted in the classroom interaction by the students as well as by the teacher? What kind of understanding of the ambiguous mathematical content becomes explicit by each participant?

### **The interface between interaction and mathematical content**

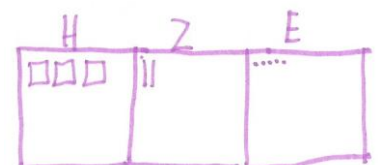
Within an interpretative analysis, neither the mathematical content nor the interaction can be considered in an isolated way. This is because of the teacher’s consideration to create learning situations by maintaining a collective focus on a critical feature of the mathematical content. Thus, the investigation is about how a joint collective focus can be conducted on the mathematical content. Questions related to this analytical focus are: Where is the interface, which determines if students follow a social or a mathematical logic in teaching-learning-processes? Where does the inquiry, which refers to the mathematical content, start; is there even a trigger? And is this particular interface a barrier, which one has to overcome? How far is it possible to interrogate, in a special manner, the fact that contention is clarified by statements related to the mathematical content? Which role do the students’ interpretations, hypotheses and reasons play?

### **ANALYSIS OF AN EXEMPLARY EPISODE: DISCUSSION ABOUT 20 TENS**

The exemplary episode originates from a small group discussion between four fourth graders (Ferdinand, Frank, Kevin and Laura) and their mathematics teacher in primary school during the research project ProFIT. For all involved participants it was the first small group discussion within the research project. The teacher has several years of teaching experience and conducted eight small group discussions during the project.

#### **Choice of the scene – epistemological description of the point of contention**

The episode was chosen because a point of contention develops during the interaction. This controversial point displays the different possibilities for the interpretation of a depiction produced by the pupils during class (see image).



The ambiguity of the representation of the number is the reason for the potential point of contention. The epistemological point of view can be interpreted, according to the point of view, as empirical-concrete or relational. If

the visual depiction with squares, strokes and dots has an *empirical-concrete* interpretation, the concrete use of the working material consisting of a hundreds board, a tens bar and a unit cube come to the fore. These materials, or rather the iconic depiction of the materials is used conventionally and “collected” in the referring boxes – labelled H, Z, E (English: H, T, U). This interpretation of the representation of the number can be described as a classified accumulation of objects in 3 boxes: 3 H-boards, 2 Z-bars and 5 E-cubes.

If the interpretation is *symbolic-relational*, the labelling is not interpreted as names for the boxes but rather depict place values in a place-value-table. In this case the decimal relation between the different place-values is important: 10 E equals 1 Z and 10 Z equals 1 H. So the quantity of symbols in one column of the place-value-table shows how many of the according place values have to be chosen.

This interpretation of the representation of the number can be described as a relational arrangement of units in three columns of place-value-tables: 3 symbols in the H-column, 2 symbols in the Z-column and 5 symbols in the E-column.

Obviously, in the symbolic-relational interpretation there might be the conflict that it has to be abandoned from the “additional” characteristics of the symbols, i.e. the iconic meaning connected with the symbols: hundreds board, tens bar and units cube.

Which forms of the mathematical point of contention can arise in this setting? First, it could be questioned which point of view is appropriate, the *empirical-concrete* or the *symbolic-relational* interpretation? (In a discussion with young pupils the differences are not named systematically and explicitly, but they have to be reconstructed afterwards from the descriptions and explanations of the pupils.) A further mathematical point of contention might be how to deal with the representation of the number as a place-value-table with filled-in symbols: Is the special iconic aspect of signification abandoned, or does it frequently play a role in the discussion and so leads to a different interpretation of the “place-value-table” to “sorting boxes”?

### The exemplary episode

Before the episode starts the children explain why this figure can only represent the number 325 and no other. They explain it only using the symbols square, stroke and dot. After that the following episode starts:

- 1 Teacher: But, like I said, this is not obvious to me. So that...without the place value table, that's clear. Over there I see, ok I know one square we said equals a hundred, doesn't it? ... A stroke you said  
#equals #and a dot equals one.
- 2 Ferdi: #equals ten. #dot. What about?

### Working Group 9

- 3 Teacher: It doesn't equal but it's similar. We agreed on this in the second or third grade. Now, if I see that, ok (*covering the place-value-table*). But now I also have my place-value-table.
- 4 Frank: (..) But there is also written tens. You can say you've got zero tens. For example that the tens mean a zero. It also might be that the tens mean 100. The number over there. (..)
- 5 Kevin: Yes, but when... This is, it is written there. One stroke is a ten and then when we, if there is a ten above it, then we know that stroke needs to be put in there, if there are tens. And therefore a stroke is one ten and so on.
- 6 Teacher: But these are twenty, right? (*pointing on the two strokes*)
- 7 Kevin: Yes, these are twenty.
- 8 Teacher: But twenty tens. And twenty tens are#
- 9 Kevin: #No. Two strokes are two, two strokes are two strokes, which are on the ten's place 20.
- 10 Teacher: But these are twenty tens. (*first pointing at the two strokes than at the Z for the tens*) 20 times 10 don't equal 200?
- 11 Kevin: This is not about multiplication.

### Description of the episode

The teacher says that for her it is not obvious to read only the number 325 in the figure and she refers to the convention introduced in the second and the third grade that one square means 100, one stroke ten and a dot one, without reference to the frame and the letters H, Z and E (in English: H, T, and U). Further, she describes this particular frame and the caption as a place-value-table and points at the combination of both. Frank and Kevin comment on the teacher's contribution. Then, she refers only to the two strokes and asks whether these are 20. Kevin agrees. The teacher refers to the place-value-table and says that they are 20 tens. Kevin negates this. The teacher repeats her statement referring to the Z as a caption for the tens and asks whether 20 multiplied by 10 equals 200. Kevin disagrees referring to the actual act of the arithmetic operation and the topic changes.

### The coping with the mathematical point of contention in the interaction during the episode – a summarizing interpretative analysis

Already at the beginning of the presented episode the teacher indicates a reasonable point and tries to conduct the collective focus to the contention of the combination of frame and letters in terms of an interpretation as place-value-table on the one-hand with the symbols inside on the other hand interactively (She says that the pre-given explanations are not obvious for her (l. 1)). By this, she reveals her own more symbolic-rational mathematical interpretation of the whole presentation, as she

labelled it as a place-value-table (l. 1). Without observing the students' reaction, probable interpretations are: The teacher tries to make her own mathematical interpretation obvious for the students or she is aware of the ambiguity of the mathematical object and tries to conduct the joint attention focus to this point. Related to the interpretation of the symbols inside, she repeats the classroom convention referring to an agreement in passed grades: "We agreed..." (l. 3).

Frank and Kevin take the teacher's statements as a cause for clarifying their interpretations in more detail and react with new explanatory statements for their interpretations (l. 4, 5). The participants do not comment on Frank's statement, wherefore it is not interpreted any more. Kevin "reads" his interpretation of symbols once again: "it is written there" (l. 5). In doing so, his interpretation is bound to everyday life situations. He probably interprets the frame as a kind of case, an assortment box, which has to be filled: "if there is a ten above it, then we know that strokes need to be put in there". What has to be put in is indicated to him by the Z. Probably frame and caption assist in the sorting. He does not indicate seeing any relation to the place-value-table.

This is followed by another question from the teacher, in which she interprets the two strokes differently. She considers the two strokes isolated and describes them as 20, instead of two tens. She asks whether the students agree that these two strokes are 20. In her statement she does not mention the two strokes explicitly, but with "these" she gives a deictic reference and points on the two strokes, so that this interpretation seems to be plausible (l. 6). Kevin agrees on the convention that one stroke is a tens (l. 5) and two are twenty (l. 7). So, from an interactional point of view, these first lines of the episode can be labelled as an inquiry context.

Related to the mathematical content, the ambiguity of such a representation becomes obvious. The teacher has a rather symbolic-relational point of view. She probably is aware of the ambiguity and tries to conduct it to the supposedly joint attention focus. Kevin has a more empirical-concrete point of view and for him only one interpretation seems to be possible: the interpretation of a kind of case, a kind of assortment box.

Initiated with the word "No" in line 9 Kevin disagrees and contradicts the teacher's statement, and discussion becomes more challenging. For him, the two strokes lying on the tens' place are 20, the quantity of strokes (two) lying on the tens' place is crucial. Here, he changes his reasons. His statement is not consistent with his statements before. He uses a more symbolic-rational understanding, without taking the place-value-table explicitly into consideration. This was not carried out at this point; the teacher does not follow this interpretation up.

The teacher's introducing "But" in line 10 also gets another, more justifying emphasis. The discussion's context changes more to the direction of an argumentative context (l. 8-11). She repeats her general statement in line 8 related to

a concrete reference to the figure (l. 10, once again deictic, supported by pointing on the figure) and connects the presentation of strokes with the Z by using multiplication, analogous to the place-value-table. She puts into question whether 20 times 10 equals 200. For Kevin it is not to be debated whether 20 times 10 equal 200, at least he does not react to it. But he does not accept the operative connection between 20 and 10 in the form of multiplication, which for him seems not to be permitted at this moment (l. 11). For him, the contention is originated in the interaction on the mathematical content. The presentation of the number 325 is not controversial for him, but the teacher's statements are worth discussing. The teacher's constant attempt to focus on another possible interpretation of the mathematical content fails (l. 11); the discussion moves onto another topic.

### **Specific communicative elements within the cause of small group interaction**

If one follows the special communication in dealing with points of contention in this setting it can be seen that the teacher tries different measures to put the point of contention into the focus of all participants; i.e. she initiates the point of contention. First, she questions the obviousness of the children's explanations (l. 1). Then, she tries to focus on the controversial point by referring to known conventions and to contrast different points of view by hiding and showing the labels (l. 3). Later, she tries to provoke the pupils with her remarks by bringing single elements of the representation into discussion again (ll. 6-10).

In the exemplary episode the pupils deal very differently with the teacher's strategies. First, they feel asked to give new explanations (Frank, l. 4; Kevin, l. 5). Next, only Kevin communicates with the teacher. Here, Kevin agrees on some of the teacher's remarks and contradicts others. They do not reach a consensus.

### **CONCLUDING REMARKS**

The interpretations at that early time of the research project led to the conjecture that it is possible to initiate and discuss mathematical contention with children at the elementary school age, but it is challenging to focus the attention on special aspects of the interpretation. In the exemplary episode, the pupils engage in a new way of communication and discussion, but deal with it in different ways. The engagement in the point of contention and the arguments seem to be difficult and unfamiliar.

The research questions mentioned in the article are the leading questions for the presented research, which cannot be answered yet. So it is still in question as to whether and to how far it is possible to interrogate in reference to the mathematical content. This barrier could not be overcome in this episode. So for the teacher's role two challenges become obvious: On the one hand the release of his/her own interpretation of mathematical signs and symbols and on the other hand the teacher's consideration to create a joint collective focus interactively, including the discourse about a cooperatively identified contention. The interest in this research has been



shown through the discussed episode and points out the difficulties of a change of the established classroom discourse to new forms of discourse. It is the concern of the research project ProFIT to work out more exactly what and where the key points discussed above are.

## REFERENCES

- Bauersfeld, H. (1980): Hidden Dimensions in the So-Called Reality of a Mathematics Classroom. In: *Educational Studies in Mathematics*, Vol. 22, No. 1, 23-41.
- Bauersfeld, H. (1988): Interaction, Construction, and Knowledge: Alternative perspectives for mathematics education. In: Cooney, T. J. et al. (Eds.): *Perspectives on Research on Effective Mathematics Teaching*. Volume 1. Reston: NCTM, 27-46.
- Bromme, R. et al. (1990): Aufgaben, Fehler und Aufgabensysteme. In: R. Bromme et al.: *Aufgaben als Anforderungen an Lehrer und Schüler*. Köln: Aulis, 1-30.
- Cobb, P. et al. (1991): Small-group interactions as a source of learning opportunities in second-grade mathematics. In: *Journal for Research in Mathematics Education* 22: 390-408.
- Cooney, T. J. (1988): The Issue of Reform: What Have We Learned from Yesteryear. In: *Math. Teacher* 81(5): 352-63.
- Kamii, C. K. (1986): *Young Children Reinvent Arithmetic: Implications of Piaget's Theory*. New York: Teachers college press.
- Mason (1987): Only awareness is educable. In: *Math. Teacher* 120: 30-31.
- Mehan, H. (1979): *Learning lessons: Social Organization in the Classroom*. Cambridge, MA: Harvard University Press.
- Miller, M. (1986): *Kollektive Lernprozesse*. Frankfurt am Main: Suhrkamp.
- Miller, M. (2006): *Dissens. Zur Theorie diskursiven und systemischen Lernens*. Bielefeld: Transcript.
- Schwarzkopf, R. (2000): *Argumentationsprozesse im Mathematikunterricht. Theoretische Grundlagen und Fallstudien*. Hildesheim: Franzbecker.
- Söbbeke, E. (2005): *Zur visuellen Strukturierungsfähigkeit von Kindern in der Grundschule*. Hildesheim: Franzbecker.
- Steinbring, H. (1994): Frosch, Känguruh und Zehnerübergang – Epistemologische Probleme beim Verstehen von Rechenstrategien im Mathematikunterricht der Grundschule. In: H. Maier et al.: *Verstehen und Verständigung*. Köln: Aulis, 182-217.

## Working Group 9

- Steinbring, H. (2005): *The Construction of New Mathematical Knowledge in Classroom Interaction. An Epistemological Perspective*. New York: Springer
- Voigt, J. (1984): *Interaktionsmuster und Routinen im Mathematikunterricht. Theoretische Grundlagen und mikroethnographische Falluntersuchungen*. Weinheim: Beltz.
- Voigt 1990: Mehrdeutigkeit als ein wesentliches Moment der Unterrichtskultur. In: *Beiträge zum Mathematikunterricht*. 305-308
- Voigt, J. (1994): Entwicklung mathematischer Themen und Normen im Unterricht. In: Maier, H. et al: *Verstehen und Verständigung*. Köln: Aulis, 77-111
- Wood, T. (1994): Patterns of interaction and the culture of mathematics classrooms. In: Lerman, S. (ed.): *Cultural Perspectives on the Mathematics Classroom*. Dordrecht, NL: Kluwer Academic Publishers, 149-168.
- Wood, T. (1998): Alternative Patterns of Communication in Mathematics Classes: Funneling or Focusing? In: Steinbring, H., Bartolini Bussi, M. G., Sierpiska, A. (eds.): *Language and Communication in the Mathematics Classroom*. Reston: National Council of Teachers of Mathematics, S. 167 – 178.
- Wood, T., Turner-Vorbeck, T. (2001): Extending the Conception of Mathematics Teaching. In: Wood, T., Nelson, B. Scott, Warfield, J. (eds.): *Beyond Classical Pedagogy: Teaching Elementary School Mathematics*. Mahwah: Lawrence Erlbaum Associates, S. 185 – 208.

# TURN-TAKING IN THE MATHEMATICS CLASSROOM

Jenni Ingram, Mary Briggs and Peter Johnston-Wilder

University of Warwick, UK.

*This paper focuses on turn-taking in secondary mathematics classrooms. Sixteen lessons in four mathematics classes for pupils aged 12-13 years were observed and video recorded and then analysed using a conversation analysis approach. The analysis of the data reveals differences in the patterns of turn-taking from those discussed in the thorough analysis by McHoul (1978). In particular, three types of situation arise where pupils self-select as next speaker: to ask a question; to initiate or perform a repair; and finally to respond to an undirected teacher's question. The last of these in particular raises some important questions about the teaching and learning of mathematics.*

## INTRODUCTION

Classroom interactions between pupils are widely held to be an important feature of the mathematics classroom, and a large number of existing studies have explored specific features such as how questions are asked and answered, and when and how pupils participate (Sfard & Kieran, 2001; Yackel & Cobb, 1996). The aim of these studies has often been to develop an understanding of the relationship between these interactions and learning. We do not yet know what interactional features of classroom discourse result in successful learning, though we may have strong opinions about the value of explanation, argumentation and discussion. We do however know that the structure of interactions play an important role in how pupils perceive and learn mathematics (Wood, Williams, and McNeal, 2006). There are also observable differences between pupils in the extent to which they participate in whole class discussions (Ball 1993; Boaler, Wiliam, & Brown, 2000). The vast majority of these existing studies focus on the content of the interactions, particularly on how teacher's support or scaffold pupil's interactions. In this study, we examine instead the structure and local management of turn-taking which constrain the content of interactions.

## THE ALLOCATION OF TURNS

Turn-taking has received considerable attention in the research literature (Maroni, Gnisci, & Pontecorvo, 2008; Sinclair & Coulthard, 1975), in particular as one of the notable differences between natural conversational discourse and classroom discourse. There are rules that govern who can speak, for how long they can speak for and also constrain what can be said. Whilst there are rules that characterise all conversational interactions, there are often more constraints on formal interactions, such as those that take place in classrooms. These rules are often unsaid, yet by analysing interactions between participants we can find evidence that these rules are

being oriented to by participants; this is particularly clear when these rules are violated (a deviant case) and participants are sanctioned.

In this paper, we particularly focus on the systematic rules that govern whole class discourse. Sacks, Schegloff and Jefferson (1974) outline the rules for the construction of turns in natural conversation and McHoul (1978) develops these for formal classroom discourse. Features common to both systems include the rules that only one speaker speaks at a time and that a change of speaker occurs and often reoccurs. Where natural conversation and classroom discourse differ is in the allocation and management of speakers.

In Sacks et al.'s (1974) rules for governing speaker change in natural conversation, at a point where it would be relevant for the speaker to change (a transition-relevance place), the current speaker can select the next speaker and thus the turn belongs to the selected participant. If the current speaker has not selected the next speaker, then another speaker can self-select. If neither of these two scenarios occur, the current speaker can continue the turn. These rules apply to any transition-relevant place. McHoul's (1978) analysis of discourse from geography classrooms results in an adaptation of these rules, highlighting the different roles of the teacher and the pupils.

When the teacher is the current speaker, then the teacher can select the next speaker. If the teacher does not select the next speaker, then the teacher must continue the turn. When a pupil has been selected by the teacher to speak and becomes the current speaker, they then can select the next speaker and the teacher takes the next turn. (It is worth noting that in McHoul's data there are no instances where the pupil selects anyone except the teacher as next speaker). If the pupil does not select the teacher as next speaker, then other speakers can select as next speaker with the teacher having the right to self-select first. If the teacher does not self-select at the transition relevance place then the current speaker (the pupil) may continue.

McHoul's systematic analysis of turn-taking makes it clear that the pupils are restricted in their roles in the local management of turn-taking and that it is the teacher who controls and manages the allocation of turns. The teacher participates in the interactions with some preformed idea of what should be said and done during the lessons; they have a lesson plan that the pupils are expected to follow. Discussion occurs within well-established norms of participation between teachers and their pupils (Yackel & Cobb, 1996). Furthermore, the interactions often begin with some additional instructions to the pupils about the nature of their participation that will be considered acceptable, such as "put up your hands" or "you need to explain why". These rules which govern turn-taking in the mathematics classroom determine the ways in which different activities are performed. For example, the teachers' control of turn-taking determines the topic of interactions.

The rules which govern turn-taking in natural conversation provide an “intrinsic motivation for listening” (Sacks et al. 1974, p.43). Participants need to monitor the interaction for possible transition relevance places for opportunities to take the next turn and for information concerning what will be acceptable in that next turn. As turns are pre-allocated, then this need to listen and monitor what is being said is lost. Teachers can circumvent this problem to some extent by nominating the next speaker at the end of their turn, but more frequently the next speaker is selected following the pupils bidding for the next turn by raising their hand.

Turns can be allocated by the current speaker nominating by name, gesture or by eye gaze. They can also be more generally solicited where the next speaker is not selected but the current turn has been constructed such that it places constraints on the next turn. In this situation, it is possible that more than one participant may take the next turn, violating the rule that only one participant speaks at a time. The number of participants in a classroom context means that some form of allocation of turns is needed. Traditionally the teacher nominates the next speaker, again often following bidding for a turn by the pupils raising their hands.

## THE STUDY

The data for this paper is taken from a collection of lessons video recorded in the UK. This collection includes a total of sixteen mathematics lessons delivered by four different teachers. For each teacher, one class of 12-13 year olds is video recorded between three and six times over a period of six weeks. The extracts presented below each feature just one of these teachers, though with only one exception, the accompanying discussion applies to all the teachers in the sample.

The videos were transcribed using Jefferson’s (2004) system, with [ ] indicating overlapping speech, (1.1) indicating a pause of 1.1 seconds and underlining indicating emphasis. These transcripts were analysed alongside the video recordings using a conversation analysis approach (Seedhouse, 2004), focusing on the local turn-by-turn organisation of the interaction. This means that the analysis is based on what is evidenced as relevant to the participants through their interactions. In other words, “analysing the interpretations that participants display, rather than creating their own” (Barwell, 2003, p. 201). Consequently, social identities and roles, such as pupil or teacher, are only attributed to speakers where it is clear from the interaction that the participants themselves are orienting themselves to these roles and identities. In this paper, the roles of teacher and pupil are used as the allocation of turns, number of turns and length of turns all indicate that the participants are adopting the roles of teacher and pupil in these interactions.

For the purposes of this paper, only whole class interactions are analysed. Interactions that occur when the pupils are working as individuals or in small groups are ignored.

## ANALYSIS

McHoul's analysis does not include pupils selecting the next speaker, as in his data this is always the teacher; McHoul also claims that a pupil only speaks when the teacher has allocated the turn to them. In this paper, we examine instances in mathematics lessons where pupils have self-selected as next speaker and this is not sanctioned by the teacher or by other pupils. Within these instances, there are also occasions where pupils have selected another pupil as next speaker.

There are three types of situation in the data where pupils self-select as next speaker. The first of these is when pupils are asking their own questions.

- 231 Teacher: one in eight. ok. if I cancel them down, that and that cancels. that and that cancels I'm left with (0.7) a tenth. so,
- 232 Pupil1: how do you know that cancels with that
- 233 Teacher: how do you know that this cancels down
- 234 Pupil1: yeh
- 235 (1.1)
- 236 Teacher: if I multiplied it out you'd see tha- that (0.3) I have a factor of eight on the top and a factor of eight on the bottom.

In these situations, the pupil is also changing the topic of the interactions. In the current data, pupils rarely ask questions during whole class interactions; the number of questions asked in these instances only ranged from zero to two. On each occasion, the pupil's question is asked at a transition relevance place, so that the teacher's current turn is not interruption. In the above example, the teacher often uses the discourse marker 'so' to mark a change in the topic of conversation, thus whilst offering a gap in which an opportunity for the current speaker to change exists, it is clear that the teacher is intending to keep the current turn.

The second situation is when the pupils are initiating or performing a repair (see McHoul, 1990 for more detail on repair in classrooms).

- 29 Teacher: one in ni:ne. so has that gone up or gone down
- 30 Pupil1: gone down
- 31 Teacher: probability's gone
- 32 Pupil1: u[p ]
- 33 Pupil: [up]
- 34 Pupil: up
- 35 Teacher: probability's gone up, it's more likely now that you're going (.) to get (0.3) the (.) red cross. so Miles (.) choose one

In the above extract, both of the teacher's turns are directed at pupil1, whose immediate answer, 'gone down', is incorrect. Whilst the teacher offers pupil1 the opportunity to self correct by re-asking the question and gesturing the correct

## Working Group 9

answer, his peers also self-select to give the correct answer. These situations are very rare in the data and the above extract is the only example where the pupils are not sanctioned for taking the turn. However, in this case, the pupil who had the right to the turn also gave the ‘correct’ answer and the teacher maintains eye contact with this pupil throughout the exchange, thus the other pupils have been ignored.

The final situation where pupils self-select is in response to an undirected question from the teacher. This occurs whenever the teacher requires a response but has not selected the next speaker, either by name or gesture.

131 Teacher: what’s the probability?

132 Pupil1: a hal[f ]

133 Pupil2: [[a ha]lf ]

134 Pupil3: [fifty fifty]

135 Teacher: fifty fifty, a ha:lf, good.

In the extract above, three pupils offer an answer, virtually simultaneously, and the teacher accepts both variations of the answer offered. In the extract below, both correct (seven) and incorrect answers are given:

176 Teacher: what total will go (5.4) ((draws diagonal boxes on the board))

177 Pupil1: seve[n ]

178 Pupil2: [si ]x

179 Teacher: what total would go diagonally across the [board]

180 Pupil3: [six ]

181 Pupil4: seven

182 Teacher: seven good. seven would be the most likely and ...

In both these situations, the pupils have the right to speak as the teacher has elicited a response yet no specific pupil has been nominated to speak. Any or all of the pupils can choose to answer, possibly simultaneously. This seemingly violates the rule that only one participant may speak at once. However, these types of teacher questions are usually followed in the data by simultaneous ‘correct’ responses suggesting that they are used when a large number of the pupils, but not necessarily all, are expected to know the answer. Consequently, if we view the interaction as between the teacher and the class of pupils as one participant, the rule is not violated. Additionally, the control over the interactions by the teacher is maintained.

Where different pupils offer different answers, as in the second extract above and the extract below, these are often given so that a different answer ‘interrupts’ a previous answer. In the extract below, the two contrasting answers result in a discussion, lasting over three minutes, in which several pupils self-select as next speaker. The pupil turns with an asterisk are the only pupil turns where the teacher has selected the pupil as the next speaker.

### Working Group 9

- 255 Teacher: which ticket would you prefer?
- 256 Pupils: second one
- 257 Teacher: why
- 258 Pupil: cause it's got
- 259 Pupil1: it doesn't [matter ]
- 260 Pupil: [it doesn't] matter
- 261 \*Pupil1: because ((inaudible)) you've still got a chance of winning, cause those numbers could come out.
- 262 Teacher: but which one's got a greater chance of winning.
- 263 Pupils: same one/they've both got the same chance/the second one/same
- 264 Pupil: they're both the same
- 265 Teacher: good.
- ...
- 270 Teacher: ok. both these this is the thing about probability people (0.4) don't necessarily always think mathematically they think about (.) what they see. these have exactly the same chance of winning on the lottery
- 271 Pupil: yeh
- 272 Pupil: yeh [but ]
- 273 Teacher: [exact]ly (.) the same chance.
- 274 Pupil2: it's not a very good ch[ance. ]
- 275 Teacher: [what?]
- 276 \*Pupil2: it's not a very good chance
- 277 Teacher: for both of them equally
- 278 \*Pupil2: for that (.) one two three four five six
- 279 Teacher: it has exactly (.) the same chance (.) of winning
- 280 \*Pupil2: but there's more chance of that one winning cause there's a different array of [numbers]
- 281 Teacher: [ok ] ((inaudible)) listen again. ...
- ...
- 296 Pupil2: that one's got more chance
- 297 Teacher: why
- 298 \*Pupil2: because it's a different variety of numbers
- 299 Pupil: and
- 300 Pupil: don't and him
- 301 \*Pupil2: you never ever get one two three four five six. [you migh]t get three [((inaudible))]



Here, the pupils are initially asked which lottery ticket they would prefer, one with the numbers one, two, three, four, five and six or one with the numbers three, seven, twelve, thirty eight, thirty nine and forty six. This initial question is not specifically directed at a particular pupil and several pupils give the answer “the second one”. The teacher’s following question, “why” continues to be directed at the class as a whole and three different pupils offer an answer.

So far, all the pupils have had the right to speak and stop talking as soon as the teacher begins to gesture to the pupil in line 261 that the turn is theirs. This offers further support for the suggestion that the rules that only one participant can speak at a time is not in fact violated if we treat the pupils in the class as one participant. Additionally, McHoul’s assumption that pupils do not self-select is also not contradicted as the question was directed to the class and the class has responded. However, in line 299 a pupil, not only self-selects as the next speaker but also her turn is directed towards pupil2 and a response is expected from pupil2. So, here is an example where a pupil has both self-selected for the current turn and selected the next speaker which is not the teacher.

This extract is then followed by several pupils both talking concurrently and directing their turns towards other pupils. Unfortunately, several pupils talking concurrently makes the content undecipherable for transcription. However, whilst these interactions clearly deviate from the rules governing turn-taking in the formal classroom environment (McHoul 1978), close examination of the video indicates that the pupils are not in fact violating the rules for turn-taking in natural conversation (Sacks et al. 1974). Whilst there are several pupils talking at once, their talk is not directed to the class as a whole but to an individual pupil elsewhere in the class.

We would argue that there are now several interactions occurring simultaneously, each with the same topic. Analysing groups of interacting pupils individually, the rules of turn-taking are adhered to. For example, observing the two pupils who spoke in lines 299 and 300, their later turns are directed towards each other, there is minimal overlap in their turns, with one cutting off their turn if they both begin talk at the same time, and the gaps between their turns are minimised. The two pupils compete for the floor and attempt to persuade the other that their answer is ‘correct’ whilst not violating the rules for turn-taking. Here the roles of teacher and pupil are not evident until the teacher takes a turn and the pupils end their turns.

In all the lessons in this study, turn-taking is tightly controlled by the teacher. All the teachers in this study control the topic through their turns and the questions that they ask and the pupils are constrained to answer these question (Barwell, 2003, p. 205). Many of the answers rely on pupils remembering facts, procedures or tasks undertaken in previous lessons. A few questions require pupils to offer explanations for their previous answers. The teacher from whom the extracts in this paper have been taken differs from other teachers in the study in that the majority of questions asked by him in the lesson are not followed by the nomination of a pupil to answer;

instead, they are directed to the class as a whole. Where questions are not directed at a particular pupil (or small group of pupils) the possibility arises for multiple respondents and multiple answers. In the lessons of the teacher discussed in this paper, when different answers are offered by different pupils an argument between pupils often ensues; the teacher often encourages these debates by asking the pupils to justify their answers. In these arguments, each pupil offers justification for their own particular answer and attempts to convince other pupils that their own answer is correct. Whilst pupils in all the lessons in the study are remembering, describing and explaining during whole-class interactions, only in these arguments are pupils using mathematical argument to justify, refute and convince others (Barwell, 2003; Mason, Burton, & Stacey, 2010). In this study, such discussions occur only in the lessons of the teacher used in this paper, though they do occur in the majority of his lessons.

We know from Mercer and Littleton's (2007) work on the *Thinking Together* approach that the classroom environment and culture have a significant impact on the extent to which pupils will dispute or disagree with their peers. Further analysis of these discussions from a conversation analysis approach is needed to identify which features of the classroom environment and culture the pupils are in fact orienting to.

## CONCLUSIONS

In this paper we have outlined three types of situation in mathematics classroom interactions which are not considered by McHoul's (1978) systematic analysis of turn-taking. All three situations involve the self-selection of a pupil as next speaker. The first occurs when pupils ask their own questions, something we as teachers ourselves value in our lessons. If we consider the self-selection of a pupil in order to ask questions within McHoul's rules of turn-taking, some interesting questions are raised about the pedagogic relationship between encouraging pupils to ask questions and the implications this might have on classroom management. Pupils also self-select to initiate or perform repairs, either on the teacher's previous turn or a peer's turn and repair in mathematics classrooms is an area of further investigation elsewhere.

Finally, pupils self-select in order to respond to undirected teacher questions. We make the suggestion that when teachers ask questions that require a response but an individual is not selected to make this response, by treating the whole class as a single participant, the rules that govern turn-taking in natural conversation are in fact not violated.

In the situations where self-selection by different pupils results in an 'argument' or a point of contention (Gellert, 2011) between pupils raises further questions. It is clear from the videos and the transcripts that there is something qualitatively different in the justification given by pupils when they are attempting to persuade a peer in an 'argument' that their answer is correct, than when a teacher has asked them to justify

their answer. The motivation to provide a justification is intrinsic to the pupil when they need to convince a peer, and is often done passionately, as can be seen from the emphasis placed on the words in the transcript above. When justifying an answer to a teacher, the motivation often appears to be extrinsic as the justification is given in response to a request from the teacher. We believe that these ‘arguments’ are a positive feature of mathematics classrooms but are they actually effective in terms of learning. Is there any difference in the understanding and the depth of mathematics considered as a result of these discussions compared to the more ‘traditional’ interactions between the teacher and their pupils?

## REFERENCES

- Ball, D.L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal* 93: 373-397
- Barwell, R. (2003). Discursive psychology and mathematics education: Possibilities and challenges. *ZDM*, 35(5), 201-207.
- Boaler, J., Wiliam, D., & Brown, M. (2000). Students’ experiences of ability grouping: Disaffection, polarisation and the construction of failure. *British Educational Research Journal*, 26, 631-648.
- Gellert, A. (2011). Contention in mathematical discourse in small groups in elementary school teaching. *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* February 9<sup>th</sup> -13<sup>th</sup>, Rzeszow, Poland.
- Jefferson, G. (2004). Glossary of transcript symbols with an introduction. In G. H. Lerner (Ed.), *Conversation analysis studies from the first generation* (pp. 13-31). Amsterdam/Philadelphia: John Benjamins.
- Maroni, B., Gnisci, A., & Pontecorvo, C. (2008). Turn-taking in classroom interactions: Overlapping, interruptions and pauses in primary school. *European Journal of Psychology of Education*, 23(1), 59-76.
- Mason, J., Burton, L., & Stacey, K. (2010). *Thinking Mathematically* (Second Edition ed.). Harlow England: Pearson Education Limited.
- McHoul, A. (1978). The organization of turns at formal talk in the classroom. *Language in Society*, 7, 183-213.
- McHoul, A. (1990). The organization of repair in classroom talk. *Language in Society*, 19, 349-377.
- Mercer, N., & Littleton, K. (2007). *Dialogue and the development of children’s thinking: A sociocultural approach*. London; New York, NY: Routledge.
- Sacks, H., Schegloff, E.A., & Jefferson, G. (1974). A simplest systematic for the organization of turn-taking for conversation. *Language*, 50, 696-735.

### Working Group 9

- Seedhouse, P. (2004). *The interactional architecture of the language classroom: a conversation analysis perspective* (Vol. 51). Oxford: Wiley-Blackwell.
- Sfard, A. & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture and Activity*, 8(1), 52-76.
- Sinclair, J.M., & Coulthard, R.M. (1975). *Towards an analysis of discourse: The English used by teachers and pupils*. London: Oxford University Press.
- Wood, T., Williams, G. And McNeal B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, 37, 222-252.
- Yackel, E. and Cobb, P. (1996) Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 2, 458-477.

# COMMUNICATING EXPERIENCE OF 3D SPACE: MATHEMATICAL AND EVERYDAY DISCOURSE

Candia Morgan<sup>1</sup> and Jehad Alshwaikh<sup>2</sup>

Institute of Education, University of London, UK<sup>1</sup> and Birzeit University, Palestine<sup>2</sup>

*In this paper, we consider data arising from student-teacher-researcher interactions taking place in the context of a teaching experiment making use of multiple modes of communication and representation to explore three-dimensional shape. As teachers/researchers attempted to support student use of a logo-like formal language for constructing 3D trajectories in a computer microworld, a system of gestures emerged to represent the various types of turn in 3d space. We focus on the ways students coordinated the multimodal resources available in the classroom. In particular, we discuss how this system of gestures was adopted and adapted by teachers and students, drawing on both mathematical and everyday discourses.*

Keywords: everyday discourse, gesture, multimodality, three dimensional space

## INTRODUCTION

As human beings living in a three-dimensional world, we continually experience shape and motion within that world. Yet the mathematical description and analysis of these aspects of our experience appear to be exceptionally difficult for learners. In particular, we note issues identified by research in relation to identification and operation with angles (Clements & Battista, 1992) and recognition of connections between the physical contexts of corner, turn, and slope (Mitchelmore & White, 2000). A common approach to explaining such difficulties has been to identify conflicts between students' intuitions, built on their everyday experiences, and formal mathematical definitions of such concepts and operations on them.

Our experience of the world, however, is not purely physical but is mediated by language and other semiotic resources (Vygotsky, 1986). Mathematics teachers attempt to structure the mediating resources available to students by providing forms of language, visual and physical resources, digital technologies, etc. that are designed to offer mathematical ways of experiencing the world. In this paper, we draw on data from a teaching experiment, part of the ReMath project<sup>1</sup>, focussing on three-dimensional shape. The sequence of lessons was designed to provide students with multiple modalities of mediating resources. We aim to consider the ways students coordinated the resources provided in the classroom with those brought from their everyday discourses. In particular, we focus on the development and use of a system of gestures to represent the various types of turn in three-dimensional space.

## **EVERYDAY AND MATHEMATICAL LANGUAGE**

Although mathematical discourse is perhaps most notably characterised by its use of formal notations and uniquely mathematical vocabulary, this is only part of the story. As in many other specialised domains, the mathematics register incorporates considerable use of what may be termed ‘ordinary’ or everyday language in order to make mathematical meanings (Halliday, 1974). While some parts of this ‘ordinary’ language are used in ways compatible with everyday usage, others diverge subtly or, in some cases, radically. Rather than interpreting failure to use such language in ‘correct’ mathematical ways as evidence of lack of mathematical understanding, it is possible to see it as a consequence of students continuing to make use of ‘everyday’ linguistic patterns in circumstances when specialised usage is more appropriate. Students draw on the resources of an everyday discourse while their teachers expect them to be situated within a specialised school mathematics discourse.

## **MULTIMODALITY – DIFFERENT MEANING POTENTIALS**

The bulk of existing research into mathematical communication has focused on the semiotic systems of language and algebraic notation. Research concerned with students’ use of diagrammatic, graphical or gestural forms has tended to approach these as vehicles for access to students’ ‘understanding’ or ‘representation’ of mathematical concepts rather than as forms of communication (though there are exceptions to this, e.g. Chapman (2003), Alshwaikh (2010)). There is, however, a major trend towards recognising the multimodal nature of communication and the importance of studying the contributions made by different modes of communication and representation. Each of the various available semiotic systems provides a different range of meaning potentials (O'Halloran, 2005). In investigating the meanings that students make within such a multi-semiotic environment, it is thus important to consider their use of all these modes and the relationships between them.

Moreover, wherever mathematical communication takes place in face-to-face contexts, gesture and actions also play a part. Kress, Jewitt, Ogborn & Tsatsarelis’ (2001) multimodal analysis of communication in science classrooms shows teachers and students making use of a “complex ensemble” of modes, including gesture alongside speech, writing, images, etc. There has been recent research interest in the use of gestures in mathematics teaching and learning. Much of this has focused on the gestures used by students, analysing the contribution made by gesture to learning and mathematical meaning making (e.g. Radford, 2009). In considering gestures used by teachers, studies have shown teachers and students making shared use of gestures initiated by student communication efforts (Arzarello, Domingo, Robutti, & Sabena, 2009; Maschietto & Bartolini Bussi, 2009) and teachers using deictic gestures as mediating resources (Bjuland, Cestari, & Borgersen, 2009).

In this article, we consider the evolution and use of a system of gestures for communication about movement in three-dimensional space, focussing on how these gestures related to other semiotic systems in use. Our particular interest is in how students adopted a system of gestures offered by the teacher/researcher team.

## EMERGENCE OF A NEW SEMIOTIC SYSTEM: PLAYING TURTLE IN 3D

The episodes we discuss here arose during a teaching experiment involving a multi-semiotic interactive learning environment, MachineLab Turtleworld (MaLT). This environment, designed by the University of Athens Educational Technology Lab (ETL) ReMath project partners, incorporates a 3D turtle geometry, driven by a Logo-like language (see Figure 1). It also includes variation tools for direct manipulation of variables, though we do not discuss this component of the software in this

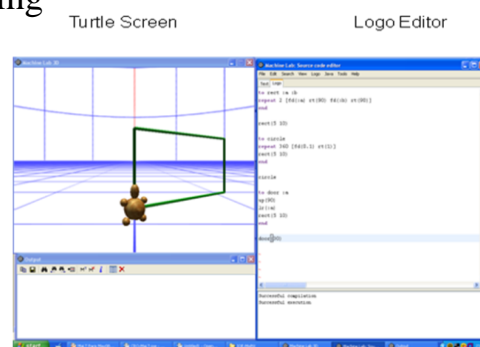


Figure 3: MaLT screenshot

paper (see Kynigos & Latsi, 2007). The pedagogical plan used in the London-based teaching experiment was designed to allow us to investigate the meanings students would make in relation to 3D geometry through their semiotic activity in the multimodal context. In addition to the resources offered by MaLT itself, the social environment of the teaching experiment was intended to allow, and indeed encourage, communication through talk and various paper-and-pencil based forms of representation and the use of physical manipulatives as well as through the computer software itself.

The teaching experiment was conducted in a state secondary school in London with a Year 8 class (aged 12-13 years). The students had no previous experience with MaLT or other forms of Logo. A sequence of nine lessons was taught collaboratively by the class teacher, the researchers and a student teacher attached to the class. In each lesson a video record was made, focusing on the teacher or researcher during whole class interaction and on a selected student or group of students during individual or group tasks. The video aimed to capture gestures and the various visual and physical resources available, including the computer screen when in use. Microphones similarly captured teacher talk and most student contributions during whole class interactions and talk within a group of students or between students and teacher during group or individual work. Episodes in which use of multiple semiotic modes was evident were selected for transcription. See Morgan & Alshwaikh (2009) for details of the transcription methods used with such multimodal data.

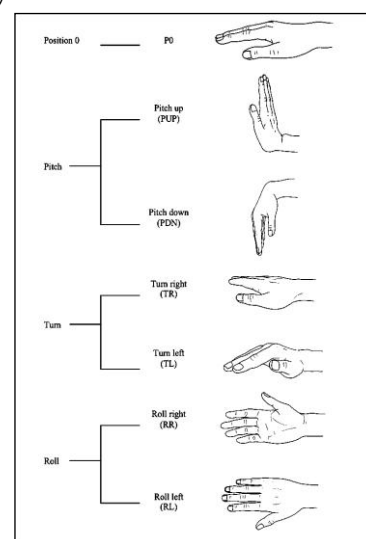


Figure 4: MaLT gesture codes

As we started to view the video data collected during use of MaLT, it was noticeable that the teachers and researchers made extensive use of gestures in an apparent attempt to support students' planning and execution of constructions. One significant type of gesture was a set of stereotyped hand and arm movements, often associated with use of the terms turn, pitch and roll and the associated Logo instructions (see Figure 2). This set of gestures constitutes a new semiotic system, linked with, but not identical to, both the linguistic description of movement and the symbolic system of Logo. They may be considered iconic gestures (McNeill & Levy, 1980; Roth, 2001), in that each bears a visual resemblance to the anticipated trajectory of an object moving in 3D space (or a turtle moving in the simulated 3D space of MaLT). Students also made use of these and other gestures to support their communication about turtle movement. Although the students used 'these' gestures to indicate that their hand and arm movements resembled those used by the teachers/researchers, we believe, as will become apparent, that the students made use of them in different ways, thus construing different meanings.

For the teachers and researchers, using these gestures as ways of thinking and communicating seemed to emerge as a natural consequence of our experience with two-dimensional versions of Logo. In Papert's seminal *Mindstorms* (1980), he argued that turtle geometry is useful for learning because it is body syntonic, "firmly connected to children's sense and knowledge about their own bodies" (p.63). This connection to personal bodily knowledge may be operationalised through 'playing turtle', either literally by walking along a path, enacting the instructions given to the turtle, or metaphorically in the imagination. Encouraging and supporting students to 'play turtle' has become a standard part of Logo pedagogy. The metaphor of 'playing turtle' thus formed part of our experience of 'Logo culture' and constituted for us a more or less implicit theory about learning with Logo.

Our partners ETL incorporated the idea of body syntonicity as an explicit theoretical justification for their own pedagogical plan, implemented in Athens, suggesting manipulating a model of an aeroplane to help students connect with the 3D logo commands (<http://remath.itd.cnr.it>). We adopted a similar initial activity in introducing MaLT to London students, using a model aeroplane to demonstrate a trajectory of turns and moves as described in Episode 1 below. We then substituted gestures (hand and arm movements without holding a model) for the movement of the aeroplane and incorporated these into our communications about three-dimensional movement throughout the teaching experiment. As students moved on to drawing 3D objects, no longer connected with the context of aeroplane trajectories, the hand and its movements came to be used as representations of the Logo turtle.


We now present two episodes from the teaching experiment in which the teachers and researchers modelled use of gestures to 'play turtle'. Then we present an analysis



of an episode of a student's use of similar gestures. Finally, we discuss differences in the ways gestures were used by teachers/researchers and by the students.


**Episode 1:** In the introductory session with MaLT, the first author (CM) introduced the notion of turtle movement using a toy aeroplane. She accompanied the physical movement of the aeroplane with a verbal description, using and stressing the terms pitch (up/down), roll (right/left) and turn (right/left) in synchrony with the associated movement. After this initial introduction, students were set the task of using the 3D Logo language to construct the trajectory of an aeroplane taking off. On observing how we and other members of the teacher/researcher team supported students' attempts at this task, we noticed that we made use of iconic gestures in which the movement of the hand resembled the desired movement of the 'aeroplane'/3D Logo turtle. Sometimes a gesture was used synchronously with an 'equivalent' word or Logo symbol; at other times a gesture appeared to be used, without equivalent verbal or symbolic language, in order to elicit such language from the student. Although this use of gesture had not been planned, in the course of the lesson a system of gestures emerged, supplementing the planned use of everyday and formal language.

**Episode 2:** In a later lesson, recognising that some students were still having difficulty distinguishing between the different kinds of turn, we planned an activity to make more explicit links between the gestures and the language of 3D movement. At this stage, the system of gestures had become a code for us, mapping each change in the relationship between hand and arm in a one-to-one relationship to the language of 3D turns and hence to the formal Logo terms as shown in Table 1.


Hand/arm relationship	3D turn language	Logo formalism
	Down (more fully 'Pitch down')	PD 90

**Table 1: Mapping gesture-language-Logo formalism**

In this activity the class teacher (GD) used her arm and hand to act out the role of the turtle drawing a 'door' under instruction from the class. Once a movement was agreed to be correct, the corresponding instruction was entered into Logo and the consequent figure displayed. GD was careful to follow the conventions of the gesture system in order to emphasise the relative nature of turtle movement.

	<p>CM Ok. Look at the way that Miss's hand is pointing. Which way has she got to turn it now?</p> <p>S1 Down</p> <p>CM OK. Would you turn your hand down please?</p>
---	--

## Working Group 9

	S1	No
	Ss	[laugh]
	CM	Was that right?
	S1	No






**Table 2: ‘Down’ doesn't mean ‘down’**

Thus, for example, she turned her hand in a pitch down gesture when given the instruction to go down, even though this resulted in her hand pointing horizontally (see Table 2). Although student S1 used a term down that was part of the code system, he used it in an everyday sense to indicate movement towards the ground. This resulted in conflict for students between their intended outcome and the visual feedback provided. The conflict was resolved as shown in the following extract.

- CM           It wasn't was it? So which? Can you think about which way to turn?
- S2           [indecipherable]
- GD           What did you say S2?
- S2           Sideways
- GD           Sideways? Which way? Right or left? [GD uses her left hand to point to the right and left sides of her right hand]
- S2           Right
- CM           Ok, everybody agree with that? [GD turns her hand TR] Does that look right?
- Ss           Yeah

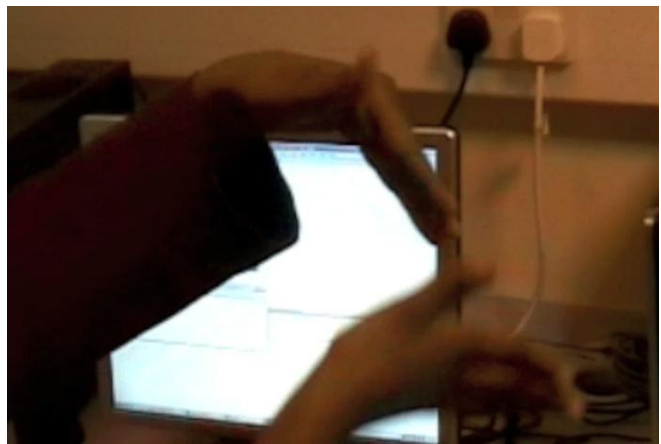
It is worth noting that student S2 again used an everyday term sideways, though in this case a term that was not also part of the formal system. The teacher GD revoiced this instruction, offering the terms right and left – terms with formal places in the code system as well as everyday meanings. GD's additional deictic gesture, pointing to the right and left sides of her hand, served to indicate the desired plane of reference for right/left. This deictic gesture might be considered to serve as a form of scaffolding for developing the formal code, as all terms in the code must be interpreted relative to the current plane of the hand. On the other hand, by offering students a choice of just two acceptable answers, their attention may be focussed very narrowly on that choice, rather than on the underlying principle.

## Working Group 9

1	here	<i>whole rt arm vertical P0, palm facing away from body, moves up in direction of fingers</i>	
2	turn here	<i>TR, arm moved in direction of fingers (maintaining TR position)</i>	
3	turn here	<i>attempt to move rt hand TR again (too difficult?)</i>	
4		<i>switch to lt hand, arm horizontal pointing rt, hand PDN (fingers pointing down)</i>	
5	turn here	<i>moves forearm clockwise, hand still PDN (fingers pointing left)</i>	
6	but I want it to come forward	<i>turns arm (awkwardly) so that, hand still in PDN position, fingers point towards body</i>	

**Table 3: T draws a wall**

**Episode 3:** T, having constructed one rectangular wall, was trying to construct a second wall perpendicular to the first. She explained what she was trying to draw using language and gesture. Her words are shown in Table 3, together with a verbal description and a sketch of the accompanying gesture. The switch (lines 3 - 4) between right and left hands appears to be a response to the physical difficulty of achieving the desired position with the right hand (see Figure 3). It allows T to maintain the direction the fingers are pointing (down). This may be taken to represent the turtle heading within the vertical plane parallel to the screen. However, in switching arms, she changes the relationship between arm and hand from a turn to a pitch gesture. We use turn and pitch within the conventions set up by the teachers/researchers and the Logo language, not to suggest that T associates her gestures with these terms. On the contrary, she does not appear to attach significance to the distinction, using turn in a generic, everyday way, focusing solely on the position of her hand and the direction in which her fingers are pointing in order to describe the intended turtle movement. While she is to some extent ‘playing turtle’ with her hand, she defines the turtle’s movements using position and heading at the corners of her imaginary wall rather than by using turn and distance as required by the Logo language. The use of the turn and pitch gestures is thus not supporting her move into using Logo code and may indeed have made her communication with teachers/researchers less effective.



**Figure 5: T switches hands**

## **CONTRASTING GESTURES: IMAGING VS. IMAGINING**

In Morgan & Alshwaikh (2008) we considered the difference between the ways in which teachers/researchers and students were using the ‘same’ gestures, and distinguished between the two notions of imaging and imagining. We defined imaging as using an iconic gesture to create an image of the construction of the turtle path. The movement of the hand mimics the movement of the turtle: the forearm is held parallel to the current heading of the turtle and the hand is moved to define the next heading. In contrast, for student T the relationship between forearm and hand did not appear to have significance: she was willing to substitute a pitch down gesture with her left hand for a turn right gesture with her right. We characterise her use of gesture as imagining, referring to her mental image of the desired outcome. Such use appears to have both iconic and deictic characteristics. In this episode, as in several other episodes within the data set, the student’s gesture points to the desired direction of movement, rather than mimicking the required type of turn. Our conclusions related to a disjunction between students’ everyday experience of 3D space and the movement of a turtle in MaLT, interpreted as a cognitive difficulty in imagining one’s body moving freely in that space. We noted at that time the possibility that students might be drawing on everyday communicative resources rather than on the formal systems proposed by the teacher-researcher team and by the Logo language. We now develop our analysis of the relationship between everyday and specialised discursive resources for describing 3D turning.

## **DISCUSSION: EVERYDAY VS. SPECIALISED RESOURCES**

When movement is restricted to a 2D space, rotations are only possible around an axis perpendicular to (and outside) the plane. Our everyday experience is most commonly confined to movement experienced as more or less within a plane (i.e. travelling on the surface of the earth) and everyday English language reflects this, using the single word *turn* to denote any form of rotation. Even when rotations out of this plane are experienced they are generally referred to using the generic *turn*, modified by a description of the sense of the rotation (e.g. *clockwise*) or of the heading following the rotation (e.g. *up* or *down*).

In contrast, in 3D space rotations are possible around any line in the space, though any rotation may be defined as a combination of rotations around a set of three mutually perpendicular axes<sup>2</sup>. Consequently, in order to specify rotations in 3D, three different words are required. In English (and Logo) the terms *pitch* and *roll* are adopted in addition to *turn*. *Turn* itself acquires a specialised use, referring to rotation around an axis perpendicular to the plane in which the moving object is currently located, while maintaining its everyday generic use. Thus it is possible to say: “In 3D space there are three types of turn – *pitch*, *roll* and *turn*.” As everyday discourse does not systematically distinguish different types of turning, it is common

to omit specific reference to the process altogether, simply providing the heading following the turn.

Another difference between everyday language and the formal language of Logo and of the introduced gesture system lies in the ways that directions are used. In Logo and the gesture system, right/left and up/down are always defined relative to the current heading. In everyday discourse while right and left are usually used in a similar, relative way, up and down are more commonly used to refer to absolute directions relative to the earth. Gestures used to indicate turns in everyday discourse may also tend to be deictic – pointing in the direction of the turn – or hybrid like student T's gestures, rather than purely iconic – mimicking the trajectory of the movement.

As students talked about their work on tasks such as drawing a room, constructing a revolving door, etc., they tended to use only directions to describe their turtle turns, omitting the verbs that would define the type of turn. Thus, rather than saying turn right or pitch down they would say simply right or down (or possibly use an indeterminate verb go right or go down). Such elision is compatible with everyday usage in which down and right are unambiguous: down towards the centre of the earth; right relative to the vertical axis of the body and the direction in which the whole body is facing. Alternatively, as seen in Episode 3, students would coordinate everyday language and gesture, using only the word turn while indicating the direction of the turn by a gesture.

A particular source of difficulty in coming to use the Logo formalism lies in the fact that the formal terms turn and roll are both modified by right and left. It was noticeable that, even as students became more familiar with the formal language, roll was used less frequently than either pitch or turn. This is consistent with the everyday focus on direction rather than type of movement. Students talked about their desire for the turtle to go up or down, right or left, then associated these directions with the formal pitch and turn but had no distinct everyday way of referring to the desired outcome direction of roll.

Unlike the situation described by Arzarello et al, where “the teacher uses the same gestures as the students and rephrases their sentences using precise mathematical language” and thus “supports the students towards a correct scientific meaning” (2009, p.106), in the situation presented in this paper the teachers/researchers themselves developed and then used a new set of gestures in an attempt to support students' development of new formal language, an attempt that appears justified according to Roth's review of studies of gesture in teaching (2001). On the other hand, Roth also suggests that students may interpret teachers' metaphoric gestures as iconic, with negative consequences for their understanding of scientific concepts (p.377). In the case presented here, students adopted the teachers/researchers' iconic gestures as if they were deictic. We have argued that this adaptation is likely to be related to the characteristics of students' everyday language and gesture use and

mismatches between these and the formal descriptions provided by the introduced system of gestures and the formal languages of mathematics and Logo. As a consequence the intended use of ‘playing turtle’ as scaffolding to support students’ development of the formal description of motion in 3D was less effective than hoped.

### NOTES

<sup>1</sup>The ReMath project (Representing Mathematics with Digital Technology), was funded by the European Commission Framework 6 Programme IST4-26751.

<sup>2</sup>In fact, two axes is the minimum required but can result in more complex sequences of turns.

### REFERENCES

- Alshwaikh, J. (2010). Geometrical diagrams as representation and communication: a functional analytic framework. *Research in Mathematics Education*, 12(1), 69-70.
- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70, 97-109.
- Bjuland, R., Cestari, M. L., & Borgersen, H. E. (2009). A teacher's use of gesture and discourse as communicative strategies in concluding a mathematical task. *Sixth Congress of the European Society for Research in Mathematics Education, Working Group 6 Language and Mathematics*, 884-893
- Chapman, A. (2003). *Language Practices in School Mathematics: A Social Semiotic Approach*. Lewiston, NY: Edwin Mellen Press.
- Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: Macmillan.
- Halliday, M. A. K. (1974). Some aspects of sociolinguistics Interactions between linguistics and mathematical education symposium. Paris: UNESCO.
- Kress, G., Jewitt, C., Ogborn, J., & Tsatsarelis, C. (2001). *Multimodal Teaching and Learning: The rhetorics of the science classroom*. London: Continuum.
- Kynigos, C., & Latsi, M. (2007). Turtle's navigation and manipulation of geometrical figures constructed by variable processes in a 3d simulated space. *Informatics in Education*, 6(2), 359-372.
- Maschietto, M., & Bartolini Bussi, M. G. (2009). Working with artefacts: gestures, drawings and speech in the construction of the mathematical meaning of the visual pyramid. *Educational Studies in Mathematics*, 70, 143-157.
- McNeill, D., & Levy, E. (1980). *Conceptual Representations in Language Activity and Gesture*. National Institute of Mental Health (DHHS), Bethesda, MD.

### Working Group 9

- Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, 41, 209-238.
- Morgan, C., & Alshwaikh, J. (2008). Imag(in)ing three-dimensional movement with gesture: 'playing turtle' or pointing? In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics* (Vol. 28(3), pp. 136-141). London: BSRLM.
- Morgan, C., & Alshwaikh, J. (2009). Mathematical activity in a multi-semiotic environment. *Sixth Congress of the European Society for Research in Mathematics Education, Working Group 6 Language and Mathematics*, 993-1002. Retrieved from <http://www.inrp.fr/editions/cerme6>
- O'Halloran, K. L. (2005). *Mathematical Discourse: Language, Symbolism and Visual images*. London: Continuum.
- Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*. New York, NY: Basic Books.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111-126.
- Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research*, 71(3), 365-392.
- Vygotsky, L. (1986). *Thought and Language* (A. Kozulin, Trans.). Cambridge, MA: MIT Press.

# **A WORKING MODEL FOR IMPROVING MATHEMATICS TEACHING AND LEARNING FOR BILINGUAL STUDENTS**

Máire Ní Ríordáin

Head of Education, St. Patrick's College, Thurles, Ireland

*The role of language in teaching and learning mathematics has been well documented amongst mathematics educators (e.g. see Barwell, 2009; Clarkson, 2007) and those in the fields of social science, sociology and linguistics (e.g. Cummins, 2000). However, there is a need for the development of a model for improving the teaching and learning of mathematics for bilingual students. A number of theoretical models have been developed (see Ellerton, 1989; Gawned, 1990) over the years that have proved useful for investigating language related issues in the teaching and learning of mathematics but the model presented in this paper is concerned with issues that mathematics teachers who have bilingual students in their classrooms need to take into consideration.*

## **INTRODUCTION**

Mathematics is made meaningful through the use of language and students should be enabled to communicate adequately the language of mathematics. Consequently, the role of language in teaching and learning mathematics has been well documented amongst mathematics educators (e.g. see Barwell, 2009; Clarkson, 2007) and those in the fields of social science, sociology and linguistics (e.g. Cummins, 2000). However, there is a need for the development of a model for improving the teaching and learning of mathematics for bilingual students. A number of theoretical models have been developed (see Ellerton, 1989; Gawned, 1990) over the years that have proved useful for investigating language related issues in the teaching and learning of mathematics but the model presented in this paper is concerned with issues that mathematics teachers who have bilingual students in their classrooms need to take into consideration. The working model presented in this paper is derived from data generated by a variety of methods in the Irish context. The authors' research is concerned with Gaeilgeoirí (students who learn through the medium of Gaeilge (Irish)) *in the transition* from Gaeilge-medium mathematics education to English-medium mathematics education in Ireland. This transition can take place at the primary to secondary interface or at the secondary to third level interface. Data generated from the second to third level transition is utilised for the development of the working model. It is anticipated that other English as an additional language (EAL) students would present comparable similarities when encountering the English mathematics register for the first time. Thus, the working model presented in this paper can provide a basis for investigation within other mathematical learning contexts in which EAL students are present. This research is the first of its kind to be undertaken in the Irish context and accordingly significant contributions have



been made both nationally and internationally to this area of mathematics education (see Ní Ríordáin & O' Donoghue, 2009).

## **THE IRISH CONTEXT**

Ireland has two official languages – Gaeilge (Irish) and English. Consequently, primary and second level education is provided through either language medium, dependent on the location of the school (all-Irish speaking region) and if it is a designated Gaelscoil (Gaeilge-medium primary level school) or Gaelcholáiste (Gaeilge-medium second level school). This rise in popularity of attending Gaeilge-medium education is significant with approximately 8% of primary school children and 3% of second level students studying through Gaeilge (Gaelscoileanna Teo., 2008). However the majority of these Gaeilgeoirí (students who learn through the medium of Gaeilge) face an imminent transition to English medium mathematics education, be it at second or third level education in Ireland. The integration of these students into English-medium mathematics classes is of concern to the author and consequently a working model has been developed exploring the issues that teachers of bilingual mathematics students need to take into consideration when engaged in mathematics education.

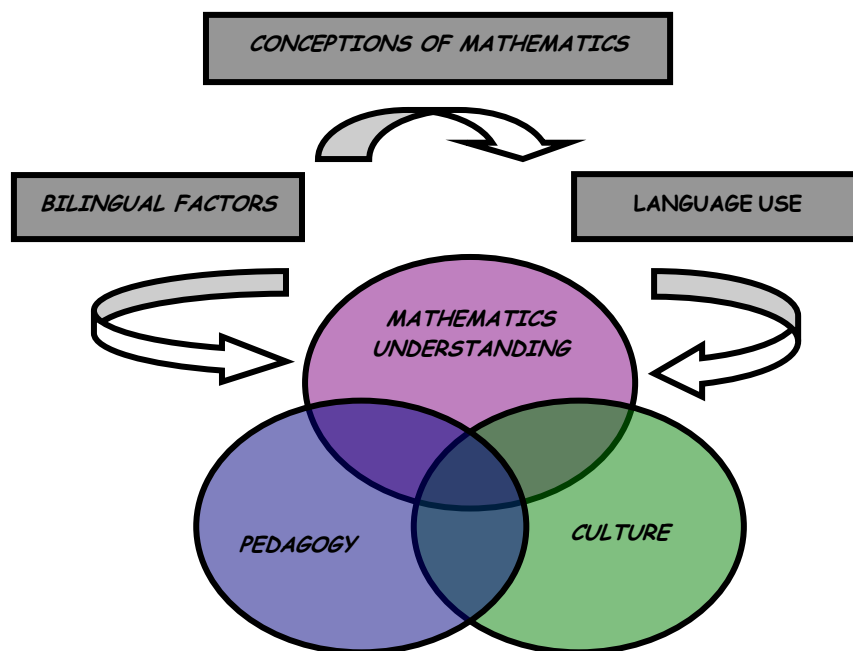
## **THE STUDY**

Because of the diverse nature of the topic, a co-ordinating framework that exposed the various components of the field was needed. Both Ellerton's (1989) and Gawned's (1990) frameworks provided a theoretical structure for the design methodology. Ellerton's (1989) model shows the need to link the various aspects of language factors in mathematics learning and it demonstrates that culture occupies the entire classroom, and that communication within this culture is of key importance. Communication and language become central factors in issues such as socio-linguistics, natural language, psycho-linguistics, problem solving and classroom discourse which intersect with each other, and with most parts of the framework. Gawned (1990) acknowledges that the language of the classroom has a very important influence on students' understanding of mathematics, and that each classroom has a unique culture of its own. Gawned (1990) also discusses the discourse patterns found in mathematics classrooms. They tend to be dominated by rules, function within strict relationships and are teacher centred. Thus, this framework reflects the nature of mathematics classrooms and how language plays a key role in learning, particularly the language of the teacher and the textbook, while also highlighting the cultural influences on mathematics education. These models establish the need to link the various aspects influencing language and mathematics learning, while depicting the centrality of the teacher.

Fifteen Gaeilgeoirí, who had studied mathematics entirely through the medium of Gaeilge (13 years of schooling) until they began third level education, took part in

the study. All Gaeilgeoirí completed a detailed questionnaire and follow-up semi-structured interviews were undertaken with seven of the Gaeilgeoirí. During the interview completed a language-use survey (Clarkson, 2007), along with responding to questions based on specific themes. The primary aim of the questionnaire was to gain an insight into how language has impacted on the transition from Gaeilge-medium to English-medium mathematics education from a pedagogical, cultural and personal perspective. The purpose of the interview was to obtain an understanding of the language employed by Gaeilgeoirí when immersed in mathematical problem solving, to establish their perceptions of mathematics and of mathematical learning, while appraising their experiences of the transition to English-medium mathematics education. Analysis of the questionnaire and interviews involved both quantitative (SPSS, Version 15) and qualitative aspects (NVivo), which is in line with the mixed methods approach employed in this study.

## DESCRIPTION OF THE WORKING MODEL



**Figure 1: A model for improving mathematics teaching and learning for bilingual students.**

The first relationships to be discussed are that of the triad of Mathematics Understanding, Pedagogy and Culture. They are examined in pairs initially with a final discussion on the group as a whole.

### Mathematics Understanding - Pedagogy

The general consensus when questioned on what they believe ‘understanding mathematics’ to be was one of an instrumental perspective (Skemp, 1978). Being able to solve mathematical problems using mathematical formulae and correct procedures reflected one’s ability at understanding mathematics. Granted

instrumental understanding plays a significant role in mathematics learning but none of the participants interviewed relayed an importance of knowing ‘why’ (Skemp, 1978). However pedagogical aspects need to be taken into consideration when engaged in a discussion on learners’ perspectives of mathematics. Through discussion it became obvious that the participants’ lecturers/tutors place an emphasis on procedural knowledge for the learning of mathematics and little time is given to the development of conceptual understanding. Because of this emphasis Gaeilgeoirí’s learning strategies revolve around the continual practice of tutorial questions and previous exam papers. Rote learning is relied on and fostered through success in examinations. So for Gaeilgeoirí, their acuity for procedural methods in learning and understanding mathematics is rewarded by a system that instinctively promotes this type of knowledge.

### **Mathematics Understanding - Culture**

Gaeilgeoirí entering Irish third level education are emerging from a learning environment immersed in the Irish language and culture. They are required to adapt to a learning environment steeped in the English language and cultural practices. Therefore Gaeilgeoirí are required to use mathematical tools within this new environment with little regard given to previous language/cultural practices. The assumption is that all students have done mathematics through English. Cultural differences played a significant role in the transition for these students. Many felt that the other students in their courses perceived them as being different

Tomás: Sometimes when in class I stand out a little bit because of my English isn’t up to their level...they’d know like because my accent and stuff that I’m different.

This in turn prevented them from engaging in discussions that arose during tutorial times. They lacked confidence in their ability to partake in mathematical discussions and ask questions through the medium of English. The lack of use of Gaeilge for academic, cultural and social purposes hindered their progression at third level education.

### **Pedagogy - Culture**

Too often mathematics students are required to use procedures of the discipline without acquiring and embracing the culture of the discipline (Brown, Collins & Duguid, 1989). All Gaeilgeoirí interviewed highlighted the didactical approach employed by many lecturers/tutors at third level education in Ireland. Gaeilgeoirí were presented with abstract concepts and independent examples. Thus Gaeilgeoirí were exposed to the procedural tools of mathematics but lacked ‘authentic activity’ in order to truly understand the conceptual tools being employed (Brown, Collins & Duguid, 1989). The culture of third level institutions is to promote independent autonomous learners which Gaeilgeoirí found difficult to adapt to and they felt that lecturers/tutors were unapproachable. Consequently, there appeared to be a fear of

being perceived as ‘different’ and ‘weak’, with Gaeilgeoirí rarely seeking help with any mathematical difficulties they were encountering.

### **Mathematics Understanding - Pedagogy - Culture**

Mathematics understanding, pedagogy and culture are interdependent in the triad – neither can be understood without the other. In order for Gaeilgeoirí to develop mathematical understanding, pedagogical practices are of key importance, which in turn are evocative of cultural influences. Mathematics is a product and a function of culture and unfolded through pedagogy.

The next sections will look at the upper section of the working model which consists of the quartet of Mathematics Understanding, Bilingual Factors, Conceptions of Mathematics and Language Use. The relationships between each pair will be discussed.

### **Mathematics Understanding – Bilingual Factors**

Colm:           Going from Gaeilge to English is hard at times...it has made the transition to college maths difficult for me but I’m getting used to it.

Although the majority of Gaeilgeoirí interviewed found the general transition from second to third level education relatively easy, having to transfer from learning mathematics through Gaeilge to learning mathematics through English impacted on their learning and understanding of the subject. The main source of difficulty was the actual ‘language of mathematics’ i.e. the mathematics register. They all referred to problems they encountered with mathematics terminology they had acquired at primary and second level through the medium of Gaeilge but were unaware of the English equivalent on entering third level. For example, basic operations such as addition (simiú), subtraction (dealú) and division (roinnt) caused problems for Gaeilgeoirí. Gaeilgeoirí were also confused by similar words in English e.g. “multiple” and “multiply” and may have been unsure of the difference in meaning. Whereas in Gaeilge two dissimilar words are used – “iolraí” (multiple) and “meadú” (multiply), thus lessening the confusion when interpreting and answering a question. Similarly, the syntax and semantics of the Gaeilge language lends itself to a clearer understanding of what is being described. For example, in Gaeilge a passage reads “It is called the Highest Common Factor the number that is highest, which is 4” (directly translated) compared to “The highest of these, called the Highest Common Factor, is 4” in an English version. Therefore, when mathematical problems are presented in ‘complex’ English it is a source of confusion for Gaeilgeoirí in this study. Gaeilgeoirí highlighted the expectation on the behalf of lecturers/tutors that all students has learnt their mathematics through English and thus their learning needs were not catered for in the transition. Clearly, Gaeilgeoirí require assistance in the initial transition and perhaps if appropriate teaching interventions were introduced this transition process may be eased as well as improving mathematical understanding. The fact that all students interviewed are relying on rote learning in

order to pass examinations and thus are seen to ‘succeed’ at mathematics, the type of understanding being developed is not the desired and one would question if this has significant implications down the line for future mathematics learning and career development.

### **Mathematics Understanding – Conceptions of Mathematics**

In a previous section the participants’ conceptions of mathematics were examined and it was found that the majority had a narrow perception of mathematics in that they strongly believe it only consists of ‘numbers’, ‘formulae’ and using ‘numbers to solve problems’. This perception partly stems from the mode of teaching employed at second and third level education. Didactical teaching is the norm where repetitive practice of questions is encouraged. Thus, Gaeilgeoirí are not gaining a deeper insight into the subject area as a consequence of the teaching methods they have encountered. What was surprising for the author was given the emphasis that Gaeilgeoirí placed on problems they encountered with mathematics terminology and the change in the language of learning, little saw a relationship between mathematics and language. If Gaeilgeoirí lack awareness of the influence of language on mathematics learning and understanding then this may have repercussions for their mathematical understanding. Language plays a key role in mathematics learning and understanding and awareness of this is crucial when immersed in a new language of learning. The author believes that this awareness of language as a source of difficulty may actually help the students’ mathematical understanding in the transition from Gaeilge-medium to English-medium education.

### **Mathematics Understanding – Language Use**

During the interview Gaeilgeoirí were asked to complete a Language Use Survey (see Clarkson, 2007). This consisted of identifying what language(s) they used in answering the individual word problems on the test instrument. Gaeilgeoirí were given the option of selecting English only, English and Gaeilge, or Gaeilge Only. Gaeilgeoirí drew on their first language of learning when answering some of the mathematics word problems even though all of the problems were presented in English. Gaeilge was used primarily for thinking out a problem and conducting mental operations such as addition and multiplication of numbers as this was what they described as ‘normal’ and ‘natural’ to them. Given their complexity at times to describe their use of languages it appears to be a subconscious action and ingrained in their process skills when engaged in mathematical problem solving and understanding.

### **Bilingual Factors – Conceptions of Mathematics**

Gaeilgeoirí’s conceptions of mathematics revolved around the belief that it consists of numbers, problem solving and using formulae. The majority saw the transition between languages as ‘relearning’ mathematical words and concepts through the new language of instruction. There was no recognition of transferring mathematical skills

from one language to another or drawing on skills developed in both languages to solve mathematical problems. This demonstrates their lack of *awareness* of the fact that they do use Gaeilge relatively often when engaged in mathematics (previous section) and therefore are drawing on both languages. Due to this lack of *awareness*, Gaeilgeoirí saw no real advantage to having two languages for learning mathematics. Perhaps if Gaeilgeoirí were more aware of the influence of language on mathematics learning this may ease the transition and improve their mathematical understanding.

### **Conceptions of Mathematics – Language Use**

Gaeilgeoirí consider language competency in one or both languages as irrelevant to mathematics learning and understanding. They view the purpose of language as solely for reading questions, but they do not see this as an important step in solving a mathematics problem. Gaeilgeoirí failed to make a connection to language as a facilitator of understanding. They considered mathematical content solely as the source of difficulty. As a consequence this may be acting as a barrier to developing their mathematical skills and understanding.

### **Bilingual Factors – Language Use**

There was a clear lack of understanding by Gaeilgeoirí of their use of their languages. It was only through probing that the students began to realise that they use Gaeilge, even if it was “*only just for simple things like adding and multiplying*” (Liam). They failed to see an advantage to having two languages for learning mathematics, but perceived it as occurring in one language or the other at a given time. The author strongly feels that this lack of awareness of language use and connection with mathematics learning may be acting as an obstacle to the development of Gaeilgeoirís’ mathematical understanding.

### **Mathematics Understanding - Bilingual Factors – Conceptions of Mathematics - Language Use**

Mathematics understanding, bilingual factors, conceptions of mathematics and language use are interdependent and neither can be understood without the other. The primary aim is to develop mathematical understanding but for Gaeilgeoirí significant bilingual factors, conceptions and language use need to be taken into consideration in the development of this understanding.

## **IMPLICATIONS FOR THE IMPROVEMENT OF MATHEMATICS TEACHING**

The focus of this investigation has been on Gaeilgeoirí in the transition from Gaeilge medium to English medium mathematics education. Clearly the teacher is going to play a significant role in facilitating this transition. The following are a number of suggestions for teachers that can be incorporated into their pedagogic practices (Anstrom, 1999), consistent with the working model developed. It is important that mathematics teachers develop innovative strategies for teaching mathematics to

Gaeilgeoirí so to challenge the high ability bilingual students, while ensuring weaker bilingual students are catered for. When in the transition between language mediums, Gaeilgeoirí will be integrated with all English students. No facilitation is made for Gaeilgeoirí e.g. separate classes, extra tuition. Thus, the recommendations suggested below are good practices that may help improve mathematics teaching across the board, not just specific to catering for bilingual students. We need to improve the quality of mainstream instruction so as to make language and mathematics content comprehensible for Gaeilgeoirí, and accordingly cater for their pedagogic, cultural, linguistic and mathematical needs.

- Teachers need to ***make mathematics accessible*** to Gaeilgeoirí and this can be achieved through introducing problem solving activities. By involving Gaeilgeoirí in solving interesting, real-life problems it will encourage critical thinking, in conjunction with basic skills development and practice and accordingly change their *conceptions* of mathematics as consisting solely of numbers, formulae, etc. These real-life problems can be directly linked to Gaeilgeoirí's cultural background thus facilitating the development of *mathematics understanding* and a more student-centred *pedagogical* approach (Buchanan and Helman, 1993). Through demonstrating to Gaeilgeoirí that their prior experiences are of importance, awareness of the Gaeilge language and its importance in developing mathematical understanding will be emphasized. Moreover, the content of mathematics should not be 'dumbed' down for Gaeilgeoirí in the transition; these students have the potential to excel.
- It is important ***to teach the language of mathematics***. From the author's findings it is clear that the language of mathematics is a source of difficulty for Gaeilgeoirí in the transition. Command of the English mathematics register will play an important role in the development of Gaeilgeoirí's mathematical ability and easing the challenges encountered with *bilingual factors* when transitioning to a new language of learning. Therefore, Gaeilgeoirí will require ample opportunities to hear, speak and write mathematically. This proposition is centred on the connection between *pedagogy* and *mathematics understanding* whereby improving Gaeilgeoirí's ability to use the English mathematics register will facilitate engagement on mathematical thinking, while encouraging students to justify ideas orally and in writing (Corasaniti Dale and Cuevas, 1992).
- Mathematics teachers should ***create language supportive environments***. Planning classroom discourse that is inclusive of Gaeilgeoirí demands that teachers create mathematical environments and instructional situations that support students' linguistic and conceptual development. By integrating reading and discussion with mathematics content, it supports the development of academic language skills and encourages greater depth in the students' understanding of the mathematical topic. By adopting such strategies in the

mathematics classroom it facilitate the development of Gaeilgeoirí *language use* and improve their *conceptions of mathematics*.

- Teachers should ***vary instructional methods***. By doing so they will provide Gaeilgeoirí with an opportunity to learn in different ways, through individual, small group and whole class work (Buchanan and Helman, 1993). These methods could include direct instruction, guided discovery, cooperative learning, and computer assisted learning. By varying the instructional methods it will allow for the improved *pedagogical* practices leading to a deeper *understanding* of mathematical concepts, while developing Gaeilgeoirí's *language use*.
- Finally, ***assessment should be authentic and meaningful*** (August and Pease-Alvarez, 1996). Naturally, assessment should have a specific and clear purpose. It may need to take place through the medium of English and in Gaeilge, depending on the language proficiency of the students and so as to truly reflect Gaeilgeoirí's mathematical ability and *understanding*. For Gaeilgeoirí, the test item should incorporate aspects associated with their *cultural* background and allow for *bilingual factors* that may influence their performance on the assessment. The teacher should aim to use a variety of measures such as portfolios, observations, anecdotal records, interviews, checklists, and criterion referenced tests (August and Pease-Alvarez, 1996). By employing a variety of methods it will allow for assessing Gaeilgeoirí's *mathematical understanding* of key concepts while examining their *language use* facility.

## CONCLUSION

Knowledge of the difficulties that Gaeilgeoirí may experience in the transition to English-medium mathematics education in the hands of a discerning teacher can prove fruitful for easing the transition for Gaeilgeoirí. Although the findings emerging from this research are specific to the Irish context, they are important because of their applicability to other bilingual contexts. The working model presented can be employed in order to investigate other EAL learning contexts in order to improve the teaching and learning of mathematics for bilingual/multilingual students. Given the increasing number of students learning in a dominant language that is not their first language, these findings are important to mathematics education (Adler, 2001).

## REFERENCES

- Adler, J. (2001). *Teaching Mathematics in Multilingual Classrooms*. Dordrecht: Kluwer Academic Publishers.
- Anstrom, K. (1999). *Preparing secondary education teachers to work with English language learners: Mathematics*, Washington, DC: Center for Study of Language and Education.



- August, D. and Pease-Alvarez, L. (1996). *Attributes of effective programs and classrooms serving English language learners*, Santa Cruz, CA: National Center for Research on Cultural Diversity and Second Language Learning.
- Bagley, T. and Gallenberger, C. (1992). Assessing students' dispositions: Using journals to improve students' performance, *Mathematics Teacher*, 85 (8), pp.660-663.
- Barwell, R. (Ed., 2009). *Multilingualism in mathematics classrooms – global perspectives*, Bristol: Multilingual Matters.
- Brown, J. S., Collins, A. & Duguid, P. (1989). Situated cognition and the culture of learning, *Educational Researcher*, 18(1), pp.32-42.
- Buchanan, K. and Helman, M. (1993). *Reforming mathematics instruction for ESL literacy students*, Washington, DC: National Clearinghouse for Bilingual Education.
- Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics*, 64(2), pp. 191-215.
- Corasaniti Dale, T. and Cuevas, G. J. (1992). 'Integrating mathematics and language learning' in P.A. Richard-Amato and M.A. Snow, (Eds.), *The multicultural classroom: Readings for content-area teachers*, White Plains, NY: Longman.
- Cummins, J. (2000) *Language, power and pedagogy: Bilingual children in the crossfire*, Clevedon: Multilingual Matters.
- Ellerton, N. F. (1989). The interface between mathematics and language, *Australian Journal of Reading*, 12(2), pp.92-102.
- Gaeilscoileanna Teo (2008). *Fás ar an nGaelscolaíocht sa Ghalltacht (Growth of Irish medium education outside of Gaeltacht regions)*. Retrieved January 10<sup>th</sup>, 2009 <http://www.gaeilscoileanna.ie/index.php?page=aims&lang=en>
- Gawne, S. (1990). An emerging model of the language of mathematics, in J. Bickmore-Brand, ed., *Language in Mathematics*, Victoria: Australian Reading Association, pp.27-42.
- Ní Ríordáin, M. & O' Donoghue, J. (2009). The relationship between performance on mathematical word problems and language proficiency for students learning through the medium of Irish, *Educational Studies in Mathematics*, 71(1), pp.43-64.
- Skemp, R. R. (1978). Relational understanding and instrumental understanding, *Arithmetic Teacher*, 26, pp.9-15.

# REVOICING IN PROCESSES OF COLLECTIVE MATHEMATICAL ARGUMENTATION AMONG STUDENTS

Núria Planas, and Laura Morera

Universitat Autònoma de Barcelona, Spain

*Type your abstract here. The purpose of this paragraph is to draw attention to the style for abstracts, which is Normal, italic, and the length is up to 10 lines. In this report we draw on interactionist theories in (mathematics) education to better understand classroom processes of collective mathematical argumentation. We discuss students' uses of revoicing in different situations of mathematical learning taken from two recent micro-ethnographic studies in Barcelona and Tarragona, Spain. We document two examples that shortly illustrate two "positive" uses of revoicing in peer interaction: i) to ensure mutual understanding; and ii) to foster more explanations. We finish with comments on what is new in our research and how it needs to go in new directions to explore other uses of revoicing that appear when considering a more critical perspective in the analysis of classroom data.*

## INTRODUCTION

Language and discursive practices shape the concepts and processes that organize much of the everyday situations in the mathematics class. In their work, Enyedy et al. (2008) refer to revoicing as a discursive practice to promote a deeper conceptual understanding of school mathematics by positioning students in relation to one another, facilitating debate and fostering mathematical argumentation. The study we present here draws on this broad notion of revoicing to examine communication and mathematical argumentation in peer interaction. We discuss students' uses of revoicing in different situations of mathematical learning taken from two recent micro-ethnographic studies in Barcelona and Tarragona, Spain. This approach is part of our more general focus on the role of language as a social resource in the construction of collective mathematical argumentation in classroom settings.

Various works have examined teachers' uses of revoicing and interpreted this practice as an essential part of what the teacher does during the process of instruction (see Krussel, Edwards & Springer, 2004; or O'Connor & Michaels, 1996, among others). So far, there has been much more empirical literature developed on teachers' revoicing than on students' revoicing in peer interaction. Our study is a contribution to the more reduced group of works on students' revoicing, specifically for the area of mathematics education. We claim that the construction of the students' mathematical discourses is highly orchestrated by what other students say and how. Our data from small groups and pairs reinforces evidence to support the importance of knowing the students' reactions to the ways in which their peers "re-tell" their words while engaged in mathematical tasks.

Before exemplifying data on some of the uses of revoicing in students' interaction, we start with theoretical considerations on the notions of revoicing and collective mathematical argumentation. We then move on to a brief summary of our methods in the analysis of classroom data, and discuss preliminary findings centered on "positive" uses of revoicing. We finish by suggesting some of the problems regarding the exclusive interpretation of revoicing as a facilitator. Our current analysis needs further examination from a more critical perspective that also signals practices of revoicing as markers of legal talk and talkers in the mathematics class.

## **THEORETICAL FRAMEWORK**

This section introduces how the notions of revoicing and collective mathematical argumentation are conceptualized in our work. We point to inspiring literature in the effort to establish empirical connections between these two interactional accomplishments. We claim that revoicing is an important part of the processes that lead to argumentation, although we recognize that the relationships between these two practices are problematical: revoicing may be used for different (social) purposes and may have different implications, some of them with no clear orientation towards mathematical learning.

### **Revoicing**

An assumption of the interactionist theories in (mathematics) education (see, for instance, Voigt, 1996; or Krummheuer, 2011) is that talk among students (and between students and teachers) needs to be analyzed as discursive practices through which (mathematical) knowledge is constructed. Some examples of these discursive practices are revoicing, questioning, requesting, telling, or managing. The practice of revoicing essentially tries to repeat some or all of what has been said in a preceding turn as the basis for a shift in the interaction. This repetition can be manifested in two forms, either as a linguistically "exact" copy or as a reformulation. Despite the linguistic possibility to repeat a sentence exactly, from a social point of view and taking into account the recursivity thesis by Giddens (1979), we understand that every instance of the use of language is a potential modification of that language at the same time as it acts to reproduce it. Thus we find it more adequate to associate revoicing to conceptual reformulation rather than linguistic repetition.

O'Connor and Michaels (1996) indicate three main uses of revoicing in teachers: 1) to position students in differing alignments and allow them to (dis)claim ownership of their position; 2) to share reformulations in ways that credit students with teachers' warranted inferences; and 3) to scaffold and recast problem-solution strategies of students whose first language is not the language of teaching. These uses have been documented by these authors as having the effect to focus group discussion and scaffold conversation on the basis of what is said, when, how, with whom... However, drawing on these three uses, the work by Forman and Ansell (2001, 2002) is focused on the examination of the students' voices. These authors analyze

conversational moves in the "follow-up" part of the Initiation-Response-Feedback sequence in mathematics lessons with frequent practices of revoicing. Although there is a clear emphasis on the social dimension of the IRF sequences and obstacles to the students' voices are recognized, revoicing is primarily seen as a facilitator in the interaction.

### **Collective mathematical argumentation**

By "collective argumentation" we mean the interactional accomplishment given by: 1) representing a task or problem alone; 2) comparing representations within a small group of peers; 3) explaining and justifying the various representations to other members of the group; 4) reaching agreement within the group; and 5) presenting the group's ideas and representations to other participants in the class to test their acceptance (see Brandt & Schütte, 2010, for a similar interpretation that expands on the idea of argumentation from an individual to a collective notion). Like Cobb (2008), we understand that situations of collective argumentation are mathematical if they are organized around specific ways in which tools and procedures are used to achieve mathematical goals. This is still a very general conceptualization of argumentation if we pay attention to the mathematics, but it becomes useful because it puts the emphasis on the processes of teaching and learning.

Sfard and Kieran (2001) have also discussed the role of the students' interaction in processes of collective mathematical argumentation. These authors interpret collective mathematical argumentations as interactive processes in the learning of how and when to participate in school mathematics discourses. They analyze the way in which students express themselves throughout their mathematical talk by means of discursive tools that help advance towards the construction of shared meanings. In particular, certain practices of revoicing in the resolution of mathematical tasks are interpreted as a social tool in the students' exploration of what counts as an accepted and "repeatable" reasoning in the mathematics classroom.

In the situated context of our work, revoicing becomes a reformulation of language to achieve new possibilities for further mathematical argumentation. The attempt is collective and entails a complex system of voices (Planas, 2011). It makes sense then to consider what it means for a student to participate in the "legal" process of constituting a culture of argumentation in the classroom. In this report, we look at the "unanimous" voice of the interaction, instead of pointing to individual voices trying to grasp what becomes necessary for them to gain membership while moving from one state of participation to another. At future stages, however, our study ventures to contribute to the much reduced group of works on the act of using revoicing socially in the delimitation of voices in the mathematics classroom.

### **EMPIRICAL CONTEXT**

Our two research projects [1] share an interest in the qualitative analysis of narrative classroom data. Since 2005, we have been collaborating with a group of mathematics

teachers [2] to develop inquiry-oriented tasks that are to be proved as facilitators of collective mathematical argumentation in small group and pair discussion. The implementation of tasks together with the characterization of situations of collective argumentation by means of the Toulmin's scheme (2007) has provided considerable information regarding students' reasoning in peer interaction. We have recently started to analyze broader discursive practices to include language and social issues in our analysis of how students help each other through talk in their joint construction of mathematical argumentations. The integration of mathematical, language and social issues guides our current process of gaining theoretical and empirical understanding.

In the context of the group of mathematics teachers and researchers, we examine data from secondary mathematics classrooms that were first chosen to validate the implementation of tasks. Up to now, we have searched for examples of students' revoicing in two main sets of data coming from two classrooms in two schools. We have had various meetings --some with the teachers-- to comment on classroom lessons represented through video data. The meetings have been oriented by three main questions: 1) What is the evidence of revoicing in this lesson (if any)? 2) In what sense are two examples of revoicing similar/different? And 3) what are the explicit uses to which different practices of revoicing are put? In what follows, we introduce two examples of peer interaction that hold the potential to make more in depth investigations of other episodes involving a variety of practices of revoicing.

## EXAMPLES OF FINDINGS

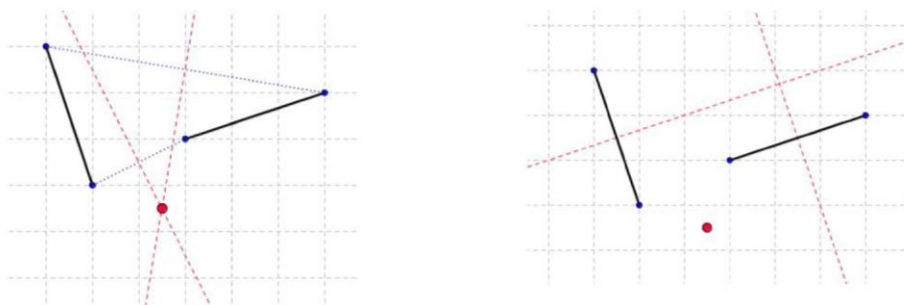
In this section, we describe processes of using revoicing as a resource. We document two types of "positive" revoicing. They are thought of as positive because they contribute to the continuity of peer interaction and mathematical argumentation. The two examples that follow illustrate two uses that prompt: i) mutual understanding, and ii) more explanations. These uses of revoicing seem to require forms of communication, argumentation, and interaction that would otherwise be difficult to achieve. Furthermore, these uses help interpret how the interaction is to be read: the degree of respect or resistance towards the students involved, with an eye to the mathematical contents or the lack there of, etc.

More generally, our examples show that mathematical learning gains "strength" when it is invoked by one student and re-invoked by other students in the context of peer interaction. Data shows different students using correct mathematical reasoning that "does nothing on its own" until it is reconstituted by others through talk. In a large variety of episodes, the argumentation is first introduced by one of the students in the small group, and then becomes an object of negotiation through conversations in which revoicing helps distribute turns among speakers. There are also episodes in which revoicing becomes a strategy to overcome occasional interruptions in some of the students' mathematical implication during the resolution of the task.

### Students' use of revoicing to reinforce mutual understanding

In group discussions among three or more students, there is a "positive" use of revoicing that contributes to making one of the student's ideas available to the rest of the members in the group, and helps reinforce mutual mathematical understanding. We have several examples of episodes in which a student partially explains an argumentation, and another student in the group uses revoicing to emphasize particular aspects of that argumentation, provide additional information, and to facilitate a more adequate mathematical understanding from her/his peers.

In the example below, documented in Morera (2010), we find four students --Elba, Joan, Carles and Uriel-- trying to find out how to transform one line segment onto another by means of a rotational superposition. The students are using a dynamic geometry package to identify the right place to construct the rotation centre. Although the four students are working together in the same small group, they are working on two different computers next to each other. The distribution into two pairs --Elba and Joan, and Carles and Uriel-- leads to the development of two different initial approaches to the problem (see Figures 1 and 2, with the given line segments printed in black).



**Figure 1. Elba & Joan's approach Figure 2. Carles & Uriel's approach**

Two approaches to the resolution of the problem can be inferred from the transcript of the group discussion: while Elba and Joan plan to construct the segments by joining the ends and then drawing the perpendicular bisectors, Carles and Uriel draw the perpendicular bisectors from the initial line segments. There is a moment when Elba and Joan realize that Carles and Uriel are not considering the ends of the given line segments. This is the starting point of the following dialogue [3] in which the two pairs bring together their ideas:

Joan: Shall we talk about what we've been doing?

Elba: I was discussing it with Joan... Maybe, if we drew the perpendicular bisectors here, they would coincide at the same point [see the red point in Figure 1].

Uriel: It's impossible for the perpendicular bisectors to coincide over there! [see the red point in Figure 2]

Joan: [to Uriel] **If we drew the perpendicular bisectors here [3], where the two points come together, not the two line segments, they would coincide at the same point.**

Carles: Ah, okay! I thought you meant this one!

Uriel: So did I!

Due to our focus on the identification of practices of revoicing in the construction of collective mathematical argumentations, we only represent the part of the transcript in which the use of a particular revoicing --with the exact repetition of a sentence in this case-- helps overcome a misunderstanding in the resolution of the problem. In the transcript above, Elba introduces a mathematical argumentation that leads to confusion as it is not clear for Carles and Uriel which line segments are for the perpendicular bisectors. Following an intervention by Uriel that confirms this confusion, Joan repeats Elba's explanation and includes a short clarification in-between --“... where the two points come together, not the two segments...”-- that helps Carles and Joan expand their understanding of the problem. The whole transcript offers more evidence of this interpretation.

This first example illustrates the collaboration among the students in the group. Joan might have completely reconstructed Elba's sentence and started with a new explanation, instead of building into his peer's explanations. More generally, this example points to the deep social dimension in the elaboration of mathematical argumentations in the classroom. Argumentation emerges in conversational contexts and is oriented toward an audience. The context and audience determine how many details and mathematical clarifications are needed to go on with reasoning, as well as to what extent certain explanations may be publicly considered as “repeatable”.

In another context, the sentence “If we drew the perpendicular bisectors here, when the two points come together, not the two segments, they would coincide at the same point”, might not represent a “good” mathematical argumentation: there is no indication as to the meaning of ‘two points coming together’, or which two line segments are being referred to. The sentence needs to be interpreted at least in relation to what has been said in previous turns, and what knowledge Elba and Joan have of their peers' reasoning. The adequacy of a mathematical argumentation in the social context of the classroom is informed by its mathematical quality, but also by the representations that the students (and the teacher) have of how mutual comprehension is facilitated.

Next, we offer a second example of collective mathematical argumentation in peer interaction with a slightly different use of revoicing that reinforces the occasions for mathematical talk, and fosters further interaction among students.

### **Students' use of revoicing to foster more explanations**

Students need ways of talking that help them deal with lack of clarity in other students' contributions in the resolution of mathematical tasks. Some students' use of revoicing acts as an enquiry for the expansion of a point that has not been fully understood. In our research, this use tends to happen in pair work situations in which a student wants another student to clarify a mathematical position and elaborate more

## Working Group 9

on a specific idea. The example below shows the collaboration between two students, Anna and Ona, to find the quantity of squares in a chessboard.

Teacher: Work in a pair and collaborate with each other, okay?

Ona: First we should reflect on the problem on our own.

Anna: Yeah, we need to know what to talk about [...]

Ona: [A few minutes later, to Anna] What are you writing here?

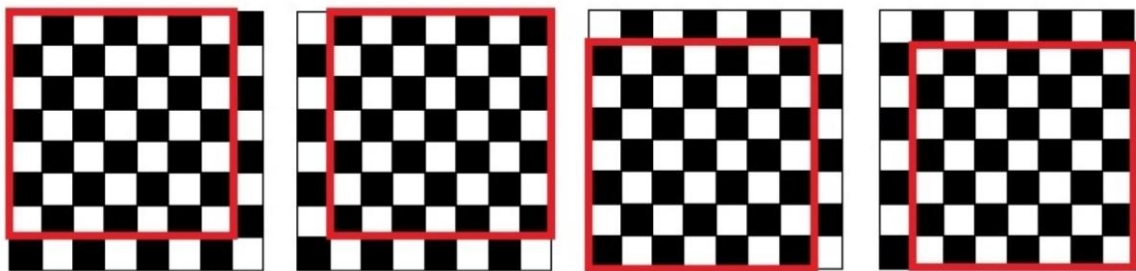
Anna: Just counting all the squares in an easy way.

Ona: Do you have the number?

Anna: It's one, four, nine, sixteen... they are always square numbers.

Ona: **So you're saying that they are always square numbers? ... And easy?**

Anna: Yeah. You know why? [She points to a page in her notebook with many numbers and her written resolution]. You have one big square with sixty- four small squares, that's eight times eight. Then you have four squares with forty-nine squares, you see, seven times seven [see Figure 3]. You see that?



**Figure 3. Anna and Ona's approach**

The excerpt above illustrates a classroom situation in which two students have been working separately for a few minutes and then come together to comment on their approaches to the problem of the squares in a chessboard. Anna has developed a complete and mathematically correct written resolution for this problem, but starts explaining it to her peer in a rather synthetic way – “Just counting all the squares in an easy way.” At the end of the conversation (the entire episode is not reproduced here), Ona comes to agree that there is an “easy” way to count all the squares in a chessboard; it is improbable that this agreement has been facilitated by Anna's interventions in which she seems to expect her peer to mathematically “read” through her words. This second example is similar to the first one, in that revoicing is used by Ona as an instrument that helps provide a way of testing Anna's claims on both the mathematics and the simplicity of the resolution method.

Anna initiates her explanations as if interacting with a mathematically “ideal” peer that would share and quickly understand her reasoning. In this context of interaction, Ona's use of revoicing acts as a way of forcing attention to who the peer is and what her specific needs are to gain agency in sharing a particular mathematical argumentation. Here, the use of revoicing facilitates Ona with the role of one who



invokes a sort of participation that controls the sharing of reasoning. Anna's notebook contains a complete resolution of the problem, and by pointing to it she evokes an "ideal" reader that might feel sufficiently satisfied with the written text. Ona's reaction, with the repetition of a sentence and the emphasis on the idea of simplicity, makes it difficult to avoid further explanations and offers the possibility in practice to develop a discourse based on the resolution of the problem.

As in the first example, cooperation among students is required for the achievement of collective argumentation. Ona's use of revoicing contributes to a new conversation with the inclusion of more explicit explanations of the resolution processes that have been followed by Anna. It is necessary that Anna accepts the new basis for this conversation. When revoicing, as an instrument, is put to work it requires the involvement of all parties. Ona, Joan or any other student, on their own, do not have enough agency to convert the reformulation of sentences into an instrument with the purpose of serving collective mathematical argumentation.

Although the empirical relationship between revoicing and collective mathematical argumentation still remains unclear in our work and interpretations of the episodes need to be reinforced with complementary perspectives, we can say a few things at this time. We have chosen two examples of "positive" revoicing for this report, but we do not affirm that revoicing either expressly leads to more argumentation or more collaboration among speakers. We have data with practices of revoicing that do not turn into "more mathematics and/or more collaboration." The status of revoicing as an instrument for the sake of mathematical conversations appears linked to the social nature of this practice. In Planas and Civil (2002), some of the social issues of influence on the interpretation of discursive practices in classroom settings were already documented, with specific attention to recognition among peers. In our second episode, for instance, Ona's revoicing is effective because Anna is willing to explain her reasoning. Nevertheless, what would happen if Anna imagined her peer as an obstacle in her learning? Would she give detailed answers to her questions?

## **FINAL REMARKS**

Together with the interest in examining relationships between revoicing and collective mathematical argumentation, a research focus on revoicing in mathematics classrooms raises many other questions. What is new in our work is the interest in the exploration of some of the critical functions that are carried out by practices of revoicing that are initially linked to "positive" uses only. Much remains to be done in this direction, and in fact, we are still at the stage of empirically illustrating "positive" uses and "generating suspicion". It is not clear whether one can critically examine revoicing in the strict context of the micro level of the small group or the whole class with no attention to the multiplicity of voices from the different and various macro levels that have an influence on how discourses are re-elaborated in classroom settings. The repetition of a sentence may serve as a strategy to foster

mutual understanding and mathematical explanations, and at the same time represent messages of incorrectness, doubt, disapproval... depending on who the speakers are.

It seems unlikely that a one-dimensional view on revoicing or any other discursive practice, based on either mathematical or social issues, helps gain a better understanding of how mathematical conversations are prompted in classroom settings. On the one hand, from the joint perspective of language and mathematics, we cannot claim that all “significant” mathematical meanings are maintained the same when a sentence is reformulated, neither can we affirm that “repetitions” always stand for evidence of learning. This uncertainty points to serious methodological obstacles, especially when trying to justify processes of individual mathematical learning that are constructed in contexts of conversation with frequent practices of revoicing. On the other hand, from the joint perspective of language and social interaction, even when a sentence is repeated exactly the same, we still cannot guarantee that the context and the interpretation have not varied. The precision of the language of mathematics and the complex social discourses around it (e.g., ‘who is considered as mathematically competent’, ‘what is expected to be included in school mathematics’) makes it difficult to answer all these questions without adopting a multi-dimensional view on how everyday situations in the mathematics classroom are organized.

## NOTES

1. The work is part of Projects ‘Estudio sobre el desarrollo de competencias discursivas en el aula de matemáticas’, EDU2009-07113, and ‘Contribución al análisis y mejora de las competencias matemáticas en la enseñanza secundaria con un nuevo entorno tecnológico’, EDU2008-01963, both funded by the Spanish Ministry of Science and Innovation. The two authors are members of the Research Group ‘Educació i Competència Matemàtica’, SGR2009-00354, recognized by the Catalan Department of Universities. The second author owns Grant BES2009-022687.
2. The Group EMAC --Catalan acronymus for Critical Mathematics Education-- is supported by Associació de Mestres de Rosa Sensat, and partially funded by Project ‘Diagnosi de necessitats socials i educatives de l’aula multilingüe: aproximació des del cas de matemàtiques’, ARFI-1-2009-00052, Catalan Government.
3. All dialogues have been translated from Catalan to English by the first author.
4. The bold format is used in our transcripts to mark the exact moment in which revoicing appears.

## REFERENCES

- Brandt, B., & Schütte, M. (2010). Collective mathematical reasoning in classrooms with a multilingual body of pupils. In C. Morgan, U. Gellert & E. Jablonka (Eds.), *Proceedings of the 6th International Conference on Mathematics Education and Society* (v. 1, 111-114). Berlin, Germany: MES.
- Cobb, P. (2008). Classroom interactions and discourse as the context for mathematical and language learning. In J. Deen, M. Hajer, & T. Koole (Eds.), *Interaction in two multicultural mathematics classrooms: Mechanisms of inclusion and exclusion* (pp. 267-280). Amsterdam, The Netherlands: Aksant.

- Enyedy, N., Rubel, L., Castellón, V., Mukhopadhyay, S., Esmonde, I., & Secada, W. (2008). Revoicing in a multilingual classroom. *Mathematical Thinking and Learning*, 10(2), 134-162.
- Forman, E. A., & Ansell, E. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics*, 46, 115-142.
- Forman, E. A., & Ansell, E. (2002). Orchestrating the multiple voices and inscriptions of a mathematics classroom. *The Journal of the Learning Sciences*, 11(2&3), 251-274.
- Giddens, A. (1979). *Central problems in social theory: Action, structure, and contradiction in social analysis*. Berkeley, CA: University of California Press.
- Krummheuer, G. (2011). Representation of the notion “learning-as-participation” in everyday situations of mathematics classes. *ZDM - The International Journal on Mathematics Education*, 43, 81-90.
- Krussel, L., Edwards, B., & Springer, G. T. (2004). Teacher's discourse moves: A framework for analyzing discourse in mathematics classrooms. *School Science and Mathematics*, 104(7), 307-312.
- Morera, L. (2010). *Momentos clave en el aprendizaje matemático en un contexto de trabajo de las isometrías usando un entorno tecnológico*. Master's Thesis. Bellaterra, Spain: Universitat Autònoma de Barcelona.
- O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning and schooling* (pp. 63-103). New York, NJ: Cambridge University Press.
- Planas, N. (2011). Language identities in students' writings about group work in their mathematics classroom. *Language and Education*, 25(2), 129-146.
- Planas, N., & Civil, M. (2002). The influence of social issues on the re-construction of mathematical norms. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th International Conference of the Psychology of Mathematics Education Group* (v. 4, pp. 73-80). Norwich, UK: PME.
- Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, 8, 42-76.
- Toulmin, S. E. (2003). *The uses of argument*. Cambridge, UK: Cambridge University Press.
- Voigt, J. (1996). Negotiation of mathematical meaning in classroom processes: Social interaction and learning mathematics. In L. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 21-50). Mahwah, NJ: Lawrence Erlbaum Associates.

# EPISTEMOLOGICAL AND SEMIOTIC ISSUES RELATED TO THE CONCEPT OF SYMMETRY

Frode Rønning

Sør-Trøndelag University College, N-7004 Trondheim, Norway

*This paper emerges from a classroom study where 8 year-old pupils are engaged in activities having to do with different aspects of symmetry (reflection and rotation). The pupils' language is analysed on the basis of an epistemological analysis of the mathematical concept isometry and certain issues that might be relevant for the pupils' conceptual development are discussed. Semiotic theory focusing on the epistemological aspect of signs and symbols plays an important role in analysing the classroom episodes.*

Keywords: Symmetry, epistemology, semiotics, signs and symbols

## INTRODUCTION

This study is based on empirical data from a three years research and development project where two mathematics educators collaborated with the teachers at a local primary school with the intention of introducing and reflecting on new ways of approaching the teaching of mathematics. In this paper I will analyse two classroom episodes from two consecutive days involving the same pupils. Both episodes can be said to relate to the concept of symmetry, albeit in somewhat different ways. In the first episode the pupils are given tasks where it is explicitly stated that the aim is to design symmetric pictures in various ways. It is implicitly understood that what is meant is reflection symmetry. In the second episode the pupils were building three-dimensional objects and they were encouraged to talk about these objects, using mathematical terms. Part of the discussion turned out to be about what the pupils could observe when the objects were rotated. In this paper I will discuss and analyse different ways that the pupils express properties of geometrical objects that in scientific terms would be labelled symmetry properties. This discussion will relate to challenges involved in linking seemingly different mathematical experiences to one common mathematical concept, namely the concept of symmetry. More precisely I will formulate the aim of the paper in the following way: Characterise the discourse that emerges when pupils investigate various aspects of symmetry, and identify epistemological issues involved in the conceptual development process that is observed.

## THEORETICAL BACKGROUND

### What is symmetry?

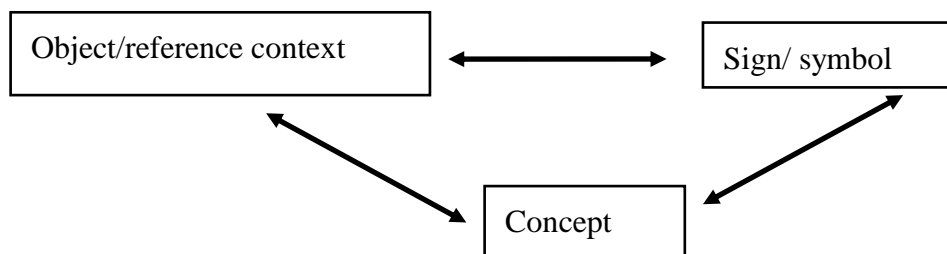
In mathematical terms symmetry is connected to the concept of isometries. An isometry is often defined as a bijective map between two metric spaces that is distance preserving (Weisstein, 2010). In particular if both metric spaces are the

Euclidian plane and the map operates on points in the plane, two figures are called congruent “if and only if one can be transformed into the other by an isometry” (Coxeter & Greitzer, 1967, p. 80). In the plane there are four isometries; reflection, rotation, translation and glide reflection. The concept of symmetry is usually connected to invariance with respect to isometries. A bounded pattern that is invariant under certain reflections or rotations is said to have reflectional or rotational symmetry. These are the only isometries for bounded patterns but for unbounded patterns all four isometries are possible. The ideas of isometries for bounded patterns in the plane can easily be extended to 3-space.

In school geometry the term symmetry is in the early years used almost synonymously with reflection, and the work with symmetry is often supported by the use of mirrors. However, one is also supposed to work with other isometries in school. In the Norwegian National Curriculum it is written under the topic Geometry: “One studies dynamical processes like reflection, rotation and translation” (Utdanningsdirektoratet, 2006, p. 59, my translation). In light of this, and the mathematical discussion of symmetry above, it can be argued that it is desirable that pupils should develop a concept of symmetry that includes all the Euclidian isometries.

### Concept development

Concept development in mathematics is in many instances a matter of making meaning out of signs. I use the term *sign* here in the sense of Peirce: “A *sign* is a thing which serves to convey knowledge of some other thing, which it is said to *stand for* or *represent*” (Peirce, 1998, p. 13, emphasis in original). A sign could therefore be a symbol, a word (written or spoken) or a gesture. In line with Peirce, Steinbring emphasises that a sign has two functions, a semiotic function – something that stands for something else – and an epistemologic function as the sign contains knowledge about that what it stands for (Steinbring, 2005, p. 21). What the sign stands for is in Steinbring’s terms called the object or the reference context. The relation between sign and object is illustrated in Steinbring’s epistemological triangle (2005. p. 22), presented in Figure 1.



**Figure 1. The epistemological triangle**

This triangle is inspired by the triangle presented by Ogden and Richards (1923/1948). They write that “[b]etween the symbol and the referent there is no relevant relation other than the indirect one, which consists in its being used by someone to stand for a referent” (p. 11, emphasis in original). Using Steinbring’s terms this means that the connection between the sign and the reference context is not a priori given but it is determined by epistemological conditions of mathematical knowledge and has to be mediated through the mathematical concept. Steinbring emphasises that the system illustrated in the epistemological triangle is not a static system. It is developing, based on actions by the learner, and interaction between teacher and learner(s) (2005, pp. 22-23). Steinbring mentions (2006, p. 136) that one can think of triangles connected in a chain to illustrate concept development. This has been explicitly used by Farrugia (2007) who uses a chain of triangles (p. 1205) to analyse how the concept of multiplication develops from addition of equal groups. This linking of triangles into chains can be seen as similar to the concept *semiotic chains* that has been used e.g. by Walkerdine (1988), and also by Presmeg who describes them as chains of ‘signifier – signified’ where “[t]he new signifier stands for all that went before” (2002, p. 302). In the context of the epistemological triangle this means that the sign/symbol in one triangle can become the object/reference context for a different sign/symbol in a new triangle, and this chaining usually leads to greater abstraction.

In this paper I will develop a new way of using Steinbring’s theoretical framework where the concept development is seen not as one chain of epistemological triangles leading to greater abstraction but instead it is seen as two independently existing chains that are merged into one, which then can be developed further. I compare this to van Hiele (1986) who, in his description of levels, states that when going to higher levels of thinking “different structures can be coordinated into one new structure, so that each of the original structures can be understood as parts of it” (p. 52). I propose here a way of using Steinbring’s theory to explain a phenomenon that is similar to what van Hiele refers to as *coordinated structures* (p. 52). I will also propose a categorisation of signs depending on which of the two chains they belong to.

## THE CLASSROOM SITUATION

The research was done as part of a project where two mathematics educators collaborated with the teachers on a primary school over a period of three years. We (the mathematics educators) participated in meetings with the teachers where lessons were planned and we also participated in some of the lessons, and in meetings afterwards where the lessons were discussed. Our role in the lessons varied, from being a passive observer to taking an active part in the teaching of the pupils. The episodes reported on in this paper took place towards the end of the three-year period.

## Working Group 9

The first episode is taken from a lesson with 8-year-old children (3<sup>rd</sup> grade, in December) where I planned a number of different activities in which the concept *symmetry* (i.e. reflection) was central. The activities were set up on different tables in a large classroom and the pupils, working in groups of four-five, moved from one activity to the other. I followed one of the groups and video recorded their work. One teacher was supervising each activity. The second episode took place in the same class on the following day. Here the pupils worked with plastic shapes that could be built together to form 3D-objects. The pupils were allowed to use the plastic shapes freely to build whatever they wanted but I engaged in conversations with them about the objects where I tried to draw attention to geometrical concepts and start a discussion about these. The episode involves two of the girls that also were on the group I followed in episode 1. The episode developed from an utterance from one of the girls, which I found interesting and which led to a conversation where the pupils and I inquire into the properties of some of the objects that have been constructed. The conversation following the utterance that caught my interest was video recorded but not the situation leading to the utterance itself. The dialogue is analysed using the epistemological triangle as a tool to gain insight into the conceptual development of the pupils.

### Episode 1

In one of the tasks in the lesson the pupils were given a large sheet of paper on which a straight line was drawn, and a selection of different plastic shapes. The task was to use the shapes to make a picture symmetric about the drawn line. Harry chooses to make his picture by placing pairs of equal shapes on each side of the given line, such that they touch the line. Eileen suggests another way of doing it by making the picture shown in Figure 2, where the shapes do not touch the line. She says “then we know that there is one like this in between”. When asked to explain what she means by this she takes another square which she places on top of the line with one half on each side (Figure 3). Then she says: “This I just take away and then I know that it is the same length between”. I interpret that she by “the same length” here refers to that the distance from the given line to the two squares is the same. Hence, she links symmetry to distance.

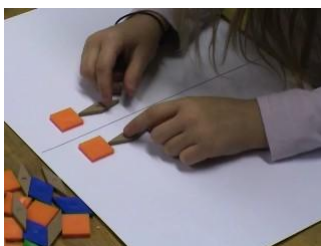
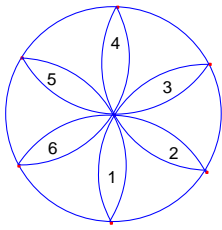


Figure 2



Figure 3

Another task was based on the figure shown in Figure 4. The task was to colour the petals of this “flower” (the petals are numbered for the purpose of this paper) to get a



symmetric picture. The pupils had mirrors available, which they could use as a tool. Eileen first colours petals 1 and 4 purple. While doing that she says “It is these two, because it is the one right across.” She uses the mirror to check and she says: “If I place the mirror here, then it must be the one right across.”

**Figure 4**



She repeats the words “right across” several times and the result is that she colours petals 2 and 5 yellow and finally 3 and 6 blue. The result is shown in Figure 5. After having completed the pictures she places the mirror in various positions and inspects the image in the mirror. In one position she observes that “this is blue [both in the mirror and on the paper], so this is correct”. However, she is puzzled by the other colours, and remarks that “something

**Figure 5**

is strange”. She realises, after several attempts, that no matter how she positions the mirror, she will not see the same picture in the mirror as on the paper. She can find a position where the blue is right, and the others are not, and similarly for the yellow and the purple. Holly is sitting beside Eileen and she has made a picture based on the same principles as Eileen. Holly makes the same observation as Eileen, that there is no position for the mirror where the picture in the mirror equals the picture on the paper. Holly finds a solution by stating “these two [petals] must swap places”. While saying this she points to two petals with different colours. Eileen follows Holly’s solution and starts “recolouring” the picture (Figure 5) so that one of the yellow petals is coloured blue and one of the blue petals is coloured yellow. The areas between the petals are recoloured in a similar way.

The two tasks described here are different in terms of what is varying and what is fixed. In the first task the reflection axis is fixed and the shapes can be chosen and placed freely. In this case both Eileen and Harry choose figures equal in shape and colour, which they place on either side of the axis. Eileen also expresses something about equal distance. In the second task the underlying structure of the picture is given, only the colours can be varied. In this context Eileen first considers pairs of petals independently and colours each pair such that the petals “right across” get the same colour. When the three pairs have been coloured she is able to see the figure as a whole. Then she observes that something is not right, and by using the mirror she corrects her first solution.



In these tasks the word symmetry was actively used, and with the understanding that it meant reflection, so I will argue that what I can see from this episode are various ways the concept reflection mediates between different signs and different reference contexts. A common feature in all reference contexts was the mirror which was available as a tool. Signs that can be observed for these reference contexts include “equal distance” and “right across”.

## Episode 2

In this episode Eileen and Holly have built a tetrahedron and a square pyramid. I ask them to describe the difference between the two objects and Holly says, pointing to the tetrahedron: “This looks the same if I tilt it but this [the pyramid] doesn’t”. Following this I initiate a conversation with the girls about the objects they have constructed. An octahedron and an icosahedron also become part of the conversation. The conversation is centred around words like “tilting” and “spinning” and to what extent the objects look the same when these operations are performed. Eileen holds the octahedron and looks at it in various positions. Then she says: “This will be the same always. No, it won’t – not always. Now it is with the tip towards me. And if I place it like this, it is like this”. She shows with her hands where the tips point in the various positions. Later the girls compare the tetrahedron and the pyramid and Eileen says that they are “almost equal”. When asked why they are only almost equal she says that “there are four here, and here there are only three” which I interpret to mean the number of triangles meeting at a corner. Holly repeats her initial observation concerning the tetrahedron: “And this one is possible to turn around without seeing any difference”.

At this stage the conversation makes an unexpected turn when I, for the first time in this episode, introduce the word “symmetry”. The following dialogue takes place.

- |         |  |
|---------|--|
| Frode:  | How about symmetries on this one? [Referring to the pyramid]   |
| Holly:  | [places her hand like a vertical plane]  |
| Eileen: | It is possible to place it there. [also showing with her hand. See Figure 6]   |
| Holly:  | It is possible to place it there. It is possible to place it everywhere. If I take ... If I take it from there .. that is also possible. |
| Frode:  | Mmm. But does it, ... if I, if I turn it around like this [rotating 90 degrees], does it look the same now, for each time I rotate it?   |
| Holly:  | For each time you rotate like this, yes, but not if you had rotated like this. [rotating 45 degrees. See Figure 7]                       |



**Figure 6**



**Figure 7**

Afterwards the rotation of both the octahedron and the icosahedron is discussed and Eileen shows among other things how the octahedron can be rotated 90 degrees and she compares it to a spinning wheel.

The important observation to be made here is that the introduction of the word ‘symmetry’ immediately leads to a shift in the discourse. The girls stop talking about rotation and both of them introduce an imaginary mirror, indicated by their hands as shown in Figure 6. Shortly afterwards they go back to rotation as soon as I use that word again and their gestures adjust to this context (Figure 7).

## ANALYSIS OF THE EPISODES

The most striking observation in Episode 2 is the impact that the introduction of the word *symmetry* has on the development of the conversation. When I ask about symmetries, the pupils link to the reference context *mirror*. When I later ask about *turning* and start rotating the object they immediately switch back to the discourse about rotation and give examples of possible and impossible rotations. The criteria for possible rotations that are applied are based on what can be done in order to keep the visual appearance of the object the same. Investigating the various objects the pupils physically move the objects to illustrate positions where they look the same and where they do not look the same, when seen from the same angle. The signs that are used by the pupils are *turn*, *spin*, *tilt*, *move* and I also use the word *rotate*. These signs are used to describe a reference context, which is about rotating the objects in 3D-space and identifying positions where the visual appearance is the same. In mathematical terms the pupils are distinguishing between operations that belong to the rotational symmetry group of the object and operations that do not belong to this group.

In the discourse involving the mirror, and invoked by using the sign (word) *symmetries*, the criteria for what is possible are based on where the mirror can be placed. The pupils use their hands as an imaginary mirror, place their hands in various positions, and say “it is possible to place it here, it is possible to place it there”. This can be seen as an exact parallel to what the same two pupils did with the star pattern in Episode 1 where they worked with a real mirror. In Episode 1 the

mirror was placed in different positions and they judged positions as possible or impossible based on the properties of the pattern. When there were no possible positions for the mirror the pattern had to be changed.

Based on these observations I will propose a classification of the signs that are used in two groups. In the symmetry discourse the signs describe properties that the object has or has not. The object may acquire the property if suitable changes are done to it but in a given state it either has or has not the property in question. I denote the signs in this category *static signs*. In the rotation discourse the signs describe actions that can or cannot be performed on the object. I denote these signs *dynamic signs*.

Table 1 shows examples of signs from the two categories that can be found in the investigated episodes.

Static signs	Dynamic signs
There is one like this in between Same length between It is the one right across This is blue, so this is correct These two must swap places It is possible to place it there	This looks the same if I tilt it This will be the same always For each time you rotate like this But not if you had rotated like this

Table 1

## DISCUSSION

The concept of isometry, containing the four operations reflection, rotation, translation and glide reflection, is constructed such that the set of isometries for a given pattern forms a *group*, hence the set should be closed under compositions (Armstrong, 1988). It is therefore possible to define the isometry concept in completely abstract terms without the need of a reference context. However, the reference context has a role as a motivation for the concept. The concept “needs this ‘reference context’ as an ‘exemplary embodiment’ of a structure or a relation” (Steinbring, 2006, p. 139). All reference contexts that are described in the episodes in this paper could have the role as “exemplary embodiments” for the concept isometry, or symmetry, as would be the term (sign) most naturally used in a school setting. However, the sign symmetry is by the pupils only connected to some of the reference contexts. The situation here is similar to what has been described by Nührenbörger and Steinbring when they observe divergent constraints between the theoretical nature of mathematical knowledge in the construction processes and the need for each individual learner to construct his/her own understanding of the mathematical knowledge (2009, p. 112). The word symmetry is in everyday language used to denote reflection. When one says that something is symmetrical it is implicitly understood that reflection symmetry is meant. When symmetry is introduced in school it is also common that the first activities that the pupils are exposed to are about reflection symmetry. This tradition was kept in the lessons

reported on here, as can be seen from Episode 1. The epistemological structure that has been established through these activities could then be described by an epistemological triangle with different signs (of a static character) and different “mirror activities” as the reference contexts and the concept that mediates between sign and reference context I call *reflection*. Similarly an epistemological structure called *rotation* could be described using the dynamic signs and their corresponding reference contexts.

When symmetry is used to denote congruence transformations in general all the activities reported in the episodes have to do with symmetry, and what is presented is a variety of signs and reference contexts connected to this concept. An epistemological triangle covering these episodes could therefore be constructed in this way (Figure 8). Here I have inserted typical examples of signs that occur in the episodes.

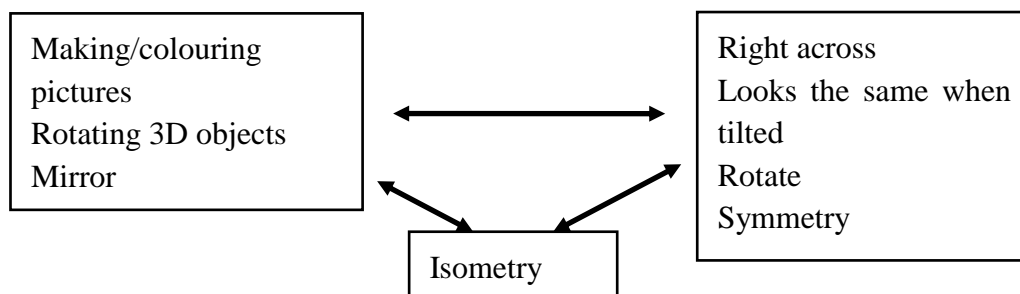


Figure 8

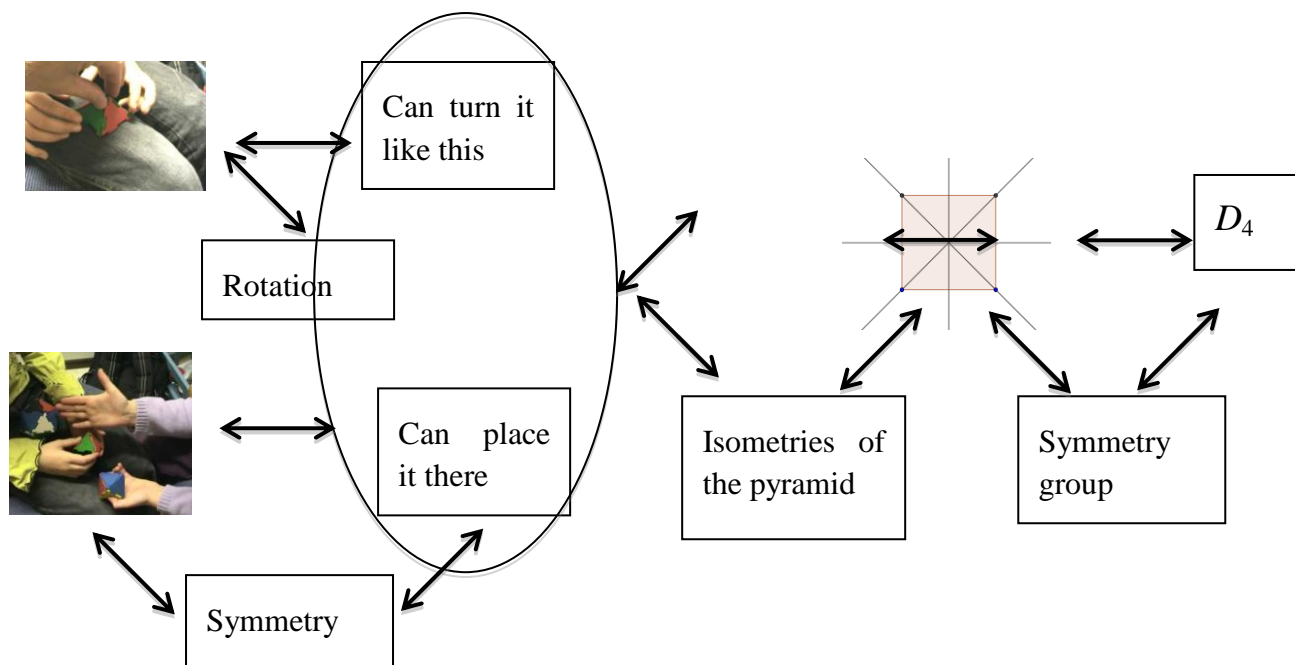


Figure 9

However, for the pupils the signs in Figure 8 belong to two different epistemological triangles, one for the static signs and one for the dynamic signs. In order to develop the general concept of isometry these epistemological triangles have to be merged into one. This triangle could further be linked to new triangles leading to further

abstraction where eventually the sign  $D_4$  (dihedral group, see e.g. Armstrong, 1988) could be a sign used for the square pyramid as a reference context, where the mediating concept is the symmetry group of the pyramid. It is important to note that there is not *one* chain leading to an abstract sign for isometry ( $D_4$ ) but a branched chain where at some point two different signs have to take the role as the reference context for the same (new) sign (see Figure 9).

## REFERENCES

- Armstrong, M. A. (1988). *Groups and symmetry*. New York: Springer.
- Coxeter, H. S. M., & Greitzer, S. L. (1967) *Geometry revisited*. Washington, DC: MAA.
- Farrugia, M. T. (2007). The use of a semiotic model to interpret meanings for *multiplication* and *division*. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the CDERME 5*, (pp. 1200-1209) Larnaca, Cyprus: ERME. Retrieved 9 September 2010 from <http://ermeweb.free.fr/CERME5b/>
- Hiele, P. M. v. (1986). *Structure and insight. A theory of mathematics education*. Orlando, FL: Academic Press.
- Nührenbörger, M., & Steinbring, H. (2009). Forms of mathematical interaction in different social settings: Examples from students', teachers' and teacher-students' communication about mathematics. *Journal of Mathematics teacher Education*, 12, 111-132.
- Ogden, C. K., & Richards, I. A. (1948). *The meaning of meaning*. New York: Harcourt, Brace & Co. (Original published in 1923)
- Peirce, C. S. (1998). *The essential Peirce. Selected philosophical writings. Vol. 2 (1893-1913)*. (Edited by the Peirce Edition Project). Bloomington, IN: Indiana University Press.
- Presmeg, N. (2002). Beliefs about the nature of mathematics in the bridging of everyday and school mathematical practices. In G. C. Leder, E. Pehkonen, G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 293-312). Dordrecht: Kluwer.
- Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction. An epistemological perspective*. New York: Springer.
- Steinbring, H. (2006). What makes a sign a mathematical sign? – An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics*, 61, 133-162.
- Utdanningsdirektoratet. (2006). *Læreplanverket for kunnskapsløftet*. Midlertidig utgave juni 2006 [National Curriculum for Knowledge Promotion in Primary and Secondary Education and Training]. Oslo: Author.
- Walkerdine, V. (1988). *The mastery of reason*. London: Routledge.
- Weisstein, E. W. (2010). *Isometry*. From [MathWorld](http://mathworld.wolfram.com/Isometry.html) – A Wolfram Web Resource. Retrieved 12 September 2010 from <http://mathworld.wolfram.com/Isometry.html>

# LANGUAGE AS A SHAPING IDENTITY TOOL: THE CASE OF IN-SERVICE GREEK TEACHERS

Konstantinos Tatsis

University of Western Macedonia, Greece

*The paper presents a part of a study that focuses on the analysis of various teachers' identities. The teachers participated in an in-service training course and their narratives were used and complemented by other data to construct – from our point of view – the various identities. Characteristic examples are presented in order to show that the notion of identity as narrative can be more operational than the traditional 'beliefs and attitudes' approach, since it may overcome the obstacles related to that approach.*

Keywords: identity, teacher training, narrative

## INTRODUCTION

Teaching mathematics and mathematics itself can be both considered as communicational activities, which would not be made possible without language. In fact, for some researchers mathematics by itself is a language (e.g. Usiskin, 1996). However, regardless of the acceptance of that 'radical' view, it is widely accepted that the way language is used in any setting – including classrooms – may affect the establishment of a community of learners (Wenger, 1998) and shape or constitute the participants' identities (Sfard & Prusak, 2005). This last view was the trigger of the study presented here. The main aim was to relate teachers' narratives with their participation in a training course. Our interest was not theory-driven; given the large number of studies on teachers' beliefs (e.g. Chamberlin, 2010; Cooney & Shealy, 1997; Franke et al., 1998; Leder, Pehkonen & Törner, 2002), we wanted to see if the notion of identity can be more useful (i.e. operational) in teacher training and what narratives would be related to the various identities.

## THEORETICAL FRAMEWORK

As mentioned earlier, the literature in teachers' and students' beliefs concerning Mathematics is vast and evolving. An important and common conclusion is that teachers' beliefs and their practice are related. It is interesting however, what Warfield, Wood and Lehman (2005) note:

... the relationship between teachers' beliefs and their instruction is not as direct as sometimes thought. Beliefs do not necessarily form a cohesive unit; it is not unusual for an individual to hold contradictory beliefs making it difficult to determine how particular beliefs influence instruction (Pajares, 1992; Pearson, 1985). (p. 442)

The idea that beliefs may not form a cohesive unit is indicative of the weaknesses of the particular approach. In our search for a theoretical and analytical tool to better

understand the hows and the whys in teachers' behaviour, we realized that the concepts of belief and attitude were not operational in Blumer's (1969) sense. Sfard and Prusak (2005) claim that "the assumption that an intention (or tendency) exists in some unspecified "pure" form independently of, and prior to, a human action was a dubious base for any empirical study" (p. 15). To put it simply, some researchers assumed that beliefs are located somewhere "inside" – or "outside" – the person and they can be merely expressed by language or other non-verbal actions; this justifies the extended use of questionnaires and interviews. But what if a person holds conflicting beliefs? How can we explain the fact that although teachers are not pleased with their practice they usually resist to reforms? The notion of *identity as a narrative* may be used to overcome such obstacles and to provide us direct access to the teaching and learning process. Sfard and Prusak (2005) differentiate between:

... actual identity, consisting of stories about the actual state of affairs, and designated identity, consisting of narratives presenting a state of affairs which, for one reason or another, is expected to be the case, if not now, then in the future. (p. 18)

Consequently, by seeing learning as the way to close "the gap between actual and designated identities" (Sfard & Prusak, 2005, p. 19) we can operationalise the notion of identity, i.e. use it as a tool to explain our teachers' practices in training and eventually teaching. The way this was done in our study is shown in the next section.

## CONTEXT AND METHODOLOGY

Fifty five in-service teachers (35 female and 20 male) had enrolled in the obligatory course named "Didactics of Mathematics", which is placed in the second – and last – year of their training. In order to participate in the training course they had to meet two requirements: over five years (and less than 25 years) of teaching in schools and passing a national exam. The duration of "Didactics of Mathematics" was one semester (three hours weekly). These teachers had little – if any – experience with enquiry classroom approaches (Cobb & Bauersfeld, 1995) or realistic mathematics (De Lange, 1999; Freudenthal, 1978), although most of them expressed their willingness to be informed about them.

The study was based on the assumption that teachers' identities (actual and designated) may account for their choices concerning their teaching practice.<sup>1</sup> The following types of data were on our disposal:

- (a) transcribed discussions and notes taken during the course,
- (b) notes made after the sessions, based on our recollection of events that occurred before, during or after the sessions,
- (c) teachers' handwritten texts taken during sessions,
- (d) teachers' texts produced as part of their assessment.

We initially assumed that our main source of data would be (a) and (b), while (c) and (d) could be used complementarily. Soon we realised that (c) and especially (d) were a reliable source of teachers' designated identities, although sometimes the relation was implicit. In the present paper, however, we focus only on data (a) and (b), in order to demonstrate the flow of our research. Following Sfard and Prusak's (2005) categorisation, three types of identifying stories may be used:

${}_AA_C$  = an identifying story told by the identified person herself. This story we call A's *first-person* identity (1st P).

${}_BA_A$  = an identifying story told to the identified person. This story we call A's *second-person* identity (2nd P).

${}_BA_C$  = a story about A told by a third party to a third party. This story we call A's *third-person* identity (3rd P). (p. 17)

Our main aim was to use the above data to identify – from the researcher's point of view – the teachers' emerging identities. Our analysis was guided by the following principles:

- The teachers' identities came from *our* interpretations of the data at hand.
- No predetermined categories were used or invented for the identities, thus all titles used from now on were put *after* the analysis.
- The data did not come from our explicit request to the teachers to talk about themselves (or somebody else), but are selected from fragments when a person is explicitly or implicitly referring to herself or somebody else.<sup>2</sup>
- Since it was impossible to obtain all types of data (e.g. 1st P, 2nd P and 3rd P stories) about all participants our analysis is far from providing a 'complete' account of all identities – if such an account can ever exist.
- The fact that the teachers knew that they are assessed by their participation (which included many assignments and a final one that was given to them three weeks before the end of the course) is taken as a contextual element (thus, it cannot be isolated in order to study its effect).

The analytical scheme is summarised in Table 1, where (a), (b), (c) and (d) stand for the types of data mentioned before. As shown in Table 1 we were interested in actual and in designated identities. According to Sfard and Prusak (2005) "Designated identities give direction to one's actions and influence one's deeds to a great extent, sometimes in ways that escape any rationalization". (p. 18)

	Data types	Story types	Formulations
Actual Identity	(a) (b) (c) (d)	${}_AA_C {}_BA_A {}_BA_C$	Assertion
Designated Identity	(a) (b) (c) (d)	${}_AA_C {}_BA_A {}_BA_C$	Wish, commitment, obligation, necessity

**Table 1: Methodological scheme**



For brevity reasons, we have chosen particular examples to demonstrate our analysis. These examples are not supposed to reflect by any means a ‘general trend’ among the teachers; they rather represent some of the most characteristic identities that emerged during the training course.

## **RESULTS OF THE STUDY – EXAMPLES OF TEACHERS’ IDENTITIES**

Most teachers participated actively in the training course, which seemed a good opportunity for them to:

- a) familiarise themselves with group work and contemporary approaches in mathematics education, and
- b) express their views on (their) teaching as a complex and demanding task.

It is obvious that the teachers’ identities were highly related to their occupational demands. This is not to say that there were not any narratives related to other aspects of their lives; however, these were significantly few, appeared rarely and most of the times not during the session hours. This is totally justifiable by the context of the situation: the teachers participated in a training course provided by an instructor they had never seen before, thus they were reluctant to talk about other things than their work. Next we present some characteristic identities that we have encountered.

### **The politically-active teacher identity**

The excerpt that follows is taken from a task that the teachers were asked to work on and comment. The task started by showing the advertisements of the three major Greek mobile telecommunication companies; their common element was the focus on the ‘population coverage’ which “is over 99%” or “touches 100%”. The focus from a teaching point of view was intended to be the use and misuse of percentages, together with the notion of limit. But once the task was presented a teacher reacted and the following discussion took place between him, another colleague and the instructor (and author of the paper). Our notes are in brackets, where we also put a code to signify the type of story told:

Teacher A: I am sorry [for the interruption], but I believe that you [the instructor and at the same time designer of the task] shouldn’t use the companies’ names in this task. [B<sub>AA</sub>]

Instructor: Why?

Teacher A: Because it’s like ‘pushing’ the students to buy or use a mobile phone from these companies. [B<sub>AA</sub>]

Teacher B: But there are no other companies in Greece!

Teacher A: Yeah, but it’s still like hidden advertisement! These tasks shouldn’t be in the textbooks...

## Working Group 9

Instructor: But the names [of the companies] are there because they are taken from real advertisements. I didn't want to delete the names, or put something like: "Company A" and "Company B". [AAc]

Teacher A: ... and it reminds me of some other tasks with hidden advertisements – but I don't remember now. Anyway, it's a pity that the whole educational system is working with market rules. I believe that in a while companies will take over schools. [BAc] And we [teachers] will be just their employees... I can see it coming. [AAc]

Some teachers showed their agreement with Teacher A's remarks; others said things like: "Come on, don't be so stuck with your politics!". So, he continued by saying:

Yeah, yeah... You know what? I will never agree to such kind of policies! [AAc]

Later on, during the break we had the chance to discuss a bit on his views on teaching and the current situation in Greek educational system. His main point was that teachers were trying hard to overcome their financial problems (due to their low salary) but the government does nothing to assist them. Actually, according to him every reform is to the wrong direction:

And every now and then they [the government] come up with a new big plan. And they've never been in a real classroom, [BAc] where I have to teach 25 to 30 students. [AAc] And some of them [children of immigrants who sometimes do not have the chance to attend extra Greek language courses] don't even speak Greek! [BAc] And what do I do? [AAc] Did you hear the story about that teacher who was giving extra-school Greek language lessons to children? She got herself into real trouble! [BAc] That's why we need to support each other! [AAc]

The previous transcripts are rich in the actual and designated identities involved. And actually, one can find data not only for teachers' identities, but also for the identities of the instructor and the policy maker(s). By focusing on Teacher A we can firstly locate the utterances which are directly related to his identity:

Actual identity:

- where I have to teach 25 to 30 students. [AAc] And what do I do? [AAc]

Designated identity:

- And we [teachers] will be just their employees... I can see it coming. [AAc]
- Yeah, yeah... You know what? I will never agree to such kind of policies! [AAc]
- That's why we need to support each other! [AAc]

From the above it is obvious that Teacher A's identity does not comprise of only the above short narratives, but should be enriched by his narratives about the other participants. Thus, we may conclude that this identity includes narratives about

teachers who feel on the one hand helpless in their own classrooms and on the other hand threatened by the reforms who – according to them – are driven by ‘market’ criteria (e.g. of competitiveness or productivity). This usually leads them to organise themselves around politically-oriented syndicates, which are most of the times sceptical to any reform movement.

### **The intelligent teacher identity**

There was only one teacher’s narratives related with the particular identity – we will name him Teacher C. In the second session of the course this teacher – who was always sitting alone in the first line of desks – invited me to visit his personal website. It turned out that the site was part of the Mensa members, i.e. the people whose IQ was measured within the upper 2% of the general population. Thus, the first narrative of that teacher already had a title – “Mensa member” – and some electronic content (mainly ‘artistic’ landscape pictures shot by him). During a break of the third session we talked about the site and he was eager to talk about his main interest, photography:

I hope you enjoyed it. It’s my big passion [photography]. Sometimes it takes me hours to get a proper shot. I prefer shooting landscapes. [A<sub>A</sub>C]

During all sessions he showed great interest in all tasks, especially those that included non-standard solution paths. Usually, he was the first that completed the task; this fact irritated some of his colleagues:

Teacher D: Come on, you always finish first! Give us some time too! [B<sub>A</sub>A]

Teacher C: I never asked you to hurry up! [A<sub>A</sub>C]

Instructor: It’s okay, you have as much time as you need.

Teacher E: Yeah, because he is so intelligent he thinks we’re all the same! [B<sub>A</sub>C]

Instructor: Please...

Teacher C: Come on, colleague...

At the end of the course he was one of the few who came individually to express his gratitude for the organisation and the realisation of the course and his views on teaching:

I think we should take advantage of all opportunities for training. And our teaching should not be based on the ‘average student’, but on each person’s characteristics, which make him unique. [A<sub>A</sub>C] Something which is not easy at all. And you also have the head of the school who puts pressure on you... [A<sub>A</sub>C] [B<sub>A</sub>C]

Until the present moment he keeps sending informative emails about his new collections of photographs. However, it is worth mentioning that his final assignment was far from showing signs of uniqueness or originality (Tatsis, in press). The above transcripts can be categorised as follows:

Actual identity:

- It's my big passion [photography]. Sometimes it takes me hours to get a proper shot. I prefer shooting landscapes. [A<sub>A</sub>C]
- Come on, you always finish first! Give us some time too! [B<sub>A</sub>A]
- I never asked you to hurry up! [A<sub>A</sub>C]
- Yeah, because he is so intelligent he thinks we're all the same! [B<sub>A</sub>C]
- And you also have the head of the school who puts pressure on you... [A<sub>A</sub>C]

Designated identity:

- I think we should take advantage of all opportunities for training. And our teaching should not be based on the 'average student', but on each person's characteristics, which make him unique. [A<sub>A</sub>C]

The intelligent teacher identity is comprised of two components, stories of the 1st P type and stories of the 2nd and 3rd P types. The 1st P stories talk about a teacher who has more interests than teaching (in our case photography) and seems to be aware of the fact that his abilities should not raise a barrier between him and his colleagues. Actually, he was participating in all discussions, trying – like all the rest – to express his view and eventually convince the other participants. His view on the teacher's positioning between policy makers, school authorities, parents and students was not much different than the one presented before: the teacher is most of the time helpless when s/he has to confront most classroom and out-of-classroom problems.

### **Other teacher identities**

Other identities that have emerged include the insecure teacher identity (in two different manifestations) and the passive teacher identity (expressed as an indifference to participation).

The insecure teacher identity in the first manifestation comprised of short narratives on the teacher's inability to cope with mathematical tasks; there was no explicit attempt to justify this fact, only continuous requests for clarification concerning the tasks contained in the session. The teachers related to this identity relied highly on the instructor's expertise to evaluate their work and they rarely showed any initiative during group work. However, they seemed to be working hardly with their colleagues, always striving to deliver their work on time.

The insecure teacher identity in the second manifestation comprised of teachers' narratives on their insecurity and how it is justified on the grounds of improper mathematical background. These narratives included stories about a mathematics teacher who – in a certain moment of their school life – halted their learning of mathematics by his attitude towards teaching and towards “those who were not so good in maths” (according to a female teacher who was explaining her attitude

## Working Group 9

during a whole group discussion. The teachers related to such narratives usually delivered poor assignments and always asked for more ‘loose’ assessment.

The passive teacher identity was manifested as an indifference to active and meaningful participation; in other words, those teachers were usually more interested in the deadline for the next assignment than its content. Some of them arrived late for the session or even asked to leave earlier. There was one teacher who asked to deliver an assignment in handwritten form, because she doesn’t have a PC at home and she doesn’t have time to work in the PC laboratory that was available to them (note that all teachers in the particular course had already at least one course in new technologies). Most of them were asking for extending the deadline for giving the assignment, because they were busy (although during the training course they are free from their teaching duties). Most of their narratives were focused on their difficulty in managing all factors of the Greek educational system. Some narratives were focused on particular issues, like the new textbooks:

Teacher F: Especially the 5th grade book is so hard. You know, they [the authors] have put so hard maths in it! And it’s so dense! Children don’t understand what I’m talking about! For example, once we were dealing with decimals and I had to go over and over again the same chapter... [A<sub>A</sub>C]

Teacher G: Yeah, and the 6th grade book is also hard...

Teacher H: The previous ones were better...

## CONCLUSIONS

By observing and analysing the teachers’ identities we have found some commonalities, which may be attributed to their common experiences gathered from years of teaching in the Greek schools – and from years of practice in the Greek educational system. One such commonality is the ‘resistance’ expressed by most teachers to employ any ‘novel’ approaches, which is related to an actual identity-narrative of a ‘helpless’ teacher caught between the various forces that are active in the system classroom-school-society. These narratives were complemented with narratives about policy makers (eventually personalised in the face of the ministers of education or university professors holding decision-making governmental positions). At the same time, these teachers sometimes used their teaching experience to justify their insistence on ‘traditional’ teaching approaches or to criticize the new textbooks – which proved to be one of their favourite and most provocative topics of discussion.

Concerning our methodological scheme, which was based on Sfard and Prusak’s (2005) approach, we realised that in order to efficiently describe teachers’ identities we need first, second and third person narratives, together with other elements related to their participation in a community of learners (in our case the community of the training course), e.g. their willingness to adhere to the norms established.

Few teachers' narratives included elements from their out-of-school life; these were related to their hobbies or to general topics related to politics or current significant events.

Some teachers' designated identities-narratives included statements like "I hope that my (teaching) practice will be improved once I get back to school", but it was obvious that they were mostly interested in making their teaching *effective* but at the same time *easier*, i.e. without conflicts (e.g. with parents or the school principle), tensions (e.g. related to classroom management issues). This was apparent in their participation and their work, which sometimes could not move beyond what has been discussed or suggested during the session.<sup>3</sup> This is not in line with the view that learning should pose challenges to the learner in order to stimulate his/her interest. Finally, returning to our initial aim, the notion of identity as a narrative seems more operational for the purpose of analysing teachers' participation in a learning process. The sources of these narratives can vary from verbal interactions to written assignments; the more varied the spectrum of data, the closer the researcher can get in the actual and designated identities of any participant.

## NOTES

1. When we talk about teachers' practices or behaviours we refer to them as either observed by the researcher or expressed in teachers' narratives about them.
2. This is the major difference between our approach and Sfard and Prusak's (2005) approach.
3. The content of the teachers' work is not the focus of the present paper. What can be noted is that sometimes when the teachers were asked to design tasks for their classrooms they merely reproduced (by slightly changing the data) the tasks that were given to them during the sessions.

## REFERENCES

- Blumer, H. (1969). *Symbolic interactionism: perspective and method*. Englewood Cliffs, NJ: Prentice-Hall.
- Chamberlin, S.A. (2010). A review of instruments created to assess affect in mathematics. *Journal of Mathematics Education*, 3, 167-182.
- Cobb, P., & Bauersfeld H. (Eds.) (1995). *The emergence of mathematical meaning: interaction in classroom cultures*. Hillsdale NJ: Lawrence Erlbaum Associates.
- Cooney, T.J., & Shealy, B.E. (1997). On understanding the structure of teachers' beliefs and their relationship to change. In E. Fennema, & B.S. Nelson (Eds.), *Mathematics teachers in transition* (pp. 87-109). Mahwah, NJ: Lawrence Erlbaum Associates.
- De Lange, J. (1999). *Framework for classroom assessment in mathematics*, National Center for Improving Student Learning and Achievement in Mathematics and Science. Madison, WI: Freudenthal Institute.

## Working Group 9

- Franke, M.L., Carpenter, T., Fennema, E., Ansell, E., & Behrend, J. (1998). Understanding teachers' self-sustaining, generative change in the context of professional development. *Teaching and Teacher Education*, 14, 67-80.
- Freudenthal, H. (1978). *Weeding and sowing. Preface of a science of mathematics education*. Dordrecht: Reidel Publishing Company.
- Leder, G.C., Pehkonen, E., & Törner, G. (2002). *Beliefs: a hidden variable in mathematics education?*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Pajares, M.F. (1992). Teachers' beliefs and educational research: cleaning up a messy construct. *Review of Educational Research*, 62, 307-332.
- Pearson, J. (1985). Are teachers' beliefs incongruent with the observed classroom behavior? *The Urban Review*, 17, 128-146.
- Tatsis, K. (in press). Assessing in-service teachers' modeling activities: issues of content and complexity. In *Proceedings of the 17th annual conference of adults learning mathematics (ALM) – a research forum. Maths at work – mathematics in a changing world*. Vox, Norwegian Agency for lifelong Learning. Oslo, Norway.
- Usiskin, Z. (1996). Mathematics as a language. In P.C. Elliott (Ed.), *Communication in mathematics, K–12 and beyond (1996 Yearbook)* (pp. 231-243). Reston, VA: National Council of Teachers of Mathematics,.
- Warfield, J., Wood, T., & Lehman, J.D. (2005). Autonomy, beliefs and the learning of elementary mathematics teachers. *Teaching and Teacher Education*, 21, 439-456.
- Wenger, E. (1998). *Communities of practice: learning, meaning and identity*. Cambridge: Cambridge University Press.

# **INTRODUCTION TO THE PAPERS OF WG10: DISCUSSING DIVERSITY IN MATHEMATICS EDUCATION FROM SOCIAL, CULTURAL AND POLITICAL PERSPECTIVES**

Paola Valero, Sarah Crafter, Uwe Gellert, and Núria Gorgorió,  
in collaboration with Alexandre Pais

## **SCOPE AND FOCUS**

Working Group 10 discussed *research* that addresses how *diversity* influences *possibilities for practice* in mathematics education. The three elements highlighted within the group were as follows. By *diversity* we understand that students, teachers, parents and many other participants in mathematics education, as well as the contexts where mathematics education takes place, are becoming more complex and varied. Diversity might be expressed in terms such as gender, ethnicity, culture, language, social and socio-economic status, disability, qualification, life opportunities, aspirations and career possibilities, etc. Contexts are diverse in terms of the variety of sites where mathematics education takes place, and the differences in the organization and structure of practice in such contexts—schools, homes, workplaces, etc. By *possibilities of practice* we understand that, in most concrete situations of mathematics education, the multiple diversities mentioned above intersect, posing challenges to actual learning and teaching practices, as well as to their improvement. Finally, when talking about *research*, we focus our understanding on the systematic reflection, of either empirical or theoretical ways in which diversity affects possibilities of mathematics education practice.

We were particularly interested in theoretical, methodological, empirical or developmental papers focusing on the social, cultural and political challenges/issues of mathematics education. The enlarged focus of the group (in relation to previous CERMEs) meant that cultural diversity was only one of the possible focuses for the current conference. Thus our appeal was also directed towards researchers who work with sociological, anthropological, discursive, political and philosophical perspectives to read mathematics education practices. We accepted for presentation and publication fifteen papers and four posters. The array of themes, methodologies and theoretical frameworks was diverse, which allowed a rich discussion among the participants.

## **THE PAPERS DISCUSSED**

Some of the papers attempted a reconceptualisation of some of the core notions that underpin much of the research in mathematics education, particularly in issues concerning the social, cultural and political dimensions of mathematics education. Tine Wedege articulated Ole Skovsmose's notion of foreground to Pierre Bourdieu's theory of habitus. Annica Anderson and Eva Nóren proposed to articulate the work of Pierre Bourdieu and Gert Biesta to current research in mathematics education,



namely by exploring the notion of agency. Annica Andersson also took on the notion of agency to discuss an analytical framework for understanding interplays between contexts and students' agency. David Kollosche invited us to posit mathematics and its education in the social arena, where school mathematical practices can be associated with processes of alienation. Charoula Stathopoulou, Karen François and Darlinda Moreira made a critical review of the way European researchers are dealing with the insights coming from ethnomathematics. Troels Lange and Tamsin Meany, examining the discourse around national testing in Australia, showed how discourses and practices outside schools pose severe restrictions to the teaching and learning of mathematics. These papers were essays with theoretical discussions addressing central dimensions connected to the relationship between society, politics and mathematics education.

Papers involving empirical work enabled the participants to reflect about different national realities regarding the teaching and learning of mathematics in relation to cultural, social and political issues. Uwe Gellert and Hauke Straehler-Pohl, in a German context, discussed innovative ways to address issues of differential access to powerful mathematics knowledge. Núria Gorgorió and Montserrat Prat, in a Spanish context, addressed the impact of teachers' social representations on the immigrant students' learning of mathematics and their identity. Andualem Tamiru Gebremichael, Simon Goodchild and Olav Nygaard showed students' perceptions about the relevance of mathematics in Ethiopia. Behiye Ubuz discussed the current status and future pathways for doctoral programs in mathematics education in Turkey, and how these programs contribute to a diversity of views on what counts as mathematics education research in this country. Richard Barwell and Cristine Suurtamm show how, in a Canadian context, it is possible to articulate important current social concerns (in this case, climate change) with the teaching and learning of mathematics.

We also had a cluster of papers addressing the relationship between school mathematics and out-of-school mathematics. Toril Rangnes explored the collaboration between school and a building company regarding mathematics learning conversations among 8<sup>th</sup> grade pupils. Javier Díez-Palomar and Sandra Torras-Ortín analysed the process of attribution regarding the relationship between school and family mathematics. Sarah Crafter and Guida de Abreu examined how teachers make sense of embedded everyday mathematics at home in relation to parents' practices. Richard Newton and Guida de Abreu, in a similar way, discussed parent and child interaction when completing primary school-style mathematics. These papers addressed the issues of transitions between different contexts of practice when a diverse set of participants meet in various sites of mathematics education. We also had four posters for presentation. Joana Latas and Darlinda Moreira described an ethnomathematical approach in a regular public Portuguese school. Petra Sevensson addressed students' foregrounds and rationales for the

learning of mathematics. Javier Díez-Palomar and Sandra Torras-Ortín described the research project FAMA, which seeks to relate family mathematics with adult education. Finally, Ana Mesquita addressed issues of parental involvement in children's mathematical achievement.

## **DIVERSITY AND THEORETICAL PERSPECTIVES**

In order to deal with the diversity of papers, the group felt the need to recurrently discuss some core notions that underpin the research presented. Some of these notions are agency, identity, and the complex relation between the individual learner and his/her social environment. This discussion was enriched by the diversity of theories deployed by the participants in their research. Most of these frameworks come from fields such as contemporary philosophy, sociology, anthropology and cultural psychology. The group were challenged by different terminologies used by the participants in their research and an attempt was made not to unify them, but to find a common ground where different notions were enacted in a clear way. There was agreement on the idea that, rather than a fixed and unified understanding of a notion, we should highlight the differences and the common points of each one. The need for further efforts concerning the theoretical and philosophical strengthening of the core notions we use in our work was enhanced by the variety of disciplinary backgrounds represented in the group participants.

## **ORGANIZATION AND FUTURE**

The leading team chose a different way of organizing the presentation of papers this year. Instead of having the author of the paper presenting his or her own work, we decided that the presentation of one's papers would be done by another author, who had 5 minutes to briefly present the paper and raise some initial questions to motivate the next 25 minutes of discussion. This strategy proved to be a productive way of organizing the sessions, and all participants agreed that the discussion which followed went further than if the own author had presented his or her paper. It also encouraged more in-depth reading of the papers in the group.

In the final session the group discussed its 'spirit' and role in CERME. There was consensus that this working group provides a forum for delegates to present research addressing less mainstream concerns of relevance for understanding the social, cultural and political constitution of mathematics education practices. The broader theoretical frameworks to study mathematics education practices, focusing on their political, cultural and sociological dimensions, were another trend that united the diversity of papers. Diversity was understood in the wider sense of diverse theories and methodologies, but also diversity of concerns and practices that become visible when we displace mathematics education from the classroom, the framework where we usually conceptualize it in, to a multiplicity of contexts where it is being practiced and made sense of. This way, diversity has the power to 'estrangle' us to the self-evidence of mathematics education as an established field of research.

# AGENCY IN MATHEMATICS EDUCATION

Annica Andersson, Eva Norén

*Aalborg University and Malmö University, Stockholm University*

*In this paper we elaborate on the notion of agency. We relate agency to Skovsmose's and Biesta's frameworks respectively. Both Skovsmose and Biesta are concerned with citizenship education, mathematics education and the purpose of education from a critical position. We explore if and how Skovsmose's and Biesta's frameworks respectively relate to agency.*

Key words: agency, citizenship, empowerment, mathematics education

## INTRODUCTION

The purpose of this paper is twofold, to widen our understanding of different approaches to the notion of agency in relation to mathematics education, and to explore the compability of Skovsmose's and Biesta's frameworks respectively in relation to agency and to each other (Wedegé, 2010). As a starting point we understand agency in a dialectic relationship to structure (Roth, 2007) and as a dynamic feature<sup>42</sup> of human beings to act independently and to make choices. Sometimes the choices are conscious; however at some times we act as agents not being aware of our options (Cohen, 1994). Agency is not just individual; it is exercised within social practises. As Holland, Lachicotte, Skinner, & Cain (2003) put it: "Agency lies in the improvisations that people create in response to particular situations" (p. 279).

In this paper firstly we investigate the notion of agency in relation to earlier research addressing agency in mathematics education. We thereafter relate agency to Skovsmose's theories of critical mathematics education (1994, 2005). The third part of the paper discusses Biesta's (2009) and Biesta and Tedder's (2006) theoretical framework for understanding agency in mathematics education. In the last section of the paper we discuss how the different frameworks and agency may add to our understanding of mathematics education practices.

As we both authors use the notion of agency in our research respectively (Norén, 2010, Andersson & Valero, 2009; Andersson, 2010 forthcoming), and both Skovsmose and Biesta theoretically have inspired our different research projects we find it fruitful to explore and elaborate on the notion of agency cooperatively in this paper. One argument is that understandings of agency in Skovsmose's philosophy of critical mathematics education and in Biesta and Tedder's (2006) may enable a way to use the theories and hence a way forward in analyzing agency in discursive practices in mathematics classrooms.

---

<sup>42</sup> Not to be seen as something situated in a person from birth

## AGENCY IN MATHEMATICS EDUCATION RESEARCH

During the last years there have been increasing attentions in mathematics education research addressing the notion agency. For example Boaler (2002), Wagner (2007) and Grootenboer and Jorgensen (2010) all refer to Pickering's (1995, p. 21) metaphor *the dance of agency*. Pickering has elaborated on scientific practices and looked at science as practice and culture. To him the 'dance of agency' takes the form of a "dialectic of resistance and accommodation" (p. 22):

Within an expanded conception of scientific culture, however – one that goes beyond science-as-knowledge, to include the material, social, and temporal dimensions of science – it becomes possible to imagine that science is not just about representation. /.../ /.../. But there is quite another way of thinking about science. One can start from the idea that the world is filled not, in the first instance, with facts and observations, but with *agency*. The world, I want to say, is constantly *doing things*, things that bear upon us not as observation statements upon disembodied intellects but as forces upon material beings (p. 5f).

As people we respond to material agency such as in winds, heating or winter. Pickering goes on describing how humans as agents seem to be different from non human agency like: "the weather, television sets, or particular accelerators" (p. 15). Humans are active and intentional beings. Pickering links Foucault's elaboration on temporal emergence and the displacement of the human subject (Foucault, 1977) via the notion of agency. According to Pickering (1995) human agency has an intentional and a social structure. The 'dance of agency' manifests itself at the human end in the intertwining of free and forced moves in practice.

Boaler (2003) uses the 'dance of agency' metaphor when illustrating the importance for mathematics learners to have an empowering identity in relation to school mathematics. To know when to draw on mathematical ideas and to be able to solve mathematical problems is a critical part of the dance of agency according to Boaler. Grootenboer and Zevenbergen (2007) note that mathematics teachers have to engage in a 'dance of agency' when to decide to encourage students' own agency as mathematicians or rearrange to the requirements of standard procedures or forms of representation. Wagner (2007) investigated students' voice in utterances, he wanted to discuss with the students who had agency in the discourse and who had control in the classroom communication. Grootenboer and Jorgensen (2010) combine the work of Boaler (2003) and the work of Burton (2001) to illustrate how teachers work together to solve mathematical problems. Teachers' sense of agency allowed them to expand their sense of learning and achievement through the solving of mathematical tasks, relying on the members of the group, their individuals' knowing, and the collective knowing of the group. Powell (2004) uses the notion of agency and motivation to avoid deterministic theories and to resist deficiency explanations of African-American students' failure in mathematics in the US. Powell's research study among 24 sixth graders gave "evidence of the mathematical achievement of

students of colour as a byproduct of their engagement of their agency” (p.10). Powell found that the students initiated investigations, reasoned and progressed in building foundational understanding of certain mathematical ideas. He continues saying that understanding agency “is particularly important since both failure and success can be located within the same set of social, economic, and school conditions that usually are described as only producing failure” (p. 6).

The last example we present comes from a Danish context, where Lange (2010) in his study on 10-year old children concludes that children seem to be suspended between two conflicting experiences: from the practical and creative school subjects and the school subjects, like mathematics, not so creative but important for their future. In the practical and creative school subjects students experience they have more space for agency than in mathematics classrooms<sup>43</sup>.

From this we infer that the research mentioned above seem to draw on differently theoretical standpoints such as socio cultural ones (research referring to Pickering’s ‘dance of agency’ and Lange) and critical theories (Powell).

## **SKOVSMOSE’S PHILOSOPHY OF CRITICAL MATHEMATICS EDUCATION**

In this part of the paper we explore the notion of agency in relation to Skovsmose’s work on critical mathematics education.

### **Towards a philosophy of critical mathematics education, including agency**

Skovsmose developed his philosophy of critical mathematics education based on the Frankfurter school. The influence of Habermas and the philosophy of Critical Theory can be traced in his work “Towards a philosophy of critical mathematics education” (1994).

Critical Theory has changed its emphases since its beginning with the Frankfurter school. Today it has included contributions from structuralism, feminism and lately postmodernism and post colonialism (Popkewitz, 1999). The different perspectives have various assumptions regarding definition of power and the self. One influence of post modernity is from Foucault and his conception of power as productive and positive, not repressive and negative. Power is then conceptualised as working in two directions and not as a one-way surveillance technique of power. The late Foucault (1980, 1982) saw discourse as a medium through which power relations produced speaking objects – in our view this relates to the concept of agency even though Foucault did not discuss agency, but related to human beings as agents

Critical mathematics education emphasizes social justice issues and students empowerment through mathematics education. In Skovsmose’s work, (1994), a basic

---

<sup>43</sup> Though we don’t perceive mathematics as non creative, Lange relates to what students told him

assumption is that implicit as well as explicit functions of mathematics education are of importance for society and democracy (see also Skovsmose, 1998; Skovsmose & Valero, 2001). According to a thesis of mathematics as a formatting power Skovsmose finds mathematics as “an essential instrument when technological authority is exercised. Mathematics is part of technological empowerment” (1998, p. 201). When he spoke about the formatting power of mathematics Skovsmose says it was a way to try to address the relationship between mathematical knowledge and power (2005). Skovsmose also articulates that mathematics education serves as a gatekeeper, to who will get and who will not get access to the information and communication structures in society (1994, 1998, and 2005). He concludes saying that the learner is a member of society and mathematics can be a source for decision-making and action makes mathematics education a critical feature in society.

When conceptualising Critical Theory “as an interdisciplinary attempt to raise awareness of problematic socio-political states of affairs” (Skovsmose 2005, p. 130) Skovsmose relates to post-modernism. He also relates to Foucault and his description of technologies of the social, the connections between power and knowledge, and to discourse. To Skovsmose it seems obvious that knowledge can be expressed in ways of acting.

### **Is agency part of Skovsmose’s writings?**

Certain forms of acting and communicating in the mathematics classroom may support the development of citizenship. According to Skovsmose (1998) citizenship is about to face the “output” from authorities, but also to provide an “input” to authority. Education for citizenship “also presupposes participation” (p. 199). Skovsmose states that “mathematics education could play an important role in developing critical citizenship” (2005, p. 132). To Skovsmose “empowerment” seems to refer to a person in an informal meaning; as to have the capacity to speak for oneself. We believe this reasoning of Skovsmose applies to the notion of agency in terms of students’ capacity to act independently and to make personal choices in a situation. Intentionality and action presupposes agency.

Skovsmose (1994) does not use the concept agency explicitly, in his writings it is an evasive concept that conceals behind expressions as empowerment, intentionality, action and choice. He writes:

Actions cannot be described in mechanical or in biological terms; and if a person’s behaviour can in fact be described in such a way, then behaviour is not a part of his or her actions. It is not a personal action to breathe or to let one’s hair grow. This I see as the first essential condition for performing an act: indeterminism must exist, or, the acting person must be in a situation where choice is possible. The person acting must have some idea about goals and reasons for obtaining them (p. 176).

Skovsmose writes that it does not make sense to talk about human action when a person is forced to do something or when a person is doing something out of a habit

or as a reflex, like when combing your hair. To be called action a person's intentions must be present in what is done. "Intentions are examples of intentionality directed towards action" (p. 177). But a person may not always be aware of her/his intentions. Intentions are grounded in a "landscape of pre-intentions or *dispositions*" (p. 178). Skovsmose divides dispositions in background and foreground. Background belongs to the history of a person, and foreground to the possibilities a certain social situation makes available for the person to perceive as her/his possibilities. The dispositions of a person reveals when a person comes to action. Skovsmose sees learning as caused by the intentions of the learner and learning has to be performed by the learner:

Students will enter school with ideas, hopes and expectations. Intentions are inherent within every human being. /.../. But the demands of the situation in school too often result in *broken* or *ignored intentions*. When students' intentions are ignored, it seems impossible for students to perform actions, which could fulfil negotiated intentions (p. 187).

Skovsmose (2005) articulates that possible structures behind social events may be much more complex than those explanatory principles which are conceptualised within contemporary sociology. He writes:

Social practices, or collective actions, can appear to be so complex that no 'acting subjects' (a person, a group of persons, an institution, a government, an organisation) can be identified. The very existence of an acting subject may appear impossible. Such actions I will refer to as *happenings*. A happening is certainly not a natural phenomenon and it cannot be explained within a framework borrowed from the natural science. Happenings are social constructions and achievements, which pack together a density of contingencies. In a happening, the involved persons are doing something, but it seems out of control as of what this could imply (p. 135f).

As an example of a happening he addresses Woodstock, the music festival. Persons taking part in a happening may not be aware of their role in it, and they may not have any control. In a happening there is no defined acting subject and a happening cannot be explained as a sum of human actions. But, we inquire, if agency is looked upon as a result of social practices, can a happening then be explained in the terms of agency?

There is a possibility to see happenings and intentions as intersecting. When intersecting the notions converge close to the notion of agency; as the capacity of human beings to act independently and to make choices of their own, though they not always may be aware of it, and as a result of social practices, as stated in the beginning of this paper.

Another way of understanding Skovsmose and the notion of agency is when Skovsmose suggest students and teachers to work within an investigative landscape, in contrast to the exercise paradigm. Skovsmose writes that working within the

investigative paradigm provides recourses for working with investigations as a learning milieu. He finishes his article with the following sentence:

My only hope is that finding a route among the different milieus of learning may offer new resources for making the students both acting and reflecting and in this way providing mathematics education with a critical dimension (2001, p. 131).

We conclude that Skovsmose within a critical paradigm concerns democratic aspects of mathematics education, part of that is the intentions of students and their role as acting and reflecting subjects in mathematics classrooms. To us the notion of agency seems to work well with Skovsmose's critical mathematics education.

### **BIESTA'S PHILOSOPHY OF (MATHEMATICS) EDUCATION**

In this part of the paper we explore the notion of agency in Biesta's (2009) and Biesta's & Tedder's (2006) writings. Biesta's (2006) philosophy of education is influenced by philosophers such as Dewey and Derrida.

Biesta (2009) discusses the purposes of education against a background of, what he understand as "the new language of learning" (p.6), that is e.g. the rise of theories emphasising teachers more facilitating role in relation to the active role students' play in their construction of knowledge, the shift of responsibility for education turning education "from a right into a duty" (p.5). He concretizes his reasoning with examples from citizenship education and mathematics education and we find these examples interesting to emphasis in relation to Skovsmose's writings and the concept of agency.

Biesta's (2009) way of understanding the purpose of education as such he describes with the qualification, the socialisation and the subjectification functions. The qualification purpose provides students with skills and knowledge required for particular professions, further studies or more general as an introduction to modern culture. Biesta argues that the qualification function is obviously a major function of schooling. The socialisation function has to do with the purposes to "become members of and part of particular social, cultural and political 'orders'" (ibid p.40). Biesta elaborates this purpose further:

*But even if socialisation is not the explicit aim of educational programs and practices, it will still function in this way as, for example, has been shown by research on the hidden curriculum. Through its socializing function education inserts individuals into existing ways of doing and being and, through this, plays an important role in the continuation of culture and tradition – both with regard to its desirable and its undesirable aspects. (ibid p. 40)*

The last purpose of education Biesta refers to is the subjectification process. Biesta writes that education has an impact on the processes of becoming a subject. In education newcomers do not only get inserted into existing orders, they also get to know how to become independent of such orders. The subjectification process,



understood as a process of becoming thus relates to a way of independence and being agentic. An example of the subjectification process is Biesta's reasoning about a citizenship education taking political agency seriously:

Political knowledge and understanding (qualification) can be an important element for the development of political ways of being and doing (subjectification), just as a strong focus on socialisation into a particular citizenship order can actually lead to resistance which, in itself, can be taken as a sign of subjectification (p. 42).

We agree with Biesta when he concludes that whether all education actually contributes to subjectification of students is debatable. However, any education worthy of its name should contribute to processes of subjectification that allow those educated to become more autonomous and independent in their thinking and acting; thus becoming agentic in our understanding of agency. He continues discussing the subjectification function in mathematics education as raising possibilities for students becoming a person who "through the power or mathematical reasoning is able to gain a more autonomous or considered position towards tradition and common sense" (p.43). To us the last quote seems to be an argument for agency as a notion relating to Biesta's philosophy. Biesta exemplifies his arguments with e.g. exploring moral possibilities of mathematics, e.g. dealing with division in relation to sharing – suggestions we think connects very well with Skovsmose's theory about critical mathematics education.

### **Agency in Biesta and Tedder's writings**

Biesta and Tedder (2006) put forward two key ideas for understanding agency, theoretically mainly building on the work by Emirbayer & Mische (1998). The first idea is that agency "should not be understood as a capacity, and particular not an individual's capacity, but should always be understood in transactional terms, that is, as a quality of the engagement of actors with temporal-relational contexts of action" (p.18). They refer to an ecological understanding of agency, "i.e. an understanding that always encompasses actors-in-transaction-with-context, actors acting by-means-of an-environment rather than simply *in* an environment" (ibid p. 18). The second key idea is that agency should not be understood as a possession of the individual, rather that something that is achieved" (ibid p. 18) in relation to the particular context. They continue: "the idea of achieving agency makes it possible to understand why individual can be agentic in one situation but not in another. It moves the explanation away, in other words from the individual and locates it firmly in the transaction (ibid p.19).

Concluding, Biesta suggests that we engage in a discussion about the purposes of education, where he sees the notions of qualification, subjectification and socialisation as important and interrelating components. His examples from citizenship education and mathematics education highlight possibilities for further discussion within these areas. As we understand agency in Biesta's and Tedder's

words as achieved in relation to/transaction with time and context, narrow in the focus, from the larger purpose of education to the individual within education. We see it as one way to further elaborate on relations and intersections between the individual, society and mathematics education.

## CONCLUSIONS

What we intended is to widen our understanding of different approaches to the notion of agency in relation to mathematics education, and to investigate the notion of agency in relation to Skovsmose's and Biesta's writings respectively. We find that Skovsmose, Biesta, and Biesta and Tedder have established cores grounded on basic democratic concerns, citizenship and empowerment. Though Skovsmose's writings are explicitly addressing mathematics education Biesta and Tedder's are not. In line with Skovsmose we believe that certain forms of communication in the mathematics classroom may support the development of citizenship. We think the "certain form of communication" may enhance students' space for agency, and vice versa, students' agency may support the "certain form of communication".

Biesta and Tedder are implicitly concerned with agency when discussing subjectivity and the individual becoming agentic. We find Skovsmose's and Biesta's frameworks compatible to some extent (Wedge, 2010) but we need to explore this further. One reason for that is that the framework of Skovsmose is grounded in his many writings since a long time back, Biesta and Tedder's work is not.

In mathematics education research, agency can be used both as a tool for locating certain forms of communication in the mathematics classroom and for locating students' activity and intentions in the communication. An example is when students' agency change directions of teachers' already planned lessons. Also the empowerment of learners as individuals and as citizens in today's society can be discussed when relating to agency. The notion of agency can add to our understanding of mathematics education practices.

As learners' intentions and their role as acting and reflecting subjects in mathematics classrooms can be discussed when relating to agency some questions arise. The questions are concerned with whether agency is something a learner can attain or achieve? Is agency already there? Can mathematics education enhance agency?

## REFERENCES

- Andersson, A. (2010). Can a critical pedagogy in mathematics lead to achievement, engagement and social empowerment? *Philosophy of Mathematics Education Journal* (25).
- Andersson, A., & Valero, P. (2009). Mathematics education giving meaning to social science students. A case from Sweden. Accepted for proceedings at II Congreso

Internacional de Investigación, Educación y Formación Docente in Medellin, Colombia 26-28 Aug 2009.

- Biesta, G. (2006). *Beyond learning: Democratic education for a human future*. Boulder: Paradigm Publishers.
- Biesta, G. (2009). Good education in an age of measurement: on the need to reconnect with the question of purpose in education. *Educational assessment, evaluation and accountability* 21(1) 33-46.
- Biesta, G., & Tedder, M. (2006). How is agency possible? Towards and ecological understanding of agency-as-achievement. Learning lives, Working paper 5. University of Exeter. Retrieved from [www.tlrp.org](http://www.tlrp.org)
- Boaler, J. (2002). *Experiencing School Mathematics. Revised and Expanded Edition. Traditional and Reform Approaches to Teaching and the Impact on Student Learning*. London: Lawrence Erlbaum Associates, Publishers
- Boaler, J. (2003). Studying and capturing the complexity of practice. The case of “Dance of Agency”. *Paper presented at the 27th International Group for the Psychology of Mathematics Education Conference Held Jointly with the 25th PME-NA Conference (Honolulu, HI, Jul 13-18, 2003)*, v1 p3-16.
- Boaler, J., & Greeno, J. G. (2000). Identity, Agency, and Knowing in Mathematics Worlds. In J. Boaler (ed). *Multiple Perspectives on Mathematics Teaching and Learning*. Westport, London: Ablex Publishing.
- Burton, L. (2001). Research mathematicians as learners – and what mathematics education can learn from them. *British Educational Research Journal*, 27(5) 589-599.
- Cohen, A. (1994). *Self Consciousness: An Alternative Anthropology of Identity*. London: Routledge
- Emirbayer, M. & Mische, A. (1998). What is agency? *American Journal of Sociology*. 103(4), 962-1023.
- Foucault, M. (1977). *Discipline and Punish*. New York: Vintage Books.
- Foucault, M. (1980). *Power/knowledge: Selected Interviews and Other Writings 1972 - 1977*. London: Harvester.
- Foucault, M. (1982). Afterword: The subject and power. In H.L. Dreyfus & P. Rabinow. *Michel Foucault: Beyond Structuralism and Hermeneutics* (2nd ed., pp. 208-216). Chicago: University of Chicago Press.
- Grootenboer, P., & Jorgensen, R. (2009). Towards a theory of identity and agency in coming to learn mathematics. *Eurasia Journal of Mathematics Science and Technology Education* 5(3), 255-266.
- Grootenboer, P., & Zevenbergen, R. (2007). Identity and mathematics: Towards a theory of agency in coming to learn mathematics. *Mathematics: Essential research, essential practice*.

## Working Group 10

*Proceedings of the 30th annual conference of the mathematics education research group of Australasia Tasmania, Vol. 1* (pp. 335-344). Adelaide: MERGA.

Holland, D., Lachicotte Jr, W., Skinner, D. & Cain, C. (2003). *Identity and Agency in Cultural Worlds*. London, Cambridge: Harvard University Press.

Lange, T. (2010). "Tell them that we like to decide for ourselves". Children's agency in mathematics education. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.). *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2587-2596). Lyon: Institut National De Recherche Pédagogique 2010.

Norén, E. (2010). Discourses and Agency in a Multilingual Mathematics Classroom. *Nordic Educational Research Working-paper series*. Stockholm University: Stockholm.

Pickering, A. (1995). *The mangle of practice: Time, agency, and science*. Chicago: University of Chicago Press.

Popkewitz, T. (1999). A social epistemology of educational research. In T. Popkewitz & L. Fendler (Eds.). *Critical theories in education* (pp. 17-42). London: Routledge.

Powell, A. (2004). The Diversity backlash and the mathematical agency of students of colour. In M. Johnsen Høines and A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen: Bergen University College

Roth, W.M. (2007). Identity in science literacy. In W. M. Roth and K. Tobin (Eds), *Science, learning, identity: Sociocultural and cultural-historical perspectives* (pp. 147-185). Rotterdam: Sense Publishers.

Skovsmose, O. (1994). *Towards a Philosophy of Critical Mathematics Education*. Dordrecht: Kluwer Academic Press.

Skovsmose, O. (1998). Linking Mathematics Education and Democracy: Citizenship, Mathematical Archeology, Mathematicy and Deliberative Action. *Zentralblatt fur Didaktik der Mathematik/International Reviews on Mathematics Education*, 30(6), 195-203.

Skovsmose, O. (2001). Landscapes of Investigation. *ZDM* 33(4), 123-132.

Skovsmose, O. (2005). *Travelling Through Education. Uncertainty, Mathematics, Responsibility*. Rotterdam: Sense Publishers.

Wagner, D. (2007). Students' critical awareness of voice and agency in mathematics classroom discourse. *Mathematical Thinking and Learning*, 9(1), 31-50.

Wedegé, T. (2010). Connecting theories in mathematics education: from bricolage to professionalism. *Nordic studies in Mathematics education*, 15(1), 59-78.

# INTERPLAYS BETWEEN CONTEXT AND STUDENTS' ACHIEVEMENT OF AGENCY.

Annica Andersson

Aalborg University, Denmark, and Malmö University, Sweden

*The purpose with this paper is to explore, within a socio-cultural theoretical landscape, an analytical framework for understanding interplays between contexts and students' achievement of agency in mathematics education. However, it became apparent that account also had to be taken for relationships between students' identities and their achievement of agency as they impact, together on the students' decisions to participate in mathematics education and hence in their learning of mathematics. A case study from an upper secondary critical mathematics innovation setting provides an empirical example of how the analytical framework was used.*

Keywords: critical mathematics education, agency, identity, context, student.

## INTRODUCTION

The students in focus of this research are a group of the increasing number of upper-secondary students who either just dislike mathematics, or who objectify (Sfard, 2008) themselves as e.g. “math-haters”, or remedial students who are present in classrooms but not obviously participating in the learning activities. The students referred to are not students with specific learning difficulties in mathematics, the concerns are for students who in a mainstream way take compulsory mathematics classes but whose well-being “diminishes when they are asked to engage with mathematics learning” (Clarkson, Seah & Bishop, 2010, p.1), whose attitudes to mathematics has fallen as they progressed through school (Beswick, Watson & Brown, 2006) and who have not experienced feelings of inclusiveness in mathematics education (Solomon, 2009). My concern is for these students who have not had the opportunity to experience a prior mathematics education that made sense to them.

For the empirical part of my Ph.D.-study a critical mathematics innovation was arranged for Swedish upper secondary social science students' first compulsory mathematics course. The pedagogy was deeply inspired by concerns raised in critical mathematics education (Skovsmose, 2005) and connected mathematics to society as intended within the domain of sociomathematics (Wedegé, 2010). The innovation was a serious attempt to consider possibilities for a pedagogy acknowledging concerns in critical mathematics education although within the frames of national curriculum and assessment qualities. A focus on individual's voices in this study provided a way to understand students' shifts in participation and changes in identities during this particular mathematics course. The purpose of this paper is to present the analytical framework which assisted explaining the interplays between contexts and students' achievement of agency. Under consideration are different

levels of context that impacted on the classroom work; the socio-political school context (Valero, 2002), the situation contexts within the classrooms and the task contexts expressed in e.g. textbooks exercises and through the developed pedagogical projects (Wedge, 1999). My underlying expectation in this paper is to open up for critique and scrutiny of the analytical framework.

This paper is built on two assumptions. First, mathematics learning can be a good thing and may contribute to empowerment and emancipation, even if that not always seems to be the case today (D'Ambrosio, 2007; Skovsmose, 2005). Second, there is a strong belief that students' expect and enjoy to be able to make decisions on their personal learning, that they want to have their ideas valued, and that they enjoy being treated as responsible young adults.

## **THE THEORETICAL FRAMEWORK**

The theoretical framework that guided the Ph.D.-study was grounded within a contemporary social-cultural perspective. Learning was thus viewed as a social activity, implying that learning processes "are constituted in the encounter between contextualised, historically grounded human beings and their activity in particular settings and spaces that are socially structured" (Valero, 2004, p.10). Learning in a social-cultural perspective is not just about getting to know, learning is also about becoming someone (Radford, 2008), thus a movement between present and designated identities (Sfard & Prusak, 2005; Solomon, 2009). This view of learning provided a lens for understanding relationships between contexts, identities and students' achievement of agency in mathematics education at particular historical times during the course. The coming section is divided into three parts. First, theoretical concerns raised in critical mathematics education are presented. Second, an analytical framework for understanding agency is put forward and third, relationships between achievement of agency, context, identity change and learning are discussed.

### **Theoretical concerns raised in critical mathematics education**

The developed pedagogy was deep inspired by critical mathematics education theory. Critical mathematics education is not to be understood as a special way of teaching or branch of mathematics education, on the contrary it is raised by concerns that need to be accounted for with reference to the particular context where the education takes place (Skovsmose, 2005). Concerns raised by Skovsmose addresses issues as how mathematics education can be stratifying and legitimising inclusion and exclusion. While Skovsmose writes about students' inclusion and exclusion in a wider socio-political context, Solomon (2009) engages in questions about inclusion and exclusion within mathematics classrooms. She points to reasons for students' identities of exclusion to be "a product of their particular educational histories and the ways in which they have responded to the ascribed or designated identities carried in repeated discursive positioning." (Solomon, 2009, p. 137)

Critical mathematics education is also concerned with the nature of the competencies supported in mathematics education, as e.g. if learning mathematics can support empowerment. Students' development of competencies as mathemacy (Skovsmose, 2005), or mathematical literacy (Solomon, 2009) in a way that supports critical citizenship can be seen as empowering as expressed by Ernest (2002, p. 1-2):

Social empowerment through mathematics concerns the ability to use mathematics to better one's life chances in study and work and to participate more fully in society through critical mathematical citizenship. Thus it involves the gaining of power over a broader social domain, including the worlds of work, life and social affairs.

Critical mathematics education also raises concerns and awareness of the students' whole situation. They have different foregrounds, understood as those opportunities that the social, political and cultural situation provides for the individual and backgrounds in relation to mathematics (Skovsmose, 2005). Regarding the students' intentions to participate in mathematics education, understanding participating and learning as action, Skovsmose (ibid, p. 20) continues:

Intentions of a person refer not only to his or her background, but also to the way he or she experience possibilities. Intentions express expectations, aspirations and hopes.

Indeed, critical mathematics education raises concerns about inclusion and exclusion, considers the competences that are learnt and take students intentionality seriously in mathematics education. That is why a critical mathematics innovation became the context, the background setting, for the empirical part of my Ph.D. thesis and thus the mathematics education context for the students referred to in this study.

### **Agency**

Learning is a social constructed activity and requires an agent, a committed human being who makes the decision to engage herself in the activity of learning (Valero & Stenoft, 2010). Human agency denotes the faculty to act deliberately according to one's personal will and by that make free choices (Johnson, 2000). A person's agency can be understood as initiating ideas, agreeing with others, to elaborate and critique, and questioning or disagreeing with others (Gresalfi et al, 2009, p. 53). There is an obvious dialectic relationship between agency and structures (e.g. Holland et al. 1998). Concerns about agency in education research relates, from the socio-cultural theoretical perspective, to the empirical conditions of agency as when and in what way agency is possible (Biesta & Tedder, 2006). Building mainly on Emirbayer and Mische's work (1998), Biesta and Tedder put forward two key ideas for understanding agency. They first suggest that agency should be understood in an ecological way, i.e. strongly connected to context and second, they implicate that agency should be seen as achieved and not as an individual's capacity:

... agency should not be understood as a possession of the individual, but rather as something that is *achieved* in and through the engagement with a particular temporal-

relational situation. The idea of *achieving agency* makes it possible to understand why individuals can be agentic in one situation but not in another. It moves the explanation away, in other words, from the individual and locates it firmly in the transaction (which also implies that the achievement in one situation does not mean that it will necessarily be achieved in other situations as well). (Biesta & Tedder, 2006, p.18-19)

Agency is thus not about *how* we act in particular situations; the agentic dimension “lies in the ways in which we have control over the ways in which we respond to the situation” (p. 20-21). Within a socio-cultural theoretical framework, regarding learning as a social activity, a definition of agency in line with Biesta and Tedder’s ecological understanding of agency fits particular well.

Biesta and Tedder (ibid) argue that a dimension of agency can be understood as ways in which actors bring their past experiences and future orientations to bear on the present situation, resonating with Skovsmose’s (2005) understanding of students’ backgrounds and foregrounds as reasons for students’ intentions for engaging in mathematics learning. Another aspect Biesta and Tedder (2006) address relates to “the extent to which people are able to distance themselves from their agentic orientations, i.e. make such orientations the object of reflection and imagination” (p. 21). This way of reasoning resonates with the definition of identity as the reified, endorsed and significant narratives told about a person as suggested by Sfard and Prusak (2005) and thus firmly connects the concept of achieving agency with changes of identity. This connection will be elaborated further in the next section.

### **Identity-agency dialectic relationship**

The relationship between achieving agency and changes of identity needs to be elaborated further as there are other aspects of importance to emphasise. One aspect relates to objectification processes (Sfard, 2008). As a student, being objectified, labelled from experiences in the past, possibly has an impact on how the student act and behave in the future. The labels originate in what Sfard recognise as objectification processes, initiated by our way of transplanting words from one discourse to another. Sfard identifies *metaphors of object* as a special figurative expression with “roots in our tendency for picturing the perceptually inaccessible world of human thinking in the image of material reality” (p. 42). These metaphors can obviously be useful or potentially harmful depending on whose actions is objectified and in what way they are objectified in the mathematics education language discourse.

By comparison, another way of understanding the identity-agency relationship is through Boaler and Greeno’s (2000) discussions of students’ experiences of agency with reference to the notion of *figured worlds* (Holland et al, 1998), that is “places where agents come together to construct joint meanings and activities” (p. 173). Agency is here conceived in terms of authorship and as a prime aspect of identity. A mathematics classroom may be thought of as a socially and culturally constructed



figured world. Boaler and Greeno's (2000) research showed that the figured worlds of many mathematics classrooms are unusually narrow and ritualistic, leading able students to reject the discipline at a sensitive stage of their identity development: "traditional pedagogies and procedural views of mathematics combine to produce environments in which most students must surrender agency and thought in order to follow predetermined routines" (ibid, p. 171) and thus lead to exclusion in the way Solomon (2009) addresses the problem. Boaler and Greeno argue that capable students discard mathematics as the views of mathematic education run counter to the students developing identification as responsible, thinking agents. They argue that students do not just learn mathematics in school classrooms, they also learn to be and thus they move between the present and designated identities as described by Sfard and Prusak (2005). Boaler and Greeno's results suggest that many students find the narrowly defined roles they are required to play within mathematics education incompatible with their developing identities.

## A CASE STUDY

In this section four 'critical moments' from Sandra's<sup>44</sup> course trajectory will exemplify how the analytical framework was used. The information was collected with ethnographic methods throughout my participation in two social science students' mathematics classes. In Sandra's case, the data consisted of several spontaneous conversations (Kvale & Brinkman, 2009) and written documents as classroom blog comments, evaluation sheets, logbooks and my field notes. The data was primarily coded in line with Sfard and Prusak's (2005) suggested analytical framework defining identities as reified, endorsed and significant narratives. In this way, the analysed stories were the stories that were significant for the students themselves at those times they were told. To clarify *why* these particular narratives were told/written at specific historical times, all narratives were arranged in chronological order on a timeline that became a 'storyline', one 'storyline' for each student. Events and incidents at the school and the teacher's assessments comments and test results were added on the 'storylines'. At last, students' comments referring to individual or group action as e.g. "I decided to..." or "we went to the bank", or "fucked algebra today" finalised the 'storylines'. In these 'storylines', clusters, which were labelled 'critical moments', emerged in different ways and numbers for different students however with family resemblances. The 'critical moments' illumined changes in the students' narrations of themselves and how the transactions with contexts impacted on the students' identities through changes in their expressed narratives at particular historical times.

In this paper four 'critical moments' from Sandra's mathematics course trajectory provided a frame within which to consider interplays between agency and context.

---

<sup>44</sup> All names are pseudonyms. The data was analysed in Swedish and here translated by the author.

They were, in chronological order: 1) Sandra's narratives told in the transition phase between lower secondary and upper secondary schools 2) Sandra's narratives during the project '*Making your dreams come true*' 3) Sandra's narratives during a teaching sequence organised with textbook work and 4) Sandra's narratives after the larger cross-subject project '*Ecological footprints*'. I refer to Andersson (2010) for an in-depth description of the development of the teaching sequences and the projects within their societal-, and school contextual background.

**First 'critical moment'**. Sandra initially shared with me that she had always disliked mathematics because she had "mathematics anxiety". This label was Sandra's way of objectifying herself (Sfard, 2008) and causing her "not wanting to spend more time with mathematics than was absolutely needed". That is the reason for why she did not want me to interview her, which would, as she said, "result in more mathematics related time". However, I was very welcome to read her blog comments, evaluation sheets and logbook and to talk with her in the classroom.

Sandra told me she desperately wanted to pass the mathematics course, as it was required for her future university studies. Foregrounding herself as a university student became her intentionality for attending and passing the mathematics courses that were required by society. The socio-political context constrained Sandra's achievement of agency; she could not decide to not participate, as her designated identity was to become an university student. Within the situation context, objectifying herself with the label 'having math-anxiety' seemed to impact on her decisions on how to act within the classroom (e.g. spending a minimum of time with mathematics). Sandra's agentic orientation might be characterised as iterative at this time; she effected action consistent with schemas derived from prior personal experiences.

**Second 'critical moment'**. This moment occurred during a two-week project where the students got high possibilities for deciding on task contexts, personal time and work distribution. Sandra evaluated her project work in the following way:

We distributed the time well, I think. [...] The group worked well. We were good at different things, and helped each other. I am proud of the work I have done as I felt I could contribute a lot in the beginning when we talked about borrowing money and interest rates. To plan time and content self got me to feel it was related to me. I think mathematics has been a little more fun than usual. [...] I feel the project has been meaningful and to look at mathematics from different angles (vända och vrida på matematiken) was positive. But I would have liked more time for explanations from the teacher, as mathematics is difficult for me. (Sandra, evaluation sheet, 10-2009)

During this project Sandra achieved agency in relation to task context and situation context. Her personal influence on content, time and work distribution impacted on her decisions to engage in the classroom activities in a different way than she

intended at the course start. In addition she experienced feelings as ‘a little fun’ and mathematics as ‘meaningful’. At this time Sandra took a projective action for learning differently to the initially intended and got rewarded with feelings as “being proud” of her work. However, even if she was proud of her work and actually passed this sequence with distinction (teacher, results sheet), the last sentence indicated that being objectified with ‘mathematics anxiety’ still implied her wishing for extra help and support from her teacher.

**Third ‘critical moment’.** In the middle of the semester the students were expected to work with textbook algebra exercises during two separate weeks. In contrast to the second critical moment, during these weeks Sandra only made two blog comments. Sandra’s first entry emphasised Sandra’s worries and feelings of stress for not passing a coming test: *I am currently worried about the test. I have received help with things I need help with. Stress. Stress. (Sandra, blog, 10-2009).*

In class she repeatedly asked the teacher about what would happen if she did not pass the test, and she asked for advice on exercises that was ‘extra smart to calculate’ when preparing for the test (Annica, field notes). She worried, and her achievement of agency was restricted to do what was required for just passing the test she was so anxious about. Not passing the test would impact on her designated identity. At this time her label ‘mathematics anxiety’ could be interpreted as ‘test anxiety’, however she never used this word herself.

Her second blog comment during this period was: *Quiet, concentrated and do my best (Sandra, blog, 11-2009)*

Sandra’s positive experience of the prior project appeared to have vanished. The tensions between task contexts, restricted to advised exercises in algebra, “something I don’t understand why I have to learn” (Sandra, classroom conversation), the situation context with expected quiet individual textbook work and her foreground to become a university student were obvious in her actions. Her “math anxiety”, resulting in her imagining herself not passing and thus not becoming what she wanted, became problematic and restricted her achievement of agency at this particular time.

**Fourth ‘critical moment’.** At this time a larger cross-subject project commenced themed “Students’ Ecological footprints on earth”. At that time Sandra’s logbook was rich with comments regarding hers and her work-friend’s collaborative work. This excerpt exemplifies her reflections on her mathematics learning during the project:

During the project I have learnt about different diagrams. E.g. I did not know about histograms before the project. I think it has been really interesting with manipulated diagrams and results – now I will be more observant when reading newspapers etc!

What surprised me most though was how important role mathematics plays when talking about environmental issues. With support of mathematics we can get people to react and

stop. [...] I am so interested in environmental questions and did actually not believe that maths could be important when presenting different standpoints.

I have probably learnt more now than if I had only calculated tasks in the book. Now I could get use of the knowledge in the project and that made me motivated and happy! I show my knowledge best through oral presentations because there you can show all the facts and talk instead of just writing a test. To have a purpose with the calculations motivated me a lot. (Sandra, logbook, conclusions).

The teacher assessed her with the highest grade, implicating she presented her statistical investigation with correct mathematical language using appropriate concepts and terminology, that she had chosen appropriate diagrams and arithmetic mean values and could argue for her choices, and that she reflected on possible sources of error and how these could have been prevented. Sandra was convinced that she had not been able to account for these criteria on a written mathematics test. However orally she clearly in a correct and convincing way presented her results and answered questions in front of an audience of 50 students, two teachers and one researcher (Annica, fieldnotes).

Sandra's actions voiced as e.g. "I will be more observant...", "I have learnt more than..." and "I could get use for..." evidently expressed her achieved agency and the relationship between her present identity as a responsible and thinking agent, achieved agency and her learning of mathematics at this particular time. Her expressed narratives, oriented both to the past, future and present is an important factor for actual agency according to Biesta and Tedder (2006). Her learning is expressed both in relation to the subject mathematics and the power of mathematics in society. There seems to be an indication of a changing agentic orientation in this particular situation context; e.g. her 'mathematics anxiety' identity is not expressed at all at this point.

### **Concluding remarks regarding Sandra's achievement of agency**

Sandra's change of identities expressed different qualities of her achieved agency that impacted on her classroom engagement and her learning of and accountancies for mathematics. The implications of this case suggest that Sandra's possibilities for achieving agency is an important component for her learning of mathematics, her accountancy of mathematics and in the end – even for her marks. Maybe Sandra's stories indicate that possibilities for achieving agency are an important feature when constructing mathematics learning environments.

## **CONCLUDING REMARKS**

The purpose of this theoretical paper was to present the analytical framework which explained the interplays between different contexts and students' achievement of agency in this particular setting. The analytical framework put forward by Biesta and Tedder (2006) understanding agency as achieved in-transaction-with-context at

particular historical times opened up possibilities to consider Sandra's different narratives and actions in the figured world of a particular mathematics classroom. I want to stress that it was Sandra's personally told or written stories that counted as her identities within this analytical framework. Hence, what she considered significant for her to tell at different times became the data in this research. To clarify *why* these particular narratives were told/written at specific historical times the framework suggested by Biesta and Tedder (2006) was combined with Sfard and Prusak's (2005) operational definition of identities and Skovsmose's (2005) concept of intentionality understanding learning as action. This was done in parallel with considering the relationships between achieved agency and different contexts. Analysing Sandra's achievement of agency in relation to the obvious task-context could have given information on how this student achieved agency when conducting project work, compared with mathematics textbook work. However, by also taking the wider socio-political context and class-room situation context into consideration gave a deeper understanding of Sandra's achievement and indicated why she acted, or achieved agency in some situations but not in others. Further elaboration of this framework might open up possibilities to recognise students as agents of their learning of mathematics in mathematics education research.

## REFERENCES

- Andersson, A. (2010). Can a critical pedagogy in mathematics lead to achievement, engagement and social empowerment? *Philosophy of Mathematics Education Journal* (25).
- Beswick, K., Watson, J., & Brown, N. (2006). Teachers' confidences and beliefs and their students' attitudes to mathematics. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, cultures and learning spaces* (pp. 68-75). Sydney: Mathematics Education Research Group of Australasia.
- Biesta, G., & Tedder, M. (2006). How is agency possible? Towards an ecological understanding of agency-as-achievement. *Learning lives, Working paper 5. University of Exeter*. Retrieved from [www.tlrp.org](http://www.tlrp.org)
- Boaler, J., & Greeno, J.G. (2000). Identity, agency and knowing in mathematics worlds. In J. Boaler (Ed) *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). USA: Greenwood Publishing Group, Inc.
- Clarkson, P.C., Bishop, A., Seah, W.T. (2010). Mathematics education and student values: The cultivation of mathematical wellbeing. In T. Lovat, R. Toomey, N. Clement (Eds.). *International research handbook on values education and student wellbeing* (pp. 111-135). Dordrecht: Springer.
- D'Ambrosio, U. (2007). Peace, social justice and ethnomathematics. In B.Sriraman (Ed.) *International perspectives on social justice in mathematics education* (pp. 25-34). Charlotte: Information Age Publishing, Inc.

## Working Group 10

- Emirbayer, M., & Mische, A. (1998) 'What is agency? *American Journal of Sociology*. 103, 962-1023.
- Ernest, P. (2002). Empowerment in mathematics education. *Philosophy of mathematics education journal*. 15 (1),1-16.
- Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: an analysis of students' participation in the activity systems of mathematics classrooms. *Educational studies in mathematics* 70 (1), 49-70.
- Holland, D., Lachicotte, W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Cambridge: Harvard University Press.
- Johnson, A.G. (2000). *The Blackwell dictionary of sociology. A user's guide to sociological language, second edition*. Cornwall: TJ International.
- Radford, L. (2008). The ethics of being and knowing: towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.) *Semiotics in Mathematics Education: Epistemology, History, Classroom, and Culture*, (pp. 215–234). Rotterdam: Sense Publishers.
- Sfard, A., Prusak, A. (2005). Telling identities: In a search of an analytical tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 2-24.
- Sfard, A. (2008). *Thinking as communicating*. New York: Cambridge University Press.
- Skovsmose, O. (2005). *Travelling through education: Uncertainty, mathematics, responsibility*. Rotterdam: Sense Publishers.
- Solomon, Y. (2009). *Mathematical literacy. Developing identities of inclusion*. New York: Routledge.
- Valero, P. (2002). The myth of the active learner. In P. Valero, & O. Skovsmose (Eds.) *Proceedings of the Third International Mathematics Education and Society Conference, Second edition* (pp. 542-553). Copenhagen: Centre for Research in Learning Mathematics.
- Valero, P. (2004). Socio-political perspectives on mathematics education. In P. Valero, & R. Zevenbergen (Eds.) *Researching the socio-political dimensions of mathematics education. Issues of power in theory and methodology* (pp. 5-24). Dordrecht: Kluwer Academic Publishers.
- Valero, P., & Stenoft, D. (2010). The 'post' move of critical mathematics education. In H. Alrø, O. Ravn, & P. Valero (Eds.), *Critical Mathematics Education: Past, Present and Future* (pp. 183-195). Rotterdam: Sense Publishers.
- Wedeg, T. (1999). To know or not to know mathematics, that is a question of context. *Educational Studies in Mathematics* (39), 205-207.
- Wedeg, T. (2010). Sociomathematics: A subject field and a research field. In U.Gellert, E. Jablonka., & C. Morgan (Eds). *Proceedings of the Sixth International Mathematics Education and Society Conference* (pp. 171-200). Berlin: Freie Universität Berlin.

# CLIMATE CHANGE AND MATHEMATICS EDUCATION: MAKING THE INVISIBLE VISIBLE

**Richard Barwell & Christine Suurtamm**

*University of Ottawa, Canada*

*Mathematics is crucial for describing, predicting and communicating climate change. In this paper, we use the perspective of critical mathematics education to examine how mathematics is constructed in reporting climate change. We focus particularly on accounts of mathematical modelling. Our discussion is illustrated by reference to two texts: a newspaper report and a policy document. We argue that mathematics, human mathematical activity and the role of human activity in causing climate change are all largely invisible in such texts. Mathematics education has an important role to play in making the invisible visible.*

Keywords: climate change, critical mathematics education, mathematical modelling

Climate change is perhaps the most urgent challenge facing human society. There is little doubt that climate change is a human-induced phenomenon with potentially disastrous consequences over the next century and beyond (IPCC, 2008). Human understanding of climate change, both collective and individual, is almost entirely mediated by mathematics: without mathematics we would have little sense of how the climate is changing and no idea how it might change in the future. Given the urgency, the scale of the challenge and the importance of mathematics it is surprising to observe that mathematics educators appear to have paid little attention to climate change in their work. One purpose of this paper is to argue that this situation needs to change. Mathematics educators have an important role to play.

Climate change is now reported, discussed and debated in a wide variety of genres, including scientific papers, newspaper reports, television reports and documentaries, policy documents, treaties, websites and blogs. We are interested in how the mathematics of climate change is represented within these different genres. An understanding of the representation of mathematics in these various kinds of text will enable us to identify what mathematics is relevant, how it is relevant and to consider ways of enhancing public understanding of climate change, whether through schooling or other means. Our discussion is informed by critical mathematics education, as we set out below. To focus this paper, moreover, we restrict our attention to the role of mathematical modelling in reporting and understanding climate change. To illustrate our discussion, we examine two examples of publicly available texts concerning climate change: a newspaper report and a policy document. We argue that, despite its centrality, the role of mathematics is largely invisible in public discussion of climate change, such as in these two texts.

## MATHEMATICS, MODELLING AND CLIMATE CHANGE

Mathematics is implicated in three aspects of climate science: description, prediction and communication (Barwell, 2010). The description of climate change largely involves measurement (of temperature, rainfall, sea level etc.) and the statistical analysis of these data. The prediction of climate change involves more advanced mathematics, such as mathematical modelling, non-linear systems and stochastic processes (McKenzie, 2007, pp. 22-23). The communication of climate change involves the production and consumption of information about climate change in the form of written texts, graphs, charts, diagrams etc. Clearly these three dimensions of the mathematics of climate change are inter-related. The interpretations made by consumers of the communication of climate change are influenced by these consumers' understanding of the mathematics of description and prediction. [1]

In this paper, we focus in particular on the role of mathematical modelling in predicting climate change. Earth climate models have been developed over many years, are highly complex and draw on expertise in many different domains of research. Weaver (2008) summarises how they work as follows:

A climate model starts with a set of equations governing the dynamics of the climate system and translates those equations into a model grid that represents the Earth. Each of the subcomponents (ocean, atmosphere, land surface, cryosphere) interacts and exchanges heat, moisture, and momentum. The resulting system is then driven by specified radiative forcings, including energy from the sun and emissions of human produced greenhouse gases (p. 183).

To evaluate the model, certain starting conditions are specified and it is then run until an equilibrium is reached. The equilibrium state is then compared with recorded data of the Earth's climate with the aim that the model corresponds reasonably well with observed conditions. This process takes several weeks of 24-hr computer processing (Weaver, 2008, p. 189). An example of the atmosphere component of a climate model is shown in Figure 1:

$$\begin{aligned} \partial \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} &= -\nabla p + g\mathbf{k} + \mathbf{F} + \mathfrak{J}(\mathbf{u}) \\ \partial \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ p &= \rho R T; \rho = f(T, q) \\ \partial T + \mathbf{u} \cdot \nabla T &= SW \\ &\leftrightarrow \\ &+ LW \\ &\leftrightarrow +SH + LH + \mathfrak{J}(T) \\ SW &= f(\text{clouds, aerosols} \dots) \\ LW &= f(T, q, \text{CO}_2, \text{GHG} \dots) \\ \partial q + \mathbf{u} \cdot \nabla q &= \text{Evap} - \text{Condensation} + \mathfrak{J}(q) \end{aligned}$$

**Figure 1: Atmosphere - Equations of a typical climate model (McKenzie, 2007, p. 13)**



The models are refined until a satisfactory fit is obtained. Predictions can be generated by modifying the conditions to represent, for example, an increasing concentration of greenhouse gases in the atmosphere (Weaver, 2008).

Interest in mathematical modelling in mathematics education has been growing over the past two decades in different parts of the world (e.g. Niss, Blum, & Galbraith, 2007; Suurtamm & Roulet, 2007). One reason for this greater interest is the advances in technology that help to facilitate the creation of models by students. Mathematical modelling helps students to see the relevance of mathematics in investigating and making sense of the world, so that mathematics is not experienced as a static domain with little connection to students' lives. In particular, mathematical modelling uses real data to investigate a wide variety of issues. The discussion document for the ICMI – Study 14 on applications and modelling, describes the modelling process:

The starting point is normally a certain situation in the real world. Simplifying it, structuring it and making it more precise - according to the problem solver's knowledge and interests - leads to the formulation of a problem and to a real model of the situation. [...] If appropriate, real data are collected in order to provide more information about the situation at one's disposal. If possible and adequate, this real model - still a part of the real world in our sense - is mathematised, that is the objects, data, relations and conditions involved in it are translated into mathematics, resulting in a mathematical model of the original situation. Now mathematical methods come into play, and are used to derive mathematical results. These have to be re-translated into the real world, that is interpreted in relation to the original situation. At the same time the problem solver validates the model by checking whether the problem solution obtained by interpreting the mathematical results is appropriate and reasonable for his or her purposes. [...] At the end, the obtained solution of the original real world problem is stated and communicated (Blum, 2002, p. 152-153).

In mathematical modelling, then, students are able to reach outside the realm of mathematics and to engage in something "real in the world" and to move back and forth between mathematics and the world as a way to understand both the world and mathematics (Pollack, 2007). The iterative comparison between the model and the phenomenon requires the creator of the model to reflect on how well their model represents the actual situation, to make clear the assumptions in creating the model and to evaluate its reliability as a predictive model. Jablonka (2010) reiterates that "explicating the underlying assumptions and exploring alternative forms" (p. 95) of models is an important part of the modelling process. Mathematical modelling goes beyond simply fitting a curve to a set of data points drawn from an experiment (Galbraith, 2007).

This general account of the modelling process fits well with Weaver's summary of climate change modelling. Both accounts use the term 'translating,' for example, to describe the relationship between observed data and model outcomes. Similarly,

climate models undergo an iterative process of evaluation and refinement in relation to observed data about the climate (Weaver, 2008, pp. 188-198).

## **CRITICAL MATHEMATICS EDUCATION**

Skovsmose (1994) makes the important point that our technological society relies heavily on mathematics. Computers in particular rely on algorithms and models that are fundamentally mathematical in nature. While the role of mathematics is often rather invisible, it plays a key role in shaping society and leads to tangible social effects. Key aspects of our everyday lives, such as banking, insurance, medicine, transportation or shopping are shaped by often hidden mathematical models and procedures. As an example, he discusses how a mathematical model of airline ticket sales includes a degree of overbooking, as a result of which, on some occasions some passengers are unable to fly since there are insufficient seats on their flight (Skovsmose, 2001). He uses this example to illustrate the way that mathematics is a powerful predictive tool, but also to show that this tool has limitations and real social effects, such as some people not being able to fly. These limitations are unavoidable, since mathematics formalises a situation, effectively creating a particular mathematical interpretation, which in turn develops a social reality of its own: “mathematical models become guidelines for our design of our world and, therefore, they become not only descriptive but also prescriptive,” (Skovsmose, 1994, p. 55). These mathematical models are not necessarily good or bad; airline ticket sales models contribute to convenience and in some cases to cheaper fares. But the model does change the nature of pricing and of air travel itself. In other words the traditional principle of: ‘Do not sell any more tickets than there are seats’ becomes substituted with the much more complex principle that includes maximizing revenues and quantifying customer dissatisfaction (Skovsmose, 2006, p. 39). Hence the decisions based on complex mathematical models developed by experts are inaccessible or invisible to citizens (Skovsmose, 2006; Jablonka, 2010). This invisibility results in a society that often accepts the output of experts and feels disempowered to examine or question the conclusions of these experts or to enter into any kind of discussion or debate.

Recognizing that mathematics is powerful yet invisible suggests that mathematics education needs to equip the citizens of our society with a critical awareness of the role of mathematics and its effects. In developing a critical approach to mathematics education, Skovsmose (1994) identifies three forms of knowing: mathematical knowing, technological knowing and reflective knowing. He emphasises that the distinctions are analytical, rather than empirically distinguishable (Skovsmose, 1994, p. 115). Mathematical knowing is concerned with formal mathematics: the kind of ‘within’ mathematics procedures and thinking with which mathematicians work. Technological knowing is about the application of mathematics – knowing both how to construct a tool and how to use it. Mathematical modelling is one such tool. Knowing how to construct a model requires more than a familiarity with, for

example, different types of equations. It requires an understanding of how to construct particular equations to model a situation as well as an understanding of how to use the model (Skovsmose, 1994, pp. 98-99). Clearly there are overlaps between mathematical and technological knowing. They each give meaning to the other.

Reflective knowing builds on mathematical and technological knowing and augments that knowledge with critical awareness of the broader effects of mathematics and of its social or ethical consequences. The value of reflective knowing in mathematics is the awareness it provides of the way mathematics works to shape our lives in different ways (its 'formatting power', Skovsmose, 1994). Reflective knowing is one of the bases for critical mathematics education, the project of which is the empowerment of critical citizens (Skovsmose, 2006). Students can use mathematics as a tool for critical investigation as they draw on all three forms. Thus, students make mathematical sense of the world, while maintaining a critical, reflective orientation towards these mathematisations and the insights they bring.

The connections between mathematical modelling in mathematics education and critical mathematics education are clear. In a critical mathematics education situation, Skovsmose (1994) highlights the following 'tasks' of reflective knowing:

*to make explicit the preconditions of a modelling process which become hidden when mathematical language gives it a neutral cover. (p. 106)*

*to address problems and uncertainties connected with transitions between the different types of language game involved in the mathematical modelling process. (p. 111)*

*to identify the formatting power of mathematics. (p. 114)*

This reflective knowing is very similar to the processes suggested by Blum in his description of mathematical modelling: both suggest making explicit and visible the assumptions and limitations that are inherent in the creation of the model. Jablonka (2010) suggests that genuine examples of mathematical models can provide a source for understanding and assessing the use of mathematics within different areas of society. Several researchers provide examples of engaging students in mathematical modelling of social justice issues such as racial profiling (Gutstein, 2006) or issues of equity (Makar & Confrey, 2007). Such work shows how students' interest in mathematics can be enhanced through engaging in mathematical modelling of issues that concern them. At the same, students become more aware of the issues they are exploring. Such work helps students to develop a critical stance that helps them to understand, evaluate, and respond to technological and social issues (Jablonka, 2010). We believe that this approach is applicable to the issue of climate change.

## **CRITICAL MATHEMATICS EDUCATION AND CLIMATE CHANGE**

Climate change models open a space for technological imagination and hypothetical reasoning which allows society to make predictions about events in the future. The

mathematical modelling of climate change is used to justify certain actions, for example concerning the development of renewable energy sources or the construction of sea defences, despite the invisibility of the mathematics (see Skovsmose, 2006). This situation is apparent in the communication of climate change. In our initial survey of articles on climate change we have observed that while scientific principles are often referred to and explained, explicit reference to the role of mathematics is much less common. To illustrate and discuss this invisibility, we refer to two examples of texts about climate change: the first is a newspaper report; the second is a government policy document.

### **Climate predictions in the news**

In August 2010, a report headlined ‘Climate scientists forecast more heat, fires and floods’ appeared in Canada’s Globe and Mail newspaper. [2] The article is typical of the proliferation of news reports about climate change, often prompted by extreme weather events. Here are three extracts:

Floods, fires, melting ice and feverish heat: From smoke-choked Moscow to water-soaked Pakistan and the High Arctic, the planet seems to be having a midsummer breakdown. It’s not just a portent of things to come, scientists say, but a sign of troubling climate change already under way. [...]

The UN’s network of climate scientists – the Intergovernmental Panel on Climate Change – has long predicted that rising global temperatures would produce more frequent and intense heat waves, and more intense rainfalls. In its latest assessment, in 2007, the Nobel Prize-winning panel went beyond that. It said these trends “have already been observed,” in an increase in heat waves since 1950, for example. [...]

The WMO [World Meteorological Organization] did point out, however, that this summer’s events fit the international scientists’ projections of “more frequent and more intense extreme weather events due to global warming.”

The report refers to several events from the preceding weeks: the heatwave and fires affecting Moscow, floods in North West China and a huge ice island that had recently calved from a Greenland glacier. The main thrust of the report is that these events ‘fit patterns predicted by climate scientists’. In the latter part of the report, these three events are linked to specific predictions from the IPCC’s latest synthesis report.

Despite the relatively detailed nature of the report, none of the three dimensions of knowing proposed by Skovsmose (1994) are particularly apparent. The report involves only a minimal level of mathematical knowing, mostly through the statistics that are quoted (e.g. “Moscow temperatures topping 37.8 degrees for the first time”) and the use of rather vague comparisons like ‘more’ or ‘increase’. Only very general references are made to the process of modelling (i.e. technological knowing), such as, for example: *modellers of climate systems are “very keen” to develop*

*supercomputer modelling that would enable more detailed linking of cause and effect as a warming world shifts jet streams and other atmospheric currents.*

Models are constructed as simply being ‘developed’ by ‘modellers’. In terms of reflective knowing, the report does not explain how the IPCC’s predictions were made, how trends are established or how projections are arrived at. The construction of climate models involves human actors making assumptions (Jablonka, 2010) but this aspect of the process is not apparent. Predictions and trends are attributed to ‘scientists,’ ‘climate scientists’ or ‘climatologists’ but their role is rather unclear. Similarly, the role of technology is apparent but only in general terms, through references to ‘supercomputers’. Such references tend to give a gloss of reliability and obscure the role of the modeller in constructing the model: both mathematics and the human nature of mathematics are largely invisible. Interestingly, the human role in climate change is also not apparent. The article makes no explicit reference to specific human actions that have been identified as contributing to climate change. There is a discussion of cause and effect, but it concerns the question of whether climate change ‘causes’ extreme events like heat waves.

### **Climate change in public policy**

In June 2009, the UK Government published ‘Adapting to climate change: UK Climate Projections,’ a document setting out likely effects of climate change on the UK, discussing various options for ‘mitigation and adaptation’ and summarising policy and action at various levels of regional and national government. In the opening part of the chapter that presents the UK Climate Projections, there is a paragraph of explanation of climate models:

Climate models are computer simulations of the way the Earth’s climate works. Beginning with the laws of physics, they represent the characteristics of air, ocean water, ice, and crucially, heat around the Earth. They model chemical, biological and physical processes in the atmosphere, oceans and on land. Ever more sophisticated models and increasing computer power enable climate scientists to understand our climate better and study a range of possible future climates. Model results are checked in part by simulating past and present climate observations and seeing how well they perform. All of the models used in UKCP09 are internationally recognised and peer reviewed. These climate model results, together with observations of climate change, form the basis of the overwhelming consensus there now is in the scientific community and international bodies (including the Intergovernmental Panel on Climate Change – IPCC) that the world’s climate is changing quickly and that this is mainly as a result of our actions.

The challenge now, and one which the UK Climate Projections will help us meet, is to use this enhanced sophistication and the variety of models to provide us with a more detailed picture of the nature and probability of various possible outcomes.

This will inform practical decisions, helping our society to deal with the risks from climate change. (DEFRA, 2009, p. 3)

As with the newspaper report (and somewhat ironically), this explicit discussion of mathematical modelling does not involve much mathematics. The quote makes only passing reference to any kind of mathematics (e.g. probability). The ‘technological knowing’ dimension is confined to a rather general account of the modelling process. While this account conforms to a certain extent with the process as understood in mathematics education (Blum, 2002), it emphasises the ‘laws of physics’ and ‘chemical, biological and physical processes’ rather than mathematical processes. In terms of reflective knowing, the human role in constructing the models, making assumptions and evaluating and applying the models is largely absent. They are computer simulations rather than human mathematical creations. Indeed in some parts of the texts, the models themselves have agency: they ‘provide us with a more detailed picture.’ Humans act only secondarily through ‘peer review’, for example. Unlike the newspaper report, there is some reference to human activity as a cause of climate change, but only in very general terms. Indeed, throughout the document, human or organisational activity is discussed mostly in terms of mitigation of climate change (e.g. constructing flood defences) rather than as contributors to the problem. The description of modelling, moreover, does not explicitly include human activity in its summary of what is modelled.

## CONCLUSION

Our discussion of these two texts illustrates the idea that mathematics and human mathematical activity are largely invisible in the reporting of climate change. Furthermore, the role of human activity in causing climate change also seems to be heavily obscured. These observations are consistent with claims in the critical mathematics education literature that mathematisation results in dehumanisation (e.g. Jablonka, 2010; Ernest, 2010). Clearly there is a role for mathematics education to make the invisible visible. Making mathematics visible, however, risks contributing to this dehumanization particularly in relation to the human role in climate change. This is not to say that mathematics should be left invisible, but that a mathematics that includes reflective knowing is necessary. Consumers of reports about climate change need to be aware of the assumptions and the iterative process of working with models and their various variables. As Jablonka (2010) suggests:

‘model-oriented reflection’ helps at evaluating whether a mathematical model is likely to fulfil the specific purpose for which it is constructed. Explicating the underlying assumptions and exploring alternatives forms an important part of this activity. (p. 95)

The emergence of human agency as a key issue in our examination of these texts is related to critiques of discourses of ‘the environment’ in contemporary society. In particular, such discourses construct humans as distinct from the environment, over

which they exercise control and can exploit and manipulate to better their own (i.e. human) welfare, rather than as just one part of an ecosystem (see, for example, Bowers, 2001). The mathematical treatment of climate change risks perpetuating this discourse through its quantification of human activity: once the situation is mathematised, human actions are replaced by variables in an equation. Mathematics, in its quantifying mathematical ‘mesh’ (Ernest, 2010), seems to filter out human activity. Mathematics helps us to become invisible.

Our discussion of these texts is *not* intended to shed doubt on the claims that are made about climate change. Rather, we argue that the full modelling process needs to be more visible, since particularly the assumptions that are made mathematise climate processes and human behaviour. Such a task is not straightforward: one might not expect a newspaper report to explain in any detail the nature of the mathematical models involved. The question arises, therefore, of where the consumers of such texts may learn about such things. Moreover, not everyone can be expected to work with the mathematical model equations shown in figure 1. Nevertheless, a more widespread public understanding of the role of mathematics in our understanding of climate change would allow for a more informed discussion of the issues and a stronger and more effective societal response. Mathematics educators have an important role to play in responding to these issues, whether through schooling or broader forms of public education. Climate change is a serious threat to our ways of life. Mathematics education can play its part in addressing this threat. We would like to see more research, more teaching and more action on climate change and mathematics education.

## NOTES

1. The nature and effect of this influence is an empirical question that remains to be investigated.
2. ‘Climate scientists forecast more heat, fires and floods’ by Charles J. Hanley of the Associated Press, published in the *Globe and Mail* on 12 August, 2010. The report can be viewed at <http://www.theglobeandmail.com/news/world/climate-scientists-forecast-more-heat-fires-and-floods/article1671364/actions.jsp>.

## REFERENCES

- Barwell, R. (2010) Climate change and mathematics education: a critical mathematics education perspective. In Pinto, M. M. F. & Kawasaki, T. F. (Eds.) *Proceedings of PME33*, vol. 2, pp. 193-200.
- Blum, W. (2002) ICMI Study 14: Applications and modelling in mathematics education-discussion document. *Educational Studies in Mathematics* 51, 149-171.
- Bowers, C. (2001) How language limits our understanding of environmental education. *Environmental Educational Research* 7 (2) 141-151.
- Department for Environment, Food and Rural Affairs (UK) (DEFRA) (2009) *Adapting to Climate Change: UK Climate Projections*. London: DEFRA.

## Working Group 10

- Ernest, P. (2010) The scope and limits of critical mathematics education. In Alrø, H., Ravn, O. & Valero, P. (Eds.), *Critical Mathematics Education: Past, Present and Future*, pp. 65-88. Rotterdam: Sense Publishers.
- Galbraith, P. (2007) Authenticity and goals – overview. In Blum, W., Galbraith, P., Henn, H-W., & Niss, M. (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study*, pp. 181-184. New York: Springer.
- Gutstein, E. (2006) *Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice*. New York, Routledge.
- Intergovernmental Panel on Climate Change (IPCC) (2008) *Climate Change 2007 Synthesis Report*. Geneva: IPCC.
- Jablonka, E. (2010) Reflections on mathematical modelling. In Alrø, H., Ravn, O. & Valero, P. (Eds.), *Critical Mathematics Education: Past, Present and Future*, pp. 89-100. Rotterdam: Sense Publishers.
- Makar, K. & Confrey, J. (2007) Moving the context of modelling to the forefront: preservice teachers' investigations of equity in testing. In Blum, W., Galbraith, P., Henn, H-W., & Niss, M. (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study*, pp. 485-490. New York: Springer.
- McKenzie, D. (2007) *Mathematics of Climate Change: A New Discipline for an Uncertain Century*. Berkeley, CA.: Mathematical Sciences Research Institute.
- Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In Blum, W., Galbraith, P., Henn, H-W., & Niss, M. (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study*, pp. 1-32. New York: Springer.
- Pollack, H. (2007). Mathematical modelling – A conversation with Henry Pollak. In Blum, W., Galbraith, P., Henn, H-W., Niss, M. (Eds.) *Modelling and Applications in Mathematics Education: The 14<sup>th</sup> ICMI Study*, pp. 109-120. NY: Springer.
- Skovsmose, O. (1994) *Towards a Philosophy of Critical Mathematics Education*. Dordrecht: Kluwer.
- Skovsmose, O. (2001) Mathematics in action: a challenge for social theorising. In Simmt, E. and Davis, B. (Eds.) *Proceedings of the 2001 Annual Meeting of the Canadian Mathematics Education Study Group*, pp. 3-17. Edmonton, AB: CMESG.
- Skovsmose, O. (2006). Challenges for mathematics education research. In Maasz, J. and Schloeglmann, W. (Eds.) *New Mathematics Education Research and Practice*, pp. 33-50. Dordrecht: Springer.
- Suurtamm, C., Roulet, G. (2007). Modelling in Ontario: Success in moving along the continuum. In Blum, W., Galbraith, P., Henn, H-W., Niss, M. (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study*, pp. 491-496. New York: Springer.
- Weaver, A. (2008) *Keeping our Cool: Canada in a Warming World*. Toronto, ON: Penguin.



# TEACHERS DISCUSSIONS ABOUT PARENTAL USE OF IMPLICIT AND EXPLICIT MATHEMATICS IN THE HOME

**Sarah Crafter & Guida de Abreu<sup>45</sup>**

*University of Northampton & Oxford Brookes University*

*Embedding everyday mathematics in home learning is considered by educationalists as one of the best mediators for learning. There is also evidence that everyday mathematics, such as cooking or the use of board games, is not used as a resource for learning by all parents. This paper examines how teachers make sense of embedded everyday mathematics at home in relation to parents' practices. The theoretical concepts of Boundary Crossing and Implicit/Explicit practice will form the basis for this paper. Data comes from interviews with eight teachers who work in culturally diverse school settings. The analysis focuses on teachers' narratives about the complexities of shared boundary crossing and home constraints which make everyday mathematics learning problematic<sup>46</sup>.*

Key words: everyday mathematics, teachers, boundary crossing, implicit/explicit

## EVERYDAY MATHEMATICS

Embedding everyday mathematics into learning in the home has been considered beneficial in both academic circles (see Young-Loveridge, 1989) and educational policy within the UK (The Williams Report, 2008). We argue, like Street, Baker, and Tomlin (2008) that embedding everyday mathematics at home is a complex business as it involves more than the inclusion of mathematics activities and procedures. Numeracy practices are enabled or constrained by sociocultural contexts, values and representations (Gorgorió & Abreu, 2009), social and institutional relations (Street, Baker, & Tomlin, 2008) and personal histories (O'Toole & Abreu, 2005). In culturally diverse home and school settings the use of mathematical practices and the resources parents draw on to make sense of their children's learning can be contradictory and complex (Crafter, 2009). This paper asks, what are teachers understandings of parents home mathematics practices? This paper uses two theoretical constructs to attempt to understand how teachers makes sense of embedded home mathematics learning – 1. Boundary Crossing and 2. Implicit/explicit numeracy practices.

---

<sup>45</sup> The authors are members of the EMiCS group – Educació Matemàtica i Context Sociocultural (Mathematics Education and its Sociocultural Context) – granted by the Direcció General de Recerca of the Generalitat de Catalunya (the research office of the Catalan autonomous government) (grant: 2009SGR-00590)

<sup>46</sup> The research reported is partially supported by the Foundation *Obra Social "la Caixa"* and ACUP (Catalan Association of Public Universities) within their program RecerCaixa 2010

## **BOUNDARY CROSSING AND IMPLICIT/EXPLICIT LEARNING**

Until recently, both traditional and sociocultural traditions in cognition have focused on learning progression within particular communities. The situated cognition tradition for example, centred on movement from periphery to full legitimate participation in a particular community (Lave & Wenger, 1999). Developmental psychological research investigating the interplay between culture and cognition has looked at the change of mathematical practices over time (Saxe & Esmonde, 2005). Movement of knowledge between or across communities has been of interest for some time, first in the form of transfer which suggested that knowledge used in one context is utilised in another (Thorndike & Woodworth, 1901). While this idea has been taken up in areas like mainstream cognitive psychology, it has been widely recognised that knowledge, including mathematical knowledge, is culturally situated to particular contexts (Abreu, 1995; Nasir, 2008).

The change of focus from the concept of transfer to transition proved much more helpful for those of us interested in studying the cultural nature of mathematical knowledge. Transitions could come in different forms – some transitions are consequential, because not only can they be a struggle, but also they have the potential to alter ‘one’s sense of self’ (Beach, 1999, pp. 114). In other words, they usually have an impact on the individual and the social context that they inhabit. The type of transition that a child makes between home and school is called a ‘collateral transition’ where, historically speaking, activities are taking place simultaneously. The child is in a continuous process of moving between these two major communities of practice and therefore the construction of meaning is ongoing for all the key players of those communities.

More recently, attention has (re)focused on the notion of boundary crossing. It is still not clear to us what distinguishes the forms of transition mentioned by Beach (1999) from boundary crossing. Perhaps it is that boundary crossing encompasses more than strategic knowledge to include symbolic resources and representations (Zittoun, 2010). For Wenger (1998) boundary crossing, seems to emphasise the practices themselves as units of analysis whilst for the purpose of this paper, the representation of the practice takes centre stage. However, Wenger’s (1998) conceptualisation of boundary crossing is useful in that it addresses the continuities and discontinuities to the forms of practice which are enacted when moving between one community and another.

Some forms of mathematics knowledge have greater power and status than others (Nasir, 2008). Some practices are deemed more worthy than others (e.g. boards games are valued mathematics practices by the school while dress making and carpentry largely go unnoticed – see González, Andrade, & Civil, 2001). The mathematical value of dress-making might not be recognised by school (the more powerful community). When certain practices are reified they are imbued with greater status than others. Boundary crossing from a cognitive perspective connotes a

shared knowledge (Akkerman, et al., 2007) but we question where this shared knowledge begins and ends.

Using the notion of implicit/explicit practices may provide a useful mechanism for looking at the boundaries of mathematical practice across the communities of home and school. Tomlin, Baker and Street (2002) explore in their research those practices which are more visible or explicit, and are recognised by all concerned as improving mathematical skills. However, some mathematical practices are viewed as less salient, or are more implicit, because they often go unrecognised as contributing to the mastery of mathematical skills. There are a number of crossovers between schooled mathematics and out-of-school mathematics practices such as working on number bonds, times tables, dates, measuring, money and playground games. Other practices such as homework and shop bought textbooks also transcend both contexts. Out-of-school mathematical practices like laying the table, counting stairs, setting the video and producing calculations from looking at car number plates reveal how varied numeracy learning can be. These examples further highlight how much the uses of home mathematical practices are reliant on the social characteristics of engaging in numeracy.

This has led to some questions - what is constrained or facilitated in the boundary crossing? What forms of mathematical knowledge (implicit/explicit) make it possible to address the continuities or discontinuities across communities? This paper explores how teachers talk about tensions and expectations on home mathematics learning across the boundaries of home and school when thinking about parents' practices.

## THE EMPIRICAL STUDY

To examine teachers' representations of parents uses of everyday numeracy practices we draw selectively on findings from interviews with eight teachers who participated in a wider ethnographic study exploring home and school mathematics learning. The teachers taught children from two different primary schools situated in the same town, known as school A and school B (all catering for pupils aged between 5-11 years). In school A the proportion of ethnic minority pupils could be described as 'culturally mixed' and school B was mainly white school. At the point of data collection pupils, parents and teachers were sampled from the highest and lowest achievement groups in the year. The exception being Richard's class in school B (see table) which had a policy of mixing different achieving children.

School	Teacher	Year Group	Achievement Group
School A	Catherine	Ages 6/7 years	High
School A	Jane	Ages 6/7 years	Low
School A	Anna	Ages 10/11 years	High

### Working Group 10

School A	Mary	Ages 10/11 years	Low
School B	Richard	Ages 6/7 years	Mixed
School B	Chris	Ages 10/11 years	Low
School B	Susan	Ages 10/11 years	High

The empirical data we use in this paper to illustrate our thinking was collected using the episodic interview and analysed using an episodic analytic technique (Flick, 2000). Selections of questions which are pertinent to the data explored in this paper are:

In your view, how important is it that parents are involved in their children's school learning? Could you tell me about a situation around that?

Do you think that practical [mathematics] work is more beneficial at home than the more traditional academic work, can you describe how you feel about it?

### SHARED BOUNDARY CROSSING

Everyday mathematics which forms that grey area between implicit and explicit mathematical practices was a central feature of the teacher narratives in this study. In a previous paper we argued that there are some home mathematical activities which are obviously explicit – homework, mimicked school-like activities like shop-bought books (O'Toole & Abreu, 2003), etc... This first quote from Jane highlights what she perceives as some of the discontinuities between her own perspective/practices and those of some parents. She has been shown a vignette from another teacher who argues strongly in favour of everyday embedded mathematical activity as a learning tool. There is not space to put the whole vignette but it provides a general idea:

I have a real antipathy towards homework for children on this age. I would much rather that the parents were helping the children in general ways, to learn. For example, if they wanted their maths to improve, I'd much rather see them taking their children shopping and talking to the children about, 'oh, you know, these apples are £1.50 and kilo, you know, how many do you think we're going to get', and just bringing mathematics into their everyday life. I tend, I do find in this school, and I don't know whether it's a generalisation, you know, or whether its true everywhere, but the parents tend to want a sheet of sums,

She responds:

Jane: I agree with her that maths should relate to the everyday life, I don't dispute that. And again, there's a big difference, and it's been studied with my own

## Working Group 10

son – stairs, count them as you go up and as you come down, pre-school thing. There are parents who don't believe unless you have a sheet, because they remember their own homework but what they're remembering is high school homework. Unless you've got a sheet folded in half, sums down there and sums down there.

This was not necessarily an opinion shared by her colleagues. Just to give you a flavour of variation around the boundaries of implicit/explicit mathematics practices another teacher says:

Susan: ... I think a lot of parents just do what she's saying automatically, and kids these days will, you know 'just whip round the corner shop and get me a loaf of bread and a pint of milk and make sure you get the right change' and they can, I think they're far more streetwise these days than they've ever been, so they're getting that kind of experience. We laugh about the traveller kids who are not necessarily that good at maths but if you give them a money problem and they've got it like that, because they've got the experience of actually going out and spending money and dealing tangibly with coins.

Her mention of the traveller community and "kids these days" is interesting in that it shows recognition for both the cultural situatedness of everyday mathematics but also alludes to the historical aspect of boundary crossing which will be addressed more deeply in a moment. Jane raises an aspect of the implicit/explicit dilemma of mathematical practice which was a feature of most teacher narratives – laying the foundation of number in pre-school.

### **Foundation laying**

From the interviews with these teachers it would seem that implicit mathematical practices are most valued when children are young, prior to coming to school. As such, parents are ideally expected to lay the foundations of their child's learning in the first five years of life. The expectation though, is that this is a 'natural' activity that is implicitly embedded in everyday practice. This foundation then sets the child up for the boundary crossing into school. Catherine is responding to the quote from the teacher about everyday mathematics mentioned already:

Catherine: I mean the idea is that parents give them support. I find that most of the parents do give them support but I think this what she's talking about here, about real life maths, like going shopping, then I think hopefully the parents are doing that anyway as well. And have been doing that ever since they were, I mean that's what makes these children better mathematicians, better at anything. That from day one when they were tiny babies you start counting their fingers and counting their toes, and doing things like that. The parents either do that naturally or they don't. They don't teach them to do it. And the

## Working Group 10

ones that started with the baby counting its toes will be at the shop saying ‘well, you tell me what the change might be’ ...It has an impact on how they come into school, and what they can do when come into school. I mean, some come into school really quite numerate and literate really. Knowing lots of nursery rhymes and all these sort of things. And other children come to school knowing absolutely nothing. And they’re not necessarily less intelligent at the end of the day, but just have not had five years of education, or whatever, that their parents would have given them.

Work from the ‘funds of knowledge’ research suggests that parents do not always “naturally” undertake mathematical practices valued by the school (see Gonzalez, Moll, & Amanti, 2005 in the US or Andrews & Yew, 2006 in the UK). Wenger (1998) talks about boundary peripheries – the trajectory where a newcomer to a community becomes a full member of that community with time and experience with the practice. In a school setting it is almost impossible for a parent to become a full member of the community because their connection through a third person (the child) keeps them on the periphery. The direction of knowledge is assumed to go from school to home, not the other way (Gonzalez, Moll, & Amanti, 2005). Yet in the case of foundation laying, parents are expected to develop the right kind of knowledge prior to their child arriving in the school community:

Mary: Children really from year dot should be learning how to speak, and if it isn’t -you know, your phone number, your door number, numbers all around you, I think that helps a lot. And then they’re not frightened of it and don’t look at it and think ‘ah, it’s totally scary’.

Foundation laying was in some cases recognised as a historical activity within the home community:

Catherine: I think its one of these things, you know, if your dad is really good at maths or your mums really good at maths it will show through in the child.

Sociocultural theories and Communities of Practice recognise the importance of mutually engaging histories by members of communities or societies. Parents own mathematical past experiences are recognised as having an influence on their current mathematical endeavours (O’Toole & Abreu, 2005).

### **Resisted boundaries**

Some parents whose knowledge in one community is enough that it can carry to another community may resist institutional boundaries placed on them. Parents who are themselves teachers have full participatory knowledge in one context which crosses the boundary to another. Anna discusses her resistance to some mathematical practices sent home from school, which she subsequently reconstructs:

Anna: though my son is only five, during the Easter holidays and half holiday, this list came; he has this maths booklet at home, which I

must admit we very rarely look at. And there was this list saying do page fourteen, fifteen, sixteen, you know, about ten different pages. And I looked at that and I thought, and I did look them up and see what areas it was and then I thought no, he's not doing that we're not going to sit and he doesn't have to write it or anything or take it back to school. But I just thought I'm not going to slog through this during his holidays, but I thought we could spend our time much more valuably. And so we did, he had 50p during the holiday and we took him to the shop 'right ok, you can buy yourself a treat', 'how much have I got mum?' 'well you've got 50p there' and it was about his change or whatever. And then another time we were cleaning out the car and it was all the money that we found underneath the seat.

As a teacher Anna's insider knowledge allows her greater freedom in terms of the boundary crossing between home and school with her own children. Her expert knowledge means she can be resistant to formal mathematical practices sent from school to home. With explicit knowledge this is less complex than implicit knowledge which is largely culturally and experientially driven.

## **HOME CREATES CONSTRAINTS**

Teachers were very conscious of the constraints to mathematical practice in the home. Some of those constraints came about because of boundary crossing with other extracurricular activities such as swimming clubs, sport activities, Mosque or dance. Some of the teachers spoke of the other difficulties in explicitly embedding everyday mathematical activities in the home. Catherine narrates the difficulties in sending practical work home for children to do:

Catherine: Do you think that practical work is more beneficial at home than the more traditional academic work?

Catherine: So what would you put in that, as practical work?

Interviewer: Um, what's been mentioned to me before are things like measuring, going in and weighing tins, rather than your sheet of sums

Catherine: Which you can end up with a lot of problems giving homework like that, because they can come back and say 'well, I haven't got a tape measure at home or I haven't got a ruler or I have no scales in the kitchen

Interviewer: So it excludes some of the children

Catherine: It does, definitely. I mean you'd be surprised what they say. I gave them some colouring, well it was a homework which said 'colour all these squares blue' or whatever, and I had a child who

## Working Group 10

said ‘but I’ve got no colours at home’, so I had to give her some. But you can’t give them scales, and you can’t give them tape measure, rulers. I mean, even, um, when we did some measuring a couple of weeks ago and one girl in my group came back and said ‘well I couldn’t do my measuring because my sister took her ruler to school’ so we did it in school with her instead. But the point is, they can’t all...we even get the trouble with clocks, some children don’t even have clocks at home. They’ll have plenty of videos and microwaves and things, with digital time on them, but they will not have an analogue clock in the house.

We have argued elsewhere that the resources parents use to understand their child’s mathematics learning can be in symbolic form, such as the representations of child development (Crafter, 2009). Resources in the form of artifacts also have the potential to reify participation between two Communities of Practice (Wenger, 1998). In relation to boundary crossing, Wenger (1998) discusses how artifacts (such as colouring pencils or clocks) help organise the interconnections between communities. As Catherine points out, culturally diverse settings can create disconnections which arguably can increase the gap between children who do, and do not engage in implicit mathematical practices at home.

Anna spoke of another form of constraint on children’s home and school mathematics learning

Interviewer: Why do you think some children do better in their mathematics than other children? Can you tell me about it?

Anna: Quite possibly the attitude of their household to mathematics. If you have parents who always say ‘oh, I was no good at maths at school, oh I found it difficult, oh not maths, don’t ask me’ I think it tends to show the children that there is something hard about it. Whereas, I actually think that with a positive attitude most children actually enjoy maths. But I suppose it’s with any subject you know, some children will find learning numeracy more difficult, but then you’ll find other children who find reading more difficult but they’re very good at numerical problems.

Chris alludes to a similar issue to that raised by Anna.

Interviewer: Do you think there are any aspects of the home background, which may affect their mathematics? Can you describe anything?

Christ: Again, I think attitudes towards numeracy in general. Again, there doesn’t seem to be any shame about not being able to do maths and it’s either joked about, whereas you’d never joke about not



being able to read, or write. I think that's probably the biggest issue to get over, that it's not ok, not to be able to do maths.

## SOME CONCLUDING THOUGHTS

This paper has attempted to make sense of teachers' understandings of parents' everyday mathematics in the home through the theoretical lens of boundary crossing and implicit/explicit mathematical practices. Understanding episodes where implicit/explicit use of mathematics has crossed boundaries seems to be important. Grossen (2010) talks about the ways in which a piece of text in one 'sphere of experience' is incorporated in the social and cultural experience in another 'sphere of experience'. Can we focus on mathematical practice in a similar way? If so, we could perhaps help avoid the tendency that teachers and schools have to 'impose upon' the home the 'right kind' of mathematical practice. Teachers' lack of knowledge about the diversity of home mathematics encourages notions like 'foundation laying' which are imbued with notions of 'naturalness.' Foundation laying obviously works well but it is also narrow in its scope in that only certain practices are included in that. While teachers know that parents' mathematical insecurities can pass on to their children there appears to be very little 'mathematical identity' work which goes on in practice. One might imagine that this is largely because a) they are not trained to address the psychology of identity and b) there is little time and space in the delivery of the curriculum to do so.

## REFERENCES

- Abreu, G. de (1995). Understanding how children experience the relationship between home and school mathematics. *Mind, Culture, and Activity* , 2 (2), 119-142.
- Akkerman, S., Van de Bossche, P., Admiraal, W., Gijsselaers, W., Segers, M., Simons, R.-J., et al. (2007). Reconsidering group cognition: from conceptual confusion to a boundary area between cognitive and socio-cultural perspectives? *Educational Research Review*, 39-63.
- Andrews, J., & Yee, W.C. (2006). Children's 'funds of knowledge' and their real life activities: two minority ethnic children learning in out-of school contexts in the UK. I, 58(4), 435-449.
- Beach, K. (1999). Consequential transitions: A sociocultural expedition beyond transfer in education. *Review of Research in Education* , 24, 101-139.
- Crafter, S. (2009). Parental resources for understanding mathematical achievement in multiethnic settings. *Proceedings from the 6th Conference of the European Society for Research in Mathematics Education*, (pp. 14-22). Lyon, France.

- DCSF (Department for Children, Schools and Families) (2008) Independent Review of Mathematical Teaching in Early Years Settings and Primary Schools, *Williams Report*, DCSF.
- Grossen, M. (2010). Relating school and everyday life: A resource for learning? Paper presented at SIG 21 Learning and Teaching in Culturally Diverse Settings (2-3 September, 2010) Utrecht, The Netherlands.
- González, N., Andrade, R., & Civil, M. (2001). Bridging Funds of Distributed Knowledge: Creating Zones of Practices in Mathematics. *Journal of Education for Students Placed at Risk*, 16(1&2), 115-132.
- Gonzalez, N., Moll, L., & Amanti, C. (2005). *Funds of Knowledge: Theorizing Practices in Households, Communities, and Classrooms*. New Jersey: Lawrence Erlbaum Associates .
- Gorgorió, N., & Abreu, d. (2009). Social representations as mediators of practice in mathematics classrooms with immigrant students. *Education Studies in Mathematics*, 72 (1), 61-76.
- Lave, J., & Wenger, E. (1999). *Situated Learning: Legitimate Peripheral Participation* (8th ed.). Cambridge: Cambridge University Press.
- Nasir, N. (2008). Culture and Mathematics in School: Boundaries Between "Cultural" and "Domain" Knowledge in the Mathematics Classroom and Beyond. *Review of Research in Education*, 32, 187–240.
- O'Toole, S., & Abreu, G. de (2005). Parents' past experiences as resources for mediation in the child's current mathematical learning. *European Journal of Psychology of Education*, 75-89.
- O'Toole, S., & Abreu, G. de (2003). Investigating parents' explicit and implicit home numeracy practices in multiethnic contexts. 3<sup>rd</sup> Conference of the European Society for Research in Mathematics Education (Bellaria, Italy).
- Saxe, G. B., & Esmonde, I. (2005). Studying cognition in flux: A historical treatment of Fu in the shifting structures of Oksapmin mathematics. *Mind, Culture, and Activity*, 12(3/4), 171–225.
- Street, B., Baker, D., & Tomlin, A. (2008). *Navigating Numeracies: Home/School Numeracy Practices*. UK: Springer.
- Thorndike, E., & Woodworth, R. (1901). The influence of one mental function upon the efficiency of other functions. *Psychological Review*, 8, 247-251, 384-395, 553-564.
- Tomlin, A., Baker, D., & Street, B. (2002). Home and school numeracy practices: Where are the borders and overlaps? In P. Valero & O. Skovsmose (Eds) *Proceedings of the 3<sup>rd</sup> International MES Conference*. Copenhagen: Centre for Research in Learning Mathematics, (pp.1-10).

### Working Group 10

- Wenger, E. (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.
- Young-Loveridge, J. (1989). The relationship between children's home experiences and their mathematical skills on entry to school. *Early Child Development and Care, Vol. 43*, 43-59.
- Zittoun, T (2010). Uses of symbolic resources in, out and through the sphere of school experience in adolescent life. Paper presented at SIG 21 Learning and Teaching in Culturally Diverse Settings as part of the European Association for Research in Learning and Instruction. (2-3 September, 2010) Utrecht, The Netherlands.

# STUDENTS PERCEPTIONS ABOUT THE RELEVANCE OF MATHEMATICS IN AN ETHIOPIAN PREPARATORY SCHOOL

**Andualem Tamiru Gebremichael, Simon Goodchild and Olav Nygaard**

*University of Agder, Norway*

*This paper presents results from a pilot study into students' perceptions of the relevance of mathematics. Cultural historical activity theory is used as an analytical perspective. Data are collected through interviews supported by classroom observation. Convenience and purposive sampling were used to select the school and students. Findings indicate that students perceive that only basic mathematics is used in their everyday activities whereas the mathematics they are learning at the moment: has an indirect relevance through other subjects or used by professionals; has use in their future studies; has an exchange value in the market place of joining the university and getting job; and in mathematics they experience a sense of identity, empowerment, spirituality, and trust in the curriculum and their teacher.*

Key words: context, preparatory school, perception, relevance

## INTRODUCTION

The students in this study are preparing for higher learning institutions (hence the school designation 'preparatory'). The first author's experience as a mathematics teacher has revealed that there are many who consider mathematics as an academic exercise rather than a social activity; he has observed that even some teachers are challenged when they encounter real-life problems that could be solved using mathematics. He has observed that students pay attention to what they perceive as relevant to them. The textbook rarely includes practical problems; for example, the topic limit, which is said by teachers to be difficult for students to understand, consists of 177 exercises and 62 examples, of which there is only one practical exercise. Literature indicates that mathematics with real world connections makes learning mathematics more effective (e.g. Mason & Spence cited in Even & Tirosh, 2008, Goldin, 2008) and that the perception of students about mathematics is important for success (e.g. Even & Tirosh, 2008, Mulat and Arcavi, 2009). Motivation – "the inclination to do certain things and avoid doing some others" – is an important construct that affects performance in mathematics (Hannula 2006, p. 165). According to Roth and Lee (2007) identity, motivation and emotion are integral to the activity of learning. They contend that emotion as well as identity – the sense that the individual has about who she/he is with respect to others and with respect to the activity of learning – are central in the student's motivation to engage.

A key term in this study is perception, which means, "a result of perceiving" where to perceive means "to attain awareness or understanding of ... to regard as being such" ('Dictionary and Thesaurus – Merriam Webster', n.d.). Our study seeks to expose the students' understanding of the relevance of mathematics she/he is

learning. Thus, the term that fits the purpose here is perception. Mulat and Arcavi (2009) also studied the perception of high achieving students about what fosters “their mathematics and academic trajectory” (p. 77). In their study ‘perception’ referred to the students’ understanding of the factors that enable or constrain learning and achievement of mathematics. The purpose of this paper is to report a characterization of Ethiopian students’ perception of the relevance of mathematics with respect to their learning goals as well as about the relevance of mathematics to their society and real life situation, which we believe exposes the perception in a peculiar context and contribute to the improvement of learning mathematics in Ethiopia. The principal research question addressed in this paper is ‘what are Ethiopian students’ perceptions of the relevance of mathematics and how are they characterized’. The paper is structured in such a way that the theoretical perspective and methodology are discussed first, followed by the analysis, presented in eight themes that emerged through the analytic process.

## **THEORETICAL PERSPECTIVE**

The theoretical perspective that guides this study is cultural historical activity theory (CHAT), which is one strand of sociocultural theory that has evolved from the work of Vygotsky (Roth & Lee, 2007). Sociocultural theory is about people’s active involvement in cultural practices and the inherently social nature of mental processes (Cobb, 2007; Lerman, 1996). Proponents of sociocultural theory contend that there are many things that will remain obscure if the focus is only on the individual (Lerman, 1996; Wundt, in Cole & Engstrom, 1993). A basic tenet of cultural historical activity theory is that knowledge appropriation is a social process and learning doesn’t take place in the mind only but distributed in the activity system, which is mediated by cultural tools such as language, with human activity as the unit of analysis (Lerman, 1996). Central to CHAT is the notion of agency, that people can choose their goals for action and this object oriented action is mediated by cultural tools, community, rules and division of labour (Roth & Lee, 2007). Engström’s expanded mediational triangle (Cole & Engström, 1993) can describe the situation, which models the individual’s activity within which perception is mediated by the tools, rules, division of labour, and the community, as well as contradictions that might exist among and/or within them (Roth & Lee 2007).

The students have been involved in the activity of learning mathematics long before they were enrolled in preparatory schools. In this activity which has a historical dimension independent of the individual student, i.e. it has no definite beginning nor an end, with the ultimate motivation of survival of the society and the individual in the society (Roth & Lee, 2007), students are involved in a variety of activity systems, for instance, the school and the local community. The school rules enforce the curriculum and exams, and the textbooks are mediating artefacts. The teachers are supposed to follow the textbook strictly, and they take it to the classroom with them and give class work and home work from it. The medium of instruction is English

language which the student uses at school only. The experience of the first author indicates that whereas the students were all learning in English and interacting with each other in Amharic, they had different mother tongues. These students are living in an impoverished economic circumstances. Teachers are less paid than other professions as a result of which teaching is considered to be of less social value. On the contrary, the teacher is generally perceived as the authority when it comes to knowledge, and in Ethiopia it is popularly said, “The teacher is the father of knowledge”. The students’ perceptions are mediated by these elements. It is in such a framework that such behaviours could be interpreted (Roth & Lee, 2007).

## **METHOD AND METHODOLOGICAL REFLECTIONS**

The study is situated in the interpretivist paradigm. The intention is to understand students’ perceptions. The research question and the theoretical framework lead to the development of a qualitative research methodology because perceptions are behaviours that “can only be understood within [their] environment, which needs to be explored and explained” (Burton, 2002, p. 8), and it allows the investigation of the situation from the perspective of the participants. It is a case study design recognising that the knowledge gained is influenced by the peculiar culture of the context. The data are collected through interview of students from one chosen school which has its own peculiar features although ‘typical’ of Ethiopian preparatory schools. Since the purpose is to see the relevance of mathematics through students’ eyes the appropriate way to do it was holding an interview because it enables one to hear their own account. Particularly focus group discussion was used where students were provided discussion points to expose their opinions, and probe with further questions to enrich the responses. This was intended to create a situation where the students feel secure among their own classmates, and probe each other as well as to engage with more informants in the time available. Classroom observation supported probing during the interviews and exposing features of the mathematics classroom. The first author had taught for more than three years in the selected school and there are former colleagues who helped the selection of students. The topics and students’ experiences might vary across gender, streams, grade levels, and achievement level. The department head selected 4 classes – 2 from social science and 2 from natural science (i.e. one from each of 11<sup>th</sup> and 12<sup>th</sup> grades) – where the mathematics teachers were homeroom teachers (the teachers having first line of responsibility for the students in a class) because they have better experience of the students and their academic standing. Then each teacher selected 3 female and 3 male students who were identified as low, medium, and high achievers. A total of 24 students were selected. Students of same sex from the same class were interviewed in the same group which was intended to create a situation where the students felt more comfortable and the relative freedom to express their ideas and probe each other. Themes emerged from the analytic process, and each theme is analysed using CHAT as a framework for identifying key features.

## DATA ANALYSIS

In this section the data analysis is presented as a descriptive account of students' perceptions of relevance, under eight themes which are 'grounded' in the analysis.

### **Mathematics is relevant because it is used in every day activity**

The students are engaged in the activity of learning mathematics and they are also participating in the day to day activities of the society. Some students perceive that mathematics is relevant to every day activity Azenegash says, "how far I am coming from home. ... Those who work in the Edir [1] should know mathematics". Her perception is mediated by the artefact (the road) and the community in the local activity. Their mathematical knowledge is valued by their community. Ruth says, "When my mother wants to calculate something she calls me to do it for her; if she was educated she could have done it herself [2]". Her perception is mediated by her role in the local activity – she is involved in the budgeting of the family's expenditure. However, Ibrahim perceives that the knowledge being used in every day activity is that which they learnt at primary school because the society is "not a developed society" but he mentions that, "the ball should be spherical so that it can roll". His perception is mediated by the local community and the cultural artefacts. On the other hand, Beza mentions an example where an outcome of the school activity was used as a tool at the workplace, "population size, proportion, death rate, average, etc. are useful in society. ... My father works for the statistics authority". Her perception is mediated by the local community, through her father. Fisiha works in a wood work shop with his brother. He says, "For example, in my job, I measure, I should read the number; in order to cut accurately you should learn number". His role in the activity of the local community mediates his perception.

### **Mathematics is relevant because it is used in other subjects**

Some students perceive that mathematics is relevant to other subjects they are learning, Mekia says, "mathematics is useful for physics. ... In chemistry we have mathematics". Beza gives a specific example. She says, "Log, we learnt, is applied in bacterial growth; so it is used in biology". Habtu provides another example, "physics involves number e.g. vector", but he comments that "those [they] are learning now are rarely used". Their perceptions are mediated by the school curriculum. Debesh sees two distinct mathematics. He says:

There is calculation in Geography, Business; not the [subject] mathematics but the calculation in these subjects is useful. ... I think there are subjects which are related to mathematics. Those subjects have societal values. Thus, your knowledge of mathematics will help you for dealing with those subjects.

His perception is mediated by the local community and the school rules. Fikru says, "In 7<sup>th</sup> or 8<sup>th</sup> our teacher said 'mathematics is the king of all subjects'. ... it has use in chemistry; rector scale in geography; it is related with all other subjects". His

perception of relevance is mediated by the school community and rules. On the other hand, Meseret's perception of relevance is associated with her stream. She says, "We are social, we don't use much calculation... Other subjects are to be learnt by heart; I take break with mathematics". Her perception is mediated by the other subjects, the school rules and the school community.

### **Mathematics is relevant because it is useful in an unknown future**

The students are preparing for university studies. Hence their perception of the relevance of mathematics is mediated by their future goal. Habtu says, "I want to study astronomy and my brother told me that in addition to mathematics, physics is the base". The local community mediates his perception of relevance. Yirdaw says:

What we are learning now, I don't see its application. ... But in offices I think they use it. ... We are in the process of development. It is useful for what will learn in the future, I think. So, we must learn it.

His perception is mediated by the local community as he perceives that it is being used in the work place. On the other hand, some students explain the future based on their experience of the interdependence of the current activities of learning, Abebe says:

I don't know the detail about astronomy and how much mathematical capacity it requires. Since mathematic is important in our every day activities, it would be the same at that level. I think it would be important.

Whereas Meseret considers a specific case and projects it to the future. She says, "[mathematics] is a mother tongue. ... In economics there is slope. We learnt it in 7<sup>th</sup> or 8<sup>th</sup>. We didn't know then that it has this use". Hayal who wants to study Medicine or Chemistry remarks its relation with science "Science without mathematics? I don't believe that". Whereas, Makida says, "In books we don't see where to apply [it]. [It] has relation with other subjects and we apply it on them... at tertiary level". Thus, the other subjects, the school rules and the tools at school mediate their perceptions.

### **Mathematics is relevant because it gives an identity**

Students form their identity in relation to mathematics. Debesh wants to "study Banking and Insurance because it has mathematics" and he "like[s] mathematics ... it is not difficult for [him]". He perceives himself as someone who can do mathematics well, and this sense of identity is a motivational factor towards making a decision about what he has to study in the future. Essayas, on the other hand, perceives himself as someone who does not want to deal with mathematics; he says, "I want to study law because my brother told me that it doesn't involve mathematics ... economics, but [it] has mathematics; so I don't like". His perception of relevance is mediated by the local community, and the emotion towards mathematics. His decisions about the future are mediated by his identity and emotion. These students are learning in English which they only use at school. The students' perceptions



might be related to this nature of mathematics. Debesh says, “I like all but word problems are tricky. Use difficult words ... or they are difficult to understand”. In addition to the mathematical concepts, he is facing difficulty in understanding the English words embedded in the mathematics problems. Abebe says,

I like word problems, because it involves critical thinking and analysing ... it has to be related to our society, things that we know and experience in our lives. Not in some other society; the names when related to our situation then we do it with interest. When it talks about some world we don't know – names and places we don't feel that we have any concern about it – then it is done while we didn't understand the use

His identity is a motivation to his being engaged in learning mathematics. The students' perceptions of relevance to their identity might also be understood in a dialectical relationship between the individual and the collective. Ruth says, “Most social science students don't like mathematics. Only few students work hard. Thus our teacher always advises us”. She is high achieving and in other discussions as well she refers to the whole of students and the social science when locating herself in the mathematics classroom. There could also be a shift in students' perceptions. Beza says, “When I saw that I scored B in mathematics in 10<sup>th</sup> I didn't expect and I believed that I can perform well”. Her perception changed because of her score. The school rules which insist on examination success in learning and the examination score used as a tool to evaluate learning mediated her perception. It is noted here that Beza's perception is related to her motivation to study mathematics. The student's role as a mathematics learner also mediates perception of relevance, Azenegash says:

... But now I departed from my friends. ... They are in another school [3] now. ...there were clever students [who] explain to us. We used to discuss while walking home. ...I don't score in it and when I miss something I don't get back to see it again.

The change in Azenegash's activity system had an impact in her learning of mathematics. She used to pay attention to her surroundings with respect to mathematical meaning, and relevance of mathematics she mentioned before was about distance from home to school. She had formed an identity as a member of that group which shifted when she departed. Her identity with respect to that group was a motivational factor for her engaging in mathematics, which later changed as a result of the change in her identity.

### **Mathematics is relevant because it empowers one to make informal decisions**

Some perceive that mathematics and the other subjects are there for them to expose their talents. Erikihun says:

I want to study language or philosophy. ... I am doing well in language. ... Mathematics and most of the subjects we are learning now might not be related to what we learn in the future. But, they help us to identify/know our interest and direct us to the future. We used to learn music; it is not important but if you have the interest then you will know. Some of us may end up in a field that doesn't involve mathematics at all but others may need it.

He perceives that the other subjects are competing with mathematics for students' choice or attention. Since he made other choices of social science, he perceives that he doesn't need mathematics, but learns it because others in his group need it. His perception is mediated by the other subjects, the school rules and community. We also see that his perception of relevance is a motivational factor, and negative one in relation to mathematics. Other students perceive that mathematics is there to broaden their mental capacity, Netsanet says,

In sequence, we use formulas. But before we use formulas, there are items which we do simply by observation, by looking at it attentively. That helps you to think and analyse; it broadens your mental capacity.

Her perception is mediated by the school rules. Her perception of relevance about the mathematics she is learning seems to motivate her to work on mathematics.

### **Mathematics is relevant because the student trusts the curriculum**

There is a sense of trust in the curriculum. Asad perceives that they learn mathematics because "[they] should learn it". Ruth trusts the curriculum because "the teacher tells [them]" and her perception is mediated by the school community. Azenegash says:

[The teacher] is our eye. ... If it were not relevant we wouldn't have been taught ... my teacher was telling me, now I realized that it was right: 'when you are walking, it is the shortest distance to travel on the straight path'. He did it for himself.

The teacher is the origin of perception for trusting the curriculum, and the division of labour in the school activity also mediates her perception. She didn't see how the teacher did it and her role was to listen. She puts trust in the teacher that what is taught would have meaning in her life to come. On the other hand, Debesh says, "We don't see. But, the teacher tells us; for example, log in earthquake measuring, in chemistry, though the concepts are difficult for us". In one of my classroom observations in Debesh's class (a social science 11<sup>th</sup> grade classroom) the teacher was providing an example about earthquake and pH value in a lesson about logarithmic function. The mathematics teachers have natural science background, which might influence their practices in implementing the school curriculum and they use the same textbook for both streams, which mediates students' perceptions of relevance.

### **Mathematics is relevant because it has exchange value**

These students are supposed to score a qualifying grade to be admitted to the university, and Ethiopia is a poor country in which success in education and securing a job relates to sustaining the life of the individual as well as parents. Beza says, "We used to hear that 10<sup>th</sup> is the turning point for life. ... [Studying] any social science would be ok [to be a hostess]. ... Mathematics is compulsory". She is sure that success in mathematics is the gate keeper to joining the university and to achieving

the goal of securing a job – becoming a hostess. This perception of relevance she attaches to mathematics motivates her engaging in mathematics. The division of labour in the local community also mediates students' perceptions. For example Ruth says, "Earlier I wanted to study law but it is 5 years. [I] study economics ... then I can help my parents. ... If I don't have the basis in mathematics I can't do [economics]". Thus, she learns mathematics to sell it at the marketplace of learning economics so as to get job at the end which enables her to sustain her family and study law which she really likes to study. The division of labour – her responsibility to help her parents – mediates her perception of relevance. Her perception of relevance is a motivational factor for being engaged in mathematics. The tools in the school activity mediate Yirdaw's perception in his endeavour to become "a private accountant". He says, "Mathematics books from abroad are better at applications than domestic ones. ... I prefer the [latter] for success in exams. But, for my interest I prefer the [former]". Here we see that this perception is a development which resulted from the contradictions between the mediating artefacts and the school rules. Ahadu wants to "become a [medical] doctor", and he thinks that "whether one becomes a medical doctor or something else learning mathematics is part of the process". His perception is mediated by his future goal. Emotion also mediates students' perceptions. Alewi says, "I am not interested in it but it is required ... I liked polynomial at the beginning. ... when I scored poor at the first test, I turned my back to it again". The contradiction between her emotion – that she liked it at the beginning – and the school rule – that she should succeed in exam – led her to change her perception of relevance, and consequently, dropping it as irrelevant.

### **Mathematics is relevant because it gives a fresh perspective of life**

Some students give spiritual meaning to the mathematical concepts they are learning. Hayal says:

I brought the idea of limit to my life and interpreted it as, there is time when life ends; all the things that bother me together with my life in this world, come to an end, and begin the new life in heaven.

She is looking at her life as function of time whose limit at a certain time gives the static life – life after death which her society believes in. Her perception is mediated by the rule in the local activity – the religious teachings. In the same interview group, another student, Makida says:

Similar to what is mentioned by Hayal, when I learnt sequence, I learnt that things are in order they don't occur/happen randomly. It is as the saying goes 'there is time set for something' [Solomon's saying from the Bible]. For example, we can't say 1 then 5; in life also we can't walk immediately after we were born; it goes in steps. Things in life are ordered or they happen sequentially. This is what we learn indirectly.

Since the school curriculum did not provide direct applications the school rule mediated these perceptions of relevance.

## CONCLUSION

The participant students perceive that mathematics in general is relevant to their real life situation. But, they perceive that the mathematics dominantly in use is that which they learnt at primary school and some concepts they are learning at the current stage have got some uses in other subjects or workplaces. This relates to their access to material and technology which might involve application of higher mathematical concepts. They perceive that the mathematics they are learning has use in some unknown future which is mediated by their future goals; and others do not see any relevance to their future studies. They see it as a means of access to the university and getting a job, which relates to the economic context of Ethiopia as well as the responsibility of the students to support their family/parents. Some put trust on the curriculum and the teacher, who is thought to be a big authority. They exhibit identity related to mathematics which sometimes changes, resulting in change in motivation. At times we see the students struggling with the contradictions between the languages at home and in school (textbooks and lessons are in English which students begin to experience as of lower secondary). The students face difficulty in understanding word problems, which are the only ways practical problems are meant to be experienced by this students. In some cases the students make informal decisions about the use of mathematics. Others attach a spiritual meaning which relates to the peculiar features of Ethiopian society: the belief in life after death which is static, and use of sayings from the Holy Book. This indicates that being situated in a mathematics classroom where the textbook, and hence the lessons, rarely present practical problems (and not in their mother tongue), the students need to produce their own contexts in order to understand the mathematical concepts they are learning. It is the gate keeper to their access to the university, hence to the better life they aspire for themselves and their parents. It becomes apparent that students are under enormous pressure to succeed in mathematics. The students exhibited motivation to engage in mathematics when they perceived that it is relevant to them in some way. Addressing the question, whether there is any relationship between the different categories of students included in the sample and different characterizations of perceptions discussed above is left to the next stage of the project.

## NOTES

1. EDIR is social institution in the Ethiopian culture where members gather to discuss about issues related to social problems. They contribute money every month, and on the death of a member or siblings, they arrange a mourning ceremony including the funeral which lasts for three days. Members earn money as compensation on the death of a member or siblings. It is led by an elected board members consisting of chairman and a secretary. In some cases there are EDIRs for women only.
2. It is a common phenomenon in Ethiopia that mothers didn't go to school. Some might have gone to the traditional school (called Kes Timihit literary means Priest School) where they learn the Geez alphabet (an old language now used in the Ethiopian Orthodox Church only and Amharic uses the same alphabet) and Ethiopian numerals which range from 1 to 1000 only, after 1000 it is elph (which means infinity).
3. In the Ethiopian education system students are enrolled to preparatory schools if they qualify in a national exam at the completion of 10<sup>th</sup> grade. Then, they will be assigned to the preparatory schools where they

learn for two years and take another national exam which screens those who will be eligible for university studies.

## REFERENCES

- Burton, A. (2002). Methodology and methods in mathematics education research: Where is “the why”? In S. Goodchild & L. English (Eds.), *Researching mathematics classrooms: A critical examination of methodology* (pp. 1-10). Westport, CT: Praeger.
- Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3-38). Charlotte, NC: Information Age Publishing.
- Cole, M., & Engström, Y. (1993). Cultural-historical approach to distributed cognition. In G. Salmon (Ed.), *Distributed cognitions: Education and educational considerations* (pp. 1-43) Cambridge, UK: Cambridge University Press.
- Dessalegn, K.G, & Adem, D. (2006). *Mathematics student text: Grade 12* (Rev. ed.), Addis Ababa, Ethiopia: Aster Nega.
- Dictionary and Thesaurus – Merriam-Webster Online* (n.d.) Retrieved from <http://www.merriam-webster.com/>
- Even, R. & Tirosh (2008). Teacher knowledge and understanding of students’ mathematical learning and thinking. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education (2nd Ed.)* (pp. 202-222). New York, NY: Routledge.
- Goldin, G. (2008). Perspectives on representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education (2nd Ed.)* (pp. 176-201). New York, NY: Routledge.
- Hannula, M.S. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics*, 63, 165-178. doi: 10.1007/s 10649-005-9019-8
- Lerman, S. (1996). ‘Intersubjectivity in mathematics learning: A challenge to radical constructivist paradigm?’ *Journal for Research in Mathematics Education*, 27, 133-150. Retrieved from <http://www.jstor.org/stable/749597>
- Mulat, T. & Arcavi, A. (2009). Success in mathematics within a challenged minority: The case of students of Ethiopian origin in Israel (SEO). *Educational Studies in Mathematics*, 72, 77-92. doi: 10.1007/s 10649-009-9186-0
- Roth, W-M., & Lee, Y-J. (2007). “Vygotsky’s neglected legacy”: Cultural-historical activity theory. *Review of Educational Research*, 77, 186-232. doi: 10.3102/0034654306298273

# **DIFFERENTIAL ACCESS TO VERTICAL DISCOURSE – MANAGING DIVERSITY IN A SECONDARY MATHEMATICS CLASSROOM**

**Uwe Gellert & Hauke Straehler-Pohl**

*Department for Educational Studies and Psychology*

*Freie Universität Berlin, Germany*

*Crucial to the reading of our paper is the assumption that students enter school with diverse experiences with language and diverse experiences with construing meanings through language. We will argue that teacher-student interaction in secondary mathematics classrooms often is realised to the effect that only for those students, who have already been introduced to the linguistic qualities of academic or institutionalised discourse, further access to it is provided in schooling. Central to our argument are the notions of vertical/horizontal discourse, of contextualised/decontextualised language and of grammatical metaphor. We discuss these concepts before illustrating their power by an analysis of a transcript on the introduction to algebra in a sixth grade mathematics classroom. The focus of the analysis is on the strategies by which the teacher manages to cope with the diversity of students' sociolinguistic orientations to meaning.*

Key words: teacher-student interaction, language, vertical discourse, access, algebra

## **CHARACTERIZING VERTICAL DISCOURSE**

As has often been said, schooling acts as a mediator of two major societal functions. Schooling provides access to particular discourses and hence to particular forms of knowledge while, at the same time, regulating this access and hence socialising its students into differing positions of power. In this context, some groups of students have been described as privileged and others as marginalised – although in many cases the marginalised build the majority. In research in mathematics education, the issue of privilege and marginalisation has been discussed in terms of social class (e.g., Cooper & Dunne, 2000), race (e.g., D. Martin, 2010), and migration (e.g., Gorgorio, Planas & Vilella, 2002). For many students, these conditions coincide and their marginalisation is exacerbated by an interaction of migration, race and social class issues. In many situations it is difficult to adequately express the complexity and diversity of marginalisation. Thus our focus is on the commonalities and, as we will argue, on the theoretical core of privilege and marginalisation: the differential access to a particularly powerful discourse – vertical discourse.

The particularities of the discourses of power to which students are differentially introduced in school have been described differently, though mostly in form of dichotomies. We will briefly reconsider the most relevant concepts for our analysis that have been constructed in Sociolinguistics and Systemic Functional Linguistics

(SFL). For further concepts of other academic disciplines, that we consider relevant, but that we are not actively applying in this paper, see Cummins (1996) and Koch and Oesterreicher (1985). All concepts taken together, serve as the theoretical grounds for the subsequent analysis of an interactional mechanism that regulates the differential access to powerful mathematics provided by instructional practice:

As a sociologist of education, Bernstein (1999) is concerned with the differences between horizontal and vertical discourse, where the concepts of discourse and knowledge are closely interrelated. Horizontal discourse “is likely to be oral, local, context dependent and specific, tacit, multi-layered, and contradictory across but not within contexts. However [...] the crucial feature is that is it [sic] segmentally organized” (p. 159). Knowledge and strategies of the horizontal discourse have the aim to maximise encounters with persons and with habitats. Vertical discourse, in contrast, “takes the form of a coherent, explicit, and systematically principled structure, hierarchically organized as in the sciences, or it takes the form of a series of specialised languages with specialised modes of interrogation and specialised criteria for the production and circulation of texts, as in the social sciences and humanities” (p. 159). The contemporary dominant ideology of the life-long learner, making use of knowledge in differing contexts, is privileging vertical forms of knowledge and hence privileging those with access to vertical discourses. But how can access to vertical discourse be provided? On what kind of students’ resources can access to vertical discourse be based? It is a core problématique of pedagogy to recontextualise horizontal discourse in school as a means to make institutional, vertical discourse more accessible for all.

Hasan (2001) draws on Bernstein’s distinction of horizontal and vertical discourse. For Hasan, horizontal and vertical discourses differ mainly in their relation to contexts. She sees the natural condition of human discourse as being contextualised language, concise a language with a close connection to the material situational setting of the interactants. Decontextualised language in contrast has a loosened or even detached connection to the material setting. However, it is decontextualised language, which is connected to positions of power: “what is remarkably pervasive today is the kind of language use that is known as context independent, disembedded or decontextualised, especially in the sorts of societies spawned by the so-called progressive Western world. [...] After all, among other things, decontextualised language is the voice par excellence of official ideology” (Hasan, 2001, pp. 48-49). The distinction between contextualised and decontextualised language is organised along the terms of actual and virtual reference. Actual references have the potential of being physically sensed by the interactants. These references may be immediate as well as displaced in time or space, however they need to be potentially sens-ible. Virtual references lack this potential. They are “non-material and removed from situational realities, they simply cannot be directly and physically experienced: they are intellig-ible, not sens-ible” (p. 54). For participation in horizontal discourses,

contextualised language may well be a totally sufficient base. However, in the vertical discourse knowledge is not structured through the sens-ible context, but through the internal logic of a specialised practice. It is obvious that this internal logic is only intelleg-ible and far beyond material situational realities. In the case of academic mathematics, it is evident, that we deal with a highly intelleg-ible discourse, far from being sens-ible. No matter how much contemporary school-mathematics is organised around sens-ible actual experiences, its end is the vertical discourse of virtual ideas: “the mastery of disembedded language will consist in feeling at home with reality that is not sensuously mediated” (p. 57). Hence, the orientation towards decontextualised language is a crucial condition for participation in the mathematics classroom.

As Bernstein and Hasan, J.R. Martin (2007) sees horizontal discourse as the original and intuitive mode of discourse. However through the perspective of SFL his concerns include the lexico-grammatical qualities of discourse, that is the make-up of utterances. He describes horizontal discourse as characterised by a harmony of the semantic and the lexico-grammatical stratum (see Fig. 1): Taking the sentence “I love my mummy and my mummy loves me.”, semantics and grammar are in complete harmony: participants (mummy, I, me) are described by (pro)nouns, processes by verbs (love(s)), and logical relations by conjunctions (and). Presumably, someone with a more elaborated use of language would rather express the same feelings in a sentence like: “My mother and I have a good relationship.” Now semantics and grammar have created a tension: the noun “relationship” is not expressing a participant, but a process (loving each other). Moreover, it gains further meaning, as through social everyday discourse different kinds of qualities implicitly got attached to it. Martin calls this tension of the semantic and lexico-grammatical stratum *grammatical metaphor*. However, although being more elaborated, the sentence quoted above remains part of horizontal discourse. Applying Hasan’s perspective, the orientation to meaning is still contextualised. As we will argue, the key to vertical discourse lies in grammatical metaphor acting on decontextualised language, or as Martin puts it, in “abstractions acting on abstracts” (p.54).

As can be seen in Figure 2, the major characteristic of grammatical metaphor is a tendency to express all kinds of semantic categories in nouns, a process Martin calls *thingification*. From a multisemiotic perspective, O’Halloran (1999, p. 382) concludes: “The analysis of mathematical pedagogical discourse indicates that nominalization and extended nominal group structures are a feature of mathematical discourse.” Martin (2007), summarising extensive research on both scientific and human-scientific texts, claims: “if no grammatical metaphor, then no verticality” (p. 54). Concerning the social ramifications of grammatical metaphor, Martin holds, “from a functional linguistic perspective, access to vertical discourse is bound up with control of grammatical metaphor, which in western societies students are expected to master in secondary school. Failure to access this recourse entails



exclusion from [academic] knowledge structures. Here lies the social semiotic nub of institutionalized learning, educational failure and the distribution of knowledge in our expiring world” (p. 55).

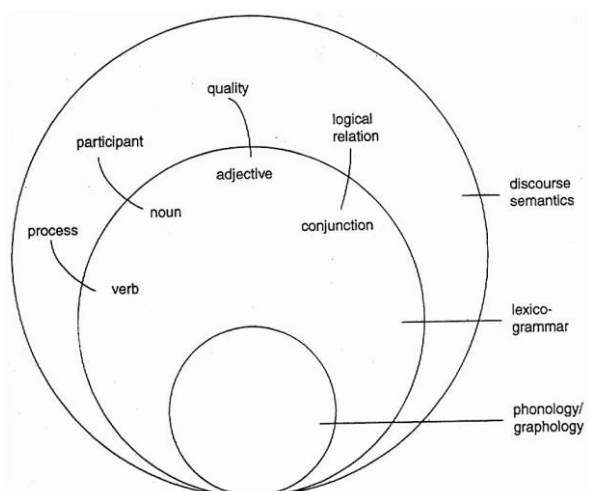


Fig. 1 Stratal harmony – grammar matching semantics

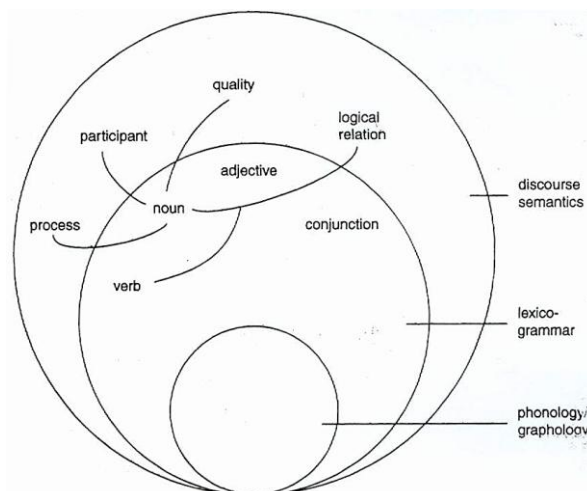


Fig. 2 Grammatical metaphor as stratal tension

## NEGOTIATING MEANING: A MECHANISM OF APPEASEMENT

Our issue is interaction in the mathematics classroom. We focus on the interactive mechanism by which one teacher deals with the diversity of his students in terms of their access to mathematics related vertical knowledge. As we will see, the concepts of (de)contextualised language and grammatical metaphor provide powerful tools for analysing the amplification of differential access to the vertical discourse. For a detailed description of the empirical research, see Knipping et al. (2008).

The setting is a 6<sup>th</sup> grade mathematics class in Nova Scotia, Canada. It is the very first lesson after the summer holidays, in which the teacher and the students engage in teacher-student interaction. The 6<sup>th</sup> grade is the beginning of secondary schooling in that region, thus no hierarchy of achievement has yet been established among students. It is a rural area and the social background of students is quite diverse.

Kevin's Grade	Alice's Age
6	4
7	5
8	6
9	7
?	?
?	?
?	?

Kevin uses a pattern. He predicts how old his sister will be during each of his school grades.

Kevin	Alice
6	4
7	5
8	6
9	7
10	8
11	9
12	10

Fig. 3 Mathematics task

Fig. 4 T-table at the whiteboard

The students are sitting at group tables. At home they have solved the task displayed in Fig. 3. Our analysis starts in the moment when a student is filling her solution into a T-table provided by the teacher at the whiteboard (see Fig. 4). We split the following ten minutes of classroom interaction into six episodes, which we consider as functionally distinct for the interactional mechanism. We shall demonstrate each episode and its relevance for the negotiating of meaning in that classroom.

### **Episode 1: Introducing the context (10:32-12:38)**

In the first episode Alicia is filling the missing numbers in the T-table at the whiteboard. Her classmates are encouraged by the teacher to help her out. Regarding the use of language, the whole episode is characterised by harmony of semantic and lexical strata, hence no grammatical metaphor is employed. Further, there is an extensive presence of physical resources (e.g. the teacher invites his students to refer to the T-tables they have in their textbooks). The language is contextualised and the discourse tends to be horizontal. The function of this episode may be best described as a smooth introduction into the emerging discourse, emphasizing the affective security necessary for the students to actively participate.

### **Episode 2: Common negotiation of orientation (12:38 – 13:45)**

10 12:38 T Is there anybody from her group, as well as Alicia, who can tell us how those numbers fit in the way they do? What did you do?

In (10) the teacher, for the first time, goes beyond discussing particular numbers and by his elicitation indicates that there is a pattern, a principle structuring the T-table. He does that by using the conjunction “how”. The semantic and grammatical strata are still in harmony, though at the same time a first insertion of decontextualised language can be observed: “How” is accompanied by a (semantic) metaphor “fit in the way they do”. Here, “Fit in” does not mean a potentially sens-ible way of fitting. It refers metaphorically to the above-mentioned structuring principle. However, this question is followed by an alternative one: “What did you do?” This refers back to previous actions and illustrates the ambivalence of the word “how”, as the students may either describe their experience or their reasoning. Hence, the teacher provides two different discourses within one utterance: a vertical discourse of reasoning and a horizontal discourse of experience. At this time, no preference can be observed: both discourses appear legitimate.

In the following, a dialogue evolves between the teacher and Mike. Mike seems to have identified the discourse as a discourse of reasoning and explains his considerations, using the conjunctions “because” and “so”. The teacher in reaction always links Mike’s answers back to the particular numbers on the board and hence seems anxious about keeping the discourse overt for horizontal discourse and contextualised language. In summary, this episode is characterised by its tendency to verticality through a smooth and partial introduction to decontextualised language remaining, at the same time, open for contextualised meanings. It is remarkable that

Mike is the propulsive force in the negotiation and, further, that he is the only student taking part in it.

### **Episode 3: Practice of a vertical discourse (13:45-14:30)**

After having – under apparent leadership of Mike – negotiated the orientation towards a more decontextualised language and a more vertical discourse of reasoning, the teacher now comes to the core of vertical discourse: He introduces the term of “relationship”.

27 13:45 T I have a question. This can come, the answer may come from any group. You may look at the T-table here or you may look at the one you’ve created in your notebook. Can anybody figure out or tell me the relationship between the left side of this T-table and the right side of the T-table.

28 (Mike is the only student who raises his hand.)

29 14:14 T OK.

30 M The difference between the numbers, there’s a difference of two on each number.

31 T A difference of two. How do you mean difference?

32 M There is, one is two higher.

33 14:28 T So in other words, this one is two higher.

By this he is making use of a grammatical metaphor. Semantically, “relationship” is not a participant. In contrast, it rather expresses qualities, a process and logical relation. Hence, there is a tension of the semantic and the grammatical stratum; qualities, process and logic relation are *thingified*. Accordingly, to successfully understand the teacher’s elicitation one has to decode both grammar and context. The discourse has reached verticality in its linguistic orientation and the term “relationship” entails the mathematical core of the talk. What students can learn here is the fact that T-tables materialise relationships, as the teacher quotes later on (48, episode 6). As expressed by Martin (2007), the core of vertical discourse goes along with the use of grammatical metaphor. But again – as in episode 2 – it is only Mike who is able to recognise and realise this orientation. As by “difference” he autonomously employs a grammatical metaphor, he seems comfortable with its use. All other students remain silent, hence are not yet actively participating in the vertical discourse.

### **Episode 4: Re-linking to the horizontal discourse**

In line 31 the teacher starts coming back to a more harmonic use of semantics and grammar. There, he is demanding to express logic relations by the use of the conjunction “how”. In line 32 and 33 Mike and the teacher are stepwise coming back to the visible and particular T-table, thus orienting towards contextualised language

and horizontal discourse. The function of this short episode may well be assumed as making the discourse more accessible to all.

### **Episode 5: A second try of common negotiation of orientation**

- 35            **T** I have a question. How do you go from this number to this one?  
Remember you said that we added down or you folks added down.  
How do we get from this side if you were looking at these numbers  
and if you say they sort of, they sort of seem to match up in a way?  
How do we get from this side to this side? Karsten, can you figure  
it out?

In some way, this elicitation resembles utterance 10 in episode 2. The teacher is reconstituting the harmony of semantics and grammar. His tool for asking for logical relation again is the conjunction “how”. This is accompanied through processes (go, added, get, etc. ) which are expressed through verbs. But as the “how” is clearly related to processes, it - this time - is not ambivalent (compare to episode 2). He is really asking for “How do you go from this number to this one?” instead of “why”. Hence the use of language is more contextualised, more bound to the experiences, the students have made in their work. However, he is not entirely coming back to a contextualised discourse, but still offering a decontextualised alternative, asking for how “they sort of seem to match up in a way”, which is equivalent to “how those numbers fit in the way they do?” (10). While marking horizontal discourse as legitimate, there remains an implicit tendency towards the vertical decontextualised discourse. But again, he is offering alternatives and accordingly giving apparent control to the students over which discourse they like to refer to. The function of this episode can be regarded as another strive for a common negotiation of orientation. However, this strive remains unsuccessful and the teacher goes on to revise his strategy.

### **Episode 6: Apparent unification of horizontal and vertical discourse**

- 37            **T** Is there anybody else or is there anybody else who can see  
anything else here that goes from here to here as far as  
relationship? How do we compare this number with this number?  
(*T waits two seconds.*)

Similar to Episodes 2 and 5 the teacher is offering two alternative questions differing in their degree of verticality. Apparently, the teacher still follows his strategy of negotiation. However, a more detailed look at the first and more vertical of the two questions indicates a modification of the strategy. He is firstly asking for a process expressed in a verb (goes) and then links it in some unhandy way to the grammatical metaphor of relationship. The use of “as far as” implies that there is something beyond the demanded process which is not expressible in a way that students can access. The difference to line (10) and (35) is in the direction towards which the discourse is oriented. While in line (10) and (35) the aim could be expressed as

drilling the seeds for the orientation towards the decontextualised vertical discourse, the aim now is to re-establish the students' feeling of comfort, neglecting the relevance of participating in vertical discourse. The following four and a half minutes of the discussion confirm this view. The teacher is limiting teacher-student-interaction on arithmetical questions as:

53            **T**    Wayne, are you with us son? What's ten minus two buddy?

54            **W**    Eight.

The focus is on the affective outputs of the discussion rather than on the content. There are several examples of the teacher's strategy of re-establishing the students' comfort in the discussion. An analysis of one of these examples may provide a good insight in the function of this strategy.

48            **T**    Because guess what, a lot of T-tables work in a pattern something like this where you can fill in a little tiny equation. If you understand that with this one you'll understand most of what happens in most of the rest of the T-tables. See this little equation here? It gets a little bit harder but they work basically the same way.

The teacher highlights the exemplarity of the T-table and outlines the relevance of the discussion in the vertical discourse. In addition with his behaviour of eliciting simple arithmetic results and positively evaluating the answers, he is establishing a straight logical chain from filling out spaces in this particular T-table to understanding algebraic structures in T-tables in general, that is, a (false) progression in verticality: If you are able to answer "ten minus two buddy?", then you "understand that with this one" and "you'll understand most of what happens in most of the rest of the T-tables." His use of the notion "little tiny equation" is illustrative for this strategy of appeasement: Through the qualities of "little" and "tiny" the term "equation" shall lose the scare of an academic grammatical metaphor.

50            **T**    Now all of a sudden you are into real simple arithmetic. You did this ages ago so guess what? We made the math look a little bit hard, now we're trying to make it look easy.

Of course, the teacher does not have the power to tear down the boundaries between horizontal and vertical discourse. Arithmetic serves as a tool in algebra, but factually it will remain a different discourse. Hence, the teacher is only able to mask the boundaries and consequently to render them invisible for the students.

## DISCUSSION

The analysis illustrates how our chosen theoretical framework allowed us to identify an interactional mechanism, which bares the potential of amplifying the students diversity resulting from their differential linguistic socialisation in early childhood and

primary school. Bernstein, Hasan and Martin all make us aware that this differential socialisation is closely connected to the issue of privilege and marginalisation. The ongoing research indicates that the mechanism observed is not a singular phenomenon.

The interactive mechanism the teacher is using to negotiate mathematical meaning on two different discursive levels effects a differential provision of access to valued forms of mathematical knowledge. As Martin (2007) and O'Halloran (1999) have argued, without grammatical metaphor and decontextualised language there is no vertical school mathematics discourse. Those students who are already prepared for decoding and using grammatical metaphor experience opportunities to further access the legitimate discourse of secondary school mathematics. Those who are only used to horizontal discourse are not challenged by new forms of knowledge. There is no attempt to make the different orientations to meaning visible. Instead of generating linguistic conflicts, the teacher establishes horizontal discourse as legitimate and blurs the boundaries between horizontal and vertical discourse. Hence, the teacher appeases rather than challenges those students who most need an explicit introduction into vertical discourse.

Atweh, Bleicher and Cooper (1998) report on the differences of register of two mathematics teachers working in schools with disparate student population. Where students are expected to aspire future university studies the teacher challenges them constantly with decontextualised language. In contrast, in a working class suburb the focus of the mathematics teaching was to develop skills useful in a consumer society, resulting in a more intuitive and less systematic use of language. Here again, we see how the teacher's perception of low educational ambitions and aspirations limits the students' access to valued forms of mathematical knowledge.

However, these teachers work with either positively pre-selected or socially marginalised groups of students. Their discursive practices are characterised by a high degree of internal consistency: either aiming at vertical *or* at horizontal discourse. The interactive mechanism of negotiating meaning on two different discursive levels *within one classroom* seems to be particularly important in unstreamed and inclusive school systems as student heterogeneity with respect to orientation to meaning is greater. In inclusive school systems, all students *potentially* access vertical discourse. In our illustrative case, however, interactional mechanisms translate student diversity into disparities of achievement. *Actually*, through appeasement, the characteristics of vertical discourse remain masked for the non-privileged.

## NOTES

The research has been supported by grants from the Alexander von Humboldt-Foundation and the Social Science and Humanities Research Council of Canada.

## REFERENCES

- Atweh, B., Bleicher, R.E., & Cooper, T.J. (1998). The construction of the social context of mathematics classrooms: A sociolinguistic analysis. *Journal of Research in Mathematics Education*, 29(1), 63-82.
- Bernstein, B. (1999). Vertical and horizontal discourse: An essay. *British Journal of Sociology of Education*, 20(2), 157-173.
- Cooper, B., & Dunne, M. (2000). *Assessing children's mathematical knowledge: Social class, sex and problem solving*. Buckingham: Open University Press.
- Cummins, J. (1996). *Negotiating identities: Education for empowerment in a diverse society*. Los Angeles: California Association for Bilingual Education.
- Gorgorió, N., Planas, N., & Vilella, X. (2002). Immigrant children learning mathematics in mainstream schools. In G. de Abreu, A.J. Bishop & N.C. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 23-52). Dordrecht: Kluwer.
- Hasan, R. (2001). The ontogenesis of decontextualised language: Some achievements of classification and framing. In A. Morais, I. Neves, B. Davies & H. Daniels (Eds.), *Towards a sociology of pedagogy. The contribution of Basil Bernstein to research* (pp. 47-79). New York: Peter Lang.
- Knipping, C., Reid, D.A., Gellert, U., & Jablonka, E. (2008). The emergence of disparity in performance in mathematics classrooms. In J.F. Matos, P. Valero & K. Yasukawa (Eds.), *Proceedings of the Fifth International Mathematics Education and Society Conference* (pp. 320-329). Lisbon: Universidade de Lisboa.
- Koch, P., & Oesterreicher, W. (1985). Sprache der Nähe – Sprache der Distanz. Mündlichkeit und Schriftlichkeit im Spannungsfeld von Sprachtheorie und Sprachgeschichte. In O. Deutschmann et al. (Eds.), *Romanistisches Jahrbuch, Band 36* (pp. 15-43). Berlin: de Gruyter.
- Martin, D.B. (2010). Not-so-strange bedfellows: Racial projects and the mathematics education enterprise. In U. Gellert, E. Jablonka & C. Morgan (Eds.), *Proceedings of the Sixth Mathematics Education and Society Conference*, 2<sup>nd</sup> edition (pp. 57-79). Berlin: Freie Universität Berlin.
- Martin, J.R. (2007). Construing knowledge: A functional linguistic perspective. In F. Christie & J.R. Martin (Eds.), *Language, knowledge and pedagogy: Functional linguistics and sociological perspectives* (pp. 34-64). London: continuum.
- O'Halloran, K.L. (1999). Classroom discourse in mathematics: A multisemiotic analysis. *Linguistics and Education*, 10(3), 359-388.

# MATHEMATICS TEACHERS' SOCIAL REPRESENTATIONS AND IDENTITIES MADE AVAILABLE TO IMMIGRANT STUDENTS<sup>47</sup>

**Núria Gorgorió & Montserrat Prat<sup>48</sup>**

*Universitat Autònoma de Barcelona, Catalonia (Spain)*

*Immigrant students in mainstream schools join in social practices that are already structured by social representations. Representations have a crucial impact on how immigrant schoolchildren construct their learning experiences and the ways in which they make sense of themselves as learners. Immigrant students' transition processes are co-constructed with other social actors. In the school context, the teacher is a key person in helping them on creating new meanings, constructing new knowledge and establishing new relationships. Mathematics teachers' social representations, however consciously or not, mediate not only the actual content immigrant students may learn but also the identities they develop.*

Keywords: immigrant students, transitions, mathematical identities, social representations

## **IMMIGRANT STUDENTS CONSTRUCTING MATHEMATICAL IDENTITIES**

### **Immigrant students' learning mathematics as a transition process**

Immigrant students' processes of learning mathematics in mainstream schools may be understood as transition processes. Being an immigrant student in a "foreign" mathematics classroom implies a new context of mathematical practice, different relationships with people and knowledge, and different understandings of the actions and interactions that take place.

According to Zittoun (2007), transitions involve changes in the social, material or symbolic spheres of experience of the person and imply processes of relocation in all of them. These changes in position convey new expectations and new possibilities but also constraints on action and losses. Transitions also imply reconstruction of identities and require new forms of knowledge and skills and bring the need to engage in meaning-making to confer sense to what happens to the person. Zittoun establishes that 'in youth, learning difficulties are often linked to the fact that the person feels his/her identity put at stake or cannot find a personal sense in the learning situation' (op. cit., p. 196).

---

<sup>47</sup> The research presented here has been partially supported by a private foundation –Fundació LaCaixa & ACUP (RecerCaixa 2010) – and the Spanish Science and Innovation Ministry of (DGI-EDU2010-15373)

<sup>48</sup> Both authors are members of the EMiCS group –Mathematics Education in its Sociocultural Contexts–(2009 SGR 00590) granted by the Direcció General de Recerca of the Catalan Government.



We argue that transition processes require transactions of meaning between people and across contexts and are therefore limited and shaped by social representations about the role and achievements of the person as part of a group to which s/he belongs or identifies with. Transition processes are co-constructed with other social actors. In the classroom the teacher is a key person in establishing new meanings and constructing new knowledge, but also in making available to his or her students certain mathematical identities.

Transitions originate in changing contexts of social practice, changes in persons, or changes in the relations between persons and objects (Zittoun, 2007). Transitions require processes of adjustment to new life circumstances and involve multiple changes in frames of reference and meanings, and in relations with people. These changes require people to modify routines and interpretations, explore new possibilities, and develop new ways of acting and interacting. Social representations, as a means of constructing reality, have a special impact on the transition processes of immigrant students, in particular on their processes of learning mathematics and the construction of mathematical identities.

### **Mathematical identity**

The limited extension of this paper does not even allow for a short account of the work done in mathematics education in relationship to mathematical identity. For an initial definition, we want to note that Martin (2007) establishes that mathematics identity (as he calls it) encompasses the dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity involves a person's self understandings as well as how s/he is constructed by others in the context of doing mathematics. Therefore, a mathematics identity is expressed in narrative form as a negotiated self, a negotiation between our own assertions and the external ascriptions of others. Mathematical identities are always under construction.

For the purposes of this paper, by the students' "mathematical identity" we mean the academic identity that they develop as mathematics learners and users. Mathematical identities include how students view their own aptitude for mathematics and how they see themselves as users of mathematical knowledge both in school and beyond. Students develop their mathematical identity through their participation in mathematical activities, their interpretation of their own classroom experience, their expectations about future (mathematics) education and about their uses of mathematics both in school and outside. Students' mathematical identities are dynamic rather than static, and are bound up in other social or cultural identities they may develop.

Mathematics lessons are patterned activities organized with reference both to social norms and values and to mathematical concepts and rules. Therefore, students'

mathematical identities also include their sense of affiliation with the mathematical practices in their particular classroom and their identification with the norms and values regulating it. The development of mathematical identities is shaped by students' social positions in the mathematics classroom, their construction of mathematical knowledge, and how students understand their experiences as mathematics learners.

Students' mathematical identities are constantly being reconstructed in relation to students' perceptions of themselves as mathematics learners and how they are seen by significant others involved in mathematical activities. How students see themselves as mathematics learners, however, is as important as how they are defined by others, especially their mathematics teachers.

Cobb and Hodge (2002) established that the gatekeeping role that mathematics plays in students' access to educational opportunities includes the difficulties that students experience in reconciling their views of themselves and who they want to become with the identities that they are invited to construct in the mathematics classroom.

### **Identities in transitions**

Immigrant students' transitions have the potential to change the ways in which they interpret themselves and their roles both in school and outside. The ways in which they experience and have experienced their learning of mathematics in the different contexts and how they interpret these experiences impact on their practices and in the identities they develop. The identities they develop as mathematics learners have also to do with their sense of affiliation with the mathematical practices in the classroom they attend, their identification with the norms and values regulating it and their possibilities to participate in the mathematical practices.

According to Abreu and Cline (2003) there are three complementary processes of identity development: a) identifying the other – how the individual understands the social identities of “others” that are dominant in the context of specific practices; b) being identified – how the individual understands the identity extended to them by “others”; and c) self-identification – how the individual internalizes and takes positionings in relation to identities that had previously existed in the social sphere. These three complementary processes are not fixed but evolve through the interaction between the person and the sociocultural world. Identities are constantly reconstructed by engaging in a practice and belonging to a group, but also by wanting to engage in real or imagined practices and belong to real or imagined worlds.

Crafter and Abreu (2010) reveal how processes of self-identification, identification of others, and identification by other significant people play a crucial role in transitions and that such processes are linked with social representations. Self-identification coexists with identification extended by others and one process is a reaction to the other. “Being identified” is a process whereby individuals understand

the identity extended to them by others. According to these authors, it is not uncommon that “others”’ identification of oneself is based on dominant representations, which can be seen as part of the social context of practices.

Our focus is on immigrant students’ identity development in relationship with school mathematics. We are interested in identities associated with learning mathematics as part of a transition process and on how this development is influenced by social representations of immigrant children as mathematics learners held by their mathematics teachers.

## **IDENTITIES MADE AVAILABLE TO IMMIGRANT STUDENTS**

### **Social representations and identity construction**

The purpose of Moscovici’s theory of social representation is to explain a process whereby individuals and groups can manage to construct a stable and predictable world out of the diversity of persons, attitudes and social phenomena (Moscovici, 1984). This diversity is organized through social representations that carry previously constructed meanings concerning the past, and make these meanings available for new applications.

Social representation theory offers a way of understanding the social construction of reality that takes into account both the cognitive and the social dimensions of this construction (Ibáñez Gracia, 1988). Identifying the representations surrounding social phenomena is an approach that allows us to understand how persons both construct and are constructed by social reality through processes of communication and interaction. Social representation theory is particularly useful for understanding phenomena related to the teaching and learning of mathematics in classrooms in which immigrant pupils are present (Abreu & Elbers, 2005; Gorgorió & Abreu, 2009).

Among the functions of social representations Abric (1994) includes the following: a) knowledge of reality through an integration of information into a common frame of reference that is consistent in the values, social norms and practices of the group; b) definition of identity and group belonging, and identification and positioning in relation to other groups; c) guidance for forms of action and social practices through definition of the purpose of a given situation, production of expectations and anticipations, and definition of what is normative and counter-normative; and d) justification of opinions and actions in regard to people and objects and, on a more general level, the maintenance of social differentiation.

The learner in transition belongs at the same time to different groups and participates in different practices. As in Duveen (2001), in the account of identity development we consider a fundamental idea that the individual in transition enters a social practice that is already structured by social representations of the specific community. In particular, teachers’ representations about immigrant children as

mathematics learners will result in actions, discourses, and relationships that make available certain identities to immigrant students. This occurs in a context, the school, where power is not homogeneously distributed.

However, identities are not only constituted by labels that people place on themselves and others. Identity is about how people become who they are and how they come to understand themselves (Urrieta, 2007). It is about how they come to figure who they are, through the worlds they participate in and through how they relate to others within and outside these worlds.

In the following two sections, we illustrate how mathematics teachers, however consciously or not, make available certain mathematical identities to their immigrant students through the actual practices they promote in their classrooms. Through the opportunities they offer to their students, teachers contribute to shaping how immigrant students become who they are as mathematics learners, and how they come to make sense of themselves. For that purpose, we will draw selectively from two different ongoing studies.

### **Marta and Ronnie**

Marta and Ronnie provide us with our first example. Ronnie is one of the participants in an ongoing study aimed at understanding the transition processes of immigrant students learning mathematics in Catalan schools (see Costanzi, Gorgorió, & Prat, in press, for more details). To date we have interviewed 33 Ecuadorian students in compulsory secondary school. In one of these schools, we have worked with 15 boys and girls, Ronnie among them. We have also worked with their mathematics teachers, and we have recorded several interviews with Marta, Ronnie's teacher.

Marta's representations about mathematics being a universal school subject and learning depending solely on cognitive abilities, led her to attribute to her students low achiever identities. During the different interviews, she told us repeatedly that she did not consider her students' place of origin to be relevant information for her teaching of mathematics. In fact, one third of the students in her class were immigrant, and she could not tell us how many of them were from Ecuador. She insisted that to her all students are equal. However, when asked, it was clear that she had a prevailing view of students from Ecuador as "working below grade level", a fact that had important consequences for her immigrant students' chances to participate in mathematics tasks with a high level of requirement.

Marta: There's no room for activities that are challenging, mathematically speaking. They're too weak; we can only do exercises if our goal is to get the students to pass. We can't do problems.

In fact, during classroom observations, we never saw any problem solving situation, and during the interviews, her students confirmed this absence.

However, Marta's students told us that she was a good and caring teacher. Ronnie is one of them. At age 16, Ronnie was in the third year of the four years of compulsory secondary education, one year behind where he should be. He told us that *"when I got here I couldn't talk"*, by which we understand that he means he could not speak Catalan, the language of instruction in Catalonia. He also told us that *"since I was not up to grade level in mathematics because I came from Ecuador I was placed in a class a year behind"* where he should have been according to his age.

Now, he speaks Catalan well and says that he wants to go to university, although he understands that it will require hard work. Despite this, although he still has one more year of compulsory education and two of baccalaureate to complete, he also believes that he will not have access to the university system because of his being weak at mathematics.

Ronnie:       since I'm not good enough at math, I won't be able to pass the entrance exam to go to university.

Ronnie is one of the cases that show how a student has limited possibilities as a mathematics learner because of what is offered to him as school mathematics. It also illustrates that he has accepted the identities made available to him, to the point that his narrative could be that of his mathematics teacher instead of that of a student who wants to succeed.

### **Carles' and his students**

The other example, that of Carles' lessons, comes from Prat (2009). Carles is the mathematics teacher of a group in the second year of compulsory education, with students aged around 13. We have observed the development of the mathematics lessons during several weeks and interviewed the teacher, and the students have answered an open questionnaire about the social organization of the mathematics lessons.

During our first meeting, when Carles was asked about his students in class in order to organize the videorecording of the lessons, he referred to those having learning difficulties or attitudinal problems. He only talked about the good ones if they had a behaviour that made them *"too obvious"*. When asked about the three immigrant students in his class, he told us:

Carles:       (...) the three of them follow the lessons with no problem, they are good students. Other years, I've had some immigrant students that were not so ...

From all the conversations we have had with him, it seems clear to us that he is ready to give opportunities to all his students, regardless of their place of birth. He commits to complete the prescribed curriculum each year and scaffolding learning is to him the basic strategy of mathematics teaching. From the very first interview he expresses that to him the person is more important than the mathematics content he is teaching.

## Working Group 10

Carles: When we teach, we are facing people that have to learn something, each one with their own ways of being and doing, (...) the basic issue is that they are human beings. (...) the most important thing is to educate them, even before teaching them mathematics.

Next, we want to share with the reader some moments in Carles' lessons. We want to insist that, once again, our interpretation is based not only on the short vignettes that we are presenting now, but on a whole process of analysis of an ample set of data.

In one of the sessions we observed, Carles' students were solving problems. The students were facing the blackboard and the teacher was the one leading the task. They were solving the following problem:

In a rectangle, the length is 3cm longer than the height. The area is 80cm<sup>2</sup>. Determine its dimensions.

Carles: *(to all the students)* What's the area in a rectangle?

10 (JR): *(not asked individually)* Length times height

Carles: *(to 10(JR))* Before answering, you have to be asked.

We want to note that Carles had described 10(JR) as "*a bright student that likes to be paid attention too much*".

Later on, when using the formulae to solve the problem, 10(JR) prompts, without being asked:

10(JR): when one of the solutions is negative, we cannot use it as a solution, since lengths cannot be negative.

Nobody is paying attention to 10(JR) except the teacher that tells him that his thinking is correct. Then, the teacher explains it to the rest of the group. As observers, we noticed that the time for the lesson was running short, and Carles wanted a right answer to finish the problem before all left.

In fact, we observed a repeated pattern in the interaction between Carles and 10(JR). When the teacher wanted an efficient answer, he would ask him or allow him to prompt his answer. However, he would not accept his non-invited contributions, even when they were right and useful, unless he was short of time. It was like on the one hand, he wanted to teach him that there were norms on how to contribute. However, on the other hand, by asking him when efficiency was needed, he was pointing him out as an able student.

From the interaction with another student we recorded the following:

Carles: *(to all the students)* It's always important to clearly establish the "x". If you know what "x" means here, raise your hands.

## Working Group 10

Carles: (several hands are raised) 16(DM), tell us.

16(DM): the length.

Carles had described 16(DM) as a student whose “*attitude... well, it has to improve, right? But, well, he is starting.*” From classroom observation and our conversations with the teacher, we understand that the teacher’s intention was to have the student involved in the lessons’ development. However, the pattern we observed was that he only asked him simple questions, like the one above. This was pointing the student out as one that was less able than average and that could only be asked about simple facts. This is especially obvious in the vignette, when he pretends the question to be an important one.

Later on, during the same lesson, he gave them a list of equations to be solved. He told them that if anyone has a doubt s/he had “*to stand up and come to ask me*”. Once they were finished, and before correcting them, he gave them the solutions of the equations and asked “*who got all the answers right? Raise your hands!*” In other occasions he asked them to raise their hands when they had the exercise or the problem wrong. This was a pattern that repeated itself throughout the time, to the point that the students described it as a natural part of the lesson.

According to his explanations about how he organized the lessons, the teacher’s intention was to follow both each student individually and the whole group. However, requiring them to go and ask him when they had any doubt or raise their hands whether they have the answers right or wrong, was also a way to distinguish those who do well from those that do not.

## DISCUSSION

Immigrant students’ transitions imply processes of identity reconstruction that could afford as well as constrain. How students see themselves as mathematics learners is as important as how they are defined by others, especially their mathematics teachers. Grades are an obvious mechanism through which students come to figure out how good they are thought to be at mathematics. Explicit praise or criticism is also a way to let students know whether they are doing what they are expected and how much they are expected to do. However, there are more subtle ways through which teachers extend identities to their students.

In Ronnie’ and Marta’s case, we have seen how identities were made available to students, as a group, through everyday classroom practices, by leading them to construct a very restricted kind of mathematical knowledge, a fact that suggested constraints in their possibilities for their academic future. How could they become good at mathematics, or how could they think they were, or would be good, at school mathematics if they were offered no mathematical challenge? Marta, with an honest intention to make them feel that they succeeded at the tasks she offered them, she

only proposed routine exercises, reinforcing her *a priori* image of them as low achievers.

Through Carles' case, we have seen how identities were extended to individual students, individually, through an honest effort to follow each student's progress. We have seen how, in everyday classroom practices, the way the teacher orchestrated participation in the classroom, and his pursuit of scaffolding learning provided evidence to all of who needed help and who succeeded without it. This way, in the actual developing of classroom practice, students could make sense of themselves as good or bad in mathematics.

Our work is still a work in progress. In this paper, we have neither discussed how acceptance or rejection of the attributed identities take place, nor how students position themselves in relationship to normative identities in class. There are many other questions still open, such as what is the role played by agency and power in the realisation or contestation of social representations, or how students negotiate with themselves and others the different identities they construct through their participation in different practices.

We have illustrated how social representations play a role in the attribution of identities through the mathematical tasks that the students are offered to participate in, and through the interactions that take place between teacher and students. It could be asked whether there is a way for teachers not to attribute identities to their students. We are convinced that the answer to this question is no. In the same way that we all have representations of the world around us, when relating to others we all identify them, while being identified by them and take positions in relationship to them.

At the beginning of the paper, we said that we would present our argument to show how mathematics teachers' social representations, *however consciously or not*, mediate what the students learn and the identities they develop. The crucial issue here is the awareness of this mediation. As mathematics teachers it is our responsibility to engage in reflexive practices to critically examine our social representations and how they impact on the identities we make available to our students.

## REFERENCES

- Abreu, G. de, & Cline, T. (2003). Schooled mathematics and cultural knowledge. *Pedagogy, Culture and Society*, 1(1), 11-30.
- Abreu, G. de, & Elbers, E. (2005). Introduction: The social mediation of learning in multiethnic classrooms. *European Journal of Psychology of Education*, XX(1), 3-11.
- Abric, J. C. (1994). *Pratiques sociales et représentations*. Paris: PUF.



- Cobb, P., & Hodge, L. L. (2002). A relational perspective on issues of cultural diversity and equity as they play out in the mathematics classroom. *Mathematical Thinking and Learning*, 4(2), 249.
- Costanzi, M., Gorgorió, N., & Prat, M. (In press). Pre-service teachers' representations of school mathematics and immigrant children. In E. Hjärne, G. van der Aalsvoort & G. de Abreu (Eds.), *Learning, social interaction and diversity – exploring school practices*. SENSE publishers.
- Crafter, S., & Abreu, G. de (2010). Constructing identities in multicultural learning contexts. *Mind, Culture & Activity*, 17(2)
- Duveen, G. (2001). Representations, identities, resistance. In K. Deaux, & G. Philogène (Eds.), *Representations of the social* (pp. 257-270). Oxford: Blackwell.
- Gorgorió, N., & Abreu, G. de (2009). Social representations as mediators of practice in mathematics classrooms with immigrant students. *Educational Studies in Mathematics*, 72(1), 61-76.
- Ibáñez Gracia, T. (1988). *Ideologías de la vida cotidiana*. L'Hospitalet: Sendai.
- Martin, D. B. (2007). Mathematics learning and participation in the African American context: The co-construction of identity in two intersecting realms of experience. In N. Nasir, & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 146-158). New York: Teachers College Press.
- Moscovici, S. (1984). The phenomenon of social representations. In R. Farr, & S. Moscovici (Eds.), *Social representations* (pp. 3-69). Cambridge: Cambridge U.P.
- Prat, M. (2009). *Un model per a l'estudi de l'organització social de l'aula de matemàtiques*. Unpublished PhD Dissertation. U. Autònoma de Barcelona, Spain.
- Urrieta, L. (2007). Identity Production in Figured Worlds: How some Mexican Americans become Chicana/o Activist Educators. *The Urban Review*, 39(2), 117-144.
- Zittoun, T. (2007). Symbolic resources and responsibility in transitions. *Young*, 15(2), 193-211.

# SOCIAL FUNCTIONS OF SCHOOL MATHEMATICS

David Kollosche

*Universität Potsdam, Germany*

*In this essay, I explore the question which social functions school mathematics might hold. After presenting a criticism of prescriptive functions, the function of imparting mathematical knowledge and its boundaries are presented. A discussion of logic thinking, of alienation in modern societies, and of the functions of school mathematics in technocratic societies in general is presented that broadens the understanding of which social functions school mathematics might hold, leaving many open questions to explore.*

## RAISING THE QUESTION

Mathematics education is compulsory for all children in modern societies. What should be learnt and why it should be learnt are central questions of mathematics education research worldwide. Answers to these questions may vary in many aspects, but most mathematics education researchers agree on one point: routine calculations are over-represented and mathematics lessons should be more mind-challenging. Depending on perspective, »mind-challenging« may include focusing on creativity, problem solving, proofs and argumentations, applications and modelling, historical and social issues of mathematics, and so on; but, no matter the perspective, most suggest alternatives to routine calculations.<sup>49</sup>

The questions *why to teach* and *what to teach* are explicitly addressed throughout the mathematics education literature and in national standards documents. Heinrich Winter's (1995) essay "Mathematikunterricht und Allgemeinbildung"<sup>50</sup> and Hans-Werner Heymann's thesis (1996) *Allgemeinbildung und Mathematik*<sup>51</sup> are the most popular German examples of the first; the *Principles and Standards for School Mathematics* (2000) – published by the National Council of Teachers of Mathematics in the USA – and the *Bildungsstandards im Fach Mathematik* (2004) –

---

<sup>49</sup> This paper will not discuss the corresponding concepts and show in what way they criticise too much routine calculations. For the purpose of this paper, it suffices to see that mathematics lessons comprehend routine calculations and that alternative concepts, in advocating different activities, are therefore directed against routine calculations. However, this is not to mean that mathematic educators or even mathematic teachers condemn routine calculations altogether.

<sup>50</sup> *Allgemeinbildung*, literally translated meaning *general education*, is a highly influential concept in German pedagogy without any conceptual equivalent in the English speaking world.

<sup>51</sup> Available to English readers as Heymann (2003) *Why Teach Mathematics? A Focus on General Education*.

published by the German Standing Conference of the Ministers of Education and Cultural Affairs and influenced by Heymann's work – are exemplars of the later.

Addressing what should be done implies that what is being done is not satisfactory. Winter and Heymann certainly belong to those who criticise too much calculating. But while the prescriptive concepts of Winter and Heymann suggest what should be done, they are unable to explain what is being done. Therefore, I ask: Is there any sense in having children master the masses of routine calculations, which we – the community of mathematics education researchers – might regard as being over-represented? Are there any reasons for the contemporary state of school mathematics? Can these explain why the abovementioned »mind-challenging« alternatives are not implemented on a large scale? And would not the answers to these questions strongly influence our ideas of what school mathematics should be like?

Pointing to tradition does not help here. Tradition may show us how the situation came to be, but it does not explain why some things changed while others have not. Therefore, I propose to address the issue in a broader context. Identifying school mathematics as an organ in our interacting, organic society, I raise the question: What are the social functions of school mathematics?

## DISCUSSING NAÏVE ANSWERS

Curricular research offers a division between *material education*, i. e. mainly imparting knowledge, on the one hand and *formal education* on the other hand. My thesis is that the role of imparting mathematical knowledge is over-estimated and that further functions must be analysed to develop a comprehensive understanding of school mathematics.

A first approach to these functions might be a sceptical discussion of the concept of *competences* which curricular standards often use. These competences may already point to social functions of school mathematics, but starting with them creates problems. First, it is yet unclear how (or if) the demanded competences are indeed learned by children – especially when curricular concepts are used as tools for curricular reforms as is the case in the *Bildungsstandards im Fach Mathematik* –, and how (or if) these competences are indeed used outside school. Second, the focus on these competences might mask other functions of school mathematics that could be more central. Therefore, I suggest a different approach: After a discussion of the boundaries of imparting mathematical knowledge, I will elaborate on insightful connections between mathematical education on the one hand and Aristotelian logic, alienation and technocracy on the other hand. These are the points my studies concentrated on so far. In each case we can ask: What is the social impact of these and how does mathematical education contribute to them?

## MATHEMATICAL KNOWLEDGE AND ITS BOUNDARIES

A function of school mathematics could be to have children impart certain mathematical knowledge in order to master certain situations that arise in society. Central questions are: What are these situations? What knowledge suits them? Is this knowledge indeed acquired in school?

A first set of situations that can be mastered mathematically is located in *private life*. Popular examples are cooking, shopping and trading, investments, or painting walls. A second set could consist of situations from *work life* that are not mathematics-intensive. Assuming that the mathematical knowledge required to master these situations is acquired in school mathematics, we nevertheless have to admit that the better part of school mathematics used in these situations has been taught after 7 or 8 years of school. For most people, quadratic, exponential, and trigonometric functions are not tools needed in mastering everyday situations in private or work life; neither are linear equation systems, calculus, conditional probability, and so forth. Heymann (2003, p. 104) argues:

In their private and professional everyday lives, adults who are not involved in mathematics-intensive careers make use of relatively little mathematics. Everything beyond the content of what is normally taught up to 7<sup>th</sup> grade (computing percentages, computing interest rates, rule of three) is practically insignificant in later life.<sup>52</sup>

After comparing several studies exploring the uses of mathematics in private and work life, Heymann outlines the mathematical concepts that are frequently used (2003, pp. 88-89):

*Arithmetic*: counting; mastery of basic arithmetical operations ('in one's head' or with paper and pencil, depending on the complexity); calculating with quantities, knowledge of the most important units of measurement, making simple measurements (primarily of time and distance); calculating fractions with simple denominators in unambiguous contexts; calculating decimal fractions; computing averages (arithmetic mean); computing percentages; computing interest rates; using the rule of three; completing arithmetical operations with a pocket calculator; basic skills in estimating and making rough calculations.

*Geometry*: familiarity with elementary regular figures (circle, rectangle, square, etc.) and objects, as well as with elementary geometrical relationships and properties (perpendicularity, parallelism); ability to interpret and draw simple graphic representations of quantities and their relationships (charts, diagrams, maps) and the relationships between given points using Cartesian coordinate systems.

---

<sup>52</sup> Heymann's thesis was followed by a vivid public discussion when the German red-top newspaper *Bild* (1995), disregarding the context of Heymann's work, printing the title »Professor: Too Much Maths is Nonsense« and stated, that »The mathematics adults need has been learned after 7 years of school.«

Mathematics certainly *is* used to master situations in private and work life, and this mathematics *is* being taught in school. Here, I must make two points, however. First, only until the seventh year of school, imparting mathematical knowledge can be regarded as a social function for people in non-mathematical jobs. Second, it is yet unclear whether the mathematical knowledge used in private and work life is indeed learned in school.

It is at least doubtful whether the mathematical knowledge used in private and work life is indeed learned in school. In her influential publication *Cognition in practice*, Jean Lave (1988) presents studies of the everyday use of mathematics in private and work life conducted in Liberia and the USA. She doubts whether »schooling is a font of transferable abilities« (p. xiii) and develops her thesis that mathematical knowledge needed to master situations in private and work life is learned »in practice« rather than in school. More recent studies, for example studies on the numeracy of nurses in the UK (see Coben, 2010, p. 14), support Lave's thesis. Heymann shares this view (2003, p. 98):

A number of factors indicate that specific vocational mathematical qualifications tend to be learned more implicitly on the job and that thus the persons involved often remain unaware of them.

The thoughts presented above leave only a relatively small group of people engaged in mathematics-intensive professions, for whom higher mathematical qualification in school might be useful. Interpreted from a social perspective, it is possible that a function of school mathematics is to prepare as many children as possible for mathematics-intensive professions. It then would be reasonable to teach all children mathematics beyond their seventh year of school, attempting to maximise the number of children entering mathematics-intensive professions.

## ENCULTURATION BEYOND KNOWLEDGE

Imparting mathematical knowledge is not as dominant a social function of school mathematics as might be expected. A critical examination of the sets of competences that curricular standards want children to learn in school mathematics suggests that competences such as »solving problems«, »modelling«, »using formal aspects of mathematics« or even »thinking logically« point at the nature of our engagement with the world, of our thinking. Our worldviews and the nature of our thinking depend on the society (or culture) in which we learned thinking and perceiving the world. This learning process can be called *enculturation*<sup>53</sup>.

---

<sup>53</sup> »Enculturation« as a terminology, derives from cultural studies. Sociologists prefer to speak about »socialisation« while educators often prefer »education«, although the latter has a strong intentional meaning.

Imparting mathematical knowledge is a part of enculturation, as it enables and encourages people to perceive and approach the world in a certain way, namely mathematically. Unfortunately, there is little literature on enculturation in school mathematics, though the work of, e.g., Roland Fischer, Alan Bishop and Ole Skovsmose is well acknowledged. However, Fischer's work (e. g. 2006) is very fragmentary and does not draw a comprehensive picture. Bishop's (1988) chapters on the »Values of Mathematical Culture« and on »Mathematical Culture and the Child« are highly important for my issue, but I do not want to discuss them in this paper. Skovsmose's work (2005) raises inspiring questions about the social functions of school mathematics, but he does not come to convincing answers.

Further related work are the analyses (and criticism) of »rationalism« and the role of mathematics in modern society in the work of Max Weber (1921/2008), as well as Max Horkheimer and Theodor W. Adorno (1944/1997), the sociologic analysis of mathematicians in practice by Bettina Heintz (2000) and the critical study concerning the legitimacy of modern mathematics by Philipp Ullmann (2008). Horkheimer and Adorno analyse and criticise how the ideas of Enlightenment shape our thinking and organise our society. Ullmann's work is strongly based on that of Horkheimer and Adorno but lays more emphasis on mathematics and its applications in society. Heintz' studies come to the conclusion that the modern mathematician is characterised by his will to avoid contradictions and that he therefore chooses a method that is intended to avoid contradictions, namely the logical proof. However, her work hardly draws attention to the social and educational implications of her results.

I do not want to discuss or present this literature in any more detail. Instead, I present my current, ever-evolving thinking of further social functions of school mathematics.

### **HIERARCHY AND LOGIC**

The first thought I offer reaches back into the depths of history on human culture, specifically, to the development of hierarchical thinking. Roland Fischer (2006, pp. 133–141), building on the work of the Austrian philosopher Gerhard Schwarz (2007), describes *hierarchy* as a certain system of relationships between people of a certain community. In the history of human culture, the genesis of hierarchies can be observed wherever a community transitioned from a nomadic "tribe" to a fixed "state". The organising principle for tribes was based on kinship, setting a limit to the growth of a community. So, on the one hand, social growth and the development of states were only possible where hierarchies were established. On the other hand, the idea of organising communities hierarchically could only spread because the developing state communities were successful.

Living in a society that is ordered hierarchically is a powerful everyday experience: People learn that for every (but some) person(s) there is a person who decides what to do and what not to do; that actions are either allowed or forbidden; and that everyone is held responsible for his obedience or disobedience, neglecting the situation that led to her or his actions.

Living in such a society leaves its marks not only on the principles of our everyday actions but also on the principles of our everyday thinking. The idea of hierarchy became an element of thinking, of perceiving the world. For example, today we consider it normal to think of a husky as a dog, of a dog as a mammal, and so on. In fact, the biological classification is totally hierarchical.

In ancient Greece, scholars became aware of the principles of the thinking invoked by hierarchies. The *logic* of Aristotle is a description and analysis of this *logical thinking*. He postulates, for example, that every statement is either *true* or *false*. This may mean: allowed or forbidden in thinking. Furthermore, for every (but some) statement(s) we have statements on the basis of which we can decide whether the first statement is true or false. Eventually, the truth or falsity of a statement depends on the system of logic only, neglecting any (e. g. everyday life) connotations the statement may have.

While logic thinking was already a topic of ancient Greece philosophy, it was not until the beginning of modernity that logical thinking became conventionalised as the only “right” thinking with mathematics as its purest manifestation. René Descartes, the French mathematician and philosopher, may be considered the founder of this modern rationalism. In his *Rules for the direction of the mind*, he states, that »arithmetic and geometry alone are free of any error of falsity or uncertainty«<sup>54</sup> (1629/1959, pp. 8–9) and

that those who seek the right way to truth must not engage with any matter that does not allow them to obtain a certainty comparable to that of arithmetic and geometric proofs.

This kind of thinking features a worldview that relies on antagonisms, causalities, and pre-determined, static concepts. It has certainly helped mankind to increase its understanding and possibilities of handling the world, but at the same time, it has shaped our thinking and perceiving the world in a certain way, leaving black spots and possibly discrediting those who think differently.

Logical thinking is not without alternative; it is not the only way of making sense. This alternative becomes clear when we look at non-Western societies (see Bishop 1988 for examples) or acknowledge that people had indeed thought in tribal communities before hierarchies and logic thinking evolved. The relentless division

---

<sup>54</sup> Translated into English from the German translation of the Latin original *Regulae ad directionem ingenii* from 1629 by D. K.

into true or false has even been criticised from within mathematics. For instance, at the beginning of the twentieth century, the Dutch mathematician L. E. J. Brouwer (1918) began to create mathematics without the assumption that every statement must be either true or false.<sup>55</sup>

Connecting my aforementioned (somewhat oversimplified) analysis of logic to school mathematics, we might ask: Is logical thinking represented here, *more* than in any other school subjects, maybe even *only* here? Does school mathematics prepare children to think and act in a logically thinking and acting society? Extending the connection more critically, we might ask: What worldview do we create by teaching the dominance of logical thinking? And eventually, what does it mean for children who develop alternatives to “our” logic?

## MODERNITY AND ALIENATION

At the verge to modernity, industrialisation made everyday life change dramatically. The medieval man (or woman) was a peasant or a craftsman, subsisting on what he produced. Although committed to kin, church, and state, he was the sovereign of his everyday life, making nearly every decision, especially the economic ones, on his own. This personal freedom was lost when more and more people began working in factories, where they had to perform prescribed repetitious work at a prescribed time of the day without causing any problems that might interfere with the production of the factory.

But it would be short sighted to assign the qualities of obedience, punctuality, and reliability in doing repetitious work to the factory worker of early industrialism alone. Contemporary work life requires the same qualities, and the modern employee must be enculturated to think, feel and act accordingly. The essence of this performance, which can be named *alienation*<sup>56</sup>, is that a person must not act according to his actual feelings and wishes. Alienation is necessary for cooperative work where the work of many depends on the cooperation and reliability of the individual.

Primary and middle schools that emerged at the time of industrialisation took over a function of enculturation, preparing children to endure the alienation necessary for factory work. School mathematics was included from the very beginning and might have a particular role in the process of alienation so typical for the modern man and woman: Do the command-like masses of mathematics exercises drill obedience (cf. Skovsmose 2005)? Does the lack of individualisation in the mathematics classroom – in the process of teaching as well as in the nature of the answers expected from

---

<sup>55</sup> In Brouwer’s logic, statements can be neither true nor false. But still, they cannot be both true and false.

<sup>56</sup> Alienation here is understood in a slightly broader sense than in Marxian terms.



children – represent the factory's disregard for individual concerns? And to return to the beginning of this essay: Do routine mathematics calculations serve a social function, e. g., developing the ability and willingness to perform repetitious routine tasks whose broader sense might not be understood and/or favoured?

## MODERN GOVERNMENT AND TECHNOCRACY

Modern government<sup>57</sup> has often been interpreted under the term *technocracy*. Technocrats (i.e., scientific specialists of a certain domain) name and determine the urgent questions of our time, planning work, health systems, education, economy, and so forth. Considering the aforementioned discussion, we may register that technocracy features a certain way of thinking; that is, logical thinking, and requires that people perform in a predictable, alienated fashion. Specifically, a technocracy requires people who follow rules which are not set up by themselves but by experts. (i. e., the technocrats).

Technocratic decision-making depends a lot on mathematics. Mathematical models are used to describe, prescribe, and predict technical, economical, and social matters. For example, medical studies claim that the effect of a new medication is twice as high as the old, the 2% increase of the GDP shows that the economy is doing well, or income taxes must be raised because the costs of the health system exceed the budget by 2 billion Euros. We accept these decisions, although we do not fully understand the justifications used.

But technocracy is nothing *imposed* on people; it is *lived* by people. Technocracy requires people to trust in it and it requires technocrats. Concerning the issue of trust, we might ask: How do people come to trust in mathematical justifications? Do people consider mathematics especially trustworthy?<sup>58</sup> And if so, do they develop this trustworthy attitude in school mathematics?

Concerning the issue of technocrats, another function of school mathematics can be identified. School mathematics might not only practice logical thinking, it might also select those children able to thinking logically, allocating the special few to technocrat positions in society. Ole Skovsmose (2005), in his book *Travelling Trough Education: Uncertainty, Mathematics, Responsibility*, raises the corresponding questions (p. 11):

Could it be that mathematics education in fact acts as one of the pillars of the technological society by preparing well that minority of students who are to become 'technicians', quite independent of the fact that a majority of students are left behind?

---

<sup>57</sup> »Government« here means any form of decision-making that other people depend on, not only in the executive of a state.

<sup>58</sup> Here, I omit an excessive, yet illuminating, discussion about the certainty and legitimacy of mathematics (cf. Skovsmose, 2005; Ullmann, 2008).

Could it be that mathematics education operates as an efficient social apparatus for selection, precisely by leaving behind a large group of students as not being ‘suitable’ for any further and expensive technological education?

## FINAL THOUGHTS

In this essay, I have argued that school mathematics might have the social function of *identifying*, *selecting*, and *allocating* children, as well as preparing children for the contemporary predominant society in terms of

- developing children’s *mathematical knowledge*,
- shaping children’s *thinking* towards a form we may call logical, and
- shaping children’s *feeling* towards a form that supports technocracy and living in a society that requires alienation.

As the works cited suggest (e. g., Skovsmose, 2005), many of these points have been discussed in the literature. These discussions are limited and often isolated, and fail to draw a cohesive picture of the social functions of school mathematics. Moreover, the discussions are highly evaluative, especially when it comes to people who suffer from mathematics education and are interpreted as being suppressed by a reign of technocrats. Although our own feelings and ideals *are* important, I am afraid that a perspective that places emphasis on the ethics of mathematics education might silence possible explanations that are necessary for a comprehensive understanding of the social functions of school mathematics.

Moreover, in this essay, I raise more questions rather than provide answers. The purpose of the essay, however, was to only mark the trajectory of my research project. My project aims not only to determine probable answers to the questions raised but also to develop a cohesive understanding of the social functions of school mathematics. This project requires not only further research on the functions discussed but also the development of deeper understandings of the intellectual concepts on the basis of which functions of school mathematics might be discussed.

## REFERENCES

- Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education*. Mathematics education library: Vol. 6. Dordrecht: Kluwer.
- Brouwer, L. E. J. (1918). *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten: Erster Teil. Allgemeine Mengenlehre*. Koninklijke Akademie van Wetenschappen te Amsterdam, Verhandelingen, (5), 1–43.
- Coben, D. (2010). “At the sharp end of education for an ethical, equitable and numerate society: working in a safety-critical context – numeracy for nursing”. In U. Gellert, E. Jablonka, & C. Morgan (Eds.) *Proceedings of the sixth international*

- mathematics education and society conference* (Vol. 1, pp. 9–22). Berlin: Freie Universität Berlin.
- Descartes, R. (1959). *Regeln zur Leitung des Geistes*. Hamburg: Meiner (Original work published 1629).
- Fischer, R. (2006). *Materialisierung und Organisation: Zur kulturellen Bedeutung der Mathematik*. München: Profil.
- Heintz, B. (2000). *Die Innenwelt der Mathematik: Zur Kultur und Praxis einer beweisenden Disziplin*. Wien: Springer.
- Heymann, H. W. (1996). *Allgemeinbildung und Mathematik*. Weinheim: Beltz.
- Heymann, H.W. (2003). *Why Teach Mathematics? A Focus on General Education*. Dordrecht: Kluwer.
- Horkheimer, M., & Adorno, T. W. (1997). *Dialectic of enlightenment*. London: Verso (Original work published 1944).
- Lave, J. (1988). *Cognition in Practice: Mind, Mathematics and Culture in Everyday Life*. Cambridge: Cambridge Univ. Press.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, Va.: National Council of Teachers of Mathematics.
- Professor: “Zu viel Mathe ist Quatsch” (1995, October 6). Bild.
- Schwarz, G. (2007). *Die „Heilige Ordnung“ der Männer: Hierarchie, Gruppendynamik und die neue Rolle der Frauen*. Wiesbaden: VS Verlag für Sozialwissenschaften (Original work published 1985).
- Skovsmose, O. (2005). *Travelling Through Education: Uncertainty, Mathematics, Responsibility*. Rotterdam: Sense Publ.
- Ständige Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland (2004). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss*. [Beschluss vom 4.12.2003]. Beschlüsse der Kultusministerkonferenz. Neuwied: Luchterhand.
- Ullmann, P. (2008). *Mathematik, Moderne, Ideologie: Eine kritische Studie zur Legitimität und Praxis der modernen Mathematik*. Konstanz: UVK.
- Weber, M. (2008). *Economy and society*. London: Routledge (Original work published 1921).
- Winter, H. (1995). „Mathematikunterricht und Allgemeinbildung.“ In: *Mitteilungen der Gesellschaft für Didaktik der Mathematik*, (61), 37–46.

# **BECOMING DISADVANTAGED: PUBLIC DISCOURSE AROUND NATIONAL TESTING**

**Troels Lange and Tamsin Meaney**

*Charles Sturt University*

*The importance of mathematics or its alter ego ‘numeracy’ has been cemented in the mind of the public with the instigation of national, high-stakes testing in Australia. The discourse around these tests illustrates how the process of social valorisation operates. Using press releases, online news articles and online public comments, we show how politicians, parents, teachers and the general public discuss ideas about disadvantage in relation to national testing of numeracy. In these discussions, deficit language labelled some children as being less likely to perform well because of their backgrounds. In contrast, poor results for individual schools were seen as contributing to the wider community labelling their students as disadvantaged.*

Keywords: public discourse, disadvantage, social valorisation, national high-stakes testing

## **NATIONAL TESTING, MATHEMATICS AND DISADVANTAGE**

In a broader research program, in which we investigated the public discourse around the *National Assessment Program – Literacy and Numeracy* [NAPLAN], the theme of disadvantage was prominent. In this paper, we concentrate on Australians’ conceptions of disadvantage as being complex and sometimes contradictory.

For some time, mathematics has been considered a gate-keeping subject in relation to people’s opportunities to take up further study and work opportunities (Davis, 1996; Quintos & Civil, 2008). In Australia, national testing has solidified this role for numeracy; with literacy the only other subjects being tested and therefore considered valuable in judging the ability of a school to deliver education to students (Donnelly, 2009). The NAPLAN website describes the background of the tests in this way:

The content of each test is informed by the National Statements of Learning in English and Mathematics which underpin state and territory learning frameworks. Test questions cover aspects of literacy (Reading, Writing, Spelling, Grammar and Punctuation) and numeracy. Questions are either multiple-choice or require a short written response (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2010b).

The tests determine whether Australian students reached minimal standards at Years 3, 5, 7 and 9 (ACARA, 2010a). Four months after the children sit the tests, parents and caregivers are sent a report about their child’s performance against the standards and in relationship to other children. In early 2010, the Federal Labor Government opened a website, called *My School*, which compared different schools’ results on the NAPLAN tests, thus very quickly making them high stakes (Lingard, 2010).

The reasons given for requiring students to complete these tests are similar to those in other English-speaking countries and are linked to ‘raising standards’, particularly for students from disadvantaged backgrounds (Donnelly, 2009). As Lingard (2010) stated, this reasoning “has become a globalized educational policy discourse” (p. 131). Researchers, such as Hursh (2008), have linked this discourse of raising standards for students under-achieving at school to neoliberal beliefs about the need to hold schools accountable and privatise them, along with other institutions. The results of tests purportedly allow individuals to make choices so they can maximise the benefit of schooling in an education market. Thus, a discourse of accountability of teachers, schools and education systems runs in tandem with one about raising standard for at risk students (Lingard, 2010). As suggested by Gutiérrez and Dixon-Román (2011) little has been done to reconcile the two discourses:

[A]lthough many use “the achievement gap” as an important call for school accountability around needed resources and additional support for marginalized students, (e.g., Education Trust, 2005), such discourse has done little more than replace “the culture of poverty” in the latest of deficit frameworks. That is, while equity issues are becoming more mainstream in the mathematics education community, theoretical framings continue to reflect equality rather than justice, static identities of teachers and students rather than multiple, fluid, or contradictory ones (Gutiérrez, 2007; 2002; Martin, 2009) and schooling rather than education (p. 21).

The consequence of these types of discourses saturating educational debates is that the discursive field pervading mathematics teaching and learning becomes highly charged. Inherently, a discursive field attributes value to some phenomena whilst depriving others of value or visibility. The result is that the discursive field facilitates only some views becoming acceptable. Summarising their findings from several studies, Abreu and Cline (2005) described this process as social valorisation:

We have evidence from previous studies that the social valorisation of practices is a key element in a person’s representation of these practices. Studies with Brazilian schoolchildren (Abreu, 1995) and with British schoolchildren (Abreu & Cline, 1998) showed that they had developed an understanding of how specific forms of mathematical knowledge were socially marked and that this enabled them to construct categories, to compare them and to relate these to given social identities (Abreu & Cline, 2003). (p. 699).

As gravitational and magnetic fields define directions in the physical world, a discursive field assigns what is ‘up’ and ‘down’, ‘south’ and ‘north’, ‘along’ and ‘against’ thus constituting what could be termed a force field of social valorisation. The force field affects the saying, doings and relatings (Kemmis & Grootenboer, 2008) of students, teachers and the general public, and it shapes backgrounds and foregrounds of students and their dispositions to engage in learning mathematics (Alrø, Skovsmose, & Valero, 2009; Skovsmose, 2005). The force field of this discursive world is inescapable to its ‘inhabitants’ in the same sense that we cannot

escape the gravitational field. It does not mean that people cannot talk or think in ways not aligned with the valorisations of the discursive field; rather, it is more appropriate to consider them as always being affected by it. The discursive field, in which 'raising standards for those at risk' and 'accountability' are included, pervades public discussions about NAPLAN and My School. Whilst participants are constrained by this force field, they also contribute to its perpetuation by their acceptance that children can be judged as potential workers and contributors to society as a result of their mathematical achievement on NAPLAN.

In this paper, we explore how 'disadvantaged' is used within public discourse and how it is compared and contrasted with discourses around accountability and raising standards. The use of 'disadvantaged', with its meaning of something or someone being less privileged, presents an opportunity to examine how status is added to certain types of mathematical achievement. Being disadvantaged or attending a disadvantaged school impacts on the social identity of students, their teachers and their parents. Measuring and broadcasting the mathematical achievement of students provides a focus in discussing what it means to be disadvantaged. Through an examination of data from publicly available news items about NAPLAN and My School, we show there are differences in how being disadvantaged is conceived. In some situations, it appears that children's backgrounds contribute to them being considered disadvantaged. At other times, it is the process of valorisation through the labelling of some schools as failures which results in their students being considered disadvantaged.

## METHODOLOGY

The data set consisted of five interview transcripts and four media releases from the Department of Education, Employment and Workplace Relations (DEEWR), and 19 online news articles, five of which included public comments. This data set captures the views of the Minister for Education at the time, Julia Gillard, and also how this information was received by journalists, academics, the teacher union, the association of school principals and members of the general public who contributed publicly to the discussions. Wherever possible we took news articles from national sources such as the *Australian Broadcasting Corporation* (ABC), the online journal *Inside Story* or the newspaper *The Australian*. However, at times articles from the Brisbane-based *The Courier Mail* and the Adelaide-based *The Advertiser* were used because they included online comments from the public. There was over 230 pages worth of data.

The data were collected between September 2008 and March 2010. Although the first NAPLAN tests were conducted in May 2008, we chose to start from September 2008, when parents were about to receive information about their children's achievements for the first time. Our cut-off date in March 2010 meant that most of the discussion was captured about the release of the school results for NAPLAN on

the My School website, but avoided discussion of the controversial 2010 NAPLAN tests. Although not all the public discourse was documented, the data set has sufficient breadth to reflect the variety of views of people involved in these discussions at this time.

Initially, both authors and a research assistant combed through the data, developing classifications and categories with an open mind, and finally identifying eight themes. In this paper, we report on the theme concerned with disadvantage. To this end, the data were re-examined for discussion dealing with or referring to disadvantage. We choose to focus on this theme because we felt that the complexity of the ideas about disadvantage that were discussed illustrated how a discursive field operated. In discussing disadvantage, people appeared to be aware of the process of social valorisation and as a consequence often tried to divert responsibility from themselves for disadvantaging others in the system. At times, the government's system of high stakes testing was blamed for contributing to the disadvantaging process.

As all documentation was publicly available, the names used by contributors, including those used to comment on online news articles, are provided in the quotes below. We have also left the spelling and grammar as they were in the contributions.

## **WHO IS DISADVANTAGED AND IN WHAT WAYS?**

In the data set, being disadvantaged was presented in several ways. Sometimes the term 'disadvantage' was used, whilst at other times, it was implied by suggesting that some children had less opportunity for learning or fewer life chances. The backgrounds of some groups of children meant that they were expected to do poorly in the NAPLAN tests. As a result, they or the schools that they attended were likely to be described as disadvantaged. Some children were positioned in the discourse as disadvantaging other students' learning because of their disruptive behaviour. Finally and in contrast, being disadvantaged was discussed in relationship to how the results of the tests meant that certain schools were labelled as failures, with the children who attended them gaining the same label. Thus, there was no consistency in descriptions of the cause of the disadvantage and the effect of the disadvantaging process.

The Minister of Education consistently identified students who were most likely to do poorly on the NAPLAN tests as those who came from disadvantaged backgrounds. Their enrolment at a school contributed to the school being known as disadvantaged. The label of disadvantaged schools has a long history in Australia. In the 1980s, schools became disadvantaged when they fulfilled a number of criteria such as enrolling students from low socio-economic areas. These schools received extra funding. Testing students did not contribute to this identification of disadvantaged schools and so funding was not tied to improving test results. The first quote seems to hark back to these earlier beliefs about disadvantaged schools.

## Working Group 10

Julia Gillard: The National Assessment Program will help us identify schools that aren't reaching the kind of standards that we want kids to get to. And there are other things that can tell us about disadvantage in schools—number of Indigenous children enrolled, for example; number of children with disabilities (DEEWR 10/9/08).

Julia Gillard: But it remains of great concern that the data shows that Indigenous student achievement is significantly lower than non-Indigenous students in all areas tested and all jurisdictions (DEEWR 19/12/08).

Julia Gillard: It's about lifting standards for every child in every school and making a huge difference for those kids most at risk of being left behind, who are our kids from our poorer households in this country (DEEWR 10/11/09).

For Julia Gillard the two discourses of 'raising standards' and 'accountability' seemed to be complementary. 'Raising standards' presupposed identification of schools with sub-standard results, in order to hold them accountable for those results, and also initially to provide money so they could improve those results. However, in presenting a case that NAPLAN would support all children to gain appropriate outcomes in literacy and numeracy, the Minister contributed to a deficit discourse. Indigenous students, students with disabilities and those from poor homes contributed to schools being disadvantaged. Simultaneously, it was predicted these students' backgrounds would lead to them performing poorly on NAPLAN. By having inappropriate backgrounds, students were to blame for their poor performance whilst at the same time it was up to schools to overcome these backgrounds. If they did not do so then they also were to blame. The inconsistencies and circularity in these arguments are not acknowledged, nor problematised.

There were few comments by the general public in the data set, which identified specific groups or schools as disadvantaged. One of these is the following comment on the ABC website to a story on teachers voting to boycott the NAPLAN tests.

Joker: And what do you purpose teachers do in indigenous communities in which you are lucky if the kids show up to school for 2 days a week???

Stop thinking about your own immediate area and start thinking about the whole of Australia.

I don't agree with this site [My School] at all. It gives an inaccurate reflection. How can you possibly compare the education standards of a remote community to say a private school in inner Melbourne where each student has a laptop and a remote community is lucky to have a reliable computer????? (Rodgers 28/1/10)

In this comment, the disadvantages faced by Indigenous students were not discussed. Instead, the writer seemed to suggest that it was the schools in Indigenous communities, which were disadvantaged because irregularly-attending students were unlikely to perform well on tests. At the same time, there was an awareness of how



## Working Group 10

My School contributed to value being awarded to some schools which may already have been privileged. The responsibility for this disadvantaging of the school seemed to be shared between the children and the system which made such rankings.

On the other hand, teachers and their spouses were much more likely to see disadvantaged students as those who were disrupted by their badly behaving peers. The parents of misbehaving children were often positioned as being responsible for the poor behaviour and thus for other students becoming disadvantaged.

Johnny Unimpressed of Adelaide: My wife gets constant abuse from parents for handing out detentions to kids who misbehave, abuse, distract and bully other students, or they simply write notes to the school making up stories about why their precious angel can't do the detention (Kenny 11/11/09).

Skip of Brissy: Parents need to be more accountable and make thier little darlings work. I came from a tough up bringing and the wrong side of the rails. My parents valued education and I have done reasonably well. Comparing schools makes no sense when it is the same trashy kids at each. My wife is a teacher, you wont turn Chaff into Bread, no hope. If parents had to do more, then educational standards would improve over night. The fault may well be two way, but with out proper and useful parental support then I am afraid we will continue to dumb down (Kenny 11/11/09).

For some time, it has been known that teachers commonly blamed factors within the child or the family if the child failed to learn at school (Hempenstall, 2009) and the comments by teachers in the public discussions, like those of their spouses above, were often of this ilk.

Parents often felt that some students were disadvantaged at school. However, they were most likely to mention children with disabilities or who were labelled gifted and talented. As exemplified below, they felt that these children were disadvantaged because their legitimate needs were not met within the schooling system. Although there was some teacher blame in these comments, parents were more likely to blame the system because of under-resourcing. As described by Gutiérrez and Dixon-Román (2011), these types of comments are connected to an accountability discourse around making schools or education system provide for marginalised students.

Bullfrog: Whilst there are some sociological advantages in classes of mixed ability, unless the resourcing model is vastly changed, the current set up disadvantages non-normal learning kids, both the less capable, and more capable (Woodley 17/11/09).

Bernard Wood of Modbury: Many kids I've met with ASD [autism spectrum disorders] don't meet the requirement for a special class but they can not handle mainstream and unfortunately mainstream teachers are not experts in teaching these children and they get suspended [temporarily barred from school] etc. therefore the kids suffer (Kenny 11/11/09).

## Working Group 10

Adelaide: The Victorian Affiliated Network of Gifted Support Groups estimates that 75 per cent of gifted students are underachievers and as many as 40 per cent leave school before the end of year 12. Dr McGuigan reported research showing that some 15 per cent of children of high intellectual potential drop out of school before completing year 12 (Kenny 11/11/09).

If NAPLAN or My School were mentioned in parents' comments about their disadvantaged children, then it was generally to dismiss them. Some commentators connected the government's discourse to one about accountability and unlikely to contribute to a more just society. In the few comments of this kind, the two kinds of discourse tended to be juxtaposed as contrasting, rather than complementary as had been the case in Julia Gillard's comments.

Sjames: Another example of a government harrasing poorly funded and undersupported schools, teachers, children and their communities. The Labour Govt has inherited its 'wisdoms' from the hyper-rationalism of the Liberals - its all about accounting (Woodley, 17/11/2009)

Nevertheless, NAPLAN and My School could not be ignored. Perceptions of a school as being disadvantaged were recognised as having a long-term impact on children's life chances.

Dan: I have pretty much no choice where my kids go to school given the zoning rules, so to me all this does is perpetuate and exacerbate the discrimination my very young children are already subject to. That is, because of where they live, they have to go to a fairly low performing school. Because of that, they will be considered to be low performing students whether or not they are. Because of that they may have more difficulty finding a job and because of that, they may not be able to afford to live in a wealthier suburb and send their kids to a private school either.....and so on (Woodley 17/11/09).

Matt: And what does the parents 'higher' education have to do with children learning to count and read. Whilst there may be statistical relationships there. Simply learning the times tables and reading are something that needs to be put into perspective. What you are saying is that children from lower socio economic backgrounds are almost destined to be failed by the education system. Instead of bleating about the additional information needed in these reports to make you feel comfortable, how about offering constructive view on how we fix the system so those children are not failed by the education system (Rodgers 19/1/10).

It would seem that not everyone accepted the complementary story that the politicians told about how increasing accountability would support the aims of social justice, through 'raising standards'. Instead the two discourses were seen as being in conflict, with contributors calling for a resolution of the differences. Parents' concern for their own children influenced their understanding of how NAPLAN

affected children being or becoming disadvantaged, especially when they felt that the schooling system was not supporting their children. They drew on this understanding when they evaluated the stories told by Julia Gillard and other politicians.

## CONCLUSION

The concept of disadvantage and how children became disadvantaged was a much more complicated issue for the general public than it was for the Minister. Although she did not explicitly label groups of students as being disadvantaged, by linking them to disadvantaged schools, she suggested that they were disadvantaged by association. Teachers, or their spouses, were more likely to consider parents to be the cause of disadvantage. By not teaching their children appropriate behaviours, these parents were directly responsible for other children becoming disadvantaged because they were unable to take up learning opportunities. This could be rectified by parents taking more responsibility in bringing up their children. On the other hand, parents saw schools and the education system as being responsible for the disadvantages that their children suffered in schools. This was either because the schools were unable to provide adequate support for their learning and so their children's life chances were restricted or because the publication of NAPLAN results resulted in the labelling of a whole cohort of students as being poor achievers whether this was the reality or not. Discourses around 'raising standards' and 'accountability' brought out many different conceptions of who was disadvantaged and in what ways. These had an impact on the lives of those who were considered disadvantaged and those who worked with them.

Children's possibilities for their future lives can be severely limited by the general public's acceptance that a school's NAPLAN results indicate the worth of its students as potential workers and citizens. In the same way, teachers' professional careers are discussed and dissected within the public discourse making them more or less likely to teach in particular ways, depending upon how much their social identities are marked by this discourse. The public discourse analysed here illustrated how the discursive field added value to some phenomena whilst making other phenomena invisible. In discussions about NAPLAN and My School, mathematical achievement in the tests added value to children, their schools and by implication their teachers. At the same time, ability to work as part of a team, for example, which may also be considered to be a worthy attribute for good workers and citizens, is not seen as valuable in these discussions. A discussion around raising standards in relation to team work is hard to imagine in this discourse.

Mathematical achievement and how it is measured is not a neutral activity. It is part of, contributes to, reinforces and is thinkable within the force field of social valorisation. The way that mathematics achievement is valorised provides an indication of what is 'up' and what is 'down'. Having good NAPLAN results was connected to having opportunities for a good future life. On the other hand, the

linking of some students to disadvantaged schools who had poor NAPLAN results was likely to affect how teachers, students and their parents viewed the teaching and learning of mathematics. If NAPLAN is considered to be the determiner of improved life chances then, the type of mathematics assessed in these tests will be the kind valued by teachers, children and their parents in schools which are most likely to achieve poorly on these tests. They will be fobbed off with this low quality schooling whilst other schools with 'good' NAPLAN results have more opportunities to widen their focus and so provide a richer mathematics education for their students.

On the other hand, our examination of the public discourse around NAPLAN and My School suggests that within the cacophony of the discussion, there were dissenting voices. Some contributors saw as contestable the suggestion that the two discourses around 'raising standards' and 'accountability' were complementary. Their comments showed an awareness of how the process of social valorisation following from the accountability discourse contributed to some children being labelled as failures. The determination of what is 'up' and what is 'down' were not fixed for these people. They were affected by the discourses, but were not determined by them. For the parents whose children's social identities were likely to be marked because of the publication of NAPLAN results on the My School website, the process of social valorisation became obvious. Often references to being disadvantaged were located within discussions about who was responsible. Although this often referred to children, their families and their teachers, it also enabled an identification of government ministers and their policies as those who were enforcing this social valorisation of a particularly limited kind of mathematical knowledge. Thus, while the discursive field can be considered as constructing what is 'up' and what is 'down', at the same time people's awareness of it enables dissent and an unpacking of how it developed. This can open up possibilities for changes.

## ACKNOWLEDGEMENT

We wish to thank Lisa Pearson who was our efficient and very tolerant research assistant for this project.

## REFERENCES

- Abreu, G. d. & Cline, T. (2005). Parents' representations of their children's mathematics learning in multiethnic primary schools. *British Educational Research Journal*, 31(6), 697-722.
- Alrø, H., Skovsmose, O., & Valero, P. (2009). Inter-viewing foregrounds: Students' motives for learning in a multicultural setting. In M. César & K. Kumpulainen (Eds.), *Social interactions in multicultural settings*. Sense Publishers.
- Australian Curriculum, Assessment and Reporting Authority [ACARA] (2010a). *Frequently asked questions*. Retrieved 11 Sept. 2010a, from ACARA: <http://www.naplan.edu.au/faqs/faq.html>

- Australian Curriculum, Assessment and Reporting Authority [ACARA] (2010b). *National minimum standards*. Retrieved 11 Sept. 2010b, from ACARA: <http://www.schools.nsw.edu.au/learning/7-12assessments/naplan/nms/index.html>
- Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland Publishing, Inc.
- Department of Education, Employment and Workplace Relations [DEEWR] (2008, December 19). *NAPLAN national report released* [Media release]. Retrieved 12-Sep-2010, from: [http://www.deewr.gov.au/Ministers/Gillard/Media/Releases/Pages/Article\\_090105\\_133929.aspx](http://www.deewr.gov.au/Ministers/Gillard/Media/Releases/Pages/Article_090105_133929.aspx)
- Department of Education, Employment and Workplace Relations [DEEWR] (2008, September 10). *The Hon Julia Gillard MP: NAPLAN, OECD Report, teacher quality, age pension* [Transcript]. Retrieved 12-Sep-2010, from: [http://www.deewr.gov.au/Ministers/Gillard/Media/Transcripts/Pages/Article\\_081009\\_185213.aspx](http://www.deewr.gov.au/Ministers/Gillard/Media/Transcripts/Pages/Article_081009_185213.aspx)
- Department of Education, Employment and Workplace Relations [DEEWR] (2009, November 10). *The Hon Julia Gillard MP: Interview on Lateline (ABC) regarding My School website; class size; Teach for Australia* [Transcript]. Retrieved 12-Sep-2010, from: [http://www.deewr.gov.au/Ministers/Gillard/Media/Transcripts/Pages/Article\\_091112\\_092857.aspx](http://www.deewr.gov.au/Ministers/Gillard/Media/Transcripts/Pages/Article_091112_092857.aspx)
- Donnelly, K. (2009). *Australia's education revolution: How Kevin Rudd won and lost the education wars*. Ballan, Vic.: Connor Court Publishing.
- Gutiérrez, R. & Dixon-Román, E. (2011). Beyond gap gazing: How can thinking about education comprehensively help us (re)envision mathematics education. In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping quality and equity agendas in mathematics education* (pp. 21-34). New York: Springer.
- Hempenstall, K. (2009). Research-driven reading assessment: Drilling to the core. *Australian Journal of Learning Difficulties*, 14(1), 17-52.
- Hursh, D. (2008). *High stakes testing and the decline of teaching and education*. Lanham, MD: Rowman & Littlefield Publishers.
- Kemmis, S. & Grootenboer, P. (2008). Situating praxis in practice. In S. Kemmis & T. J. Smith (Eds.), *Enabling praxis: Challenges for education* (pp. 37-64). Rotterdam: Sense Publications.
- Kenny, M. (2009). Gillard urges parents to confront schools. *The Advertiser*. Retrieved 12-9-2010, from: <http://www.adelaidenow.com.au/news/south-australia/gillard-urges-parents-to-confront-schools/story-e6frea83-1225796261111>
- Lingard, B. (2010). Policy borrowing, policy learning: testing times in Australian schooling. *Critical Studies in Education*, 51(2), 129-147.

## Working Group 10

- Quintos, B. & Civil, M. (2008). Parental engagement in a classroom community of practice: boundary practices as part of a culturally responsive pedagogy. *Adults Learning Mathematics: An international Journal*, 3(2a), 51-71. Available from: [http://www.alm-online.net/images/ALM/journals/almij-olome3\\_2\\_a\\_nov2008.pdf](http://www.alm-online.net/images/ALM/journals/almij-olome3_2_a_nov2008.pdf)
- Rodgers, E. (2010). Teachers vote to boycott school tests. *ABS News*. Retrieved 12-9-2009, from: <http://www.abc.net.au/news/stories/2010/01/19/2795904.htm>
- Rodgers, E. (2010). Website reveals remote schools struggling. *ABS News*. Retrieved 12-9-2009, from: <http://www.abc.net.au/news/stories/2010/01/28/2804040.htm>
- Skovsmose, O. (2005). Foreground and politics of learning obstacles. *For the Learning of Mathematics*, 25(1), 4-10.
- Woodley, N. (2009). Schools unite against ranking website. *ABS News*. Retrieved 12-9-2009, from: <http://www.abc.net.au/news/stories/2009/11/17/2744666.htm>

# **PARENT-CHILD INTERACTIONS ON PRIMARY SCHOOL-RELATED MATHEMATICS**

**Richard Newton and Guida de Abreu**

Oxford Brookes University, UK

*This paper reports some initial results and findings of a research project investigating parent and child interaction when completing primary school-style mathematics. It suggests that through using a sociocultural lens and a theoretical and analytical structure based on activity goals we can study how parents and children interact and co-construct learning and conceptual development in primary school-related mathematics. The paper also sketches out how a wider study into the milieu of parent-child interaction on primary school-related mathematics could reap interesting and insightful findings in the UK context.*

Key Words: parent-child interaction, school-mathematics, co-construction, goals

## **INTRODUCTION**

How a parent supports their child's learning impacts upon that child's attainment in primary school (Morrison, Rimm-Kauffman, & Pianta, 2003). A comprehensive review of contemporary literature on parental involvement, carried out for the UK government by Desforges (2003), showed that the quality and character of parent-child interaction plays a significant role in attainment in primary school. This is supported by large-scale statistical studies (Duckworth, 2008; Peters, Seeds, Goldstein & Coleman, 2007) and UK government policy (DfES, 2007). These suggest that in the UK, attainment at the end of primary school is more closely correlated to types and qualities of parental involvement than social class, income, maternal educational level, or the school attended. Whilst some research in the UK has focused on school- and home-mathematics practices (Abreu & Cline, 2005; Street, Baker & Tomlin, 2006), a limited amount has addressed the dynamics of primary school-related mathematics in the home contexts. In order to address this gap a research project was formed to specifically investigate parent-child interactions on primary school-related mathematics in the UK.

This paper presents a theoretical framework emerging from the study and an initial analysis of a single event of parent-child interaction on primary school-related mathematics. It sets out to begin to answer two research questions: (1) How do parents and children interact and co-construct learning on primary school-related mathematics? (2) How do parents support children's development of conceptual understanding of primary school mathematics? The paper tackles this by analysing an event of parent-child interaction. It concludes by proposing a wider study into some of the factors influencing parent-child interaction.

## THEORETICAL FRAMEWORK

In this project the parent-child interaction is located in sociocultural theories of learning and development, primarily within the work of Vygotsky (1978), Leont'ev (1981), Wertsch (1985) and Saxe (1991). Such a position assumes that learning takes place on the social plane before it is reproduced within the individual. This viewpoint has been used successfully to study the social interaction between parents and children and the resultant co-construction of mathematical knowledge (Anderson & Gold, 2006; Hyde, Else-Quest, Alibali, Knuth, & Romberg, 2006; Saxe, Guberman, & Gearhart, 1987).

Vygotsky's (1978) ideas on mediation, internalization and a zone of proximal development are relevant to developing a theoretical framework to study parent-child interaction. Vygotsky rejected the idea that development was driven by any single factor, and so can not be explained by any single corresponding principle (Wertsch, 1985). The idea that psychological processes develop through 'culturally mediated' activity is at the heart of Vygotskian theory (Cole, 1996). Vygotsky (1978) was primarily preoccupied by the role of language in mediation. He saw language as facilitating social connections and cultural behaviour (Vygotsky, 1997). It is social connection, interaction and transmission of culture that allows the internalization of higher psychological functions. Internalization is not just a mental function it is the formation of a mental plane (Leont'ev, 1981). This formation occurs through cooperation and social interaction (Tharp & Gallimore, 1988). The process of internalization is critical in Vygotsky's (1978) 'general law of cultural development', which states that learning takes place on the social plane before it is reproduced within the individual. In order to ascertain learning and development Vygotsky (1978) developed the concept of the 'zone of proximal development' (ZPD). This allows us to study the difference between assisted and unassisted performance, in other words processes which are advancing or maturing but have not yet been finalised or completed. Because the ZPD is a social and contextual concept it involves some form of negotiation (McLane, 1987). This negotiation takes place between the more capable 'expert', and a less capable 'novice'. Using and interpreting theories of mediation, internalization and the ZPD to study development in cultural contexts is not new, however it is a difficult proposition.

A solution to this problem can be found by utilising elements of Activity Theory (AT) which has been used to operationalise both Vygotsky (Wertsch, 1985) and sociocultural studies in mathematical understanding (Beach, 1995). AT can be traced to the work of Leont'ev (1981). It is a complex entity and difficult to apply in its entirety. Of the many elements within AT, Leont'ev argued for a focus on goal-directed activity as a mechanism for understanding culture and cognition (Nasir, 2002). The centrality of goals to AT is expounded by Nasir and Hand (2006, p.460)

"Activity theory presupposes that all activity is goal directed. These goals, or objectives, manifest differently depending on the level of analysis; taking the activity as the



fundamental unit of analysis, these objectives appear as motives. Moving to an individual or group level, motives become directly aligned with conscious goals. Although often explicit, these goals generally emerge over the course of activity”

Saxe et al. (1987) studied the relationship between numerical goals and social and cultural processes. Their basic assumption, which is adopted by this paper, is that

“...children’s numerical understandings are their goal-directed adaptations to their numerical environments, therefore, the study of number development should entail coordinated investigations of children’s emerging abilities to generate numerical goals and the shifting sociocultural organization of their numerical environments” (Saxe et al., 1987, p. 4)

This supports the idea that negotiation, interaction and goal-construction plays an important role in emergent and situated cognition. Saxe (1991) shows cognitive developments are enacted through efforts to accomplish numerical goals. He developed a framework for studying the components of these emergent goals at the microgenetic scale. Goals are emergent in the sense that they alter and shift in response to: (1) activity structures, the goals that are formed in the practice; (2) social interactions, where goals are modified and through negotiation take form; (3) artefacts/conventions; and (4) prior understandings. This is termed the four parameter model. This approach has been used in a number of research studies (Guberman & Saxe, 2000; Nasir, 2000, 2002; Saxe, 2002; Saxe and Guberman, 1998). If we accept, as Saxe does, that goals are a reflection of situated cognition, then by studying the goals of parents and children we can study co-construction of knowledge and conceptual understanding in mathematics.

## METHODOLOGY

In this paper an instance of parent-child interaction is analysed using the earlier theoretical framework. The participants were a 40 year-old British female and her 10 year-old son. The dyad completed a 30-minute mathematics task which involved a number of subtractive calculations and word problems. This topic was chosen as a particular focus since professional experience, and academic research (Barnby, Bilsborough, Harries, & Higgins, 2009), suggests that children can struggle with different elements of subtractive understandings. Teaching of subtraction has evolved greatly over the past 10-15 years, which means parents may well have different experiences and mathematical representations to their children. The word problems tackled different elements in subtraction and presented different subtractive structures in order to elicit a range of conceptualisations. The task was similarly designed to allow elements of ‘expert-novice’ communication and co-construction of mathematical knowledge. Research on word problems informed the production of the task (Fuson, 1992) as did research on calculation (Anghileri, 2006). The task was designed to replicate the schoolwork parents and children regularly complete together. Whilst this is not a study of actual homework practices,

it does look at how parents and children negotiate and co-construct mathematical understanding, and begins to highlight how this interaction is shaped by social and cultural forces. The dyad was video recorded as they completed the task. This video was then transcribed and analysed qualitatively using NVivo 8.

## ANALYSIS

The video recording presented a highly complex and rich corpus of data that could have been investigated from a range of directions. This analysis concentrates on the co-construction of mathematical learning evidenced by language use and behaviour. It approaches this from three tiers of complexity. These progressively narrow the focus on the analysis, but in doing so lose elements of their wider interconnectedness. This approach was both emergent, in the sense that it was informed by the data, and theoretical, in the sense that it was informed by relevant research literature.

### **First tier: Descriptive analysis of mathematical operations and thinking**

This first tier of analysis looks at the interaction globally to begin to address the research questions of this project on parent-children co-construction of learning and understanding. In this case it interprets the utterances of the dyad in accordance with theories of goal-directed activity and mathematical principles and understandings. For example, in the following passage the dyad is trying to find the difference between 86 and 64, M refers to the mother and C to the child.

- C: Okay, so, count on from 64 to 86 because you add 6 it gets to 70 another 10 so that's 16.
- M: Sorry?
- C: 16 I think.
- M: You think 16.
- C: What do you think?
- M: 64, she's got 64 but she had 86...
- C: Yeah, Yeah.
- M: ...so I would kind of...I'd look at my 64 and I probably turn it...I...I'd add it up rather than try to take that figure away.
- C: I know that's what I just did.
- M: So that's what you've done. So if you have your 64 how many do you need on...4 to make 6?
- C: 2 [M writes down on sheet]
- M: How many from 6 to make 8?
- C: 2 [M writes down 22 on sheet]

The child has a number of different strategies available given the operation and the numbers involved. He decides to use a complementary addition and add, in steps, from 64 to 70 and 70 to 80. However he does not use a third step and add from 80 to 86. This means he reaches an answer of 16 rather than 22. He then follows a procedural objective of seeking M's confirmation of his correctness. His mother confirms the appropriateness of his strategy and that she would similarly use complementary addition. However, whilst he appears to use a mental number line to count in steps between 64 and 86, she seems to use a mental imaging of a column subtraction. This entails counting the difference between 4 and 6 (64 and 86) then writing 2, and 6 and 8 (64 and 86) writing another 2 to make the number 22. This shows that the two have a different understanding of what it means to 'add up' to 'find' a difference. This could be linked to contrasting school experiences.

### **Second tier: Evidence of practice-linked goals through the analysis of emergent goal construction**

This second tier looks deeper to try to highlight the parameters linked to the 'emergent' goals (Saxe, 1991) formed in this cultural practice. It uses Saxe's four parameter model to study and explain the practice-linked goals constructed by the dyad. In this case instances of each parameter in the transcript were coded using a simple framework and linked to potential explanations.

The **prior understandings** that are brought to a cultural practice both enable and constrict emergent goals (Saxe, 1991). So children and parents could be expected to construct different goals since they are utilising different mathematical experiences and representations. This assertion is supported by data from the parent-child task. M had a very different primary mathematics experience to her son. This is displayed in the strategies she uses in the task and the barriers she appears to face regarding a familiarity and understanding of the mathematical methods that her son uses. Of the four parameters prior understanding is perhaps the most difficult to determine through the study of interaction alone, initiatives to address this shortcoming are discussed later.

Cultural practices, in this case the **activity structure** of the task, are defined by the motives required to complete them. The goals of one practice may be different from the goals of another. Within the interaction it is possible to see a great deal of evidence of the role that the activity setting has on practice-linked goal formation and the objectives the mother and child pursue. This is shown below in the following subtraction calculation activity. Here the dyad answered a question by following the practices ingrained within the school mathematics-related activity: reading the question, answering the question, and explaining reasoning.

M: Alright... right, let's have a look. Have we read the question?

C: Yeah. Can you solve these subtraction calculations, show your workings. 40 minus 21 equals... 20 away from 40 is 20, and take away 1 is 19.

## Working Group 10

- M: Well ok. Start by writing that out then, so how did you get to that? So how did you first do it? [C writes an explanation, M checks]
- C: That's super... right... just pop your answer there, so you got 19.

The task gave a great deal of information about the way in which **social interaction** impacts upon goal construction. There were several cases when one participant suggested a procedure which altered the goal of the other. This usually led to a phase of negotiation around the appropriateness of the procedure. These examples showed mother and child playing out of Vygotskian roles of 'expert' and 'novice' in setting and amending practice-linked goals. The task also provided numerous examples of M explaining or modelling strategies and concepts to scaffold onto C's mathematical understandings.

The dyads' practice-linked goals constructed within the activity are also influenced by the **artefacts and conventions** enmeshed within this cultural practice. Calculations and word problems, similar to those used in the classroom, triggered a certain style of response and practice-linked goal structure (as evidenced in the above example). There was evidence that mathematical artefacts, such as algorithms for subtraction, influenced goal construction in the dyad.

### **Third Tier: Evidence of the negotiation of mathematical goals**

This final tier delves deeper into the interaction to observe how mathematical goals are negotiated, formed and operated. Within the task the dyad appeared to operate through negotiation. There was little conflict or disagreement. There was however several instances of M prompting different mathematical goals and of C needing to reason and justify his choices. A coding framework, informed by the background literature and instances within the transcript, was used to study these negotiation processes within the social interaction.

**Table 1 Codes used to study the negotiation of mathematical-linked goals**

<i>Code</i>	<i>Description</i>	<i>Code</i>	<i>Description</i>
C1	Agreement with a statement	C11	Abandoning a previous answer or approach
C2	Disagreement with a statement	C12	Appearing to mathematically reason
C3	Probing understand/action	C13	Setting a new mathematical goal
C4	Prompting understanding/action	C14	Accepting the mathematical goal of the other party
C5	Confusion	C15	Rejecting the mathematical goal of the other party
C6	Checking the reasoning of the other party	C16	Abandoning their mathematical goal
C7	Suggesting an answer to a mathematical operation	C17	Suggesting a different mathematical goal
C8	Providing an explanation or model		
C9	Responding to a question or prompt		
C10	Asking the other party whether an argument is right or wrong		

## Working Group 10

These codes break down the interaction into smaller components in order to view the building blocks of the co-construction of mathematical goals. From this we saw that the parent M tended to allow her child to formulate and operate his own mathematical goals, but she would intervene if she thought his reasoning was flawed or a more efficient method existed. In the following episode, which includes my reflections, we see M and C negotiating how to solve a two-step word problem: *Josh had 307 stamps. He gave 118 stamps to Katie. He lost another 43 stamps. How many stamps does Josh have now?* The problem can be solved in two different forms:  $307 - 118 = 189$ ,  $189 - 43 = 146$ ; or  $118 + 43 = 161$ ,  $307 - 161 = 146$ . M favoured following a goal leading to the first, whilst C preferred the latter goal. Through probing and prompting, giving and answering questions and apparent reasoning we can see how one goal was accepted and another rejected.

Dialogue	Interpretation	Codes
C: How many stamps does Josh have now. So it's 307 take away 118.	<i>C recognises that the answer can be found by <math>307-118=x</math> then <math>x-43=y</math>.</i>	C12
M: Yeah C: So basic...so then he lost 43 so... 118 add 43.	<i>C sees that he could add the two subtractive elements (118 and 43) then subtract the answer from 307.</i>	C1 C11 C12
M: I'll tell you what...what we'll do...yeah, you can add your...you could start with the sum and take the 118 from the 307 couldn't you	<i>M agrees with C but proposes <math>307-118=x</math> then <math>x-43=y</math> suggesting that C could break it down or (partition) prior to subtraction.</i>	C15 C4
C: Yeah. M: And you could say...could say right well we'll break that down we'll take the hundred off the 300 first and then we'll take the 18 off... C: That'd be 161 mum wouldn't it? [C points to sum on the paper?]	<i>C suggests the answer to <math>118+43</math> and seeks confirmation.</i>	C1 C12 C15 C7 C10
M: ...and then we could add the 7 back, yeah. Or you can do it...yeah...you can do it this way, do you find it...	<i>M recognises that compensation and partitioning does not work well with these numbers. This leads her to think through C's method.</i>	C12
C: That's 161 [C points to calculation on paper] M: If he's lost 43 and he's given this amount away as well, add those together, and then take that figure off the 307.	<i>M discusses C's method and recognises that it would work.</i>	C7 C12 C16 C14 C11
C: So add those two its 161...	<i>C seeks confirmation of his answer.</i>	C7
M: Well we'll add it, we'll work it out here, write it down...write it down. Put your 118 and put your 43 underneath, do it as a sum write it as a sum like you would do at school. Yeah.	<i>M accepts C's goal and rejects her own.</i>	C1 C4 C13

## Conclusion and ideas further research

In this paper we can see different mechanisms inherent within parent-child interaction, even though these can be difficult to untangle and classify. By operating from a sociocultural viewpoint in terms of study design and analysis we can attempt to answer our two research questions regarding interaction, co-construction and the development of conceptual understanding. This paper, nonetheless, only presents a single story limited to two characters. A much wider study of more parent-child dyads is needed to see if the ideas and findings from this one case are comparable to others and whether any similarities or differences exist. In addition, a focus on the interaction alone is not enough to explain the interaction. Whilst we understand a great deal about how children are taught mathematics we have little indication of parental experiences or mathematical identities. Nasir (2002) has shown how prior experience, motive and identity are important in goal construction. Her model allows the paralleling of the microgenetic study of goal-directed activity with an ontogenetic study of identity and motive. This can be incorporated by episodic interviewing (Flick, 2000) of parents, allowing a greater awareness of the milieu of the parent-child interaction and richer answers to our main research questions. Since research shows that parent-child co-construction of school-related mathematical knowledge is also influenced by factors such as the level and quality of communication between home and school (Hughes & Pollard, 2006) this should also be taken into account. This too can be accomplished through interviewing parents.

These points present a model for the next stages of this inquiry and a way forward to further study some of the elements of parental involvement which have been shown to play such a key role in attainment.

## NOTES

1. This research was sponsored by the Doctoral Training Programme on Children and Young People, Oxford Brookes University, UK.
2. Professor Guida de Abreu is a member of the EMiCS group – Educació Matemàtica i Context Sociocultural (Mathematics Education and its Sociocultural Context) – granted by the Direcció General de Recerca of the Generalitat de Catalunya (the research office of the Catalan autonomous government) (Grant: 2009SGR-00590) whose aim is to develop and explore the explanatory potential of theories that enable a better understanding of the experiences of immigrant schoolchildren learning mathematics.

## REFERENCES

- Abreu, G. d., & Cline, T. (2005). Parent's representations of their children's mathematics learning in multi-ethnic primary schools. *British Educational Research Journal*, 31(6), 697-722.
- Anderson, D., D., & Gold, E. (2006). Home to school: numeracy practices and mathematical identities. *Mathematical Thinking and Learning*, 8(3), 261-286.

- Anghileri, J. (2006) *Teaching number sense* (2<sup>nd</sup> ed.) London: Continuum.
- Barmby, P., Bilsborough, L., Harries, T., & Higgins, S. (2009). *Primary mathematics: teaching for understanding*. Maidenhead: Open University Press.
- Beach, K. (1995). Activity as a mediator of sociocultural change and individual development: The case of school-work transition in Nepal. *Mind, Culture and Activity*, 2(4), 285-302.
- Cole, M. (1996). *Cultural psychology: a once and future discipline*. Cambridge: Belknap Press.
- Department for Education and Skills. (2007). *Every parent matters*. Nottingham: DfES Publications.
- Desforges, C., & Abouchar, A. (2003). *The impact of parental involvement, parental support and family education on pupil achievements and adjustment: a literature review (Vol. Research Report RR433)*. Nottingham: DfES Publications.
- Duckworth, K. (2008). *Influences on attainment in primary school: interactions between children, family and school context*. London: Institute of Education, University of London.
- Flick, U. (2000). Episodic interviewing. In M. Bauer & G. Gaskell (Eds.), *Qualitative researching with text, image and sound: a practical handbook*. London: Sage.
- Fuson, K. C. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.) *Handbook of research on mathematics teaching and learning: a project of national council of teachers of mathematics*. New York: Macmillan 243-275
- Guberman, S. R., & Saxe, G. B. (2000). Mathematical problems and goals in children's play of an educational game. *Mind, Culture and Activity*, 7(3), 201-219.
- Hughes, M., & Pollard, A. (2006). Home-school knowledge exchange in context. *Educational Review*, 58(4), 385 - 395.
- Hyde, J. S., Else-Quest, N. M., Alibali, M. W., Knuth, E., & Romberg, T. (2006). Mathematics in the home: homework practices and mother-child interactions doing mathematics. *Journal of Mathematical Behavior*, 25, 136 - 152.
- Leont'ev, A. N. (1981). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology*. New York: M. E. Sharpe.
- McLane, J. B. (1987). Interaction, context, and the zone of proximal development *Social and Functional Approaches to Language and Thought*, 267 - 285.
- Morrison, E. F., Rimm-Kauffman, S., & Pianta, R. C. (2003). A longitudinal study of mother-child interactions at school entry and social and academic outcomes in middle school. *Journal of School Psychology*, 41, 185 - 200.

- Nasir, N. S. (2000). "Points ain't everything": emergent goals and average and percent understandings in the play of basketball amongst African American students. *Anthropology and Education Quarterly*, 31(3), 283-305.
- Nasir, N. S. (2002). Identity, goals, and learning: mathematics in cultural practice. *Mathematical Thinking & Learning*, 4((2&3)), 213-247.
- Nasir, N. S., & Hand, V. M. (2006). Exploring sociocultural perspectives on race, culture, and learning. *Review of Educational Research*, 76(4), 449 - 475.
- Peters, M., Seeds, K., Goldstein, A., & Coleman, N. (2007). *Parental involvement in children's education 2007 (Vol. DCSF-RR034)*. London: Department for Children Schools and Families.
- Saxe, G. B. (1991). *Culture and cognitive development: studies in mathematical understanding*. Hillsdale: Lawrence Erlbaum.
- Saxe, G. B., & Guberman, S. R. (1998). Studying mathematics learning in collective activity. *Learning and Instruction*, 8(6), 489-501.
- Saxe, G. B., Guberman, S. R., & Gearhart, M. (1987). Social processes in early number development. *Monographs of the Society for Research in Child Development*, 52(2, Serial No. 216).
- Street, B., Baker, D., & Tomlin, A. (2008). *Navigating numeracies: home/school numeracy practices*. Dordrecht: Springer.
- Tharp, R. G., & Gallimore, R. (1988). *Rousing minds to life: teaching, learning, and schooling in social contexts*. Cambridge: Cambridge University Press.
- Vygotsky, L. (1978). *Mind in society: the development of higher psychological processes*. Cambridge: Harvard University Press.
- Vygotsky, L. (1997). *The collected works of L. S. Vygotsky: Volume 4. The history of the development of mental functions* (M. J. Hall, Trans.; R. W. Reiber, Ed.) New York: Plenum Press (Original work written in 1931)
- Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. London: Harvard University Press.



# **SOCIO-CULTURAL ROOTS OF THE ATTRIBUTION PROCESS IN FAMILY MATHEMATICS EDUCATION**

**Javier Díez-Palomar and Sandra Torras Ortin**

*EMiCS – Department of Mathematics and Science Education – Universitat Autònoma de Barcelona*

*This paper explores the process of attribution drawing on a socio-cultural approach. Scientific literature has widely demonstrated that attribution may have a crucial impact on individuals' beliefs to learn mathematics. The process of attribution is always present within individuals' interactions. In addition, previous researches also suggest that many attributes assumed by learners may be grounded on social representations. We build on a European research study about family mathematics education. In this paper we want to give some examples to describe the kind of beliefs family members have that may impact on their own attitudes towards mathematics, as well as their children's ones. We explore how these beliefs may have a social origin.*

**Key words:** beliefs, attitudes, social representations, mathematics, family involvement.

Mathematics is a place either for enthusiastic attitudes or negative feelings. Individuals may look at mathematics very differently according to their experience with this subject. When children ask to their parents (or other relatives) for help to do their homework of mathematics, this “previous experience” emerges and mediates parents' disposition to help them. According to previous researches, the type of memories underpinning family members' background may explain their attitudes towards mathematics. Positive reactions to this subject (the “aha!” experience of successful problem solvers) clearly have a different impact than negative experiences drawn on frustration of getting stuck on mathematical activities. However, we think that sometimes these reactions towards mathematics are not just individual responses to people' ability to solve problems and other types of activities; there is also a social (and cultural) component embedded. Individuals develop their particular identity as mathematics doers because their own ability to use them. But they also develop this identity as a consequence of the role acquired within the group. Interactions with other persons should promote either positive or negative attitudes towards mathematics depending on whether they are supportive or discouraging peers. The final result of these interactions (in terms of identity) may have a crucial influence on individuals' achievement. In fact, there is a prominent body of research suggesting that self-confidence as mathematics doers have a heavy impact on students' performances in mathematics. With this paper we aim to introduce a discussion with the scientific community about the usefulness of the attribution theory to provide some light to understand how attitudes and emotions that impact on individuals' feelings towards mathematics may be socially rooted.

## THEORETICAL FRAMEWORK

Since 1970s many researchers in mathematics education had studied the impact of beliefs, attitudes and emotions on learners' performance (Bachman, 1970; Evans, 2000; Fennema & Sherman, 1976; Gómez Chacón, 2000; Hannula, 2002; House, 1975; Lester, Garofalo, & Kroll, 1989; McLeod, 1994; Zan, Brown, Evans, & Hannula, 2006). There are many researches on this field that place assumptions such as "in order to learn mathematics you need to do a number of routine mathematical tasks", "mathematics is memorization", or "mathematics is difficult" under the label "beliefs". Researchers have developed many scales to study how these assumptions impact on students' mathematics learning<sup>1</sup>. When authors talk about the idea of "confidence about learning mathematics", there is a disagreement on whether using the term "beliefs" or "attitudes" to refer to this idea.<sup>2</sup> Although this divergence on the use of the terms, all studies proved a relation between the affective and the cognitive domains. According to Hart (1989), interactions of students and teachers in mathematics classrooms are strongly influenced by confidence and other beliefs. Bachman (1970) noticed that self-concept is a salient aspect emerging from the study of attitudes and beliefs that influences on students' performance in mathematics. This notion (self-concept) was used later on to explain adult learners' anxiety towards mathematics (Evans, 2000). This is also one of the more salient reasons underpinning the difficulties that family members should face when helping their kids with mathematics (Díez-Palomar, J., Menéndez, J.M., Civil, M., forthcoming). Anxiety explains some parents' (and other family members) reluctance to do mathematics themselves, hence they use to prefer other strategies to get involved in their children's learning, such as send them to after-school programs, to academies, or hiring personal teachers to support their children at home (Hoover-Dempsey & Sander, 1995).

According to prior researches, there is a relation between mathematics anxiety and learning. Clute (1984) found that anxious students do less well in discovery lessons than with expository teaching. Zan and his colleagues (2006) stated that anxiety inhibits cognitive processes. They affirmed that learners experiencing stressful situations within the classroom of mathematics perform worse than other students. Adult learners use to experience this feeling (Evans, 2000). According to previous researches (Díez-Palomar, 2004), anxiety and other negative attitudes may be rooted on personal past experiences. Here there is place for socio-cultural explanations. In fact, drawing on the literature review, we can find (at least) two main bodies of research: (a) the one based on individual experiences, and (b) the one grounded on social (and cultural) characteristics.

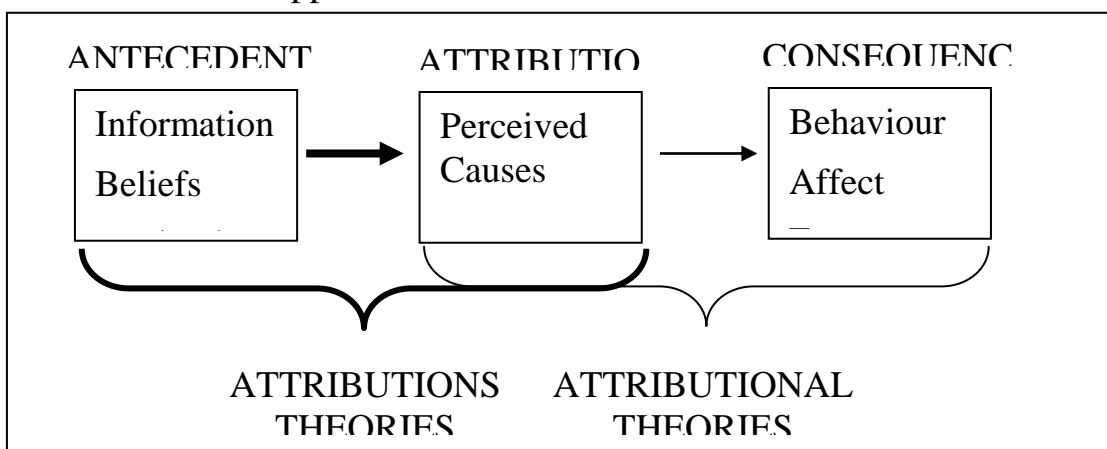
Op't Eynde and his colleagues (2006) affirm that the affect domain is primarily rooted in the social context. This is not new for the sociologists. Mead (1934), for example, developed a theory on how persons build their (individual and social) identity. Mead (1934) distinguished between I, self, and me, and progressive forms

of the individuals' identity. He found that self and me are built versus the "others". A particular individual defines his/her identity according others' image of him/her. This person may be the "coolest pal", "the best student", "the faster problem solver", as far as his/her peers (family, relatives, siblings, friends, teachers, and other persons from his/her environment) use these terms to refer him/her.<sup>3</sup> Op't Eynde et al. (2006) claim that individuals' understanding of mathematics is "a function" of the interplay between the person they are (their identity) and the specific classroom context. By "specific classroom context" they mean the type of interactions that individuals establish with the teacher and their mates. Previous researchers have already demonstrated that interactions grounded on high expectations and trust towards learners prompt them to high-scores more likely than low-expectations approaches / attitudes (Flecha, Garcia, Gomez, & Latorre, 2009).

That means emotions are social in nature. Op't Eynde et al. (2006) also argue that emotions are socio-historical context rooted. They mean that children' emotions towards mathematics nowadays are slightly (or even heavily) different from the ones experienced by their parents years ago.

Drawing on these arguments presented up to this point, we may think that family members' attitudes and beliefs towards mathematics somehow are rooted on social (and cultural) features. Similarly, those attitudes and beliefs mediate or intervene on individuals' behavior towards mathematics. When a parent escapes his/her kid's request for support, it may happen because the parent does not feel comfortable with the kind of mathematics embedded in his/her children's notebook. This happens for a number of reasons (lack of confidence, forgetfulness due to the time going by, fair, hard past experiences regarding mathematics, etc.).

In this paper we focus our discussion around the connection of attitudes and beliefs, with attributions and social representations to deeply study the reasons parents (and other family members) may have to support their children (or to avoid them). We use the socio-cultural approach to frame our discussion.



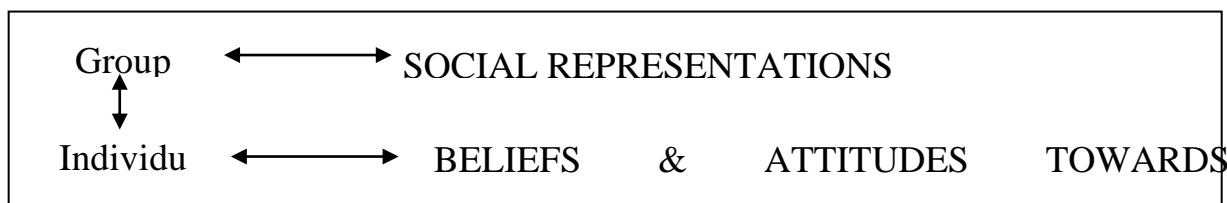
**Figure 1. General model of the attribution field (Kelley & Michela, 1980, p. 459)**

We also take a few elements from the attribution theory to provide some light to our analysis. According to Kelley and Michela (1980), attribution explains individuals' behavior in terms of its causes. They use a schema to explain how attribution works (see figure 1).

Drawing on Kelley and Michela's (1980) schema, we suggest that a plausible explanation to understand why some family members feel confident dealing with mathematics whereas others see themselves as not good at math at all may have (at least) two different elements: (a) the content (the intrinsic difficulty of mathematics), and (b) individuals' attitudes/beliefs towards mathematics.

Regarding the second element, we already know that other's opinions have a strong impact of how a particular individual perceive him/herself, which also affects his/her attitude and behavior (Kelley & Michela, 1980; Asch, 1946). This is not new. We also know that beliefs have a crucial impact on individuals' self-concept. What we suggest is that those beliefs (or attitudes according to some authors) are socially (and culturally) rooted.

De Abreu & Gorgorió (2006) suggest that behavior and actions may be explained due to social representations. They claim that social representations are interpretative frameworks to explain social and cultural phenomena. These representations somehow are embedded on individuals' beliefs, which mediate their behavior and actions (see figure 2). Drawing on these arguments, we suggest that family members' beliefs towards mathematics are constructed based on social representations. In addition, we also suggest that the personal experiences of individuals are both rooted socio-culturally and socio-historically.



**Figure 2. Relationship between beliefs & attitudes and social representations towards mathematics.**

## METHODOLOGY

The data used in this paper comes from a study funded by the European Commission (Grundtvig Program).<sup>4</sup> This research project aims to build a European network of good practices and resources to promote family mathematics education in schools over Europe. We would provide adults with a pathway to improve their knowledge and competences in Mathematics, while promoting also the development of innovative practices in adult education to transfer them to schools all over Europe. We collected data using interviews, questionnaires and focus groups in the five countries involved in this research study. These instruments have been applied to

students, teachers and families. In this paper, we build on Spanish' data collection. We collected 355 questionnaires (n=253 students, n=96 families and n=6 teachers); and conducted 27 interviews (n=10 students, n=13 families and n=5 teachers); and 4 focus groups (n=1 students, 2=families and n=1 teachers). To analyze the data we used discourse analysis techniques (Gee, 1999) and critical communicative methodology (Gómez González & Díez Palomar, 2009; Gómez, 2006).

## DISCUSSION

We organize our discussion along three cases illustrated by three different persons involved in our study. Drawing on their interviews, we discussed some salient topics to provide some light to the relationship between attitudes, beliefs, social representations and mathematics.

### Case 1. Yazmín

Yazmín is a nurse assistant.<sup>5</sup> She does administrative work in the hospital. She is married and she has two daughters, 12 and 7 years old. She is very demanding of her children education. She does not participate in the school family association, although she usually chats with other parents at the school front door almost every afternoon. When we asked her about her daughters, she explained that the older one is "useless for mathematics". She described her as: "useless, useless. This [Mathematics] is the most important difficulty that she has faced ever". Yazmín explained how her 12-years-old daughter struggles with problem solving. "When you explain to her an equation and... Of course, is the issue of reasoning, to say... well, I'm going to solve this problem... And you see her like... [making a gesture and a sound indicating embarrassment]. That's it! She had no idea what to do, she had not." Her daughter's attitude towards mathematics was really negative. According to Yazmín, her daughter has a low self-esteem that discourages her to afford any kind of mathematical activity: "She has a very low self-esteem in the sense that she says why I am going to do that if I'm going to fail it? Then she refused to do it: she said that she was not aimed to do it [the exercise] because it will not be worth it at all." Even teachers had a negative concept of Yazmín older daughter. When the girl was in fourth grade (elementary education), the teacher meet Yazmín to tell her that her daughter somehow had phobia against Mathematics. She always was reluctant to solve the mathematical activities.

We notice that there is a strong belief that Yazmín's daughter lacks the ability to solve mathematical problems underpinning in these comments. Drawing on our data we know that Yazmín's daughter is not confident with mathematics at all. This attitude is related to the content, but some details also suggest that there is a negative process of attribution that could be also part of the girl feelings towards Mathematics. Yazmín declares that she feels bad regarding how she has managed her older daughter education. She affirms that maybe there still are some room for doing things better: "Because you think, OK, maybe because I was working [and I had no

time] I felt stressed, so I did not do my best with her". But, although this feeling of fault, she has strong beliefs regarding learning. She declares, "A wrong understanding of a problem always produces more troubles, of course". She affirms that this [belief] was her biggest problem when helping her daughter with Mathematics. We may conclude that achievement somehow is related to the children context. It seems reasonable to suggest that beliefs may impact on attitudes towards mathematics, which also have some kind of influence on students' performance.

## **Case 2. Carlos.**

Carlos is a risk analyst graduate. He became unemployed and started to work at her wife's business, a printing office. Now he works as a sales representative while helping her wife with the printing office. He (with his wife) has two children, a 14 years old daughter, and a 12 years old son. He is really involved in the school family association: he is the person in charge to organize the after school program. He is also an active member in his neighborhood. He participates in a range of different associations and people define him as a pro-active person who always is engaged in social life activities.

When he talks about his children's education [in Mathematics] he is very critic. He does his best helping them with their homework. However, he has strong social representations about Mathematics. "I'm telling my children, one plus one equals two; this is Mathematics". He has his own way to solve the mathematical activities (drawing on his memories and experience), thus he use to feel upset with teachers' methods or strategies to solve them. "You see it either easy or difficult, because they use more steps than needed... [to solve the exercise]. He also has his own beliefs about what does it mean to learn mathematics: "you get them [Mathematics], or you don't"; "if you get them, then you need to fully understand it".

Carlos attitudes towards his children when helping them with Mathematics are strongly mediated by his own beliefs on the subject. He also explains how he argues with his children's teacher about how to teach [Mathematics]. They are old mates, thus Carlos feels confidence enough to talk openly with her (the teacher) about didactics. He is very reluctant against "new" ways to teach Mathematics (more comprehensive, less mechanical), since according to him, mathematics involves "lots of practice [routine work]". He is also very critic with his children (with adolescents in general). He claims that, "children now are in that moment like they are in a "colorful World", they do not read well... they have their mind focused in other things, so it is difficult to focus them".

Drawing on this data, we may suggest that parents' involvement is really mediated by parents' attitudes and beliefs. Carlos is a pro-active man. He is really devoted to support his children. However, his radical beliefs about Mathematics lead him to confront teachers' methods, which may be a starting point for a conflict with his children (or his children with the teacher).

**Case 3. Jordi.**

Jordi is a Pharmacy graduate. He works in a Pharmaceuticals multinational company. He is married and he has two children: a 19 years old daughter and a 14 years old boy. His wife is a Nursing graduate. Both of them are members of the school family association. Both belong to a “middle class” group, and they come from well-off families. They bring their children to a religious school (catholic). They are also involved in activities for families (family involvement, teaching for families, etc.). Jordi is a person with great amounts of curiosity and he is very supportive.

He has strong beliefs around Mathematics and its teaching and learning. According to him, “I think of Mathematics that if you already have understood it when you are solving the problems, there is no need to go back... you are already in a good position. Then, you need to practice a lot with the activities, etc.”. Jordi really believe that in order to be proficient in Mathematics, there is a need of intense practice. He assumes that children need to solve series of routine activities to perform well in Mathematics assessments. This assumption suggests that somehow Jordi shares the social representation of Mathematics as a set of routine practices / methods. His belief (based in that social representation) mediates his relation with their children at home, since he asks them to practice seriously Mathematics by doing and doing exercises and using algorithms methodically.

According to him, helping children at home in some way is natural for parents, “Then, the fact that parents want to spend their time helping their children is something innate, isn’t it?” However, he believes that “not everybody can do it [help children with Mathematics] equally”. Jordi has a strong feeling on discrepancies (even inequalities) between individuals, thus some are better than others to teach Mathematics.

“...We also have extra abilities that are based on genetic issues, what “nature” has given to us, so look, you may take advantage of it or you may not, according to your [level] of understanding; but you already have it. With parents is the same... to have in mind this extra support, as a regular part of their [children] learning, maybe is discriminatory for children, because not all of them [the parents] have the same level. I’m very close to some parents of my son’s peers and, I don’t know, I’m able to imagine that for some things that I explained to my son some of them may be better than I [in explaining these concepts], but others, poor people, they will find lots of troubles to explain it or even they will not be able to do it because of their own lack of understanding themselves, or because they never studied that”.

Jordi believes profoundly that there are differences between individuals’ capacities to deal with Mathematics. According to him, parent involvement is an action that may produce streaming consequences between children. Base on that, Jordi claims that he prefers teachers to be the ones responsible for children’s learning. He thinks that teachers may have a deep understanding of the subject (Mathematics), but they also need to be able to teach it in an interesting way for children. This is another

“social representation” shared by parents and other community members, which emerges from Jordi’s arguments.

## **FINAL REMARKS AND FURTHER RESEARCH**

Drawing on our data, we may suggest that some of the family members’ beliefs towards Mathematics are grounded on social representations (specially the ones shared with other parents). These beliefs crucially impact children’s achievements in Mathematics. Family members may project over their children their own attitudes (phobia, frustration, anxiety, lack of self-confidence) towards Mathematics (based on their past experiences). These practices could label children (as in the case of Yazmín’s daughter) and may be an exclusory fact that makes it harder for children to perform well in Mathematics. Base on our data, social context emerges as a crucial element explaining children’s achievements in Mathematics. The kind of interactions that children maintain within their everyday life could help to understand why some children always perform well in Mathematics, whereas others use to fail again and again. As previous research suggested, attitudes towards Mathematics may not be just individually based: there is room for the impact of social context. Attitudes also play a crucial role in self-concept and self-esteem, which are central aspects of the self-confidence in getting good scores in mathematical assessments. In this paper we slightly have started to analyze this kind of relationship. More research is need in order to provide more evidences and a full understanding of how these variables are related within the learning processes.

We also need more research to analyze how are “attitudes”, “beliefs”, “attribution”, and “social representations” connected to each other in the frame of family involvement practices. What type of interactions we may observe if we had the opportunity to share time with parents and children at home? An ethnographical line of research may provide more light to this question.

Finally, drawing on our data we see how some parents have beliefs about education and learning processes, which are not coherent with the findings demonstrated by the scientific community. This is the example of Jordi, who thinks that parent involvement may produce streaming, when we already know from a large body of research that it is the opposite. This is also the example of Carlos and Jordi, when they affirm that Mathematics education should be based on routine rather than other type of methods. Why do these attitudes take place, and how? What kinds of consequences have for children, and how can we manage to transfer the research findings to society? These questions also open another line for further research.

## **NOTES**

1. The Fennema-Sherman scale (1976), the Mathematics Anxiety Rating Scale (Richardson & Suinn, 1972), the Enjoyment of Mathematics and the Value of Mathematics scale (Aiken, 1974), the Mathematics Attitude Inventory (Sandman, 1980), etc.



## Working Group 10

2. While authors such as Fennema and Sherman used the term “attitude” to refer a set of feelings that individuals may have towards mathematics, such as “I like mathematics”, or “mathematics is a boring issue”, other researchers prefer the use of the term “beliefs”, such McLeod (1994). Fennema and Sherman (1976) developed the Mathematics Attitudes Scale including different sub-scales (values, beliefs, confidence in learning mathematics, math anxiety and disposition towards active problems solving). McLeod (1994) proposed a range between beliefs and emotions, with attitudes in between, assuming that “emotions” are more intensive and less stable than “beliefs”. “Attitudes” are in between these two extremes.
3. In 1946 Asch published a study about how individuals create an “image” of somebody (Asch, 1946). He chose two sets of university students. He gave them a list of personal features from an unknown person. Both list included the same adjectives (smart, handy, decisive, practical, and prudent), but one. Asch added “warm” to the first list, and “cold” to the second one. Students that received the first list described the unknown person as somebody “generous”, “kind” and “happy”. The second set of students came with adjectives such as “miserly”, “unhappy” and “unpopular”. This study suggested that identity is a result of a social construction.
4. FAMA: Family Math for Adult Learners. Number of reference: 504135-LLP-1-2009-1-ES-GRUNDTIG-GMP. This project has been funded with support from the European Commission. This publication paper reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.
5. In Spain to become a nurse assistant is not required to go to the University to get the certificate. There is a vocational training course to obtain this professional accreditation.

## REFERENCES

- Asch, S. E. (1946). Forming impressions of personality. *Journal of Abnormal and Social Psychology*, 41, 258-290.
- Bachman, A. M. (1970). The relationship between a seventh-grade pupil's academic self-concept and achievement in mathematics. *Journal for Research in Mathematics Education*, 173-179.
- Clute, P. S. (1984). Mathematics anxiety, instructional method, and achievement in a survey course in college mathematics. *Journal for Research in Mathematics Education*, 15(1), 50-58.
- de Abreu, G., & Gorgorió, N. Social representations and multicultural mathematics teaching and learning, *CERME 5*. Larnaca. Cyprus.
- Díez-Palomar, J. (2004). *La enseñanza de las matemáticas en la educación de personas adultas. Un modelo dialógico*. Universidad de Barcelona. Unpublished PhD dissertation.
- Díez-Palomar, J., Menéndez, J.M., Civil, M. (forthcoming). Learning mathematics with adult learners: Drawing from parents' perspective. *RELIME. Revista Latinoamericana De Investigación En Matemática Educativa*,
- Evans, J. (2000). *Adults' mathematical thinking and emotions: A study of numerate practices*. London, UK: Falmer Press.
- Fennema, E., & Sherman, J. A. (1976). Fennema-sherman mathematics attitudes scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for Research in Mathematics Education*, 7(5), 324-326.

- Flecha, A., Garcia, R., Gomez, A., & Latorre, A. (2009). Participation in successful schools: A communicative research study from the included project. *Cultura y Educacion*, 21(2), 183-196.
- Gee, J. P. (1999). *An introduction to discourse analysis: Theory and method*. Psychology Press.
- Gómez Chacón, I. M. (2000). *Matemática emocional :Los afectos en el aprendizaje matemático*. Madrid: Narcea.
- Gómez González, A., & Díez Palomar, F. J. (2009). Metodología comunicativa crítica. *Teoría de la Educación*, 10(3), 103.
- Gómez, J. (2006). *Metodología comunicativa critica*. Esplugues de Llobregat, Barcelona: El Roure.
- Hannula, M. S. (2002). Attitude towards mathematics: Emotions, expectations and values. *Educational Studies in Mathematics*, 49(1), 25-46.
- Hart, L. E. (1989). Classroom processes, sex of student, and confidence in learning mathematics. *Journal for Research in Mathematics Education*, 20(3), 242-260.
- Hoover-Dempsey, K., & Sander, H. (1995). Parental involvement in children's education: Why does it make a difference. *The Teachers College Record*, 97(2), 310-331.
- House, P. A. (1975). Learning environments, academic self-concepts, and achievement in mathematics. *Journal for Research in Mathematics Education*, 6(4), 244-252.
- Kelley, H. H., & Michela, J. L. (1980). Attribution theory and research. *Annual Review of Psychology*, 31(1), 457-501.
- Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. *Affect and Mathematical Problem Solving: A New Perspective*, , 75-88.
- McLeod, D. B. (1994). Research on affect and mathematics learning in the JRME: 1970 to the present. *Journal for Research in Mathematics Education*, 25(6), 637-647.
- Mead, G.H. (1934). *Mind, self, & society*. Chicago: The University Chicago Press.
- Opt' Eynde, Corte, E. D., & Verschaffel, L. (2006). "Accepting emotional complexity": A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, 63(2), 193-207.
- Zan, R., Brown, L., Evans, J., & Hannula, M. S. (2006). Affect in mathematics education: An introduction. *Educational Studies in Mathematics*, 63(2), 113-121.

# **BETWEEN SCHOOL AND COMPANY: A FIELD OF TENSION?**

**Toril Eskeland Rangnes**

*Bergen University College*

*This paper is based on a school/company collaboration that involves mathematics learning conversations among 8<sup>th</sup> grade pupils. It illustrates how the different goals for the use of mathematics in these fora, create a field of tension in which the pupils encounter different languages and modes of thought. Taking mathematics conversations in and out of school as a starting point, the polyphony of this field of tension is analysed and discussed on the basis of Bakhtin's ideas of dialogism. In addition I will consider how the encounter with different voices influences the pupils' positioning and strengthens their participation in making decisions.*

Key words: Conversations, realistic mathematical education, pupils' positioning, polyphony.

## **INTRODUCTION**

In the context of a more general debate in Norway on the theorisation of schools, mathematics as a subject has been criticised for its distance from mathematics in everyday life. The politicians have encouraged practical mathematics, with a focus on literacy (reading, writing and numeracy) which includes “to identify, to understand, to interpret, to create and to communicate” (Utdannings- og forskningsdepartementet, 2004, p. 33) in the Norwegian National Curriculum. Oral skills are emphasized, as well as the ability to apply problem solving and investigation on the basis of practical, everyday situations (LK06, Norwegian National Curriculum for Knowledge Promotion in Primary and Secondary Education and Training). LK06 mirrors OECD's (2006) perception of mathematical literacy as a positive and necessary skill if one is to succeed in society and participate as a democratic citizen. Cooperation outside of school (e.g. with local companies) is encouraged, in order to learn and be motivated (LK06). Although more practical approaches and outside contact are encouraged, there has been little research into the learning that takes place when pupils in lower secondary school cooperate with institutions outside school to learn mathematics.

The purpose of this paper is to examine conversations in which 8<sup>th</sup> grade pupils (ages 13–14) encounter different mathematical practices and how this influences their positioning and language usage. This is also an exploration of how conversations can close or open up possibilities for further discussion and consideration of options, as well as for critical reflection about school mathematics vs. company mathematics.

Research in this field is limited, although there are studies of conversations in which everyday discourse and school mathematics discourse meet (e.g. Rønning, 2009). There have also been studies of adults attending courses to develop numeracy associated with work and everyday life (e.g. Wedege, 2010). Essential to this paper

is that the sequences have been taken from a project with the aim of allowing secondary school pupils to learn mathematics in the context of a company they initially know little about. It was not part of the pupils' everyday life, and it is different from the school mathematics practice they knew. The teacher and the pupils engaged in developing a new practice together.

In this case study the pupils meet two kinds of practices; those of the school, governed by learning goals in mathematics as defined in the curriculum and effectuated by the teacher; and those of the building company, guided by production goals, efficiency and profitability. This creates a field of tension where participants are confronted with different goals, practices and language and where they have to cope with the challenges this entails. Johnsen-Høines (2010) describes the pupils' movement between school and company as a learning loop. This movement is not confined to location – rather, it is about how the *moving between* is present in conversations both in school and in the building company.

The pupils were told beforehand what the mathematical and social goals were. The assignment given to them by the teacher and the building company was to construct 3D models of a *rorbu*, a combined boathouse and seaside cottage, which is popular in this coastal region. Initially, the building company sent the pupils several construction drawings of different size *rorbu*. These were to provide a basis for the pupils' own construction drawings and suggestions for possible room plans. The group were to take their construction drawing to the company, and discuss their drawing with a carpenter. Back in the classroom, the pupils produced 3D models at a scale of 1:25 based on their construction drawings.

## THEORETICAL FRAMEWORK

Wedeg (2006) describes the differences between working with mathematics at work and mathematics in school on the basis of the experience of her adult students. For instance, she notes that in professional life, one has to find the relevant information oneself, whereas in school, one is given problems cleansed of unnecessary information. According to Wedeg even “reality” has a different function. In professional life, reality provides opportunities to use mathematical ideas and techniques, and solutions have practical consequences; whereas in school, it serves as pretence for using mathematics, and the results usually have no practical consequences. Furthermore, the tasks in professional life are governed and structured by technology, whereas in school, the mathematical problems structure the teaching (Wedeg, 2006, p. 217).

The benefits achieved from boundary crossings between school and company settings with different cultures, is a moot point. To understand and to question one's own culture, outsideness can be a most powerful factor, according to Bakhtin (1986, p. 7). The dialogue between cultures, does not result in merging and mixing, he says,

since “each retains to its own unity and open totality, but they are mutually enriched” (Ibid, p.7).

In Bakhtinian dialogism, the utterance is the unit of analysis. An utterance, which can be a word, a sentence, a drama or a dissertation, can never be studied in isolation, but must be seen in relation to the preceding utterance and its continuation in the utterance that follows. Utterances must, according to Bakhtin, be seen in light of time; present, past and future. They must be seen in relation to context: Social, cultural and historical (Bakhtin, 1986).

Bakhtin never explicitly defines polyphony (Morson & Emerson, 1990) but rather describes it as a space in which different opinions, understandings and linguistic settings are expressed (Bakhtin, 1981; 1986). The voices in a polyphony can be identified through choice of theme, expressivity and purpose. An utterance may contain several voices. In order to create understanding, however, an utterance has to be more than different voices; there is also a need of tension and struggle between them (Dysthe, 1999, p. 76).

A dialogue which opens up for polyphony can be contrasted with monologic talk, in which one person or group has the power to decide the topic and the mode. Polyphony opens up for different possible positions including critique and negotiation of power. This is in accordance with the perception of “power as a relational capacity of social actors to position themselves in different situations and through the use of various resources of power” (Valero, 2004, p. 15). In this case study the Bakhtinian theory of dialogue, polyphony and positioning is a tool for describing pupils’ movement between different forms of argumentation which can be positioned in a school mathematics discourse or in a company discourse.

The form of the conversation is related to how power is divided between the participants. An inquiry dialogue is considered a conversation in which there are symmetrical relationships between the participants and the participants investigate each other’s perceptions (Alrø & Skovsmose, 2002). Lindfors (1999) stresses that the object of an inquiring utterance must be an authentic wish to seek others’ help to investigate what lies beyond that which one understands. Through an inquiring attitude one also shows what one knows. To ask in order to invite others involves risk taking since one demonstrates one’s need for the other’s good will, as they must listen and interact (Ibid). However dialogue is not only about question/answer; it is about the construction of meaning and is included in a social practice (Bakhtin, 1981, p. 121).

With Lindfors’ (1999) inquiry dialogue and Alrø & Skovsmose’s (2002) inquiry co-operation model (IC-model) as a background, I will in this paper describe dialogues in which the participants demonstrate an intention to listen and contribute, as in an “inquiry dialogue”. In such a dialogue, in which awarenesses meet, there lies an opportunity for change in the participants’ awareness. (Lindfors, 1999, p. 150). In

this respect, dialogical conversations stand in contrast to monologic ones where an authority holds the truth and tries to persuade or guide other people's choices.

These perspectives give me a basis for answering the questions: *What are the characteristics of the pupils' conversations when they encounter in school and workplace? What different positions do the pupils take in the conversations as they move between different and to some degree contradictory voices?* Implicitly, this will highlight how relational power moves between the participants in conversations.

## METHOD

The research project "Learning Conversation in Mathematics Practice" (LCMP) [1] of which this study is a part, cooperated with teachers in a municipality which intended to focus on the aim of carrying out practical learning in mathematics in cooperation with local enterprises. In this project student teachers were active in developing the teaching in this program. After graduating, one of these students wanted to try cooperating with an enterprise in her first year as a teacher. Her argument for doing this was that some of her pupils prefer practical work, and most of her teaching was theoretical. At the teacher's request, I was her conversation partner in the project. The teacher had the main responsibility for the design of the teaching plan and the implementation of the teaching. My role as a researcher was explained to the pupils; they were told that it was OK to communicate with me, but the teacher was in charge of the project.

The teacher and the researcher met the company carpenter for an initial clarifying conversation, in which it was made clear that it was their practical application of mathematics in the company that was of interest. The carpenter's role in the company is, among other things, to advise customers and to plan annexes and minor new buildings.

Gert Hana (researcher in LCMP) and I followed two groups with video cameras and sound recorders for seven sessions, one of which was at the company. The members of groups we followed were selected by the teacher. The groups were not based on mathematics ability; the only criteria were that the participants' were able to participate in conversations and that they had consented to take part.

In order to study both the form and the content of the conversations, the analysis has been inspired by both conversation analysis (Nielsen & Nielsen, 2005) and pragmatics (Svennevig, Sandvik & Vagle, 1995). In this paper I present three conversation sequences from one group of five pupils. In the conversations presented the active participants are two girls (Anne and Hilde), two boys (Daniel and Jonas), the teacher and the carpenter. The sequences have been selected because, according to Bakhtin's description of polyphony, they illustrate meetings between different voices based on the contextual dissimilarities represented by school and company. There is, of course, always a risk when interpreting conversations that one finds what one is looking for. To minimize this danger, possible interpretations have been

discussed in the research group that I am a member of and with the teacher. Alternate interpretations have been continually tested against each other; due to limitations of space, this material cannot be included in this paper.

## CONVERSATIONS RELATED TO CHOICES OF *RORBU* MODEL

### Sequence 1: Establishing the parameters

Anne was not present when the group made their sketch and needs information about the construction drawing of the *rorbu*. She asks Hilde about the plan floor and is told that there will not be two floors [2]; they will just plan the upper floor.

Anne: Won't it look like this, then, with only one floor? (Forms a one-floor structure with her hands) and not like *this*? (Then forms a two-floor structure with a slanting roof.)

Hilde: The teacher said not to make a roof. She said we could just... imagine a shoebox, she said (Forms an imaginary shoebox with her hands.)

Anne: OK. I'll imagine a shoebox. (Forms a shoebox with her hands.)

This conversation between Hilde and Anne must be seen in light of the conversation with the carpenter which they are preparing for. Anne's question reflects her experience with her family's *rorbu*. Her question, "won't it look like this with only one floor?" as she demonstrates the shape, can be seen as an argument in a discussion. Her realistic *rorbu* model with two floors and a roof becomes a counterargument to the group's artificial model with only one floor and without consideration of the slanting roof. Hilde responds by repeating the teacher's simplification: No roof, imagine a shoebox. I interpret this as a counterargument based in a school context. The choice between the two models has implications related to floor space and determines how they will proceed. Hilde refers to the teacher's utterance by saying "teacher said" or "she said", thereby implicitly giving her argument authority. The tension is articulated in Anne's utterance, which is positioned in a realistic out of school context and Hilde's utterance positioned in a school context. The utterance with reference to the teacher, the external authoritative voice, halts further discussion. Anne chooses to accept the school context.

### Sequence 2: Exploring alternative solutions

Shortly after, the whole group talks with the carpenter about their construction drawing:

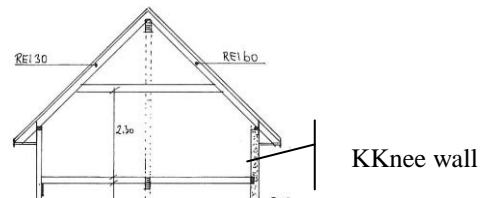
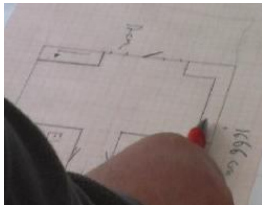
Cptr: What have you got?

The carpenter reads out loud the different rooms the pupils have drawn in, among them a room for computer games. The pupils have had heated arguments about whether there should be such a room in a *rorbu*. Floor use, space limitations, and special interests like computer games have been elements in the discussion. As the carpenter listens the pupils take up this discussion, speaking loudly and showing

## Working Group 10

deep disagreement. The carpenter does not enter this conflict but directs his attention to the kitchen:

Cptr: You have placed the kitchen cabinets mainly on the knee wall. You can have base cabinets there, but not top cabinets.



**Fig 1: Carpenter points to the base cabinet on the knee wall**

Daniel: [It] was a bit like that. (Points to construction drawing from company)

Anne: Yes, we looked at that. (Refers to a building plan from the company)

Cptr: The base cabinets will protrude 60 cm. (Takes his folding rule and measures 60 cm from the knee wall under the slanting roof in the room they are in.) So you can stand there doing the dishes for a while (addressed to Anne, who is standing closest to the knee wall). But not for very long. So the kitchen counter should probably be placed a bit more, so if you turned it.

Jonas: Couldn't we do it more like this, then... (He points along an inner wall.)

Cptr.: Yes, there (takes his pencil). I think I would take, put the kitchen cabinets along here (Same place as Jonas suggested where the counter is turned 90°)

The carpenter opens by asking what the pupils have to show him. This question is open and focuses on the pupils' contribution. He then looks over the plan quickly before focusing on the kitchen. This communication is both oral and written, one might say that the carpenter paraphrases the pupils' written contribution and in this way shows interest in and reinforces the pupils' contribution to the dialogue. Rather than following the pupils as they argue about who is responsible for a controversial room they have included, the carpenter zooms in on a problem he notices: The pupils have not considered the slanting roof and knee walls. This reveals his realistic approach to the model; it should be functional as if it were to be built in reality, in full scale. This supports Anne's realistic thinking. In a response to the carpenter's utterance, Daniel and Anne refer to a building plan they had been given by the company. I assume they are trying to give the company some of the responsibility for what they have done, through this reference. They defend what they have drawn and are retrospective in their positioning. The carpenter does not pick up what they say; he just goes on, communicating through a combination of words and actions. The proposed kitchen cabinet will protrude 60 cm from a knee wall. Using a folding rule, he demonstrates physically how low the ceiling will be there; a grown up will not be



able to stand upright. His expression and use of tools to demonstrate the consequences of the pupils' drawing, is polyphonic; on one hand, he instructs with the authority of a professional, concretizing to someone who is learning as in a master–apprentice relationship; on the other hand, he points out the consequences of choices in planning layout, as in a consultant–customer relationship. He indicates a possible solution: The kitchen counter could be turned. The carpenter's statements continue to be solution-oriented; he demonstrates what he would do. He uses many words to soften his statements. The kitchen counter “should probably” be placed “a little more”. When he then says they could turn it a little, Jonas is quick to suggest a concrete placement that involves turning the kitchen counter 90 degrees. The carpenter agrees with this proposal with a simple “yes” and adds “I think I would...” which suggests that others may have a different opinion. The pupils in this sequence choose to position the utterance in different ways; most of them are defensive and preoccupied with the distribution of responsibility and with explaining their choices; only Jonas chooses, like the carpenter, to look ahead towards possible solutions.

### Sequence 3: Considering other perspectives

This last conversation sequence was recorded about a month after the visit to the company. In the meantime, the pupils had, among other things, practiced constructing a 90° angle using a compass and ruler, and measuring and drawing angles using a protractor.

In this sequence, Jonas and Daniel are discussing whether to cut the inner walls according to a shoebox model (wall height in real life: 1.4 m) or whether to allow for knee walls and full ceiling height, as in a realistic model.



**Fig. 2: "If you have a cabinet there, right?"**



**Fig. 3: "It slants"**

Daniel has arguments for both the models, and uses a practical form of argumentation. In fig. 2, he points out that a cabinet will look rather odd against the inner wall in a shoebox model. He then balances this by adding that this kind of a shoebox model is easier to make. They then refer to what the teacher has told them to do, which is the shoebox model. Nevertheless, Daniel returns to the cabinet argument (fig. 3).

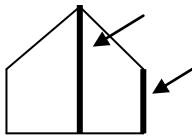
Daniel: But then they (inner walls) must slant, then we have to make the correct angle on all of them. That's a drag.

Jonas asks me what they should do. I tell them I don't know what teacher has said. Daniel comments: “No, well she says different things”. Teacher comes in:

Jonas: What should the inner walls be like?

## Working Group 10

- Daniel: Should we make them with a slanting ceiling?
- Jonas: Should there be a slanting ceiling here or should it just go straight along?
- Daniel: That'd be a bit silly 'cause if you have a cabinet, right, it should fit with the slanting ceiling.
- Teacher: What do you want to do?
- Daniel: It will be easier if ... but a slanting ceiling is better.
- Jonas: No, it's difficult (Daniel signals his agreement through sounds and gestures)
- Teacher: Is it difficult?
- Daniel: Yeah, you have to know the angle of...
- Teacher: Yes, or the height here and here, and then it will slant automatically.



**Fig. 4. Teacher points out the heights the pupils need**

The boys get going at once. Their concern has been addressed.

In their conversation with the teacher, Daniel and Jonas first clarify the question they are asking, they refer to the possible choices they are facing. In so doing, they also demonstrate their insight into the problem area. They take the pupil role, asking the teacher as an authority. At the same time, Daniel is arguing against the teacher's shoebox model. In his arguments, he draws freely on the carpenter's realistic model, which problematized the location of the kitchen counter and lack of room for cabinets – the cabinet will be too tall in relation to the walls. A field of tension develops between Daniel's argumentation and the teacher's open follow-up questions, making room for polyphony. The teacher empowers the pupils when she asks them what they prefer. In her response to the pupils' problem with the angles, she gives them an alternative way to find out the slant of the walls. The teacher's statement is also polyphonous. It can be interpreted from the perspective of a practical context, in which she gives the pupils a practical solution to the immediate problem, and can also be interpreted from the perspective of a school mathematics, in which she opens for a new field of learning, with regard to the characteristics of similar triangles and the connection with trigonometry. The result of teacher's statement is that Daniel and Jonas are given a real choice as to how advanced they are going to make their model. Their voices and choices have been clarified in the field of tension between school and company.

## CLOSING REMARKS

Returning to the first sequence, the conversation between Anne and Hilde appears to start as a balanced conversation between two pupils who position themselves

differently. Anne has “real-world” knowledge about *rorbu* since her family owns one, and Hilde has the knowledge gained in the discussion with the group and with the teacher. The balance was altered when Hilde positioned herself in a school context and referred to an external authority, which left little room for negotiation. In the second sequence, between the pupils and the carpenter, the conversation is dominated by the carpenter, who shows genuine interest in the drawing the pupils have made. The pupils are on the carpenter’s home ground, discussing their drawing with an expert. In many ways the carpenter speaks to the pupils as he would to customers, pointing out possible solutions. However, the pupils do not have the real power a customer/buyer would have, so there is little symmetry in the conversation. I have chosen to call this interaction pattern a master/apprentice conversation. In the third sequence, the conversation between Jonas, Daniel and the teacher, I would characterize the interaction as an inquiry dialogue, in which the participants are exploring each other’s perspectives. Here, the pupils’ argumentation is given room; they are allowed the opportunity to explain their arguments and what they consider difficult.

These three sequences have different purposes and qualities. I have identified how the participants, both the pupils and the teacher, make use of voices from both school practice and “real world” in their argumentations. In this case, placement in a field of tension where the participants are confronted with different goals for the use of mathematics (Wedge, 2006), seems to open up for polyphony (Bakhtin, 1981) and make the polyphony more visible. The differences between school and company give the participants experience related to argumentation, language usage and choices. In this way, polyphony opens up for the pupils’ voices to be used and strengthened. The pupils’ participation in polyphone dialogues gives reason to believe there is potential for developing critical reflectiveness about the practice of mathematics in and outside school.

## NOTES

1. This is part of my ongoing ph.d.-study in the research project Learning Conversation in Mathematics Practice (LCMP, leader: Marit Johnsen-Høines). The study is financed by the Research Council of Norway (NFR) and Bergen University College.
2. In this area a new *rorbu* always have two floors.

## REFERENCES

- Alrø, H., & Skovsmose, O. (2002). *Dialogue and learning in mathematics education: Intention, reflection, critique*. Dordrecht: Kluwer Academic Publishers.
- Bakhtin, M. (1986). *Speech genres and other late essays*. Austin: University of Minnesota press.
- Bakhtin, M. (Ed.). (1981). *The dialogic imagination*. Austin: University of Texas Press.

- Dysthe, O. (1999). Dialogical Perspective and Bakhtin. In O. Dysthe (Ed.), *The dialogical Perspective and Bakhtin*. PLF Report 2-99. (pp. 73-88). Bergen: Program for læringsforskning, Universitetet i Bergen.
- Johnsen-Høines, M. (2010). Interpretative Research as Collaborative Inquiry. In B. Sriraman, C. Bergsten, S. Goodchild, G. Palsdottir, B. D. Søndergaard, & L. Haapasalo (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education*. Charlotte NC: Information Age Publishing.
- Lindfors, J. W. (1999). *Children's inquiry. Using language to make sense of the world*. New York: Teachers college press.
- LK06. (2006). *Læreplan for grunnskole og videregående opplæring*. Retrieved from: <http://www.udir.no/grep>.
- Morson, G. S., & Emerson, C. (1990). *Mikhail Bakhtin: creation of a prosaics*. Stanford, Calif.: Stanford University Press.
- Nielsen, M.F., & Nielsen, S. B. (2005). *Samtaleanalyse*. Roskilde. Samfundslitteratur.
- OECD (2006). *Assessing scientific, reading and mathematical literacy: A framework for PISA 2006*. Organisation for Economic Co-Operation and Development.
- Rønning, F. (2009). *Tension between an everyday solution and a school solution to a measuring problem*. Paper presented at the CERME 6. Lyon.
- Svennevig, J., Sandvik, M., & Vagle, V. (1995): *Tilnærminger til tekst – Modeller for språklig tekstanalyse*. Oslo. Cappelen.
- Utdannings- og forskningsdepartementet (UFD). (2004). Kultur for læring. St. Meld. Nr 30. (2003–2004).
- Valero, P. (2009). Socio-political perspectives on mathematical education. In P. Valero, & R. Zevenbergen. (Eds.), *Researching the socio-political dimensions of mathematics education: issues of power in theory and methodology*. Boston: Kluwer Academic Publishers.
- Wedge, T. (2010). *Researching workers' mathematics at work*. In A. Araújo et al. (Eds.), EIMI 2010 Conference: Educational interfaces between mathematics and industry. Proceedings, Lisbon - April, 19-23, Portugal (pp. 565-574). Lisbon: Centro Internacional de Matemática. <http://eimi.glocos.org/>
- Wedge, T. (2006). Menneskers matematikholdige kompetencer. In O. Skovsmose & M. Blomhøj (Eds.), *Kunne det tænkes? - om matematiklæring* (pp. 208-227). Copenhagen: Mallings Beck.

# ETHNOMATHEMATICS IN EUROPEAN CONTEXT

**Charoula Stathopoulou\*, Karen François\*\* and Darlinda Moreira\*\*\***

\*Univ. of Thessaly, \*\*Free Univ. Brussels, \*\*\*Univ. Aberta UIDEF- U. Lisbon

*Through a literature review we document the fact that ethnomathematics in Europe—in comparison with other places like America or Africa—are of less application. In this paper we are discussing the reasons for this late development of ethnomathematics in Europe. We explore the importance of an ethnomathematical approach to contribute and inform mathematics education of every minority and culturally different group. We intent through an ethnomathematical perspective to a mathematical education without any distinguish and which will be addressed to all students independently cultural or any other kind of differences.*

Keywords: Cultural diversity, Ethnomathematics, Eurocentrism, Mathematics education.

## INTRODUCTION

The research question we are focussing on is if the situation concerning ethnomathematics in a European context differs from other parts of the world, and if so why and how we can identify and explain this differences. We plead for the importance of an ethnomathematical perspective as a critical way of doing mathematics education and as an opportunity to improve mathematical education for ALL. Our analysis of the ethnomathematical perspective is mainly based on the theoretical framework of D'Ambrosio (1985, 1992) the intellectual father of ethnomathematics and Vithal & Skovsmose (1997) who analyzed the concept of ethnomathematics in a critical sense, considering that these perspectives are complementary. The research program of Ethnomathematics has been changed over the last decades. Firstly, ethnomathematics research has been associated with the mathematical practices of particular tribes or indigenous, 'primitive' peoples, as well as those of a nation and/or human race. In recent times, under the impulse of an encompassing research programme, the meaning of the concept of ethnomathematics changed and has received a much broader interpretation (François & Van Kerkhove, 2010). Looking at the description of D'Ambrosio, one can observe a rather broad meaning of the concept. D'Ambrosio speaks about Ethnomathematics as "[t]he mathematics which is practiced among identifiable cultural groups, such as national-tribal societies, labor groups, children of certain age brackets, and professional classes." (1985, p. 45) and as "[t]he arts or techniques developed by different cultures to explain, to understand, to cope with their environment" (1992, p. 1184). In his ICEM3 presentation D'Ambrosio (2006) defines ethnomathematics as "[...] a research programme in the history and philosophy of mathematics, with pedagogical implications, focusing the arts and techniques (*tics* [from *technē*]) of explaining, understanding and coping with (*mathema*) different socio-cultural environments

(*ethno*)”. Although D’Ambrosio does not restrict the application of ethnomathematics to indigenous cultures, ethnomathematics (as explicit labeled) found much more fertile ground in non-western societies. We will argue this claim by, firstly presenting the results of our literature review of both the Ethnomathematics Digital Library and the International Study Group on Ethnomathematics. From this first results on the very narrow interpretation of ethnomathematics (as explicit labelled) we will also argue that research in the field of ethnomathematics developed later in Europe and that there *seems* to be less research (or less interest) in this field. These phenomena could be explained by the concept of Eurocentrism on which we will elaborate at the end of the first part of this paper. In the second part of the paper we will investigate the notion of ethnomathematical research in a European context based on a broader perspective of the concept of ethnomathematics as related to the notion of cultural diversity. This includes the ongoing research on the relation between mathematics education and the diverse backgrounds of the students (e.g. Moreira, 2002, 2007, 2009; Stathopoulou, 2005, 2006a) and adults education which involves ethnomathematical practices (e.g. Evans, 2000; Wedege, 2010).

## LITERATURE REVIEW

The method used in this explorative study is a screening of the research output of three sources: the Ethnomathematics digital library, the International Study Group on Ethnomathematics (ISGEm), and the conferences on ethnomathematics organized by ISGEm. Sources have been selected on the basis of research communities who are labelling themselves as doing research as explicit referred to as ethnomathematics research. The explorative screening of the output is based on articles and papers presented at the respective conferences. In this investigation we use the concept ethnomathematics as explicit labelling itself as ethnomathematical research. From this investigation we will argue that research in the field of ethnomathematics developed later in Europe and that there seems to be less research (or less interest) in this field. At the end of this section we will explain our findings from a sociological perspective that is dealt with in terms of Eurocentrism.

### The Ethnomathematics Digital Library (EDL)

The majority of the researchers listed at the EDL at <http://www.ethnomath.org> belong to places outside Europe and very often European researchers conducted their research in places outside Europe. Our observations are highlighted in table 1. Within the European countries, the ethnomathematical research is limited and a great part of it deviates from what is considered “genuine ethnomathematical research”. Looking at the examples of Greece, Italy and Portugal almost all papers concern historical issues and thus could hardly be categorized as ethnomathematics according to the four strands of ethnomathematics proposed by Vithal & Skovsmose (1997, p. 134-135). The first strand deals with historical aspects in non-western mathematics;

the second analyses the original mathematical practices of traditional cultures (mostly of indigenous peoples); the third explores the mathematics of different groups in everyday settings showing that mathematical knowledge is generated in a wide variety of contexts by both adults and children. The fourth strand focuses on the relationship between ethnomathematics and mathematics education.

Place	Number of references	Content
USA	83	Majority indigenous, part multicultural
Australia	31	Majority indigenous (Aboriginals)
New Zealand	33	Majority indigenous (Maori)
Spain	7	2 crafts, 1 Bask, 1 immigrants
Italy	7	6 historical
Greece	10	9 historical
Portugal	2	1 historical

**Table 1: Overview of Ethnomathematics Digital Library**

### **International Study Group on Ethnomathematics (ISGEem)**

On the main page of the ISGEem website hyperlinks/sites are sorted according to ethnicity/geography: African mathematics, Native American mathematics, Math in Euro-American culture, Pacific Islander mathematics, African American mathematics, Latino mathematics, Middle Eastern mathematics, Asian mathematics. The research belonging to the category of Math in Euro-American culture, also hardly could be considered as ethnomathematics according the four strands mentioned above. In regard to the late development of Ethnomathematics in Europe, the situation has been changing slowly. In Portugal and Greece, e.g. several researchers have been dedicated to integrating the perspectives of Ethnomathematics in their research, e.g. Moreira, (2002); Pires (2005); Candeia (2006); (Moreira & Pires, 2006), Stathopoulou (2005; 2006a). These investigations are not included in the ISGEem website. Considering that most of it is not published in English leads us to the question of the language of publication.

### **Conferences of Ethnomathematics**

At the 1<sup>st</sup> conference of ethnomathematics (Granada/Spain) the emphasis was on theory of ethnomathematics (e.g. Philosophy of ethnomathematics, ethnomethodology), on issues of mathematics teaching/ learning and on the connection of ethnomathematics to critical mathematics education. The main part of the research was on indigenous population while even European researchers had conducted their research out of Europe (e.g. F. Favilli's research was on Somalia). At the 2<sup>nd</sup> conference (Ouro Preto/Brazil) there was a specificity regarding the structure. The majority of the researches were presented in poster's form. Apart from it there were round tables and a few lectures. The opening lecture was Freire's contribution on Ethnomathematics, the closing concerned an overview of ethnomathematics. The other two focused on philosophy of ethnomathematics and the Eurocentrism of

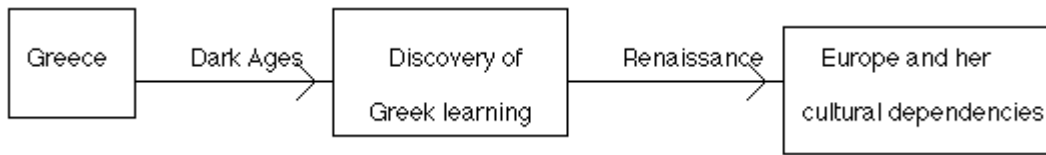
mathematics. Ethnomathematics and indigenous people, ethnomathematics and rural education, ethnomathematics and its theory, ethnomathematics and teacher's qualification, ethnomathematics and its history were the focus of the round tables. At the 3rd conference (Auckland/New Zealand) the main part of the papers concerned indigenous cultures. Another part concerned theoretical issues of ethnomathematics as well as issues of more general interest. There were four papers that came from European countries; two from Greece (Stathopoulou, 2006b; Stathopoulou & Chaviaris, 2006), one from Norway (Onstad, 2006) and one from Sweden (Norén et. al, 2006). At the last conference (July 2010 at Towson-Baltimore/USA), more percentage than in previous conference concerned indigenous cultures. Also issues of indigenous teachers and teachers in general were discussed. Political and methodological aspects of ethnomathematics occupied a considerable part of the presentations. There was one panel on Ethnomathematics in a European context in which the authors participated.

### **Eurocentrism**

By ethnomathematical research and approach we referred to that literature which is deliberately labelled as ethnomathematics. Based on this narrow interpretation of ethnomathematics research we used in our literature review we can observe that studies in ethnomathematics in Europe are scarce. The outcomes of our non-exhaustive review are indicative of the progress of the ethnomathematics research in Europe. We dare an interpretation for this late development of ethnomathematics to Europe. The fact that ethnomathematics to a large extent, emerged as a contradiction to the domination of western mathematics seems to be one of the main reasons that ethnomathematics were developed later in Europe. Powell & Frankenstein (1997) suggest that ethnomathematics could contribute to the elimination of many false dichotomies such as 'practical, everyday knowledge' versus 'abstract, theoretical knowledge'. What is mostly discussed here is the fact of the domination of western mathematics and in general, the western culture that, using an evolutionist schema evaluates any other culture in comparison to itself. Greenhalgh & Megaw (1978) commenting the European interest about anything non-European, reveal that European people always tend to assimilate the various and independent art traditions of the other cultures through a diffusionist schema in the center of which there are their own experiences. Correspondingly, mathematics of other cultures are approached and evaluated in comparison to western mathematics. For a long time, western mathematics are appeared as an authoritative knowledge, as a corpus of cognition and perceptions that is preferential to others, since it is considered that are more close to the 'absolute truth'; the non-western mathematics are "measured" by the meter of western mathematics. There is no scientific perspective or any other that has not its roots to particular stories and prejudices; something that is imminent in human beings (Erickson & Murphy, 2002: 203). Concerning mathematics, Ghevarughe (1987) suggests that "there exists a widespread Eurocentric bias in the



production, dissemination and evaluation of scientific knowledge." He diagrams the "classical" Eurocentric approach as follows:



**Figure 1: Eurocentric Approach on the Development of Mathematics**

Ernest (2008), in this spirit, notes that many histories of mathematics, such as Eves (1953), promote a simplified Eurocentric view of its development:

Typically such accounts identify Mesopotamia and Egypt as the sites of preliminary work that provided the raw materials for mathematics. Based on this, the flame of 'real' mathematics was lit by the Ancient Greeks, kept alight by the Arabs during the Dark Ages, until when passed on like an Olympic torch, it blazed anew in modern Europe and her cultural dependencies (Ernest, 2008, p. 93).

A perception like this ignores the connection of mathematics with the corresponding culture, as well as the fact of major contributions to the corpus of academic mathematics of non-western cultures as the invention of the decimal place value system with zero in India is (Ernest, 2008). Ghevarughese (1987) claims that this Eurocentric approach served as a "comforting rationale for an imperialist/racist ideology of dominance" and has remained strong despite evidence that there was significant mathematical development in other places. This imperialistic/racist perspective of western mathematics has not only its consequences for non-western contexts. A lot of European cultures –we dare to use the plural since there is no single and unique culture in Europe- face these consequences. Although the differences in European classrooms are not so obvious, comparing with indigenous people in e.g. USA. European classrooms have to deal with a lot of students coming from minority and marginalized groups. These challenges have their impact in math classes since mathematics education is an acculturation procedure (Stathopoulou, 2006a). All groups that are marginalized and oppressed by the dominant culture and the educational system are facing more problems since their culture is contemned. Thus, teachers—teachers of mathematics— face the challenges of teaching in multiethnic and multilingual classrooms that includes students belonging to the above groups. Apart from the minority students coming from immigrants and refugees, traditional cultural groups, such as Romany students do also contribute to the formation of current classrooms. In the following part of the paper we will investigate the notion of ethnomathematical research in a European context bases on the broader perspective of the concept of ethnomathematics.

## **TOWARDS AN ETHNOMATHEMATICAL PERSPECTIVE**

The increasing cultural diversity in Europe has been changing the landscape in the European classrooms. The debate on the concept of cultural diversity leads to the

question of equity and social justice that has been pointed out as one of the main challenges in the (research) field of mathematics education. In Europe equity in mathematics education has become an important issue since basic education is mandatory and mathematical literacy has been seen as a human right. However institutional education (e.g. schools, curricula, ...) results in exclusion of a large number of students that do not succeed in schools, being most of them from cultural minority groups. Moreover student population in Europe is not only a much more cultural diverse group; their diversity has also a strong correlation with the achievements in education. The same observation has been done in the USA by Suárez-Orozco & Suárez-Orozco (2002). They observe that “today’s immigrants are a much more diverse group than ever before in terms of educational background and skills” (2002, p. 56).

Due to the shifted multicultural settings in schools and the increased variety of cultural diversity, questions and challenges in the (research) community of mathematics education has been changed. One important challenge is the background of the students; the valorization of students’ socio-cultural roots and their previous knowledge. It is important to deal with this starting position because it contributes to the future learning of the students and it contextualizes the act of learning. Indeed, the heterogeneity of the school population is expressed in various ways: language, behavior, habits, ways of enhancing the own knowledge, ways of giving meaning to school contents. In addition, when we talk about education we consider that educational processes happen within particular socio-cultural contexts, being school one of them. Indeed education is a vast process with the presence of several protagonists who use different strategies and learning technologies which are located mainly in the family, school and community (Pinxten, 1997; Moreira, 2007). Hence, an important issue for current thinking about educational inclusion is located at the epistemology of social groups. Since learning and its specific processes are socially and culturally situated, social group theory of learning and knowing emerges as essential to frame the educational content and to give meaning to social practices –being it school or communities based practices. To the extent that education conducted by the school is based on assumptions and educational processes different from those usually carried out in domestic groups –which are reflected particularly in the substantial differences between rationalities, discourses and practices– children from social groups that are more familiar with the school’s body of knowledge and artifacts have a higher probability of school success. Both the ethnographic research and theoretical/critical reflection have been showing how school achievement is related in many different ways to the cultural background of students, which withdraws or is legitimated in schooling. Furthermore, the different life histories of the students are immersed in memories, affections and knowledge that demonstrate the existence of different relationship to knowledge and how learning are processed in different ways, based as they are in their own cultural epistemologies. In this sense, one of the educational function of schooling is seen as a way to connect and transfer discourses

and practices among different social groups, being necessary to bear in mind that different types of knowledge are embedded in their own contexts, which imply the presence of cognitive processes, forms of thinking, teaching and transmitting knowledge to new generations, leading even applications, objects, problems, technologies, and particular professions (Moreira, 2009; Stathopoulou, 2002).

Research in the field of ethnomathematics –even in Europe– is clearly showing that communities are locus of mathematical knowledge much of which is yet to be considered and legitimized in schools. Acting as a strong source of criticism of how the mathematical activity of different groups has been erased or ignored by schools, ethnomathematics has been questioning the disjunctions between home based and school-based mathematical practices. Ultimately the findings from ethnomathematics show how we need to go beyond universalist and essentialist notions of mathematics and the need to build a mathematics that is based on everyday experience of mathematics, opening the door to the wealth of knowledge of various social groups. It is however necessary to think about how to interpret such perspective in the context of each particular educational setting because each one possess its own diversity and gives a meaning to it according to one experiences of the complex social fabric of present societies. Each classroom and each school imprints a particular dynamics that requires a different knowledge about the setting, the people and the culture. As Chuche points out, “in cultural construction, what comes first is the culture of the group, the local culture, the culture that joins individuals with immediate interaction with each other, and not the global culture of the larger collectivity.” (1999, p. 87). Also Pinxten (1997) reminds us that:

[A]n educational program will most probably be more efficient if it draws on the native strategies for thinking and learning than when simply implementing the western (or for that matter any other) way. (...) [T]he particular classifications and notions of a culture will in all probability constitute the best material to work within an educational setting (Pinxten, 1997, p. 135-136).

From an ethnomathematical perspective, the local references and practices have to be taken into account as a starting point of the educational process. This is, an ethnomathematical perspective that leads to an innovative reflection in educational settings and thus can change the traditional educative paradigm with its depreciation of the experiences and knowledge that students bring from their culture and daily life.

We are indeed aware of the pits and falls and the pedagogical implications of the ethnomathematical perspective. In recent years strong criticisms are made to such implications (Skovsmose & Vithal, 1997; Rowland & Carson, 2002; Adam, Barton & Allangui, 2003; Knijnik, 2006; Duarte, 2006; Domite and Pais, 2009). These studies make aware of a counter effect by implementing ethnomathematical ideas into school curricula (e.g. social exclusion, de-substantialization of Other's culture, essentialist approaches to culture). This observation gives rise to a central question if

the import of the concept ‘ethnomathematics’ benefits a European critical math education that is looking for social justice and math education for *all*. In the frame of this paper we can only mention this topic, knowing that deepening this issue will be the focus of further research.

## CONCLUSION

In this paper we focused on the late development of ethnomathematics in Europe and the reason thereof. Based on a literature review of three sources from the ethnomathematical research community (as officially labeled) we can observe indeed a different ethnomathematical investigation in a European context. Part of the reason we explained by the concept of Eurocentrism, which includes a perspective on ethnomathematics as the study of mathematical practices of non-western cultures based on the cognitive categories of European research. This narrow interpretation of ethnomathematics gave rise to an imperialistic view on western mathematics which is not restricted to a non-western context. The European landscape changed into a multicultural society with a rich variety of diversity and even in this European context of diversity, the imperialistic perspective comes into play where school curricula do not deal with the diverse background of students. In this paper we referred to undertaken and ongoing ‘ethnomathematical’ research in a European context that focuses on this topic. More research has to be done in this area and there is a need to bring more results of this research to a broader public by translating it into the lingua franca.

## REFERENCES

- Adam, S., Alanguai, W. & Barton, B. (2003). A comment on: Rowlands and Carson “Where would formal, academic mathematics stand in a curriculum informed by Ethnomathematics? A critical review”. *Educational Studies in Mathematics*, 56, 327-335.
- Candeia, C.J.M. (2006). *Etnomatemática: o cálculo mental na comunidade cigana*. Braga: Universidade do Minho. Master dissertation.
- Chuche, D. (1999). *A Noção de Cultura nas Ciências Sociais*. Lisboa: Fim do Século.
- D’Ambrosio, U. (1985). Ethnomathematics and its Place in the History and Pedagogy of Mathematics. *For the Learning of Mathematics*, 5(1), 44-48.
- D’Ambrosio, U. (1992). Ethnomathematics: A research Programme on the History and Philosophy of Mathematics with Pedagogical implications. *Notices of the American Mathematics Society*, 39, 1183-85.
- D’Ambrosio, U. (2006). Ethnomathematics: The scenario 30 years after. Plenary presentation, ICEm3: Cultural Connections and Mathematical Manipulations, Auckland/ New Zealand, 12-16 February.
- Domitte, M. C. & Pais, A. (2009). Understanding Ethnomathematics from its Criticisms and Contradiction. In Proceedings of CERME6 (pp. 1473 – 1483), January 28th February 1st 2009, Lyon France: © INRP2010.

## Working Group 10

- Duarte, C. (2006). Implicações curriculares a partir de um olhar sobre o “mundo da construção civil”. In Knijnik, G, Wanderer, F, & Oliveira, C. [orgs], *Etnomatemática: currículo e formação de professores* (pp. 53-69). Santa Cruz do Sul: Edunisc.
- Ernest, P. (2008). What does the new philosophy of mathematics mean for mathematics education, *Proceedings of 25th Panhellenic Congress in Mathematics Education of Hellenic Mathematical Society*, 21-23, 60-91.
- Evans, J. (2000). *Adults Mathematical Thinking and Emotions: a Study of Numerate Practices*. London: Routledge Falmer.
- Eves, H. (1953). *An introduction to the history of mathematics*. (3<sup>rd</sup> ed.). Holt: Rinehart and Winston.
- François, K., & Van Kerkhove, B. (2010). Ethnomathematics and the Philosophy of Mathematics (Education). In Benedikt Löwe & Thomas Müller (eds.) *Philosophy of Mathematics: Sociological Aspects and Mathematical Practice. Series: Texts in Philosophy*. (pp. 121-154), London: College Publications.
- Ghevarughese J.G. (1987). Foundations of Eurocentrism in Mathematics. *Race & Class*, 27(3), 13-28.
- Greenhalgh, M. & Megaw, V. (1978). *Art in Society*. London: Duckworth.
- Knijnik, G. (2006). Etnomatemática e educação no Movimento Sem Terra. In Knijnik, G, Wanderer, F, & Oliveira, C. [orgs], *Etnomatemática: currículo e formação de professores* (pp. 53-69). Santa Cruz do Sul: Edunisc
- Moreira, D. (2002). *Contas da vida: interação de saberes num bairro de Lisboa*. Lisboa: Instituto Superior das Ciências do Trabalho e da Empresa (ISCTE). Departamento de Antropologia. PhD dissertation.
- Moreira, D. (2007). The discourses and practices of social groups outside of school as developing student identities related to schooling and mathematics. In *Proceedings of the Second Socio-Cultural Theory Conference*.
- Moreira, D. (2009). Etnomatemática e mediação de saberes matemáticos na sociedade global e multicultural. In M. C. Fantinato (Org.), *Etnomatemática, novos desafios teóricos e pedagógicos* (pp.57-66), Niteroi-RJ: Universidade Federal Fluminense
- Moreira, D. & Pires, G. (2006) Aspectos Etnomatemáticos na Educação das Crianças Ciganas. *Actas das Jornadas Caminhos da Alteridade: Os Ciganos na Península Ibérica*. (pp.123-143) Lisboa: Câmara Municipal de Lisboa.
- Norén, E., ter Vehn, M., Rönnerberg, I. & Rönnerberg, L. (2006). Reflecting the cultures and languages in Swedish mathematics classrooms. In *Proceedings of ICEm3* <http://www.math.auckland.ac.nz>.
- Onstad, T. (2006) Is the mathematics we see the mathematics they do? In *Proceedings of ICEm3* <http://www.math.auckland.ac.nz>.

## Working Group 10

- Pinxten, R. (1997). *When the Day Breaks. Essays in Anthropology and Philosophy.* Frankfurt: Peter Lang Verlag.
- Pires, G. (2005). *Cálculo mental das crianças ciganas. Ideias silenciosas a serem ouvidas.* Porto: Universidade Aberta. Master dissertation.
- Powell, A. B., & Frankenstein, M. (1997). *Ethnomathematics: Challenging Eurocentrism in Mathematics Education.* Albany: State University of New York.
- Rowlands, S. & Carson, R. (2002). Where Would Formal, Academic Mathematics Stand in a Curriculum Informed by Ethnomathematics? A Critical Review of Ethnomathematics. *Educational Studies in Mathematics*, 50, 79-102.
- Stathopoulou, Ch. (2002). *Use of informal cognition in teaching mathematics.* In Proceedings of the 2nd International Conference on the Teaching of Mathematics (in electronic format). 1 – 6 July 2002, Crete, Greece: <http://www.math.uoc.gr/~ictm2/Proceedings/pap505.pdf>
- Stathopoulou, C. (2005). *Ethnomathematics: Exploring the Cultural Dimension of Mathematics and of Mathematics Education.* Athens: Atrapos.
- Stathopoulou, C. (2006a). Cultural and Cognitive Conflicts in School Context: Mathematics Education as an Acculturation Process for Romany Students. *Themes In Education*, 7(2), 131-143.
- Stathopoulou, C. (2006b). Symmetry and gold section: two mathematical ideas incorporated in the tradition of Xysta (Pyrgi-Chios-Greece). In *Proceedings of ICEm3* <http://www.math.auckland.ac.nz>.
- Stathopoulou, C. & Chaviaris, P. (2006) Sociocultural view on relationship between school failure and mathematics education: a case study. In *Proceedings of ICEm3* <http://www.math.auckland.ac.nz>.
- Suárez-Orozco, C. & Suárez-Orozco, M. (2001). *Children of Immigration.* Cambridge: Harvard University Press.
- Vithal, R., Skovsmose, O. (1997), The End of Innocence: A Critique of ‘Ethnomathematics’. *Educational Studies in Mathematics*, 34( 2), 131-157.
- Wedeg, T. (2010). Ethnomathematics and mathematical literacy: People knowing mathematics in society. In C. Bergsten et al. (eds.), *Mathematics and mathematics education: Cultural and social dimensions. Proceedings of MADIF 7.* (pp. 31- 46), Stockholm: Linköping: Skrifter från SMDF, Nr.7, Linköping Universitet.

# DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION: CURRENT STATUS AND FUTURE PATHWAYS FOR TURKEY

Behiye Ubuz<sup>1</sup>, Erdinç Çakıroğlu<sup>2</sup>, Ayhan Kürşat Erbaş<sup>1</sup>

<sup>1</sup> Middle East Technical University, Department of Secondary Science and Mathematics Education

<sup>2</sup> Middle East Technical University, Department of Elementary Education

*The purpose of this paper is to review the current state of doctoral programs in mathematics education in three Turkish universities. In this context, we first provide brief background information about Turkish higher education system and teacher education policies in Turkey. Then the major national initiatives towards future faculty development efforts are explained. Finally, the nature and components of mathematics education doctoral programs in Turkey is provided. In doing so, we provide information about admission procedures, requirements for coursework and dissertation, and employment opportunities for those with a doctoral degree in mathematics education.*

Keywords: doctoral programs, mathematics education, higher education in Turkey

## INTRODUCTION

Mathematics education in Turkey is relatively young compared to other educational areas such as educational administration and educational measurement. Undergraduate programs focusing specifically on mathematics teacher education in Turkish universities began to appear after the establishment of *Yüksek Öğretim Kurulu* (YÖK) [The Turkish Council of Higher Education] in 1981. While establishing these programs, the staff requirements were filled mostly by mathematicians, not by mathematics educators. Moreover, faculty in educational sciences has supported these programs in the pedagogical sense, without focusing much on mathematics. As Shulman (1986, 1987) and many others (e.g., Ball, Thames, & Phelps, 2008; Hill et al., 2008) have either suggested or provided evidence that simple combination of expertise in mathematics and educational sciences do not guarantee a sound base for quality instruction in mathematics. Instead, a pedagogical basis specific to mathematics needs to be established. Such a perspective calls for an audit of mathematics education programs and draw attention to the need for doctoral degrees in mathematics education for supplying quality staff for these programs.

Our goal in this paper is to describe the current state of the mathematics education doctoral programs in Turkish universities as every country is diverse and unique in its own way. For such a young field in Turkey, doctoral programs play a crucial role in the future of mathematics education. In fact, doctoral programs create contexts in which traditions of research, approaches to issues about school mathematics, and philosophical stands are produced and re-produced. In this sense, the nature of these

programs and the discourse growing in these environments will have direct and indirect impact on cultural and political issues about mathematics education in Turkey through the graduates of these programs and work produced by the people involved. On the other hand, this paper contributes to our understanding of diversity in mathematics education doctoral programs on a global level in terms of components of these programs (e.g., program activities, students, faculty and/or staff, outcomes) and the contexts they function in various countries.

In Turkey, mathematics education research on graduate level has been conducted in various programs, such as early childhood education, primary education, and the educational sciences. However, our discussion in this paper is limited to the programs offering degrees in elementary and secondary mathematics education. A better understanding of the mathematics education doctoral programs is possible when they are situated within the larger context of higher education and teacher education policies in Turkey. Thus, we will begin with a brief background of Turkish higher education system and teacher education policies. Then we describe faculty development attempts before discussing specifically the nature doctoral programs in mathematics education.

## **HIGHER EDUCATION SYSTEM AND MATHEMATICS TEACHER EDUCATION IN TURKEY**

In Turkey, there is a unified system of higher education under the surveillance of the Council of Higher Education. Currently, in this system, there are 93 state and 38 private universities throughout Turkey (YÖK, n.d. -a). Each university consists of faculties offering undergraduate programs. Admissions to these programs are centralized and based on a nation-wide examination conducted by *Öğrenci Seçme ve Yerleştirme Merkezi* [the Student Selection and Placement Center]. Graduate programs for master's and doctoral degrees, on the other hand, are offered under the graduate schools in universities. According to the Council of Higher Education, in the 2004-2005 academic year, the total number of graduate students in Turkish universities was about 120 000 (YÖK, 2007). Of this number, 92 600 students were in master's and 27 400 in doctoral programs. No data were available for the graduate programs offering degrees specific to mathematics education.

Four year undergraduate mathematics teacher education programs were established within the faculties of education for initial training of teachers, since the establishment of the Council of Higher Education. Until 1998, mathematics teacher education programs trained teachers for both middle and high schools (grades 6 through 11). After that time, the programs were re-established so that the training of teachers for middle (grades 6 through 8) and high schools (grades 9 through 12) were separated. In fact the changes in 1998 were beyond this separation, as teacher education programs in Turkey had undergone a major reform movement. The changes include, (i) shifting the focus in teacher education to the quality of teacher, (ii) focusing more on the middle and elementary grade levels, (iii) developing



master's level programs for teacher education, (iv) focusing more on the methods of teaching relevant to specific subject matter, and (v) meeting the shortage of faculty in these programs (Simsek & Yildirim, 2001).

Although this reform movement aimed to improve teacher education programs throughout Turkey, the faculty development aspect of it had significant impact on research and graduate programs in the coming years. Increased number of faculty members in mathematics education resulted in developments in graduate programs.

## **FACULTY DEVELOPMENT**

Although there have been some efforts to increase number of graduates having doctoral degrees, universities in Turkey still experience the problem of faculty shortage for teaching and research. There is an acute shortage of faculty members in terms of both quality and quantity in higher education institutions and the doctoral programs currently exist in the system is insufficient to supply that demand (YÖK, 2007, p. 132). In response to such shortage, bodies organizing and coordinating higher education such as the Council of Higher Education and the Ministry of National Education (MoNE) have taken certain actions to support students in graduate level studies according to Law numbers 2547 and 1416 respectively (Government of Turkey, 1981, 1929). Within this context, supporting students to study their graduate education abroad is the most important initiative for faculty development. Following this attempt, another initiative for addressing shortage of faculty members in higher education is a doctoral scholarship program for studying in Turkish universities.

Both the Council of Higher Education and the Ministry of National Education have been sending selected students abroad mostly to get their doctoral degrees in various disciplines. Every year these institutions determine the number of doctoral degrees to be pursued abroad. Candidates are selected based on their academic records, language proficiencies, and the priority of the field in that year. Candidates are also required to have an acceptance from the host university. Those students must successfully finish their studies and come back to Turkey to claim a position for which they have been sent for—a compulsory academic service. The amount of compulsory services is about two years' service for every year s/he received the scholarship. In the case of no-returns or failure to finish studies successfully, the recipients must pay back the amount received with interest (Tansel & Güngör, 2003).

Between 2003 and 2008, 300 students who were sent abroad with the MoNE scholarship returned to Turkey after completing their studies successfully. Half of these students were education majors (e.g., elementary education, social sciences education, mathematics education, and science education—including physics, chemistry, biology education) and received their degrees from the universities in the United States ( $n = 106$ ), France ( $n = 21$ ), and the United Kingdom ( $n = 23$ ). Twenty-one of the education majors received their graduate degrees in mathematics

education (7 from France, 6 from the United Kingdom, and 8 from the United States). The people who completed their doctoral studies abroad are currently working in Turkish universities. On the other hand, as of December 2008, 940 graduate students are studying abroad (mostly in the United States, 74.7%) with a scholarship provided by the MoNe. Hundred-and-twenty of those are in education related fields. Among them, about 23 to 27 of the graduate students are studying in the area of mathematics education.

Like the MoNe, between 1987 and 2008, a total of 3899 graduate students were sent to abroad to thirty different countries by the Council of Higher Education. In 1996, the Council of Higher Education has established a board, the Board for the Training of Academic Staff and Researchers, to coordinate this program. A great portion of the scholarship students has been sent to the United States ( $n = 1941$ ) and the United Kingdom ( $n = 1454$ ). A great majority of these students ( $n = 2485$ ) finished their graduate education and started to serve as faculty in various Turkish universities. Among them 16 has received their doctorates in mathematics education, 7 from the United States and 9 from the United Kingdom. While 297 of the scholarship students are still in progress, the remaining students either did not finish their studies ( $n = 386$ ) due to reasons such as failure and health; or did not return to Turkey (i.e., brain drain) and/or resigned from the scholarship ( $n = 731$ ). The statistics about the scholarships provided by the Council of Higher Education and the Ministry of National Education were obtained from these institutions through official communication, and based on unpublished data. Some published statistics covering the years up to 2005 was provided in a report by *Türkiye Bilimler Akademisi* (TÜBA) [Turkish Academy of Sciences] (TÜBA, 2006).

The second leg of the faculty development efforts is the scholarship opportunities for seeking doctoral degrees in Turkish universities. These efforts have two legislative bases: first one is the Article 35 of the Turkish Higher Education Law No 2547 and the second one is the “Faculty Development Project.” The Article 35 of the Turkish Higher Education Law No 2547 (Government of Turkey, 1981) concerns the needs of teaching staff of the higher education institutions. According to that law, all higher education institutions, whether established or yet to be established, are responsible for educating future faculty members. Within this context, the Council of Higher Education designated more research assistantship positions to universities. In case of lack of faculty members and doctoral programs in a particular university, the assistantships in that particular university are allowed to be transferred to another university offering a doctoral program. Students who complete their doctoral studies return to their own universities to carry out compulsory service for a certain period of time.

*Öğretim Üyesi Yetiştirme Programı* (ÖYP) [Faculty Development Project] was initiated in 2001 with the financial support of *Devlet Planlama Teşkilatı* [State Planning Agency] in order to meet the growing need for quality faculty members in

Turkish universities. The difference of ÖYP from the above article 35 of the Higher Education Law numbered 2547 is that besides getting their salaries for the assistantship positions, graduate students are supported financially to conduct their research and to study abroad for about two semesters. Middle East Technical University (METU) was the first to offer doctoral education for ÖYP program to meet the faculty needs in other four partner universities. Until 2006, 19 other partner universities have joined the program and 562 ÖYP students were accepted for Ph.D. programs in 43 different disciplines. Currently there are eight ÖYP doctoral students studying in mathematics education at METU for different partner universities.

## **DOCTORAL PROGRAMS IN MATHEMATICS EDUCATION**

In this part, we will explain the nature of the mathematics education doctoral programs in Turkey. In doing so, we will explain general characteristics of the program of studies - including admission procedures and various requirements-, and employment opportunities for those with a doctoral degree in mathematics education. Describing doctoral programs throughout Turkey is a challenging task for several reasons. First of all, doctoral programs in different universities have different characteristics and expectations from students. Second, the characteristics of doctoral programs are rapidly changing and new programs are being established. Finally the information about doctoral programs is available in different levels – departments, graduate schools, the Council of Higher Education– which sometimes may not be aligned with others. Despite these difficulties, we attempt to summarize general characteristics of doctoral programs by providing examples from some selected doctoral programs. More specifically, although our search in dissertation database of the Council of Higher Education has revealed that 10 universities have mathematics education doctoral programs throughout Turkey, we will only focus on the programs in three universities: Middle East Technical University (METU) in Ankara, Karadeniz Technical University (KTU) in Trabzon, and Atatürk University (AU) in Erzurum, Turkey. Considering the number of dissertations completed and their geographical regions, we consider these programs represent the diversity in mathematics education doctoral programs in Turkish universities.

The information provided in this section was obtained through the Web pages of universities and a survey sent to program chairs by e-mail to respond, which consisted of a questionnaire that included open-ended questions regarding the number of dissertations completed so far, the type of courses offered during the program, etc.

### **Admissions to Doctoral Programs**

In Turkey, all doctoral programs are considered under Ph.D. Students can apply to a Ph.D. program after having either a bachelor's or a master's degree. Doctoral programs do not require teaching experience prior to admission. Admission to doctoral program is based on applicants' cumulative grade point average (GPA) in

undergraduate and/or the masters program (if attended), their *Akademik Lisansüstü Eğitim Sınavı* (ALES) [Academic Graduate Education Exam] scores or equivalent international examination scores, such as the USA-based Graduate Record Examination, level of English language proficiency and the evaluation of other criteria required and announced by the relevant department administration, such as recommendation letters or letter of intentions. ALES is given twice a year by the *Öğrenci Seçme ve Yerleştirme Merkezi* (ÖSYM) [Student Selection and Placement Center]. This exam measures verbal reasoning and quantitative reasoning. The skills measured in verbal reasoning include the test taker's ability to analyze and evaluate written material and synthesize information obtained from it, to analyze relationships among component parts of sentences, and to recognize relationships between words and concepts. The skills measured in quantitative reasoning include the test taker's ability to understand basic concepts of arithmetic, algebra, geometry, and data analysis, to reason quantitatively, and to solve problems in a quantitative setting. Applicants' level of English proficiency is evaluated based on the result of either the university's English proficiency examination or the equivalent exams such as *Üniversitelerarası Kurul Yabancı Dil Sınavı* (ÜDS) [Interuniversity Foreign Language Examination] or Test of English as a Foreign Language (TOEFL). ÜDS is given twice a year by ÖSYM. For admission into a graduate study program, the acceptable score on these exams is determined by the recommendation of the department administration and the acceptance of the Administrative Board of the Graduate School.

### **Number of Faculty Members and Graduate Students**

Institutions vary greatly in the number of faculty members as well as the number of graduate students. While AU has 14 faculty members in mathematics education program, METU and KTU have 7 and 4 faculty member respectively. The main reason for AU having more faculty members is that they offer both mathematics and mathematics education courses, which is not the case in both METU and KTU. Therefore, the program at AU employs faculty members with PhD's either in mathematics education or mathematics. While approximately half of the faculty in AU hold doctoral degrees in mathematics education, the rest have degrees in mathematics. This combination of mathematicians and mathematics educators in AU results in some doctoral students doing research in pure mathematics. For instance, only 9 out of 19 doctoral graduates in AU did their dissertation research on mathematics education (see Table 2).

Currently mathematics education doctoral programs in these three universities have 110 students in total (see Table 1). The distribution of genders of the current students and doctorates granted vary across universities. In total about half of the current doctorate and doctorate granted students were females (see Table 2). Thirty-seven of the current doctorate students have positions in the universities as research and teaching assistant. Of these 37 students, 9 are in ÖYP program and 17 are benefitting

## Working Group 10

from article 35. Taking together ÖYP and Article 35, twenty-six doctoral students will return to different universities that have been assigned previously. Considering the number of dissertations completed until 2008, and the number of doctoral students currently progressing, the mathematics education will develop in the next decade in terms of the number graduates and the research. However, the rapid increase in the demand of doctoral degrees in mathematics education will eventually reach to saturation in terms of the need of faculty in the coming decades. Therefore, we believe that, it is just the time for policy makers to think about planning the supply and demand for mathematics education doctoral degrees, by considering all employing alternatives.

Universities	Faculty Members		Students progressing in Ph.D.	Assistantships Status of Current Students			Dissertations Completed as of 2008
	Ph.D. earned in Turkey	Ph.D. earned abroad		Offered by Program	ÖYP	Article 35	
METU	2	5	52	4	9	6	15
KTU	2	2	25	-	-	10	12
AU	13	1	33	7		1	19

*Note.* The numbers are as reported by the universities.

**Table 1: Number of faculty members and graduate students as of early 2009**

	Doctorates Granted		Doctorates in Progress	
	Female	Male	Female	Male
METU	11	4	37	15
KTU	6	6	6	19
AU	3	16	13	20
Total	20	26	56	54

**Table 2: Number of doctorates (by gender) in mathematics education**

### The Content and the Demand for Coursework

The course of studies in doctoral programs has two tracks; one for students with bachelor's degree and the other for students with a master's degree. Students who hold a masters degree must complete at least 7 courses – not being less than 21 credits in total – a doctoral qualifying examination, a dissertation proposal, and a dissertation. For those who have been accepted with a bachelor's degree, this program is comprised of a minimum of 42 credits or 14 courses, a doctoral qualifying exam, a dissertation proposal, and a dissertation. The maximum period to complete the course work is 4 semesters for students holding a master's degree, and 6 semesters for students accepted with a bachelor's degree. One credit hour for graduate courses represents an hour of lecture or two hours of laboratory work per week. Each academic year has two semesters. At the end of four or six semesters,

CGPA must be at least 3 out of 4. In addition, dissertations need to be completed in four semesters. If necessary, students may use extra four semesters to complete their dissertations.

Consistent with the faculty in the program, the courses offered vary greatly in the range of mathematics, mathematics education and other topics (research courses and/or general education courses). While in KTU there are no compulsory courses, at METU some research courses such as statistics and research methods and at AU courses in science ethics and computer are compulsory. The rest of the courses are electives that are selected either by the students or by the recommendation of the supervisor. Mathematics education faculty at KTU and METU do not offer any mathematics courses, as the mathematics departments offer these courses. The official language in the universities apart from METU is Turkish. However, doctoral students at KTU are required to complete at least two courses conducted in English. All courses and dissertations at METU are conducted in English.

### **The Process and Nature of Doctoral Qualification Examination and Hereafter**

Upon the completion of the coursework, students in each university need to take the doctoral qualifying examination. Doctoral students holding a master's degree must take this exam within their fifth semesters, and students enrolled with a bachelor's degree within the seventh semester at the latest. The doctoral qualifying examinations committee established with five members, one being the student's dissertation supervisor, are responsible to prepare and administer the qualification examinations. Committee members are required to have a doctoral degree. The doctoral qualifying examination consists of a written and oral examination to evaluate students' skills in conducting independent research and their understanding of major concepts and issues in the field.

Doctoral students conduct their dissertation research under the supervision of a faculty member whom has the position as assistant professor or above. If needed a co-supervisor may be appointed. A dissertation supervising committee consisting of three faculty members is also appointed upon successful completion of the doctoral qualifying examinations. Within six months after the qualification exam, each doctoral candidate needs to prepare and defend a dissertation proposal to a committee consisting of three members including the dissertation advisor himself/herself. Candidates are expected to prepare a doctoral dissertation demonstrating somehow an original contribution to the field of mathematics education by using appropriate methodologies.

### **Graduates with Mathematics Education Doctoral Degrees**

The majority of doctoral graduates in mathematics education seek positions in the higher education. For example, based on our survey, we have found out that among 46 doctoral graduates of KTU, METU, and AU in mathematics education programs (see Table 1), 42 work in higher education and others are employed in test

development companies and in schools. Those employed in higher education assume a range of teaching responsibilities including teaching mathematics and/or mathematics education courses offered in the program and conduct research in the field. Considering the number of students progressing in doctoral studies (see Table 1), we assume that the graduates of doctoral programs will contribute not only to higher education, but also to other institutions, such as the Ministry of National Education, schools, or private companies in the future.

## CONCLUSION

In Turkey, about 35 million people (about half of the population) is under the age of 28 (Türkiye İstatistik Kurumu [Turkish Statistics Institute], 2009). Considering that the population growth rate is about 1.013% per year, the need for education is a growing demand in Turkey. In recent years the government and the Council of Higher Education have a determined policy to increase the number of higher education institutions in Turkey. For instance, in 2006, fifteen new public universities were established throughout Turkey. Currently, 62 public universities and 5 private universities have faculties of education. With the trend of establishing new universities, there is an increasing need for faculty development. In this sense, the demand for staff is still evident for the coming few decades. Since some universities are still in the process of gathering together their staff, most of which have the potential to employ the graduates of mathematics education doctoral programs. The statistics we provided in this paper demonstrates that, in recent years, there is an increasing number of students pursuing a doctoral degree in mathematics education. However, considering the rapid increase in the number of higher education institutions, we argue that more efforts are needed to meet the need of growing faculty need. One pathway may be to find ways to enhance the possibilities for pursuing a doctoral degree in mathematics education. Another one is to explore the benefits of doing national or international collaborations in developing new programs or improving the existing ones.

While dealing with the quantity issue, another challenge for the doctoral programs in Turkey is to maintain and improve the quality. Considering the varieties in workplace of graduates, universities should try to increase the diversity in the coursework and graduate research within the mathematics education doctoral programs. In addition, doctoral programs should also try to put more efforts in offering a program of study that reflect the nature of mathematics education in variety of ways. Finally, creating a productive research community of mathematics educators that is well integrated with their international counterparts should be a major goal for the doctoral programs.

Even the Council of Higher Education has general criteria to open graduate programs in universities (YÖK, n.d. -b), the principles to guide the design and implementation of doctoral programs in mathematics education could be established to provide a number of ideas and suggestions regarding doctoral programs in mathematics

education. Besides, national conference on doctoral programs in mathematics education could be set to develop the ideas and suggestions regarding doctoral programs in mathematics education.

## REFERENCES

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Government of Turkey (1929). Ecnebi memleketlere gönderilecek talebe hakkında kanun, 8 April, No. 1416, Ankara: Government of Turkey.
- Government of Turkey (1981). Yüksek öğretim kanunu, 4 November, No. 2547, Ankara: Government of Turkey. Retrieved March 19, 2009 from <http://www.yok.gov.tr/english/law/content.html>
- Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430-511.
- METU Araştırmalar Koordinatörlüğü. (2007). Öğretim üyesi yetiştirme programı v1.0 istatistikleri (Faculty development project statistics). Ankara: METU.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Simsek, H. & Yildirim, A., (2001). The reform of pre-service teacher education in Turkey. In R. G. Sultana (Ed.), *Challenge and change in the Euro-Mediterranean region* (pp. 411–430). New York: Peter Lang.
- Tansel, A., & Güngör, N. D. (2003). ‘Brain drain’ from Turkey: Survey evidence of student non-return. *Career Development International*, 8(2), 52-69.
- Türkiye İstatistik Kurumu. (2009). Adrese dayalı nüfus kayıt sistemi [Address based population registration system]. Haber Bülteni [News Bulletin], Issue 14.
- Türkiye Bilimler Akademisi. (2006). Türkiye’de doktora eğitiminin durumu üzerine görüşler. Retrieved March 19, 2009, from <http://www.tuba.gov.tr/userfiles/image/haber/kurumdan/rapor-1.doc>.
- Yüksek Öğretim Kurulu. (2007). Türkiye’nin yükseköğretim stratejisi. Ankara: Author. Retrieved February 5, 2009, from [http://www.yok.gov.tr/duyuru/yok\\_strateji\\_kitabi.pdf](http://www.yok.gov.tr/duyuru/yok_strateji_kitabi.pdf)
- Yüksek Öğretim Kurulu. (n.d.-a). Üniversiteler [Universities]. Retrieved March 19, 2009, from [http://www.yok.gov.tr/universiteler/uni\\_web.htm](http://www.yok.gov.tr/universiteler/uni_web.htm)
- Yüksek Öğretim Kurulu. (n.d.-b). Yükseköğretim kurulu doktora programı açmak için gerekli ölçütler ve başvuru format. Retrieved March 20, 2009, from [http://www.yok.gov.tr/egitim/lisans\\_doktora/doktora/basvuru\\_formu.htm](http://www.yok.gov.tr/egitim/lisans_doktora/doktora/basvuru_formu.htm)



# CONNECTING THE NOTION OF FOREGROUND IN CRITICAL MATHEMATICS EDUCATION WITH THE THEORY OF HABITUS

Tine Wedege

*School of Education, Malmö University*

*The dialectics between individual and structure is an important issue in any sociomathematical study of students' learning conditions in mathematics education. On the basis of a conception of learning as action and intentionality as a basic element in any action, Skovsmose introduced the notion of the student's foreground as an element in critical mathematics education. The intention is to make visible learning obstacles as a political instead of an individual phenomenon based only on the student's social and cultural background. In this paper, a discussion is initiated to re-establish the significance of students' background by integrating the notion of foreground with Bourdieu's theory of habitus as systems of dispositions as principles of generating and structuring practices and representations.*

Keywords: connecting theories, critical mathematics education, foreground, habitus,

## INTRODUCTION

In mathematics education research, the grounding questions concern people's cognitive, affective and social relationships with mathematics. Conditions for students to learn mathematics is one of the key issues to be studied whether the focus is learning environments established in the mathematics classroom; e.g. didactical situations (Brousseau, 1986) or landscapes of learning (Alrø, Skovsmose & Valero, 2007); or the focus is students' motives for learning mathematics; e.g. motivation (Wæge, 2010) or instrumental and social rationale (Mellin-Olsen, 1987). In sociology, the grounding questions concern the connection between people and society or, from a philosophical point of view, the dialectics between individual and structure. In a sociomathematical study of learning conditions, this dialectics is an overarching theme because the societal context for teaching, learning and knowing mathematics is taken seriously into account (Wedge, 2010).

In a recent overview of the sociomathematical research field it is stated that students' positioning may cause structural disadvantage for learning mathematics:

It has long been recognised that neither education systems in general nor mathematics education in particular is neutral in terms of learners' positionings with respect to class, gender, "race", ethnicity and global position. With respect to each of these (and other) positionings, some learners are systemically, structurally disadvantaged. (Povey & Zevenbergen, 2008, p. 4)

Skovsmose (2005) has pointed out that learning obstacles are often identified in students' social and cultural background and thus, in his understanding, "individualised". Skovsmose's countermove is to introduce the notion of students'

foreground but I find it important analytically to connect people's motives for learning – or not learning – mathematics to their lived lives in order to investigate the dialectics between individual and structure. During my first reading of Skovsmose's (1994) "Towards a philosophy of critical mathematics education", I wondered why he did not have any reference to the Bourdieuan concept of habitus when the term "dispositions" and the meaning attached to this term through his definition of foreground point in the same direction: "Dispositions are grounded in the social objectivity of the individual, and simultaneously produced by the individual, partly as a consequence of the actions performed by the individual" (p.180), and the future of different social groups of students "is present in the dispositions of the students" (p. 191).

The purpose of this paper is to initiate a discussion about the possibility of integrating locally a concept of foreground in the theory of habitus. I will do this by presenting and discussing the compatibility of the notion of foreground in critical mathematics education respectively the concept of habitus in Bourdieu's sociology. As a part of this, I will try, in an analysis of a narrative interview, to link habitus and foreground of a Swedish student in vocational education.

## THE NOTION OF FOREGROUND

Intentionality was the pivotal point when Skovsmose (1994) introduced the notion of foreground in his book "Towards a philosophy of mathematics education" where three notions are interconnected: learning as action, dispositions and intentions. His point of departure is that knowledge development or learning is an act and, as such, it requires indeterminism: the acting person must be in a situation where choice is possible. To be an action, an activity must be related to an intention. A person acting must have some idea about goals and reasons for obtaining them. Skovsmose sees learning as caused by the intentions of the learner, thus, he does not see enculturation and socialisation as learning. Dispositions are seen as resources for intentions: "Intentions are grounded in a landscape of pre-intentions or *dispositions*" (p. 179). As Skovsmose does not see the background (the socially constructed network of relationships and meanings which belong to the history of the person) as the only source of intentions he divides the dispositions into a "background" and a "foreground". He finds the *foreground* equally important and, in 1994, defined it as

the possibilities which the social situation makes available for the individual to perceive as his or her possibilities. (...) The foreground is that set of possibilities which the social situation reveals to the individual. (Skovsmose, 1994, p. 179)

Skovsmose stresses that the background as well as the foreground are interpreted and organised by the individual. However, at first, the foreground of a person was

defined as the opportunities in future life made available to her/him by society. In 2002<sup>59</sup>, Skovsmose clarifies the functioning of the individual:

By “foreground” of a person I understand the opportunities, which the social, political and cultural situation provides for this person. However, not the opportunities as they might exist in any “objective” form, but the opportunities as perceived by a person. (Skovsmose, 2005, p. 6)

In this article, the notion of foreground is presented as the pivotal point in the introduction of learning obstacles as a political phenomenon. And foreground becomes the key word in one of the principles for the pedagogy of critical mathematics education. “Third, critical mathematics education must be aware of the situation of the students. (...) A way of establishing this awareness is to consider not only the background of the students but also their foreground” (Skovsmose, 2006, p. 47).

Foreground is introduced – and used – by Skovsmose (1994, 2005, 2006) as a notion not as a concept, i.e. an element of a theory. But students’ foregrounds have been investigated empirically in two doctoral thesis which have fleshed out the notion (Baber, 2007, Lange, 2009). In the publication “Inter-viewing foregrounds”, Alrø, Skovsmose and Valero (2007) have continued the work by giving a “conceptual definition” of students’ foregrounds. They stress that the concept allows linking two of the key conceptual elements of educational theory, learning and meaning, and that foreground is a concept emphasizing the socio-political nature of education and learning. It is actually the notion of dispositions – defined by Skovsmose (1994) as pre-intentions –, which disappeared from his writing (2005, 2006), that links foreground with learning. Alrø, Skovsmose and Valero (2007) point to the basic principle in the theory of learning-as-action, which presupposes the person’s readiness to find motives for engaging in action; i.e. the person’s *dispositions*:

Dispositions can be seen as the constant interplay between a person’s *background* and *foreground*. The background of a person is the person’s previous experiences given his or her involvement with the cultural and socio-political context. In contrast to some definitions of context which see background almost as an objective set of personal dispositions given by one’s positioning in different social structures, we consider background to be a dynamic construction in which the person is constantly giving meaning to previous experiences, some of which may have a structural character given by the person’s positioning in social structures. The foreground, as previously defined, is also an element in the formation of dispositions. The person is all the time finding reasons to get engaged in learning activities not only because of the permanent reinterpretation of his or her background, but also because of the constant consideration of his or her foreground. That is, the person connects previous experiences with future possible

---

<sup>59</sup> The article “Foregrounds and politics of learning obstacles” was published 2002 in a preprint: Publication no 35, Centre for research in learning mathematics, Roskilde University.

scenarios for action (pp. 7-8).

The authors see a person's dispositions as readiness to engage in intentional practice or action and they associate them selves from understanding the background as decisive. However, the awareness is present of students' positioning resulting in structural and systematic disadvantages, as well as advantages, in mathematics education. "Dispositions", which are objectively rooted but mediated by the individual, thus expressing subjectivity (Skovsmose, 1994, p. 179), is the term making it relevant to think about connecting foreground and habitus. However, the very idea of integrating foreground and habitus is based on the central place of action in both frameworks and the related critique of structural determinism.

## THE THEORY OF HABITUS

"Socialization" is a key term – and concept – in sociology meaning the process of internalizing or of incorporating norms, traditions and ideologies which provides people with habits and dispositions necessary for participating within their culture and society. Like this, socialization is one of the mechanisms ensuring the reproduction of the society. In Danish and Swedish, a distinction is made between socialization as a process (socialisering) and socialization as a result (socialisation). Using the term "habitus", Bourdieu has conceptualised the result of socialization.

Many theories of socialization are based on a fundamental dichotomy: out there in society there are norms which are internalized in the individual. In Bourdieu's sociology people are most often agents in the etymological meaning of the word (Lat.: *agens*, *agere* = act). His project has consisted in combining studies of human experience with studies of the objective condition under which the same people live (Broady, 1991). Thus instead of "internalization", Bourdieu (1980) employs the term "incorporation", and the theory of habitus is incompatible with the idea of people as "bearers" of social structures and norms. In his work, according to Broady (1991) there is no direct, unmediated influence from social structures and norms to individuals. At this point, it is notable that Bourdieu's notion of socialization is consistent with the idea of social background in Critical Mathematics Education as presented above.

*Habitus* is the concept developed and employed by Bourdieu for a system of dispositions which allow the individual to act, think and orient her or himself in the social world. People's habitus is incorporated through the life they have lived up to the present and consists of systems of durable, transposable dispositions as principles of generating and structuring practices and representations:

The conditionings associated with a particular class of conditions of existence produce *habitus*, systems of durable and transposable *dispositions*, structured structures predisposed to function as structuring structures, that is, as principles generating and organising practices and representations which can be objectively adapted at their aim without presupposing the conscious aiming at goals and without the express mastery of

the operations necessary to attain them. Objectively “regulated” and “regular” without being, by no means, the product of obeying rules they are collectively orchestrated without being the product of the organising action of a concert leader. (Bourdieu, 1980, pp. 88-89, my translation)

The term “dispositions” is only defined implicitly by Bourdieu. According to the dictionary it means “ability to”, “instinct”, “taste”, “orientation” etc., but, as it appears from the definition of habitus, it is not a case of innate, inherited or natural abilities. To make this visible, I have chosen to translate “disposition” into the Danish word “tilbøjelighed” (En.: inclination). The term “system” stands for a structured amount which constitutes a whole. Habitus (as a system of dispositions) contributes to the social world being recreated or changed from time to time when there is disagreement between the people’s habitus and the social world. The dispositions which constitute habitus are “durable” (Fr.: durables). This means that although they are tenacious, they are not permanent. Bourdieu (1994) has discussed precisely these two matters in an answer to attacks on him by critics for determinism in his theories.

There are several reasons for importing habitus as an analytical concept in mathematics education and trying to connect foreground with the habitus theory:

- The theory of habitus has to do with other than rational, conscious considerations as a basis of actions and perceptions, and it provides a theoretical starting point for criticism of the ideology of inherited abilities.
- Habitus is durable but it undergoes transformations. Dispositions point both backwards and forwards in the current situation of the individual.
- The concept of habitus aims at an action-orientation anchored in the individual and can simultaneously explain non-actions. Furthermore, habitus “allows for economy of intention” (Bourdieu, 1980, p. 97) (see Wedege, 1999).

I would claim that the notion of foreground, developed and belonging in critical mathematics education can be integrated as a theoretical element with habitus in a *problematique* of mathematics education. Bourdieu (1994) emphasises that the theory of habitus is not “a grand theory”, but merely a theory of action or practice. The theory has to do with why we act and think as we do. It does not answer the question of how the system of dispositions is created, and how habitus could be changed in a (pedagogical) practice<sup>60</sup>. There is no sense in seeing habitus as the

---

<sup>60</sup> Note that at the end of the 60s the term, habitus, achieved a central place in Bourdieu's terminology, where it is presented as product in the pedagogical activity in the book “La reproduction” from 1970 about the function of the educational system in social reproduction. Here a durable formation and habitus achieve equal status (Bourdieu & Passeron, 1970, pp. 46-47). Several references in educational literature refer to this work and thus deal with habitus as a result of formal education.

result of an isolated pedagogical activity (a product of learning). But it is fruitful to employ the concept of habitus in the work of descriptive analysis about the conditions for people learning mathematics, precisely because habitus is formed through impressions and acquisition, either directly where the objective structures are experienced and leave traces, or indirectly when we are exposed to and engaged in activities that make impressions (see Wedege, 1999).

Bourdieu has not studied people's sense of doing mathematics (Fr: *le sens de pratique mathématique*) and, thus, he has not developed a concept of "mathematical habitus" a notion introduced by Zevenbergen in a study of implications of ability grouping in school (the middle years). Zevenbergen (2005, p. 608) proposes that when the practice of ability grouping "is enacted in mathematics classrooms it can create a learning environment that becomes internalized as a mathematical *habitus*." However, this structuralistic interpretation of habitus is neither compatible with the understanding of the dialectics between individual and structure in Critical Mathematics Education nor in the work of Bourdieu. Furthermore, Zevenbergen presents the mathematical habitus as a product of school mathematical practices alone. The data from interviews with 96 students from six schools serving upper-, middle- to working-class families were explored in terms of gender, school and year-level, not in terms of social class. Thus, I do not find that this notion of mathematical habitus resonates with the sociological theory of habitus.

## **LINKING BEN'S FOREGROUND FOR LEARNING MATHEMATICS WITH HIS HABITUS**

As a part of an essay, one of my students, Jonas Lovén (2010) did a narrative interview with a male student at the vocational programme at higher secondary school in Sweden. The purpose of doing a mathematical life history interview was to test the analytical power of combining the concept of habitus with the notion of foreground. Carrying out the interview, Lovén followed the methodology of the narrative biographic interview as developed by the German sociologist Fritz Schütze (Andersen & Larsen, 2001). The interview with Ben, as Lovén has named the student in his essay, was taped and transcript and they have both approved my use of the transcript for further analysis. The disadvantage of this procedure is, among other things that I did not have the opportunity to follow up the interviewee's narrative. But the advantage of a young Swedish teacher student as interviewer is a reduction of the built-in asymmetry in any inquiry and hence a diminution of what Bourdieu (1993) has called symbolic violence. However, the mere fact that Lovén has a position as a future mathematics teacher and as such has been a trainee in the mathematics classroom of Ben seems to have an impact on his discourse when he – as an interviewee – answered the question about mathematics in his life, in a very favourable way.

## Working Group 10

The initial question put by the interviewer is “Could you please tell about mathematics in your life? Quite simply – you may begin precisely where you want to and try to recount what comes into your mind” (l. 6-7). Ben seems to join the mathematics teacher discourse of “mathematics is everywhere”:

One uses math, yeah, every day – in principle. (I: Mmm) Yeah, where is it (Pause) Yeah, it does not work without math – nothing works. It is something you have to know and just carry on. Start in an early age. (Pause) Yeah ... (Pause). (I: Yes, precisely.) Yeah, later on it is often in the shops; these unreliable shop assistants and so on. It is fantastic being able to think and to do the sums rapidly. If they take one or two Krona from you. Not much – maybe, but (...) Quite often I am surfing on my mobile. Then it is good to calculate how much the cost is a minute if you do not have free surf. Which I do not have. Then I have to calculate a little, and eh.. you are on Facebook every day so.. So it is good to know it ... that the money does not flow away just like that. What more can one tell? Yeah a great hobby, I am playing golf (...) (l. 41-63)

And Ben continues by telling about the scoring in golf and again about not being cheated, this time by his father. Ben’s narrative takes off when he was “a little boy” just at the school start with supportive parents at home: “At that time, it was very cool. Everything was pretty simple, at the beginning. But after some years. Everything new is difficult. (Pause)” (l. 12-13). A central figure in Ben’s narrative is his grandfather, who also supported him in mathematics. He is introduced like this: “Even my grandfather [helped me]; he is a genius in mathematics. So already as a small kid I started to calculate” (l. 17-18, [my insertion]).

“Yes, OK, I ... Yeah, one has been doing mathematics for 11 or 12 years now”, Ben states (l. 32). School mathematics has been a part of his lived life over a long period and, together with the social interaction in out-of-school situations, influenced the socialization process resulting in his dispositions for doing mathematics today and tomorrow. In Ben’s biographic narrative about mathematics, two persons are important: First his grandfather, who supported him also by serving as a great example, and second Magnus, who owned a store where he had a job as a 15-years old kid. Together they did a piece of joinery:

I think that it was much fun. Then I decided to aim for joiner and to apply for this vocational school to be a joiner, but later on you circle around – you have to try everything from construction work to house painting. And I fell for the sheet metal work (...) New exciting stuff, and more great challenges and I have nothing against solving difficult problems (l. 134-139)

Ben tells that he had some difficulties with mathematics in grade 9 but the grandfather helped him and later on his uncle, who is graduate engineer and has a “sharp brain”. “Unfortunately”, Ben just passed in mathematics at the end of lower secondary:

... but I knew that I could do better and then I came here in August 2009 and started with

the mathematics here. And I do not find it difficult at all because it is mostly repetition from lower secondary. (...) But when you are in the workshop it is not only  $1 + 1 = 2$ . As I told you before it is diameter multiplied by pi. And how many degrees you have to twist a disc wind (...) It is cool, really cool. (l. 145-150)

In Ben's narrative, the link between his habitus and his foreground for learning mathematics is visible. His lived life resulting in habitus acts as the background for the interpretation of his future life (foreground). Likewise, his experiences in the vocational school opens up for a foreground with sheet metal work. When Ben at the end of the interview is asked if he has any plans for a higher education, he refers to the fact that many of his friends have already left school:

... and mathematics above all because they think that it is damned boring. But I have nothing against it. I am doing fine. It is showing up at the test, you have to learn, it's just like that. Yes ... no, I do not care what others are thinking. It is my life. (...) I do not think that university is something for me. In fact, I have never been considering it, I think ... No ...” (l. 263-270).

In the Swedish society, the possibility of a higher education is available for Ben but this is not a part of his foreground.

## **INTEGRATING FOREGROUND AS A CONCEPT IN THE THEORY OF HABITUS**

With this paper, I hope to initiate a discussion of possibility and potential of connecting the notion of foreground as a theoretical element in Critical Mathematics Education with the theory of habitus. I have argued for the compatibility and the connecting strategy suggested is *integrating locally*, i.e. some elements from one theoretical structure are integrated in a more elaborated theory, and the aim is theory development (see Wedege, 2010).

At first, the notion of students' foregrounds was based on anecdotic evidence (Skovsmose, 1994). Later it is given a conceptual definition based on qualitative empirical studies (Alrø, Skovsmose, Valero, 2007). Broady (1991) has argued that the key concepts in Bourdieu's sociology should be regarded as research tools or condensed research programmes. They get their full meaning when they are set in motion as tools in investigations. The notion of foreground has inspired research within Critical Mathematics Education. I claim that the concept of students' foregrounds locally integrated in the theory of habitus should be regarded as a research tool and I see the possibility of further theoretical development based on a combination of future large scale quantitative studies and qualitative studies in mathematics education.

When theories are imported from sociology, psychology, anthropology etc. into mathematics education they are adapted and reconstructed, in time. The notion of habitus has guided some studies in mathematics education (e.g. Gates, 2003;



Wedge, 1999; Zevenbergen, 2005). I hope that local integration of foreground, which originates from the “homebrewed” theory of Critical Mathematics Education, into the theory of habitus can strengthen both concepts as research tools in mathematics education.

## REFERENCES

- Alrø, H.; Skovsmose, O.; & Valero, P. (2007). *Inter-viewing foregrounds*. Aalborg: Working papers on learning 5, Aalborg University
- Andersen, A. S.; & Larsen, K. (2001). *Det narrative livshistoriske interview: Introduktion til Fritz Schützes teoretiske og metodologiske arbejde*. Roskilde: Småskrift nr. 11, Livshistorieprojektet, Roskilde University.
- Baber, S. A. (2007). *Interplay of citizenship, education and mathematics: Formation of foregrounds of Pakistani immigrants in Denmark*. (Doctoral thesis). Aalborg: Department of Education, Learning and Philosophy, Aalborg University
- Bourdieu, P. ; & Passeron, J.-C. (1970). *La reproduction : Éléments pour une théorie du système d'enseignement*. Paris: Les Éditions de Minuit.
- Bourdieu, P. (1980). *Le sens pratique*. Paris: Les Éditions de Minuit.
- Bourdieu, P. (1993). Comprendre. In Bourdieu, P. et al., *La misère du monde* (pp. 1388-1421). Paris: Editions du Seuil.
- Bourdieu, P. (1994). *In other words*. Cambridge: Politic Press.
- Broady, D. (1991) *Sociologi och epistemologi: Om Pierre Bourdieus författarskap och den historiska epistemologin*. Stockholm: HLS förlag.
- Brousseau, G. (1986). Fondements et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques*, 7(2), 33-115.
- Gates, P. (2003). Excavating and mapping the social landscape of beliefs. In P. Valero; & O. Skovmose (Eds.), *Proceedings of the Third International Mathematics Education and Society Conference, Helsingør, Denmark, 2002* (pp. 317-325). Roskilde: Centre for Research in Learning Mathematics, Roskilde University.
- Lange, T. (2009). *Difficulties, meaning and marginalisation in mathematics learning as seen through children's eyes*. (Doctoral thesis). Aalborg: Department of Education, Learning and Philosophy, Aalborg University.
- Lovén, J. (2010). *”Ja man har ju hållit på med matematik i 11 eller 12 år nu” – En elevs matematiska livsberättelse utifrån begreppen habitus och förgrund*. (Examensarbete). Malmö: Malmö Högskola, Lärarutbildningen.
- Mellin-Olsen, S. (1987). *The politics of mathematics education*. Dordrecht: Kluwer Academic Publisher.

Povey, H.; & Zevenbergen, R. (2008). Mathematics education and society: An overview. Paper presented at ICMI, Rome 2008, WG3.

<http://www.unige.ch/math/EnsMath/Rome2008/>

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer Academics Publishers.

Skovsmose, O. (2005). Foregrounds and politics of learning obstacles. *For the Learning of Mathematics*, 25, 4-10.

Skovsmose, O. (2006). *Travelling through education: Uncertainty, mathematics, responsibility*. Rotterdam: Sense Publishers.

Wedge, T. (1999). To know or not to know - mathematics, that is a question of context. *Educational Studies in Mathematics Education*, 39(1-3), 205-227.

Wedge, T. (2010). Connecting theories in mathematics education: From bricolage to professionalism. *Nordic Studies in Mathematics Education*, 15(1), 59-78.

Wedge, T. (2010). Sociomathematics: a subject field and a research field. In U. Gellert, E. Jablonka & C. Morgan (Eds.), *Proceedings of the sixth international Mathematics Education and Society conference, Berlin, Germany, 20-25 March 2010* (pp. 449-458). Berlin: Freie Universität Berlin.

Wæge, K. (2009). *Elevenes motivasjon for å lære matematikk og undersøkende matematikkundervisning*. (Doctoral thesis). Trondheim: Department of mathematical sciences, Norwegian university of technology and science.

Zevenbergen, R. (2005). The construction of mathematical *habitus*: Implications of ability grouping in the middle years. *Journal for Curriculum Studies*, 37(5), 607-619.

# **INTRODUCTION TO THE PAPERS OF WG 11: COMPARATIVE STUDIES IN MATHEMATICS EDUCATION**

Eva Jablonka<sup>\*</sup> and Paul Andrews<sup>\*\*</sup>

<sup>\*</sup>Luleå University of Technology, Sweden, <sup>\*\*</sup>University of Cambridge, UK

## **COMPARATIVE STUDIES**

As we interpret “comparative” in a broad sense, including studies that document, analyse, contrast or juxtapose similarities and differences across all aspects and levels of mathematics education, some time has been devoted to discussions focussing on the identity of the group “comparative studies in mathematics education”. As comparative studies are dependant on the changing theoretical and methodological standards established in research in mathematics education in general, delineation will always be in flux. However, the view was shared that there are two “branches”, one deriving from documentation, description and analysis of classroom and school practices, and another stemming from large-scale international achievement surveys. Classifying a study as “comparative” means that the comparative method is emphasised, but comparison cannot be a goal in itself.

The papers and posters contributed to the group revealed a wide range in the aspects of mathematics education they addressed. These included teaching practices related to a particular mathematical topic, metaphors used by pre-service mathematics teachers, groups of low-achieving and high-achieving students, a teacher training course implemented in different countries, test items used in different countries, task representations in calculus textbooks, conceptions of problem-solving, theories originating from different cultural traditions, teacher education and student teaching in different countries, and the relationship between students’ self-regulation, self-efficacy and mathematical competence.

## **CULTURALLY SENSITIVE RESEARCH**

One issue discussed throughout the sessions was how “culturally sensitive research” can be achieved in comparative studies. The notion is taken from Linda Tillman (2006), a US-based sociologist who in the context of her work describes how she sets out to understand the lives of African-American women. According to her, culturally sensitive research possesses five key characteristics. Firstly, it employs culturally congruent research methods. That is, a range of qualitative methods to “investigate and capture a holistic contextualized picture of the social, political, economic and educational factors” (p. 269) influencing participants’ experiences. Secondly, it is sensitive to the culturally specific knowledge of the researched. That is, the “researcher is committed to and accepts the responsibility for maintaining the cultural integrity of the participants and other members of the community” (p. 269). Thirdly, it seeks cultural resistance to theoretical dominance in its attempts to

counter unequal power-relationships. It privileges the voice of the cultural group over more general and frequently dominant theories and, in so doing, “claims of neutrality and objectivity are questioned” (p. 270). Fourthly, it attempts culturally sensitive data interpretations in its positioning of the experiential knowledge of participants at the heart of the research. Fifthly, culturally sensitive research aims to produce culturally informed theory and practice. That is, “Researchers rely on participants’ perspectives and cultural understandings of the phenomena under study to establish connections between espoused theory and reality and then generate theory based on these... perspectives” (p. 271). In our discussions we found that not all our work would be amenable to Tillman's framework, but it helped to raise questions concerning how we undertake comparative research, whether it be a comparison of two classrooms in the same school or two different educational systems. In particular, we asked how we would be able to suspend our own culturally constructed values and expectations when we examine the classrooms of “others”.

We distinguished between form and function and noted that while insiders can recognise the function of events in a practice (culture), outsiders first of all attend to the form. One group member came up with the example of school uniforms as a form with different functions, such as covering poverty in the Philippines, and conveying an aura of discipline and learning in England. We acknowledged the value of both, the insiders’ and outsiders’ perspectives, and noted that as insiders we share blind spots due to cultural blindness.

We discussed, also, notions of “culture” and “multicultural society”. We concluded that cultural homogeneity cannot be assumed in national societies, even if there are shared political institutions. We employed the term identity rather than the term cultural affiliation, and acknowledged that individuals have multiple identities.

### **SPECIFIC TOPICS**

The sessions were structured around possible common points of discussion thought likely to arise from the points made in the papers and poster presentations grouped together for a particular session.

The first topic concerned the development of “culturally sensitive research instruments”, as one of the contributions dealt with item performance in different cultural contexts. We acknowledged the necessity of exploring the validity of the test items imported from another culture. Also, we discussed the conditions under which such an import would make sense at all. We also discussed the labels used in research reports for outcomes related to a group. For example, the data produced in a study may be from Hungary, but if all the schools are in Budapest this label would seem too general.

More generally, the cultural bias of concepts in a research study was discussed. As an example, we used the cultural bias in answers to question about “self-concept” and “self-efficacy” and the relation between attitudes and performance that is often

investigated in large-scale studies. As reported in such surveys, some Asian countries' students' show a low mathematical self-concept but high performance. This apparently inverse relation between the two constructs can be an outcome of a value attributed to unassumingness in some cultures. Consequently, such measures cannot be taken as culturally neutral.

In another session, we discussed the role of cultural traditions in theories and didactical principles. We asserted a central role to metaphors in a culture and suggested to ask whether there are structural differences in the metaphors used in different cultures. Further, the role of metaphor as fundamental to learning as well as more local metaphors in mathematics classrooms were explored, such as the use of the balance scale.

A more general issue in this context is whether we can successfully use a local home-grown theory from one culture to interpret data or ask questions in another culture. The French anthropological theory of didactics served as an example. We considered examples in which the use of theoretical frameworks linked to researchers' cultural heritage is a strength and examples where it is a limitation. Limitations could be overcome by cultural exchange between researchers, such as through the group meetings at the CERME. We found it valuable to treat theories comparatively as culturally situated.

### **ISSUES TO BE PURSUED FURTHER**

In the course of the meetings, we identified some issues that are worth pursuing in the working group on comparative studies in mathematics education. Having noticed an extraordinary range of comparative studies, from very small to the very large, the potential of comparison should be further explored. We noticed that the purpose of comparison might not always be immediate, but it should be clearly identified in the beginning of a study in order to ensure coherence and permit planning. The issue of the overall design of comparative studies needs further attention.

We agreed that a comparison often opens our eyes to old problems of teaching and learning mathematics in classrooms, but that comparison can also assist in basic, fundamental research that opens up new issues. Generally, comparison can be seen as a tool for remaining a reflective stance in research activities. We agreed that the role and potential of comparative approaches in theory building needs to be explored further.

Although matching an appropriate question to an appropriate unit of analysis is a major problem in all research studies, we agreed that it comes out more easily in comparative studies because the sites are so different. Consequently, methodological questions related to the unit of analysis in comparative studies deserve more discussion.

## Working Group 11

Further, methodological issues concerning adaptations of research instruments and translation of data and outcomes deserve more attention in comparative studies. Large-scale comparative surveys were mentioned (such as TIMSS and PISA) in a sporadic discussion about making instruments culturally more sensitive. We agreed that the goals and political influence as well as issues of validity and reliability of these studies should be discussed further.

### REFERENCES

Tillman, L. C. (2006). Researching and writing from an African-American perspective: reflective notes on three research studies. *International Journal of Qualitative Studies in Education*, 19(3), 265 - 287.

# THE CASE OF CALCULUS: COMPARATIVE LOOK AT TASK REPRESENTATION IN TEXTBOOKS

Emmanuel Adu-tutu Bofah, Markku Hannula

Department of Education, University of Helsinki, Finland

*Abstract: Many mathematics teachers, across the world, rely heavily on textbooks in their day-to-day teaching. This study compared the mathematical tasks in the chapters on calculus (derivatives) in two mathematics textbooks adopted for the International Schools International Baccalaureate (IB) in Finland: a calculus textbook adopted for examination at the University of Helsinki and a Ghanaian Senior Secondary textbook. Eight classification schemes were used for the study and each mathematics task was coded into one of the categories.*

Key words: Textbooks, representations, problem solving, comparative studies.

## INTRODUCTION

Students' understanding of calculus concepts lays a foundation for their future study of advanced mathematics, science, and engineering. Calculus textbooks show a very definite attempt to stress the idea of meanings from the very beginning and reviewers of textbooks have made reference to the simplicity and understanding with which the subject is presented (Dragoo, 1945).

Textbooks are an important part of teaching mathematics in classrooms in Finland and Ghana as well as many other countries. It seems natural that textbooks are designed for the purpose of helping classroom teachers to organize their teaching. In this sense, school mathematics textbooks are used in varying ways and to varying degrees by teachers and schools. However, a textbook should be a tool, not a plan, but in reality many teachers rely heavily on textbooks to provide sample problems, diagrams, worked examples, and homework assignments. . The influence of mathematics textbooks upon what is taught in school mathematics has been highlighted in research into teachers' use of curriculum materials.

## THE AIMS OF THE STUDY

The aim of the study was to compare the tasks in the calculus chapters (derivatives) in two Finnish mathematics textbooks for the International Schools (International Baccalaureate (IB)), an adopted calculus textbook for examination at University of Helsinki and Ghanaian Secondary School calculus textbook chapters. This study is not a country comparison but was influenced by the fact that, while existing textbook studies have been focused on content analysis (Schmidt, Mcknight, Valverde, Houg, & Wiley, 1997; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002), less attention has been placed on the analysis of tasks presented in textbooks (Stigler, Fuson, Ham, & Myong, 1986; Zhu & Fan, 2006). There have been other studies on use and roles of mathematic textbooks (Reys, Reys, & Chávez, 2004) and textbooks and the

intended curriculum (Johansson, 2003; Chávez-López, 2003), but very few studies on secondary school advanced mathematics task representations in textbooks. This study therefore is important in that it raises the issue of textbook use again, but in previously unexamined circumstances.

During the past few decades, there has been much research on student achievement in calculus and students' understanding of calculus concepts. These studies have focused on students' understanding of concepts such as function and limit (Tall & Vinner, 1981; Vinner, 1989), calculus problem solving (Selden, Selden, Hauk, & Mason, 2000), student understanding of the derivative and integral concepts (Dragoo, 1945; Ferrini-Mundy & Gaudard, 1992; Häikiöniemi, 2006; Tall, 1996), teaching of calculus (Dragoo, 1945), and calculus reform (Ganter, 2001). However, little research has been undertaken on the ways in which problems in calculus textbooks have been represented. The following research question was thus formulated:

33 How are mathematical tasks represented and distributed in the selected textbooks?

## **SIGNIFICANCE OF THE STUDY**

This study will offer insights into existing textbooks' representation of calculus-related tasks, particularly from the hitherto unexamined perspective of comparing European and African textbooks.. In so doing the study will highlight the diversity of tasks presented to students located in different educational systems. To achieve these objectives the study combines Tall's (1996) spectrum of representations in functions and calculus and the coding system from Zhu and Fan (2006) to provide a common coding framework system for analysing tasks in calculus textbooks.

## **METHODOLOGY**

### **Selection of textbooks**

The study draws on two mathematics textbooks used in Finnish IB schools; Coad, et al. (2004), used in the Helsinki IB and abbreviated as HIBTB and Cirrito (2000), used in the Turku IB and abbreviated as TIBTB. It draws also on one calculus textbook adopted for the entrance examination at the University of Helsinki (Mendelson, 1997), abbreviated HUTB, and the national mathematics textbook used in Ghana (Eshun, et al., 1992), abbreviated as GTB. The full references to the selected textbooks are in Appendix A.

Schaum's Outline of Beginning Calculus (Mendelson, 1997) was selected for the study because it was a calculus reform textbook and was used by the University of Helsinki as a reference book for entrance examination. The IB books selected for the study were books adopted for use by two leading international schools in Finland, Turku International School and Helsinki International School. The Ghanaian



textbook is the approved ministry of education textbook. All the textbooks are intended for mathematics teaching and learning at the same secondary level. Most students using the textbooks are between the ages of 15-18 years and, calculus is introduced at that level. The decision to focus on the presentation of mathematical tasks in textbooks was based on the fact that solving textbook mathematical tasks is a typical element of the learning of the corresponding mathematical content.

In this study, only those mathematical tasks that did not have add-on solutions or answers presented were analysed. The tasks mostly appear within or immediately after the topic contents, and they often appear under such headings as: exercise, review set, investigation, past examination questions, examination style questions, self-assessment test, supplementary problems and discussion.

### Data Collection and coding of problem types

In this study, we have adopted the definition of a mathematical problem from Zhu and Fan (2006) as a situation that requires a decision and /or answer, no matter if the solution is readily available or not to the potential problem solver. However, we prefer to call these exercises or tasks rather than problems. This definition suits the study and operates in textbook problem analysis. To solve the ambiguity in the terms mathematical problem, mathematical task and exercise, in this study they have been used interchangeably.

The problem classification is based on the categories described by Zhu & Fan, (2006). In addition we used Tall's (1996) spectrum of representations in functions and calculus. The following eight coding systems were used for the study.

*Routine Task (RT)*: Is one for which solvers can follow a certain known algorithm, formula, or procedure to obtain the solution, and usually, the path to the solution is immediately evident. They involve the use of knowledge previously acquired and practiced, have a low degree of mathematical complexity and do not ask for arguments in the response. Their interpretation is straightforward and they do not require the use of different kinds of representations.

*Example:* Find  $\frac{dM}{dt}$  if  $M = t^3 - 3t^2 + 1$  (HIBTB, p.627)

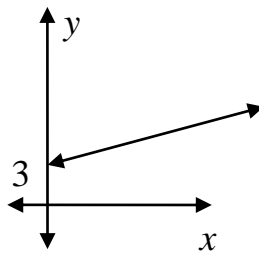
*Non-routine Task (NRT)*: Is a task that cannot be solved by merely applying a standard algorithm, formula, or procedure, which is usually readily available to problem solvers. They require the establishment of relationships or chains of reasoning, procedures, or computations and ask for a certain level of interpretation. They may include a request for justification or a simple explanation. They tend to have a closed structure and to be presented in a familiar or almost familiar context. They are mostly to introduce ideas, to deepen and extend understandings of algorithms, skills, and concepts and to motivate and challenge students.

*Example:* The tangent to the curve  $y=x^3+bx$  at the point where  $x=2$  passes through the point  $(-1, 11)$  and  $(3,-29)$ . Find the values of the constant  $b$ .

(GTB, p.128)

*Visual/graphic/tabular task (VT):* Mostly concerned with the task of pictorialization, i.e. visualizing, graphing, and tabulation; (needs graphs or pictures to solve the task).

*Example:* Find the intervals where  $f(x)$  is (i) increasing (ii) decreasing.



(HIBTB, p. 622)

*Numerical Task (NT):* Mostly concerns problems of estimating, approximating, most but not all questions demand numerical absolute answers.

*Example:* The radius of a circle is 5cm. Find the change in the area if the change in the radius is (a) 0.1cm (b) 0.03cm (c) 0.5cm (GHTB, p. 135).

*Symbolic Task (ST):* Mostly concern with tasks of manipulating, and limiting. These types of tasks mostly demand symbolic representative answers. This includes tasks that involve the use of manipulation skills in arithmetic and algebraic procedures.

*Example:* find, from first principles, the derivatives of  $f(x)$  where  $f(x)$  is:

a)  $\frac{1}{x^2}$       b)  $\frac{3}{x^3}$       □

(HIBTB, p. 614).

*Non-Traditional Task (NTT):* These are problem posing problems, project tasks, activity tasks, investigation problems and derived problems. This task normally leads students to predict, derive, and comes out with concrete understanding of a concept. It also includes activity problems linked to the CD attached to the textbook.

*Example:* A farmer wishes to fence off a rectangular paddock using an existing stretch of a river as one side. The total length of wiring available is 100m. Let  $x$ m and  $y$ m denote the length and width of this rectangular paddock respectively, and let  $A$ m<sup>2</sup> denote its area. Obtain an expression for  $y$  in terms of  $x$ .

Find an expression for  $A$  in terms of  $x$ , stating any restrictions on  $x$ .

Determine the dimensions which will maximize the area enclosed by this rectangle (HUTB, p. 112).

## Working Group 11

*Formal/Purely Mathematical Task (PT):* Deduction, defining, deducing; questions which need purely calculus concepts to solve the problem. They are more complex and require a high level of interpretation and reasoning, ask for a solution that involves the coordination of several steps, and often demand a response with some written communication and argumentation. Their structure is often open or semi open and they are generally presented in less familiar situations.

*Example:* For each of the functions,  $f$ , given below, find the gradient of the secant joining the points  $P(a, f(a))$  and  $Q(a + h, f(a + h))$  and hence deduce the gradient of the tangent drawn at the point  $P$ .

$$f(x)=x, \quad (b) f(x)=x^2, \quad (c) f(x)=x^3, \quad (d) f(x)=x^4$$

Hence deduce the gradient of the tangent drawn at the point  $P(a, f(a))$  for the function  $f(x) = x^n$ ,  $n \in \mathbb{N}$  (HUTB, p.125).

*Calculator/PC Task (CT):* It includes questions that students have been specifically instructed to use calculators or computers to solve them. These include some interactive problems on student CD's attached to the textbook. Questions are mostly linked to the textbooks own specially designed graphing packages, calculus packages and more.

*Example:* Use technology to help sketch the graph of  $y = 4x^2 + \frac{600}{x}$  (HIBTB, p. 635).

## Procedure

Utilizing the conceptual framework above, each task in the selected textbooks was examined against each of the classifications and coded into one of the categories.

Inter-rater reliability of the coding was checked between the researcher and an external scholar for 126 (10%) randomly selected tasks from each chapter according to the conceptual framework described earlier. There was an agreement on 96% of all the codings.

## RESULTS AND ANALYSIS

The chapters that represented calculus and their respective percentage per textbook are depicted in table 1 below.

The total number of tasks across all the four books varied considerably. The Ghanaian textbook had the highest number of tasks. This was due to the fact that the Ghanaian textbooks give reference to one question under a different content analysis and use the same question stem to create new questions without the need to write a completely new question.

## Working Group 11

In terms of task location and distribution, there were no considerable differences between the books . All tasks were located just after the content analysis with an average of three worked examples before the tasks. Table 2 shows the total number of tasks across the textbooks and their corresponding percentages.

Textbook	Chapters	%	N
HUTB	11,12,13,14,15,18,19,20	18.40	236
HIBTB	19	22.30	286
TIBTB	22,23	21.40	274
GTB	9	37.90	486
TOTAL		100	1282

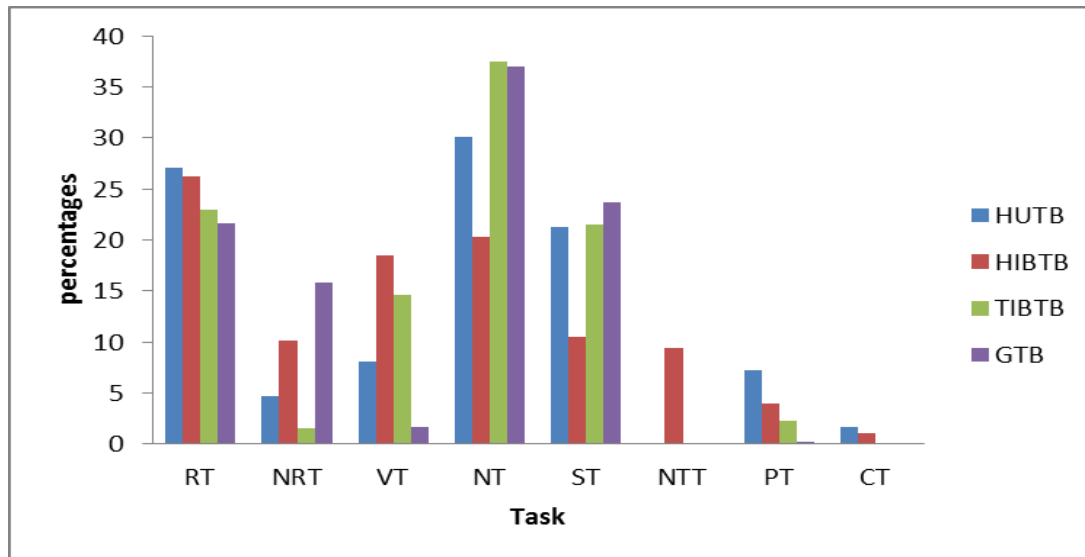
**Table 1: Problem Distribution Across Adopted Textbooks. N=Total number of questions in each textbook.**

	RT %	NRT %	VT %	NT %	ST%	NTT%	PT%	CT%
HUTB	20.9	9.1	15.8	17.2	19.7	0	48.6	57.1
HIBTB	24.4	24	44.2	14.1	11.8	100	31.4	42.9
TIBTB	20.5	3.3	33.3	25	23.2	0	17.1	0
GTB	34.2	63.6	6.67	43.69	45.3	0	2.9	0
N	307	121	120	412	254	27	35	7

**TABLE 2: Percentage Distribution of Task Across Textbook. RT=Routine task, NRT=Non-routine task, VT= Visual/Graphical/Tabular task, NT= Numerical task, ST= Symbolic task, NTT=Non-Traditional task, PT=Formal/Purely Mathematical task, CT=Calculator/PC task, N= total number of tasks in each book.**

The distribution of the eight question types across the three books is depicted by the histogram in figure 1 and the line graphs in figure 2. It is obvious from figure 1 that the textbooks are different from each other, numerical tasks being typical for all the textbooks and calculator /computer aided tasks being non-existent for others. Also non-traditional tasks appear only to exist in the Helsinki IB Textbook.

## Working Group 11

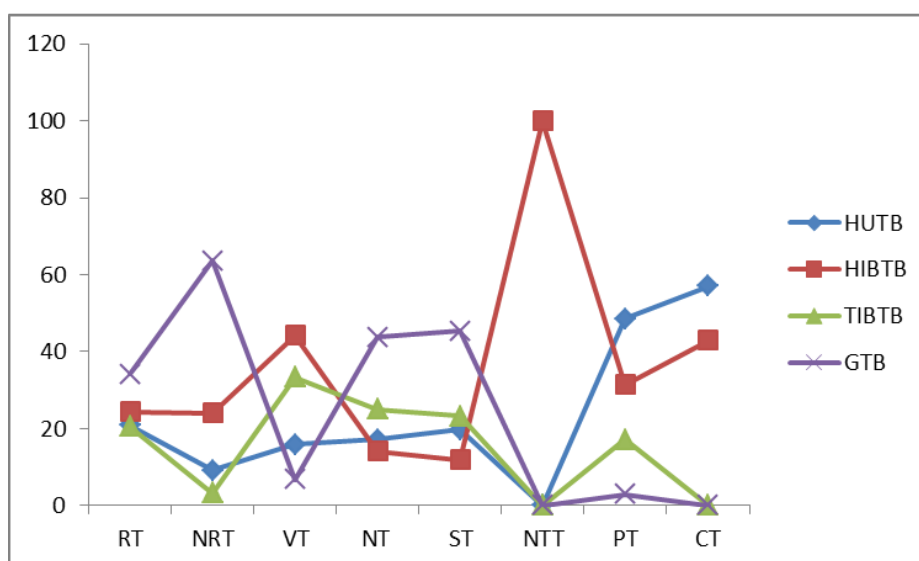


**Figure 1: Percentage Distribution of Task Across Textbook**

The Ghanaian Textbook has an interesting pattern in terms of task distribution. It has the highest number of a some task types and the lowest number of other task types, as shown in the zigzag line graph in figure 2. This suggests that the Ghanaian textbook has the most uneven distribution of tasks followed by the Turku IB textbook (TIBTB). Both the Turku IB Textbook and Ghanaian Textbook (GTB), have peaked and uneven graphs, reflecting a highly uneven distribution of task types.

The data also revealed that there is a uniform distribution of routine tasks. The Helsinki IB textbook has a fairly evenly distributed pattern of tasks with the exception of being the only textbook in the study that has any non-traditional tasks.

Also the line representing University of Helsinki textbook (HUTB) is relatively flat, depicting an even distribution of tasks according to the type of task.



**Figure 2: Percentage Distribution of Task Types Across Textbooks**

Numerical problems were the main focus for all the textbooks, and perhaps the textbooks give not enough attention to non-traditional, calculator problems. Students using HUTB are exposed to a more even distribution of different types of problems, than students using the other books. Students using GTB are exposed to different exercise types with respect to the other book problems but less exposed to VP, PP or non-exposure to NTP and CP.

In HIBTB all the calculator tasks were designed for students to use two specific calculator types. There were illustrations on how to use these calculators throughout the chapters. However, the students need to buy their own calculator to be able to use the book conveniently. While calculator problems in HIBTB were clearly marked, none of the tasks in HUTB was clearly marked.

Reasons for inclusion of the few calculator tasks in HIBTB and HUTB was that Finnish students are allowed to use calculators for their final examinations so the textbooks need to design some tasks that will prepare students for that challenge.

### CONCLUSION

According to the study, the analysed textbooks have few non-traditional and calculator tasks although these tasks may suit present day challenges and the new reform curriculum. For instance, there is the need for textbooks to include more calculator tasks to help students adjust to present day technology use in mathematics education. As Tall, (1996, p. 318) puts it, “Calculus has broadened in its meaning from traditional symbolic techniques to a wider science of how things change, the rate at which they change, and how their growth accumulates”.

Textbooks may help students realize how useful mathematics can be in their lives, but if the link between a mathematical concept and the corresponding real life situation is not made clear, students may not be able to completely grasp the mathematical concept.

It is not clear if the students using the Ghanaian textbook are exposed to more questions than the other students and gain more problem solving skills than their Finnish counterpart. However, the researcher thinks the reason for repetitive and exploratory activities/tasks in the Ghanaian textbook is to help the lower achieving students. At the same time, there are higher-level tasks to challenge and motivate the more able ones.

Furthermore, there is the need for further studies about textbooks among or across different educational systems, by comparing the various features of the mathematical tasks and the content presentations. Such a study will be useful in furthering our knowledge of the effects textbooks have on classroom instruction and student's mathematics achievement.

In conclusion, it appears that all the studied textbooks provide a wealthy source of problems for students to develop their ability in problem solving. As argued by Zhu

and Fan (2006), students' exposure to many problems has a substantial influence on students' learning and their achievement because tasks provide a rich learning environment in which students can have plenty of opportunities to experience problem solving.

## APPENDIX A: TEXTBOOKS ANALYSED IN THIS STUDY

Mendelson, E. (1997). Schaum's Outline of Beginning Calculus, 2ed. Published by Schaum's Outlines. Chapters 11, 12, 13, 14, 15, 18, 19, 20. University of Helsinki entrance textbook. Abbreviated in the study as HUTB.

Coad, M., Whifren, G., Owen, J., Haese, R., Haese, S., Bruce, M (2004). Mathematical Studies – Standard Level: International Baccalaureate Diploma. Published by Haese & Harris Publications .Chapter 19; Introductory Differential Calculus, pages 601- 638. Finnish IB mathematics textbook(Helsinki). Abbreviated in the study as HIBTB.

Cirrito, F. (2000). Mathematical Studies: Standard Level. Published by Schoenhof's Foreign Books. Chapter 22 & 23; Introductory Differential Calculus, Pages 434-495. IB Finnish Mathematics Textbook(Turku). Abbreviated in the study as TIBTB.

Eshun, A., B., Amisah S. E., Appiah-Danquah K., Gordor, B. K., Wilson, G.B (1992). Mathematics for Senior Secondary School, Ministry of Education. Published by H. Gangaram & Sons. Chapter 9; Differentiation, pages 119-135. Ghanaian mathematics textbook. Abbreviated in the study as GTB.

## REFERECES

Chavez-Lopez, O. (2003). From the textbook to the enacted curriculum: Textbook use in the middle school mathematics classroom. ProQuest, UMI Dissertations Publishing. *ProQuest Dissrtatoons and Theses*. Retrieved from <http://search.proquest.com/docview/288060282?accountid=11365>

Dragoo, R. C., (1945). Teaching the calculus. *National Mathematics Magazine*, 19(4), 186-193

Ferrini-Mundy, J. & Gaudard, M. (1992). Secondary school calculus: Preparation or pitfall in the study of college calculus? *Journal for Research in Mathematics Education*, 23(1), 56-71

Ganter, S. L., (2001). Changing calculus: a report on evaluation efforts and national impact from 1988 to 1998. Mathematical Association of America

Gray, E. M. & Tall, D. O. (1994). Duality, ambiguity and flexibility: a “proceptual” view of simple arithmetic. *Journal for Research in Mathematics Education*, 25(2), 116-140

- Hähkiöniemi, M. (2006). The role of representations in learning the derivative. Doctoral dissertation. University of Jyväskylä, Finland. Retrieved from <http://www.math.jyu.fi/research/reports/rep104.pdf>
- Johansson, M. (2003) *Textbooks in mathematics education: a study of textbooks as the potentially implemented curriculum*. Licentiate Thesis, Lulea University of Technology, Sweden. Retrieved from <http://epubl.luth.se/402-1757/2003/65/LTU-LIC-0365-SE.pdf>
- Reys, B. J., Reys, R. E. & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership*, 61(5), 61-66
- Schmidt, W. H., McKnight, C. C., Valverde, G. A., Houn, R. T. & Wiley, D. E. (1997). *Many visions, many aims: A cross-national investigation of curricular inventions in school mathematics*. Dordrecht, The Netherlands: Kluwer Academic Publishers
- Selden, A., Selden, J., Hauk, S. & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education IV* (pp. 128-153). Providence, RI: American Mathematical Society.
- Stigler, J., Fuson, K., Ham, M. & Myong Sook, K. (1986). An Analysis of Addition and Subtraction Word Problems in American and Soviet Elementary Mathematics Textbooks. *Cognition & Instruction*, 3(3), 153
- Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169
- Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. & Houang, R.T. (2002). *According to the book: Using TIMSS to investigate the translation of policy into practice in the world of textbooks*. Dordrecht, The Netherlands: Kluwer Academic Publishers
- Vinner, S. (1989). The avoidance of visual considerations in calculus students. *Focus on Learning Problems in Mathematics*, 11(1-2), 149-156.
- Zhu, Y. & Fan, L. (2006). Focus on the representation of problem types in intended curriculum: A comparison of selected mathematics textbooks from mainland China and United States. *International Journal of Science and Mathematics Education* 4(4), 609-626.



# THE TEACHING OF LINEAR EQUATIONS: COMPARING EFFECTIVE TEACHERS FROM THREE HIGH ACHIEVING EUROPEAN COUNTRIES

Paul Andrews

University of Cambridge

*On various international tests of achievement Finnish, Flemish and Hungarian students have been amongst the more successful in Europe. Linear equations, a topic students traditionally find difficult, is a key topic in the transition from mathematics as inductive and concrete to deductive and abstract. This paper, by means of an analysis of video-taped lessons taught by case study teachers, one from each of Finland, Flanders and Hungary, examines comparatively how teachers defined locally as effective construct opportunities for their students to learn the mathematics of linear equations. The findings show that all three teachers acted in ways contrary to received research wisdom, exploiting the balance scale as the key metaphor for inducing students into the solution processes of algebraic equations.*

## INTRODUCTION

The transition from arithmetic to algebra is problematic due, not least, to ambiguities with regard to the role and meaning of symbols of mathematics in general and the equals sign in particular. On the one hand it is a command to execute an operation, reflecting *procedural* (Kieran, 1992) or *operational* (Sfard, 1995) expectations. On the other, it is as an object on which other operations may be performed, reflecting *structural* (Sfard, 1995) expectations. Related to such concerns is the distinction between arithmetical equations and algebraic equations. The former, with the unknown on one side only, are generally assumed to be susceptible to undoing (Filloy & Rojano, 1989). However, the latter, with unknowns on both sides, cannot be solved by arithmetic-based approaches and require not only that the learners “understand that the expressions on both sides of the equals sign are of the same nature (or structure)” (Filloy & Rojano, 1989, p. 19) but also that they are able to operate on the unknown as an entity and not a number. In this manner, arithmetic equations are procedural while algebraic equations are structural (Kieran, 1992; Boulton-Lewis, Cooper, Atweh, Pillay, Wilson & Mutch, 1997). However, many students fail to navigate this transition and are “reduced to performing meaningless operations on symbols they do not understand” (Herscovics & Linchevki, 1994, p. 60). This failure has been described as either a *cognitive gap* (Herscovics & Linchevski 1994) or a *didactic cut* (Filloy & Rojano, 1989), although Pirie and Martin (1997) argue it is more likely to be the responsibility of inappropriate didactics than cognitive inadequacies.

Research shows that the use of different embodiments or representations can create the potential for new concepts, entities and operations to become endowed with

meaning (Filloy & Rojano, 1989). They can facilitate the link between concrete and abstract thinking by acting as analogues for the intended abstractions (English & Sharry, 1996; Warren & Cooper, 2005). With regard to equation solving, one of the most frequently used, and criticised, embodiments is the balance scale. Its advocates argue that it both helps students understand equations as entities rather than computational instructions and supports those symbolic representations that underpin algebraic formalisms (Filloy & Rojano, 1989; Warren & Cooper, 2005). Its critics argue that it cannot represent negatives in anything but a contrived way and is unfamiliar to modern students (Pirie & Martin, 1997).

Despite such criticisms several studies have examined the efficacy of the balance. Warren and Cooper (2005) found it helped children not only solve algebraic equations but also understand the equals sign as representing equivalence between entities. Vlassis (2002) found that although students understood both the conceptual and procedural role of the balance many experienced difficulties with negatives, irrespective of whether an equation was arithmetical or algebraic. Boulton-Lewis et al. (1997) found students preferred to use inverse approaches rather than the concrete representations taught them. Such examples highlight the diversity of findings with respect to the use of the balance, although their respective research designs may have contributed significantly to the findings. Warren and Cooper (2005) and Vlassis (2002) invoked the balance as a means of solving algebraic equations, while Boulton-Lewis et al. (1997) did so with arithmetic equations only.

In the following, the teaching of linear equations by three teachers, one from each of Finland, Flanders and Hungary is examined. Analyses of such teaching, located in countries typically shown to be more successful than England on various TIMSS and PISA assessments, present opportunities for an evaluation of the adaptive potential of the culturally located practices of one culture for another (Clarke, 2004). That is, such analyses have the potential to inform curriculum and teacher education development in less successful but culturally similar countries like England.

## METHOD

This paper draws on data from the EU-funded, Mathematics Education Traditions of Europe (METE) project. Based at Cambridge, England, the project, which ran from 2003-2005, examined aspects of mathematics teaching in Flanders<sup>61</sup>, England, Finland, Hungary and Spain. This paper draws on analyses of video recordings of sequences of lessons taught by teachers defined locally as competent in the manner of the Learner's Perspective Study (Clarke, 2006). Thus, while it is not possible to

---

<sup>61</sup> From the perspective of the METE study, Flanders, the autonomous Dutch-speaking community of Belgium, is construed as a country. This distinction is well known in the literature with, for example, Flanders being reported as a different educational system from Wallonia, the French-speaking community of Belgium.

comment on the attainment of project students, it would be reasonable to assume that project teachers would be as successful as any of their national peers. Four sequences of five lessons were filmed on the same topic in each country during the academic year 2003-2004. One of the topics was linear equations, chosen because of its importance in the transition from arithmetic to algebra. Videographers focused on the teacher whenever they were speaking. Teachers wore radio-microphones while a static microphone was placed strategically to capture as much student talk as possible. For each sequence, according to agreed project procedures, the first two lessons were transcribed and translated by English speaking colleagues working in the project universities. The translations, which were verified by Finnish-, Hungarian- and Dutch-speaking graduate students, enabled the creation of subtitles to facilitate a foreigner's (my) viewing and comprehension of the lessons.

It is important to acknowledge that although data were collected by local academics familiar with the teachers' contexts, the analyses presented here represent one cultural outsider's attempts to understand how participants enact their roles in culturally diverse classrooms. In such circumstances, where researchers are cultural outsiders, there is a danger of inaccurate reporting due to incomplete understanding of the cultural issues underpinning participants' beliefs and actions (Liamputtong, 2010). Such concerns invoke a need for culturally sensitive research (Tillman, 2002, p.6), whereby researchers accept and maintain the "cultural integrity of the participants and other members of the community" by privileging the voice of the cultural group over more general and frequently dominant theories. Moreover, culturally sensitive research aims to produce culturally informed theory by drawing on participants' culturally located perspectives on the phenomena under scrutiny. The manner in which data were collected, involving academics based at the partner universities and local teachers well known to them, was managed with appropriate regard to cultural sensitivity, as were the analyses described below.

All videos, with and without subtitles, were viewed by me several times and a narrative for each lesson constructed in which participants' actions and utterances were recorded in as much detail, and with as little interpretation, as possible. Where they were available, narratives were developed alongside transcripts in such a way that sections of dialogue were annotated with additional details relating to participants' actions. Where transcripts were unavailable, the narrative comprised my attempt to describe what was seen. All narratives ran to several pages of prose. These narratives were read against repeated viewings and increasingly refined. Eventually, having viewed and reviewed every lesson several times, a tentative understanding emerged with respect to the key elements of each teacher's conceptualisation and presentation of linear equations. Throughout this process, and conscious of the need to achieve culturally sensitive analyses, one was conscious of the need to avoid evaluation of teachers' actions alongside the desire, in accordance with conventional

case study practice, to provide as thick a description of events as possible. The following reflects these ambitions.

## RESULTS

The teachers were between 29 and 33 years of age with between six and eight years' teaching experience. Each was a graduate of the project partner university and was known locally to have a commitment to both continuing professional development and school-based teacher education mentoring. All three teachers, two women, Pauline in Flanders and Emese in Hungary and one man, Sami in Finland, worked in unremarkable schools in provincial cities that were homes to the partner universities.

The analyses indicated that all three sequences comprised four phases that I have come to call, definition, activation, exposition and consolidation phases. In general, the definition phase introduced students to the notion of an equation and, either implicitly or explicitly, presented a definition. The activation phase alerted students to or revisited intuitive procedures for solving arithmetic equations. The exposition phase, through an initial presentation of an algebraic equation, exposed the inadequacies of intuitive approaches and warranted the introduction of the balance. Lastly, the consolidation phase enabled students to exploit their newly acquired skills. In the following each phases is discussed against, due to space limitations, a selection of the available evidence.

### The definition phase

Lasting up to one lesson, this phase saw Sami in Finland, Emese in Hungary and Pauline in Flanders introducing and defining the topic.

#### Finland

The first Finnish lesson found Sami writing on the board that an equation was “two expressions denoted as being of equal magnitude”. He then wrote six “sentences”, as he called them, on the board:  $5$ ,  $x - 1$ ,  $x = 3$ ,  $5 + 3 = 7$ ,  $3x - 1 = 4$ ,  $x^2 = 8$ , before asking students to decide which were equations and which were not. Through constant reference to the definition, the “sentences” were categorised. Those accepted as equations were then discussed from the perspective of truth and a classification emerged that equations could be conditionally true, always true or always false.

#### Hungary

Emese exploited several open sentences to revisit the role of the basic set in determining a statement's validity. This was followed by a discussion through which an equation was defined as comprising two expressions connected by an equals sign. She asserted, through her questioning, that equations may or may not contain variables or unknowns depending on circumstances and that they were always true, sometimes true or never true. Finally she operationalised her definition through an

exercise in which three open sentences,  $5 - \square = 8$ ,  $5 - \square > 6$  and  $\square.2 = 7$ , were solved in relation to the basic set  $-3 \leq \square \leq 3$ .

#### Flanders

Pauline posed a problem involving characters from the cartoon series, the Simpsons: if Bart, Lisa and Maggie, are 7, 5 and 0 years old respectively and their mother, Marj, is 34, in how many years would the sum of the children's ages equal their mother's? She drew a two-rowed table before completing, collaboratively, the first three columns. Individual completion of the remaining columns was followed by a discussion leading to the solution of 11 years. Pauline then discursively introduced the unknown and the equation,  $34 + x = 12 + 3x$ , appeared, after which she sketched the straight line graph for each row in the table to highlight the intersection.

Summarising, Sami and Emese presented explicit definitions, although Sami's, which was presented both orally and in writing, involved no student input. His definition and the tripartite classification were operationalised collectively by means of his six sentences, while Emese's definition was discursively derived and operationalised. Her use of inequalities provided a more general entry to equation solving and facilitated an awareness of the three categorisations. Pauline exploited a *realistically*-derived equation to define, implicitly, both equation and equation solving. The latter was achieved with reference to both the table and graphs. She made no allusion to equation types.

#### The activation phase

This second, activational, phase saw all three teachers, as preparation for their main presentations, activate material covered earlier in their students' careers to both contextualise and facilitate the material that followed.

#### Finland

Sami began his second lesson by asking for a conditional equation. One student suggested  $x + 5 = 2$ , with a second offering  $-3$  as a solution. Sami then introduced  $x/8 + 1 = 4$  and invited mental solutions. After a minute, despite a student offering a correct solution, he demonstrated a *covering up* method and, through closed questions and his board sponge, confirmed 24 as the solution.

#### Hungary

Emese began by posing oral problems like, "Kala is twice as old as her sister; the sum of their ages is 24, how old are they?" Each was solved individually before solutions were shared. Next, the class was split into four groups with each given a superficially different word problem for translating into an equation. One group's problem was: "Some friends went on a trip. The first day they covered just 2km. The second day they covered  $2/10$  of the remaining journey. If they covered 6km on the second day, how long was their journey?" After several minutes the group representative explained how its equation had been derived and wrote  $0.2 \cdot (x - 2) = 6$

on the board. Lastly, a volunteer, exploiting a *thinking backwards* strategy, obtained a solution of 32, which Emese checked against the text of each problem.

Flanders

Pauline modelled, through discussion, several sketches and an introduction to the balance, an analytical solution to  $x + 7 = 9$ , which was then summarised symbolically before  $x - 2 = 10$ ,  $3x = 8$  and  $x/3=7$  were managed in the same way. This was followed by her summarising the relationship between each of her four exemplars and their respective formalisations. For example, in relation to  $x+7=9$  she wrote  $a = b \Rightarrow a + c = b + c$ . The lesson ended with her setting a homework whereby solutions to equations like  $x - 3 = 10$ ,  $200 - x = 20$  were placed in a crossword grid. The following lesson answers were shared with particular attention being paid to  $3/2x = 30$  and how division by  $3/2$  was equivalent to multiplying by its inverse.

Summarising, Sami invited his students to solve intuitively two equations with the unknown on one side and used the latter to introduce the cover up method that he never again mentioned. Emese exploited various word problems; initially to revisit the processes of *undoing* and latterly to derive arithmetic equations from *realistic* contexts and solve them with a thinking backwards strategy. Pauline privileged an explicit revision of arithmetical structures and their role in the solution of less straightforward equations. In so doing, she made an explicit reference to the balance.

### The exposition phase

All three teachers began their formal exposition by presenting their students with an algebraic equation, seemingly in the knowledge that intuitive methods would fail.

Finland

Sami wrote  $5x + 3 = 2x - 8$  and invited solutions. Once it became clear that this was too difficult, spoke of balance scales and how the same operation applied to both sides would retain the balance. Throughout he used outstretched arms to demonstrate the effect of different actions on the scales while commenting that “an equation is like scales... in principle, if you have it in balance, the equation is true”. Returning to the equation, he asked what could be subtracted from both sides of the equation. Someone suggested  $x$  and Sami, without comment, wrote  $4x + 3 = x - 8$ . Another volunteer suggested subtracting  $2x$ , at which point Sami wrote, with little student input:

$$\begin{array}{r|l} 5x + 3 = 2x - 8 & -2x \\ 3x + 3 = -8 & -3 \\ 3x = -11 & \end{array}$$

After some student uncertainty with regard to the next step, Sami, having asserted that they should divide by three as division is the opposite of multiplication, led the class to the solution  $x = -11/3$ . Lastly, individual seatwork was set from a text book.

## Hungary

Emese began her second lesson with a word problem, “On two consecutive days the same weight of potatoes was delivered to the school’s kitchen. On the first day 3 large bags and 2 bags of 10kg were delivered. On the second day 2 large bags and 7 bags of 10kg were delivered. If the weight of each large bag was the same, what weight of potatoes was in the large bag?” Soon a volunteer wrote  $3x + 20 = 2x + 70$ . Then, having established that intuitive strategies were now insufficient, Emese drew a picture of a scale balance with the various bags represented on both sides. Drawing on a student’s suggestion Emese erased two small bags from each side, leaving a representation of  $3x = 2x + 50$ . Next she erased two large bags from each side to show one large bag balancing 5 small. Then, in response to her request, students volunteered sufficient for her to write alongside her drawings:

$$\begin{array}{rcl} 3x + 20 = 2x + 70 & & | \quad -20 \\ 3x = 2x + 50 & & | \quad -2x \\ x = 50 \text{ kg} \end{array}$$

Finally, Emese reminded her class of the importance of checking and did so.

## Flanders

Midway through her second lesson, Pauline wrote  $6(x - 5) - 8 = x - 3$  on the board and began a formal treatment in which the algebra, including actions, was written on the left side of the board and justificatory annotations on the right. Throughout the process, which lasted more than twenty minutes, Pauline questioned continuously. Space prevents a detailed account, although what follows represents a fragment of what was written.

$$6(x - 5) - 8 = x - 3 \quad (1) \text{ Eliminate brackets}$$

At this point, Pauline drew from her students notions of associativity and commutativity before settling on distributivity as the warrant for what she was about to do.

$$6x - 30 - 8 = x - 3 \quad (2) \text{ Calculate if possible}$$

$$6x - 38 = x - 3$$

Eventually, after obtaining a solution and discussing its uniqueness, Pauline undertook a check.

Summarising, all three teachers presented equations with the unknown on both sides with, it seemed, the intention of creating contexts in which intuitive approaches could not be exploited. All three teachers invoked the balance as an underlying principle although the extent to which it was sustained varied. Sami, having introduced the balance, made little use of it during his rather directed exposition. Moreover, despite an implicit acceptance of his first student’s subtraction of  $x$ , his

subsequent actions indicated that he had a clear view as to what was acceptable. His solution was annotated conventionally although he invited no student input into its introduction. Emese exploited a realistic word problem to warrant the construction of her equation. She sustained the balance throughout her presentation, made explicit the relationship between her sketches and the symbolic representation and questioned her students constantly. Pauline offered the most complex of equations, deliberately provoking a frisson of excitement in her students. Her solution process, which was driven by many questions, was very formal and invoked a number of concepts studied earlier to highlight inter-topic and structural links. Both Sami and Pauline operated in exclusively mathematical worlds although it was Pauline and Emese who included checks at the conclusions of their expositions.

### **The consolidation phase**

The fourth phase, lasting two or three lessons, provided various opportunities for students to consolidate earlier learning and further develop both conceptual and procedural equations-related understanding. All three teachers set increasingly complex exercises, all involving algebraic equations incorporating brackets and both negative and fractional coefficients. Sami and Pauline located all their exercises within mathematics-only worlds while Emese integrated *realistic* word problems. Sami and Emese derived additional insights from the tasks set, while Sami introduced the *change the side change the sign* rule and, essentially, prescribed a preferred approach. Emese invited multiple solution strategies, discussed notions of efficiency and elegance, and constantly solutions. Pauline included a test. The manner in which tasks were completed and solution shared varied with Sami and Pauline sharing solutions after several problems had been completed while Emese always shared solutions after each problem had been solved individually.

## **DISCUSSION**

All three sequences shared common structural - definition, activation, expositions and consolidation - characteristics. Such similarity is unsurprising as comparative studies that adopt broad and inclusive variables tend to find similarity rather than difference; as in LeTendre, Baker, Akiba, Goesling & Wiseman's (2001) analysis of teachers' self reported use of core instructional practices that included, for example, seatwork and whole class instruction. In other words, such broad categories tend to be inclusive and reflect patterns of instruction commonly found across cultures.

However, within these macro-level similarities were important similarities and differences. With respect to similarities, all three teachers offered definitions, either explicitly or implicitly, which were operationalised through problems and exercises. All three, having activated students' knowledge and skills, provoked analytical solution methods by posing an algebraic equation that could not be solved by intuitive methods. All, as is discussed below, based their expositions on the balance, and all three offered extensive opportunities for consolidation that incorporated



expectations of students' managing brackets and different forms of coefficient alongside particular privileged additional insights.

In respect of differences, several issues of interest emerged. Despite similarities with respect to the balance, the manner of its introduction and maintenance varied greatly. Pauline offered only a scant allusion; Sami enacted bodily its characteristics but made no further reference once the first expository example had been solved. Emese, through bodily enactment and sketches, made explicit the link between the embodied and symbolic forms of equation, a link sustained through several examples. Interestingly, and confounding Pirie and Martin's (1997) scepticism, not only did all students appear familiar with the balance but also, once introduced, negatives. The examples and exercises exploited by Sami were located entirely within a world of mathematics. Pauline, having kick-started the topic with a single word problem, behaved similarly. Interestingly, while the tasks set by Sami and Pauline were generally complex, task difficulty was so teacher-managed that students experienced few teacher-independent opportunities to engage with non-routine problems. In this respect, the data suggest that both teachers had been slower than systemically desired to incorporate problem solving into their repertoires, whether in Flanders (Verschaffel, De Corte & Borghart, 1997) or Finland (Pehkonen, 2009). Emese, on the other hand, exploited both mathematical and word problems throughout and, in accordance with earlier studies, provided her students with constant opportunities to mathematise and solve text problems (Andrews, 2003). With regard to classroom norms, Emese engaged her students in collective activity focused on students' awareness and acquisition of diversity of mathematical thinking. Pauline had clear objectives that were explicitly addressed by means of extensive but tightly focused bouts of public questioning. Sami, having operationalised his definition, exploited extensive bouts of teacher telling, interspersed with exercises, from which students were expected to infer meaning. In conclusion, although space prevents a detailed summary, such teacher behaviours, whether similar to or different from those found elsewhere, are likely to reflect characteristic patterns indicative of a national mathematics teaching script.

## REFERENCES

- Andrews, P. (2003). Opportunities to learn in the Budapest mathematics classroom. *International Journal of Science and Mathematics Education*, 1(2), 201-225.
- Boulton-Lewis, G., Cooper, T., Atweh, B., Pillay, H., Wilson, L., & Mutch, S. (1997). Processing load and the use of concrete representations and strategies for solving linear equations. *Journal of Mathematical Behavior*, 16(4), 379-397.
- Clarke, D. J. (2004). *Issues of voice and variation: The problematics of international comparative research in mathematics education*. Paper presented to discussion group 11: International comparisons in mathematics education at the Tenth International Congress on Mathematical Education, Danish Technical University.

- Clarke, D. J. (2006). The LPS research design. In D. Clarke, C. Keitel & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 15-36). Rotterdam: Sense.
- English, L. D., & Sharry, P. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, 30(2), 135-157.
- Fillooy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, 9(2), 19-25.
- Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics*, 27(1), 59-78.
- Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: Macmillan.
- LeTendre, G., Baker, D., Akiba, M., Goesling, B., & Wiseman, A. (2001). Teachers' work: Institutional isomorphism and cultural variation in the U.S. Germany, and Japan. *Educational Researcher*, 30(6), 3-15.
- Liamputtong, P. (2010). *Performing qualitative cross-cultural research*. Cambridge: Cambridge University Press.
- Pehkonen, E. (2009). How Finns learn mathematics: What is the influence of 25 years of research in mathematics education? In M. Lepis (Ed.), *Teaching mathematics: Retrospectives and perspectives* (pp. 71-101). Tallinn University: Institute of Mathematics and Natural Sciences.
- Pirie, S. & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations. *Educational Studies in Mathematics*, 34(2), 159-181.
- Sfard, A. (1995). The development of algebra: confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, 14 (1), 15-39.
- Tillman, L. C. (2002). Culturally Sensitive Research Approaches: An African-American Perspective. *Educational Researcher*, 31(9), 3-12.
- Verschaffel, L., De Corte, E., & Borghart, I. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems. *Learning and Instruction*, 7(4), 339-359.
- Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, 49(3), 341-359.
- Warren, E., & Cooper, T. J. (2005). Young children's ability to use the balance strategy to solve for unknowns. *Mathematics Education Research Journal*, 17(1), 58-72.

# **EXPLORATORY DATA ANALYSIS OF A EUROPEAN TEACHER TRAINING COURSE ON MODELLING**

Richard Cabassut\* & Jean-Paul Villette\*\*

\*University of Strasbourg, France, LDAR, Paris 7 University

\*\*University of Strasbourg, France

*The European project LEMA experimented a teacher training course on modelling in four different countries. We present here an exploratory analysis to evaluate the effect of the training course. From the teachers' answers to a questionnaire on beliefs before the beginning of the training course, the teachers are split in different clusters. With the answers to the questionnaire after the training course, we observe how in every cluster the answers change. We will show how the national groups can be taken into account. Then we will point to questions and challenges related to comparative studies.*

## **THE EVALUATION OF A TEACHER TRAINING COURSE**

### **The LEMA project context and the focus of the paper**

The project LEMA [1] has from 2006 to 2009 developed a teacher training course on mathematical modelling. This project was not a research project trying to answer a research question but has tried to evaluate the training course. The course was implemented in some partnership countries. Before attending the course the teachers have answered a questionnaire. This questionnaire deals with their biography, with their interest in modelling, with their beliefs about mathematics, and with their beliefs about their ability to teach modelling. The same questionnaire had to be answered at the end of the attended training course. The aim of this paper is not to present the teacher training course, which does not mean that this presentation would not be interesting. A description of the LEMA project is made in (Cabassut & Mousoulides, 2009) and a description of the questionnaire is made in (Maaß & Gurlitt, 2009). The aim of this paper is not to discuss the relevancy of this questionnaire, which does not mean that this discussion is not of interest. A discussion on qualitative and quantitative methods in comparative research is made in (Cabassut, 2007). An example of qualitative evaluation of the teacher training course is offered in (Cabassut & Mousoulides, 2009) and in (Schmidt, 2009). The main aim of this paper is to present an exploratory analysis of the data, produced by the answers to the questionnaires, in order to help to answer the questions: How to evaluate the LEMA teacher training course? How to integrate in this evaluation the comparative approach between countries? A secondary aim is to present a comparative method of analysis of data from different countries in order to contribute to the debate on comparative methods. Let us consider the theoretical framework related to this exploratory analysis.

## **The theoretical frameworks**

Different theoretical frameworks are used in this paper. For the study of modelling we use the theoretical framework of PISA, based on the mathematisation cycle inspired by the works of (Blum, 1996) and described in (Cabassut & Mousoulides, 2009). The questionnaire takes into consideration works of Grigutsch, Raatz and Törner (1998) on teachers' beliefs in school mathematics, of Bandura (1997) on teachers' self efficacy and of Kaiser (2006) on knowledge and beliefs on modelling: the theoretical framework of the questionnaire is well described in (Maaß & Gurlitt, 2009). We describe with more detail the theoretical framework related to the main object of this paper: the exploratory data analysis. We propose to present this method of analysis because first we did not find so many studies in mathematical education using exploratory data analysis and secondly this data analysis looks as fitting with a comparative approach as illustrated later.

The exploratory data analysis was developed in (Tukey, 1977) in order to analyse data, and to formulate hypotheses based on this analysis. These hypotheses could be confirmed with a confirmatory analysis. In the case of exploratory analysis, descriptive statistics is used to observe the data without preconceptions and without formulating hypotheses in advance. In our example, the data are composed of a whole population of 83 teachers and their answers to a questionnaire. The descriptive statistics on the whole population will allow an inductive approach to describe the population and to formulate hypotheses on the structure of the population depending on the results of the exploratory data analysis. It is possible to try to evaluate assumptions made on the population.

On the contrary, confirmatory analysis uses inferential statistics to test, in a deductive approach, the hypotheses formulated in advance. Confirmatory analysis works on a sample of a population, using hypothesis tests and confidence interval estimation. The assumptions have to be accepted and are not testable: only the hypotheses are tested. Confirmatory data analysis is used in the LEMA project as reported in (Maaß & Gurlitt, 2009). In this confirmatory analysis two teachers' groups are compared: the intervention group did attend the teacher training course and the control group did not. The data analysis tests the research question: Has the intervention group outperformed the control group over time (by comparing the responses to the questionnaire before the training course and after the training course)? Means and standard deviations are used to summarize the answers to the different parts of the questionnaire (questions on beliefs, self-efficacy and pedagogical content knowledge) for the intervention group and for the control group. A two-factorial analysis of variance is used (factor intervention/control group and factor pre/post test). An F-test is used to test the hypotheses. Let us present now the methodology used in our exploratory data analysis.

## Methodology

The population studied is composed of 83 teachers. The questionnaire is composed of questions (variables) with multiple-choice qualitative answers and questions with quantitative answers. All quantitative variables are reconditioned in two intervals by using the median to separate the classes. We split now the variables in three parts: the biographical variables are the questions related to the teachers' biography (country, age, gender, type of school ... ) that do not change between the pre and the post questionnaires; the active variables are the questions of the pre-questionnaire, except the biographical variables; the variables of the post-questionnaire are considered as illustrative variables.

We begin with a multiple correspondence analysis (MCA) on the active variables (36 variables with a two choices answer, which means  $2^{36}$  possibilities). Then we apply a hierarchical ascendant classification (HAC or cluster analysis) by using the distances measured on the first coordinates between teachers on the teachers' first coordinates on the factorial axes determined by the MCA. SPAD software is used. There is a cluster of three teachers who have too many no answers. We decide not to take these three teachers into consideration for the cluster analysis. After the cluster analysis each of these three teachers joins his nearest cluster. In this new cluster analysis there is a one teacher cluster. We repeat the precedent procedure and we get eventually 79 teachers offering with the active variables a cluster analysis with four clusters [2].

For every cluster we consider the **splitting active variables**. A splitting active variable is an answer for which the percentage of the answer in the cluster is very different than in the whole population. In the next paragraph, every cluster will be described through these splitting variables. In the post-test, the answers to the same questions are considered as illustrative-supplementary variables. Those for which the percentages are very different in the cluster than in the population are split by clusters. The biographical variables (gender, type of school, nationality, level of studies ...) are also split by clusters. Let us describe the clusters.

## TEACHERS ' BELIEFS BEFORE THE TRAINING COURSE

To describe the whole population, we observe a variety of positions about beliefs and self confidence: a majority can agree for some items, or disagree for other ones or being divided between agreement, disagreement or neutrality. To describe every cluster we look at the difference in the percentage of an answer between the studied cluster and the whole population.

### First cluster

The first cluster contains 13 teachers with the following splitting answers, much more answered than in the whole population. They agree strongly that every student can create or recreate parts of mathematics; that there is usually more than one way

to solve mathematical tasks and problems at school; that students with the right age are able to solve the proposed modelling task and that this task does not take too much lesson time; and that if students get to grips with mathematical problems they can often discover something new (connections, rules and terms). They disagree strongly that to solve a mathematical task at school, one has to know the one and only correct procedure or you are lost; that school mathematics is the memorizing and application of definitions, formulas, mathematical facts and procedures; and that school mathematics is a collection of procedures and rules which determine precisely how a task is solved. The teachers look less confident than the whole population for all the items, and specially to give effective verbal feedback to groups and individual students to assist them with modelling, or to support students in developing competencies in arguing related to modelling tasks.

The teachers of this cluster look positive at the teaching of modelling, expressing a need to support students in modelling and having an open-minded view on school mathematics beliefs, with mostly strong positions on these points (strongly agree or disagree).

### **Second cluster**

The second cluster contains 31 teachers with the following splitting answers, much more answered than in the whole population. Most of the teachers feel less confident than the whole population for all the items to teach modelling, and specially they feel less confident to be able to design modelling lessons that help students overcome difficulties in all modelling steps (e.g. problems in validating); to use students' mistakes to facilitate their learning in modelling; to effectively assess students' progress as they work on modelling tasks; to adapt tasks and situations in text books to provide realistic open problems; and to design their own modelling tasks. In this cluster the teachers look less confident to teach modelling.

### **Third cluster**

The third cluster contains 14 teachers with the following splitting answers, much more answered than in the whole population. They agree strongly that school mathematics is the memorizing and application of definitions, formulas, mathematical facts and procedures. They strongly disagree that school mathematics is useful in helping individuals to become critically aware citizens; that it is possible for students to discover and try out many things in school mathematics; and that school mathematics helps to understand phenomena from various areas of our society. Much more than in the whole population, they are neutral to agree that mathematics is of general and fundamental use to society, and that there is usually more than one way to solve mathematical tasks and problems at school or that school mathematics helps to solve daily tasks and problems. About self confidence, there is a variation depending on the items, sometimes they are less confident than the whole population, sometimes more confident, without strong difference. The teachers of

this cluster look conservative on school mathematics and are less open to application of school mathematics in life.

#### **Fourth cluster**

The fourth cluster contains 21 teachers. For all the items of self confidence, these teachers are much more confident than the whole population, and specially to design modelling lessons that help students overcome difficulties in all modelling steps (e.g. problems in validating); to design their own modelling tasks; to effectively assess students' progress as they work on modelling tasks; to develop detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks; to use students' mistakes to facilitate their learning in modelling; to support students in developing competencies in arguing related to modelling tasks; and to give effective verbal feedback to groups and individual students to assist them with modelling. They strongly agree to use the modelling approach in their future teaching. The teachers of this cluster look very self-confident to teach modelling.

#### **TEACHERS' BELIEFS AFTER THE TRAINING COURSE**

We observe that for the answers to self-confidence variables the average and median are increasing after the training course, which looks quite normal.

For the variables on beliefs about mathematics there are small variations with some exceptions. Teachers agree much more that school mathematics is useful in helping individuals to become critically aware citizens. Maybe the training has offered examples or situations to illustrate this possibility. There is also a big increase in the probability to use a type of modelling task in teaching, or to disagree that the students will not be able to solve a given modelling task.

To analyse the change in every cluster, we look after the training course what are now the answers with the biggest differences between the percentage of the answer in the cluster and the percentage of the answer in the whole population.

#### **First cluster**

More than in the whole population the teachers strongly disagree that school mathematics is the memorizing and application of definitions, formulas, mathematical facts and procedures. They strongly agree that every student can create or recreate parts of mathematics and that there is usually more than one way to solve mathematical tasks and problems at school. They disagree that central aspects of school mathematics are flawless formalism and formal logic. For these split variables the difference with the whole population is smaller than it was before the training with the splitting variables. All these changes are positive to teach modelling.

Similarly, before the training course, for all items of the self-confidence variables, the teachers were less confident than in the whole population. After the training we observe that the level of confidence has increased in the population and in the cluster

and there are no more strong differences with the whole population for the self-confidence variables. More precisely, for the two items where the self-confidence was splitting before the training course, after the training course we have a change of the confidence. Before the training course a majority of the cluster had a degree of confidence under the median of the whole population for both items. It is the contrary after the training course, even if the median of the whole population has increased after the training course.

More generally after the training course, the difference between this cluster and the whole population is weaker.

### **Second cluster**

Much more than in the whole population, the teachers are neutral to think central aspects of school mathematics are flawless formalism and formal logic, disagree that doing mathematics at school involves innovative thinking and new ideas, and agree that there is usually more than one way to solve mathematical tasks and problems at school. These split differences are difficult to interpret because the differences observed are reduced. The case is particularly true for the confidence items. The teachers of the cluster keep less confident than in the whole population, even if the level of confidence has increased in the population and in the cluster. But there is no more big difference.

After the training course, in this cluster, there are not so much split answers and the differences are considerably reduced.

### **Third cluster**

More than in the whole population the teachers disagree that every student can create or recreate parts of mathematic; that school mathematics helps to understand phenomena from various areas of our society; and that it is possible for students to discover and try out many things in school mathematics. For these three variables we observe after the training course a decrease of strongly disagreement and an increase of disagreement. The differences with the whole population are reduced in comparison with the splitting differences before the training course. The splitting variables of strong disagreement and agreement, present before the training course, are no more present as split variables after the training course. Even if the teachers of the cluster keep some conservative beliefs as illustrated above, their position is more moderated and less splitting than before the training course.

### **Fourth cluster**

For all items of self-confidence the average has increased in the cluster and in the whole population. The median for all items has increased in the whole population. In the cluster for all items the majority of the answers keep in the interval over the median. Even if this percentage has decreased, for every item teachers are more confident in the cluster than in the whole population, and specially to develop



detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks; to adapt tasks and situations in text books to provide realistic open problems; to support students in developing competencies in arguing related to modelling tasks; and to effectively assess students' progress as they work on modelling tasks and to select appropriate tasks suitable for a modelling approach to teaching. The teachers of this cluster keep more confident than the whole population even if the difference is reduced.

## **BIOGRAPHICAL VARIABLES AND CLUSTERS**

We can observe now how the biographical answers (gender, age, country...) are split in the clusters. We use the same method as for the answers after the training. We look what are the biggest differences between the percentage of the answer in the cluster and the percentage of the biographical answer in the whole population.

We will try to interpret the relation between biography and clusters. But it is clear that you can find the same age or the same country or the same type of school split in different clusters. It means that we want to break the idea to define a cluster with the biography answer. A biography answer, for example a given country like France, can be an explanation factor but neither a necessary factor, neither a sufficient factor to belong to a cluster, it means to fill the profile of a cluster.

### **First cluster**

Younger teachers and French teachers are more numerous in this cluster than in the whole population. On the contrary Hungarian teachers, secondary school teachers, older teachers, teachers with a high number of years as teacher are less numerous.

The teachers of this cluster look positive to teach modelling, expressing a need to support students in modelling and having an open-minded view on school mathematics beliefs, with mostly strong positions on these points (strongly agree or disagree). Young teachers could be more sensible to open-minded beliefs because their teacher education was more focused on didactics and pedagogy than older teacher education. In France problem-solving plays a main role in mathematics teaching focused on mathematical content. The French teachers of the training course were from primary school where every day life problems are very important in the official syllabus (Cabassut & Wagner, 2009). Hungarian teachers are less present maybe because their school system is more traditional (Vancso & Ambrus, 2009). Secondary school teachers are also less present maybe because they are more focused on mathematical content than on modelling activities.

### **Second cluster**

Secondary school teachers, German teachers, and older teachers are more numerous in this cluster than in the whole population. On the contrary primary school teachers, and younger teachers are less numerous.

## Working Group 11

The teachers of this cluster look less confident to teach modelling, and moderately open-minded to modelling. The secondary school teachers are maybe more focused on mathematical content than primary school teachers and have a pressure to achieve their syllabus. German teachers have seen a big change in their curriculum in 2009 where modelling becomes a leading idea (Garcia, Wake, & Maaß, 2007). This official change could make them open to modelling but less confident because it is a new idea in the curriculum. Older teachers could also feel uncomfortable to change their teaching if modelling corresponds to a new teaching.

### **Third cluster**

Older teachers, teachers with a high number of years as teacher, Hungarian teachers, teachers who have studied mathematics at university level, and secondary school teachers are more numerous in this cluster than in the whole population. On the contrary German and Spanish teachers, teachers with a low number of years as teacher, and younger teachers are less numerous.

The teachers of this cluster look very conservative on school mathematics and are less open to application of school mathematics in life. Older teachers with long experience could have a more conservative behaviour than young and less experienced teachers. Hungary has a traditional and theoretical mathematics teaching that can explain this position (Vancso & Ambrus, 2009).

### **Fourth cluster**

Younger teachers, Spanish teachers, and primary school teachers are more numerous in this cluster than in the whole population. On the contrary older teachers, teachers with a high number of years as teacher, German teachers, and secondary school teachers are less numerous.

The teachers of this cluster look very self-confident to teach modelling. Younger teachers or teachers from Spain are maybe more confident (Garcia et al., 2007). Primary school teachers, being used to multi-subject activities, are more confident to teach modelling. We have explained in the second cluster why German teachers, older teachers or secondary school teachers could be less confident to teach modelling. With the fourth cluster we observe that young teachers or primary school teachers seem to be more open to modelling.

## **DISCUSSION**

Teachers can be split in different groups depending on their mathematical beliefs (conservative or open minded) and on the degree of confidence to teach modelling. The main difference after the training course is the increase of the level of confidence. The consequence is that the level of confidence is no more a split variable for the two first clusters. It keeps to be a split variable for the fourth cluster gathering teachers much more confident than the whole population.

After the training course, the variables on the beliefs about mathematics are no more strongly split variables: the training course seems to homogenize and to moderate the beliefs. The training course has effectively taken into consideration advantages and disadvantages, difficulties and interests, needs and potentialities of the teaching of modelling. The third cluster keeps on characterising the teachers by conservative points of view on mathematics teaching, even if they look more moderate after the training course. It means that a big change in the beliefs seems difficult to be achieved as expressed in (Maaß & Gurlitt, 2009). If one of the aims of a future course is to change the teachers' beliefs, the course will have to focus on this aim.

We have offered this example of comparative study to contribute to the discussion on comparative studies in mathematics education. We do not know why the exploratory data analysis is not so popular in mathematical education in comparison with other fields. A reason could be related to the fact that mathematical education is more controlled by mathematicians than other fields of knowledge. For a mathematician confirmatory data analysis uses a deductive approach and inferential statistics while exploratory analysis uses an inductive approach and descriptive statistics. In the exploratory analysis, clusters enable to zoom in the whole population by aggregating individuals through their answers and not through their biographical values. They enable to build types of teachers to whom targeted training can be addressed.

In some international comparative studies, a descriptor aggregates the answers of a country in order to compare it with other countries, ranking the countries with the value of the descriptor. For example, the performance of Germany at PISA 2000 shows that Bavaria is significantly above the OECD average and Brandenburg is significantly below. Considering the German performance is a zoom-out where the difference between Bavaria and Brandenburg is not taken in consideration, this zoom-out leads to identify for every country a coherent body of practices. The hit is to define the best country - Finland for example in PISA 2000 - and to propose this country as a model for the others.

The aim of the example developed in this study is to show that other ways are possible. The diversity of the practices in a country shows that the country contributes to different clusters and that teachers from different countries can be gathered in the same cluster. The information got from a zoom-in can be as much interesting as that from a zoom-out. The challenge is to develop the comparisons among different countries taking into account the diversity of the practices and of the beliefs inside every country.

## NOTES

1. LEMA means “Learning and Education in and through Modelling and Applications”. This project is funded by the European Union and is described on the project site: [www.lemma-project.org](http://www.lemma-project.org)
2. The data analysis (MCA and HCA) is made for the statistical part under the control of J.-P. Villette, and for the didactical interpretation of the clusters under the control of R. Cabassut.

## REFERENCES

- Bandura A. (1997). *Self-efficacy: The exercise of control*. New York: Freeman and Company.
- Blum W. (1996). Anwendungsbezüge im Mathematikunterricht in der didaktischen Diskussion. *Mathematische Semesterberichte*, 32(2), 195-232.
- Cabassut R., & Mousoulides N. (2009). Theoretical considerations for designing and implementing a teacher training course on mathematical modeling: Insights from a French-Cypriot Comparison. In A. Gagatsis et al. (Eds.), *Cyprus and France research in mathematics education*. Lefkosia: University of Cyprus.
- Cabassut R., & Wagner A. (2009). *Roles of knowledge in the teaching of modelling at primary school through a French-German comparison*. Paper presented at the 14<sup>th</sup> International Conference on the Teaching of Mathematical Modelling and Applications, July 2009, University of Hamburg.
- Cabassut R. (2007). Examples of comparative methods in the teaching of mathematics in France and in Germany. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the CERME 5* (pp. 2423-2432). Larnaca: University of Cyprus.
- Garcia F.J., Wake G., & Maaß K. (2007). *Theory meets practice: working pragmatically within different cultures and traditions*. Paper presented at the 13<sup>th</sup> International Conference on the Teaching of Mathematical Modelling and Applications, July 2007, University of Illinois.
- Grigutsch, S., Raatz, U., & Törner, G. (1998). Einstellungen gegenüber Mathematik bei Mathematiklehrern. *Journal für Mathematikdidaktik*, 19(98), 3-45.
- Kaiser, G. (2006). The mathematical beliefs of teachers about application and modelling - results of an empirical study. In J. Novotna et al. (Eds.), *Proceedings of the PME 30* (Vol. 3, pp. 393-400). Prague: Carles University.
- Maaß K., & Gurlitt J. (2009). Designing a teacher-questionnaire to evaluate professional development in modelling. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the CERME 6* (pp. 2056-2065). Lyon: Institut National de Recherche Pédagogique.
- Schmidt B. (2009). Modeling in the classroom – motives and obstacles from the teacher's perspective. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the CERME 6* (pp. 2066-2075). Lyon: Institut National de Recherche Pédagogique.
- Tukey J. (1977). *Exploratory data analysis*. Reading, MA: Addison-Wesley.
- Vancsó Ö., & Ambrus G. (2009). *Teaching mathematical modeling in Hungarian schools based on some national traditions*. Paper presented at the 14<sup>th</sup> International Conference on the Teaching of Mathematical Modelling and Applications, July 2009, University of Hamburg.

# COMPARISON OF ITEM PERFORMANCE IN A NORWEGIAN STUDY USING U.S. DEVELOPED MATHEMATICAL KNOWLEDGE FOR TEACHING MEASURES

Arne Jakobsen, Janne Fauskanger, Reidar Mosvold and Raymond Bjuland

University of Stavanger [1]

*In the Learning Mathematics for Teaching (LMT) project, measures were developed in order to gauge growth in teachers' mathematical knowledge for teaching (MKT) and to learn if and how such knowledge contributes to students' achievement. This paper documents the results from using adapted U.S. developed measures in a pilot study involving 142 Norwegian teachers. Psychometric analyses were performed on the Norwegian measures, and for some item parameters, we find strong correlation to the item performance characteristics found in the U.S.*

## INTRODUCTION

The study of mathematics teachers' knowledge and practice has been an active field of research for several decades (Ponte & Chapman, 2006). Inspired by Shulman's (1986) work, a group of researchers at the University of Michigan have developed a theory of "mathematical knowledge for teaching" (MKT). Part of this research included development of items used to measure teachers' MKT and to understand the effect such knowledge has on student achievement. It is also used to study and to compare outcomes of professional development of teachers and to improve teacher education. Research shows that teachers with high MKT score can be positively associated with increased learning by their pupils (Hill, Rowan, & Ball, 2005) and with the mathematical quality of instruction (Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, et al., 2008). While some researchers claim that teaching is a cultural activity (Stigler & Hiebert, 1999), little is known to what extent this also applies to the MKT construct.

The measures under investigation were created on the basis of qualitative studies of mathematics teaching in U.S. classrooms and designed to reflect U.S. teachers' knowledge about the content taught as well as pedagogical content knowledge (Ball, Thames, & Phelps, 2008). Although the items were never made for use outside of the U.S., there have been several attempts to adapt and apply the MKT measures in other countries (e.g. Delaney, Ball, Hill, Schilling, & Zopf, 2008; Mosvold, Fauskanger, Jakobsen, & Melhus, 2009; Ng, 2009). The challenges of adapting and validating these items for use in Norway relate to issues of translation (Mosvold et al., 2009) as well as issues concerning teachers' experiences and reflections after having worked individually on the measures (in Norway, see Fauskanger & Mosvold, 2010; Bjuland, Mosvold, & Fauskanger, in progress). As part of this ongoing research, we want to see how the U.S. developed MKT measures perform in Norway and compare with performance in the U.S. This is important if one wants to rely on research and

development of MKT theory done in other countries. In this paper, we discuss how the items performed in a pilot study involving 142 primary and lower secondary school teachers in Norway. Our results are compared to results obtained in the U.S. (Hill, 2007). We focus on psychometric properties of the items by addressing the following research question:

Do the U.S. developed mathematical knowledge for teaching items perform in the same manner in Norway?

## **THEORETICAL BACKGROUND**

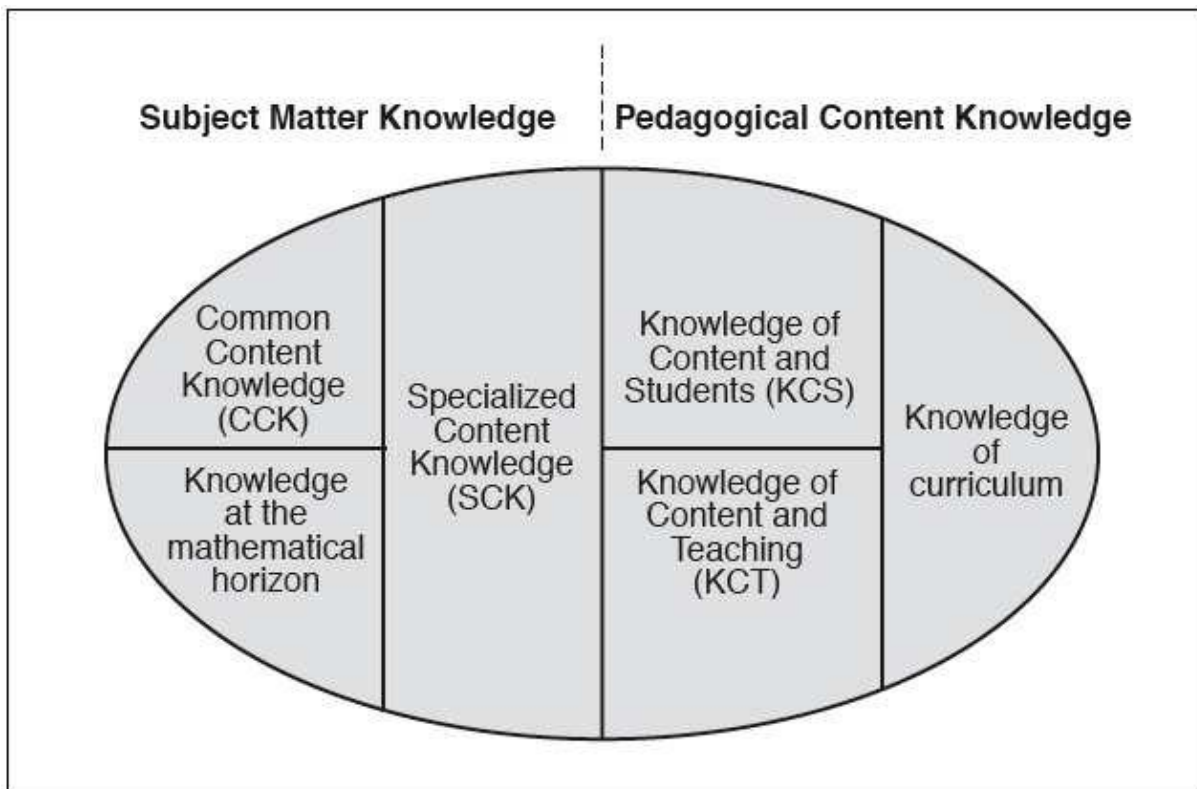
In recent years, effective professional teacher development has been studied extensively (e.g. Garet, Porter, Desimone, Birman, & Yoon, 2001). According to these authors, however, literature in this field provides little direct evidence of positive outcomes for the participating teachers and their students. Some promising work has been carried out by researchers at the University of Michigan. Their analyses of teachers' MKT demonstrate that teachers' MKT made a difference in teachers' mathematical quality of instruction (Hill et al., 2008) and in pupil's achievement in mathematics (Hill et al. , 2005).

Theoretically, the MKT construct follows Shulman's (1986) work and the categorization of the various components of teacher knowledge that has evolved from Shulman's original proposal. The work done at the University of Michigan resulted in the model of MKT presented in Figure 1, a model still under development.

The MKT items were developed based on studies of videos from classroom practice, and the domains have been identified both in the U.S. (Ball et al., 2008) and in Norway, where Drageset (2009) has verified the existence of the constructs specialized content knowledge (SCK) and common content knowledge (CCK).

Although the items focus on important tasks of teaching (e.g. presenting mathematical ideas), which are supposed to be of a universal nature (Ball et al., 2008) they may not perform as intended in other countries (e.g. Delaney et al., 2008). This indicates that the translation and adaptation of the MKT measures into a different language (and culture) is not straightforward and requires careful scrutiny and different methodological approaches in order to be successful (e.g. Mosvold et al., 2009).

Investigating what in-service mathematics teachers know is uncommon in Norway, and according to Lysne (2006), assessment in education is a controversial issue in many western countries. In addition, the multiple-choice format is seldom used in Norway, but this seems to be changing (Sirnes, 2005). Reutzel and his colleagues (in press) claim that the measurement of practicing teachers' knowledge is not widely accepted, and other assessments than written are highlighted (Baxter & Lederman, 1999).



**Figure 1: Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403. The domains are defined and discussed in the same reference).**

We recognize that no assessment is perfect, and all measurement instruments have their advantages as well as disadvantages. The MKT is no exception (e.g. Kane, 2007; Schoenfeld, 2007). It is clear from the writings of Hill and her colleagues (Hill, Sleep, Lewis, & Ball, 2007) that the goal is to move the debate on assessment of teachers from a debate of argument and opinion to one of professional responsibility and evidence. These authors claim that there is a need for assessment instruments that are designed to “*investigate* what teachers know, and to associate that knowledge with their professional training and their instructional effectiveness” (ibid. p. 112). From this perspective, it is important to develop different approaches to measure teachers’ MKT. The measures in focus in this paper represent one such attempt.

Schilling and Hill (2007) describe their work on validating the MKT measures, and even when building on their work, we need to be aware of the fact that researchers believe that more efforts need to be made concerning the work of validation (Schoenfeld, 2007) in general and more specific the use of psychometric models as the IRT (e.g. Kane, 2007).

## METHODS

Efforts have been made to translate and adapt the 2004 elementary form A (MSP\_A04, see LMT 2010) of the MKT items into Norwegian (Mosvold et al., 2009). After the translation phase, a pilot study was organized in order to add to the

process of validating the translation and adaptation of the MKT items. The overall aim of the study was to investigate whether and how the MKT measures could be used in a Norwegian context (Fauskanger & Mosvold, 2010). The study includes a quantitative part where 142 teachers' MKT were measured and a qualitative part where a selection of teachers were interviewed in five focus groups. In this paper, we analyze data from the quantitative data only, using item response theory (IRT) models as an approach.

The form that was used consisted of two parts. Part 1 included a total of 61 items (30 item stems). Part 2 consisted of some questions related to the teachers' gender, teaching experience and background education in mathematics. Part 1 is the focus of attention here. Figure 2 illustrates the nature of the items. This example asks teachers to respond to a mathematical task situated in a teaching context.

11. Students in Mr. Hayes' class have been working on putting decimals in order. Three students — Andy, Clara, and Keisha — presented 1.1, 12, 48, 102, 31.3, .676 as decimals ordered from least to greatest. What error are these students making? (Mark ONE answer.)

- a) They are ignoring place value.
- b) They are ignoring the decimal point.
- c) They are guessing.
- d) They have forgotten their numbers between 0 and 1.
- e) They are making all of the above errors.

**Figure 2: Example from the set of released items (Ball & Hill, 2008).**

### **Item Response Theory (IRT) models**

Since MKT is not directly observable, the MKT items are meant to relate to the construct and can be viewed as one possible operationalization of the construct. Many measurement models could serve as a link to the observed latent world, and item response theory (IRT) is one such model (Edwards, 2009). The LMT project used IRT models to learn more about item performance characteristics, and in order to compare item performance in Norway and the U.S., we have followed the same approach.

A basic idea in IRT is that an observed item response is a function of person properties and item properties (Edwards, 2009). We have used two IRT models in the analysis of our data. A two-parameter model was used first, making it possible to identify items with high slope and item difficulty. The higher the slope is, the more variability in items responses is attributable to differences in the underlying construct. Item difficulty indicates the point on the ability axis where an individual would have a 50% chance of endorsing a particular item. Item difficulty is reported



in standard deviation units, and 0 is the average teacher ability. Items with negative difficulty indicate easier items, whereas items with positive difficulty indicate more difficult items.

Edwards (2009) found that adequate IRT parameter recovery is possible from as few as 200 respondents. However, a general rule is that the bigger the sample size is, the better the estimates get. The quality of the data also determines the number needed for adequate parameter recovery. Since the number of respondents in our sample is lower than 200 ( $n=142$ ), we also compared the results coming from a one-parameter model. Missing data is not used in parameter estimation in both models.

In addition to the psychometric analysis, we also present the test information curve and the reliability of the assessment. The test information curve shows how much information items provide for the individual teacher along the ability axis.

## RESULTS

In this section, the psychometric analyses are presented and discussed in order to determine if the U.S. developed measures perform well in Norway. We have used the program BILOG-MG (Zimowski, Muraki, Mislevy, & Bock, 2003) for the estimation and testing of item response theory (IRT) models.

Difficulty	$< -2$	$[-2, -1>$	$[-1, 0>$	$[0, 1>$	$[1, 2>$	$> 2$
Norway	9	14	22	9	5	2
U.S	5	19	20	12	2	3

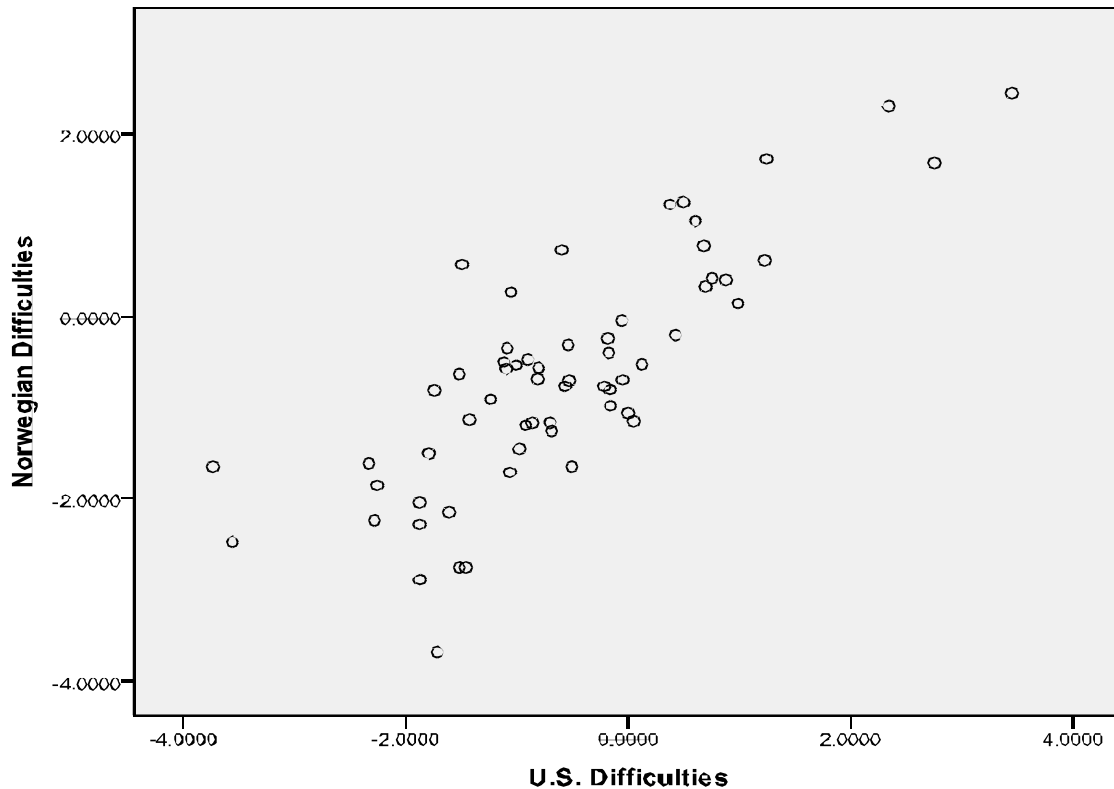
**Table 1: Numbers of items within different ability intervals in Norway and U.S.**

In order to study item performance characteristics in Norway, we first looked at the output from BILOG-MG using a two parameter IRT model. We used the same item names as in the original U.S. form. Data coming from the Norwegian sample show that one item has a negative point-biserial correlation of -0.116 (Item 17c). This indicates that respondents who answered other items correctly would most likely give the wrong answer to this item, which again indicates that the item is not working well in Norway and should be removed. The average difficulty for the Norwegian items was -0.649, and standard deviation 1.267 (average standard error 0.152), whereas in the U.S. the average item difficulty was -0.573, i.e. slightly higher. In both countries the scale used consists of a distribution of items of difficulty across the ability spectrum (see Table 1).

Figure 3 displays a scatter plot of the Norwegian item difficulties relative to item difficulties found in the U.S. study (LMT 2004). The correlation between the relative item difficulties is relatively strong (0.804 and  $p\text{-value} < 0.0005$ ). Similar strong

### Working Group 11

correlation is also reported in an Indonesian study where two MKT geometry measures were adapted and used (Ng, 2009).

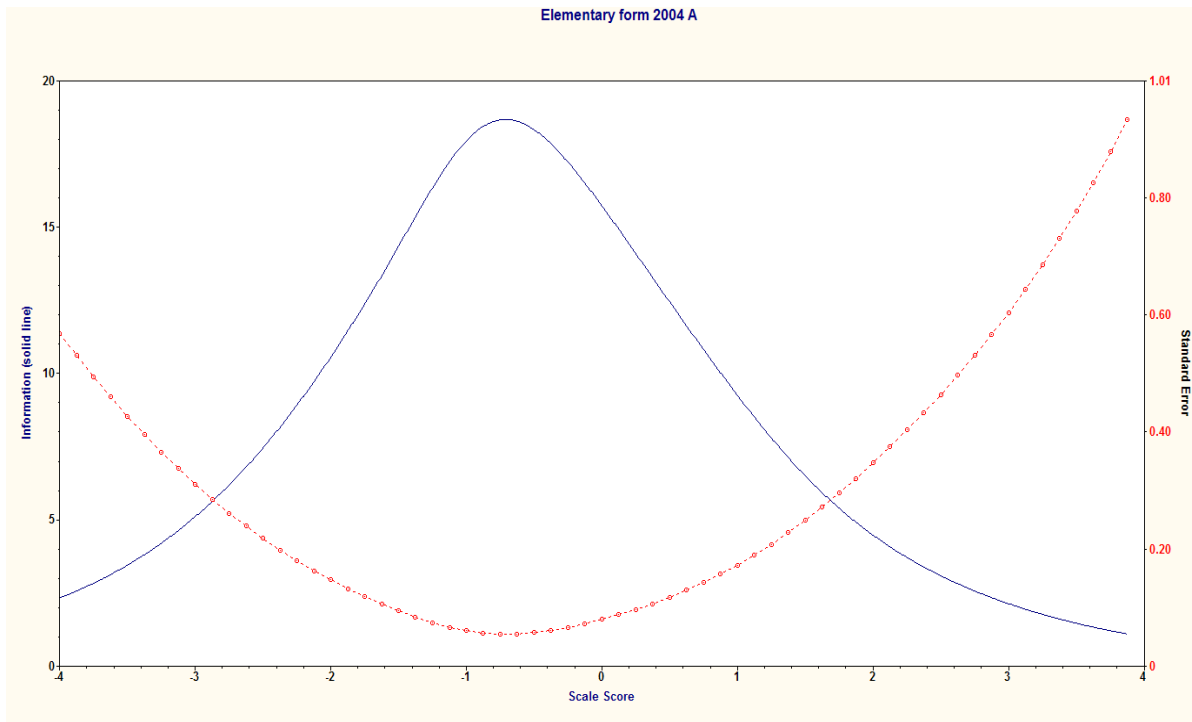


**Figure 3: Scatter plot of the relative difficulties of items in the Norwegian adapted form and the U.S. version of the same form, using a two-parameter model.**

The average Norwegian item slope was 0.688 with standard deviation 0.287 (average standard error 0.152), while the average item slope found in the U.S. was 0.533. We found the correlation between U.S. slopes and the Norwegian slopes to be rather low (0.375). This can be explained by the low sample size.

Items with slopes lower than 0.5 are normally considered as problematic because they do not discriminate between teachers with high mathematical knowledge for teaching and those with lower mathematical knowledge for teaching. We find that the majority of the items have slopes higher than 0.5, both for items in the adapted form used in Norway (51 out of 61 items) and for the original U.S. form (41 out of 61 items). In both countries, only two items had slopes below 0.3.

Each item also has its own information function that is calculated from the item's parameters, and the peak of the function is a value of the item difficulty. The item slope determines how peaked the test information curve is, and the higher the slope value is, that item will provide more information around this difficulty level. To understand how our test is functioning as a whole, the items information's function can be combined into a test information function. The test information curve shows how the test measures teachers across the ability levels.



**Figure 4: Test information function (solid line) and standard error (dotted line) curves for the Elementary 2004 A form.**

In Figure 4 we have plotted the test information function for the adapted Norwegian form. 0 is the mean teacher's ability score. The peak of the test information function is at -0.75, and this test measures best individuals between -2.5 and 2 standard deviations from the mean ability level. The corresponding standard error for this range is below 0.3, which is under a third of a standard deviation. Reliability for the two-parameter IRT model was 0.9145, higher than what was found in the U.S. study.

Due to the relatively small number of respondents in Norway, a one-parameter IRT analysis was also performed. However, since the U.S. item characteristics are coming from a two-parameter model, we will not compare item characteristics coming from two different models (and for slopes it is meaningless). We observed that the correlation between the U.S. difficulty and the Norwegian difficulty was less strong using a one-parameter model (0.775 compared to 0.804). For the one-parameter model the reliability index of the test was 0.899, and maximum information for -0.8750.

## CONCLUSIONS

During the translation of the items and in focus group interviews with teachers, we became worried if the measures would function in Norway. Several teachers expressed concern about some of the items, finding them difficult and hard to answer. Some teachers had expected a test with questions similar to what can be found in a textbook for their own students, and being challenged with the

understanding of e.g. nonstandard student solutions in a test situation was frustrating (Bjuland, Mosvold, & Fauskanger, in progress). In our analysis of the data, we find that the scale was composed with items with difficulties over a broad ability range (from -3.683 to 2.443) in the same manner as in the original U.S. scale (from -3.734 to 3.454). We also find that there is a strong correlation between the item property characteristics found in Norway and in the U.S. The reliability index is high (0.9145), and from only looking at the item property characteristics, we argue that the adapted measures are working well in Norway and that the measure performs as intended. The scale is performing well for an individual with the ability between -2.5 and 2 standard deviations from the mean ability level. However, this study has several limitations. First, the sample size is low, making the use of a two-parameter model questionable. Second, our study only examines the content knowledge domain, and we would therefore like to do further studies involving the knowledge of content and students (KCS) and the knowledge of content and teaching (KCT) domains. Third, we also believe that observations in classrooms need to be done in order to investigate if the mathematical quality of instructions is linked to MKT measures in the same way as found in the U.S. For items with difference in item performance, we are also looking into the content of the items trying to see if for instance cultural differences can explain this difference.

### NOTES

1. Our research project has been supported by OLF, The Norwegian Oil Industry Association.

### REFERENCES

- Ball, D.L., & Hill, H.C. (2008). Mathematical Knowledge for Teaching (MKT) measures. Mathematics released items 2008. Retrieved 12th of September, 2010, from [http://sitemaker.umich.edu/lmt/files/LMT\\_sample\\_items.pdf](http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf)
- Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Baxter, J.A. & Lederman, N.G. (1999). Assessment and measurement of Pedagogical Content Knowledge. In J. Gess-Newsome & N.G. Lederman (Eds.), *Examining Pedagogical Content Knowledge* (pp. 147-161). Dordrecht: Kluwer Academic Publishers.
- Bjuland, R., Mosvold, R., & Fauskanger, J. (in progress). "I think it should be on the level I normally teach!" – How teacher bias might influence the internal validity when measuring MKT.
- Delaney, S., Ball, D.L., Hill, H.C., Schilling, S.G., & Zopf, D. (2008). "Mathematical knowledge for teaching": adapting U.S. measures for use in Ireland. *Journal of Mathematics Teacher Education*, 11(3), 171-197.

- Drageset, O.G. (2009). Exploring mathematical knowledge for teaching. In M. Tzekaki, M. Kaldrimidou & C. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 473-480). Thessaloniki, Greece: PME.
- Edwards, M.C. (2009). An introduction to Item Response Theory using the need for cognition scale. *Social and Personality Psychology Compass*, 3(4), 507-529.
- Fauskanger, J., & Mosvold, R. (2010). Undervisningskunnskap i matematikk: Tilpasning av en amerikansk undersøkelse til norsk, og læreres opplevelse av undersøkelsen. *Norsk Pedagogisk Tidsskrift*, 94(2), 112-123.
- Garet, M.S., Porter, A.C., Desimone, L., Birman, B.F., & Yoon, K. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal*, 38(4), 915-945.
- Hill, H.C. (2007). Technical report on number and operations content knowledge items – 2001-2006. Learning Mathematics for Teaching project, University of Michigan.
- Hill, H.C., Sleep, L., Lewis, J.M., & Ball, D.L. (2007). Assessing teachers' mathematical knowledge. What knowledge matters and what evidence counts? In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-156). Charlotte, NC: Information Age Publishing.
- Hill, H.C., Blunk, M.L., Charalambous, C.Y., Lewis, J., Phelps, G.C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430-511.
- Hill, H.C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Kane, M. (2007). Validating measures of mathematical knowledge for teaching. *Measurement: Interdisciplinary Research and Perspectives*, 5(2), 180-187.
- LMT (2010). *Learning Mathematics for Teaching Project (LMT)*. Retrieved 12th of September, 2010, from <http://sitemaker.umich.edu/lmt/about>
- LMT (2004). *Frequencies, slopes and difficulties*. Statistics from NCOP and PFA Equations forms (GEO stats from 2002), A04. Learning Mathematics for Teaching project, University of Michigan.
- Lysne, A. (2006). Assessment theory and practice of students' outcomes in Nordic countries. *Scandinavian Journal of Educational Research*, 50(3), 327-359.
- Mosvold, R., Fauskanger, J., Jakobsen, A., & Melhus, K. (2009). Translating test items into Norwegian - without getting lost in translation? *Nordic Studies in Mathematics Education*, 14(4), 9-31.

- Ng, D. (2009). *Translating and adapting the mathematical knowledge for teaching (MKT) geometry measures for Indonesia*. Paper presented at the Annual Meeting of the American Educational Research Association, San Diego, CA.
- Ponte, J.P., & Chapman, O. (2006). Mathematics teachers' knowledge and practice. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education - past, present and future* (pp. 461-494). Rotterdam: Sense Publishers.
- Reutzel, D.R., Dole, J.A., Read, S., Fawson, P., Herman, K., Jones, C.D., et al. (in press). Conceptually and methodologically vexing issues in teacher knowledge assessment. *Reading & Writing Quarterly*.
- Schilling, S.G., & Hill, H.C. (2007). Assessing measures of mathematical knowledge for teaching: A validity argument approach. *Measurement: Interdisciplinary Research and Perspectives*, 5(2&3), 70-80.
- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Schoenfeld, A.H. (2007). The complexities of assessing teacher knowledge. *Measurement: Interdisciplinary Research and Perspectives*, 5 (2&3), 198–204.
- Sirnes, S.M. (2005). *Flervalgsoppgaver - konstruksjon og analyse*. Bergen: Fagbokforlaget.
- Stigler, J.W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
- Zimowski, M.F., Muraki, E., Mislevy, R.J., & Bock, R.D. (2003). BILOG-MG 3 for Windows: Multiple-group IRT analysis and test maintenance for binary items [Computer software]. Lincolnwood, IL: Scientific Software International, Inc.

# **BELGIAN AND TURKISH PRE-SERVICE PRIMARY SCHOOL MATHEMATICS TEACHERS' METAPHORICAL THINKING ABOUT MATHEMATICS**

Çiğdem Kiliç

Mersin University, Education Faculty

*The aim of this study was to examine Turkish and Belgian pre-service primary school mathematics teachers' perceptions of mathematics. 42 Belgian and 41 Turkish mathematics teacher students participated in the study. They were asked to indicate their metaphors about mathematics and explain the reason. The results of the study revealed that Belgian pre-service teachers tended to use action metaphors while Turkish pre-service teachers tended to use emotion metaphors for explaining mathematics. It is concluded that cultural factors and the teacher education system are the likely causes of such different metaphors.*

Key words: teacher education, pre-service primary school mathematics teacher, metaphor, comparative studies in mathematics education

## **INTRODUCTION AND THEORETICAL FRAMEWORK**

Beliefs are an interesting and powerful research object. In order to develop effective teacher training, the belief systems of pre-service teachers need investigating and metaphor, in this respect, is an important tool (Reeder, Utley and Cassel, 2009; Schinck et al., 2008). As indicated by Presmeg (1998), metaphor is derived from the Greek, *metaphora*, meaning to 'transfer' or 'carry over' and is an implicit form of analogy. According to Leavy, McSorley and Bote (2007),

Metaphors have a coherence and internal consistency, which provide insights into ideas that are not explicit or consciously held. They can also be evocative, stimulating both self and others to tease out connections which might not be made use of by direct questions (p.1220).

Metaphor, long thought to be just a figure of speech, has recently been shown to be a central process in everyday thought. Metaphor is not a mere embellishment it is the basic means by which abstract thought is made possible (Lakoff and Nunez, 2000). Reasoning with metaphor is considered a fundamental way of human thinking and communication, as can be seen in our everyday use of abstract concepts (English, 1997). Metaphors have many advantages for educators and learners. Metaphors link abstract ideas to concrete images, thus evoking an experiential connection. Metaphoric thought supports embodied knowing and is not merely a communication or visualization device (Sternberg, 2008). Metaphor is seen as a strategy for explicating peoples' views about a phenomenon (Sternberg, 2008). It is a research tool (Reeder, Utley and Cassel 2009) showing relationships and focusing on similarities.

Metaphors are explicit explorations of personal views of mathematics and understanding new images of mathematics can contribute to the explication of teachers' views of mathematics (Sterenbergh, 2008). If our conceptual systems and thought processes are largely metaphorical (Lakoff & Johnson, 2003) analyzing peoples' metaphorical thinking is a good way to understand what happens in their mind. We understand metaphor as finding a mapping between the target domain, that is, the topic of metaphor, and the source domain (English, 1997).

Metaphor-related educational research has been undertaken on specific content like mathematics (Sterenbergh, 2008; Schinck et al., 2008), more broadly on teaching and learning (Saban, Kocbeker and Saban, 2007; Mahlios and Maxson, 1998; Leavy, McSorley and Bote, 2007) and in juxtaposition, the teaching and learning of mathematics (Reeder, Utley and Cassel, 2009). Research on use of metaphors to describe images of mathematics shows diversity. For example, Lim (1999) categorized mathematics as a journey, a skill and a game or a puzzle. Sterenbergh (2008) categorized the metaphors produced by pre-service teachers into mathematics as a battle, a mountain, a bridge and a language. Noyes (2006) explored pre-service mathematics teachers' beliefs about mathematics and the teaching and learning of mathematics. Among participants' metaphoric descriptions of mathematics were found four metaphors like structure, journey, language and toolkit. In the study of Reeder, Utley and Cassel (2009) production, journey and growth were identified as pre-service teachers' metaphors about mathematics teaching and learning.

The aim of this study was to examine Belgian and Turkish pre-service primary school mathematics teachers' beliefs about mathematics. The beliefs of participants were revealed using metaphorical expressions. There were two main questions in the study;

- 1) What kind of metaphors are used by Belgian and Turkish participants?
- 2) Are there any differences between two countries' participants' metaphors for mathematics?

## **METHOD**

### **Participants**

The participants of the study were 42 Belgian and 41 Turkish pre-service primary school mathematics teachers. All participants were enrolled in mathematics method and basic mathematics courses and were able to articulate effectively their metaphorical thinking about mathematics. In Belgium a primary school mathematics teacher trainee program is three years, in Turkey it is four. Demographically, all participants were aged between 20 and 22.

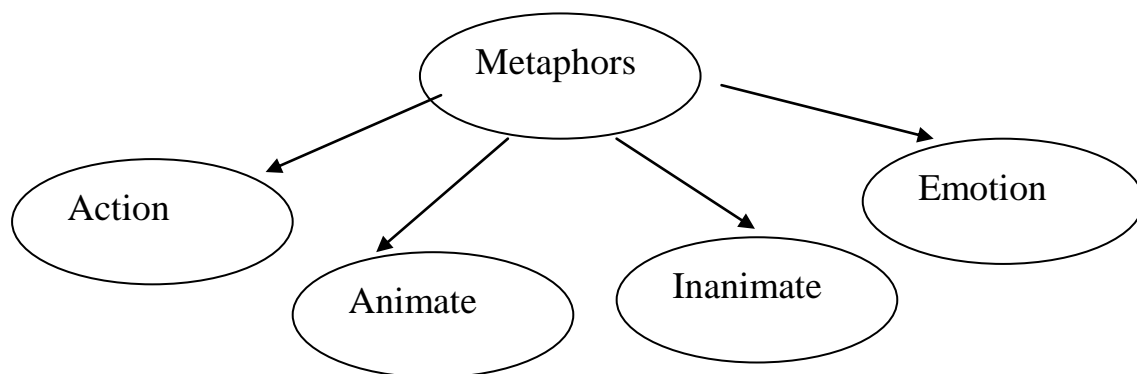


## Data Collection

Data were collected during university sessions. Metaphor was explained to participants before they were presented with a written form, “*Mathematics is like... Because....*” Participants were then invited to write their images of mathematics and their justification for them. The data were collected during an Erasmus Staff Mobility project in March 2010 in Belgium and at the beginning of April in Turkey.

## Data Analysis

Students’ written metaphoric expressions were read, listed, organised, coded and labelled according to their common characteristics. One colleague also helped the researcher with data analysis. The colleague and researcher coded metaphors separately. First, all metaphors were listed and similar metaphors were combined. Listed metaphors were reorganized. All metaphors were coded and similar coded were combined. Coded metaphors were labelled (if, for example, a metaphor had been coded as human or plant it would have been characterised more generally as animate and so forth). In this manner, metaphors were categorized in four different clusters based on the patterns that emerged as animate, inanimate, emotion and action (see in Figure 1). Inter-rater reliability was calculated and found to be 98%. When a repeated metaphor was found, a (+) sign was written beside it to show its frequency.



**Figure 1: Metaphors that emerged in the study**

## RESULTS

In the study animate, inanimate, action and emotion are the main metaphorical themes, although in all categories there were sub-categories. In Table 1 can be seen the metaphors produced by participants for mathematics, while in Table 2 sub-categories and frequency of metaphors are given.

## Working Group 11

**Table 1: Metaphors for mathematics produced by participants**

<b>Turkish</b>	<b>Belgian</b>
A star in sky that you know the name	A toy
Reality itself	Building a house
Depression	A ghost
A woman	The roots of a planet
A ladder that you never reach summit	A tree +++
A puzzle +++	A puzzle +++++++
A life ++++	Learning how to drive
An ocean+++++	A volleyball
A cyclic chain	A computer
A buckle	A chain
Dividing Turkish bagel equally	A circle
A infinite road	A spoon
Returning earth	A summer
A game	A pyramid ++
A matryoshka baby	A maze
Universe +++	A game
Knowing everything	An infinity
A pencil	A fairy tale
A maze	Climbing a mountain ++
A wardrobe	A flower ++
A jinn Ali figure	An air
A chocolate	Gardening
Erupted corn	Playing the piano
Lifeblood	Riding a bicycle
Growing a human	Running a marathon
A kite	A magic box ++
An eye	The world
A ball of yarn	Playing an instrument
An artist painting	A house

**Table 2: Frequency of metaphors for mathematics produced by participants**

Country	Animate		Inanimate		Emotion		Action	
	Human	Plant	Thing	Place	Abstract	Concrete	Action	Game
Belgium	-	5	5	2	5	3	8	11
Turkey	2	-	8	-	5	14	6	5

## Working Group 11

As seen from the figures of Table 2 sub-categories of emotion- and action-related metaphors were used by both two groups. Some differences can be seen among the animate and inanimate categories.

### Animate

In the animate category human and plant sub-categories were used. Human metaphors were used by Turkish and plant metaphors by Belgian participants. Some examples and participants' reasons are given in Table 3.

**Table 3: Participants' animate metaphors for mathematics**

Animate	Country	Mathematics is like....	Because...
Human	Turkey	an eye.	when you look you see the original.
		a woman.	to understand mathematics look like to understand a woman, in fact you can't understand both of them easily.
Plant	Belgium	a flower.	mathematics needs to grow. A flower needs water to grow. We need mathematics (questions, exercises) to grow. Also the children need to grow, so we give them water (theory, exercise).
		a tree.	you start with a base. No knowledge, when you start, but the longer you study math, the more you discover.

### Inanimate

In this category place and object sub-categories were used by Belgian and Turkish participants. Although Belgian participants used object and place metaphors, Turkish participants used only object metaphors for expressing mathematics. Some examples can be seen in Table 4.

**Table 4: Participants' inanimate metaphors for mathematics**

Inanimate	Country	Mathematics is like....	Because...
	Belgium	a computer.	you really have to learn to use it and when you understand how it works, you can use it to do a huge amount of things.
		a pyramid.	in primary school you learn the principles of maths those are the fundamentals of math. When one brick placed wrong, the pyramid will fall

### Working Group 11

Object			down. After a time you will have build a pyramid.
	<i>Turkey</i>	a wardrobe.	every topic of mathematics is clothes and mathematics looks like a wardrobe which contains all of them.
		a chocolate.	what you feel while occupying mathematics looks like the taste of chocolate while you are eating it.
Place	<i>Belgium</i>	a house.	first of all you learn the fundamentals and later you learn difficult things.
		the world.	you have all kinds of people. In mathematics there are so many parts, you can't learn them all.

### Emotion

In the emotion category abstract and concrete sub-categories were found. Both groups used this metaphor for explaining mathematics. In Table 5 some examples from participants are given.

**Table 5: Participants' emotion metaphors for mathematics**

Emotion	Country	Mathematics is like....	Because...
Abstract	<i>Belgium</i>	air.	mathematics is everywhere, you can find it in the nature, in stars...
		a circle.	everything is constructing.
	<i>Turkey</i>	a ladder that you never reach its summit.	the more you climb up stairs, the more you meet new stairs. You never reach its summit.
		a depression.	when you think you solve the problem, you meet new problem. After a while you confront a more difficult problem.
Concrete	<i>Belgium</i>	a magic box.	you always get surprised when you work with it.
		the roots of a plant.	there are many ways to reach solution a problem.
	<i>Turkey</i>	a star in the sky that you know the name.	you know only the things that you probe, identify. You don't know the things that you never gain, set forth.
		an ocean.	there is no border of mathematics. The more you put out to sea, the more you can lose your way. In it, there are a lots of wind and wave.

## Action

In this category action and game sub-categories were found. Some examples from participants' produced metaphors are given in Table 6.

**Table 6: Participants' action metaphors for mathematics**

Action	Country	Mathematics is like....	Because...
Action	Belgium	playing an instrument.	there is a lot to learn about and one small mistake can get the entire exercise wrong.
		building a house.	first you need to have a good fundaments, bricks to build, a roof and so on, later on you can give the house your own style.
	Turkey	painting an artist.	a painter paint freely, mathematics also is the same. It is a kind of information network.
		dividing a Turkish bagel equally.	in mathematics there is an equity, an order and an aesthetic.
Game	Belgium	a game.	you play with numbers. There are certain rules you have to follow.
		a puzzle.	you have to put together all the pieces before you really understand the whole picture.
	Turkey	a game.	there are definite rules of game and so is mathematics.
		a puzzle.	in mathematics you feel you are in dilemma and try to find the exit.

## DISCUSSION AND CONCLUSION

Participants produced animate, inanimate, emotion and action metaphors for mathematics with, in every category, sub-categories. Analyses of participants' metaphors for mathematics indicated that Turkish and Belgian participants had different images of mathematics; a finding likely to have several causes. A number of writers argue that different views of mathematics, as reflected in the metaphors presented, are linked to interpretations of how mathematics is played out in their own lives, both within and outside of school (Schinck et al., 2008; Noyes, 2004; Soto-Andrade, 2007). Indeed, as indicated by Stigler and Hiebert (1999), in Noyes (2004),

the way mathematics duplicates the text through which is taught varies according to location, both geographical and cultural, at international and national levels. When the reasons indicated by participants were scrutinized it can be concluded that differences in Belgian and Turkish pre-service primary school mathematics teachers' metaphoric expressions may stem from participants' cultural background, individual experiences about mathematics within and outside of their school and also educational system.

Saban, Kocbeker and Saban (2007) emphasized that metaphors are selective. When asked to complete the statement *Mathematics is like... Because...* their participants' metaphoric expressions about mathematics were different from each other. In this study, however, some similar metaphors were used by Belgian and Turkish pre-service primary school mathematics teachers. This was seen in the game sub-category with words like *puzzle, maze and game*. Both Turkish and Belgian students exploited puzzle metaphors to reflect their thoughts about mathematics. In similar vein, the metaphorical categories of emotion and action were used predominantly by participants from both countries. That can be explained by the characteristics of mathematics itself. Participants explained the abstract nature of mathematics by using both abstract and concrete phenomena. As indicated in the study of Lakoff and Nunez (2000) one of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts.

Comparing different countries' pre-service mathematics teachers' metaphorical mathematics-related expressions may help us to understand the extent to which such devices are culturally located. The evidence above suggests that the more we use metaphor analysis as a research and evaluation tool, the more we will understand the belief and thought systems of people.

## REFERENCES

- Soto-Andrade, J. (2007, February). *Metaphors and cognitive modes in the teaching-learning of mathematics*. Paper presented at the Fifth Congress of the European Society for Research in Mathematics Education (CERME5), Larnaca, Cyprus.
- English, L. (1997). Mathematical reasoning: Analogies, metaphors and images. In English, L., & D. Mahwah (Eds.), *Analogies, Metaphors, and images: Vehicles for mathematical reasoning*. New Jersey: Lawrence Erlbaum Associates Publisher.
- Lakoff, G., & Nunez, R.E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York: Basic books.
- Lakoff, G., & Johnson, M. (2003). *Metaphors we live by*. Chicago: University of Chicago Press.
- Leavy, A.M., Mc Sorley, F.A., & Bote, L.A. (2007). An examination of what metaphor construction reveals about the evolution of preservice teachers' beliefs about teaching and learning. *Teaching and Teacher Education*, 23, 1217-1233.

## Working Group 11

- Lim, C.S. (1999). Using metaphor analysis to explore adults' images of mathematics. *Philosophy of Mathematics Education Journal*, 12.
- Mahlios, M., & Maxson, M. (1998). Metaphors as structures for elementary and secondary preservice teachers' thinking. *International Journal of Educational Research*, 29, 227-240.
- Noyes, A. (2004, July). *The poetry of the universe: New mathematics teachers' metaphoric meaning-making*. Paper presented at the 28th conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.
- Noyes, A. (2006). Using metaphor in mathematics teacher preparation. *Teaching and Teacher Education*, 22, 898-909.
- Presmeg, N.C. (1998). Metaphoric and metonymic signification in mathematics. *Journal of Mathematical Behavior*, 17(1), 25-32.
- Reeder, S., Utley, J., & Cassel, D. (2009). Using metaphors as a tool for examining preservice elementary teachers' beliefs about mathematics teaching and learning. *School Science and Mathematics*, 109, 290-297.
- Saban, A., Kocbeker, B.N., & Saban, A. (2007). Prospective teachers' conceptions of teaching and learning revealed through metaphor analysis. *Learning and Instruction*, 17, 123-139.
- Schinck, A.G., Neale, H.W., Pugalee, D.K., & Cifarelli, V.V. (2008). Structures, journeys, and tools: Using metaphors to unpack student beliefs about mathematics. *School Science and Mathematics*, 108, 594-599.
- Sterenbergh, G. (2008). Investigating teachers' images of mathematics. *Journal of Mathematics Teacher Education*, 11, 89-105.

# PROBLEM SOLVING AND OPEN PROBLEMS IN TEACHERS' TRAINING IN THE FRENCH AND MEXICAN MODES

Alain Kuzniak\*, Bernard Parzysz\*, Manuel Santos-Trigo\*\*, Laurent Vivier\*

\* Laboratoire de Didactique André Revuz, Paris-Diderot University, France

\*\* Center for Research and Advanced Studies, Cinvestav-Mexico

*Mathematics education may differ in traditions and theoretical approaches throughout countries, but it is generally acknowledged that problems and tasks play an important role in learning. Two research teams, one in Mexico and the other in France, decided to work together in giving mathematics teachers in both countries the same task to solve during a training session. This paper summarizes the results of this study. It shows that in the two countries the reference to different theoretical frameworks results in putting emphasis on different but complementary aspects of teaching.*

## INTRODUCTION

In different countries, mathematics teachers' education and research programs in mathematics education exhibit differences both in principles and ways to implement them. There are also different research traditions or paradigms to frame mathematics education projects, but all of them recognize that mathematical problems or tasks play an important role in fostering the development of teachers' and students' mathematical knowledge. Contrasting different systems and approaches, in this study French and Mexican, may be of interest to identify features, common goals, and differences associated with their traditions in education. Our research interest is to analyze and document both the processes put into play by pre-service teachers while working directly with problems and later the way they use these problems in their teaching practices.

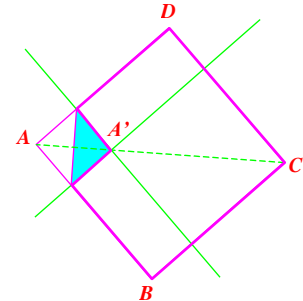
Specifically, the purpose of our common project is to investigate ways in which pre-service and in-service teachers work on a series of problems in order to prepare, organize, schedule and implement their lesson plans. Moreover, the development of technologies in class is quickly changing teaching practices and it appears necessary to take it into account. During their interaction with the tasks, pre-service teachers were encouraged to use computational tools.

In the present contribution, we will focus on the initial step of the project in which a common problem was used as a teachers' training tool in both systems of education. Our main interest was to see the implementation in contexts in which "solving a problem" has not necessarily the same meaning. Introduced by Polya (1945), problem solving constitutes the traditional framework for teachers' training in Mexico. In France, the traditional framework for teachers' training relies much more on Brousseau's Theory of Didactical Situations (Brousseau, 1998).



Taking into account the differences in the referent frameworks in France and in Mexico, we chose the following mathematical problem in both countries for teachers' training to compare how it was used. Other problems were discussed, but not chosen by the French team because of the difficulty to use them in secondary teaching.

The task: A square piece of paper  $ABCD$  with side  $l$ , has a white front side and a blue back side. Corner  $A$  is folded over point  $A'$  on diagonal  $AC$ . Where should point  $A'$  be located on this diagonal (or: how far is  $A'$  from the folding line) in order to have the total visible area half blue and half white? (Based on Carlson & Bloom, 2005, p.71).



For each country, we expose briefly the main features of the theoretical framework used and describe and analyze teachers' training scenarios based on this problem. We start with teachers' training in Mexico City, then with teachers' training in Blois (France). Furthermore we analyze the main elements of the comparison.

## TEACHERS' TRAINING SESSIONS IN MEXICO

### Problem solving

Mathematical problem solving perspectives have framed and oriented the development of multiple research programs and supported curriculum proposals in mathematics education in the last four decades (Schoenfeld, 1985, NCTM, 2000). In general, a problem-solving route to learn mathematics relies on an inquiry or inquisitive approach to deal with mathematical ideas or problems (Santos-Trigo, 2007); however, there are multiple factors around its identity and practical applications. "The patterns that form a problem-solving identity are complex, involving varied motivational patterns, affective reactions and cognitive and social engagement in different circumstances both within a given task and across tasks" (Lesh & Zawojewski, 2007, p. 776).

In teachers' education, it is important to address issues regarding the nature of problems, the rationale for using them, and the goals that are intended to be achieved during the problem solving sessions. Similarly, it is also important to elaborate on ways to organize and develop the problem solving sessions to be held with pre-service teachers. For that, it is important to discuss the dynamic of the problem solving scenarios in which teachers and students will work on the problems.

### The context

The problem was given on a sheet of paper to a group of 14 pre-service teachers. Afterwards, they were asked to read and make sense of the problem individually, and later they worked on the problem in pairs. Then, each pair had the opportunity to present their approaches to the problems to the others. At this stage, all the

participants and the instructor could ask for concept explanation or clarification. At the end of the session, the instructor encouraged the participants to contrast the different models used to solve the problems. In general terms, the sessions' dynamics fostered the participants' inquiring approach to the problems. For example, when the participants introduced an idea or a problem representation, the instructor questioned them and encouraged them to reflect on their ideas and other related concepts. Similarly, when the participants ran out of ideas the instructor either oriented the discussion (through questions) or asked other participants for suggestions to be considered. The aim was to encourage all the participants to formulate and then go deeper into the questions. In this perspective, the organization of the sessions is consistent with the activities recommended by Mason and Johnston-Wilder (2006) to have students participate during problem-solving discussions. In particular, the authors identify four ways to organize students' participation in developing and discussing their mathematical knowledge:

*Individual* work allows learners to review, consolidate, and develop their facility, as well as to reconstruct for themselves.

Work in *pairs* allows learners to try out ideas on each other before offering them to a wider group; it also provides an opportunity for learners to consider something that has happened or been said, and to generate more ideas about this [the problem] than an individual is likely to produce when working alone.

Work in *small groups* allows a multitude of ideas to be generated, and also allows a large task to be split up amongst several people; with discipline, small groups can provide a forum for discussing ideas, modifying conjectures, and coming to a consensus with supporting reasons and justifications.

*Collective* and *plenary* work allows everyone to hear about novel ideas and approaches, and to see teachers or peers displaying their mathematical thinking (p. 52, italics in the original).

It is important to mention that the pre-service teachers were taking a problem-solving course when they worked on the task. The aim of this course was to use different computational tools in problem solving activities. The participants worked on the task during two sessions (2 hours each) held at the end of the semester. The goal of this report is not to analyze individual approaches to the task but to focus on characterizing mathematical behaviors exhibited by the group of participants as a whole. The solution process is structured into episodes which distinguish key principles associated with the finding of solutions (Santos-Trigo & Camacho; 2009).

### **First Episode: An overarching inquisitive principle**

The core aspect in this approach is to conceptualize and examine the contents of the problem (definition, theorem, etc.) in terms of questions or dilemmas that need to be explored. An overarching principle that permeates the entire problem solving process is that teachers and students should transform the problem statement into a set of

meaningful questions to be examined. Thus, the questions addressed by the group included: What are the relevant properties of a square? What does it mean to fold a vertex over to a point on the diagonal? And also questions to examine properties of the involved figures: What properties does the folding line hold? What figures are formed in blue and white colors? What happens to both areas when point  $A'$  is close to point  $A$  or close to point  $C$ ? What type of triangle is formed (blue region) when point  $A$  is reflected with respect to the folding line to determine  $A'$ ? How can we calculate the areas of the blue and white regions? In general, the participants observed that for different positions of point  $A'$  (*i.e.* when  $A'$  is moved along the diagonal) the area of one region increases while the other decreases. Then, they agreed that there would be a position for point  $A'$  where both areas were equal. At this stage, two main approaches to solve the task appeared: one focusing on representing the task algebraically and the other relying on constructing a dynamic model of the problem.

### **Second Episode: An exploration principle**

The figure provided in the problem statement (some labels were added) was used by some participants to represent the triangle area as  $x^2/2$  and the other as  $l^2 - x^2$ , which led them to solve the equation  $l^2 - x^2 = x^2/2$  in order to find the position of point  $A'$ .

The dynamic approach carried out by some participants required thinking of the problem in terms of mathematical properties and software commands. This approach allowed them to visualize and quantify the area variation of both regions as a result of moving a point on the diagonal.

During the plenary discussion, the participants acknowledged that the two approaches complement each other because they gave them an opportunity for examining the task from various perspectives. While the algebraic method gives a general solution to the problem, the dynamic approach offers not only the possibility of exploring visually the area variation of both regions; but also to graphically interpret the solution achieved algebraically (intersection point of curves).

### **Third Episode: The principle of extension and generalization**

The participants asked whether the methods used to compare the area variations of the square could be extended to the case of a rectangle. They recognized that in the process of solving a problem or understanding a mathematical concept or idea it is always important to reflect on the scope of the solution or applications of that concept. In this context, the method used to explore the square case dynamically was adjusted to deal with the rectangle case.

### **Fourth Episode: Reflection Principle**

Participants generally recognized that a problem solving approach for learning mathematics involves the construction of mathematical representations of concepts, situations or problems, in order to find and explore mathematical relations. In this

process, any mathematical statement (problem, definition, statement, content) is conceptualized as a starting point for teachers, students, or problem solvers in general to search for different ways of solving and extending a problem. In this perspective, the use of technological tools offered the participants the opportunity to explore some parameters of the task from varied angles and as a consequence, they could relate different ways of reasoning about the problem. For example, the construction of a dynamic model requires from the problem solver a functional approach without defining explicitly a function which is necessary with an algebraic approach. However, both models complement each other in terms of making sense, visualizing, and generalizing a set of relations emerging during the solution process.

### **Conclusion**

In this approach, teachers tend to guide their practices through problem solving principles that lead them to search and explore diverse ways to solve the task. This particular process takes a long time for the participants to get appropriated and exhibited in their regular practices. However, they clearly recognized it as powerful approach to develop a deep comprehension of mathematical concepts and solving problems. Thus, the emphasis in teachers' education is to conceptualize the problem solving process as an opportunity for them to look for various ways to approach the tasks and to search for relations that emerge as a result of using different tools.

## **TWO TEACHERS' TRAINING SESSIONS IN FRANCE**

### **Problem situation, open problem and research narrative**

The notion of "problem situation" appeared in France in the 1980s in Brousseau's TDS, which is based on a socio-constructivist conception of learning. A problem situation is a learning situation aiming at fostering the acquisition of a new knowledge by the students. Its setting up implies identifying previously erroneous or weak conceptions among the students by analysing their errors. On this basis the teacher conceives of and sets up a situation presenting some specific features, namely: 1) be relevant for the cognitive objective aimed at; 2) have a meaning for the student; 3) allow him/her to begin the search for a solution; 4) be rich (in terms of mathematical and heuristic contents); 5) be possibly formulated within several conceptual "frames" (Douady, 1986).

The notion of "open problem" was introduced at about the same time (Arsac *et al.*, 1988, Arsac & Mante, 2007). In comparison with the problem situation, the aim of an open problem is methodological rather than cognitive. The students are induced to implement processes of a scientific type, *i.e.* experimenting, formulate conjectures, test them and validate them. The problem must belong to a conceptual domain in which students are somewhat familiar with, the wording (statement) has to be short and induce neither a solution nor a solving method.

The French official curricula for junior high school (BOEN 2008) integrated recently – though without naming them – open problem and problem situation, which refer to two complementary sides of mathematical work:

- in the case of an open problem the question is to find a genuine, personal solution, with one's own means, the general solution being out of reach;
- in the case of a problem situation the question is, starting from a specific problem, to elaborate a more general knowledge (concept, process...) which is intended to be institutionalised, socially acknowledged and mastered by all.

The notion of “research narrative”, which is explicitly linked with those of open problem and problem situation, appeared in France some twenty years ago (Bonafé *et al.*, 2002). It involves asking the student to write an account of the thought processes he/she has undertaken in order to solve a given problem, pointing out his/her ideas, successes, failures, etc. The features of the problem are the same as for an open problem, but it has often several questions and the student must be able to start a research, test his/her results and validate them. And, if possible, different solutions need to be considered.

### **Implementation in high school teachers' training in Blois**

Sessions on the folded square problem were organized with two groups of pre-service teachers: a “standard group” included 18 trainees in their first year of professional training and a “special group” included 6 trainees who had not been appointed to a permanent post at the end of the normal training year.

Besides this different context, the tasks given to the standard and special groups are quite the same. A material bi-colored square folded along its diagonal is used to explain the problem. The trainees have to compare both areas using different tools: software, spreadsheet, calculator, paper and pencil. They were asked to write a research narrative of their exploration and solution of the problem with the specific tool used and then to prepare a classroom session for their own students, to make explicit their mathematical aims and the material they would use with their students.

### **The standard group**

The described session is a 3 hour lecture, part of a course focused on theoretical issues aiming at initiating the trainees to Brousseau's theory and the pedagogical use of open problems. They were also initiated to pedagogical uses of Dynamic Geometry Software (DGS) and spreadsheet. They generally mastered very well DGS but not spreadsheet, which is not a common tool for mathematics students at university. No particular mathematical framework was privileged and students had to use some specific artefacts to solve the problem. The 18 trainees were split into 5 teams: 2 teams worked with *Geogebra*, 1 team with a spreadsheet, 1 team with a hand-held calculator and 1 team with paper and pencil. When they had completed

## Working Group 11

work, all the teams presented it to the others. After the presentations, they were asked to prepare a classroom session for their secondary students.

**Team with DGS.** The task is very open and dependent on knowledge about the software. Normally, the problem must be handled a bit more in the geometric frame than with other tools. Students had indeed a very good knowledge of the software and they immediately solved the problem by linking it to the curriculum for grade 9. We can follow the methods from the narratives:

- Drawing the folded square, linked with the area of both parts. The dragging point is point  $A'$  on the diagonal.
- Drawing two curves in a Cartesian system (one representing the area of the blue triangle and the other the area of the white hexagon). The link between the point and the area is made by using the *track* mode (not *locus*). The intersection point is reached by approximation.

Both teams indicated the objective of exploring the problem as an example of modelling. They retained that modelling and experimental approach in official documents must rely on the computer. These teams wanted to support students by giving them a rich environment favouring an exploration of the problem. They gave the same problem with the same instruction as that received during the course (for them the use of digital tools is not a problem for the high school students). In a second stage, they intended to work on the difference between exact and approximate values: the exact value controls the different results depending on the square size.

**Team with calculator.** The team focused on an approximate resolution of the equation of two curves and followed this method: free hand drawing modelling the situation, where  $x$  is a side of the right angle in the folded triangle; observation that it is impossible to find an exact geometrical construction of the point equalling the two areas (hence they decided to approximate the result with a calculator); plotting curves representing the areas in a particular case; approximating the result with a table of values and using the calculator zoom; solving algebraically a quadratic equation to get the exact value.

This team's objectives were using the calculator and writing out equations. They therefore used the activity to improve their students' knowledge of calculator. They retained the instrument for the course, using it to work graphically on the variation of two functions and the intersection points of curves.

**Team with spreadsheet.** This team started the research with a bi-colour paper and modelling. Students chose the height of the blue triangle ( $AA'/2$ ) as a variable. After discussion, they agreed to make the values vary on a range equivalent to the diagonal. Then they used a spreadsheet to increment the variable with a given step

## Working Group 11

and calculated the areas of the blue triangle and white hexagon, and their ratio. An approximate value was reached when this ratio was 1.

To determine objectives for the class, members of the team discussed sharply and did not agree, some wishing to continue with the spreadsheet, others finding no interest in this activity with the spreadsheet.

**Team paper-and-pencil.** This team changed its name into Paper-and-Brain Team. They immediately solved the problem by using two general equations.

Only junior high school teachers participated in this team (grades 7 and 8) and they concluded that the activity was too difficult for their students. They made a proposal for grade 10 students based on the resolution of a quadratic equation. They remained in a paper-and-pencil environment and did not plan to use software (the institutional pressure for using technologies is weaker on junior than on senior high school teachers). They spent time writing instructions for the students with no reference to an actual folding.

### The special group

Each trainee received by mail an envelope containing a square sheet of paper and the task they had to perform. The trainees were isolated from each other and the exchanges took place only during the two sessions. They had to undertake a research of the different possible solving strategies, choosing two environments (paper and pencil; calculator; DGS; spreadsheet) and to write a research narrative. They also had to write a complete course for one of their secondary classes and really teach this course. They had to write down their work and present it to the group (the first session was mainly devoted to mathematical research and the second to the lesson).

All trainees used a paper-and-pencil environment, which is not surprising. Calculator was only used by one trainee (approximate values were obtained with *GeoGebra* by visual adjustment of the areas, sometimes with the *slider* tool, in order to get better values); spreadsheets were not used at all. All trainees used the *GeoGebra* software for modelling the folding; no other dynamic geometry software was used.

Only the two trainees who taught in senior high school used the software for solving the problem by considering intersections of function graphs. It seems that the syllabus plays a great role for these two trainees, since the senior high school curriculum emphasises functions and graphs. This link between teacher's research and topics he teaches at school also appeared with a grade 8 teacher who focused his research on geometrical proofs, which are of great importance at grade 8.

Unsurprisingly, the tasks given to their students confirm this last point, since teachers care a lot about the subjects they have to teach. In spite of this obvious fact, a detail has to be mentioned: only the two senior high school teachers proposed a comparison of the areas to their students; all of the others changed the question into

“when are the areas equal?”. It is a fact that the comparison of functions by means of graphic strategies is an important subject at grade 10.

### **Conclusion**

In both French groups, the grade, or the teaching institution, in which a teacher teaches seems to influence not only the transposition of the problem for the students, but also his/her research when he/she solves a problem for himself. And this could be observed whether trainees knew that they had to set up a course for their students (special group) or not (standard group).

The software chosen in the special group was still *GeoGebra*, one could think that there was in fact no choice of the software. This is linked with software currently used: the standard group is richer for DGS and especially *GeoGebra*. We explain this in this way: *GeoGebra* is a free software, both a dynamic and analytic geometry software which can even replace a calculator, and for that reason the training and school institutions possibly put the stress on it. But this does not explain why spreadsheet was left aside (maybe because of a lack of teachers' competences?).

We also noted the importance of the artefact used during the session on the trainees' school planning: they stayed very close to the experience they lived during the training. This argues for repeating this kind of training sessions but changing the tools used by the teams.

### **COMPARISON OF FRENCH AND MEXICAN SESSIONS**

There are a lot of similarities between teachers' training in France and Mexico. Obviously, we can first mention structural organization similarities: number of trainees in a course, duration of sessions, trainees teaching in a real class. There are also some pedagogical organization similarities, such as sharing courses into individual work, work by groups (even by pairs) and plenary sessions. These similarities show that the way to plan teachers' training is quite equivalent from a pedagogical point of view.

The emphasis on the use of new technologies is another similarity. Obviously it relies on didactics and the interest for teaching mathematics, but we could also see an external pressure of society and more specifically curriculum demands. Nevertheless, spreadsheet seems to be left out in France which is not possible in Mexico because of the problem solving framework which demands to explore all solving possibilities.

The main difference is clearly the role of real secondary classes. In France it appears that the level at which the trainees teach has a great influence on their behaviours and, in teachers' training, teaching at secondary level is one of the main aims (virtually for the standard group and really for the special group). In Mexico, real classes seem to be out of the training since there is no reference to them in problem solving sessions. Obviously this is not the case because one of the aims of the Mexican problem solving sessions is that teachers would be able to do the same



activities with their students. But this is not explicitly worked, contrary to French teachers' training.

Another difference could be related to the difference in didactic frameworks reference. In France the trainees' work began with a material square whereas in Mexico the situation was modelled under the form of a geometrical diagram.

## CONCLUSION

These examples from two different countries with different cultures and traditions about mathematics education show that in each of them the stress is put on different points: In Mexico solving strategies and associated scenarios or episodes, in France the influence of tools and implementation of the problem situation in a classroom. Both aspects are important for the training of mathematics teachers: On the one hand they must be prepared to understand, assess and support any solution proposed by their students, and on the other side they need to identify the contribution of the situation to the students' knowledge and consider how to bring it into play in a real class. We believe that the development of such cross studies can have an influence on the training of pre- and in-service teachers and finally help them to develop a more holistic view of teaching.

## REFERENCES

- Arsac, G., Germain, G., & Mante, M. (1988). *Problème ouvert et situation-problème*. Ed. IREM de Lyon.
- Arsac, G. & Mante, M. (2007). *Les pratiques du problème ouvert*. Ed. IREM/CRDP Lyon.
- Brousseau, G. (1998). *Théorie des situations didactiques*. Ed. La Pensée sauvage.
- BOEN (Bulletin officiel de L'Education nationale) (2008). *Programme de mathématiques du collège*. Bulletin spécial 6. Ed. Ministère de l'Education nationale.
- Bonafé, F., Chevalier, A., Combes, M.-C., Deville, A., Dray, L., Robert, J.-P., & Sauter M. (2002). *Les narrations de recherche de l'école primaire au lycée*. Ed. IREM/APMEP 151.
- Douady, R. (1986). Jeux de cadres et dialectique outil-objet. *Recherches en Didactique des mathématiques* 7/2, pp. 7-31. Ed. La Pensée sauvage, Grenoble.
- Carlson, M.P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem solving framework. *Educational Studies in Mathematics*, 58(1), 45-75.
- Schoenfeld, A. (1985). *Mathematical problem solving*. New York: Academic Press.
- National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston VA: The Council.

### Working Group 11

- Lesh, R. & Zawojewski, J. (2007). Problem solving and modeling. In F. Lester, Jr. (Ed.). *The second handbook of research on mathematics teaching and learning*, (pp. 763-804). Charlotte, NC: Information Age Publishing.
- Mason, J., & Johnston-Wilder, S. (2006, 2<sup>nd</sup> edition). *Designing and using mathematical tasks*. St Albans: Tarquin.
- Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.
- Santos-Trigo, M. (2007). Mathematical problem solving: An evolving research and practice domain. *ZDM - The International Journal on Mathematics Education*, 39(5-6), 523-536.
- Santos-Trigo, M. & Camacho-Machin, M. (2009). Towards the construction of a framework to deal with routine problems to foster mathematical inquiry. *PRIMUS*, 19(3), 260-279.

# WHAT KINDS OF TEACHING IN DIFFERENT TYPES OF CLASSES?

Céline Maréchal

Equipe DiMaGe, Université de Genève

*In our research we examine the forms of teaching found in three structurally different primary classrooms in Geneva (“ordinary” classes, specialized classes, and schools with classes dealing with children with personality and learning difficulties). The aim is to determine whether the ecology of the didactics in those three types of classes obstructs, even prevents, the achievement of certain didactical goals, in our case, the introduction of addition.*

Key-words: teaching practices, mathematic organizations (MO) and didactic organizations (DO), conditions and constraints

## INTRODUCTION

In this paper, the comparative aspect on which we focus deals with groups of students, which share specific characteristics in primary school. We distinguish “ordinary” classes, specialized classes, and schools with classes dealing with children with personality and learning difficulties. One of the main aspects of the study is to determine if the teaching is dependent on the type of class structure. Indeed, we hypothesise that the institutional conditions and constraints particular to each of those three types of classes will have an impact in terms of didactical and mathematical organizations. “It is in the classroom that we can best discover those conditions and constraints that make up the specific ecosystem in which the teacher has to handle knowledge” (Chevallard, 1989, p.62). In this research we are interested in the case of the introduction of addition in first year of primary school in Geneva.

## BACKGROUND

Our investigation takes place in a primary school in Geneva where there is a policy of structural differentiation, which is defined by Doudin and Lafortune (2006) as the creation of various types of classes within the same school system. Each type is supposed to accommodate a certain profile of pupil defined principally by his school qualification, level and/or problem behaviour in class. In this research, we observed three “ordinary” classes (OC), three specialized classes (SC) and three classes with children with difficulties with personality and learning (TC).

One particularity of the French-speaking Swiss context is that there is a single common official set of pedagogical material for mathematics teaching, including text-books and files for students and a book for teachers with didactical commentaries. However, teachers in special education do not have to use those official documents, while it is more or less compulsory in “ordinary” ones. We also know that teachers in special education usually have more liberties than in

“ordinary” education, which will necessary influence their practice. For example, they are neither obliged to follow the official curriculum nor to evaluate their pupils, through the official tests designed for “ordinary” classes.

## METHODOLOGY AND CONSTRUCTION OF A TYPOLOGY OF TASKS

For our research, we have collected scenarios from our 9 classes during a school year. We compared the time of effective teaching of addition during one year, the frequency of use of official documents and we analyzed the types of tasks and register of ostensive involved. These various elements allowed us to bring out the mathematic organizations and didactic organizations in each type of institutions (at the *regional*<sup>[1]</sup> level). From those, we were able to define the MO and DO typical for each type of institution.

To analyze each activity about addition proposed by teachers to their pupils during one year, we needed to construct a typology of tasks. This allowed us to categorized all the activities according to the types of tasks implied.

Our research focuses on teaching practices. This is why we use the ATD – the anthropological theory of the didactic (Chevallard, 1992) – to analyze praxeologies, which are available in one particular type of classes. Therefore, we consider those three types of classes as three different *institutions* that offer us the possibility of systemic analyses. A “praxeology” is the basic unit into which one can analyze human action at large. In our research we started by a categorization of mathematical praxeologies. This offers tools to analyze institutional practices instead of one single person practice. Any praxeology defines itself by the following quadruplet:  $[T/\tau/\theta/\Theta]$ . This grouping defines a system of types of task (T) to carry out with a technique ( $\tau$ ) that must be validated by a technology ( $\theta$ ), which requires a theoretical justification ( $\Theta$ ). The first block  $[T/\tau]$  defines a know-how which is a matter for the practice (praxis) while the second block  $[\theta/\Theta]$  is from a reasoned speech (logos). To study teacher’s work, the ATD proposes two interdependent components which are mathematical organizations (MO) and didactic organizations (DO). It allows us to examine teachers’ work by means of two questions “what does he teach?” and “how does he teach it?” As mentioned by Chevallard, the analysis of these two components cannot be independently undertaken because of their co-determination.

The tools offered by this theory allow us to bring out the mathematical and didactical organizations set up by the teachers. We thus are interested in the block "praxis", defined by Chevallard, that focuses on the types of tasks and associated techniques . For this purpose we introduce a typology of tasks with two levels of specification that we explain below.

### The numerical calculus

At first, we look at the numerical calculus involved in each activity proposed to pupils. In an on-line addition of type  $a + b = c$ , we distinguish three different

possibilities to code the activity: 1)  $a$  and  $b$  are given and  $c$  is to be found; 2)  $a$  (resp.  $b$ ) and  $c$  are given and  $b$  (resp.  $a$ ) are to be found; 3) Only  $c$  is given and some or all possibilities for  $a$  and  $b$  are to be found (additive decomposition). We also differentiate with the symbol (+) and (−) whether the activity involves an addition or a subtraction<sup>[2]</sup>.

Below we present the coding of the various possibilities we distinguished for the analysis of the activities proposed in the classes:

1° Mathematic : (numerical calculus)		
$T_1$	(+) $a + b = \dots$	(−) $a - b = \dots$
$T_2$	(+) $a + \dots = c$	(−) $a - \dots = c$
$T_3$	(+) $\dots + \dots = c$	(−) $\dots - \dots = c$

Figure 6 : First level of specification: numerical calculus

### The registers of ostensive

Then, we added a second level of specification. Indeed, our first categories did not allow us to differentiate the coding of certain activities collected, which were however very different. We used the registers of ostensive introduced by Bosch and Chevallard (1999). Ostensive objects are defined as handleable objects which have a handleable reality by the subject. Non ostensive objects, on the contrary, are neither "seen", nor "perceived", nor "heard". They need ostensive objects to appear. For example, the notion of addition (non ostensive), needs ostensive objects (such as the manipulation of tokens or codes of type  $a + b = c$ ) to emerge.


We distinguished six different registers of ostensive, that we present in figure 2 below:

**3° Registers of ostensive** (Bosch & Chevallard, 1999)

**1 – effective body movements**

Jeu impliquant des manipulations

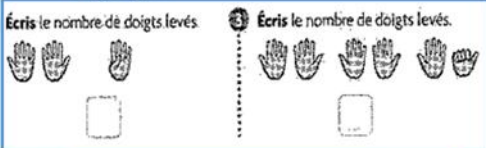
Écris combien il y aura de jetons dans la boîte.



**3a- figurative body movements**

**2-graph**

Écris le nombre de doigts levés.



Écris le nombre de doigts levés.

Pierre a 2 bandes. Il en achète encore 5. Combien en a-t-il ?

**3b- written problems**

**3c- orals problems**

**4a – written formalism**

**4b- oral formalism**

Calculs

$1 + 3 = 4$	$3 = 1 + 4$
$3 + 3 = 6$	$4 = 2 + 2$

Figure 2: Third level of specification: registers of ostensive

The first register of ostensive corresponds to a task which involves an effective situation involving pupils. This task allows a material validation by manipulation (counting collections of concrete objects representing quantities). In the second register of ostensive, the task represents a fictitious situation where the manipulation is no longer possible, but the result can be reached by counting collections of figurative objects. The register of ostensive 3a represents, through a pictorial representation, a fictitious situation, yet the manipulation is not possible any more. The pupils have to reconstruct mentally the operations to be made, but the image facilitates this organization. The registers of ostensive 3b and 3c correspond to written and oral problems where a fictitious situation is described through writing or oral speech. The pupils have to reconstruct mentally the operations to be made, but there is no image anymore to facilitate the understanding of the described situation. Finally, in the registers of ostensive 4a and 4b, there is no longer any reference situation, this is purely formal, only written or oral numerical operations are conveyed.

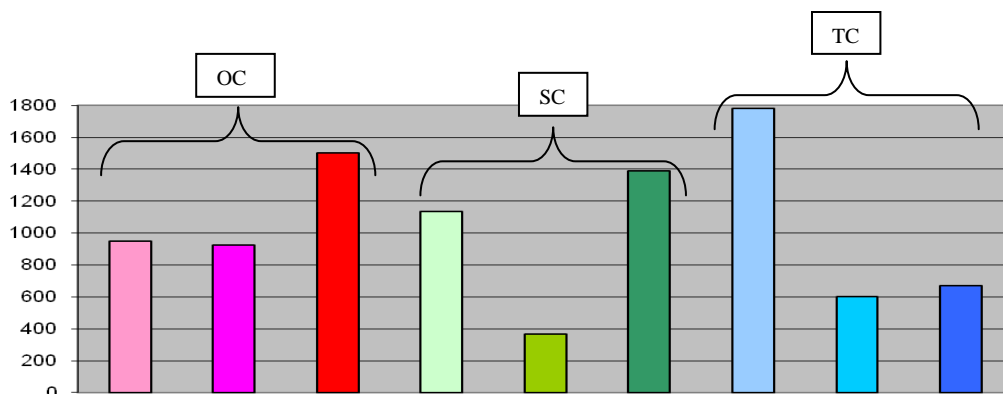
This second level of categorization informs us about a hierarchy in possible techniques to solve the types of tasks proposed previously in figure 1. We present, below, the analyses we developed on the basis of this typology of tasks.

## INVESTIGATION

For the nine scenarios of teaching, we established a set of data corresponding to the time of effective teaching about addition during one year (DO), the frequency of use of official documents proposed in Geneva (DO) and an analysis of the types of tasks (MO) and register of ostensive involved during one year of teaching (DO). Those different data bring to light the mathematical and didactic organizations set up about the teaching of addition by the nine teachers.

### Time of effective teaching addition

We begin with the graph below which indicates for every class the time (in minutes) that was assigned to the teaching of addition during the year.



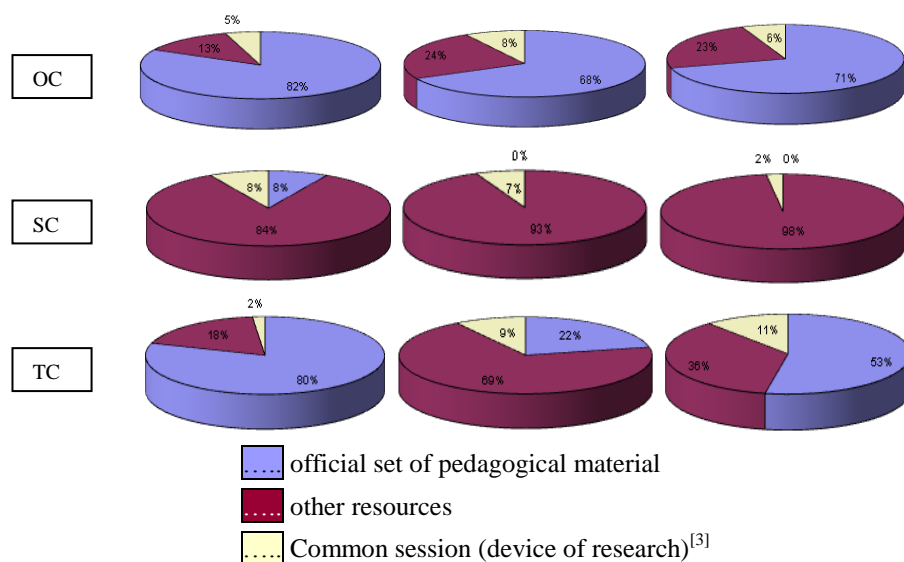
**Graph 1: Time (in minutes) which was assigned to the teaching of addition during one year in the 9 classes**

## Working Group 11

First of all, we notice a clear disparity within the nine classes. However, the values are more homogeneous in the OCs. This fact can be related to the strong constraint of the program, which constrains teachers in the ordinary classes. In the case of the SCs, one of the teachers dedicated a particularly low time to the teaching of addition. In fact, this teacher has chosen to interrupt her teaching concerning addition in the course of the year. Indeed, she considered this notion too complex for the only pupil of the class for whom the introduction of addition was appropriate. Even if this case is extreme, it shows that the teachers of the SCs are not forced, unlike the teachers in the “ordinary” classes, to follow the official program. Concerning the two other classes, the values indicate a slight overinvestment with respect to “addition” compared to the OCs’ average. This seems to be due to the importance of the numerical domain in the specialized context, as several research works have already indicated (Conne, 2003, Cherel & Giroux, 2002). For the three classes of the TC’s institution, we notice two very low values and a very high value (in fact the highest of all 9), in a class, where the teacher has chosen to teach almost only addition in mathematics over the whole year. In the two other classes, the teachers teach all mathematical modules during the year and consequently devote a more restricted time to work on addition.

### Frequency of use of official set of pedagogical material

The following graphs represent the use of the official pedagogical material by the teachers of the three types of institutions OC, SC and TC:



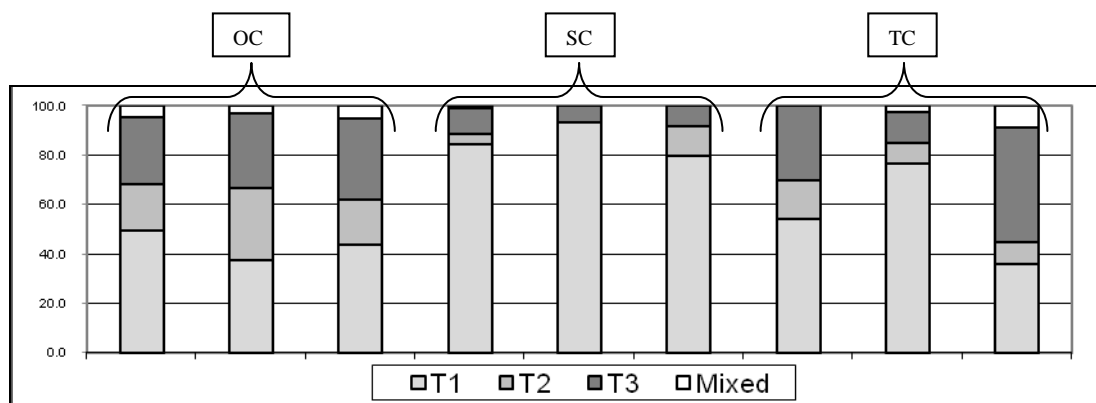
**Graph 2: Use of the official pedagogical material by the teachers of the three types of institutions OC, SC and TC**

In the three OCs, we notice homogeneous scenarios with a net tendency for teachers to use extensively the official pedagogical materials. This fact is not surprising considering the strong constraint that represent those documents. On the contrary,

teachers of the three SCs use little, if at all, the official material. In fact, these teachers do not even employ replacement textbooks, but dig into a reserve of activities that they accumulated over the years. Therefore they are more involved in the process of didactical transposition (Chevallard, 1991) through a necessary adaptation of the knowledge to the specificities of their pupils. This work is normally executed by the *noosphere* and thus demands a reflection on the contents of an upper level.

### Analysis of types of tasks

Let us look in what follows the distribution of the types of tasks T1, T2 and T3 during the year of our observations in each of the nine classes:



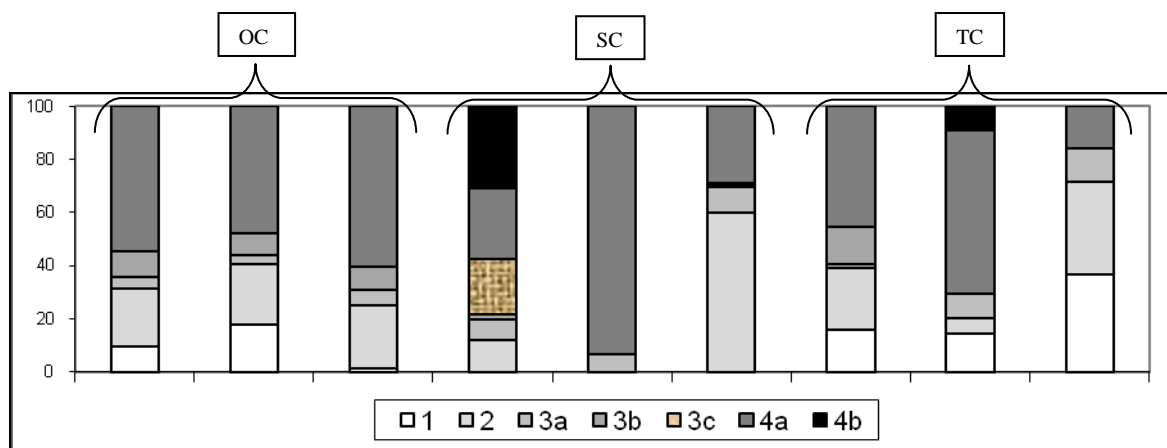
**Graph 2 : Distribution (in percentages) of the types of tasks T1+, T2+ and T3+ during one year<sup>[4]</sup>**

This graph shows homogeneity within ordinary classes. We note a distribution more or less balanced by the three types of tasks, with however a majority of activities of type T1, then T3 and T2. In the specialized classes, homogeneity is also noticed. However, there is, in the three classes, a substantial overinvestment in type T1 tasks to the detriment of the two other. This result is certainly due, among other factors, to the fact that teachers do not use the official material or any other textbook, and implies a "thoughtful" progression of the content of teaching. On the other hand, for the TCs' institution, no homogeneity is noticed between the three classes. The first one gets closer to the functioning of the "ordinary" classes, the second to the specialized classes and the last one has a functioning rather original. It is the only class, where type T3 tasks are the most represented.

### Analysis of registers of ostensive

Finally, let us look at the distribution of the registers of ostensive during the year of our observations of the nine classes (graph 3):





**Graph 3 : Distribution (in percentages) of the registers of ostensive during one year**

Again this graph indicates homogeneity within the OCs that we can attribute to the use of the official pedagogical material by the teachers and to the strong constraint of the program. A certain variety of registers of ostensive is represented in these three classes with, however, a majority of activities involving the register of ostensive 4a. The techniques of calculation, even counting, are thus facilitated, even if this is contrary to the fact that at the end of the primary school such strategies of “enumeration” should be overcome. In the specialized classes a “certain” homogeneity is noticed, because the three teachers chose to introduce a large number of “formalized” activities and the register of ostensive 1 (allowing the counting of collections of concrete objects) is absent. In the TCs, we notice again three different scenarios, representative of a large heterogeneity in these places. The first class has results close to the OCs’ and the second close to the SCs’.

## CONCLUSION

Our various analyses quickly presented above show that the different constraints, which weigh on the three types of institutions are not the same and engender the activation of different praxeologies. The teachers of the “ordinary” classes are confronted with strong constraints such as the use of the official pedagogical material and the “strict” follow-up of the proposed program. This results in relatively homogeneous MO and DO in this institution, with a large variety of different types of tasks relative to addition and also a variety of registers of ostensive.

In specialized classes, the constraints are numerous<sup>[5]</sup> and come along, according to the classes, with more “local” constraints such as the behavior disorders of pupils or school heterogeneousness of the classes. The activated praxeologies are thus more or less homogeneous with, in particular, a massive accent on activities “formalizing” and implying essentially the type of tasks T1. This practice seems to be the result of the fact that teachers do not use reference textbooks. So teachers propose activities “valued” by numerous actors of the school, even the more general society, and

overinvest the numerical domain (of which “addition” is part) at the expense of the geometrical domain or of measure. Furthermore, the fact that specialized classes are located in the same building as “ordinary” classes occasion a certain connection to the “ordinary norm” and “pressure of reinstatement” of pupils in the “ordinary” network, which influences the choices of MO and DO. In this case, we can notice a preference for the type of tasks T1 and registers of ostensive 4a, with a lot of “formalized” activities. In the interview we had with the teachers, one of the three specialized teachers discussed proposing a large number of “formal” activities to his pupils to prepare them for a possible reinstatement in the “ordinary” circuit.

The fact that the TC obtained heterogeneous results can be explained by the large autonomy of the teachers in this institution. They can focus on more “local” constraints to activate their praxeologies, which results in more varied cases.

Our study showed that the differences in teaching in “ordinaries” classes, specialized classes, and schools with classes dealing with children with troubles in personality and learning can be to a certain point explained by the differences in the constraints that weigh on these 3 types of institutions. Our work helped at explaining this fact and sorted out the different effects.

The activated praxeologies thus depend on the conditions and on the institutional constraints, which weigh on the teachers of every type of institutions. It stands out from this research work that the didactic ecology in the SCs’ institution may not be optimal and may give rise to scenarios of repetitive and impoverished teaching, which do not coincide with the initial intention to introduce addition. So, the teachers who are due to have a more active role in the process of didactic transposition are not equipped didactically to adapt their practice. On the other hand, the teachers of the TC’s institution have a larger space of freedom that explains why they can focus on the more particular context of their class (local constraints).

## NOTES

1. Concerning more particularly the study of the OM, several levels are distinguished in the TAD (*punctual, local, regional* and *global*). The regional organization corresponds to a whole sector of the mathematics, as for example the notion of arithmetic operation represented by the sign  $+$  or  $-$  (Chevallard, 2002).
2. In our whole research, we also took into account if the unknown was  $a$  (initial value) or  $b$  (Vergnaud, 1981). However, we do not present these results in this article, because they bring few significant elements.
3. Session which we do not discuss in this article.
4. We do not consider the subtractive types of tasks. Indeed, during the first introductory year of addition, no subtractive activity is proposed in the official pedagogical material. However, we noticed that only the classes of the special education (SC and TC) proposed subtractions during the year of our collection of data.

5. For a more detailed analysis of the constraints appropriate for every studied type of institutions, refer to Maréchal (2010).

## REFERENCES

- Bosch, M. & Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs. *Recherches en Didactique des Mathématiques*, 19(1), 77–123.
- Cherel, C., Giroux, J. (2002). Intégration d'élèves en difficulté : une problématique didactique, *Revue Instantanés Mathématiques* 39, 37–48.
- Chevallard, Y. (1980, éd. 1991). *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble, La Pensée Sauvage.
- Chevallard, Y. (1989). On Didactic Transposition Theory : Some Introductory Notes. Communication à l'*International Symposium on Selected Domains of Research and Development in Mathematics Education* (Bratislava, 3-7 août 1988), pp. 51–62.
- Chevallard, Y. (1992). Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique, *Recherches en Didactique des Mathématiques*, 12(1), 73–111.
- Chevallard, Y. (2002). Organiser l'étude. 3. Ecologie & régulation. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, R. Floris (Eds.), *Actes de la 11<sup>e</sup> école d'été de didactique des mathématiques* (pp. 41–56). Grenoble: La Pensée sauvage.
- Conne, F. (2003). Interactions de connaissances et investissement de savoir dans l'enseignement mathématiques en institutions et classes spécialisées [version électronique]. *Education et francophonie*, 31(2), (pp. 82–102).
- Doudin, P.-A., & Lafortune, L. (2006). Une vision de l'aide aux élèves en difficulté entre inclusion et exclusion. In P.-A. Doudin et L. Lafortune (Eds.), *Intervenir auprès d'élèves ayant des besoins particuliers. Quelle formation à l'enseignement ?* (pp.45–74). Québec: Presse de l'Université du Québec.
- Maréchal, C. (2010). *Effet des contraintes institutionnelles sur les pratiques enseignantes dans l'enseignement spécialisé. Une analyse didactique à partir du cas de l'introduction à l'addition*. Thèse de l'Université de Genève. Genève: Université de Genève
- Vergnaud, G. (1981/1994). *L'enfant, la mathématique et la réalité*. Berne: Editions Peter Lang.

# COMPARING THE CONSTRUCTION OF MATHEMATICAL KNOWLEDGE BETWEEN LOW-ACHIEVING AND HIGH-ACHIEVING STUDENTS – A CASE STUDY

Ingolf Schäfer\*, Alexandra Winkler\*\*

\*Georg-August-University of Goettingen, \*\*University of Bremen

*The case study reported in this paper investigates whether there is a difference in the way high-achieving and low-achieving students construct mathematical knowledge and, if there is, how it might look. Furthermore, it investigates possible differences in metacognitive actions between these groups. For this, we study how pairs of high-achieving and pairs of low-achieving students deal with a problem-solving task about the divisibility of sums, using the theory of abstraction in context and a category scheme for metacognitive activities. This paper is part of a larger ongoing project that compares knowledge construction of high-achieving and low-achieving students using different tasks.*

## INTRODUCTION

When it comes to understanding why some students are low-achieving in mathematics there are many different approaches to the problem, including a focus on basic arithmetical difficulties often discussed in the context of dyscalculia (cf. Moser Opitz, 2007, for an overview), emotional aspects like, in the extreme case, math anxiety (Ashcraft & Moore, 2009) or motivational aspects like self-esteem (Pendlington, 2006). Research also indicates metacognitive actions to be very influential in mathematics achievement (Cohors-Fresenborg et al., 2010; Wang, Haertel & Walberg, 1993). Other factors like social class or cultural background play an important role, too (Cooper & Dunne, 2000).

We restrict ourselves to looking at the process of knowledge construction while separately taking metacognitive actions into account as a complementary view. This approach is done because it is an open question whether there are structural differences in the way low-achievers and high-achievers construct mathematical knowledge and if there are whether this is mainly due to differences in metacognition as indicated in (Cohors-Fresenborg et al., 2010).

## THEORETICAL BACKGROUND

### Abstraction in Context

The theory of abstraction in context (Hershkowitz, Schwarz, & Dreyfus, 2001; Dreyfus, Hershkowitz, & Schwarz, 2001), rooted within activity theory, is a model for the process of knowledge construction that has been applied to low-achieving students (Schäfer, 2009) and processes of knowledge construction that were only partially correct (Ron, Dreyfus, & Hershowitz, 2006). In these studies the main

benefit over other theories concerning epistemic processes is that the epistemic actions defined below are observable actions – usually verbal – that allow insight into the internal process of abstraction.

Defining abstraction as “an activity of vertically reorganising previously constructed mathematical knowledge into a new structure”, Dreyfus et al. (2001) propose that the process of abstraction consists of three phases. In the first phase a *need for a new construct* arises, followed by a sequence of epistemic actions in an actual *construction* phase. Finally there can be a phase of *consolidation* of the construct.

In the construction phase three epistemic actions can be found. *Recognising* existing mathematical structures, *building-with* those structures, e.g., combining recognised artefacts to justify a particular claim, and *constructing* a new structure. These epistemic actions are nested, i.e., constructing incorporates building-with and recognising actions, and building-with incorporates recognising actions.

### Layers of relation to objects

One of the directions in which activity theory has been developed is given by Oerter’s (1982) theory of action, which we will use to refine the three epistemic actions. Oerter uses the notion of action as the foundation of this theory and postulates that any interplay between individual and environment is only possible through actions. These actions, whether they are mental actions or physical actions, are always done with respect to some object, which may be physical or mental. An individual’s relationship to objects may only be changed by performing actions.

Oerter describes three different layers of relation to an object, which we summarise briefly in the following table.

Layer	Description
Singular	Object is only existent in the course of the action
Contextual	Object is discernible, but only inside a specific context of use
Formal	The formal structure of the object without its use

**Table 1: Oerter’s Layers of Relations to Objects (Oerter, 1982, p. 114)**

The singular layer is defined as the layer where subject and object cannot be distinguished by actions. Oerter gives the example of a child that uses a stick as a “sword” in playing knights. If the object “sword” is not existent anymore when the child has finished playing, then the object is only existent in the course of the action. If this playing knights is repeated more often and done with other children, the “sword” will exist beyond the actual situation, but still inside the contextual layer of play. The reader should be warned that the notion of contextual layer refers to similar situations and is different from the context in the sense of abstraction in a context which includes personal history or the classroom setting for example. Thus, we

always state explicitly which notion of context is meant in the following. The abstract layer is the equivalence class of all object uses which means that only the abstract formal structure of the object is left, e.g. the mathematical definition of a triangle is located here. For further details on Oerter's framework and the interplay with the theory of abstraction in context we refer to the considerations in (Schäfer, 2009).

## **Metacognition**

The concept of metacognition, introduced by Flavell (1976), refers to the knowledge, active monitoring and controlling by the individual of her or his cognitive activities. As is shown by Schneider and Artelt's (2010) recent overview of research in mathematics education and psychology on metacognition, the concept has developed much since then. We follow the approach of the Osnabrück school (Cohors-Fresenborg et al., 2010; Cohors-Fresenborg & Kaune, 2007) and study the concrete metacognitive actions of participants in mathematics learning processes. They see metacognitive actions as a triple with the components *planning*, *monitoring* and *reflecting*. Planning refers to actions like organising or anticipating, monitoring includes regulating, verifying and checking, and reflecting comprises evaluating, assessing and judging. Cohors-Fresenborg and Kaune (2007) suggest the additional component of discursive actions for the analysis, but we restrict ourselves to the three principal components. Based on these components Cohors-Fresenborg and Kaune (2007) empirically developed a categorisation scheme for metacognitive and discourse actions. Although this scheme was developed for classroom situations in which algebraic questions are discussed on the basis of a discursive classroom culture, it is applicable to our situation as well, since the metacognitive actions themselves should not differ as long as the setting gives room for discourse.

We choose this framework for our metacognitive analysis, because it relies on an action perspective that is very fitting with Oerter's approach to express everything in terms of actions.

## **On the interplay of metacognition and abstraction in context**

It is not our aim to have a unified theory of abstraction in context with epistemic and metacognitive actions, because some metacognitive actions can be part of the epistemic actions and the need. We would rather see our approach as looking at the same problem with different lenses in order to contrast and compare the findings with different theoretical perspectives (Prediger, Bikner-Ahsbahr & Arzarello, 2008).

## **Working environment: Divisibility of sums**

The main task described below is concerned with the question whether, in a set of  $k$  natural numbers, there can always be found  $n$  numbers, such that their sum is divisible by  $n$ . This kind of question has been used in problem solving tasks for some

time now, e.g. it was the problem of the week no. 80 from Harvard Physics Department (2004), which also contains a general solution with proof.

Depending on the numbers  $n$  and  $k$  the solutions of the problem can be very different. We restrict to the two cases which we used. Case 1 is  $k=13$  and  $n=4$ , which was given to the low-achievers, case 2 for the high-achievers was  $k=17$  and  $n=5$ . Bardy (2007, p. 72-91) used the same problem for a case study [1] on gifted students in primary school.

In both cases the fundamental insight required is that it does not matter which concrete number is given in the set of  $k$  numbers, but only the remainders matter. After restricting to the remainders each case is solved by arguing how many numbers there have to be in each remainder class. A priori we expect to find the constructs “residue class” and “Dirichlet’s box principle”.

It may seem that case 1 is more complicated because recognising the different remainder classes seems more obvious with respect to five, but that does not take into account the effort of handling an additional number in the subset. On the other hand in case 2 the combinatorics for the residue classes are more complicated and involve a case-by-case analysis.

### **The notions “low-achieving” and “high-achieving”**

The school system in Bremen comprises the Gymnasium and two forms of secondary school (Realschule and Hauptschule). Each type has its own curricula, final exams, lessons per week and years to exam. While 15-year old students in the Gymnasium perform above the international average in tests like PISA 2003, the students in the Hauptschule perform below average (with over 50% at risk). The difference in performance between Hauptschule and Gymnasium students is equivalent to a difference in 3-4 grades on the average (PISA Konsortium Deutschland, 2005).

For our purpose we state that those children in Hauptschule are low-achieving whose achievement in school is significantly below their peers and who have been identified by school tests as in need of additional support in mathematics. On the other hand, we state that students in Gymnasium who achieve significantly better in mathematics than their peers and who successfully take part in mathematics problem solving competitions are high-achieving.

### **RESEARCH QUESTIONS**

We are interested in answering the following questions:

1. Are there different patterns in the use of epistemic actions between high-achievers and low-achievers?
2. Are there different patterns in the use of metacognitive actions between high-achievers and low-achievers?

## METHODOLOGY AND DESIGN

The students are presented the task on paper in form of a dialog between two children. One fictional child named Kathy says she had found out about an interesting problem in a riddle magazine and goes on (translation of German original for the low-achievers; the high-achievers had the same text with the parameters 5 and 17 and the different example sequence (22, 7, 2, 4, 6, 6, 9, 18, 6, 12, 17, 6, 11, 6, 20, 5, 16)).

*Kathy: You take 13 natural numbers, e.g., 7, 2, 4, 5, 5, 9, 14, 5, 10, 1, 5, 11, 3.*

*Lars: Ok, I understood. And what is so interesting about them?*

*Kathy: Within those 13 numbers you can always find 4 numbers, whose sum is divisible by 4!*

*Task:*

- 1. Verify Kathy's claim in her example. Find as many subsequences as possible of 4 numbers whose sum is divisible by 4.*
- 2. Is it really always true what Kathy claims?*

The students are asked to discuss these problems with each other and write down a solution. They are given no additional information or help, with the exception of certain prompts from a field manual that should be given to the low-achieving students at certain times to ensure that they recognise certain phenomena, e.g., one of the authors might ask the students to explicitly compare two subsequences they found in order to help them recognise the fact that it is possible to replace a number with another representative of the same equivalence class without contravening divisibility. The task was mainly chosen because it does not depend on special knowledge other than basic arithmetic skills (other tasks are planned in future). No such prompts are provided for the high-achievers. The whole process is then videotaped and transcribed. The data is analysed in two separate turns: Each utterance in the transcript is coded according to the coding guidelines for the metacognitive actions as described in Cohors-Fresenborg & Kaune (2007). Separately, the analysis according to the RBC-model was done in a sequence analysis using an interpretative approach to the text utilising the theory of speech acts (cf. Bikner-Ahsbabs, 2008).

## FINDINGS

Our case study was conducted with 2 pairs of high-achieving grade 6 students in Gymnasium and 2 pairs of low-achieving grade 9 students in Hauptschule meeting the definition. The difference in grades is supposed to roughly compensate for the difference between the school types as explained above. Due to space limitations we can only give a short summary here.



### **The case of Alice and Betty**

Alice and Betty [2] are in the low-achievers group. They look for subsequences for question 1 more or less by trial and error and do not make much use of additional structure besides using the sum of the subsequences for ordering and comparing, which they talk about explicitly but use rarely. After a prompt by the interviewer they realize that certain numbers belong to the same remainder class, but it seems that they do not come to a more concrete understanding of the concept. Other than stating that they would need to find other examples, since one example is no proof and guessing that the product of the numbers in the subsequence might be involved, they do not come to arguments for question 2.

### **The case of Carl and Dan**

Carl and Dan are also in the low-achievers group. For question 1 they struggle with the formulation of the task where Carl initially understands “subsequences of four numbers” as “four subsequences of four numbers”, which is clarified by Dan and the interviewer, but 30 minutes later Dan shows the same misunderstanding. Due to a prompt by the interviewer (“in a subsequence a 5 can be replaced by 1 without changing the divisibility of the sum”) they are able to use the concept of remainder class and Dan even finds the general definition of equivalence. Although they make some considerations on the divisibility of even and odd numbers, they do not achieve a proof by themselves.

### **The case of Eric and Fred**

Eric and Fred are high-achievers. They work separately for most of the time while still informing each other about their respective findings and apply some strategies in finding examples including Eric doing a systematic case-by-case testing for subsequences of length 4, which he stops at some point. Fred tries to assemble a counter example and realises that it is possible to restrict the numbers from 1 till 9 for the sequence. Eric builds on this and introduces remainder classes. This restriction, in combination with Fred’s combinatorial considerations on the way to construct a counter example, leads to the proof given above in the mathematical considerations.

### **The case of Greg and Hank**

Greg and Hank are high-achieving students. In answering question one they use many different strategies to find subsequences, most prominently a replacement strategy where they start with a given subsequence and replace numbers in it such that the sum remains a multiple of 5. They explicitly call these “rules” and give example subsequences with rules to modify them. Working with those rules they find the concept of remainder classes, writing down the first ten elements of each class. Restricting to the representatives 1 to 5 they give the proof outlined in the mathematical considerations.

### Similarities and differences in the RBC-model

For all groups we notice many instances of recognising and building-with actions, but constructing mainly occurs among the high-achievers. Regarding the quantity of epistemic actions we note that the high-achievers perform a greater number of such actions in a fixed time span than the low-achievers. But one should keep in mind that although we tried to compensate by using smaller numbers in the task for the low-achievers, they spend much time on arithmetical actions, mainly adding the numbers, leaving less time for epistemic actions. Most of the epistemic actions happen at the situational or contextual layer in all groups and we could reconstruct three common contexts in the sense of Oerter's theory of action among the groups:

1. *Arithmetics*, in which the sums of the subsequences are calculated and some important properties can be recognised, for instance the invariance of the divisibility under replacement of a number by another representative of the same remainder class.
2. *Combinatorics*, in which possible subsequences are explored, e.g., how many pairs of numbers sum up to 12, or how many numbers have to be at least in each remainder class.
3. *Understanding of the task*. Some recognition actions are about what the task is and what it is not, i.e., how many subsequences must be found, how to deal with duplicate numbers etc. Since these actions are on-going for a longer period of time (a couple of minutes) and in all the groups, we consider them contextual rather than situational.

The low-achievers spend most of their time on situational epistemic actions or epistemic actions on the contextual layer regarding arithmetics or understanding the task and show very little formal actions, the high-achievers show more formal actions and also are less occupied with the arithmetical contextual layer.

We want to focus on a specific example of epistemic action which illustrates the difference between low-achievers and high-achievers. At some point the students in every group have found a subsequence of  $n$  identical numbers, but this does not lead to the same kind of epistemic actions. When confronted with the specific case of the number six while dealing with question 1, Greg and Hank immediately contextually recognise that this is  $n$  times six, i.e., multiplication and thus, divisible by  $n$ . Later when they deal with question 2 they recognise this case on the formal layer.

Hank:            So, if you got five of one [number], you can divide it by five in any case.

Eric and Fred deal with the example of 6, 6, 6, 6, 6 in no special way in question 1. They just treat it as an example like the others, but in dealing with question two, they recognize on the formal layer that five times an identical number is divisible by five. For the low-achievers this piece of knowledge has probably not previously been constructed in a way that they could recognise it. In the case of Carl and Dan, they

construct multiple examples of sequences of identical numbers and Dan calculates the sum of four of them to check, whether they are divisible by four. This means that they are building-with the idea that the sum of four identical numbers is divisible by four contextually and seem to be unable to recognise the structure. Alice and Betty do not find the example of 5, 5, 5, 5 in working on the first question. When they work on the second question they follow a prompt by the interviewer to look at sequences containing only the numbers 1 to 4 first. Betty suggests to look at sequences containing only one number, for example the number 1.

Alice: Yes, then I can take – then – I can take four times one. Yes, four.

Then they try four times two and four times three by calculating the sum and checking whether it is a multiple of four. In the case of four times four she says

Alice: ... and four times four is – works anyway.

which may indicate that she has recognised some structure here. But as the episode shows, they do not recognise on the formal layer, but rather recognise divisibility of each result in the first three examples in the contextual layer.

There is also a difference between high-achievers and low-achievers regarding falsely recognising a mathematical property. This happens for all groups, but the difference is in the way it evolves: In the case of the high-achievers they drop those ideas when they find a counterexample, the low-achievers are usually willing to keep them for at least another counterexample.

In summary, we uncover several differences regarding Oerter's layers between the groups, e.g., the high-achievers may "easily" recognise a formal property, while low-achievers have to built-with it contextually. We do not find a more systematic pattern in our cases.

### **Similarities and differences in the metacognitive actions**

All four groups use many different metacognitive actions in all of the three components, but there are differences in the use of certain subcategories. No actions occur in the subcategories related to different representations, which is due to the fact that all groups solely used the representation by numbers. Note, that the letter and the number in curly brackets refer to the metacognitive actions in the scheme of Cohors-Fresenborg and Kaune (2007).

Regarding the planning component there are few differences between the two groups, but in the monitoring and reflecting component, we see differences. The high-achievers reflect on the notions used {R1} and change their point of view numerous times {R4}, the low achievers do not reflect on the notions and change the point of view only in one case. But even for those metacognitive actions that are common to both groups there are differences in quality and use. Those differences can be described by three themes, which we illustrate by means of specific examples.

## Working Group 11

One of the themes is the *broadness of the content* to which the metacognitive strategies are applied. For example in the monitoring of deficits in understanding or planning {M5, M6}, the low-achievers only apply this to each other's direct actions and utterances or in understanding the questions.

Dan:           What does she claim then? [regarding Kathy in question 1]

In contrast, the high-achievers also monitor with regard to the purpose of certain steps or intermediate results.

Eric:           What do we get out of it, when we know how many groups there are?

The second theme concerns *the interplay of metacognitive actions*. All groups actively search for divisible subsequences and monitor their actions with respect to the goals {M7}, but the low-achievers mainly utilise calculating monitoring strategies for this {M1}, while the high-achievers plan in advance {P2, P3}, e.g., Eric fixes three numbers, states possible numbers to be added in general and controls which of these numbers are occurring in the example and thus allowed.

The last theme relates to differences in the *perseverance of metacognitive actions*. While high-achievers and low-achievers alike monitor their actions with respect to content and goals {M7}, the low-achievers seem unable to maintain the level of monitoring all the time, e.g., both groups of low-achievers at first calculate the total sum of a sequence of 13 identical numbers, when working on question 2, instead of calculating the sum of only 4 numbers. The high-achievers also study the example of sequences where each number is identical, but correctly use only partial sums.

In summary, our cases indicate a difference between the groups with regard to metacognition, but more about the quality than about simple occurrences.

## SUMMARY AND OUTLOOK

The above analysis indicates that it is worthwhile comparing the processes of knowledge construction between low-achievers and high-achievers. In the cases studied there seem to be great differences already in the recognising actions, which then in turn affect the possibility of building-with and constructing due to the nested nature of the epistemic actions. The role of contextual actions and the different themes in using metacognitive actions also seem to be influential. But, as was mentioned above, the arithmetical side of the problem was considerably more time-consuming and therefore probably harder for the low-achievers. Since this may account for some of the differences between the groups regarding the process of knowledge construction, we are in the middle of performing two additional tasks – one related to graph theory and another to geometry – with the students to see how much is related to these arithmetical problems.

## NOTES

1. The question and the transcript itself is not part in the book. We thank Peter Bardy for sharing them with us.
2. The names of the participating children have been replaced by arbitrary chosen names.

## REFERENCES

- Ashcraft, M. H., & Moore, A. M. (2009). Mathematics anxiety and the affective drop in performance. *Journal of Psychoeducational Assessment*, 27(3), 197-205.
- Bardy, P. (2007). *Mathematisch begabte Grundschüler: Förderung und Diagnostik*. Munich, Germany: Spektrum Akademischer Verlag.
- Bikner-Ahsbahr, A. (2008). Erkenntnisprozesse – Rekonstruktion ihrer Struktur durch Idealtypenbildung. In H. Jungwirth & G. Krummheuer (Eds.), *Interpretative Unterrichtsforschung im deutschsprachigen Raum* ( pp. 105-144). Weinheim, Germany: Beltz.
- Cooper, B., & Dunne, M. (2000). *Assessing children's mathematical knowledge*. Buckingham, UK: Open University Press.
- Cohors-Fresenborg, E., & Kaune, C. (2007). *Kategoriensystem für metakognitive Aktivitäten beim schrittweise kontrollierten Argumentieren im MU*. Osnabrück, Germany: Forschungsinstitut für Mathematikdidaktik
- Cohors-Fresenborg, E., Kramer, S., Pundsack, F., Sijts, J., & Sommer, N. (2010). The role of metacognitive monitoring in explaining differences in mathematics achievement. *ZDM*, 42, 231–244
- Dreyfus, T., Hershkowitz, R., & Schwarz, B. (2001). Abstraction in Context II: The case of peer interaction. *Cognitive Science Quarterly*, 1(3/4), 307-368.
- Flavell, J. (1976). Metacognitive aspects of problem solving. In L. Resnik (Ed.), *The nature of intelligence* (pp. 231-236). Hillsdale, USA: Lawrence Erlbaum.
- Harvard Physics Department (2004). *Problem of the week*. Retrieved August, 24th from <http://www.physics.harvard.edu/academics/undergrad/problems.html>
- Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in Context: Epistemic actions. *Journal for Research in Mathematics Education*, 32(2), 195-222.
- Moser Opitz, E. (2007). *Rechenschwäche/Dyskalkulie*. Bern, Swiss: Haupt.
- Oerter, R. (1982). Interaktion als Individuum-Umwelt-Bezug. In E. D. Lantermann (Ed.), *Wechselwirkungen* (pp. 101-127). Göttingen, Germany: V&P.
- PISA-Konsortium Deutschland (Eds.) (2005). *PISA 2003*. Münster: Waxmann.
- Pendlington, S. (2006). Mathematics isn't easy. *Mathematics Teaching Incorporating Micromath*, 197, 3-8.

### Working Group 11

- Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches. *ZDM*, 40(2), 165-178.
- Ron, G., Dreyfus, T., & Hershkowitz, R. (2006). Partial constructs for the probability area model. In J. Novotná et al. (Eds.), *Proceedings of PME 30* (Vol. 4, pp. 449-456). Prague, Czech Republic.
- Schäfer, I. (2009). The potential to act for low achieving students as an example of combining use of different theories. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1635-1644). Lyon: Institut National de Recherche Pédagogique.
- Schneider, W., & Artelt, C. (2010). Metacognition and mathematics education. *ZDM*, 42, 149-161.
- Wang, M., Haertel, G., & Walberg, H. (1993). Toward a knowledge base for school learning. *Review of Educational Research* 63(3), 249-294.

# CONCEPTUAL METAPHORS AND “GRUNDVORSTELLUNGEN”: A CASE OF CONVERGENCE?

Jorge Soto-Andrade<sup>62,\*</sup> and Pamela Reyes-Santander<sup>1,°</sup>

\* University of Chile, ° Universität Augsburg.

*We compare the Metaphor Approach to mathematics teaching and learning and the Basic Ideas (“Grundvorstellungen”) approach of the German school of didactics of mathematics. We find close connections as well as some differences: “Grundvorstellungen” tend to be more “top-down” from the student’s viewpoint than metaphors, which operate in a more “bottom-up” and “poietic” (concept generating) ways.*

Keywords: Metaphor, Basic Ideas, Grundvorstellungen, fractions, stochastic thinking.

## INTRODUCTION

In this paper we compare two theoretical approaches to the didactics of mathematics, which have evolved independently, in different periods of time, although they address the same didactical problems in closely related ways. They could be described as the “Basic Ideas Approach” and the “Metaphor Approach”. Here “Basic Ideas” (even better, “Basic Notions”) is an approximate English rendering of the German “Grundvorstellungen” (vom Hofe, 1998), which alludes to the basic ways we imagine or represent something rather abstract to ourselves. “Metaphor” means in fact “Conceptual Metaphor” in the sense of Lakoff and Núñez (2000).

Although metaphors can be traced back to Aristotle’s Poietic, the cognitive and didactical role of (conceptual) metaphors in mathematics has been only recently and progressively recognized (English, 1997; Lakoff & Núñez, 2000; Presmeg, 1997; Sfard, 1997; Soto-Andrade, 2006, 2007). On the other hand, the origin of the “Grundvorstellungen” approach, developed by the German school of didactics of mathematics over more than two centuries - see vom Hofe (1995, 1998) for a detailed account - as a means of grasping and making sense of abstract mathematical concepts, may be traced back to Euler, who already in 1766 interpreted negative numbers as debts, pointing out that “to take away a debt amounts to make a gift” (v. Hofe, 1995, p. 19). German didacticians, however, do not mention metaphors and researchers working on the metaphor approach have completely ignored the “Grundvorstellungen approach”, perhaps because “Metapher” in everyday German is typically understood as a rhetorical device and contributions of the German school, mostly written in German, were scarcely known outside Germany, until vom Hofe’s paper on the generation of basic ideas and individual images (vom Hofe, 1998).

---

<sup>62</sup> Supported by PBCT- CONICYT, Project CIE-05 and FONDEF Project D06i1023.

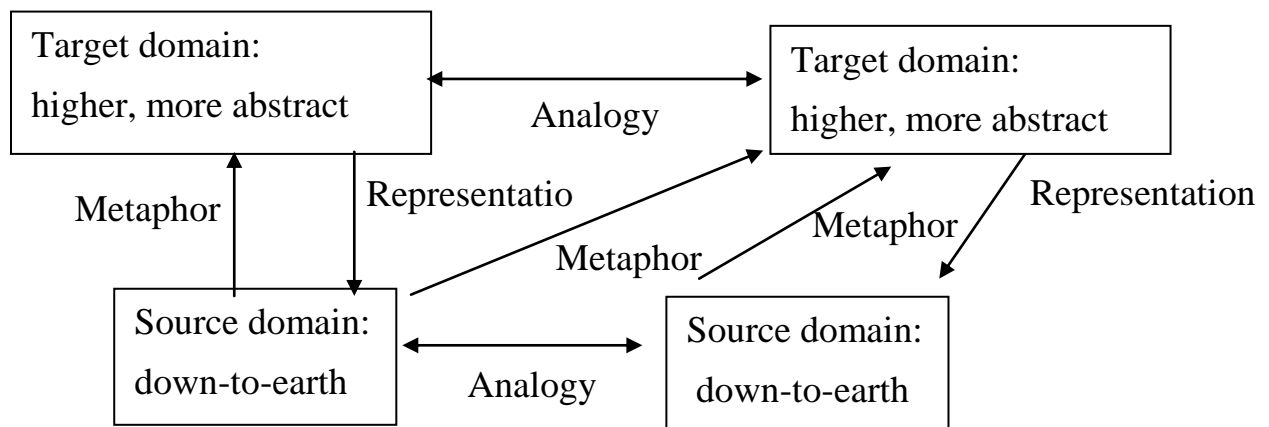
In what follows, after reviewing the main ingredients of the metaphor and ‘Grundvorstellungen’ approaches, we argue that conceptual metaphors, in the sense of Lakoff and Núñez (2000), play essentially the same cognitive role as “Grundvorstellungen” in vom Hofe (1995, 1998), although their didactic implementation may differ. We support our claim by comparing concrete examples of both approaches, taken from the literature and from our own teaching experience. We further discuss the ‘poietic role’ of both Metaphors and “Grundvorstellungen”.

## THEORETICAL FRAMEWORK

In what follows, we adhere mainly to the theoretical framework laid by Presmeg (1997), Sfard (1997) and Lakoff & Núñez (2000) for metaphors, Breidenbach (1967), Oehl (1967), Griesel (1971), vom Hofe (1995, 1998, 2003) and Prediger (2008) for Grundvorstellungen.

### Nature and Role of Metaphor

Metaphors are not just rhetorical devices, but powerful cognitive tools that help us to build or grasp new concepts, as well as solving problems in efficient and friendly ways (Lakoff & Núñez, 2000; Soto-Andrade, 2006, 2007). In the literature, often “metaphor” means “representation”, “analogy”, “model”, “image”, etc., but we intend here to be more precise, as in the diagram in Fig. 1, where metaphors “go up”, representations “go down” and analogies, “go horizontally” both ways.



**Figure 1. A spatial metaphor for metaphors, representations and analogies.**

Here analogy means “simile” (an explicit comparison between two different things), symmetric in nature, and it is not an umbrella concept embracing metaphors, representations, similes, etc. So our viewpoint is closer to Sfard’s (1997) than to Presmeg’s (1997).



### Example: Metaphors for multiplication

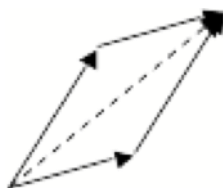
Most primary teachers are well acquainted with the “area metaphor” for the product, but somewhat less with the “grafting metaphor” (Soto-Andrade, 2006), illustrated below in Fig. 2, where both metaphors help us to visualize that  $2 \times 3 = 3 \times 2$ .



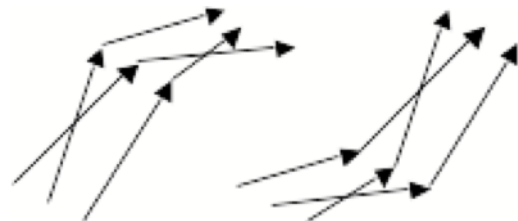
**Fig. 2. Two metaphors for the commutativity of multiplication.**

In the first, commutativity is perceived simply as invariance of area under rotation in one fourth of a turn. So you “see” that  $2 \times 3 = 3 \times 2$ , without counting and knowing that it is 6. In the second, commutativity is less obvious: you need to be familiar with trees for this metaphor to become a “met-before” in the sense of Tall (2005). Notice that these tree diagrams, drawn upside down, also suggest an ‘hydraulic metaphor’, which helps us to understand multiplication of fractions: A litre of water will drain evenly, by gravity, from the tree root, through the pipes, splitting into two and then into three in the left tree. So  $1/6$  appears not only as  $1/3$  of  $1/2$  but also as  $1/2$  of  $1/3$ .

This illustrates the importance of having metaphors of various scopes for the same mathematical object and of being able to transit between them. Indeed, from the grafting metaphor for multiplication we may transit to a deeper metaphor for multiplication: “*multiplying is concatenating*”. Fig. 3 below suggests commutativity if you look at the resulting diagonal arrow, but not so if you consider the concatenated pathways, which are different (The mathematical notion of category emerges here!). If we concatenate now triple arrows (which could be realized as triple threads or braids) we end up with “products” which depend on the order of concatenation (see Fig. 4).



**Fig. 3. Concatenation metaphor for multiplication (1 arrow)**



**Fig. 4. Concatenation metaphor for multiplication (3 braided arrows).**

Finally, note the connection with  $2 \times 3$  and  $3 \times 2$ : if we concatenate an enlargement of ratio 2 with one of ratio 3, we get one of ratio 6, and the other way around.

### Metaphors in mathematics education

Mathematicians have been aware for a long time of the key role of analogies and metaphors to achieve understanding and insight (Sfard, 1994). Indeed, Euler himself

may be seen as a forerunner of this approach, when he suggests considering negative numbers as debts (Euler, 1802, p. 12). Thom, the creator of Catastrophe Theory, besides complaining that since the advent of positivism analogy is considered as a remnant of magical thinking to be condemned absolutely and at present is thought to be just a figure of rhetoric (Thom, 1994), uses even animistic metaphors: “On tue la topologie de la variété en l’appliquant sur l’axe réel, mais la topologie résiste, elle ‘crie’, et ses cris se manifestent par l’existence de points critiques” (Thom, 2002).

After Euler, the use of metaphors in mathematics education proper began rather late and heterogeneously. They were employed first mainly by mathematicians turning to didactics, like Dienes (1960) or Tall (2005), among others. It may be argued that when Dienes and Bruner (Dienes, 1960; Bruner & Kenney, 1964) posited that abstract concepts and processes may be apprehended by manipulating various embodiments, they were in fact using various metaphors for the same concept, as in “factorizing is assembling a puzzle” and “factorizing is balancing a family of weights against a single weight in a balance beam with hooks” (Bruner & Kenney, 1964).

In general, educational promotion of metaphors has been intermittent, with mixed results. In the U.K. the systematic use of the number line metaphor as a “key classroom resource” promoted by the National Numeracy Strategy since 1999 for primary school, was clearly not a success (Doritou & Gray, 2007). Apparently not enough attention was paid to preparing the soil first, where metaphors may grow. In Chile, the explicit use of metaphors, like the hydraulic one, was exemplified in textbooks distributed by the Ministry of Education to public primary and secondary schools during 2000-2008. However, their exploitation in the classroom remained limited, because most teachers had the feeling that to rely on metaphors is not serious, while mathematics is indeed serious stuff (Soto-Andrade, 2006, 2007, 2010). However, recent hard evidence of the positive didactical impact of the prescriptive use of metaphors in Chile is reported in Araya et al. (2010). An example of a non prescriptive use of metaphors, being tested at present in Chile, Germany and France: “the emergent metaphor approach”, is commented below (Brownie’s random walk).

### **Basic Ideas or “Grundvorstellungen” (vom Hofe, 1995, 1998)**

Since the beginning of the 19<sup>th</sup> century, German didacticians, influenced by Pestalozzi (vom Hofe, 1998), were acutely sensitive to the importance that students imagine, visualize and represent to themselves abstract mathematical concepts and processes in some concrete way, to be able to make sense of them and so gain real understanding. This approach involved the generation of “Anschauungen”, i. e. visual mental models for mathematical objects, albeit in a rather passive and merely associative way, until Kühnel (1916) emphasized the role of individual activity and insight in the learning of mathematics. He developed a course of mathematical instruction for primary schools where the generation of “Stellvertretervorstellungen” (representative notions), mediating between the world of abstract mathematical thinking and the world of “real life” played a key role. Later, Breidenbach (1957)

initiated didactical analyses of mathematical subject matter, creating the influential German school of “Stoffdidaktik” (“Stoff” = mathematical “stuff” or content), that produced teaching units for introducing different mathematical content and supporting the generation of suitable “Grundvorstellungen”. Oehl (1962), introducing the term “Grundvorstellungen” systematically, integrated the previous approaches into a coherent theory, that was later extended to the secondary school level by Griesel (1971) and others. “Grundvorstellungen” remained however used in a *prescriptive* way, as normative categories to train students to learn with understanding, until vom Hofe (1995, 1998) investigated their *descriptive* aspect, as idiosyncratic notions students actually develop, eventually inadequate from the teacher’s viewpoint. They involve (vom Hofe, 1998):

- Gv1. The constitution of meaning of mathematical concepts based on familiar context and experiences (as in metaphor = metbefore);
- Gv2. Generation of visual representations making operative thinking possible;
- Gv3. Ability to apply mathematical concepts to real life contexts.

Vom Hofe (1992) distinguishes moreover between *primary* and *secondary* “Grundvorstellungen”. The former are mainly psycho-motoric in nature, involving manipulation of concrete objects or acting in the world and entailing activation of the concrete or enactive representation mode in the sense of Bruner (Bruner, 1996; Bruner & Kenney, 1965). The latter take foothold on iconic or pictorial objects instead, like the number line, the Cartesian plane, graphs, and so on. They all correspond to metaphors, but with different kinds of source domains: concrete, iconic, abstract...

**Example 1: “Grundvorstellungen” and metaphors for multiplication of natural numbers** (Prediger, 2008; vom Hofe, 2003)

Prediger (2008) lists the following “Grundvorstellungen”:

- GM1. Iterated addition ( $3 \times 5$  as  $5 + 5 + 5$ )
- GM2. Area of a rectangle ( $3 \times 5$  as the area of a 3 times 5 rectangle).
- GM3. Multiplicative comparison ( $3 \times 5$  as 3 times as much as 5)
- GM4. Enlargement ( $3 \times 5$  as a 3-fold enlargement of 5)
- GM5. Combinatorial ( $3 \times 5$  as number of ways to combine 3 shorts and 5 shirts)

and points out that only GM2, GM3 and GM4 carry over to fractions and that on the other hand, the “*portion of - Grundvorstellung*” ( $1/3 \times 1/2$  means  $1/3$  of  $1/2$ ) for fractions does not come from the natural number case. These “discontinuities” and the required conceptual change highlight epistemological obstacles (Prediger, 2008).

In the metaphor approach however, instead of considering the combinatorial and the “portion of – Grundvorstellungen” as totally unrelated, recalling that “to multiply is to concatenate” students might look upon the trees in Fig. 2 (drawn upside down) as trees of ducts and visualize *simultaneously* multiplication of natural numbers (product = the number of ducts in the last generation) and multiplication of fractions

(product = portion of 1 litre poured at the root that will drain down to each end of the tree). In this way the “concatenation metaphor” in some sense bridges the gap between GM4 and GM5 for natural numbers and the “portion of - Grundvorstellung” for fractions, suggesting that the relation between the former and the latter is in fact one of duality...

### **Example 2: Grundvorstellungen and metaphors for stochastic thinking**

In a prescriptive way, Malle & Malle (2003), consider the following “Grundvorstellungen” for the probability of an event:

GP1. A measure of the likelihood that the event occurs.

GP2. The relative portion of favourable outcomes to the event, within the set of all possible outcomes of the corresponding experiment.

GP3. The relative frequency of the event in a series of repetitions of the experiment.

GP4. The subjective confidence in the occurrence of the event.

In a descriptive way, Wollring (1994) has investigated the emergence of animistic “Vorstellungen” in elementary school students, finding that they strongly determine their stochastic thinking. Most animistic notions, with no experimental support, concern the “will” and “consciousness” of coins or dice, the possibility of mentally influencing them and the fact that “the world is fair”.

Borovcnik (1984), who adheres, like vom Hofe (1998), to the “didactical triangle” (theory, reality, subject) where “Grundvorstellungen” live as a didactical construct, besides rehabilitating descriptive statistics as a didactically and mathematically interesting domain, points out that following the “spiral principle” one may develop at each successive level of teaching of descriptive statistics, adequate “Grundvorstellungen” for stochastic concepts. So this provides contexts where students can “mathematize” and “translate” problematic situations into mathematical models, besides developing their critical thinking in real applications, recognizing subjectivity and arbitrariness in model construction.

On the other hand, it has been shown that Bayesian “false-positive problems” in bio-medical test interpretation (“How likely is it that I am infected, if my viral test was positive?") may be solved by 6th graders, with the help of pedestrian metaphors, which allow then to calculate with natural (absolute) frequencies as in Zhu & Gigerenzer (2006) instead of fractions or percentages that confuse even experienced physicians.

### **RESEARCH QUESTIONS AND HYPOTHESES**

We have reviewed two independently developed theoretical perspectives concerning mathematics teaching and learning, the ‘Grundvorstellungen’ approach and the metaphor approach, both aiming at achieving a meaningful teaching-learning process, as an antidote to rote, mechanical learning, without understanding.

Given the high degree of influence that the former has had in German didactics and the increasing didactical attention the latter is now receiving, we would like to compare them, in terms of their theoretical cognitive and didactical background, as well as their classroom implementation with students and in-service teachers. In particular we would like to find out how they address the problematic of facilitating meaningful learning and *construction* (“*poiesis*”) of concepts by the students themselves.

Our first research hypothesis is that metaphors as well as “Grundvorstellungen” are key and necessary ingredients in a meaningful teaching-learning process that usually entails a switch between cognitive modes.

Our second hypothesis is that in almost every didactic case we could look upon “Grundvorstellungen” as metaphors and vice versa, without modifying the operational sense of the didactics, although in some cases metaphors may have a clearer poietic “bottom-up” role than “Grundvorstellungen”, which tend to be more prescriptive and “top-down” in their classroom implementation.

We intend to gather evidence in favour of these hypotheses from didactical experiments reported in the literature or coming from our own teaching experience or research. To this end, we comment below some compared examples from the viewpoint of “Grundvorstellungen” and metaphors.

## COMPARED DIDACTICAL EXAMPLES

### Example 1: Which fraction is bigger?

Using the “Grundvorstellungen approach”, Padberg (2009) points out that 6<sup>th</sup> graders have trouble in comparing fractions because of a lack of instruction in suitable “Grundvorstellungen”. They apply mechanically the standard method of calculating a common denominator, instead of flexibly using different strategies adapted to the fractions involved. He gives paradigmatic examples in context, of fractions to compare, and suggests to take advantage of the “sharing Vorstellung”, prompting the students to draw “ratio tables” to arrive at equating either the numerators or the denominators of the involved fractions. A typical example is the exercise where 5 children share evenly 3 pizzas and 8 children share other 5 identical pizzas, and it is asked in which group does each child get more pizza.

Metaphorically, one can proceed in an analogous way with the help of the metaphor “fractions are portions”, but also tackle some more unfriendly comparisons like that of  $15/25$  and  $16/26$  (loc. cit., exercise 2.b, p. 61) or say,  $17/153$  and  $18/154$ , with the help of a different metaphor. To this end, the students could be challenged to recall other metaphors for fractions, adequate to look at the unfriendly  $17/153$  and  $18/154$ , that allow for an easier denominator manipulation. They might recall the “ratio (or proportion) metaphor”. In our case, this could mean a big bag with 153 apples, 17 of which turn out to be rotten. How do we “move” to a bag with 154 apples, 18 of

which are rotten? Just adding one rotten apple to the previous bag! The “proportion” of rotten apples clearly increases! So, a shift to a more suitable metaphor enables the students to compare some unfriendly fractions in a transparent way, without even calculating.

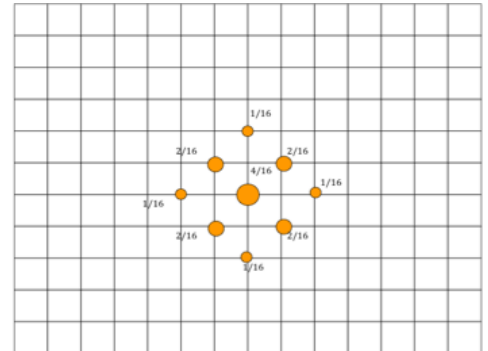
**Example 2: Brownie’s walk** (Soto-Andrade, 2006, 2007, 2010)

We claim that random walks provide a meaningful and friendly way to introduce students to stochastic thinking. We comment here on a baby avatar of Brownian motion (a natural example of randomness).

*Brownie is a puppy that escapes randomly from home, when she smells the shampoo her master intends to give her. At each street corner she chooses equally likely any of the 4 cardinal direction and runs nonstop a whole block until the next corner. Exhausted, after 4 blocks, say, she lies at some corner. Her master would like to know where to look for Brownie and also to estimate how far she will end up from home...*

This problem may be tackled in several ways:

1. Using descriptive statistics in a meaningful way (cf. Borovcnik, 1984 and “Grundvorstellung” 3 after Wollring), i.e. simulating, to begin with.
2. Applying Malle & Malle’s GP2 prescriptive “Grundvorstellung” above.
3. With an “ideal simulation” metaphor or “Vorstellung”: if we unleash a pack of  $4^4$  Brownies from home, say, they will split evenly among the 4 cardinal directions, and so on... The *pedestrian metaphor* (Soto-Andrade, 2006) may so be rediscovered!
4. In the metaphor approach, students would be also prompted to see Brownie’s random walk in a more concrete way. The *splitting metaphor* may then emerge, which sees Brownie, at each corner, splitting into 4 pieces, and so on... Or the gentler and less scary *hydraulic metaphor*, where the streets of the city become pipes and Brownie becomes a litre of fruit juice that flows equitably at each junction of pipes. This metaphor may be easily visualized or even performed with students sharing a litre of fruit juice (see figure 4).



**Fig. 4. Brownie’s splitting (2 blocks).**

Note that in the metaphor approach the students are not given a “Grundvorstellung” for probability before they address the problem. On the contrary, they are prompted to tackle the problem “bare handed” first and eventually look for a friendly metaphor for the *concrete random walk* they want to study (e.g. “Brownie splits”). When trying to give pertinent answers to the questions asked, “poietic metaphors” may emerge that enable them to construct the abstract probability concept, like “probabilities of finding Brownie at a given corner are pieces of Brownie”.

## DISCUSSION

Although Metaphors and “Grundvorstellungen” play essentially the same didactical role, aiming at giving concrete and familiar meaning to abstract concepts, we have seen that their typical ways of implementation may differ to some extent. Metaphors appear more radical and blunt, playing a poietic, bottom-up, role. “Grundvorstellungen” on the other hand, are more often used to represent in the “real world” pre-existing mathematical concepts, than to construct them; they tend to be used in a more prescriptive and top-down way, although we also have examples of descriptive research on idiosyncratic animistic metaphors in probability (Wollring, 1996). A systematic classification of “Grundvorstellungen” has been undertaken (vom Hofe, 2003; Blum & vom Hofe, 2003) and exercises for their visualization for a vast array of mathematical concepts have been proposed (Weber, 2007), but no examples of a poietic role of “Grundvorstellungen” seem to have been reported.

On the other hand, the typical contemporary metaphor approach is somewhat more optimistic, à la Dienes (1960), regarding as possible and trainable the spontaneous and autonomous emergence of suitable metaphors when students have the opportunity of tackling a problem, or a didactical situation à la Brousseau (1960), with their own means. Moreover, further research in this direction is suggested by the fact that this approach may be more naturally integrated with Brousseau’s theory of didactical situations (loc. cit.) than the more top-down “Grundvorstellungen” approach.

## REFERENCES

- Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, M.-A., Lacourly, N., Soto-Andrade, J., & Dartnell, P. (2010). The effect of analogies on learning to solve algebraic equations, *Pedagogies: An International Journal*, 5(3), 216–232.
- Blum, W., & vom Hofe, R. (2003). Welche Grundvorstellungen stecken in der Aufgabe? *Mathematik lehren*, 118, 14-18.
- Borovcnik, M. (1984). *Was bedeuten statistische Aussagen*. Stuttgart: Teubner.
- Breidenbach, W. (1957). *Methodik des Mathematikunterrichts in Grund- und Hauptschulen. Band 1 – Rechnen*. Hannover: Schroedel Verlag.
- Brousseau, G. (1998). *Théorie des situations didactiques*. Grenoble: La pensée sauvage.
- Bruner, J. (1996). *The culture of education*, Cambridge: Harvard University Press.
- Bruner, J., & Kenney, H. (1965). Representation and mathematics learning, *Monographs of the Society for Research in Child Development*, 30(1), 50-59.
- Dienes, Z.P. (1960). *Building up mathematics*. London: Hutchinson.

- Doritou, M., & Gray, E. (2007). The number line as metaphor of the number system: A case study of a primary school. In D. Pitta-Pantazi, and J. Philippou (Eds.). *Proceedings CERME 5* (pp. 111-120). Cyprus: Department of Education. University of Cyprus.
- Euler, L. (1802). *Vollständige Anleitung zur Algebra (1. Teil). Von den verschiedenen Rechen-Arten, Verhältnissen und Proportionen*. St. Petersburg: Kaiserliche Akademie der Wissenschaften.
- Griesel, H. (1971). *Die Neue Mathematik für Lehrer und Studenten. Mengen, Zahlen, Relationen, Topologie. Vol. 1*. Hannover: Schroedel Verlag.
- vom Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*. Heidelberg: Spektrum Akademischer Verlag.
- vom Hofe, R. (1998). On the generation of basic ideas and individual mages: normative, descriptive and constructive aspects. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity. An ICMI study*. (pp. 317-331). Dordrecht: Kluwer Academic Publishers.
- vom Hofe, R. (2003). Grundbildung durch Grundvorstellungen, *Mathematik lehren*, 118, 4-8.
- Kühnel, J. (1920). *Gedanken über Lehrerbildung, eine Gegenschrift*. Leipzig: Klinkhardt Verlag.
- Lakoff, G., & Núñez, R. (2000). *Where Mathematics comes from?* New York: Basic Books.
- Malle, G. & Malle S. (2003). Was soll man sich unter einer Wahrscheinlichkeit vorstellen? *Mathematik lehren*, 118, 52-56.
- Oehl, W. (1967). *Der Rechenunterricht in der Hauptschule (fünftes bis achtes Schuljahr); didaktisch-methodische Überlegungen und Hinweise für die tägliche Unterrichtsarbeit*. Hannover: Schroedel Verlag.
- Padberg, F. (2009). *Didaktik der Bruchrechnung*. Heidelberg: Spektrum Akad. Verlag.
- Prediger, S. (2008). The relevance of didactic categories for analysing obstacles in conceptual change: Revisiting the case of multiplication of fractions. *Learning and Instruction*, 18(1), 3-17.
- Presmeg, N. C. (1997). Reasoning with metaphors and metonymies in mathematics learning. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 267-279). London: Lawrence Erlbaum Associates.
- Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, 14(1), 44-54.



## Working Group 11

- Sfard, A. (1997). Commentary: On metaphorical roots of conceptual growth. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339-371). London: Lawrence Erlbaum Associates.
- Soto-Andrade, J. (2006). Un monde dans un grain de sable: Métaphores et analogies dans l'apprentissage des maths. *Ann. Didactique Sciences Cogn.*, 11, 123– 147.
- Soto-Andrade, J. (2007). Metaphors and cognitive styles in the teaching-learning of mathematics. In D. Pitta-Pantazi, & J. Philippou (Eds.). *Proceedings CERME 5* (pp. 191-200).
- Soto-Andrade, J. (2010). Ingénierie didactique ou bricolage?, Communication, to appear in *Actes de la XVe École d'Été de l'ARDM (Association française pour la recherche en didactique des mathématiques)*, Clermont-Ferrand, France.
- Tall, D. (2005). A Theory of Mathematical Growth through Embodiment, Symbolism and Proof, *Ann. Didactique Sciences Cogn.*, 11, 195-215.
- Thom. R. (1994). Spectre bord d'un centre obscur, in M. Porte (Ed.), *Passion des Formes*, (pp. 13-24). Fontenay-aux-Roses: ENS Éditions Fontenay-SaintCloud.
- Thom, R. (2002). Exposé au Colloque International de Cerisy: Logos et théorie des Catastrophes. *Annales de la Fondation Louis de Broglie*, 27(4), 575 – 596.
- Weber, C. (2007). *Mathematische Vorstellungen bilden. Praxis und Theorie von Vorstellungsübungen im Mathematikunterricht der S. II*. Bern: HEP Verlag.
- Wollring, B. (1994). Animistische Vorstellungen von Vor- und Grundschulkindern in stochastischen Situationen, *Journal für Mathematikdidaktik*, 15 (1/2), 3-34.
- Zhu, L., & Gigerenzer, G. (2006). Children can solve Bayesian problems: The role of representation in mental computation, *Cognition*, 98, 287-308.

# INTRODUCTION TO THE PAPERS OF WG12: HISTORY IN MATHEMATICS EDUCATION

Uffe Thomas Jankvist, Snezana Lawrence, Constantinos Tzanakis, Jan van Maanen

## THE BRIEF HISTORY OF THE HISTORY GROUP

The idea to have a group focussing on the *empirical side of history in mathematics education* was coined by Abraham Arcavi and Uffe Thomas Jankvist at CERME-5 in Cyprus, 2007. The proposal was made to ERME and a first call for papers was written in 2008 by A. Arcavi, U. T. Jankvist, C. Tzanakis and J. van Maanen (the latter two former chairs of HPM, the ICMI affiliated study group on the relations between History and Pedagogy of Mathematics). Fulvia Furinghetti (also former chair of HPM) chaired the group at CERME-6 in Lyon, 2009; she did so with the help of co-chairs Tzanakis, van Maanen, Jankvist, and Jean-Luc Dorier. In Lyon, 13 papers and 1 poster were presented. For CERME-8 in Rzeszów, the group had 13 papers and 5 posters. During its brief time of existence the history group has come to embrace not only the research on history *in* mathematics education, but also research on history *of* mathematics education in relation to (present) educational practices. This, together with the always relevant issue of quality versus inclusiveness at CERMEs, led to many thoughts on the actual structuring of the working group sessions. We discuss this below after presenting themes and papers.

## WG12'S MAIN THEMES AS GIVEN IN THE CALL FOR PAPERS

1.	Theoretical, conceptual and/or methodological frameworks for including history in mathematics education;
2.	Relationships between (frameworks for and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education
3.	The role of history of mathematics at primary, secondary, and tertiary level, both from the cognitive and affective points of view
4.	The role of history of mathematics in pre- and in-service teacher education, from cognitive, pedagogical, and/or affective points of view
5.	Possible parallelism between the historical development and the cognitive development of mathematical ideas
6.	Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments
7.	Surveys on the existing uses of history in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels
8.	Design and/or assessment of teaching/learning materials on the history of mathematics
9.	Relevance of the history of mathematical practices in the research of mathematics education

**PAPERS AND POSTERS PRESENTED IN WG12**

<b>Authors</b>	<b>Contribution</b>	<b>Main Themes</b>
<u>Mustafa Alpaslan</u> , Mine Isiksal, Cigdem Haser	Paper	4
<u>Kristín Bjarnadóttir</u>	Paper	9
<u>Kathleen M. Clark</u>	Paper	3, 4, (1, 2)
<u>Uffe Thomas Jankvist</u>	Paper	1, 2, 6, 8
<u>Tinne Hoff Kjeldsen</u>	Paper	1, 3, (2, 4)
<u>Panayota Kotarinou</u> , Charoula Stathopoulou, Anna Chronaki	Paper	3, (6, 8)
<u>Jenneke Krüger</u>	Paper	9
Snezana Lawrence, <u>Peter Ransom</u>	Paper	3, 4
<u>José Manuel Matos</u>	Paper	9
<u>Catarina Mota</u> , Maria Elfrida Ralha, Maria Fernanda Estrada	Paper	9
<u>Maurice O'Reilly</u>	Paper	1, 2, 3
<u>Peter Ransom</u>	Paper	3, 6, 8
<u>Constantinos Tzanakis</u> , Yannis Thomaidis	Paper	1, (2)
<u>Mária Correia de Almeida</u> , José Manuel Matos	Poster	9
<u>Ana Amaral</u> , Alexandra Gomes, Elfrida Ralha	Poster	3, 9
<u>Rui Candeias</u>	Poster	9
<u>Ersin İlhan</u>	Poster	3, 8, (6)
<u>Teresa Maria Monteiro</u> , José Manuel Matos	Poster	9

**STRUCTURE AND OUTCOMES OF THE WORKING GROUP SESSIONS**

The sessions of WG12 were organized so that every session began with two short presentations of papers. These presentations were followed by group work or reports from group work. The group work was structured according to four general topics (A, B, C, and D – listed below) and the participants discussed these topics in two smaller subgroups, the compositions of which varied according to the topics.

**Topic A: Research questions and relevance of research**

For the first sessions the two subgroups, say  $\alpha$  and  $\beta$ , were made so that subgroup  $\alpha$  consisted of the less experienced researchers in the field of history in mathematics education, who, based on the papers and posters of the WG, would discuss topic A under the guidance of a more experienced researcher and ‘subgroup manager’ (van

Maanen). Examples of questions that subgroup  $\alpha$  discussed are: *Why is your research relevant (and do you have literature references to underpin the relevance)? Do you have clearly stated research questions? How will your research questions guide you in your research – and in the choices you have to make? Is your research theory-driven or problem-driven – and how is this reflected in your research questions?* In the initial group work phase, the participants of subgroup  $\alpha$  were asked to briefly present their work and research questions if they had these formulated. This turned out to be a good, fast and efficient way of getting the ‘younger researchers’, and in particular the poster presenters, engaged in the WG discussions from the very beginning. Several participants decided to reconsider their research aim(s), formulate questions, refine formulations of existing questions, or expand their research perspectives. Also, the discussion of theory-driven versus problem-driven research led to discussions of the role of theory in (empirical) research, etc. O'Reilly presented the report from subgroup  $\alpha$ .

### **Topic B: Use of HPM theory and mathematics education theory**

Subgroup  $\beta$ , consisting of the more experienced researchers in the field, discussed topic B – use of HPM theory and mathematics education theory – based on questions such as: *What should the use of theory be in our subfield? What may we make use of from both mathematics education theory and history of mathematics theory? To what extent do we need HPM theories – and how may theses be shaped?* For a selection of the WG-papers, subgroup  $\beta$  discussed the influence of various other fields, e.g. history, history of mathematics, history of science, education and pedagogy, mathematics education, science education as well as philosophy and epistemology of mathematics and science. The following key-issues were identified as important, or crucial for the domain of history in mathematics education: *the need for developing theoretical constructs that provide some order in the wide spectrum of research and implementations done so far; to somehow check the efficiency of introducing a historical dimension, not least to convince the target population (teachers, math educators, curriculum designers, etc.); and to develop appropriate conditions for designing, realizing, and evaluating our research, including for instance the availability of useful resources, ‘worked-out’ material ready for ‘direct’ use, ‘history friendly’ teachers to cooperate in research as well as ‘history friendly’ authorities/curricula/ official regulations.* C. Tzanakis ‘managed’ and reported.

### **Topic C: Methods, data, and analysis**

For topics C and D two subgroups were again made: subgroup  $\gamma$  consisting of researchers in the area of history of mathematics education and subgroup  $\delta$  of researchers in history in mathematics education. The subgroups discussed topics C and D in turn. Examples of questions to be considered for theme C are: *What methods do you use to answer your research questions and how are these connected to your theoretical framework? What kinds of data do you gather (or have access to) and why these? How do you analyze your data and how is your analysis connected to*

## Working Group 12

*method and theory? Could you come to the same or similar conclusions using different methods, collecting different data, or analyzing those using different theoretical constructs?* Regarding data subgroup  $\gamma$  discussed, for example, the occasional scarceness of historical sources, which can make methods of ‘triangulation’ more or less impossible. Among many other things, subgroup  $\delta$  discussed the different methods related to quantitative and qualitative research, and the possibilities of combining such methods in the same study. K. Bjarnadóttir was the ‘subgroup manager’ for subgroup  $\gamma$ , and J. Krüger gave the report. T. Kjeldsen was ‘manager’ for subgroup  $\delta$  and the report was delivered by M. Alpaslan and P. Ransom.

### **Topic D: Validity, reliability, and generality of research results**

Examples of questions for topic D are: *How valid are your results? On what grounds must the validity be ‘measured’? How reliable are your results? How is this connected to method and theory (e.g. quantitative/qualitative; explain/predict)? Are your results generalizable and if so, then in what way?* For topic D, subgroup  $\gamma$  in particular, had to consider implications for mathematical practices of today. Also, subgroup  $\gamma$  spent a long time discussing the problems related to defining reliability and validity for qualitative research. Following similar discussions, embracing also reproducibility and driving forces for empirical research, subgroup  $\delta$  ended up discussing a variety of research questions that was deemed essential for the present status of the field of using history in mathematics education. And a plan was made for constructing a list of such ‘burning’ questions and publishing it once done.

### **EVALUATION AND ASPECTS TO CONSIDER FOR THE NEXT WG**

It was decided that for the next CERME the poster proposals will be sent to everyone in the group before the meeting and that the posters will be displayed *during* the sessions. Also, the chairs consider it important to maintain and even strengthen the connections between the CERME history group and the HPM group. One of the main things that were brought forth when evaluating WG12 was the friendly, inclusive and productive atmosphere, where everybody talked to and interacted with everybody. One participant expressed it like this:

A week ago I was completely scared, because I didn't know how the CERME work was done, and I didn't know how everyone in the WG would react to my work and my opinions (if I had enough courage to express them). Today I have in my memory the best conference I ever attended: a fantastic working group that made me desire for more opportunities to work with everyone.

# **USES OF HISTORY IN MATHEMATICS EDUCATION: DEVELOPMENT OF LEARNING STRATEGIES AND HISTORICAL AWARENESS**

Tinne Hoff Kjeldsen

IMFUFA, Department of Science, Systems and Models, Roskilde University

*The purpose of the paper is to present a theoretical framework for a systematic analysis and discussion of uses of history for teaching and learning mathematics, hereby proposing a didactical transposition of history from the academic research subject to history in mathematics education. The use of the theoretical framework is exemplified by an analysis of a project work on the history of Ancient Egyptian mathematics taught in a class of Danish upper secondary school students (10<sup>th</sup> grade), illustrating how uses of past mathematics can aid development of students' learning strategies and historical awareness.*

## **INTRODUCTION**

The purpose of the present paper is to develop a theoretical framework for a systematic analysis and discussion of uses of history for teaching and learning of mathematics with respect to how history benefits students' learning of mathematics, and develops students' historical awareness. Several recent papers have discussed whether these two aims pose a dilemma between genuine history and relevant mathematics for teachers who want to use or integrate history in their classrooms (Freid, 2001; Jankvist and Kjeldsen, forthcoming; Kjeldsen, forthcoming; Kjeldsen and Blomhøj, forthcoming). While these discussions have focused on transforming views of mathematics and mathematics education, their conception of history has been taken to be more or less synonymous with a traditional professional historians' approach to history – at least in the methodological approaches and the criteria for a genuine approach to history. However, perhaps we also need to broaden our view of history as well if we want history to play a more significant role for teaching and learning mathematics. In the present paper such a broadened view of history is outlined, and its implications for history in mathematics education are discussed. The aim is to develop an adequate theoretical framework for integrating history of mathematics in mathematics education that can be used to analyze specific implementations and to provide a tool for orienting the design of future implementations of the history of mathematics in mathematics education. The main focus of the paper is theoretical, but it also contains an empirical section that illustrates the theory in a carefully designed and implemented case study.<sup>1</sup>

First of all, some historiographical reflections and a position are presented. Secondly, uses of history are discussed to present a framework in which their uses for the teaching and learning of mathematics can be systematically analyzed with respect to purposes and didactical values. This discussion is based on the Danish historian

Bernard Eric Jensen's (2010) approach to history. Thirdly, the framework is adapted to mathematics education. Lastly, to connect the discussion with the practice of teaching, a project work on mathematics in Ancient Egypt is analyzed. The project work was designed by a mathematics teacher working at a Danish Gymnasium (upper secondary level) during a professional development course in "problem based project work in, with and about mathematics". The teacher implemented the project work in his own teaching practice in a class of first year Danish high school students (age 16-17) and documented his experimental teaching in a written report. His report will be analyzed to illustrate how uses of past mathematics can aid development of students' learning strategies and historical awareness, thereby substantiating some of the points raised in the present paper. The paper ends with some concluding remarks.

## **HISTORIOGRAPHICAL REFLECTIONS**

Mathematical knowledge is produced and used by humans; hence we can think of such activities as integrated elements of historical-social reality and of human life. We can perceive mathematical activities as creations of history as well as acts that create a history of mathematics. The development of mathematics and changes within our perceptions, views, and treatments of mathematics can to a certain extent be understood as realisations (intended as well as unintended) of goals set by people. If we want to understand historical-social processes in the development of mathematics as products of human activities, we must pay attention to intentions and thoughts of the actors, as well as their understanding of the subject matter and the context in which they performed and made their choices.

At a first sight it might seem that while such an approach can be used to study the history of sociological aspects of mathematics, such as the development of its profession in different countries and/or places or the history of mathematical journals, it cannot be used to study the history of the subject-matter of mathematics due to the universal character of mathematics. But if the development of mathematics is studied from its practice, where the historian focuses on concrete practices of mathematics, acknowledging that, despite its universal character, mathematical knowledge is produced by mathematicians, who live, interact and communicate in concrete social settings, the history of mathematical ideas, concepts and theories can also be pursued within such a framework.

Such a position is in accordance with recent trends in the history of mathematics that have emerged as reactions towards the well-known critic of the widely used anachronistic (whiggish) approach to history of mathematics and the methodological debate of internalism versus externalism (Epple, 2000), (Kjeldsen et al. 2004), and *Science in Context*, 2004, 17(1/2). Within the last decades many studies in the history of mathematics focus on the practice of mathematics within social, intellectual, and cultural contexts of mathematical activities. Here professional

historians of mathematics have a critical approach to source material they analyze in order to understand its significance in its proper historical context.

## USES OF PAST EPISODES

It is not the main purpose of general mathematics education to educate and train professional historians of mathematics, and in most cases mathematics teachers will not be professional historians. In some countries development of students' historical awareness is part of the curriculum, but that is not always the case, and if it is it only plays a minor part. With this in mind it seems too restrictive to require that the history of mathematics taught within mathematics education should be presented as traditional academic history. A didactical transposition is needed, just as is the case with school mathematics, which is also not identical with the discipline of (academic) mathematics. In the following, Jensen's (2010) broader view of history will be introduced along with several pairs of concepts that can be useful for a nuanced analysis and discussion of the role of past mathematical episodes for the learning and teaching of mathematics.

Jensen (2010) sees the academic research subject *history*, as professional historians think and work with it, as just one of many approaches to history. According to him, history is employed every time a person or a group of people is interested in something from the past, and uses their knowledge about it for some purpose. People use history for many different purposes and in many different connections, and consequently there are major differences between a lay person's and a professional historian's use of history. Recent investigations (Rosenzweig and Thelen, 1998) have shown that lay persons' and professional historians' conceptions of history differ in various respects and on several levels. Lay-history has a reputation of being naïve viewed from the academic discipline of history, while on the other hand lay historians view academic history as lifeless and remote from the real world. For professional historians it is important to place past episodes and artefacts in their historical contexts. Their historical awareness is conceived of as an interpretation of the past whereas lay persons view history more as a source of memoirs.

Jensen distinguishes between pragmatic and scholarly approaches to history. In a pragmatic approach the study of the past is guided by the idea that we can learn from history. The "usefulness" of history is an underlying perspective or principle in a pragmatic approach to history. The idea is that through history we can gain knowledge about our world of today, that history can teach us better ways to live our lives. In a pragmatic approach to history, past events are studied from a utility perspective. Jensen (2010, p. 51) contrasts a pragmatic approach to history with a scholarly approach, where historians retain a critical distance to past events and emphasize differences between past and present. In the professional, academic discipline of history both traditions can be found, but since the mid 19<sup>th</sup> century the scholarly approach to history has been more and more dominant.



Observer history and actor history are another pair of concepts through which we can discuss and understand uses of past events and sources. Jensen (2010, p. 41) talks about uses of the past from an actor perspective, if we use history to orient ourselves and act in a present context. He calls this an intervening use of history. If the past is viewed retrospectively with a purpose to enlighten instead of a purpose to act or intervene he talks about uses of past from an observer perspective.<sup>2</sup>

Finally, the so-called "living history" use of history is a way of using the past to help participants develop historical awareness and learning strategies. In Denmark living history takes place at some museum centres and at some yearly events. One such centre is The Medieval Centre. On their homepage (<http://www.middelaldercentret.dk/Engelsk/welcome.html>) they state that the centre: "is an experimental museum where you can experience life in a reconstructed late 14th century market town: Daily life, knights tournaments, trebuchets, canons, ships, markets, ... and a lot more...". According to Jensen (2010, p. 145) living history appeals to so many not only because the participants actively take part in the events, but also because they use other types of learning strategies where the focus can be, for example, to develop the skills of past craftsmen.

## **WHAT IS THE CONNECTION TO MATHEMATICS EDUCATION?**

These concepts of, approaches to, and thinking about history and uses of past episodes and artefacts present a framework for a refined discussion and systematic analysis of how past episodes and sources can be/are used in the integration of history for the teaching and learning of mathematics. They open up a variety of approaches to history and uses of the past for teachers who want history to play a role for teaching and learning mathematics. Which approach to choose depends on the intended learning. For example, Kjeldsen and Blomhøj (forthcoming) argue, based on Sfard's (2008) theory of thinking as communicating, that history presents itself as the obvious tool for developing students' proper meta-discursive rules, because meta-discursive rules are contingent and as such can be studied at the object level of history discourse. This presupposes a scholarly approach to history. The idea is to use past mathematical activities and sources with the intention of creating learning and teaching situations where students can experience what Sfard calls commognitive conflicts. Hence, the past is used with the purpose of intervening, and therefore the scholarly approach to history is from an actor perspective.

Kjeldsen (forthcoming) discusses the role of history for the teaching and learning of mathematics with reference to a competence based understanding of mathematics education (Niss, 2004). Here the development of students' mathematical competence is the main purpose of mathematics education along with the development of some second order competencies, including historical overview and awareness. For the development of historical overview and awareness, a scholarly approach from an

observer perspective can be chosen. For development of specific mathematical competencies, a pragmatic approach from an actor perspective might be considered.

### **AN IN-SERVICE COURSE ON PROJECT WORK**

The focus of the paper is on theoretical issues, but to illustrate the theory, a project work that was developed and implemented during an in-service course for upper secondary teachers in Denmark will be analysed. In this discussion the “living history” approach will be examined to see how it might be adapted as a way for mathematics teachers to use past episodes and sources to develop students’ learning strategies and historical awareness.

The theme for the project work was Egyptian mathematics. It was developed and tested in a classroom of students (10<sup>th</sup> graders) in the Danish upper secondary school in 2004 as part of an in-service course for mathematics. The in-service course was developed in response to a reform that was to be implemented in 2005.

Compared with more traditional ways of teaching mathematics the reform challenged the teachers in several ways: (1) Many were not used to teach either the history of mathematics or mathematical modelling, both of which having more prominent positions in the new curriculum than they had in the former curriculum; (2) they were required to bring mathematics into play in interdisciplinary projects in cooperation with other subjects, from science, from the humanities, and from the social sciences; and (3) they had to design, organise and carry out project work in their mathematics teaching. The goal was to create an in-service course where theories in didactics and pedagogy interacted with development of the participants’ own teaching practice in ways that also related to inquiry-based teaching and learning. On this basis the objective of the in-service course was to support teachers in their development as teachers, implementation in their own classes, evaluation of the project work, and documentation through a written report of a project-based and problem-oriented course in the history of mathematics or in mathematical modelling. The core element of the in-service course was the development of the teachers’ experimental practice with history of mathematics or mathematical modelling and problem-oriented project work.

The in-service course began with a three day seminar where the teachers were introduced to the history of mathematics, mathematical modelling, didactical theories, and problem oriented project work. The teachers worked in small groups developing a project-organised course in either history of mathematics or mathematical modelling of their own choice consisting of approximately 10 lessons of 45 minutes each. They decided on (1) the objectives for their own professional development, (2) their objectives for students’ learning, (3) how to “set the scene” for their own students’ project work, and (4) how to evaluate the students’ learning.

A few weeks after the seminar a first draft of the design for the project work and the materials that should be given to the students were distributed to all participants in

the in-service course. All teachers tried out their project work in their classroom. During that period there was a one day seminar to support the teachers in the documentation of their results and reflections on their experimental teaching. It all ended with a 2-day seminar, where the teachers' written reports were discussed extensively. The final reports are published on the internet together with the handout materials for the students for other teachers to use (<http://magenta.ruc.dk/nsm/uddannelser/gymnasielaerer/>).

I will not go into further detail on how we define problem-oriented project work (interested readers are referred to Blomhøj and Kjeldsen, 2006), but only emphasize that the problem that students are going to work on should function as the “guiding star” for their work. In the ideal case every decision made in the project work should be justified by its contribution to the solution of the problem. This is crucial, since engaging in decisions provides opportunities for students to work independently, to gain control, and to direct the project. In order for this to happen, though, the teacher needs to set a scene for the project work, that is to formulate the task for the work, the conditions for the working process, the time constraints, and the requirements for the end product, for example a written report or a power point presentation fulfilling some specific requirements. In this way it is possible for the teacher to have some control while at the same time to leave room for the students to take responsibility and make decisions.

The in-service course is still offered with the modification that we focus only on mathematical modelling. Therefore we only have one history project to present, but since its function here is to serve as a concrete illustration of the theoretical framework developed above, and not as documentation from an empirical experiment it can be used to characterize the suggested methodology.

### **EGYPTIAN MATHEMATICS: A PROJECT WORK IN A 10<sup>TH</sup> GRADE**

The project on Egyptian mathematics was developed and implemented in a classroom of 1. year students (10<sup>th</sup> grade, age 16) in a Danish upper secondary school in the fall term. The project work was meant to be interdisciplinary, with history about Ancient Egypt in combination with their mathematics. The mathematics teacher had no experience with project-organised teaching in mathematics, which was his focus for his own professional development. His objectives for the students' learning were to:

- a) enhance the students' competence to work in teams
- b) enhance the students' independent learning
- c) enhance the students' oral presentation skills
- d) have the students gain experiences with power point

## Working Group 12

- e) have the students appreciate that mathematics has been different from what it is today
- f) develop the students' awareness that mathematical results have evolved, that mathematics is not static, which is contrary to the way it is often presented
- g) develop the students' awareness that mathematics develops in an interplay with culture and society. (Wulff, 2004, p. 2-3; my translation)

The objectives fall into two parts that cover all three of the above listed challenges of the reform: the first four address competence in independent study, the development for which problem-oriented project work is an excellent pedagogical tool, whereas the last three concern the history of mathematics requirements of the new mathematics curriculum. Note that a)-c) and e)-g) are elaborated versions of some of the ICMI Study *whys*, see Fauvel and van Maanen (2000, pp. 205, 207, 211-212).

The teacher orchestrated the students' project work in three stages:

- (1) The first stage was an introduction to Egyptian mathematics using a text from the students' textbook (Carstensen and Frandsen, 2002), where the teacher introduced the Egyptians' method of multiplication by repeated doubling, their number symbols, and their way of formulating problems (two lessons).
- (2) The introduction was followed by eight lessons during which the students worked in teams of four, guided by a description of
  - i) the problem formulation, which was given by the teacher (see below);
  - ii) the learning objectives; iii) the product; iv) the topics for the teams.

The teams worked independently. The teacher took the role of a consultant who could be called in for advice. When that happened he focused on posing questions and challenging the teams instead of providing answers. The problem formulation for all teams was: How and why did the Egyptians calculate? Each team worked with a chapter from a textbook on Egyptian mathematics (Frandsen, 1996), seven chapters all together treating their numerals, their methods for arithmetical operations, the 2/n-table, bread and beer (Pesu) exercises, equations and geometry. To have a whole textbook on an episode from the history of mathematics in Danish is a rare circumstance, and one of the reasons why Egypt was chosen for this project work.

- (3) Each team had to present its results for the rest of the class in an oral presentation supported by a power point presentation. This took up four lessons.

The first set of learning objectives deal with issues of enhancing students' independent study skills. In his evaluation the teacher emphasized in particular that the students acquired the mathematical knowledge of the Egyptians by themselves (in contrast to ordinary teaching where he explained everything), that they "cracked the code" themselves, and that they were conscious about it. Regarding item e) and f)

of the second set of learning objectives, the teacher wrote: “they were all about gaining insights into current mathematics precisely by studying the mathematics of another time” (Wulff, 2004, p. 3), from which we can infer that the teacher used a pragmatic approach to history. He used past episodes of mathematics from a utility perspective. This also becomes clear from his description of a discussion that took place between him and the students during the introduction: “Already during the first module [the first two lessons] came the classical question, why are we going to learn this? And we had a good talk about the intended learning issues e), f), and g), during which the class apparently accepted that historical mathematics, besides being interesting as such, could contribute to a more nuanced view on current mathematics.” (Wulff, 2004, p. 5). Regarding the learning objective of realizing that mathematics has evolved over time, the teacher was rather critical, explaining that this aspect was not really complied with, since a comparison of Egyptian and modern mathematics only shows that mathematics has changed; it does not give insights into the actual process of change. Regarding the last item g) of the second part of the learning objectives, the teacher wrote in his evaluation: “here is where the subject of history can be involved. From a general knowledge about Ancient Egypt and its society, students can discuss how society and culture have been driving forces for the mathematics of that time. At the same time the historians’ method of source criticism is an essential tool for interpreting ambiguous and defective papyri” (Wulff, 2004, p. 4). In contrast to items e) and f) the teacher here takes a scholarly approach to history. The teacher used the past from an observer perspective in both approaches.

The students’ work with the sources and exercises in the textbook on Egyptian mathematics to answer the “How” part of the problem formulation can be considered a “living history” approach. They put themselves in the place of Ancient Egyptians, trying to understand and learn how they calculated, how they dealt with geometry, how they proposed mathematical problems, and so forth. The teacher reported the following situation he observed in the classroom: “Many students wondered about how “stupid” the Egyptians were. Why did they only use unit fractions? Why should a number be expressed as a sum of different unit fractions? On the other hand their methods were very difficult to understand; that is rather advanced, so in that respect they weren’t stupid at all. I think that many of the students realized that current mathematics is not “just” like today, but is a result of a long development, during which many things have been simplified.” (Wulff, 2004, p. 7). This shows a development of historical awareness among the students. That the students’ learning strategies were developed through this kind of “living history” approach can be inferred from the following observation made by the teacher: “This [that mathematics had made progress] became especially obvious when the students constantly rewrote the Egyptian notation to current notation with  $x$ ’s, formulas, etc. After they had finished an Egyptian calculation they would say: ‘but that just corresponds to ...’ followed by a solution of an equation in our way. It was very inspiring to see how students, who normally were a bit alienated towards  $x$ ’s and

equations now had taken those to themselves as their own, and all of a sudden perceived equations as an easy way to solve problems. The students became aware that modern notation makes the calculations much easier than they would have been otherwise” (Wulff, 2004, p. 7).

As mentioned above the teacher found that item g) in the list of learning objectives, which was supposed to link the development of mathematics with a scholarly approach to history, was not realized. The “why” part of the problem formulation was designed especially towards this goal. The mathematics teacher had hoped that the students would have been able to experience concrete examples of how needs of society sometimes act as driving forces for the development of mathematical ideas. This is a very ambitious goal, and since the history teacher focused more on religion and dynasties, the mathematics teacher felt that the students did not get opportunities to gain real insights into why mathematics was developed in interaction with the needs of society and culture. A less ambitious teacher would probably evaluate this part differently, pointing towards the fact that was explained above, that the students gained genuine historical knowledge about Egyptian mathematics situated in the proper historical context. Finally, the teacher concluded that the students afterwards showed signs of possessing a more mature and reflective approach to mathematics than they had before. Unfortunately, the teacher did not document this with observations from the classroom.

### CONCLUDING REMARKS

The purpose of the paper was to present a theoretical framework for a systematic analysis of the uses of history for teaching and learning mathematics in order to propose a didactical transposition of history from the academic research subject to history in mathematics education. The analysis of the teacher’s report on the project work on Ancient Egyptian mathematics with respect to the described framework of different uses of past episodes shows that in this project, history was used in different ways to provide a very rich teaching and learning environment. The teacher used different approaches to history and used past episodes from various perspectives for different purposes, thereby creating learning situations that developed students’ historical awareness and mathematical learning strategies at the same time. History was used in ways in which students gained genuine historical insights, developed learning strategies, and enhanced their mathematical problem solving skills even though they worked on mathematics that might not be part of the core curriculum.

### NOTES

<sup>1</sup> I would like to thank Costantinos Tzanakis for helpful comments on an earlier version of this paper.

<sup>2</sup> A fourth pair of concepts is identity neutral vs. identity concrete history writing, which will not be used in this paper.

## REFERENCES

- Blomhøj, M. and Kjeldsen, T.H. (2006). Teaching mathematical modelling through project work – Experiences from an in-service course for upper secondary teachers. *ZDM*, 38, 2, 163-177.
- Carstensen, J. and Frandsen, J. (2002). *Mat 1*. Viborg: Systime.
- Epple, M. (2000). Genies, Ideen, Institutionen, mathematische Werkstätten: Formen der Mathematikgeschichte. *Mathematische Semesterberichte*, 47, 131-163.
- Fauvel, J., and van Maanen, J. (eds.), (2000). History in mathematics education – The ICMI study. Dordrecht: Kluwer.
- Frandsen, J. (1996). *Ægyptisk matematik*. Viborg: Systime.
- Fried, M.N. (2001). Can Mathematics Education and History of Mathematics Coexist? *Science & Education*, 10, 391-408.
- Jankvist, U.T. and Kjeldsen, T.H. (forthcoming). New Avenues for History in Mathematics Education: Mathematical Competencies and Anchoring. Accepted for publication in *Science & Education*.
- Jensen, B.E. (2010). *Hvad er historie*. Copenhagen: Akademisk Forlag.
- Kjeldsen, T.H. (forthcoming). A multiple perspective approach to the history of the practice of mathematics in a competency based mathematics education: history as a means for the learning of differential equations. To appear in Katz, V., & Tzanakis, C. (eds.) *Recent developments on introducing a historical dimension in mathematics education*, to be published in 2010.
- Kjeldsen, T.H., Pedersen, S.A. & Sonne-Hansen, L.M. (2004). *New Trends in the History and Philosophy of Mathematics*, Odense: SDU University Press, 11-27.
- Kjeldsen, T.H. and Blomhøj, M. (forthcoming). Beyond Motivation – History as a method for the learning of meta-discursive rules in mathematics. Accepted for publication in *Educational Studies of Mathematics*.
- Niss, M. (2004). The Danish “KOM” project and possible consequences for teacher education. In R. Strässer, G. Brandell, B. Grevholm and O. Helenius (eds.) *Educating for the Future*. (pp. 179-190). Göteborg: The Royal Swedish Academy.
- Rosenzweig, R. and Thelen, D. (1998). *The Presence of the Past. Popular Uses of History in American Life*. New York: Colombia University Press.
- Sfard, A. (2008). *Thinking as Communicating*. Cambridge: Cambridge University Press.
- Wulff, P. (2004). *Ægyptisk matematik*. Report from the in-service course: Problem oriented project work in, with and about mathematics, Roskilde University.

# CLASSIFYING THE ARGUMENTS & METHODOLOGICAL SCHEMES FOR INTEGRATING HISTORY IN MATHEMATICS EDUCATION

Constantinos Tzanakis

Yannis Thomaidis

Department of Education, University of  
Crete, Rethymnon 74100, Greece

Regional Administration of Education,  
Central Macedonia, Thessaloniki, Greece

tzanakis@edc.uoc.gr

gthom54@gmail.com

*The ICMI Study volume “History in Mathematics Education”, published in 2000, includes a comprehensive list of arguments for integrating history in Mathematics Education (ME) and methodological schemes of how this can be accomplished. Recently Jankvist distinguished between using “history-as-a-goal” and using “history-as-a-tool” to classify the above arguments. Independently, Grattan-Guinness distinguished between “history” and “heritage” in the hope that –among other things- this will help to understand better which history is expected to be helpful and meaningful in ME. We attempt to connect these two conceptual “dipoles”, aiming to provide in this way a finer and deeper classification of the arguments and methodological schemes for integrating history in ME that will serve as an appropriate theoretical framework.*

**Keywords:** *History, Heritage, History–as–a-tool, History–as–a-goal, complementary*

## RATIONALE AND BASIC IDEAS

In the last decades, there is a worldwide growing interest in integrating the history of mathematics (HM) in mathematics education (ME). Several attempts have been made, education material has been produced, empirical research has been conducted, methodological schemes have been invented & implemented and arguments for this integration have been put forward to refute possible objections and/or to enhance the interest of the ME community in this direction. For a long time, there were no coherent theoretical ideas and framework to place, see and compare all these activities. A serious attempt in this direction is the comprehensive ICMI Study volume (Fauvel & van Maanen 2000). In particular, it presents a comprehensive list of the arguments for integrating the HM in ME (the whys)<sup>1</sup> and the general ways of how to accomplish this task (the hows)<sup>1</sup>, in the sense that the whys correspond to tasks that one attempts to accomplish and the hows correspond to methodological approaches one could follow.

Since then, further important work followed, however:



Jankvist (2009a, b, c, d) has reconsidered the whys and the hows. He made the distinction between two broad ways to introduce the HM in ME, namely in his terminology,

- *History as a tool* emphasising *inner issues* in Mathematics (*in-issues*), and
- *History as a goal* emphasising *meta-perspective issues* in Mathematics (*meta-issues*)

and attempted to classify the *whys* using this distinction (Jankvist 2009a §§2.3, 8.1, 8.5; 2009b §1.1; 2009c §§2, 3), which constitutes the first **conceptual dipole**, in the context of the present paper<sup>2</sup>.

Moreover, he attempted to classify the *hows* according to his 3-fold distinction of possible implementations: *illumination approaches*, *modules approaches* and *history-based approaches* and to connect them to the above conceptual dipole (Jankvist 2009a §§2.4, 2.6, 2009c §§6, 9).

Independently, Grattan-Guinness (2004a, b) introduced in a more general context the distinction between “*history*” and “*heritage*”, to interpret mathematical activities and their products, in an effort to clarify existing conflicts and tensions between a mathematician’s and a historian’s approach to mathematical knowledge. Grattan-Guinness (2004b) gives several examples by contrasting the general characteristics of the two concepts. This is our second **conceptual dipole**. It could be an important tool to revisit the issue of “which history is appropriate to ME?” (see e.g. Barbin 1997).

As we will argue, within each *dipole*, the two “*poles*” are **complementary**<sup>3</sup> to each other, in the sense that they are **mutually exclusive** for a **simultaneous** use, but none of them, taken alone, can lead to a sufficiently wide and deep enough understanding of what (a specific piece of) mathematics is; instead **both are necessary** to understand mathematics as a cultural endeavour & human intellectual activity, either didactically or/and epistemologically. We stress at this point that “mutually exclusive” should not be understood in its strictly **logical** sense, but rather in Bohr’s broader sense<sup>4</sup>. Here, “complementarity” has to do with the fact that within each dipole, each pole focus on a particular aspect of what mathematics and its development are, but none of these emphases is sufficient for that if taken alone and an extreme version of each one of them leaves no space for the others.

The key idea and aim of this paper is to **classify** the ICMI Study *whys* and *hows* in a **finer** way, by projecting them onto the 2X2 grid formed by the two dipoles, thus getting “... a clear[er] idea about **why** history should be used in a given situation; i.e. what *whys* the use of history should fulfill...” and which are suitable options of *hows* (Jankvist 2009c p.256). Connecting the two dipoles in this way may contribute to clarify their relevance to ME and provide a finer conceptual framework to integrate HM into ME. However, this **2-dimensional classification** is **tentative**, possibly subject to modifications, as **specific** realizations & examples of the *whys* and the *hows* are examined how they fit into it.

The paper is structured as follows: A concise description of each *dipole* is given in the next section, with reference to distinctions similar to the *History-Heritage* dipole found in Fauvel & van Maanen 2000. The next two sections give a concise list of the ICMI Study *whys* and *hows* (together with Jankvist's 3-fold implementations), followed by the **classification tables** with brief reference to indicative examples that support this classifications, or reveal some possibly controversial aspects of them. Some final remarks are given in the last section. Due to space limitations, only the general ideas are presented, with specific applications to illustrate their use given in another paper (Thomaidis & Tzanakis, to appear).

## THE TWO CONCEPTUAL DIPOLES

Jankvist introduced two broad ways (or purposes, as he calls them) in which HM could be helpful and relevant to ME: *History-as-a-tool* and *History-as-a-goal*, which are intimately connected with issues within mathematics (what he calls *inner issues*) and with issues that concern mathematics itself (what he calls *meta issues*). In his own words,

“History-as-a-tool concerns the use of history as an assisting means, or an aid, in the learning [or teaching] of mathematics.... in this sense, history may be an aid both...”<sup>5</sup> “as a motivational or affective tool, and... as a cognitive tool...”<sup>6</sup> “[It] concerns... inner issues, or in-issues, of mathematics [that is] issues related to mathematical concepts, theories, disciplines, methods, etc.— the internal mathematics”<sup>7</sup>.

On the other hand,

“*History-as-a-goal* does not serve the primary purpose of being an aid, but rather that of being an *aim* in itself... posing and suggesting answers to questions about the evolution and development of mathematics,... about the inner and outer driving forces of this evolution, or the cultural and societal aspects of mathematics and its history” (Jankvist 2009b §1.1). In other words, “[It] concerns ... learning something about the meta-aspects or *meta-issues* of mathematics ... [that is] issues involving looking at the entire discipline of mathematics from a meta perspective level” (Jankvist 2009c, pp239-240)<sup>8</sup>.

This description makes clear that these two ways in which HM becomes relevant to ME are mutually exclusive, in the sense that the emphasis put on each are clearly different and to a large extent mutually incompatible. Nevertheless, it should be remarked that although “history-as-a-goal ‘in itself’ does not refer to teaching history of mathematics *per se*, but using history to surface meta-aspects of the discipline ... in specific teaching situations, [it] may have the positive side effect of offering to students insight into mathematical in-issues of a specific history” (Jankvist 2009d, p.8). Conversely, using “history-as-a-tool” to teach and learn a specific mathematical subject may stimulate reflections of a meta-perspective nature extrapolated from the particular subject considered; that is, we may have a kind of *anchoring* of *meta-issues* into the *in-issues* that constitute the study of the subject (see Jankvist 2009b, §§5.3, 5.4, 6.1, 6.3 for more details and examples of this idea). These are important interrelations, stressing the indispensability of both the “history-as-a-tool” and the “history-as-a-goal” ways, which

thus constitute what we called earlier a coherent *conceptual dipole*.

Quite independently, having in mind both historians and educators of mathematics, Grattan-Guinness introduced the distinction between *History* and *Heritage*. More specifically:

The History (Hi) of a particular mathematical subject  $N$  refers to "... the development of  $N$  during a particular period: its launch and early forms, its impact [in the immediately following years and decades], and applications in and/or outside mathematics. It addresses the question 'What happened in the past?' by offering descriptions. Maybe some kinds of explanation will also be attempted to answer the companion question 'Why did it happen?'"<sup>9</sup>. "[It] should also address the dual questions 'what did not happen in the past?' and 'why not?'; false starts, missed opportunities ..., sleepers, and repeats are noted and maybe explained. The (near-)absence of later notions from  $N$  is registered, as well as their eventual arrival; differences between  $N$  and seemingly similar more modern notions are likely to be emphasized"<sup>10</sup>.

On the other hand,

The *Heritage* (He) of a particular mathematical subject  $N$  refers "...to the impact of  $N$  upon later work, both at the time and afterward, especially the **forms** which it may take, or be **embodied**, in later contexts. Some modern form of  $N$  is usually the main focus, with attention paid to the course of its development. Here the mathematical relationships will be noted, but historical ones... will hold much less interest. [It] addresses the question '*how did we get here?*' and often the answer reads like 'the royal road to me.' The modern notions are inserted into  $N$  when appropriate, and thereby  $N$  is unveiled... *similarities* between  $N$  and its more modern notions are likely to be emphasized; the present is *photocopied* onto the past" (Grattan-Guinness, 2004a, p.165). Although, Grattan-Guinness is mainly concerned with the implications of this distinction on the way past mathematics should be approached, he clearly indicates its relevance to ME (Grattan-Guinness 2004b, §1.6). He argues that ME can profit equally well from both Hi and He, urging for further exploration in this context, a fact that has not passed unnoticed (Siu 2006, p.273, Rogers 2009 pp.120-121, Schubring 2008, p.5). He also gives a detailed list of the differences between these two conceptions (Grattan-Guinness 2004b, §1.3), showing their **incompatibility**, summarized as follows:

"The distinction between history and heritage is often sensed by people who study some mathematics of the past, and feel that there are fundamentally different ways of doing so. Hence the disagreements can arise; one man's reading is another man's anachronism, and his reading is the first one's irrelevance. The discords often exhibit the differences between the approaches to history usually adopted by historians and those often taken by mathematicians." (Grattan-Guinness 2004b, p.8).

On the other hand, however, their **indispensability** in understanding the development of mathematics is clearly emphasized:

"The claim put forward here is that both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or

asserting that one is subordinate to the other, is not.” (I. Grattan-Guinness 2004b, p.8)

Thus, the above quotations show that the two concepts are complementary in the sense of the first section, constituting the two poles of a coherent conceptual dipole.

As far as HM in ME is concerned, the distinction between History and Heritage is related to the rationale underlying the distinction between pairs of methodological approaches put forward in the past, like explicit & implicit use of history, direct & indirect genetic approach, forward & backward heuristics (Fauvel & van Maanen 2000, §7.3.2 and references therein), e.g.:

When “... HM is explicitly integrated, mathematical discoveries are presented in all their aspects. Different teaching sequences can be arranged according to the main historical events, in an effort to show the evolution and the stages in the progress of mathematics by describing a certain historical period ... [When] ... HM enters implicitly, history suggests a teaching [approach], in which use may be made of concepts, methods and notations, that appeared later than the subject under consideration, keeping always in mind the general didactic aim, namely to understand mathematics in its modern form. ... [It] does not necessarily respect the order by which the historical events appeared; rather, one looks at the historical development from the current stage of concept formation and logical structuring of the subject ... they have a dual character...and both may be used.... in complementary ways ... in an explicit integration of the HM, the emphasis is on a rough, but ... more or less accurate mapping of the path network that appeared historically and led to the modern form of the subject; in an implicit integration [of the HM], the emphasis is on the redesigning, shortcutting and signalling this path network”<sup>11</sup> (Fauvel & van Maanen 2000, p.210; our emphasis).

Hence, Grattan-Guinness distinction is potentially of great relevance to ME (as he himself points out; see also Rogers 2009), serving - among other things - to contribute towards answering the recurrent question “Why history and which history is appropriate to be used for educational purposes?” (Barbin 1997).

## **A CONCISE LIST OF THE *WHYS* & *HOWS***

In this section we present concisely the list of the whys & hows according to the ICMI Study volume (Fauvel & van Maanen 2000, §§7.2, 7.3) and Jankvist (2009c, §6) that will be used in the next section.

The “ICMI Study whys”

The following are the areas in which the teaching and learning of mathematics can profit from integrating the HM in the educational process.

### **A. The learning of Mathematics**

1. Historical development vs. polished mathematics: To uncover/unveil concepts, methods, theories etc.

2. History as a re-source: To motivate, to raise the interest, to engage the learner by linking present knowledge and learning process to knowledge and problems in the past.
3. History as a bridge between mathematics and other disciplines/domains: From where and how did a great part of mathematics emerged? To bring in new aspects, subjects and methods.
4. The more general educational value of history: To develop personal growth and skills, not necessarily connected to mathematics.

**B. *The nature of mathematics and mathematical activity***

1. *Content*: To get insights into concepts, conjectures & proofs, by looking from a different viewpoint; to appreciate “failure” as part of mathematics in the making; to make visible the evolutionary nature of meta-concepts.
2. *Form*: To compare old and modern; to motivate learning by stressing clarity, conciseness and logical completeness.

**C. *The didactical background of teachers***

1. *Identifying motivations*: To see the rationale behind the introduction of new knowledge and progress.
2. *Awareness of difficulties & obstacles*: To become aware of possible didactical difficulties and analogies between the classroom & the historical evolution.
3. *Getting involved and/or becoming aware of the creative process of “doing mathematics”*: To tackle problems in historical context; to enrich mathematical literacy; to appreciate the nature of mathematics.
4. *Enriching the didactical repertoire*: To increase the ability to explain, approach, understand specific pieces of mathematics and on mathematics.
5. *Deciphering and understanding idiosyncratic and/or non-conventional approaches to mathematics*: To learn how to work on known mathematics in a different (old) context; hence to increase sensitivity and tolerance towards non-conventional, or “wrong” mathematics.

**D. *The affective predisposition towards mathematics***

1. *Understanding mathematics as a human endeavour*: To show and/or understand explicitly evolutionary steps.
2. *Persisting with ideas, attempting lines of inquiry, posing questions*: To look in detail at similar examples in the past.
3. *Not getting discouraged by failures, mistakes, uncertainties, misunderstandings*: To look in detail at similar examples in the past.

**E. *The appreciation of mathematics as a cultural endeavour***

1. *To appreciate that mathematics evolves under the influence of factors intrinsic to it:* To identify and appreciate the role of internal factors.
2. *To appreciate that mathematics evolves under the influence of factors extrinsic to it:* To identify and appreciate the role of external factors.
3. *To appreciate that mathematics form part of local cultures:* To understand a specific piece of mathematics through approaches belonging to different cultures.

### **The “ICMI Study *hows*”**

Below is a similar account of the *hows* for integrating HM in ME according to the ICMI Study volume, followed by that of Jankvist.

#### **1. Learning *history* by providing *direct historical information***

Isolated factual information; historical snippets; separate historical sections; whole books and courses on history etc.

#### **2. Learning *mathematical topics* by following a teaching *approach inspired by history***

Teaching modules inspired by history; worksheets based on original sources; historical-genetic approach; modernised reconstructions of a piece of mathematics etc.

#### **3. *Developing***

*Awareness of the intrinsic nature of mathematical activity (**intrinsic awareness**) and*

(i) The role of general conceptual frameworks

(ii) The evolutionary nature of all aspects of mathematics

(iii) The importance of the mathematical activity itself (doubts, paradoxes, contradictions, heuristics, intuitions, dead ends etc);

*Awareness of the extrinsic nature of mathematical activity (**extrinsic awareness**)*

(i) Relations to philosophy, arts and social sciences

(ii) The influence of the social and cultural contexts

(iii) Mathematics as part of (local) culture and product of different civilizations and traditions

(iv) Influence on ME through ME history.

### **“Jankvist’s *hows*”**

1. ***Illumination approaches:*** Teaching and learning of mathematics, in the classroom or the textbooks used, is supplemented by historical information of varying size and emphasis.

2. ***Modules approaches:*** Instructional units devoted to history, and often based on the detailed study of specific cases. History appears more or less directly.

3. **History-based approaches:** Directly inspired by, or based on the HM. Not dealing with studying the HM directly, but rather indirectly; the historical development not necessarily discussed in the open, but, often sets the agenda for the order and way in which mathematical topics are presented.

Both types of *hows* correspond to possible implementations of the HM into ME, of a different character, however; the ICMI Study *hows* focus on different emphases, whereas, Jankvist's focus strictly on the adopted methodologies. Both are useful in the present context and will be further considered later. Their possible interrelations is an interesting issue needing further study, not to be done here.

## THE 2-D CLASSIFICATION OF THE *WHYS* & *HOWS*

By taking into account the presentation of the conceptual dipoles above (in connection with the details provided in the corresponding references therein), a 2X2 table results, composed by the distinct elements of each dipole. Then, according to the description provided in the preceding section, each of the *whys* and *hows* can be placed in at least one cell, depending on how sharply and clearly it has been described in the corresponding references (Fauvel & van Maanen 2000, §§7.2, 7.3; Jankvist 2009c, §6).

	<i>History</i>	<i>Heritage</i>
<b><i>History as a goal (emphasis on meta-issues)</i></b> <sup>12</sup>	C.2, C.3(?); E.1, E.2, E.3	A.3; B.1, B.2; D.1
<b><i>History as a tool (emphasis on inner-issues)</i></b> [12]	A.3; C.1, C.3, C.4, C.5; D.2, D.3	A.1, A.2; B.1, B.2(?); C.2, C.3; E.3

**Table 1: The classification of the ICMI *whys***

In this way, classification scheme results, in which the two conceptual dipoles act as a “magnifying lens”, either requesting a more complete description of each *why* and *how*, or/and providing a clearer orientation of the way each *why* and *how* could be implemented. This is explained below by means of some examples.

Items appearing more than once are marked in green and those placed with reserve appear in red and an interrogation mark<sup>13</sup>. This suggests that the *whys* are not **irreducible** with respect to the two classification dipoles, but consist of **simpler** elements; hence, they should be further analysed and more sharply described. For instance, C.3 is a good example in this respect: It appears in three cells. According to the preceding section, it concerns the involvement and/or acquisition of awareness of the *creative process of “doing mathematics”*, by (i) tackling problems in historical context; (ii) enriching mathematical literacy; (iii) appreciating the nature of mathematics. Clearly, (i) is related to the *History – History as a tool* cell and (ii) to the *Heritage – History as a tool* cell (one is being involved into doing mathematics in historical context and in so doing improves his mathematical literacy, thus becoming more aware of what this intellectual activity has been through the ages). However, (iii) cannot be classified without ambiguity. Although it should be placed in the *History as a goal* row, this can be

accomplished by looking either at a subject's *History*, or *Heritage*. This argument needs further clarification. Moreover, A.4 is not easy to be placed in the table; it is necessary to be analysed further. In view of such examples<sup>14</sup>, we think that most of the ICMI *whys* could be further sharpened, so that they are decomposed into “irreducible” arguments, in the sense that they fall into only one cell of Table 1. But this remains to be shown and further work is needed (hence the tentative character of this and the other tables, mentioned in before). Similar comments hold for the other items as well. In this way, a **finer** classification of the *whys* becomes possible, to the extent that the classification dipoles have been determined as sharply as possible, of course. Clearly, this presupposes the detailed study of the *whys* and each *conceptual dipole* in the context of specific examples (cf. Tzanakis & Thomaidis, to appear). In addition, this and the following tables can be considered in relation to the **target population** to whom they are addressed; mathematics teachers; curriculum designers, producers of didactical material; mathematics teachers' trainers and advisors. That is, they can be useful to specify which entries are better suited to whom.

	<i>History</i>	<i>Heritage</i>
<b><i>History as a goal</i></b> <b>(emphasis on meta-issues)[12]</b>	Direct historical information Intrinsic awareness (ii) Extrinsic awareness (ii)	Direct historical information Extrinsic awareness (i) (iii) (iv)
<b><i>History as a tool</i></b> <b>(emphasis on inner-issues)[12]</b>	Intrinsic awareness (i) (iii) Learning mathematical topics (explicit use of history)	Learning mathematical topics (implicit use of history)

**Table 2: The classification of the ICMI *hows***

The labels (i) to (iv) in this table refer to the corresponding items in §3.2.3 and provide another example of the “irreducibility” idea mentioned above; the development of mathematical awareness has been described rather clearly in the ICMI Study volume, which allows for an unambiguous classification of its various aspects. The same holds for learning mathematical topics by following an approach inspired by history, either explicitly, or implicitly (cf. the last paragraph of the second section on the distinction between “explicit” and “implicit” use of history). However, this is not so for learning history by providing direct historical information, which is only briefly described.

	<i>History</i>	<i>Heritage</i>
<b><i>History as a goal</i></b> (emphasis on meta-issues)[12]	<i>Modules approaches</i>	<i>Illumination approaches</i> <i>History-based approaches(?)</i>
<b><i>History as a tool</i></b> (emphasis on inner-issues)[12]	<i>Illumination approaches</i> <i>Modules approaches</i>	<i>History-based approaches</i>

**Table 3: The classification of Jankvist's *hows***



The table above suggests that Jankvist's *hows* constitute an interesting and promising identification of broad categories of approaches, which can be analysed into more sharply described approaches. We think that this is already apparent in the description given in Jankvist 2009c, §6.

### CONCLUDING REMARKS

Due to space limitations, the paper is theoretical, with no illustrations of the classification schemes by means of specific examples. This is absolutely necessary, in order to check the validity of these classifications more thoroughly and to show how they can be useful in actual implementations (an example is given in Tzanakis & Thomaidis, to appear). This is a promising line of inquiry and expect that analysing specific examples in this context, the arguments for and approaches of integrating HM into ME will be sharpened and the importance and possible interrelation of the conceptual dipoles described here will be better revealed and understood. Finally, proceeding along the same lines, the **objections** against using HM in ME (e.g. as they appear in Fauvel & van Maanen 2000, §7.2, Siu 2006, §2) can be classified, as well.

### NOTES

1. In Jankvist's (2009a) terminology.
2. We use the term "dipole", instead of a more directly interpretable one, like "pair", to emphasize the deep interconnections between the two concepts, which reflects better their complementary character described below.
3. The term "complementary" is used in a way close to that introduced by N. Bohr to describe the microphysical reality and subsequently was raised to a general conceptual tool to understand reality (see Bohr 1934, 1958).
4. E.g. Understanding biological systems requires a holistic view, whereas, understanding their biochemical processes needs a reductionist approach: Both are indispensable for a sufficiently wide and deep understanding of life phenomena, but it is impossible to put **absolute** emphasis on the one, without "destroying the other (Bohr 1958, chs.2 &3).
5. Jankvist 2009b §1.1.
6. Jankvist 2009d, p8.
7. Jankvist 2009c, p240.
8. E.g. "How does mathematics evolve in time and space? What forces and mechanisms cause the evolution of mathematics? How does the evolution of mathematics interact with society and culture? Can mathematics become obsolete?" (Jankvist 2009d, p.8).
9. Grattan-Guinness, 2004b, p.7.
10. Grattan-Guinness, 2004a, p.164.
11. Careful reading of these quotations, reveals the co-existence of the two poles of each of the dipoles (History, Heritage) & (History-as-a-tool, History-as-a-goal)
12. Relating *History-as-a-goal* and *History-as-a-tool* with *Inner-issues* and *Meta-issues*, respectively is done keeping in mind the possibility of cross-interrelations mentioned in the third section!
13. This convention is applied to all tables in this paper.
14. Others could be similarly analyzed, but lack of space does not allow for that.

## REFERENCES

- Barbin, E., (1997). "Histoire et Enseignement des Mathématiques. Pourquoi? Comment? », *Bulletin AMQ*, Montréal, **37** (1), 20-25.
- Bohr, N. (1934). *Atomic theory and the description of nature*, Cambridge University Press 1934, ch.2.
- Bohr, N., (1958). *Atomic Physics and Human Knowledge*, New York: J. Wiley, ch.2 & 3.
- Fauvel J & van Maanen J. (eds). (2000). *History in Mathematics Education: The ICMI Study*, Dordrecht: Kluwer Academic Publishers.
- Grattan-Guinness, I. (2004a). "The mathematics of the past: distinguishing its history from our heritage" *Historia Mathematica*, **31**, 163-185.
- Grattan-Guinness, I. (2004b). "History or Heritage? An important distinction in mathematics for mathematics education", in G. van Brummelen & M. Kinyon (eds), *Mathematics and the historian's craft*, Springer, pp.7-21. Also published in *Am. Math. Monthly*, **111** no1, 1-12.
- Jankvist, U.Th. (2009a). *Using history as a "goal" in Mathematics Education*", PhD Thesis, Roskilde: Roskilde University, chs.2 & 8.
- Jankvist, U.Th. (2009b). "On empirical research in the field of using history in mathematics education", *Revista Latinoamericana de Investigación en Matemática Educativa*, **12** (1), 67-101, particularly §1.1
- Jankvist, U.Th. (2009c). "A categorization of the 'whys' and 'hows' of using history in mathematics education", *Educational Studies in Mathematics*, **71** no3, 235-261.
- Jankvist, U.Th. (2009d). "History of modern applied mathematics in mathematics education", *For the Learning of Mathematics*, **29** no1, 8-13.
- Rogers, L. (2009). "History, Heritage and the UK Mathematics Classroom", in *Proc. of CERME 6, Working Group 15: The role of history in mathematics education: theory and research*, pp.119-128.
- Schubring, G. (2008). "The debate on a "geometric algebra" and methodological implications". In R. Cantoral, F. Fasanelli, A. Garciadiego, R. Stein, C. Tzanakis (eds), *Proceedings of HPM 2008: The HPM Satellite Meeting of ICME 11*, Mexico City, Mexico (edition in cd form).
- Siu, M-K. (2006). "No, I don't use history of mathematics in my class. Why?". In F. Furinghetti, S. Kaijer & C. Tzanakis [eds.], *Proceedings of HPM 2004 & ESU 4* (revised edition), University of Crete, Greece, pp.268-277.
- Tzanakis C. & Thomaidis Y. (to appear). "Classifying the arguments and methods to integrate history in mathematics education: an example". In E. Barbin, M. Kronfellner, C. Tzanakis (eds), *Proceedings of ESU 6*, Vienna University of Technology, Austria

# **THE DEVELOPMENT OF ATTITUDES AND BELIEFS QUESTIONNAIRE TOWARDS USING HISTORY OF MATHEMATICS IN MATHEMATICS EDUCATION**

Mustafa Alpaslan, Mine Isiksal, Cigdem Haser

Middle East Technical University, Turkey

*This study intends developing a valid and reliable instrument in order to determine pre-service mathematics teachers' attitudes and beliefs towards using history of mathematics in school mathematics courses. The data were collected at the beginning of the fall semester of 2010-2011 academic year from 237 teacher candidates in Turkey via Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education (ABHME) Questionnaire developed by the researchers. The reliability coefficient of the ABHME was calculated as .91 and explanatory factor analysis revealed three factors: positive attitudes and beliefs towards the use of history in mathematics education, negative attitudes and beliefs towards the use of history in mathematics education, and self-efficacy beliefs towards the use of history in mathematics education.*

## **INTRODUCTION AND THEORETICAL FRAMEWORK**

The origins of the attitudes and beliefs of teachers about teaching probably endure their school years as a student (Kagan, 1992). This idea stands on the 'apprenticeship of observation' which basically means that teachers develop ideas about their profession when they were students (Lortie, 1975). In addition to these opinions, it may be claimed that pre-service training programs have a leading role in the shaping of the attitudes and beliefs at issue. Among several elements constituting the education given in undergraduate programs, history of mathematics should take place for better professionally developed future mathematics teachers. In the literature, quantitative measurement instruments for accessing beliefs and attitudes toward using history in mathematics education are somewhat limited. Some of the existing relevant instruments are on some specific issues about the use under consideration. There are also instruments comprising few items which do not seem to be comprehensive enough to see the whole picture. Developing a versatile instrument measuring attitudes and beliefs of pre-service mathematics teachers towards the historical approach is crucial, because they are going to undertake today's teachers' educational missions in the future. Thus, this study aims to develop a valid and reliable instrument for investigating elementary mathematics teacher candidates' attitudes and beliefs towards the use of history of mathematics in mathematics education. With this purpose, the research question for this study could be stated as follows:

- What could be the underlying constructs of Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education Questionnaire?

The idea of using history in the learning and teaching of mathematics is not new; as it has been discussed since the beginning of the 20<sup>th</sup> century (Fried, 2001). An historical approach enriches mathematics teachers' repertoire with its different usages (e.g., Mitchell, 1997; Tzanakis & Arcavi, 2000). It supplies meaningful examples of mathematical algorithms and methods, and offers semantic representations which enable students to see mathematical concepts from unusual perspectives (Furinghetti, 2007). It also supports the preparation of a classroom environment where students discuss the past of mathematics and mathematics as a discipline (Jankvist, 2010). Jankvist (2009) separated the reasons proposed in the literature in favour of using the historical approach into two main categories: seeing history of mathematics as a teaching tool, and seeing it as a goal in itself.

Leder and Forgasz (2002) introduced that the concepts of attitude and belief were defined by different researchers in many times, and there are not common definitions of these. To achieve this study's goal, the concept of belief was considered in respect of Törner's study (as cited in Goldin, Rösken, & Törner, 2009). From his perspective, beliefs are strongly related to affective aspects which include attitudes as an element. According to Philipp (2007), individuals' true-false dichotomies constitute their beliefs. It can be inferred that mathematics teachers' beliefs about a teaching method amount to their thinking about different components, applications and some other aspects of that method as true or false. In this study, historical approach in mathematics education corresponds to this teaching method. Philipp (2007) also stated that these ideas and dispositions of individuals are clarified by their behaviours. In other words, behaviours are reflections of individuals' emotions, actions and thoughts called as attitudes. It is possible that mathematics teachers' attitudes towards an instructional method reveal themselves as emotions, feelings, actions, and thoughts about that method. Thompson (1992) expressed that beliefs are sensed less densely, but more cognitively than attitudes. In her opinion, beliefs are structured more strongly than attitudes, so they are less changeable. It can be expected that individuals' expressions of their beliefs are more direct and certain than their expressions about the attitudes they have. With reference to Leder and Forgasz (2002), these kind of affective matters comprise the learning of mathematics as well as cognitive ones. In this sense, the teaching of mathematics could be influenced by affective domain because of that instructors also keep some thoughts, feelings, and emotions to the education which they give. Their beliefs and attitudes towards a teaching method are believed to determine the degree of benefiting from that method, and this will naturally have effects on the quality of instruction that they give.

The inclusion of the history of mathematics in pre-service mathematics teacher training programs has been supported by mathematicians, mathematics educators and mathematics historians (Schubring, 2000). There are many studies on this matter in which pre-service teachers participated. Sullivan (2000) found that a positive change

is possible in prospective secondary mathematics teachers' attitudes towards the integration of history in mathematics education. In the context of Turkey, it was shown that prospective mathematics teachers' had high positive attitudes towards the historical approach (Oprukçu-Gönülateş, 2004). There are also studies regarding historical materials' usage. Fraser and Koop (1978) determined that in-service teachers liked historical materials, a play and an article, and find them appropriate for mathematics teaching. However, they brought up some concerns about considerable time required, and stated that they would not use such materials in their teaching.

The elementary mathematics curriculum in Turkey draws attention to the importance of using history of mathematics (Ministry of National Education [MNE], 2009). It emphasizes that students must have an idea about the historical evolution of mathematics; its role on many scientific fields, its status and value in the development of human thoughts. It suggests carrying out mathematics projects on several domains of mathematics in which history of mathematics plays a part. In spite of the fact that history of mathematics has such a crucial place in the mathematics education, there are few studies conducted about it. Among these, it is necessary to consider the work of Oprukçu-Gönülateş (2004) as background to this study. She examined pre-service mathematics teachers' views about the integration of history of mathematics in mathematics courses and determined that they agreed on the benefits of the integration for getting high motivation in mathematics classes. They thought that it was more appropriate for motivational purposes than for conceptual development.

## **METHOD**

### **Participants**

The data of the study were collected from 237 pre-service elementary mathematics teacher candidates (including 45 freshmen, 52 sophomores, 96 juniors, and 44 seniors) purposefully sampled at the beginning of the fall semester of the academic year 2010-2011. The participants, who were 53 males and 184 females, have been attending elementary mathematics teacher education program in one of the large state universities in Ankara, Turkey.

### **Measuring Instrument**

The measuring instrument developed for this study is a questionnaire entitled "Attitudes and Beliefs towards the Use of History of Mathematics in Mathematics Education (ABHME) Questionnaire". The ABHME consisted of two parts: the first is related to the demographic information such as gender, and grade level; and the second part consisted of 35 Likert type (where 5 corresponds to "strongly agree" and 1 to "strongly disagree") items (including 13 negative and 22 positive expressions).

Those items were developed in order to measure pre-service teachers' attitudes and beliefs towards the use of history in the teaching of mathematics.

In the very beginning of the instrument's development process, an item pool of 220 items for the actual questionnaire was formed following a comprehensive review of the literature on the related field, and examining the existing instruments relevant to attitudes and beliefs towards the historical approach in mathematics education (e.g. Clark, 2006; Fraser & Koop, 1978; Oprukçu-Gönülateş, 2004; Percival 1999, 2004; Sullivan, 2000). During this process necessary permissions were taken from the authors mentioned above. For ensuring the content validity of the instrument, three experts from elementary mathematics education field, and one expert in elementary science education field were consulted. Necessary corrections (like combining and deleting items) were made on the items and number of the items was reduced to 40 after this process. The pilot study was performed with these 40 items which were related to teacher candidates' attitudes and beliefs towards the usability of the historical approach, its effects on the comprehension of the disciplinary structure of mathematics and the understanding of mathematical concepts, its contributions to their professional development, and their efficacy about using the history of mathematics in their own teaching. The questionnaire items were written by using statements in which the teacher candidates can choose the most appropriate response to their feelings, thoughts, and also possible actions which they would do towards the mentioned sub dimensions (such as the usability of the history in mathematics teaching, and the historical approach's contributions to their professional development) about the history use in mathematics education. The developed instrument's items which shelter strong expressions are closer to beliefs, and the items that contain less strong expressions are closer to attitudes. These were determined in accordance with Thompson's (1992) detachment approach among the two concepts as mentioned before. During this test item writing process, the difference between the concepts of attitudes and beliefs were not considered, rather these two very close concepts were dealt in the frame of the affective factors influencing the teaching and learning of mathematics as Leder and Forgasz suggested (2002). Thus, any item discrimination according to the concepts of attitude and belief was not expected to appear at the beginning of the instrument's development process as it was not the intention of this study.

For investigating the instrument's validity and reliability, a software program named PASW Statistics 18 was used. Before the validity and reliability analyses, hot deck imputation was used to fill the missing values in the raw scores. The construct validity of the instrument was gained by factor analysis which intends to reduce great variable (item) sets to smaller component groups (Pallant, 2007). Exploratory factor analysis technique with the principal components factor extraction approach was utilized to clarify the dimensions of the instrument. Varimax rotation was selected to analyse each of the factors appeared, and minimum factor loading for an item to be

placed in a component was selected as .3. Before the factor analysis was performed, negative items were recoded.

In Pallant's (2007) opinion, the number of factors can be limited if researchers think that a particular number is best describing the variables' interrelationships. Thus, the number of factors was limited to three for this study. After the factor analysis was conducted, the suitability of the data for this factor analysis was controlled by checking the assumptions, which are sample size, factorability of the correlation matrix, linearity, and outliers among cases, as Pallant (2007) offers. The underlying reason for designating the sample size is that it should be at least five times of the number of the items in the instrument. In other words, five cases are enough for each item in the questionnaire (Tabachnick & Fidell, 2007). In this study, the instrument formed of 40 items was applied to 237 participants, so this sampling is adequate in number. The factorability of the correlation matrix (strength of the relationship among items) was assessed by examining the correlation matrix, Kaiser-Meyer-Olkin (KMO) measure of sampling adequacy, and Bartlett's test of Sphericity. The correlation matrix clarified some correlations of  $r=.3$  or greater among many item (variable) pairs, Bartlett's test of Sphericity was found significant (Chi-Square  $\chi^2=3447.582$ ;  $p=.000$ ), and the KMO value was calculated as .871. These are indicators of that the data were appropriate for factor analysis (Tabachnick & Fidell, 2007). It would not be a problem that the linearity assumption was not checked because the sample size was high enough (Pallant, 2007). The data were screened for outliers, but none could be found. After the assumptions were checked and it was understood that there was no problem with conducting the factor analysis, the values concerning the factors were analysed. Moreover, factor loadings of the items were examined with the help of the rotated component matrix in order to make a decision about each component's items and determine whether any item should be taken out of the instrument or not. The values regarding the factors can be seen in Table 1 below:

Factor	Initial Eigenvalues		
	Total	% of Variance	Cumulative %
1	9.893	24.733	24.733
2	2.893	7.234	31.967
3	2.466	6.166	38.133

**Table 1:** The values related to the factors (components)

It can be inferred from the table that the three factors explain a total of 38.133 per cent of variance. Moreover, the three envisaged components before the factor analysis have appeared with 4 items (Factor 3) or more. With respect to Pallant's view (2007), at least three items loading on each component are sufficient. Furthermore, an item's maximum factor loading to a component should be at least .1 greater than its factor loading to the other components (Büyüköztürk, 2002). When

## Working Group 12

the maximum factor loadings for each item were examined, it was seen that the maximum factor loading of the item '38' was .155 to the second factor. Because of that an item's factor loading to a component was selected as at least .3 previously, the item 38 was extricated from the instrument. There are also 5 items whose largest factor loading to a component was not .1 greater than its factor loading to other components. Therefore, they were removed from the instrument, as well. The reliability was ensured by calculating Cronbach Alpha coefficient as .905. This value indicates very good internal consistency reliability for the instrument (George & Mallery, 2001).

After the problematic items were removed from the questionnaire, the same statistical process was conducted for analysing validity and reliability again. It was seen that the three factors of the finalized questionnaire (35 items composing of 3 factors) explained a total of 39.711 per cent of the variance, Bartlett's test of Sphericity was found significant again (Chi-Square  $\chi^2=2870.341$ ;  $p=.000$ ), and the KMO value was calculated as .875 this time. The Cronbach Alpha coefficient was .902, which addressed a high reliability. When the contents of the items placed in each component were examined by considering the field of history in mathematics education and some instrument development studies about social sciences (e.g., Cantürk-Günhan & Başer, 2007; Turan & Demirel, 2009), a meaningful pattern was established among the components. In respect of this pattern, the factors were defined. The first component is "Positive Attitudes and Beliefs towards the Use of History in Mathematics Education" including positive items of the instrument. The second factor is "Negative Attitudes and Beliefs towards the Use of History in Mathematics Education" corresponding to the negative items. The third and last factor is "Self-efficacy Beliefs towards the Use of History in Mathematics Education" corresponding to items about an individual's self-efficacy, which is criticism on his/her own proficiency about overcoming tasks specified beforehand (Bandura, 1997). The items related to attitudes and beliefs were not separated by these underlying factors of the instrument as the researchers expected before. Three sample items from each of these factors can be seen in Table 2 below:

Item	Name of the Factor
Learning the history of mathematics enriches teacher candidates' professional repertoire.	Positive ABHME
Pre-service teachers must be given courses about how to use the history of mathematics in mathematics education.	
The history of mathematics makes students to notice that mathematics is a universal product.	
It is <u>difficult</u> to integrate history in mathematics education.	Negative ABHME
Using the history in mathematics education causes students to <u>lose their enthusiasm</u> for mathematics.	



## Working Group 12

Including history in mathematics education <u>hinders</u> mathematics teaching.	Self-efficacy Beliefs towards the Use of History of Mathematics in Mathematics Education (SBHME)
I <u>do not</u> have an idea about how to use historical materials.	
I <u>do not</u> know how to integrate history in mathematics teaching process.	
I <u>do not</u> have enough information about the historical evolutions of the concepts which I will teach in the future.	

**Table 2:** Sample items from each of the factors

## RESULTS AND DISCUSSION

The aim of this study was to construct a valid and reliable instrument measuring pre-service elementary mathematics teachers' attitudes and beliefs towards the use of history of mathematics in mathematics education. The conducted factor analysis eliminated some of the items (1, 2, 29, 37, and 38) and terminated the number of the items as 35. It also showed that 22 items were gathered under the component 'Positive ABHME', 9 items were collected under the component 'Negative ABHME', and 4 items formed 'SBHME' component. When the possible underlying reasons of the separation of these components are thought, it can be said that a difference existed between in favour of and opposed to the attitudes and beliefs of the teacher candidates towards the expressed mathematics teaching approach, and the pre-service teachers have a perception about to what degree they are able to use and/or integrate history in mathematics education. Moreover, the instrument was found to have very good reliability ( $\alpha=.9$ ), which is an indication of high internal consistency among the items. The validity and reliability of it is open to being ensured by other researchers via following the similar procedure mentioned in the method section of this paper.

To bring in such an instrument to the literature is believed to inspire the researchers interested in this field for producing studies regarding the attitudes and beliefs towards the use of the history of mathematics for the teaching of mathematics either with pre-service elementary or secondary mathematics teachers. If it is adopted (e.g. by deleting and adding some items, changing some of the items' words written particularly for teacher candidates) properly, it can also be used with in-service mathematics teachers since the contents of the items are focused on teaching mathematics with its history. In order to advance the investigations of this field, the researchers may benefit from the instrument as an attachment of an experimental study concerned with the use of history in mathematics education in order to reveal potential effects of interventions on the attitudes and beliefs towards the teaching approach in question. Survey research also can be conducted with the scale in order to form an opinion about the attitudes and beliefs towards the historical approach usage of both pre-service and in-service teachers. Moreover, future studies will make

some important contributions to the research conducted in the field of using history of mathematics in the teaching and learning of mathematics, and take attention of those concerned with mathematics education to this approach some more. Additionally, the results gained via using the instrument by future research projects would have valuable implications for teacher educators, education policy makers and curriculum developers in designing curricula for mathematics education at different levels.

## REFERENCES

- Bandura, A. (1997). *Self-efficacy: the exercise of control*. New York: W. H. Freeman.
- Büyüköztürk, Ş. (2010). *Sosyal bilimler için veri analizi el kitabı: istatistik, araştırma deseni, SPSS uygulamaları ve yorum* (10th ed.). Ankara: Pegem Akademi Yayıncılık.
- Cantürk-Günhan, B. & Başer, N. (2007). The development of self-efficacy scale toward geometry. *Hacettepe University Journal of Education*, 33, 68-76.
- Clark, K. M. (2006). *Investigating teachers' experiences with the history of logarithms: a collection of five case studies* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database (UMI No. 3212049)
- Fraser, B. J., & Koop, A. J. (1978). Teachers' opinions about some teaching material involving history of mathematics. *International Journal of Mathematical Education in Science and Technology*, 9(2), 147-151.
- Fried, M. N. (2001). Can mathematics education and history of mathematics coexist? *Science and Education*, 10, 391-408.
- Furinghetti, F. (2007). Teacher education through the history of mathematics. *Educational Studies in Mathematics*, 66, 131-143.
- George, D., & Mallery, P. (2001). *SPSS for windows step by step: a simple guide and reference 10.0 update* (3rd ed.). Needham Heights, the USA: Allyn & Bacon.
- Goldin, G., Rösken, B., & Törner, G. (2009). Beliefs – no longer a hidden variable in mathematical teaching and learning processes. In J. Maaß, & W. Schölglmann (Eds.), *Beliefs and attitudes in mathematics education – new research results* (pp. 1-18). Taipei, Rotterdam: Sense Publishers.
- Jankvist, U. T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics*, 71, 235-261.
- Jankvist, U. T. (2010). An empirical study of using history as a ‘goal’. *Educational Studies in Mathematics*, 74, 53-74.
- Kagan, D.M. (1992). Implications of research on teacher belief. *Educational Psychologist*, 27, 65-90.
- Leder, G. C. & Forgasz, H. J. (2002). Measuring mathematical beliefs and their impact on the learning of mathematics: a new approach. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: a hidden variable in mathematics education?* (pp. 95-114). Dordrecht: Kluwer Academic Publishers.

## Working Group 12

- Lortie, D. (1975). *Schoolteacher: A Sociological Study*. London: University of Chicago Press.
- Ministry of National Education [MNE].: 2009, *İlköğretim Okulu Ders Programları: Matematik Programı 6-7-8 (Elementary Curricula Programs: Mathematics Curricula Program for Middle Grades)*. İstanbul: Milli Eğitim Basımevi.
- Mitchell, M. (1997). *Mathematical history: activities, puzzles, stories, and games*. Reston, VA: NCTM.
- Oprukçu-Gönülateş, F. (2004). *Prospective teachers' views on the integration of history of mathematics in mathematics courses* (Unpublished master's thesis). Bosphorus University, İstanbul.
- Pallant, J. (2007). *SPSS Survival manual: a step by step guide to data analysis using SPSS for windows* (3rd ed.). Berkshire, England: Open University Press.
- Percival, I. (1999). *Mathematics in history: integrating the mathematics of ancient civilizations with the grade 7 social studies curriculum* (Master's thesis). Available from ProQuest Dissertations and Theses database.
- Percival, I. (2004). *The use of cultural perspectives in the elementary school mathematics classroom* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database.
- Philipp, A. R. (2007). Mathematics teachers' beliefs and affect. In F. K. Jr. Lester (Ed.), *Second handbook of research on mathematics teaching and learning. A project of the national council of teachers of mathematics* (pp. 257-315). Charlotte, NC: Information Age.
- Schubring, G. (2000). History of mathematics for trainee teachers. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 91-142) The ICMI Study. Dordrecht: Kluwer Academic Publishers.
- Sullivan, K. M. (2000). *Preservice secondary mathematics teachers' attitudes about the history of mathematics* (Master's thesis). Available from ProQuest Dissertations and Theses database (UMI No. 1399933)
- Tabachnic, B. G. & Fidell, L. S. (2007). *Using multivariate statistics* (5th ed.). Boston: Pearson Education.
- Thompson, A. (1992). Teachers' beliefs and conceptions: a synthesis of the research. In A. D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning. A project of the national council of teachers of mathematics* (pp. 390-419). New York: Macmillan.
- Turan, S. & Demirel, Ö. (2009). The reliability and validity of the attitude scale towards problem-based learning. *Science and Education* 34(152), 15-29.
- Tzanakis, C. & Arcavi, A. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel, & J. van Maanen (Eds.), *History in mathematics education* (pp. 201-240) The ICMI Study. Dordrecht: Kluwer Academic Publishers .

# IMPLEMENTING ‘MODERN MATH’ IN ICELAND – INFORMING PARENTS AND THE PUBLIC

Kristín Bjarnadóttir

*University of Iceland – School of Education*

*‘Modern math’ was implemented in Icelandic schools at all levels in the 1960s. It was introduced to parents at meetings and by media articles, interviews and a television programme in 17 episodes. It is argued that the information was presented by unrealistic convictions of the value of the ‘modern math’ programme, that the timing of the presentations was sub-optimal, and that more information was needed when the programme had reached the majority of primary-level.*

## INTRODUCTION

In the 1950s questions arose in many countries about mathematics teaching. In November 1959, the OEEC (later OECD) arranged a meeting on reform of school mathematics in Royaumont, France (OEEC, 1961), whereby reform ideas turned into an international movement. The dominant idea was that the theory of sets, uniting the various branches of mathematics, be the basis for school mathematics. This reform movement is commonly referred to as ‘modern mathematics’.

The ‘modern math’ movement hit Iceland at a standstill state (Bjarnadóttir, 2006). The Education Act of 1946 ensured compulsory schooling of all children aged 7–15 and equal access to higher education, while the legislator failed to ensure university access for primary teachers. There was a serious shortage of secondary school mathematics teachers (OECD, 1961, p. 158; Bjarnadóttir, 2007, 184–192, 238–239). Research revealed that mathematics and science syllabi for 13–15 year old pupils, who had only eight month schooling a year, were behind those in the other Nordic countries (Björnsson, 1966). Primary level mathematics textbooks created in the 1920s were still in use. Textbooks were then, as now, produced by a state enterprise, which was in a serious financial crisis at that time.

A reason often mentioned for implementing the ‘modern math’ was the fear to lag behind other countries in a progress towards modern society, e.g. in England:

... topics as the algebra of sets or relations might be taught with a profit ... lower down the school ... In other countries they are learning how to do this, and unless we learn too we shall be left behind (*Mathematics Teaching*, 1958; Cooper, 1985, p. 76)

This view was reflected in Iceland: ‘... we have to protect our honour and interests in the cultural competition of today’ (Sigtryggsson, 1964, p. 146), and in Brazil: ‘The teaching of mathematics had to be modern, such as Brazil wanted and expected to be’ (Búrigo, 2008).

The implementation of ‘modern math’ in other countries has been examined, e.g. in Denmark, Norway and England (Høytrup, 1979; Gjone, 1983; Cooper, 1985). This

study is an extension of a larger study on the implementation of modern math in Iceland (Bjarnadóttir, 2007), focussing on its publicity, such as information offered to Teachers, parents and the public.

Questions to be explored are: What went wrong in the implementation of the ‘modern math’? Was the timing of the publicity process wrong, or was the ‘modern math’ a cause too weak to defend?

The questions are of interest, not only to understand the past, but also for the present. Much time, effort and resources are devoted to centrally organized curriculum development for the relatively small market in Iceland, with about 4000 children in a year cohort. How should the limited means be allocated? There are issues regarding the quality of content and material, and the publicity to teachers, parents and providers of funds. The study might offer some hints of relevance.

## **THE RESEARCH METHOD**

The research method is historical; searching information in newspapers, journals, leaflets, archive documents, and by personal communication in emails and telephone interviews. The interviewees are a sample of convenience. In a small society as in Iceland, it is relatively easy to contact persons, who are likely to provide relevant information, through family, colleagues and acquaintances. Widespread newspapers have been digitalized so access to printed discussion is relatively easy. The sources found may therefore reflect enthusiasm and subsequent regrets rather convincingly.

## **‘MODERN MATH’ IN ICELAND**

After the Royaumont meeting, the Nordic countries – Denmark, Finland, Norway and Sweden – established cooperation on school mathematics reform, each appointing four persons to a Nordic committee for modernizing mathematics teaching, *Nordiska kommittén for modernisering af matematikundervisningen* (NKMM) (Gjone, 1983, II, p. 78). Information from the NKMM was transmitted to Iceland through personal contacts. Mathematics professor Svend Bundgaard, the main advocate for ‘modern math’ in Denmark, was a friend of Guðmundur Arnlaugsson, associate professor and high school mathematics teacher, educated in Copenhagen; and Agnete Bundgaard, primary teacher and the author of a ‘modern math’ textbook series for primary level on behalf of the NKMM, was Svend’s sister.

Arnlaugsson became a leader of those concerned about mathematics education. His activities were in several steps: In 1964 he asked to be appointed as consultant in mathematics teaching at the Ministry of Education in a half-time position, to continue until 1966 (Minister Gylfi Þ. Gíslason, personal communication, Jan. 10, 2002). He organized week-long in-service courses for teachers on the ‘modern math’ from 1965. In 1966 he published a textbook, *Numbers and Sets*, (Arnlaugsson, 1966) on introduction to numbers and set theory. The book was intended for college bound students preparing for a national entrance examination into high schools and was

tested in 1966–1968. Finally, Arnlaugsson suggested the textbooks by Agnete Bundgaard to be translated and tested in seven groups in two primary schools in Reykjavík during 1966–1967.

The primary school project was planned with regular meetings of the two project leaders and the teachers, and with meetings to inform parents. In spring 1967 the project was presented to headmasters of primary schools in Reykjavík, who became enthusiastic with the majority wishing to adopt it in their schools. By the time the series was chosen, only the first year-course had been finalized in Danish, the second year-course been tested for two years and the third was being tested for the first time (K. Gíslason, 1978) so it was not known how the project would proceed.

The latter part of the series turned out to be highly theoretical (Høytrup, p. 59). The commutative and associative laws, Roman numerals and place-value notation to the base five, prime numbers, permutation of three digits and the transverse sum together with its relation to the nine times table were introduced in the third grade. Set theory with pairing, subsets, intersection and union, more place-value systems and geometry in a set-theoretical framework were added in fourth grade. Last but not least, there were algorithms that were different from those Icelanders were accustomed to, especially the multiplication algorithm (Bundgaard, 1967–1972).

### **PUBLIC INTRODUCTION TO THE ‘MODERN MATH’**

The first presentation of the ‘modern math’ to the public was characterized by optimism. Articles were written and a television programme was made to introduce the modern ideas, to parents in particular. Later sources indicate concerns.

An interview with one of the project leaders, K. Sigtryggsson, was published in *Foreldrablaðið / Parents’ Journal* in 1967. Only the teaching methods were being changed, not the content, he said, while some topics were introduced earlier than before. Mechanical working methods had been overly emphasized at the expense of discussions of basic mathematical concepts, training of logical thinking and accuracy in presentation. Finally, children were to have no homework (Stefánsson, 1967).

Later that same year, 86 teachers attended a course to prepare for teaching the new syllabus to the majority of first grade pupils in Reykjavík (NN1, 1968, p. 95). The leaders did not have capacity to keep in contact with all the teachers and arrange information to parents. The television programme that autumn must have been expected to reach a greater number of parents than possible otherwise.

By April 1968, around 200 secondary school pupils gathered outside the parliament building with protest banners, claiming ‘Down with obsolete textbooks’, ‘Better teaching methods’, ‘We are not parrots’, ‘No dry-book learning’ and ‘Youth today is the public tomorrow’. The protest was peaceful and Minister of Education, Gylfi Þ. Gíslason, invited members of the group to his office and discussed with them for most of the day. Benedikt Gröndal, Member of Parliament, spokesman of the

minister, recited this event in a semi-yearly general political radio discussion, admitting that much was needed in the school system, while listing what had been achieved in modernizing it. 'Modern math' was taught widely, and the public television had brought information to parents. Televised teaching was a topic of the near future (Gröndal, 1968). The government thus used the televised teaching to indicate its constructive reactions to criticism of a longstanding standstill state.

In 1970 a newspaper, *Morgunblaðið*, published an interview with the textbook author Agnete Bundgaard and her colleague, Karen Plum, who had visited Iceland to give a course to 65 teachers (NN2, 1970). By then the 'modern math' material was used in 141 classes in the first four grades in primary schools. Bundgaard stated that the main emphasis was on promoting the pupils' understanding of the nature of their tasks and on training them in using their own logical faculties. The 'modern math' had been introduced in many countries and had influenced the way of teaching mathematics. The experience of other nations suggested that its concepts and symbols would be of great use in training pupils in clear thinking and communication. The 'modern math', Agnete Bundgaard said,

is as a new language, totally different from the mathematics the parents of modern school-children learnt themselves. Many parents have a hard time in accepting being unable to know exactly what their children are working on at school and assist them. But it can have very bad consequences for the child if its parents are trying to help, more willing than able to guide the child. This can lead only to confusion. Therefore it has been decided to not to assign homework to the children and even not allow them to bring their books home. However, in order to increase parents' understanding of what their children are working on, special books have been published, admittedly not available in Icelandic, where the new mathematics is explained. It should pacify the parents until the moment, when the children have reached enough understanding of the project to be able to explain to their parents what is happening there (NN2, 1970, pp. 23–24).

Plum added that the 'modern math' had caused dispute in many places, and that

many years will pass until its advantages can be proven statistically, since all comparison is difficult. But surely the 'modern math' teaches children to think logically and ... logical thinking will always be necessary ... children ... like the 'modern math' and they show more interest than do children at the same age, who learn by the old methods ... (Ibid.)

This period was characterized by great confidence in experts and the mistrust to parents in the case of school mathematics was not unique. Parents were asked to refrain from teaching their children to read as this would be taken care of by experts in schools, and parents were e.g. not allowed to visit their hospitalized children. The request that parents did not interfere with school assignments was thus not unique.

## THE TELEVISION PROGRAMME

To promote the compulsory school ‘modern math’ project, Arnlaugsson gave a series of seventeen episodes in a television programme on school mathematics during October 1967–January 1968. Each episode lasted about 15–18 minutes. It was sent out during prime time at 20:50 after the foreign news report each Tuesday.

Copies of the programme do not exist anymore. All remarks on it are therefore from memory. There was a tradition of adult education in foreign languages on the public radio so mathematics on television was a natural analogue. In this second year of domestic television broadcasting, over 92% of 10–14 year old children in Reykjavík and Vestmannaeyjar reported in a sociological survey that their homes had a television set, while the broadcasting had not reached further than the south and south-west area of Iceland (Broddason, personal communication, Aug. 23, 2010).

The author of this paper had no TV-set and could only watch a few episodes of the series, but with great interest, as a student mathematics teacher. Most owners of TV-sets at that time and parents or teachers of school children are now past seventy and they and their then adolescent children are the few likely to remember the content. Among them are Vilhjálmur Bjarnason (personal communication, Aug. 28, 2010), Jón I. Magnússon (Sept. 2, 2010), Guðný H. Gunnarsdóttir (Sept. 3, 2010) and Jónína V. Kristinsdóttir (March 4, 2011) who agree, as this author does, that the content was closely connected to Arnlaugsson’s textbook, *Numbers and Sets*; that is



introduction to numbers, their divisibility, prime numbers and prime factoring, and number notation in place-value systems to various bases, such as 6, 8 or 2. The binary system received special attention with respect to the new computer technology. Definitions of set operations, open sentences, implications, negations, introduction to Boolean algebra, etc., followed (Arnlaugsson, 1966).

**Figure 1. G. Arnlaugsson on the TV-programme.**

The programme was well reviewed in the newspapers. The picture in Fig. 1 was attached to the television programme schedule in one of the newspapers along with the remark that Arnlaugsson’s programme had been very well received and many had expressed that they had learnt quite a lot from it. A special TV-leaflet introducing the programme remarked twice (NN3, 1967) that the episodes were excellent, Arnlaugsson was a born TV-celebrity and the explanations were clear.

Magnússon, then 14 years old, now mathematics professor, and Bjarnason, 15 years, now lecturer in economics, remember that they watched the programme closely, as did their fathers, both teachers, but their mothers did not. The programme had without doubt a good influence on a group preparing for an entrance examination into high school, using Arnlaugsson’s text on experimental basis. The group included Bjarnason, who remarked that he liked the base-two algorithms and the



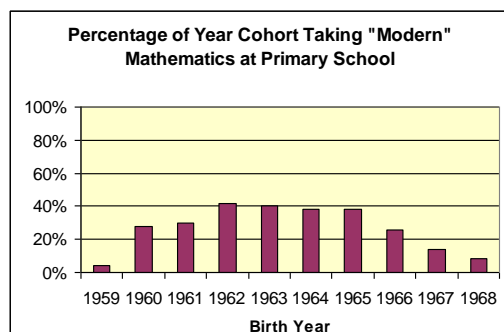
Boolean algebra, and the implications were a revelation to him. His classmate, J. V. Kristinsdóttir, now lecturer in mathematics education, noted that their teacher had discussed topics from the programme, and she, J. I. Magnússon and G. H. Gunnarsdóttir, also lecturer in mathematics education, remember also that their elder siblings, then in high school, watched the programme attentively. In general, the programme had a good reputation in their homes, theoretical as it was.

Much later Professor Magnússon (1996) wrote in memory of Arnlaugsson on behalf of the Icelandic Mathematical Society: “His programs on the television about ... ‘modern math’ are ... unforgettable, and many people were as glued to the TV-set when they were programmed”.

Different opinions were also voiced. The reporter of the foreign news report (A., personal communication, September 19, 2010), remembers a feeling of inferiority being unable to assist his children with this new math, and the managing director of the broadcasting service (G., personal communication Sept. 23, 2010), said that the series did not appeal to him, and he preferred the old math.

## REACTIONS TO THE ‘MODERN MATH’

By the time of Bundgaard and Plum’s visit, authorities had realized that things were going wrong; the mathematics teaching experiments in the primary schools had become far too extensive, too difficult to run with respect to guidance to teachers, and even in a few cases close to being disastrous (R. Bjarnadóttir, personal communication, Sept. 16, 2003). A School Research Department, SRD, of the Ministry of Education had been established. It laid down a certain procedure for adopting school reforms; i.e., to set goals, write national curricula, and from there compose learning material on an experimental basis. In the crisis that had emerged, the department decided in 1971 to skip the goal-setting and curricular-writing steps, and go directly ahead to create a new set of home-made mathematics textbooks (Ísaksson, personal communication, March 10, 2003). In their final editions, sets were hardly mentioned. Enthusiasm for the ‘modern math’ at primary level had passed its peak in Iceland before 1972.



The cohort born in 1965, entering first primary grade in 1972, was the last large cohort to use the Bundgaard material. After that authorities began gradually to withdraw it, while the new state-made material was introduced after careful testing, keeping in mind the difficulties of the rapid implementation of the ‘modern math’.

**Figure 2: Percentage of year cohort studying Bundgaard material up through grade 6.**

H. Lárusson, a mathematics consultant at the Ministry of Education knew of the problems but attempted to defend the ‘modern math’, while he himself was writing a series as a continuation of the Bundgaard material. He wrote in the teacher journal:

Among parents, teachers and others ... there have been many discussions on a new syllabus in mathematics ... People have had very different views on this new syllabus ... There has been no general publicity of the syllabus as a whole, nor of its goals, and this may partly cause the criticism which has emerged. This innovation became far more widespread far quicker than was planned ... (Lárusson, 1972, p. 9).

In 1973, Lárusson was on record stating that set-theoretical concepts were overly emphasized instead of being used to complement traditional methods. This had reduced pupils’ number skills, which was detrimental to their later studies in secondary schools and at work. Alterations made were expected to contribute to pupils’ broader perspectives and deeper understanding of mathematics at all levels and spheres of mathematics. Abroad, people were making extensive experiments on the content and presentation of textbooks with special consideration to those who have difficulty studying mathematics (Bjarnadóttir, 2007, p. 299).

An evening school for adults, Námsflokkar Reykjavíkur, reacted to parents’ need for support and offered them a course in the ‘new math’ (Árnadóttir, 1975).

## **AFTERMATH**

The adventure of the ‘new math’ lingered in people’s memory for a long time. Educational matters were brought up in parliament in December 1988. Eiður Guðnason, M.P., recalled that many reforms had gone wrong. The ‘modern math’ had been a serious mistake; whole cohorts knew little mathematics and could not do mental arithmetic (Guðnason, 1988). A governmental report on the development of education policy in European environment, dated in 2010, claimed that the ‘modern math’ of the 1960s and 1970s failed (Menntamálaráðuneytið, 2010, p. 46).

Minister of Education, Gylfi Þ. Gíslason, stayed in office during 1956–1971. He had to cope with the great increase in student population of the post-war baby boom and the rebellious currents in the late 1960s. In a personal communication (G. Gíslason, Jan. 10, 2002) he expressed a great confidence in Arnlaugsson as a school leader, but also doubts about his role in introducing the ‘modern math’. A confidential source told a story of another minister of the same government, E. Jónsson, an engineer and a renowned mathematics student during World War I, who had expressed a doubt that he would be able to solve an O-level examination equivalent of the 1970s, when the ‘modern math’ had been implemented.

Only recently a retired primary teacher in his eighties remarked: “It is always the same problem in this country. We took up this material when the Danes themselves had discarded it.” (S. Jóelsson, personal communication, Nov. 27, 2010).

## **DISCUSSION**

What went wrong in the implementation of the ‘modern math’? The problem was not in adapting something discarded by other nations, as Icelanders often fear in their linguistic and geographical isolation. On the contrary, the Bundgaard material had only been finalized for the first grade and its continuation for grades 4–6 had not even been created. The books for the first two grades were not too complicated, while the hardcore mathematics emerged in the third grade. One may clearly conclude that the consequential decision was not well grounded.

Was any serious harm done? The ‘modern math’ with its unfamiliar algorithms was an intellectual surprise to a nation that was homogeneous, educationally as well as ethnically. Everyone knew the same algorithms. If anyone had seen any different approach abroad it was much scarcer than at present times. Some pupils may not have learnt any algorithms, but pocket calculators were coming soon.

Clearly, an effort was made to publicise the ‘modern math’ in the media as a part of a modernizing process, in a ‘cultural competition’. The publicity, such as the television programme in 1967–1968, may therefore have created too great expectations. It may not have been well timed either. Arnlaugsson’s TV programme fascinated some of the 15-year old students, chosen to study his textbook on an experimental basis, while preparing for entrance examination into the upper secondary level. For parents of primary school children it might have been more appropriate in 1969–1970, when place-value notation to different bases, prime numbers etc. were introduced to the first large cohort, or the year after, when the concepts of set theory were implemented. And finally, Arnlaugsson was an academic, a professional mathematician, famous for his radio talk shows on the widely practiced game of chess, but his mathematics may have failed to reach common people.

The interview with Bundgaard and Plum had doubtlessly the aim to inform parents and the public, in 1970 when the majority of primary school children were studying their material. However, remarks that information in a foreign language was ‘to pacify the parents’ witness a lack of sense of the situation and respect for the parents, as well as the decision not to let the pupils bring their textbooks home as it ‘can have very bad consequences for the child if its parents are trying to help, more by being willing than able to guide the child’. That, and remarks that ‘the ‘modern math’ teaches children to think logically’ to a higher degree than earlier, and that parents should wait ‘until the moment, when the children have reached enough understanding of the project to be able to explain to their parents’ witness unrealistic convictions of the value of the programme. The ‘modern math’ was indeed a difficult cause to defend.

However, the ‘modern math’ programme at least inspired Prof. Magnússon and eventually led him to undertake the serious studies of mathematics. This was also the case with V. Bjarnason, G. H. Gunnarsdóttir and J. V. Kristinsdóttir who were in their teens in 1967. All of them have had a role in the field of mathematics,

mathematically related subjects or mathematics education in Iceland. In a small country, each individual weighs in this respect.

The leaders of the project experienced that they were not able to guide all the teachers involved in the curriculum reform. Many teachers gave up, reverted to the old syllabus and there is no doubt that a bad reputation has followed the ‘new math’ reform ever since. However, isolating one factor of a large study, the publicity process, as is done in this paper, can lead to oversimplification. The project may be credited for that many teachers experienced that there was more school mathematics, useful to children, than the four operations in whole numbers and fractions, as presented in the traditional national syllabus. The reform offered those teachers the training they missed by the 1946 legislation when the legislator failed to open a route for them to higher education.

A group of teachers began to create new material in the favourable climate of SDR, supported by all governments for over a decade. This environment made it possible to replace the difficult syllabus in minimal time, in contrast to decades of previous stagnated syllabus. This second reform was initiated only five year after the implementation of the Bundgaard material. The lesson was learnt to avoid too great enthusiasm for hastily chosen novelties. Resources were made available to recruit teachers to test it and implement it slowly and properly.

Now, forty years after the eventful period of the late 1960s, a new state-made syllabus has replaced the one of the 1970s. It has not been universally well received. Resources have not been allocated to test the material before distributing it to whole cohorts. Many teachers have protested indirectly by photocopying old material. In the 1960s, centralized week-long courses for teachers were organized, while presently they are offered half-day meetings, if their schools choose to allocate funds for specific subjects rather than general topics, such as class discipline. The lessons learnt from the ‘modern math’ of the 1970s have gradually faded away, while its deficiencies contribute more to its reputation than its merits.

## REFERENCES

- Árnadóttir, Th. J. (1975). Karlmenn hafa lítinn áhuga á saumanámskeiðum. *Morgunblaðið, Lesbók*, November 30, 6–7.
- Arnlaugsson, G. (1966). *Tölur og mengi / Numbers and sets*. Reykjavík: Ríkisútgáfa námsbóka.
- Bjarnadóttir, K. (2006). From Isolation and Stagnation to ‘Modern’ Mathematics in Iceland: a Reform or Confusion? *Paedagogica Historica*, 42, 4 & 5, 547–558.
- Bjarnadóttir, K. (2007). *Mathematical Education in Iceland in Historical Context*. IMFUFA tekst, Roskilde University electronic library. [http://rudar.ruc.dk/bitstream/1800/2914/1/Chapter0\\_IMFUFA.pdf](http://rudar.ruc.dk/bitstream/1800/2914/1/Chapter0_IMFUFA.pdf), accessed November 20, 2010.

- Björnsson, S. (1966). Samanburður á námi í stærðfræði, eðlisfræði og efnafræði í dönskum, norskum og íslenskum unglinga- og gagnfræðaskólum. *Menntamál* 39(2), 100–121.
- Bundgaard, A. (1967–1972). *Stærðfræði. Reikningur*. 1, 2a, 2b (Co-author E. Kyttä), 3a, 3b, 4a, 4b, 5a, 5b, 6. Reykjavík: Ríkisútgáfa námsbóka.
- Búrigo, E.Z. (2008). Modern Mathematics in Brazil: the promise of efficient and democratic teaching. A paper submitted to TSG 38, ICME 11, Monterrey, Mexico.
- Cooper, B. (1985). *Renegotiating Secondary School Mathematics. A Study of Curriculum Change and Stability*. London: The Falmer Press.
- Gíslason, K. (November 1978). *Nýja stærðfræðin*, an unprinted report to the Reykjavík Director of Education in the author's possession.
- Gjone, Gunnar (1983). “Moderne matematikk” i skolen. *Internasjonale reformbestrebelse og nasjonalt læreplanarbeid*, I–VIII. Oslo.
- Gröndal, B. (1968). A speech at the Alþingi parliament, April 18. <http://www.althingi.is/altext/gomulraeda.php4?rnr=1920&lthing=88&dalkur=2012>. Accessed September 3, 2010.
- Høyrup, J. (1979). Historien om den nye matematik i Danmark – en skitse. In P. Bollerslev (Ed.). *Den ny Matematik i Danmark*, pp. 49–65. Copenhagen: Gyldendal.
- Guðnason, E. (1988). A speech at the Alþingi parliament December 6. <http://www.althingi.is/altext/111/r0/0757.html>. Accessed September 3, 2010.
- Lárusson, H. (1972). Hugleiðingar um stærðfræði og stærðfræðikennslu. *Menntamál* 45(1), 9–18.
- Magnússon, J.I. (1996). Guðmundur Arnlaugsson. Kveðja frá íslenska stærðfræðafélaginu. *Morgunblaðið*, November 19.
- Menntamálaráðuneytið (2010). *Menntun í mótun. Þróun menntastefnu á Íslandi í evrópsku samhengi*. Education and Training. Reykjavík: Menntamálaráðuneytið.
- NN1. (1969). Frá Fræðslumálaskrifstofunni. *Menntamál*, 41(1), 94–97.
- NN2. (1970). Nýja stærðfræðin er eins og nýtt tungumál. *Morgunblaðið*, September 13, 23–24.
- NN3. (1967). Spjall. *Sjónvarpstíðindi*, 3(41), 3(45).
- OEEC. (1961). *New Thinking in School Mathematics*, 2nd ed. Paris.
- Sigtryggsson, K. (1964). Tilraunir með reikningskennslu. *Menntamál* 37(3), 146–159.
- Stefánsson, E. (1967). Yngsti skólinn: Nýjungar í reikningskennslu. *Foreldrablaðið*, 23 (1), 7–13.

# VOICES FROM THE FIELD: INCORPORATING HISTORY OF MATHEMATICS IN TEACHING

Kathleen M. Clark

The Florida State University

*Using the voices of teachers in the field, this paper describes how and what historical content secondary and post-secondary teachers decided to use in their teaching. For the small exploratory study, I analyzed survey results in which teachers reported their use of history as anecdote, biography, or interesting problems during the 2008 – 2009 academic year. Of the 32 episodes reported, 20 involved “history as anecdote”; nine episodes were characterized as a use of “history as biography”; and 15 included the use of “history as interesting problems” (with each episode having the possibility of being coded as a use of each or some subset of the three). The “history as interesting problems” episodes are provided in some detail and I end with a discussion on directions for future research.*

## SITUATING THE STUDY

In the field of mathematics education we know little about how teachers obtain historical knowledge and subsequently implement that knowledge in teaching. Experts agree with the claim, “teachers can profit from studying the history of mathematics” (Bruckheimer & Arcavi, 2000, p. 135) yet corresponding research is difficult to find, especially in the United States. More recently, Jankvist (2009; 2010) provided two purposes for using history of mathematics in teaching: (1) history as a tool, in which history of mathematics is an assisting means for teaching and learning mathematics, and (2) history as a goal, where students are taught about the historical development of mathematics, its interplay with other practice areas, and societal and cultural influences on mathematics and the other way around.

Jankvist (2009; 2010) observed that too much rhetorical and theoretical literature exists that describes the affordances that history of mathematics can lend to both classroom teachers and their pupils. Furthermore, he stated that, “despite history in math education being an old topic only relatively few empirical studies have been made” (Jankvist, 2010). While Jankvist makes the point regarding the lack of empirical studies that examine history as a tool or history as a goal in mathematics teaching and learning from a global perspective, the lack of empirical research on these same notions is even more extreme when examined within the context of the United States.

## A CURRENT PERSPECTIVE

My view regarding the direction of studying the incorporation of history of mathematics in teaching is that we must begin with teachers “friendly” to the idea.

To that end, the exploratory study discussed in this paper begins with teachers who possess some general knowledge of how the history of mathematics may be used in teaching, including the knowledge of appropriate resources. Furthermore, they expressed positive results (either for themselves or with students) from initial efforts to use history of mathematics in teaching.

The primary goal of the study was to survey a purposeful sample of secondary and post-secondary teachers and collect their reported planned and actual use of history of mathematics in teaching. The research questions were:

*What historical content (e.g., anecdotal, biographical, mathematical) do teachers incorporate in their teaching?*

*What obstacles do teachers report, from either their own or from the student perspective, as a result of incorporating history of mathematics in teaching?*

*What benefits do teachers observe, from either their own or from the student perspective, as a result of incorporating history of mathematics in teaching?*

## **RATIONALE**

Incorporating history of mathematics in teaching is problematic for several reasons. For example, obtaining knowledge of the historical development of school mathematics topics requires guidance (e.g., selecting appropriate and reliable resources) and time to read, study, and adapt historical content for use with students. These reasons are accompanied by the typical obstacles that teachers report; the primary of these is having instructional time to use history in teaching mathematics. Although beyond the scope of this paper, considerations of time obstacles often accompany teachers' views that drawing upon history of mathematics when teaching is something extra and any inclusion of history of mathematics is often viewed as supplemental to the mathematics curriculum. Consequently, this primary obstacle discourages teachers to use history as an instructional tool or history as a viable goal for instruction.

We know little about how teachers obtain historical knowledge and consequently implement such knowledge in teaching. Fauvel (1991) observed that:

making use of history...is hard for teachers – who have usually learned little or no mathematical history during their training, let alone received training on how to use history with their pupils. (p. 4)

Other experts agree with Bruckheimer & Arcavi's (2000) claim that there is something to be gained by teachers who use history of mathematics in teaching and that if we seek to humanize mathematics instruction for students "we must do it through the teachers" (Avital, 1995, p. 3).

## PARTICIPANTS

I considered the dearth of empirical studies on incorporating history of mathematics in teaching within the United States as a justification for conducting an exploratory study of teachers who indicated their interest to incorporate history of mathematics in their instructional practice. To identify my purposeful sample, I sent a survey to 26 teachers at the end of an online course, *Using History in the Teaching of Mathematics* [1]. Fourteen of the 26 teachers responded to the survey, of which 12 responded favorably to the final question, “Would you be interested in participating in a research effort designed by Dr. Clark, regarding the use of historical content in teaching mathematics?” Finally, six of the 12 teachers agreed to participate in the study beginning in August 2008. These teachers received a two-page summary of the study, including goals of the study and the approximate time commitment for completing monthly surveys. Additionally, each teacher received a copy of the survey questions so that they could use the document format to compile their responses for the Survey Monkey<sup>TM</sup> monthly survey. The monthly survey included prompts such as:

- Describe your plans for incorporating history of mathematics in teaching, including dates of instruction, the course title, the mathematical topic, and what you included in your instruction.
- Describe any obstacles that you experienced when implementing your instructional plans for including history in teaching mathematics.
- Describe any obstacles that you feel your students experienced as a result of incorporating the history of mathematics in teaching.
- Identify the benefits (to you, the teacher) that you experienced as a result of studying, planning for, and incorporating the history of mathematics in your teaching.
- Identify the benefits that you feel your students experienced as a result of your inclusion of the history of mathematics.
- Classify the overarching historical perspective from which you approached each inclusion of the history of mathematics as: history as anecdote; history as biography; history as interesting problems; or, a combination of two or more of these.
- Provide the bibliographic information for the resources that you accessed for planning and implementing the history of mathematics in teaching.
- Indicate whether you created handouts to be used with students (or that were actually used with students) or used handouts from other sources.

The six teachers (only pseudonyms are used here) were teaching in three different states. Evan, Judy, Jeannette, and Suzanne were teaching in the same state in the



## Working Group 12

southeastern United States; Maxine was teaching in a state in the Midwestern US; and Libby was teaching in the southeastern US. As an unfunded, exploratory study, the online survey was the most viable data collection method. Although I am fully aware of the problematic nature of self-report data, my original plan was to use the “Voices from the Field” monthly survey responses to describe what teachers in a variety of teaching contexts were able to accomplish, given their self-professed goals of wanting to infuse their teaching with history of mathematics. Table 1 presents the six teacher participants, their teaching assignment in 2008 – 2009, and an excerpt from their response to the question, “Do you plan on using history of mathematics in teaching in future? If so, please describe an example or two.”, from a final summary assignment during the *Using History in the Teaching of Mathematics* course.

Participant	Teaching assignment	Do you plan to use history of mathematics in the future?
Maxine	Developmental mathematics courses (community college level)	“I would like to refine the “restoration and opposition” approach so that students in the skilled trades have more facility in solving equations for a certain variable.”
Evan	Middle grades mathematics (advanced/honors level)	“Over time, I will work to integrate more history of mathematics in my teaching. More than anything, learning about mathematics history has shown me how important it is for my students to know why mathematical concepts exist.”
Judy	High school mathematics (advanced level)	“Definitely. I know I will use lessons from [the <i>Historical Modules</i> ] in every class that I teach next year.”
Suzanne	Middle grades mathematics (all levels)	“I will use the history of mathematics in my future lessons. I will use the modifications I made to the ‘determining the angle of the sun with shadows’ activity this school year. Next year, I’d like to use my module add-on lesson ‘From the Earth to the Sun’.”
Libby	High school mathematics (geometry)	“I absolutely will incorporate many historical aspects into the subjects that I teach in the future. For example, if I am teaching Algebra 1, I would certainly include the activities from the negative numbers module that we studied.”
Jeannette	High school mathematics (geometry)	“Yes, I am planning on using some of the logarithms activities with my students when I get to that section. I will be looking through the modules again over the summer to decide which ones I would like to include next year.”

**Table 1: Teacher participant data**

## DATA COLLECTION

Participants were asked to complete the “Voices from the Field” survey approximately every six weeks, or six times during the 2008 – 2009 school year. Table 2 documents each participant’s survey completion, as well as comments on issues that a participant may have experienced during data collection. Comments are provided when participants were unable to complete each of the six surveys.

Participant	Surveys completed	Comments
Maxine	Surveys 1 – 6	
Evan	Survey 2; Survey 6	Did not complete surveys when no use of history to report
Judy	Survey 1; Survey 2; Survey 6	Unable to continue with plans to incorporate history of mathematics during high-stakes test “season”
Suzanne	Survey 1; Survey 2; Survey 4	Obstacles resulting from administration intervention
Libby	Survey 1; Survey 3	Unable to continue due to graduate school and new course planning
Jeannette	Survey 1; Survey 2; Survey 3; Survey 4	Discontinued plans to incorporate history of mathematics due to high-stakes test “season”

**Table 2: Documentation of participant survey completion**

## RESULTS

For the purposes of this paper, I only address the first research question (What historical content (e.g., anecdotal, biographical, mathematical) do teachers incorporate in their teaching?). As indicated in their response to a final summary assignment during the *Using History in the Teaching of Mathematics* course, each teacher did indeed incorporate history of mathematics in their teaching. This was not completely unexpected given the positive-toward-the-power-of-using-history disposition each participant shared while taking the *Using History* course and their subsequent expression of interest to participate in the study.

Examination of the survey responses revealed that the teachers, some of whom in spite of experiencing significant obstacles, developed particular preferences for incorporating history of mathematics when teaching. As one way to describe such preferences I used the terms, “history as anecdote”, “history as biography”, and “history as interesting problems”, which were adapted from the reasons for using history in the teaching of undergraduate mathematics suggested by Siu (1997). Siu identified four such strategies (“ABCDs”):

- A is for Anecdotes;
- B is for Broad Outline;
- C is for Content; and

## Working Group 12

- D is for Development of Mathematical Ideas. (p. 144)

The modifications I developed, based upon my experience with teachers seeking to incorporate history of mathematics at the secondary level, are:

<u>Siu's strategy</u>	<u>Modification for the "Voices from the Field" study</u>
A is for Anecdote	history as anecdote
B is for Broad Outline	(none represented)
{mostly aligned with "A"}	history as biography
C is for Content	history as interesting problems
D is for Development	(none represented)

Question 10 of the survey requested that teachers classify the overarching historical perspective from which they approached each inclusion of the history of mathematics in their teaching. The question, which asked teachers to identify their "goals of using history in teaching", was developed prior to publication of Jankvist's (2009) identification of purposes, "history as a tool" and "history as a goal". With the view of these purposes, I could argue that each episode reported by the teachers in this study represents an example of "history as a tool". I would further argue, however, that elements of "history as a goal" are reflected in both teachers' descriptions of what they included in their teaching, as well as in the comments that they offered about their work in this regard. My purpose in this paper is to not so much delineate types of use of history of mathematics in teaching; instead I aim to highlight ways in which teachers are able to, and to suggest directions for future empirical work.

### **History as...**

Teachers reported that they included history of mathematics in their teaching in 32 teaching episodes, with content ranging from Grade 7 mathematics (pupils aged 12 to 13) to community college developmental mathematics (mathematics course work at the secondary level, taken after secondary school in colleges preparing students for college-level work). "History as anecdote" was identified as the overarching perspective for their use of history in 20 of the 32 episodes. "History as interesting problems" was identified in 15 episodes and "history as biography" was reported in only nine of the 32 episodes (note that any episode had the possibility of being coded as a use of each or some subset of the three types of use of history). The prevalence of use of historical anecdotes in teaching mathematics is not surprising. Sharing anecdotes can situate a new topic – both for students and teachers, can aid in humanizing mathematics, and often require the least amount of preparation and time commitment during teaching [2].

The result with significant promise, however, is that teachers chose to plan an instructional episode in which students would work with historical problems in 15 of the 32 episodes. This is significant for two reasons: a "history as interesting

## Working Group 12

problems” perspective engages students in mathematical content and this engagement allows students to investigate alternative solution methods. The 15 “history as interesting problems” episodes are described in Table 3.

Participant	Course/topic	Historical content	Participant comments
Maxine	Basic Math	Numerals in ancient number systems	Students were apprehensive about doing something new; student confusion and discomfort
Maxine	Intro to Algebra	Al-Khwarizmi’s method of completing the square	Students needed more time to understand these sorts of problems
Maxine	Intro to Algebra	Solving right triangle problems from historical documents	Students demonstrated interest and engagement, in particular because of relevance of material to students’ lives
Maxine	Intro to Algebra	Solving linear equations using Euler’s method for transforming equations	A student question prompted the teacher to introduce operations with integers using ancient Chinese methods; student exclaimed, “Wow! You should tell the whole class!”
Maxine	Intro to Algebra	Solving quadratic equations using the methods of al-Khwarizmi	Students were able to see that such problems are “mathematical ideas that people have struggled with over centuries”.
Maxine	Intro to Algebra	Theorem of Pythagoras; solving right triangles	Integration of history feels “natural” after doing so for several semesters
Maxine	Intro to Algebra	Chinese method for integer addition	Using the Chinese method reduced misunderstanding and subsequent re-teaching
Maxine	Intro to Algebra	Solving equations (Maclaurin’s transformations)	Students indicated that they really “get it”
Judy	Honors Geometry	Proofs of the Theorem of Pythagoras	Students enjoyed working on the problems, though they question themselves when working on atypical problems
Judy	Honors Geometry	Solving right triangle (historical) problems	Working with history is rewarding if students “get it”
Libby	Geometry	Research on tangrams; construction of	Learning about the background contributes to the experience; includes logic skills and spatial

## Working Group 12

Participant	Course/topic	Historical content	Participant comments
		examples	awareness
Jeannette	Geometry	Goldbach's conjecture	Historical information helps her to understand the material better
Jeannette	Geometry	Gauss' number theory	Students benefit from "knowing the people behind the math"; makes it more real for students
Jeannette	Geometry	Discussion of the Declaration of Independence	Reading is a big obstacle for the students
Jeannette	Algebra 2	Babylonian guess-and-check method for solving equations	Students get to see another side of mathematics; historical problems promote critical thinking skills

**Table 3: History as interesting problems (by teacher participant)**

Maxine's and Jeannette's "history as interesting problems" use was the most prevalent of the six teacher participants. Maxine identified actual problems she used with her students in eight of the nine teaching episodes she reported in the survey responses. In many ways, Maxine may have experienced the time obstacle more extensively than other teacher participants. Although she did not have to deal with the pressure of a high-stakes assessment of her students at the end of the year, she did have to adhere to a very full curriculum, for courses that met only two or three days per week for one semester. She mentioned that her students did not always have sufficient time to work on the mathematical ideas within the historical problems, yet she consistently incorporated them in her teaching.

Jeannette did have concerns about the high-stakes assessment and because of it she abandoned her plans to incorporate history of mathematics at the end of the school year. Judy experienced the same issue, however she did resume her plans for using history after the conclusion of the assessments. The remaining three teachers did not incorporate historical problems as frequently (Libby) or at all (Evan and Suzanne). This may be due to the level of courses they taught (Evan and Suzanne taught middle grades mathematics; Libby taught regular geometry). Follow-on interviews – along with classroom observations – may provide information to better understand this phenomenon.

## DISCUSSION

The episodic account presented here is encouraging, but many questions arise. Certainly it is encouraging that after participating in a one-semester history course (e.g., *Using History in the Teaching of Mathematics*), teachers felt empowered and able to incorporate history of mathematics into their instructional plans. This may be attributed to a variety of reasons, including the participants' predisposition to use history of mathematics or the availability of classroom activities with which they had

prior experience (e.g., access to collections of classroom materials, including *Historical Modules for the Teaching and Learning of Mathematics* (Katz & Michalowicz, 2005)). Yet the ways in which a teacher develops his or her mathematical knowledge for teaching still requires high-quality, empirical study. In particular, it is necessary to examine ways in which history of mathematics contributes to mathematical knowledge for teaching, aids in improving “students’ problem-solving skills, lays a foundation for better understanding, helps students make mathematical connections, and highlights the interaction between mathematics and society” (Karaduman, 2010).

The current study contributes in two ways. First, analyzing the use of history of mathematics employed by teachers, along with the obstacles and benefits that they observe provides a foundation from which to plan future research. For example, when and for what topics is it more beneficial to incorporate “history as interesting problems”? And, what is the value-added effect of “history as anecdote”? Second, this study contributes to the current conversation and framework-building surrounding “history as...”, as suggested by the tool I used to analyze teachers’ use of history and the approaches of Jankvist’s (2009; 2010) two purposes of using history of mathematics in teaching. Developing a framework can only result from research that proceeds with a purpose. In this case, our purpose should be focused on moving away from the rhetoric that has survived for several decades, and on moving towards implementing a research agenda that seeks to address the questions regarding the use of history of mathematics in teaching.

## NOTES

<sup>1</sup>. The course was required in an online Masters of Science in Mathematics Education program at a large university in the southeastern US. Admission to the program required students to be classroom teachers (mathematics, grades 5 – 12).

<sup>2</sup>. This is not to say that anecdotes are always used for an instructive purpose and without appropriate resources can be wildly inaccurate. See for example, Montelle and Clark (2009).

## REFERENCES

- Avital, S. (1995). History of mathematics can help improve instruction and learning. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, and V. Katz (Eds.), *Learn from the Masters* (pp. 3-12). Washington, DC: Mathematical Association of America.
- Bruckheimer, M., & Arcavi, A. (2000). Mathematics and its history: An educational partnership. In V. Katz (Ed.), *Using history to teach mathematics: An international perspective, MAA Notes #51* (pp. 135-146). Washington, DC: Mathematical Association of America.

- Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics*, 11(2), 3-6.
- Jankvist, U. T. (2009). A categorization of the “whys” and “hows” of using history in teaching mathematics education. *Educational Studies in Mathematics*, 71(3), 235-261.
- Jankvist, U. T. (2010, July). *An implementation of two historical teaching modules: Outcomes and perspectives*. Plenary lecture delivered at the Sixth European Summer University, Vienna, Austria.
- Karaduman, G. B. (2010). A sample study for classroom teachers addressing the importance of utilizing history of math in math education. *Procedia Social and Behavioral Sciences*, 2, 2689-2693.
- Katz, V. J., & Michalowicz, K. D. (Eds.) (2005). *Historical modules for the teaching and learning of mathematics*. Washington, DC: Mathematical Association of America.
- Montelle, C., & Clark, K. M. (2009). Using history to enrich undergraduate mathematics: Beyond the anecdote. In D. Wessels (Ed.), *Proceedings of the 7<sup>th</sup> Southern Right Delta Conference on the Teaching and Learning of Undergraduate Mathematics and Statistics*, (pp. 185-195). Stellenbosch, South Africa: DELTA.
- Siu, M.-K. (1997). The ABCD of using history of mathematics in the (undergraduate) classroom. *Bulletin of the Hong Kong Mathematical Society*, 1(1), 143-154.

# DESIGNING TEACHING MODULES ON THE HISTORY, APPLICATION, AND PHILOSOPHY OF MATHEMATICS

Uffe Thomas Jankvist

University of Southern Denmark [1]

*Firstly the paper raises the question of designing teaching modules focusing on elements of the history, application, and philosophy of mathematics for use within the Danish upper secondary mathematics program. The design relates to the Danish KOM-project's competence-based framework and may be seen as a way to put its three types of overview and judgment into practice. The design question is answered by a description of a specific module. Secondly the paper poses a research question of the ways in which the design of such modules may create a basis for students' development of overview and judgment. Finally the paper discusses theoretical constructs from general mathematics education research that may assist in answering this second question based on in-progress classroom implementations.*

## THE DANISH KOM-PROJECT

The Danish KOM-project [2] defines *mathematical competence* as “a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge” (Niss & Jensen, 2002, p. 43). Depending on the mathematical challenge in question, different types of mathematical competencies are called for. The KOM-project identifies eight more or less self-explanatory mathematical competencies (each referred to as a *competency*) that students are to come to possess during their mathematical training and which together span the above mentioned overall and general mathematical *competence*. These competencies are divided into two groups:

- I. the ability to ask and answer questions in and with mathematics, and
- II. the ability to deal with mathematical language and tools.

From an overall point of view, the ability to cope with and in mathematics can be said to consist of exactly these two capacities, each of which contains a set of four specific competencies. The four specific competencies which make up the first capacity (I) are: (1) mathematical thinking competency, (2) problem solving competency, (3) modeling competency, and (4) reasoning competency. The four competencies regarding mathematical language and tools (II) are: (5) representation competency, (6) symbols and formalisms competency, (7) communication competency, and (8) aids and tools competency.

Besides these eight ‘first order’ competencies, the KOM-project also mentions three ‘second order’ competencies, referred to as the three types of *overview and judgment*:

- OJ1: the actual application of mathematics in other subject and practice areas;



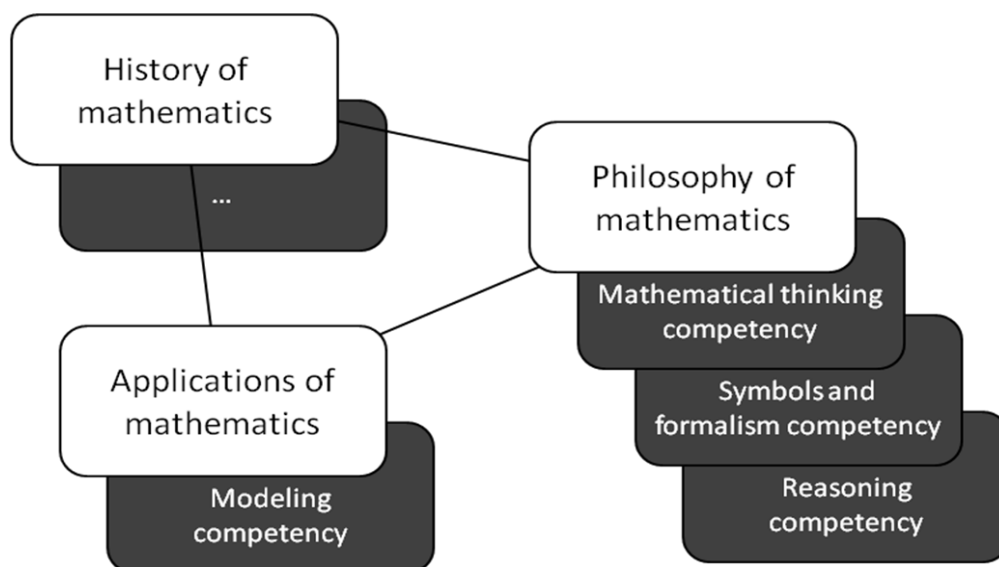
- OJ2: the historical evolvement of mathematics, internally as well as in a societal context;
- OJ3: the nature of mathematics as a subject.

Where mathematical (first order) competencies “consist in having knowledge about, understanding, being able to practice, apply, and commit oneself to mathematics and mathematical activities in a profusion of connections in which mathematics plays a part or can play a part”, or in other words a kind of “insightful readiness to act expedient in situations which hold a certain kind of mathematical challenges”, the three types of overview and judgment are “‘active insights’ concerning the character of mathematics and its role in society”. Niss & Jensen state that “these insights equip those who possess them with a set of viewpoints, which provide overview and judgment about the connection of mathematics to circumstances and allotting in nature, society, and culture” (Niss & Jensen, 2002, pp. 43, 66).

OJ1 concerns actual application of mathematics to extra-mathematical purposes within areas of everyday life, society, or other scientific disciplines. The application is brought about through the creation and utilization of mathematical models. As opposed to the so-called modeling competency, which deals with active modeling skills, this type of overview and judgment is of a more broad and generalized form, almost of a sociological or science-philosophical nature. OJ2 should not be confused with knowledge of the history of mathematics viewed as an independent topic. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and is subject to the motivations and mechanisms which are responsible for this development. On the other hand, the KOM-project says, it is obvious that if overview and judgment regarding this development is to have any weight or *solidness*, it must rest on concrete examples from the history of mathematics. The third type of overview and judgment (OJ3) concerns the fact that mathematics as a subject area and academic discipline has its own characteristics, as well as the characteristics themselves. Some of these, mathematics has in common with other subject areas, while others of them are unique. This type of overview and judgment includes several elements of more modern *philosophy of mathematics* (OJ3), and for that reason I shall also refer to it as such. For similar reasons the other two types shall also be referred to as *history of mathematics* (OJ2) and *applications of mathematics* (OJ1).

### INTERRELATIONS BETWEEN OJS AND COMPETENCIES

Although being of different ‘order’ the eight mathematical competencies and the three types of overview and judgment are related. Niss & Jensen (2002) list the interrelations which are illustrated on figure 1.



**Figure 1: Interrelations between the OJs and the mathematical competencies.**

Niss & Jensen state that a well-developed modeling competency on the one hand will contribute to a concrete entrenchment and consolidation in terms of OJ1, but on the other hand that such not automatically is a result of having a well-developed modeling competency. For OJ3 Niss & Jensen (2002, p. 69) state that if one were to point to competencies which in particular contribute to “the creation of a basis for overview and judgment when it comes to the *particular* traits of mathematics, then it must be the mathematical thinking, reasoning, and symbol and formalism competencies.” As evident from figure 1, OJ2 cannot in the same way as the two other OJs be said to have a set of corresponding competencies (although one may speak of a ‘mathematical historical competency’ as part of the overall mathematical *competence*). However, the KOM-report also makes clear that to be properly entrenched, the three types of overview and judgment need to rest on a foundation of the eight competencies. Or in other words: “to have overview and judgment regarding mathematics, it is insufficient merely to have heard (stories) of mathematical application, historical development, and its particular nature” (Niss & Jensen, 2002, p. 70).

## MOTIVATION AND A QUESTION OF DESIGN

The three types of overview and judgment play a role in the new Danish mathematics program for upper secondary school, i.e. the students have to develop these ‘second order’ competencies to some extent. Besides the fact that this can happen as part of 1/3 of the curriculum, which is free for the teachers and schools to choose themselves, neither the KOM-project nor the regulations for mathematics at upper secondary level say much about how it is to take place in practice. In a previous study [3] I considered this for the case of history (OJ2), although choosing historical cases related to actual applications of mathematics as well (OJ1) (Jankvist, 2009c). Thus, one question for this study is:

- How to design teaching modules that take into account all three types of overview and judgment – in unison – with upper secondary level as the target group?

### A ‘GUIDED READING’ OF ORIGINAL SOURCES

The main idea of the design is to have the students read and work with one original source for each of the three types of overview and judgment, all of them adhering to a common mathematical theme and/or topic. I shall illustrate this by describing a concrete module. The three texts (in Danish translation) included in the teaching material for this module are:

- Leonhard Euler, 1736: *Solutio problematis ad geometriam situs pertinentis*
- Edsger W. Dijkstra, 1959: *A Note on Two Problems in Connexion with Graphs*
- David Hilbert, 1900: *Mathematische Probleme – Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction only).

The overall theme for these are ‘mathematical problems’, which was what Hilbert addressed in general terms in the introduction of his lecture from 1900. To make Hilbert’s general observations a bit more concrete, the students are first to read the two other texts, each of which addresses a mathematical problem. Euler’s paper from 1736 is on the Königsberg bridge problem: how to take a stroll through Königsberg crossing each of its seven bridges once and only once – and today the paper is considered the beginning of mathematical graph theory. Two centuries later, with the dawn of the computer era, graph theory (and discrete mathematics in general) found new applications. Dijkstra’s algorithm from 1959 solves the problem of finding shortest path in a connected and weighted graph, and today it finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore Dijkstra also discusses a method for finding minimum spanning trees, a problem relevant for minimizing the amount of expensive copper wire to be used in the building of computers at the time, but also highly relevant today.

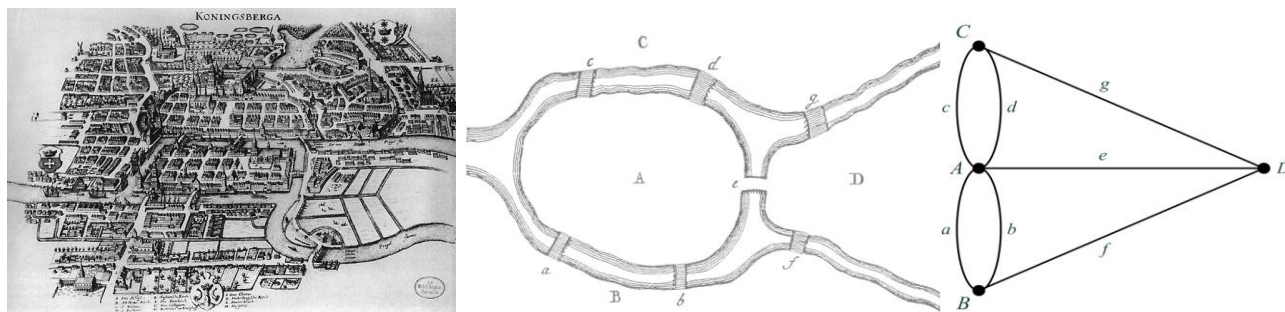
Because original sources often are difficult to access, the presentation of these are supplied with explanatory comments and tasks along the way, one purpose of these also being to bring the students up to date with modern notation, definitions, and terminology. Thus, the presentation may be considered a so-called ‘guided reading’ of the sources, very much inspired by the format developed by David Pengelley, Jerry Lodder, Janet Barnett and others related to the group at NMSU who consider the use of original sources in the classroom [4].

Practically no mathematical requirements are needed beforehand on the students’ behalf to study the text of Euler, and many of those needed for the Dijkstra text are introduced along with the Euler text. This was a major reason for choosing exactly these two texts, taking into account that the 1/3 free curriculum allowed such topics.

## MATHEMATICAL COMPETENCIES LAYING THE GROUND FOR OJS

The way of trying to create a basis for the students' development of overview and judgment is by focusing on and having the students work with the mathematical content of the original sources by Euler and Dijkstra in such a way that their work will require a development of the mathematical competencies related to the different types of overview and judgment (see figure 1). To illustrate this I shall show three exercises from the teaching material, which each focus on the development of one of the three mathematical competencies related to the type of overview and judgment on the philosophy of mathematics (OJ1), i.e. the mathematical thinking competency, the symbols and formalism competency, and the reasoning competency.

The students' way into the first original text is by looking at Euler's diagram of landmasses and rivers in Königsberg (figure 2, middle) and then verify that this is in fact an accurate representation of (or model for) the Königsberg bridge problem by comparing with a picture of the town (figure 2, left). Afterwards the students are told that in modern graph theory, landmasses are represented by vertices (or nodes) and links between them by edges. Students are asked to transform Euler's diagram into such a modern graph individually and then compare their own representation to the students next to them, this illustrating that graph representations are not unique. The idea is to have the students adapt more and more schematic representations of the Königsberg bridge problem until arriving at something looking like figure 2 (right), gradually increasing the level of abstractness.



**Figure 2: Left: A picture of Königsberg with its 7 bridges from 1652. Middle: Euler's 1736 simplification of Königsberg's bridges. Right: A modern graph representation.**

Once being familiar with the modern representation of a graph, the students are introduced to the problem of representing *multiple edges*, such as for example the two edges between vertices  $A$  and  $B$  in the Königsberg graph. These cannot be represented by only their pair,  $(A,B)$ , since this causes ambiguity (which also is why Euler named them  $a$  and  $b$ , respectively). To illustrate a formal and general way of dealing with this to the students, they were provided with the following modern definition: A graph  $G$  is a set of vertices  $V(G)$  and a set of edges  $E(G)$  together with a function  $\psi$ , which for every edge  $e \in E(G)$  assigns a pair, called  $\psi(e)$ , of vertices from  $V(G)$ . As a way of having the students develop their symbols and formalism competency, which also includes being able to go back and forth between ordinary

language and a language of symbols, they are asked to write up the sets  $V(G)$  and  $E(G)$  for the Königsberg graph and the seven function values of  $\psi(e)$ . On the one hand, the idea of this is to enable them to perceive the definition of a graph as a triplet  $G=\{V(G), E(G), \psi_G\}$ , and on the other hand to have them realize how the above definition in a general fashion resolves the problem of ambiguity when two vertices in a graph have multiple edges.

As Euler himself in his text introduces various constructs, the students are introduced to the somewhat equivalent modern terminology in the intermediate commentaries, e.g. *route*, *path*, *Euler path* (open and closed), *subgraph*, *degree* of a vertex as well as a few small theorems which Euler explicitly or implicitly uses, such as for example the *handshake theorem* [5]. At the end of his paper, Euler states his three main results (Euler, 1741, pp. 138-139; Fleischner, 1990, p. II.19, numbering is mine):

- [i.] *If there are more than two regions with an odd number of bridges leading to them, it can be declared with certainty that such a walk is impossible.*
- [ii.] *If, however, there are only two regions with an odd number of bridges leading to them, a walk is possible provided the walk starts in one of these two regions.*
- [iii.] *If, finally, there is no region at all with an odd number of bridges leading to it, a walk in the desired manner is possible and can begin in any region.*

The students are first asked to formulate these three results using the modern terminology and notation they have been introduced to. Next they are provided with a modern definition of a *connected graph*, i.e. that there exists a route between every pair of vertices, a property Euler does not state explicitly. Using this property, the three results may be reformulated as (e.g. Jankvist, 2010b, pp 17-21):

- i. *If a connected graph  $G$  has more than two vertices of uneven degree, then it does not contain an Euler path.*
- ii. *Let  $G$  be a connected graph, then  $G$  contains an (open) Euler path if and only if  $G$  contains exactly two vertices of uneven degree.*
- iii. *Let  $G$  be a connected graph, then  $G$  contains a (closed) Euler path if and only if all vertices of  $G$  have even degree.*

Most of Euler's efforts goes into proving his first result (i), and regarding the third (iii), which today is considered the main theorem of the paper, he only proves it in one direction. To develop the students' reasoning competency as well as to introduce them to the notion of if-and-only-if theorems, they are to consider result i as being of the form  $\mathbf{P} : \mathbf{A} \Rightarrow \mathbf{B}$ , and then identify  $\mathbf{P}$ ,  $\mathbf{A}$ , and  $\mathbf{B}$ . After having the students prove that  $\mathbf{A} \Rightarrow \mathbf{B} \equiv \neg\mathbf{A} \Leftarrow \neg\mathbf{B}$  (by means of a truth table), they are asked to write up  $\neg\mathbf{B} \Rightarrow \neg\mathbf{A}$  for result i, i.e. formulating the contrapositive of this theorem, which states that *if  $G$  is connected and has an Euler path (open or closed), then  $G$  has two or less vertices of uneven degree*. Since Euler has shown, in his own context of course, that a graph will always contain an even number of vertices with uneven degree, we may

distinguish between two different cases: when  $G$  has exactly two vertices of uneven degree and when it has none, i.e. when all vertices have even degree. These cases correspond to the  $\Rightarrow$ -direction in results ii and iii, respectively. Thus, by looking at Euler's original text again, the students would be able to deduce that the missing parts of the proofs are the  $\Leftarrow$ -directions for results ii and iii. For result iii this is ascribed to Carl Hierholzer (published posthumous in 1873), and the students are shown this proof. The students are then asked to prove the  $\Rightarrow$ -direction for iii and both ways for result ii using modern terminology.

To illustrate a task that is thought to develop the students' mathematical thinking competency, we move into the context of finding minimum spanning trees and shortest paths in connected, weighted graphs as discussed in the paper by Dijkstra. As an introduction to this paper, the students are provided with definitions of a *weighted graph*, a *tree*, and a *spanning tree*: *A connected graph  $T$  without any subgraphs that are circuits is called a tree, and a tree that for some graph  $G$  contains all vertices of  $V(G)$  is called a spanning tree.* In one task the students are asked to look at the Königsberg graph (figure 2, right) and find the number of different spanning trees that can be constructed from this and then explain their method for finding the answer. (Try it yourself to see the systematic approach required, when not using a formula [6].)

After the students having worked through the Dijkstra text, the commentaries to this, accompanying examples, and a modern proof of the shortest path algorithm's correctness (Dijkstra gives none), the students get to the third text by Hilbert and following this the three so-called *essay assignments*.

### THREE ESSAY ASSIGNMENTS INVOLVING THE THREE OJS

In my previous study [3] I found that having groups of students prepare small essays was a good way of bringing them to work with the history of mathematics. So the same approach is taken to bring in the two other types of overview and judgment. The module includes three essay assignments and I shall address elements of them in turn.

The first essay is on *mathematical problems*, linking the three texts by Euler, Dijkstra, and Hilbert together. Paraphrasing Hilbert roughly, he talks about that often some mathematical development is spurred on by a problem in the extra-mathematical world. Then it is drawn into mathematics and rephrased so that it is hardly recognizable anymore and embedded in a much more general context. Years later, when this has grown into a mathematical discipline, what often happens is that it may then again be used to solve some new extra-mathematical problem. This is of course the case for graph theory, spurred on by the Königsberg bridge problem, which Euler generalized so that the answer to the original problem falls out as a small corollary to his more general results. Two centuries later when we have a much clearer idea about the discipline of graph theory, Dijkstra solves the extra-

mathematical problem of shortest path (and also considers minimum spanning trees) in this graph theoretical context. For the students to realize this, they are asked to identify the criteria that Hilbert proposes for a good mathematical problem and see to what degree the problems treated by Euler and Dijkstra fulfill these, and then relate these cases to Hilbert's comments on the development of mathematics in general.

The second essay is on *mathematical proofs* and first deals with different kinds of proofs and proof techniques as well as the use and need for new signs and symbols (both arithmetical and graphical) in the development of new mathematics (concepts, definitions, etc.), something that Hilbert also addresses. The students are asked to discuss this with relation to Hilbert's text and try to draw connections to the two cases, in particular the advantages Dijkstra had in 1959 with a fully developed graph theoretical and conceptual apparatus at his disposal as compared to Euler who had to start from scratch in 1736. In the end, this essay moves into Hilbert's actual discussion of proofs and their role in solving mathematical problems as well as the role of rigor in mathematical proofs. On the overall, the idea of this is to spur some reflections on the students' behalf regarding the epistemological development of the notion of proof.

The third essay is about *mathematics' status as a (scientific) discipline*, in its own right and in comparison to other disciplines, e.g. physics. Based on their readings of Hilbert, and the two texts by Euler and Dijkstra, the students are asked to try to point out some characteristics of mathematical problems, methods, and ways of thinking as well as to say something about the types of results mathematics delivers and what they may possibly be used for. They are invited to discuss this by comparing mathematics to other academic disciplines. Then they are asked to identify what Hilbert says about the differences and connections between mathematics and other disciplines, and then discuss to what extent they agree or disagree.

## RESEARCH QUESTION AND EXPERIMENTAL SETUP

This brings us to the actual research question of this study in progress:

- In what ways may the design of the modules create a basis for students' development of overview and judgment – and perhaps even help in ensuring such development?

The approach to trying to provide an answer to this question is an empirical one. The described module together with a second one, which is in preparation, will be implemented in an upper secondary class in their first (age 16-17) and second year of high school [7]. The set up of the implementation is that student groups to the largest degree possible read and work with the material themselves, i.e. they read at home and in class they work on the related tasks, proofs, etc. in their respective groups. The teacher circles the class and provides help in the form of regulatory input and feedback when needed. In this way an attempt is made to keep the voice and discourse of the teacher to a minimum, in order to be able to assess the design discourse of the module and its material. The students also do their essay

assignments in groups and they do group hand-ins of these as well. Two sets of selected mathematical tasks from the material are however to be handed in to the teacher individually.

The experimental setup includes before, in between, and after individual questionnaires and interviews with the students and teacher following a model developed in Jankvist (2009b). One of the student groups, referred to as the focus group, is video filmed by myself during the implementations of both modules. Together with questionnaires, interviews, the hand-in essays and mathematical tasks, the video recordings provide the opportunity to do methodological triangulation between data sources in order to address the research question.

### THOUGHTS ON THEORETICAL CONSTRUCTS FOR ANALYSIS

The theoretical constructs to be used in the analysis of the data in particular include the discursive and commognitive approach of Sfard (2008) and the adaption and/or application of this to research on use of history in mathematics education (Jankvist, 2009b; Kjeldsen & Blomhøj, in press). More precisely, it may be argued that the students' discussions (and reflections) follow various different discourses: historical, cultural, sociological, philosophical, epistemological, psychological, etc. and of course mathematical. By comparing the discourses present in the students' discussions – and essays and interviews – it should be possible to say something about the connections and dependencies of these with the discourses present in the teaching material, which mainly center around an historical/cultural for OJ1, an applicational/sociological for OJ2, and a philosophical/epistemological one for OJ3.

Following the recommendations of the KOM-report, one important feature of the materials for these modules is that the treatment of overview and judgment related aspects build on and are rooted in the actual mathematics related to these. Another way of saying this is that the treatment of the meta-issues is *anchored* in the related mathematical in-issues (Jankvist, 2009a, 2009b). Therefore an important aspect of the assessment and evaluation of the modules – as well as the answering of the research question – is to see if this built-in anchoring somehow transfers to the student discussions. This may be done by searching for instances in the data where meta-issue discourses related to the OJs build upon in-issue discourses or episodes where the (historical and/or modern) mathematical in-issues are used by the students to substantiate or deepen the meta-issue reflections (Jankvist, 2009b, In Press).

### NOTES

1. This study is financed by a postdoctoral grant from the Danish Agency for Science, Technology and Innovation.
2. KOM is a Danish abbreviation for 'competencies and mathematics learning'. The report from the KOM-project has not yet appeared in English, so all quotes are my own translations from Danish.
3. The study of my PhD (Jankvist, 2009b) has, among other places, been reported in Jankvist (2009a, 2009b, 2010a).



## Working Group 12

4. Links are: [http://www.math.nmsu.edu/hist\\_projects/](http://www.math.nmsu.edu/hist_projects/) and <http://www.cs.nmsu.edu/historical-projects/> (Retrieved on November 29, 2010).
5. These terms and theorems are explained in most textbooks dealing with graph theory..
6. The number can be calculated using the so-called (Kirchhoff-Trent) *Matrix-Gerüst-Satz*. Deleting the  $i$ 'th row and column of this matrix and taking the determinant of the one dimension smaller matrix reveals it – in our case it is 21.
7. The first module was implemented in Spring of 2010, but data have not yet been analyzed. The next module, which will be on Boolean algebra and Shannon's use of it in circuit design, will be implemented in the Spring of 2011.

## REFERENCES

- Euler, L. (1741). 'Solutio problematis ad geometriam situs pertinentis'. *Commentarii academiae scientiarum Petropolitanae* **8** (1736), pp. 128-140.
- Fleischner, H. (1990). *Eulerian Graphs and Related Topics*. Amsterdam: Elsevier Science Publishers B.V. Includes original of Euler (1741) and English translation.
- Jankvist, U. T. (2009a). 'A categorization of the 'whys' and 'hows' of using history in mathematics education'. *Educ. Stud. in Math.* **71**(3), 235–261.
- Jankvist, U. T. (2009b). 'Using History as a 'Goal' in Mathematics Education'. PhD thesis, IMFUFA, Roskilde University. <http://milne.ruc.dk/ImfufaTekster/>
- Jankvist, U. T. (2009c). 'History of modern applied mathematics in mathematics education'. *For the Learning of Mathematics* **29**(1), 8–13.
- Jankvist, U. T. (2010a). 'An empirical study of using history as a 'goal''. *Educ. Stud. in Math.* **74**(1), 53-74.
- Jankvist, U.T. (2010b). *Historisk fremkomst og moderne anvendelse af grafteori – et matematikfilosofisk undervisningsforløb til gymnasiet*. Published for school use only.
- Jankvist, U.T. (In Press). Anchoring students' meta-perspective discussions of history in mathematics. *Journal for Research in Mathematics Education*.
- Kjeldsen, T.H. & Blomhøj, M. (In press). 'Beyond motivation – history as a method for the learning of meta-discursive rules in mathematics', *Educ. Stud. in Math.*
- Niss, M. & Jensen, T.H. (eds.) (2002). *Kompetencer og matematiklæring – Ideer og inspiration til udvikling af matematikundervisning i Danmark*. Copenhagen: UVM.
- Sfard, A. (2008). *Thinking as Communicating*. NY: Cambridge University Press.

# USES OF HISTORY IN MATHEMATICS EDUCATION: DEVELOPMENT OF LEARNING STRATEGIES AND HISTORICAL AWARENESS

Tinne Hoff Kjeldsen

IMFUFA, Department of Science, Systems and Models, Roskilde University

*The purpose of the paper is to present a theoretical framework for a systematic analysis and discussion of uses of history for teaching and learning mathematics, hereby proposing a didactical transposition of history from the academic research subject to history in mathematics education. The use of the theoretical framework is exemplified by an analysis of a project work on the history of Ancient Egyptian mathematics taught in a class of Danish upper secondary school students (10<sup>th</sup> grade), illustrating how uses of past mathematics can aid development of students' learning strategies and historical awareness.*

## 1. INTRODUCTION

The purpose of the present paper is to develop a theoretical framework for a systematic analysis and discussion of uses of history for teaching and learning of mathematics with respect to how history benefits students' learning of mathematics, and develops students' historical awareness. Several recent papers have discussed whether these two aims pose a dilemma between genuine history and relevant mathematics for teachers who want to use or integrate history in their classrooms (Freid, 2001; Jankvist and Kjeldsen, forthcoming; Kjeldsen, forthcoming; Kjeldsen and Blomhøj, forthcoming). While these discussions have focused on transforming views of mathematics and mathematics education, their conception of history has been taken to be more or less synonymous with a traditional professional historians' approach to history – at least in the methodological approaches and the criteria for a genuine approach to history. However, perhaps we also need to broaden our view of history as well if we want history to play a more significant role for teaching and learning mathematics. In the present paper such a broadened view of history is outlined, and its implications for history in mathematics education are discussed. The aim is to develop an adequate theoretical framework for integrating history of mathematics in mathematics education that can be used to analyze specific implementations and to provide a tool for orienting the design of future implementations of the history of mathematics in mathematics education. The main focus of the paper is theoretical, but it also contains an empirical section that illustrates the theory in a carefully designed and implemented case study.<sup>1</sup>

First of all, some historiographical reflections and a position are presented. Secondly, uses of history are discussed to present a framework in which their uses for the teaching and learning of mathematics can be systematically analyzed with respect to purposes and didactical values. This discussion is based on the Danish historian

Bernard Eric Jensen's (2010) approach to history. Thirdly, the framework is adapted to mathematics education. Lastly, to connect the discussion with the practice of teaching, a project work on mathematics in Ancient Egypt is analyzed. The project work was designed by a mathematics teacher working at a Danish Gymnasium (upper secondary level) during a professional development course in "problem based project work in, with and about mathematics". The teacher implemented the project work in his own teaching practice in a class of first year Danish high school students (age 16-17) and documented his experimental teaching in a written report. His report will be analyzed to illustrate how uses of past mathematics can aid development of students' learning strategies and historical awareness, thereby substantiating some of the points raised in the present paper. The paper ends with some concluding remarks.

## 2. HISTORIOGRAPHICAL REFLECTIONS

Mathematical knowledge is produced and used by humans; hence we can think of such activities as integrated elements of historical-social reality and of human life. We can perceive mathematical activities as creations of history as well as acts that create a history of mathematics. The development of mathematics and changes within our perceptions, views, and treatments of mathematics can to a certain extent be understood as realisations (intended as well as unintended) of goals set by people. If we want to understand historical-social processes in the development of mathematics as products of human activities, we must pay attention to intentions and thoughts of the actors, as well as their understanding of the subject matter and the context in which they performed and made their choices.

At a first sight it might seem that while such an approach can be used to study the history of sociological aspects of mathematics, such as the development of its profession in different countries and/or places or the history of mathematical journals, it cannot be used to study the history of the subject-matter of mathematics due to the universal character of mathematics. But if the development of mathematics is studied from its practice, where the historian focuses on concrete practices of mathematics, acknowledging that, despite its universal character, mathematical knowledge is produced by mathematicians, who live, interact and communicate in concrete social settings, the history of mathematical ideas, concepts and theories can also be pursued within such a framework.

Such a position is in accordance with recent trends in the history of mathematics that have emerged as reactions towards the well-known critic of the widely used anachronistic (whiggish) approach to history of mathematics and the methodological debate of internalism versus externalism (Epple, 2000), (Kjeldsen et al. 2004), and *Science in Context*, 2004, 17(1/2). Within the last decades many studies in the history of mathematics focus on the practice of mathematics within social, intellectual, and cultural contexts of mathematical activities. Here professional

historians of mathematics have a critical approach to source material they analyze in order to understand its significance in its proper historical context.

### 3. USES OF PAST EPISODES

It is not the main purpose of general mathematics education to educate and train professional historians of mathematics, and in most cases mathematics teachers will not be professional historians. In some countries development of students' historical awareness is part of the curriculum, but that is not always the case, and if it is it only plays a minor part. With this in mind it seems too restrictive to require that the history of mathematics taught within mathematics education should be presented as traditional academic history. A didactical transposition is needed, just as is the case with school mathematics, which is also not identical with the discipline of (academic) mathematics. In the following, Jensen's (2010) broader view of history will be introduced along with several pairs of concepts that can be useful for a nuanced analysis and discussion of the role of past mathematical episodes for the learning and teaching of mathematics.

Jensen (2010) sees the academic research subject *history*, as professional historians think and work with it, as just one of many approaches to history. According to him, history is employed every time a person or a group of people is interested in something from the past, and uses their knowledge about it for some purpose. People use history for many different purposes and in many different connections, and consequently there are major differences between a lay person's and a professional historian's use of history. Recent investigations (Rosenzweig and Thelen, 1998) have shown that lay persons' and professional historians' conceptions of history differ in various respects and on several levels. Lay-history has a reputation of being naïve viewed from the academic discipline of history, while on the other hand lay historians view academic history as lifeless and remote from the real world. For professional historians it is important to place past episodes and artefacts in their historical contexts. Their historical awareness is conceived of as an interpretation of the past whereas lay persons view history more as a source of memoirs.

Jensen distinguishes between pragmatic and scholarly approaches to history. In a pragmatic approach the study of the past is guided by the idea that we can learn from history. The "usefulness" of history is an underlying perspective or principle in a pragmatic approach to history. The idea is that through history we can gain knowledge about our world of today, that history can teach us better ways to live our lives. In a pragmatic approach to history, past events are studied from a utility perspective. Jensen (2010, p. 51) contrasts a pragmatic approach to history with a scholarly approach, where historians retain a critical distance to past events and emphasize differences between past and present. In the professional, academic discipline of history both traditions can be found, but since the mid 19<sup>th</sup> century the scholarly approach to history has been more and more dominant.

Observer history and actor history are another pair of concepts through which we can discuss and understand uses of past events and sources. Jensen (2010, p. 41) talks about uses of the past from an actor perspective, if we use history to orient ourselves and act in a present context. He calls this an intervening use of history. If the past is viewed retrospectively with a purpose to enlighten instead of a purpose to act or intervene he talks about uses of past from an observer perspective.<sup>2</sup>

Finally, the so-called "living history" use of history is a way of using the past to help participants develop historical awareness and learning strategies. In Denmark living history takes place at some museum centres and at some yearly events. One such centre is The Medieval Centre. On their homepage (<http://www.middelaldercentret.dk/Engelsk/welcome.html>) they state that the centre: "is an experimental museum where you can experience life in a reconstructed late 14th century market town: Daily life, knights tournaments, trebuchets, canons, ships, markets, ... and a lot more...". According to Jensen (2010, p. 145) living history appeals to so many not only because the participants actively take part in the events, but also because they use other types of learning strategies where the focus can be, for example, to develop the skills of past craftsmen.

#### **4. WHAT IS THE CONNECTION TO MATHEMATICS EDUCATION?**

These concepts of, approaches to, and thinking about history and uses of past episodes and artefacts present a framework for a refined discussion and systematic analysis of how past episodes and sources can be/are used in the integration of history for the teaching and learning of mathematics. They open up a variety of approaches to history and uses of the past for teachers who want history to play a role for teaching and learning mathematics. Which approach to choose depends on the intended learning. For example, Kjeldsen and Blomhøj (forthcoming) argue, based on Sfard's (2008) theory of thinking as communicating, that history presents itself as the obvious tool for developing students' proper meta-discursive rules, because meta-discursive rules are contingent and as such can be studied at the object level of history discourse. This presupposes a scholarly approach to history. The idea is to use past mathematical activities and sources with the intention of creating learning and teaching situations where students can experience what Sfard calls commognitive conflicts. Hence, the past is used with the purpose of intervening, and therefore the scholarly approach to history is from an actor perspective.

Kjeldsen (forthcoming) discusses the role of history for the teaching and learning of mathematics with reference to a competence based understanding of mathematics education (Niss, 2004). Here the development of students' mathematical competence is the main purpose of mathematics education along with the development of some second order competencies, including historical overview and awareness. For the development of historical overview and awareness, a scholarly approach from an

observer perspective can be chosen. For development of specific mathematical competencies, a pragmatic approach from an actor perspective might be considered.

## **5. AN IN-SERVICE COURSE ON PROJECT WORK**

The focus of the paper is on theoretical issues, but to illustrate the theory, a project work that was developed and implemented during an in-service course for upper secondary teachers in Denmark will be analysed. In this discussion the “living history” approach will be examined to see how it might be adapted as a way for mathematics teachers to use past episodes and sources to develop students’ learning strategies and historical awareness.

The theme for the project work was Egyptian mathematics. It was developed and tested in a classroom of students (10<sup>th</sup> graders) in the Danish upper secondary school in 2004 as part of an in-service course for mathematics. The in-service course was developed in response to a reform that was to be implemented in 2005.

Compared with more traditional ways of teaching mathematics the reform challenged the teachers in several ways: (1) Many were not used to teach either the history of mathematics or mathematical modelling, both of which having more prominent positions in the new curriculum than they had in the former curriculum; (2) they were required to bring mathematics into play in interdisciplinary projects in cooperation with other subjects, from science, from the humanities, and from the social sciences; and (3) they had to design, organise and carry out project work in their mathematics teaching. The goal was to create an in-service course where theories in didactics and pedagogy interacted with development of the participants’ own teaching practice in ways that also related to inquiry-based teaching and learning. On this basis the objective of the in-service course was to support teachers in their development as teachers, implementation in their own classes, evaluation of the project work, and documentation through a written report of a project-based and problem-oriented course in the history of mathematics or in mathematical modelling. The core element of the in-service course was the development of the teachers’ experimental practice with history of mathematics or mathematical modelling and problem-oriented project work.

The in-service course began with a three day seminar where the teachers were introduced to the history of mathematics, mathematical modelling, didactical theories, and problem oriented project work. The teachers worked in small groups developing a project-organised course in either history of mathematics or mathematical modelling of their own choice consisting of approximately 10 lessons of 45 minutes each. They decided on (1) the objectives for their own professional development, (2) their objectives for students’ learning, (3) how to “set the scene” for their own students’ project work, and (4) how to evaluate the students’ learning.

A few weeks after the seminar a first draft of the design for the project work and the materials that should be given to the students were distributed to all participants in

the in-service course. All teachers tried out their project work in their classroom. During that period there was a one day seminar to support the teachers in the documentation of their results and reflections on their experimental teaching. It all ended with a 2-day seminar, where the teachers' written reports were discussed extensively. The final reports are published on the internet together with the handout materials for the students for other teachers to use (<http://magenta.ruc.dk/nsm/uddannelser/gymnasielaerer/>).

I will not go into further detail on how we define problem-oriented project work (interested readers are referred to Blomhøj and Kjeldsen, 2006), but only emphasize that the problem that students are going to work on should function as the “guiding star” for their work. In the ideal case every decision made in the project work should be justified by its contribution to the solution of the problem. This is crucial, since engaging in decisions provides opportunities for students to work independently, to gain control, and to direct the project. In order for this to happen, though, the teacher needs to set a scene for the project work, that is to formulate the task for the work, the conditions for the working process, the time constraints, and the requirements for the end product, for example a written report or a power point presentation fulfilling some specific requirements. In this way it is possible for the teacher to have some control while at the same time to leave room for the students to take responsibility and make decisions.

The in-service course is still offered with the modification that we focus only on mathematical modelling. Therefore we only have one history project to present, but since its function here is to serve as a concrete illustration of the theoretical framework developed above, and not as documentation from an empirical experiment it can be used to characterize the suggested methodology.

## **6. EGYPTIAN MATHEMATICS: A PROJECT WORK IN A 10<sup>TH</sup> GRADE**

The project on Egyptian mathematics was developed and implemented in a classroom of 1. year students (10<sup>th</sup> grade, age 16) in a Danish upper secondary school in the fall term. The project work was meant to be interdisciplinary, with history about Ancient Egypt in combination with their mathematics. The mathematics teacher had no experience with project-organised teaching in mathematics, which was his focus for his own professional development. His objectives for the students' learning were to:

- h) enhance the students' competence to work in teams
- i) enhance the students' independent learning
- j) enhance the students' oral presentation skills
- k) have the students gain experiences with power point

## Working Group 12

- l) have the students appreciate that mathematics has been different from what it is today
- m) develop the students' awareness that mathematical results have evolved, that mathematics is not static, which is contrary to the way it is often presented
- n) develop the students' awareness that mathematics develops in an interplay with culture and society. (Wulff, 2004, p. 2-3; my translation)

The objectives fall into two parts that cover all three of the above listed challenges of the reform: the first four address competence in independent study, the development for which problem-oriented project work is an excellent pedagogical tool, whereas the last three concern the history of mathematics requirements of the new mathematics curriculum. Note that a)-c) and e)-g) are elaborated versions of some of the ICMI Study *whys*, see Fauvel and van Maanen (2000, pp. 205, 207, 211-212).

The teacher orchestrated the students' project work in three stages:

- (3) The first stage was an introduction to Egyptian mathematics using a text from the students' textbook (Carstensen and Frandsen, 2002), where the teacher introduced the Egyptians' method of multiplication by repeated doubling, their number symbols, and their way of formulating problems (two lessons).
- (4) The introduction was followed by eight lessons during which the students worked in teams of four, guided by a description of
  - i) the problem formulation, which was given by the teacher (see below);
  - ii) the learning objectives; iii) the product; iv) the topics for the teams.

The teams worked independently. The teacher took the role of a consultant who could be called in for advice. When that happened he focused on posing questions and challenging the teams instead of providing answers. The problem formulation for all teams was: How and why did the Egyptians calculate? Each team worked with a chapter from a textbook on Egyptian mathematics (Frandsen, 1996), seven chapters all together treating their numerals, their methods for arithmetical operations, the 2/n-table, bread and beer (Pesu) exercises, equations and geometry. To have a whole textbook on an episode from the history of mathematics in Danish is a rare circumstance, and one of the reasons why Egypt was chosen for this project work.

- (3) Each team had to present its results for the rest of the class in an oral presentation supported by a power point presentation. This took up four lessons.

The first set of learning objectives deal with issues of enhancing students' independent study skills. In his evaluation the teacher emphasized in particular that the students acquired the mathematical knowledge of the Egyptians by themselves (in contrast to ordinary teaching where he explained everything), that they "cracked the code" themselves, and that they were conscious about it. Regarding item e) and f)



of the second set of learning objectives, the teacher wrote: “they were all about gaining insights into current mathematics precisely by studying the mathematics of another time” (Wulff, 2004, p. 3), from which we can infer that the teacher used a pragmatic approach to history. He used past episodes of mathematics from a utility perspective. This also becomes clear from his description of a discussion that took place between him and the students during the introduction: “Already during the first module [the first two lessons] came the classical question, why are we going to learn this? And we had a good talk about the intended learning issues e), f), and g), during which the class apparently accepted that historical mathematics, besides being interesting as such, could contribute to a more nuanced view on current mathematics.” (Wulff, 2004, p. 5). Regarding the learning objective of realizing that mathematics has evolved over time, the teacher was rather critical, explaining that this aspect was not really complied with, since a comparison of Egyptian and modern mathematics only shows that mathematics has changed; it does not give insights into the actual process of change. Regarding the last item g) of the second part of the learning objectives, the teacher wrote in his evaluation: “here is where the subject of history can be involved. From a general knowledge about Ancient Egypt and its society, students can discuss how society and culture have been driving forces for the mathematics of that time. At the same time the historians’ method of source criticism is an essential tool for interpreting ambiguous and defective papyri” (Wulff, 2004, p. 4). In contrast to items e) and f) the teacher here takes a scholarly approach to history. The teacher used the past from an observer perspective in both approaches.

The students’ work with the sources and exercises in the textbook on Egyptian mathematics to answer the “How” part of the problem formulation can be considered a “living history” approach. They put themselves in the place of Ancient Egyptians, trying to understand and learn how they calculated, how they dealt with geometry, how they proposed mathematical problems, and so forth. The teacher reported the following situation he observed in the classroom: “Many students wondered about how “stupid” the Egyptians were. Why did they only use unit fractions? Why should a number be expressed as a sum of different unit fractions? On the other hand their methods were very difficult to understand; that is rather advanced, so in that respect they weren’t stupid at all. I think that many of the students realized that current mathematics is not “just” like today, but is a result of a long development, during which many things have been simplified.” (Wulff, 2004, p. 7). This shows a development of historical awareness among the students. That the students’ learning strategies were developed through this kind of “living history” approach can be inferred from the following observation made by the teacher: “This [that mathematics had made progress] became especially obvious when the students constantly rewrote the Egyptian notation to current notation with  $x$ ’s, formulas, etc. After they had finished an Egyptian calculation they would say: ‘but that just corresponds to ...’ followed by a solution of an equation in our way. It was very inspiring to see how students, who normally were a bit alienated towards  $x$ ’s and

equations now had taken those to themselves as their own, and all of a sudden perceived equations as an easy way to solve problems. The students became aware that modern notation makes the calculations much easier than they would have been otherwise” (Wulff, 2004, p. 7).

As mentioned above the teacher found that item g) in the list of learning objectives, which was supposed to link the development of mathematics with a scholarly approach to history, was not realized. The “why” part of the problem formulation was designed especially towards this goal. The mathematics teacher had hoped that the students would have been able to experience concrete examples of how needs of society sometimes act as driving forces for the development of mathematical ideas. This is a very ambitious goal, and since the history teacher focused more on religion and dynasties, the mathematics teacher felt that the students did not get opportunities to gain real insights into why mathematics was developed in interaction with the needs of society and culture. A less ambitious teacher would probably evaluate this part differently, pointing towards the fact that was explained above, that the students gained genuine historical knowledge about Egyptian mathematics situated in the proper historical context. Finally, the teacher concluded that the students afterwards showed signs of possessing a more mature and reflective approach to mathematics than they had before. Unfortunately, the teacher did not document this with observations from the classroom.

## 7. CONCLUDING REMARKS

The purpose of the paper was to present a theoretical framework for a systematic analysis of the uses of history for teaching and learning mathematics in order to propose a didactical transposition of history from the academic research subject to history in mathematics education. The analysis of the teacher’s report on the project work on Ancient Egyptian mathematics with respect to the described framework of different uses of past episodes shows that in this project, history was used in different ways to provide a very rich teaching and learning environment. The teacher used different approaches to history and used past episodes from various perspectives for different purposes, thereby creating learning situations that developed students’ historical awareness and mathematical learning strategies at the same time. History was used in ways in which students gained genuine historical insights, developed learning strategies, and enhanced their mathematical problem solving skills even though they worked on mathematics that might not be part of the core curriculum.

## NOTES

<sup>1</sup> I would like to thank Costantinos Tzanakis for helpful comments on an earlier version of this paper.

<sup>2</sup> A fourth pair of concepts is identity neutral vs. identity concrete history writing, which will not be used in this paper.

## REFERENCES

- Blomhøj, M. and Kjeldsen, T.H. (2006). Teaching mathematical modelling through project work – Experiences from an in-service course for upper secondary teachers. *ZDM*, 38, 2, 163-177.
- Carstensen, J. and Frandsen, J. (2002). *Mat 1*. Viborg: Systime.
- Epple, M. (2000). Genies, Ideen, Institutionen, mathematische Werkstätten: Formen der Mathematikgeschichte. *Mathematische Semesterberichte*, 47, 131-163.
- Fauvel, J., and van Maanen, J. (eds.), (2000). History in mathematics education – The ICMI study. Dordrecht: Kluwer.
- Frandsen, J. (1996). *Ægyptisk matematik*. Viborg: Systime.
- Fried, M.N. (2001). Can Mathematics Education and History of Mathematics Coexist? *Science & Education*, 10, 391-408.
- Jankvist, U.T. and Kjeldsen, T.H. (forthcoming). New Avenues for History in Mathematics Education: Mathematical Competencies and Anchoring. Accepted for publication in *Science & Education*.
- Jensen, B.E. (2010). *Hvad er historie*. Copenhagen: Akademisk Forlag.
- Kjeldsen, T.H. (forthcoming). A multiple perspective approach to the history of the practice of mathematics in a competency based mathematics education: history as a means for the learning of differential equations. To appear in Katz, V., & Tzanakis, C. (eds.) *Recent developments on introducing a historical dimension in mathematics education*, to be published in 2010.
- Kjeldsen, T.H., Pedersen, S.A. & Sonne-Hansen, L.M. (2004). *New Trends in the History and Philosophy of Mathematics*, Odense: SDU University Press, 11-27.
- Kjeldsen, T.H. and Blomhøj, M. (forthcoming). Beyond Motivation – History as a method for the learning of meta-discursive rules in mathematics. Accepted for publication in *Educational Studies of Mathematics*.
- Niss, M. (2004). The Danish “KOM” project and possible consequences for teacher education. In R. Strässer, G. Brandell, B. Grevholm and O. Helenius (eds.) *Educating for the Future*. (pp. 179-190). Göteborg: The Royal Swedish Academy.
- Rosenzweig, R. and Thelen, D. (1998). *The Presence of the Past. Popular Uses of History in American Life*. New York: Colombia University Press.
- Sfard, A. (2008). *Thinking as Communicating*. Cambridge: Cambridge University Press.
- Wulff, P. (2004). *Ægyptisk matematik*. Report from the in-service course: Problem oriented project work in, with and about mathematics, Roskilde University.

# **ESTABLISHING THE ‘METER’ AS CITIZENS OF FRENCH NATIONAL ASSEMBLY DURING THE FRENCH REVOLUTION**

Panayota Kotarinou, Charoula Stathopoulou, Anna Chronaki

University of Thessaly

*This paper discusses how a group of adolescent students has been engaged in a cross-curriculum project focusing on making explicit the historical background of establishing the ‘meter’ as a commonly used unit. The project was organised around a role-playing debate imagined to have taken place at the time of French Revolution. The potential of this particular project can be recognized at varied interrelated levels; a) in its effectiveness in students’ knowledge in Mathematics and Physics and in changing stereotypic images of Mathematic, and b) in supporting student’s creative and critical thinking in the realm of their involvement in social problems within a specific economic and historical context. Moreover we consider that students, in their involvement in this project, have the potential for cultivating communicative abilities and a stronger sense of citizenship.*

## **INTRODUCTION: HISTORY OF MATHEMATICS AND EXPERIENTIAL LEARNING**

The integration of the history of mathematics in teaching can support, enrich and ameliorate mathematics education processes as far as mathematical learning is concerned. Besides to mathematical learning, the development of beliefs (and positive attitudes) concerning the nature of mathematics and mathematical activity, as well as the recognition of Mathematics as a human cultural product can be encouraged through the history of the subject.

Tzanaki and Arcavi (2000, pp.203) refer to five main areas in which mathematics teaching may be supported, enriched and improved through integrating the history of mathematics into the educational process: a) the learning of mathematics b) the development of views on the nature of mathematics and mathematical activity c) the didactical background of teachers and their pedagogical repertoire d) the affective predisposition towards mathematics and e) the appreciation of mathematics as a cultural-human endeavour. According to Jankvist (2009), the arguments for using history are of two different kinds: those that refer to history-as-a-tool for assisting the actual learning and teaching mathematics and those that refer to history as-as-a-goal in itself which focus on the developmental and evolutionary aspects of mathematics as a discipline. Based on a review of the literature, one could point out that the historical approach in mathematics teaching, amongst other things, focused towards:

1. Disputing the mathematics myth as a sterile, static and insular science, projecting mathematics as humanitarian science whose development depends on social, historical and philosophical factors (Ernest, 1998).

## Working Group 12

2. Highlighting the interaction between Mathematics and society, revealing the ways, the norms and practices of various cultures, which influence Mathematics and vice versa how Mathematics influences the ways in which people operate, and think about, the world (Wilson, 2000).
3. Exemplifying and making explicit the genesis and origins of some ideas and procedures (Furinghetti, 98)
4. Supporting the learner to appreciate that heuristic processes, conjectures, doubts, mistakes insufficient claims and proofs, and impasses, all make constitutive pieces of Mathematics creation, and hence an inextricable piece of what Mathematics is (Tzanakis, 2009: pp. 20).
5. Stressing the close and two-way interrelation of mathematics and physics, through their historical evolution ( Tzanakis & Thomaidis, 2000; Tzanakis, 2000)
6. Showing the interdisciplinary nature of mathematics – its relevance to every aspect of human life— (Ernest 1998).
7. Integrating elements of different curricular subjects, through the interdisciplinary projects in which History of Mathematics plays an important role (Furinghetti, 97; Furinghetti, 98; Kotarinou & Stathopoulou, 2009)
8. Providing us with ways to introduce students to the beauty, utility and abstraction of Mathematics (Wilson, 2000)
9. Supporting the introduction of notions and methods, the necessity of which we find difficult to explain to the students.
10. Providing students with the opportunity to ascertain that famous mathematicians not only followed the similar way of thinking, but had done the same mistakes as students (Furinghetti, 2003). [1]

History of mathematics can also contribute towards experiential learning. Experiential learning theory defines learning as "the process whereby knowledge is created through the transformation of experience. Knowledge results from the combination of grasping and transforming experience"(Kolb 1984, p. 41, in Kolb et al, 2000). Experiential learning permits student learning in an effortless, constant and permanent way through the 'real' experience and in participating in activities that concerns him and that he finds interesting. Some ways of experiential learning through the history of mathematics, are: enactment of theatrical plays (Fraser & Koop, 1981) dramatization of different themes from History of Mathematics (Hitchcock, 1998; 1999; 2000; Lawrence 2000; Prosperini, 1999) story-telling from History of Mathematics (Selby, 2009; Schiro, 2004), educational programmes inspired from history of Mathematics in Museums, archaeological or historical places, role-playing (Perl and Christner, 1982; Hallenberg, 1995), debates (Furinghetti, 97), historical debates based in a mathematical issue (Bartolini, Bassi & Mariotti, 1999a, 1999b; Katz, 1997)

In this paper we shall provide a review of our research which aimed to explore the use of Drama in Education and specifically the role-playing debate with a theme from the history of mathematics: 1) in helping students realize the relationship of mathematics and science in a socio-cultural-historical context; and 2) in assisting students to understand the mathematical and/or scientific notions of a selected historical case. For this reason an interdisciplinary project has been conducted about the processes of establishing the ‘meter’ as a commonly used unit during the French Revolution. Among other activities a role-playing debate was carried out, which was a simulation of the confrontation in the French National Assembly during the French Revolution and was concerned with the choice of a unified unit of length measurement.

## **DEBATE AND ROLE-PLAYING: THEIR CONNECTION WITH HISTORY OF MATHEMATICS**

Drama in Education is a highly structured pedagogical activity that utilises exercises and techniques of dramatic art, in which the emphasis is on the procedure and not on the final product (Alkistis, 2000). It constitutes an experiential approach to teaching and aims at collaborative, active learning through experience while giving the child the opportunity to develop acceptance, understanding, creativity, curiosity, the ability to express themselves, self-consciousness, and skills in team work (Wagner, 1999). Among other techniques of Drama in Education is Role playing. Role playing is a technique in which students adopt a particular character putting themselves in the same position as him, trying to think what would that imaginary character have said, thought or felt (Neelands 1998).

As has been noted through bibliography, this technique is often supplemented with the use of debate. The use of debate in teaching can facilitate knowledge acquisition as well as the development of critical thinking (Huber, 2005; Snider 2006). Furthermore, participation in activities of this type familiarizes children with the holistic handling of issues, makes them able to justify their point of view, practices their mental flexibility and alertness, and enhances their ability of self-criticism and their conciliatoriness. At the same time they are given the opportunity to exercise their mind by producing and communicating ideas. Debates about mathematical discoveries can help students understand that old paradigms of thought were successful because the theories worked in the context of the old world, not because people of that day were ignorant (Snider, 2006 pp.230). Debates can also enable teachers to bring the historical contexts of mathematical theory to real life.

More specific, the role playing debate that simulates historical events, can revive the historical context within which a science theory (or technique) was developed (Snider 2006). Role-playing emphasizes the dimension of Science as “process” and as “social institution” makes Science education more effective (Alkistis, 2005). In addition, through the research required for the debate, students have the opportunity

to get to know the historical and social context within which a certain scientific theory emerged and generally comprehend that science, society and culture are interrelated.

## METHODOLOGY

The project was carried out in the 2<sup>nd</sup> State Lyceum of Ilion in Greece, in a low social and economical area of Athens, during school year 2007-8, with two classes of 11<sup>th</sup> grade [2] students (with 22 and 23 students) in order for them to approach through experiential learning an historical event concerning the establishing of the meter [3] as a unified unit of measurement. This project aimed to discuss notions in Mathematics and Physics and the connection of the two disciplines, the change of stereotypic images of Mathematics by the students, the students training in argumentation skills, the students development in creative and critical thought and the students cultivation of communicative abilities and of a stronger sense of citizenship.

We chose the model of collaborative teaching practice which was based on the reiterative cycle of planning - researching - sharing resources - teaching collaboratively - and finally assessing the outcomes of a lesson (Lawrence, 2008).

## THE INTERDISCIPLINARY PROJECT

**Historical events:** By the end of the 18<sup>th</sup> century the diversity of weights and measures in France was held responsible for great problems in economical dealings as well as the exploitation of people by feudal lords. Embezzlement, fraud, injustice, arbitrariness were ascribed to the diversity of about 2.000 different units of measurement in all France. Things were made worse because majority of the population was illiterate and thus incapable of making conversions amongst different measuring units. Due to the aforementioned reasons standardization of measuring units was one of people's basic demands. The establishment of new units consisted a political decision. From the first year, the French National Assembly voted the uniformity measurement units and sought new ones. In 1790, the French National Assembly accepted the definition of the meter as the length of pendulum that has a period of 2 sec at latitude of 45°, and asked the Academy of Science to propose the base of the metric system. The Academy responded to this request and recommended a decimal system of measurement. In 1791 a committee of the French Academy of Science—Lagrange, Laplace, Condorcet, Borda and Monge—suggested that the new definition for a meter be equal to 1/ 10 millionth part of the quadrant of the terrestrial meridian between Dunkirk and Barcelona and this was accepted by the National Assembly. The unit was given the name “meter” in 1793.

### **Placing the drama project in practice: ‘sensitive pendulum or heavy earth?’**

During this project, a role-playing debate was organized, which was a simulation of the confrontation in the French National Assembly. This particular debate entitled “The sensitive pendulum or the heavy earth?” was carried out twice with 11<sup>th</sup> grade

students of two different classes. One class had previous experience with activities in the use of Drama techniques, in contrast with the other class in which the students had had no similar experience. A class that wasn't involved in the project attended each debate. This confrontation concerned the choice of a length measurement unit, through the two aforementioned different approaches. The topic of the debate was selected to show students the confrontation, the protagonists and the historical context within which the unit 'meter' was introduced to help them comprehend that scientific theories are result of both intellectual and social interaction.

Preparing the 'debate' seven teaching hours was required over a period of three weeks. In history-class, a PowerPoint presentation concerning the establishment of the meter, helped students to realize the problems stemming from the use of many different units of measurement. Extracts from a documentary film about the French Revolution introduced them in the ambience existing before and during the first two years of the French Revolution. During the subsequent discussion, pupils started pondering why the choice of weighs and measures by feudal lords was a privilege for them. They finally found out that the change of size of the unit of volume resulted into an increase of taxes. The role of scientific unions concerning their decisions not only in scientific subjects but in social and political ones has been discussed.

During the language-class, the teacher helped students translate and understand texts concerning the origin and life, as well as the ethical and political role of scientists and other historical issues emerging in this revolutionary period.

The students were divided into 6 groups and had to locate their arguments for preparing a 'debate', reading extracts from Denis Guedj's book "*Le metre du monde*". Each group chose a representative to take part in the debate. The teachers in charge were present all the time to support the teams in their work. The students' ability to argue about the choice of the pendulum is directly correlated with their comprehension of its governing laws. Therefore, the students were taught in physics-class the pendulum principles and were also familiarized with the notion of pendulum isochronisms phenomenon. In the debate, for a suitable atmosphere to be created, simple settings and costumes have been used. In order to find out the way the members of the Assembly were dressed and how they spoke, the children watched David's famous painting 'The Oath of the Tennis Court' and dramatized scenes from a documentary with Robespierre speaking in French Assembly.

In the debate all students adopted the role of a responsible citizen, member of the French Constituent Assembly, who had to take decisions about crucial matters, in this specific historical context. Certain students played the role of historical figures as Talleyrand, Bailly, Prieur, Condorcet, Borda. Prieur spoke in favor of establishing new units; Borda presented the proposition of the French Scientific Academy, while Talleyrand spoke in favor of the pendulum and Condorcet in favor of the meridian. Some students, as members of the Assembly spoke either for the pendulum or for the meridian. During the debate students participated vividly either acclaiming or



disapproving the speakers. Before reading the real decision of the French Assembly, all of them- in roles of citizen members of the assembly- voted in favor of the meridian as the most appropriate unit of measurement, without knowing the real decision of the French Assembly.

## **PRELIMINARY DISCUSSION**

The project has been evaluated through the analysis of students' answers to a questionnaire and of particular episodes of the process as they were captured by means of videotaping the debate performance. The reason for distributing a questionnaire was primarily to identify whether the project had any impact on students' knowledge concerning the measuring units. The majority of students, even a month after the debate, answered correctly to questions involving the establishment of the meter (27 from the 45 students answered correctly about the meridian definition of the meter) and the laws of pendulum (28 students answered correctly that the period of swing of a simple gravity pendulum depends on its length and the local strength of gravity and that it is independent of the mass of the bob, while 17 gave wrong answers or didn't answer at all). The students emphasized that this activity engendered the development of critical thought, the familiarization of a more rounded approach, active participation and cooperation with each other. More importantly that what they learned they emphasized the way in which they learned: they pointed out that they all took part, worked together, took the initiative and felt enthusiastic. Furthermore they liked the seriousness with which the debate was carried out; the good organization and the role-play which made them believe that they really took part in the French National Assembly.

### **Analysis of the videotaped debate performance**

A preliminary analysis of the videotaped debate performance permitted us to stress a number of episodes related to processes of knowledge construction and values building such as; discussion of mathematics and physics, training their skills in argument, and gaining a sense of becoming responsible citizens. Each of these are briefly analysed below.

**Discussion about mathematics and Physics notions:** Students have discussed about arbitrary and conventional measures, connecting the conventional unit of length measurement with justice. From their arguments in the debate we concluded that students had understood the laws of simple pendulum as they referred - in their argumentation - to the dependence of the period of pendulum on  $\pi$  and on  $g$  - the acceleration to the gravity [4].

**Training in argumentation skills:** Both sides had to prepare strong arguments to support the definition of the meter. The team in favor of the pendulum used the argument that pendulum was a rational solution, simple, cheap in construction and easy and functional in use. The second team claimed that the definition of the meter

as a part of a meridian was a global solution, not easy and cheap but yet an accurate and reliable one, as not depending on numbers that in real life can't be calculated with accuracy.

Quotes from student-in- role of Talleyrand argumentation in favour of the pendulum

...Laws of pendulum are dictated by Newton laws and because we are now in a rationalism era we believe that this is one of the best solutions...Laws of pendulum are very simple. Its movement depends only on the length of the rope and the geographical area. The weigh that you see in the end of the rope doesn't play any role (He explains the pendulum laws pointing out on a pendulum in movement). We can then define as unit of measurement the meter, equal with the length of pendulum which does this movement forward and backward in 2 seconds. It's something very simple, it can be realised in front of people, it's very economical and easy...

Quotes from student-in- role of Condorcet argumentation in favour of the meridian

...You have mentioned that we live know in an era that we must be rationalists. Can the measurement that you have proposed, be a product of rational and realistic thought? You have mentioned some other factors to which I will now refer. More precisely you have referred to the acceleration of gravity and to the number  $\pi$ , the well-known 3,14.... I want to remind you that all these numbers have infinite digits ... Is it possible that the meter that we will use in all our life, for all the time from now and then, for all next generations, will not be an exact number? Is it allowed such a thing? How is it possible for the meter to be approximated in such a way? As citizen Talleyrand correctly mentioned before, the acceleration of gravity changes from place to place. Is this not enough for us to reject it? Will France have a different acceleration of gravity, Spain another one, and Russia too? Will we have then a different meter for every different place in the earth? How is it possible for us to accept it?

**Fostering the sense of citizenship:** Through the whole procedure, the value of public (and responsive) dialogue has been brought out. During the introduction of the debate, the necessity of innovation 'in order for the people stop being victims' has been underlined. Also, the exploitation of people by feudal lords though the use of arbitrary units of measurement has also been emphasized by the students. The need to release from king's units has been emphasized, as they were local and also being defined in an arbitrary way. Students correlated the common meter with human rights while both teams have argued that the 'meter' had to be defined in a way that can be understood by every citizen.

...I would like to agree with Condorcet aspect, as pendulum is a phenomenon, which is encountered with a lot of formulas and notions that any person in France isn't obliged to know. So, how is it possible for any simple tradesman or farmer to measure the period as it was mentioned before? How are they obliged to know the oscillation?

...Pendulum is based in a physical phenomenon. Everyone of us can use it simply, as many times he wants, quickly and with precision, something that is quite impossible with the method of meridian.

### SOME CONCLUDING REMARKS

The experiential dimension of learning, considered as a fundamental component in alternative approaches of teaching, is absent or has a decreasing role in traditional teaching. In the aforementioned project we have seen students involving in an experiential way –with role playing- in a more complicated process than a traditional lesson and we have seen them cultivate multiple aspects of their selves and their creativity. Apart from the students their facing different subjects in the same activity, they saw another image of Mathematics. Contrary to a conventional lesson which emphasises solving of formal problems, -often de-contextualised - students faced a non direct mathematical problem but a problem with mathematical notions. The social conditions determining the context have appeared to also determine decisions, while the scientists' social responsibility, as people who play active roles in society, came into the fore. According to Ernest (2008) the adoption of mathematics as a cultural construction, as much from a historical perspective as from the perspective that examines knowledge in relationship to the context, can endow a human element to school mathematics once more. We believe that with all the aforementioned activities this aspect of mathematics came to the forefront. With the role-playing the central role of historical and social context and ways mathematics could be utilized in was brought to light and students were able to experience this dimension of mathematics, not only mentally but emotionally and physically. We think that there is a need to reconsider research related to ways in which Drama in Education techniques can contribute to an experiential learning of history of mathematics and mathematics in general.

### NOTES

1. We consider that the first seven from the aforementioned benefits -for integrating the history of mathematics into the educational process - are the most relevant for the study presented in the present paper.
2. The necessary knowledge for the project -the 'simple pendulum' and 'the French Revolution' - is included in the 11th grade Physics and History curriculum.
3. The 11<sup>th</sup> grade Euclidean geometry curriculum includes the notion of a straight line segment measurement. In the school book there is a reference of the arbitrariness of the measurement unit. This reference gave us the opportunity for the project concerning the establishment of the meter.
4.  $T=2\pi(1/g)^{1/2}$ , T=The period of a simple pendulum, l = length of the pendulum

### REFERENCES

Alkistis (2000). *Drama in Education*, Ellinika Grammata, Athens.

## Working Group 12

- Alkistis, and Vavougiou, D. (2005). "Theatrical Activities and the Teaching of Science. Presentation of a dialectical relation", *Teaching of Science: Research and Practice* 14, pp.06-12.
- Bartolini Bussi, M., Mariotti, M. (1999a). Semiotic Mediation: from History to the Mathematics classroom, *For the Learning of Mathematics*, 19, 2.
- Bartolini Bussi, M., Mariotti, M. (1999b). Which is the shape of an Ellipse? A cognitive analysis of an Historical debate, *Proceedings of PME-XXII*, Stellenbosch, South Africa, July 12-17, vol. 2, pp.64-71.
- Ernest, P. (2008). What does the new philosophy of mathematics mean for mathematics education, *Proceedings of 25<sup>th</sup> Panhellenic Congress in Mathematics Education of Hellenic Mathematical Society*, Volos 21- 23, November 2008.
- Ernest, P. (1998). The History of Mathematics in the classroom, *Mathematics in school*, 27, no.4. pp. 25-31.
- Fraser, B., Koop, A. (1981). Changes in Affective and Cognitive Outcomes among Students Using A Mathematical Play, *School Science and Mathematics*, v81 n1 pp.55-60.
- Furingueti, F., Domingo, P. (2003). History as a crossroads of Mathematical Culture and Educational needs in the Classroom, *Mathematics in School*, 32 no1, pp.37-41
- Furingueti, F., Somaglia, A. (1998). History of Mathematics in School across Disciplines, *Mathematics in School*, 27, no4, pp.48-51.
- Furinghetti, F. (1997). History of Mathematics, Mathematics Education, School Practice: Case Studies in linking Different Domains, *For the Learning of Mathematics*, 17,1.
- Hallenberg, H. (1995). Mathematics in History, *Montessori Life*, v7, n2 pp.30-32.
- Hitchcock, G. (2000). A Window on the World of Mathematics, 1871: Reminiscences of Augustus De Morgan—A Dramatic Presentation, in Katz J. Victor (ed.) *Using History to Teach Mathematics: An International Perspective*, The Mathematical Association of America, pp.225-230.
- Hitchcock, G. (1999). Dramatizing the Birth and Adventures of Mathematical Concepts: Two Dialogues, in Calinger Ronald (ed.) *Vita Mathematica: Historical Research and Integration with Teaching*, The Mathematical Association of America, p.27-41.
- Hitchcock, G. (1998). Entertaining strangers: a dialogue between Galileo and Descartes, in Shaffer S. E. (Ed.) *Comparative Criticism: Philosophical Dialogues*, Volume 20, pp.63-85.
- Huber, R. (2005). *Influencing Through Argument*. New York: International Debate Education Association.
- Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education, *Educational Studies in Mathematics*, 71, pp.235-261.
- Katz, V. (1997). Some ideas on the use of History in the teaching of Mathematics, *For the Learning of Mathematics* 17, 1.

## Working Group 12

- Kolb, D., Boyatzis, R., Mainemelis, C. (2000). Experiential Learning Theory: Previous Research and New Directions, in R. J. Sternberg and L. F. Zhang (Eds.), *Perspectives on cognitive, learning, and thinking styles*, NJ: Lawrence Erlbaum.
- Kotarinou, P., Stathopoulou, Ch. (2009). “The Sand Reckoner”: an interdisciplinary project for integrating different subjects in Lykeio (Upper High School), *3<sup>d</sup> Panhellenic Congress of the Association of Researchers of Didactics of Mathematics*, Phodos, October 29-31, 2009, p.331-341
- Lawrence, S. (2008). What works in the classroom-project on the history of mathematics and the collaborative teaching practice, *Proceedings of CERME 6*, WG 15, Lyon France, pp. 2742-2751.
- Lawrence, H. S. (2000). A Visit from Pythagoras--Using Costumes in the Classroom. *Mathematics Teacher*, v93, n8, pp. 652-55.
- Neelands J., Goode, T. (1990). *Structuring Drama Work*. Cambridge University Press.
- Perl, T. H., Manning, J. M. (1982). *Women, Numbers and Dreams. Biographical Sketches and Math Activities*, National Women's History Project (ED279510)
- Prosperini, L. (1999). ‘Matematica a teatro: perché no?’, *Didattica delle Scienze e Informatica nella Scuola*, a.XXXV, n.199, pp.28-34.
- Selby, V. (2009). Storytelling adds meaning, *Mathematics Teacher*, vol.102, n.8.
- Schiro, M., Lawson, D. (2004). *Oral storytelling and Teaching Mathematics*, SAGE publications.
- Snider A., Schnurer (2006). *Many Sides: Debate Across the Curriculum*, International Debate Education Association, New York.
- Tzanakis, C. (2009). The utilization of relations between History of Mathematics and Mathematics Education, in *The exploitation of History of Mathematics in Mathematics Teaching*, Scientific Union for Didactics of Mathematics, Ziti Publications, Thessaloniki.
- Tzanakis, C., Arcavi, A. (2002). “Integrating history of mathematics in the classroom: an analytical survey”, convenor with A. Arcavi of ch.7 of the John Fauvel & Jan Van Maanen (eds) *History in Mathematics Education ICMI Study Book*, pp.201-240, Kluwer.
- Tzanakis, C., Thomaidis, Y. (2000). Integrating the close historical development of Mathematics and Physics in Mathematics Education: Some methodological and Epistemological remarks, *For the Learning of Mathematics*, 20, 1.
- Wagner B. (1999). *Dorothy Heathcote: Drama as a Learning Medium*, Heinemann, Portsmouth.
- Wilson, P., C. (2000). Who? How? What? A strategy for using History to teach Mathematics, *Mathematics Teacher*, 93, no.8.

# LESSONS FROM EARLY 17TH CENTURY FOR CURRENT MATHEMATICS CURRICULUM DESIGN

Jenneke Krüger

SLO (Netherlands Institute for Curriculum Development), Enschede, Netherlands

*The Dutch Engineering School at Leiden University (Duytsche Mathematique) is the first example in the Netherlands of a professional training for engineers and surveyors, connected to a university. The Instruction (the formal curriculum) was provided by Simon Stevin in 1600. A manuscript from 1622, 'Mathematische Wercken', presumably lecture notes by Frans van Schooten the Elder, is clearly based on the Instruction. Stevin and Van Schooten designed an up-to-date curriculum, which seems to have been successful for about 65 years. Some reasons for its success and decline are still meaningful for curriculum design in mathematics education today.*

## RESEARCH

The motivation for this research originates in the heated discussions before and during recent revision of the Dutch national study programmes for mathematics at secondary level. The upheaval was at least partly caused by differences of opinion on which topics to include and to which depth. Attempts to influence the content based on personal points of view were noticeable. An obvious question, also prompted by the recent interest in links between history and mathematics education, is whether and how history can provide inspiration and reference material for those involved in the design and evaluation of mathematics curricula. The case presented in this paper forms part of a research for a PhD thesis on the history of mathematics education.

The main research question is

*Which are the factors and actors that influence to a high degree the content of mathematics curricula?*

The following questions structure the research more specifically.

- Which motivations and whose ideals are influential on the content of the formal curriculum?
- Which factors and which people determine the interpretation of the formal curriculum and its implementation?
- Which factors and which people are important for successful implementation of a curriculum?

## Method

Analysis and comparison of three cases, considering influences on the formal curriculum and the relative success of each of them, will result in a preliminary list

of conditions, which are important for successful curriculum design and development. Comparison of these conditions with recent curriculum design and development in mathematics education should result in a list of criteria which are decisive for successful mathematics curricula today.

The three cases in Dutch history of mathematics education, in the seventeenth, eighteenth and nineteenth century are described as follows.

1.The *Duytsche Mathematique*, established as a school for Engineers at Leiden University in 1600 shows the first known example in the Netherlands of a formal curriculum. It was especially successful from 1600 - ca 1665.

2.In the second part of the eighteenth century, the three Foundations of Renswoude offered professional education to talented poor orphans. Mathematics was the main subject for all pupils at least during the first two years of their education. The focus of my research in this second case is the Foundation in Utrecht, operational from 1761. and still existing today.

3.In the second part of the nineteenth century, in 1863, the Dutch national government established the *Hogere Burgerschool* (HBS), a secondary school for children of citizens who would not enter university, but who were to take up higher positions in society or enter the Polytechnic Institute in Delft, the present Technical University of Delft.

In this paper some aspects of the first case, the *Duytsche Mathematique* are presented.

## THE DUTCH REPUBLIC

On 23 January 1579, four provinces in the Low Countries signed the Union of Utrecht, a treaty to establish a military league to fight the Spanish armies. During 1579 and 1580, more provinces and towns signed the treaty, which is considered the foundation of the Dutch Republic (the Seven United Provinces). The States General, established in The Hague, formed the government of the Republic, and the 'stadtholders', elected by their provinces, became powerful political leaders. In 1584, the leader of the rebellion, Prince William I of Orange, was assassinated in Delft, and the following year his son Prince Maurits of Orange, aged 18, became stadtholder of Holland and Zeeland. From 1590, Maurits was commander of the army of the States and successful in (re)capturing a number of fortified towns. He was interested in mathematics and its applications and was well aware of the importance of the application of mathematics to modern warfare.

From about 1590, the economic situation improved considerably: trade and navigation increased, and some towns grew to four or five times their earlier size during the early years of the seventeenth century. The political and economic situation led to a high demand for qualified surveyors and engineers, who were able to design and build modern fortifications, but who also could map the country or had

other surveying skills. Town expansions were necessary to house the fast growing population, though during the first part of the seventeenth century this was secondary to fortification (Maanen, 1987; Struik, 1981). Surveyors and engineers, both military and civil, needed mathematical knowledge and skills.

Italian architects and engineers became very influential in the Netherlands: in the first half of the sixteenth century, Italian advisors were responsible for improving the defensive walls of towns. Their ideas also became widely known through publications of Italian mathematicians such as Tartaglia, Marcaurelio de Pasino, Castriotto, Serlio, and others like Daniel Speckle and Simon Stevin (Maanen, 1987; Metin, 2007). In the *Stercktebouwing* (1594) Stevin described a system of fortification based on the Italian system with polygons as plan, and pentagonal bastions. Any engineer who applied this system had to have knowledge of geometry and trigonometry. Surveyors and engineers should also be able to measure land and determine area, subdivide land into proportional parts, produce maps, determine the height of towers, the width of water, or the volume of dykes, both on accessible and inaccessible land (Gulik-Gulikers 2005). The use of a reliable trigonometric table could improve the results of surveying techniques considerably. The extensive calculations that were necessary could be simplified using decimal fractions, which were not yet commonly used by mathematical practitioners (Struik, 1995). This is also apparent in publications and manuscripts from that time, see Sems and Dou 1600; Ceulen 1615; Marolois 1628 and manuscripts BPL 1351; BPL 1970; BPL 2084. Stevin did much to promote the use of decimal fractions in *De Thiende* (1585).

Thus in order to meet the demands of the developing nation there was a great need for good mathematical education. In the Dutch Republic there was no centralized educational system. Schools were the responsibility of town councils, and in addition there were private schools or private teachers. There was no standard curriculum, nor any form of quality control (Krüger, 2010).

### **THE DUTCH ENGINEERING SCHOOL AT LEIDEN UNIVERSITY (1600–1679)**

In 1600 a school for military engineers was attached to the University of Leiden at the request of Prince Maurits, with a separate programme, devised by Simon Stevin and taught in Dutch instead of Latin: *Duytsche Mathematique* (Dutch Mathematics) or Engineering School. The School taught both surveying and fortification. Thus, practical mathematics of high quality became available to those who did not have access to university education. Both Maurits and Stevin, but also Rudolf Snel, professor of mathematics at Leiden and at least one of the curators, Jan de Groot, valued a combination of theory and practice and thought mathematics very relevant to engineering.

The intended programme by Simon Stevin is known as the *Instruction*<sup>1</sup>. In this early example of a formal curriculum, Stevin unambiguously connected content and



## Working Group 12

working methods with the purpose of the School: training of engineers for the Dutch army.

The *auditores* will be trained to serve the country as engineers as quickly as possible. To this end they shall learn arithmetic or counting, and surveying, but of both only so much as is necessary for a common engineer. Those who have come this far are allowed to study more in depth if they wish to do so. This is the general outline, specifications are the following.

The *Instruction* of Stevin specified in more detail what the content of the programme should be.

Arithmetic, including the four operations in whole numbers, rational numbers, and decimal numbers; also the rule of three in those three types of numbers
Surveying on paper, that is calculating area with the use of decimal numbers
Measuring a circle, parts of a circle and area, ... subdividing rectilinear figures and curvilinear figures into several parts, such as triangles or other figures, checking calculations
Measurements on paper of dykes and learning how many ‘schachten’ or feet the works contain
Fieldwork, learning how to use proper tools
Mapping on paper what is measured in the field and the reverse, from a map setting out stakes in the field
Fortification, learning the names of the parts from wooden or earth models, making maps of towns, drawing the perimeters of forts or towns with four, five, or more bastions and staking them out in the field

**Table 1: Main items of *Instruction* by Simon Stevin**

The first known interpretation of the *Instruction* is a manuscript from 1622, *Mathematische Wercken* (*Mathematical Works*) by Frans van Schooten (the elder), who taught *Duytsche Mathematique* from 1611 to 1645 (Maanen, 1987). *Mathematische Wercken* is a set of lecture notes that presumably served as background for teaching. The manuscript gives insight into the level of mathematics taught and an indication of the didactical methods used at the time at *Duytsche Mathematique*.

Frans Van Schooten (1582 - 1645) assisted Ludolf van Ceulen, professor from 1600 - 1610 and continued teaching after Van Ceulen had died, though without a regular salary. He also worked as a surveyor and engineer for the army. In the archives of the University for the Period 1611 to 1614 there are references to payments made to him and to the provision of four wooden instruments for the teaching of mathematics (LdnUL Arch. Cur. 42). There are also requests from his students for him to be made

van Ceulen's successor. He finally was appointed professor in 1615 (LdnUL Arch. Cur. 42).

## MATHEMATISCHE WERCKEN: AN INTERPRETATION OF THE INSTRUCTION

**Fig. 1: Measuring the distance between two mountain tops A and B, from a base CD.** *Mathematische Wercken*, f 83<sup>r</sup>

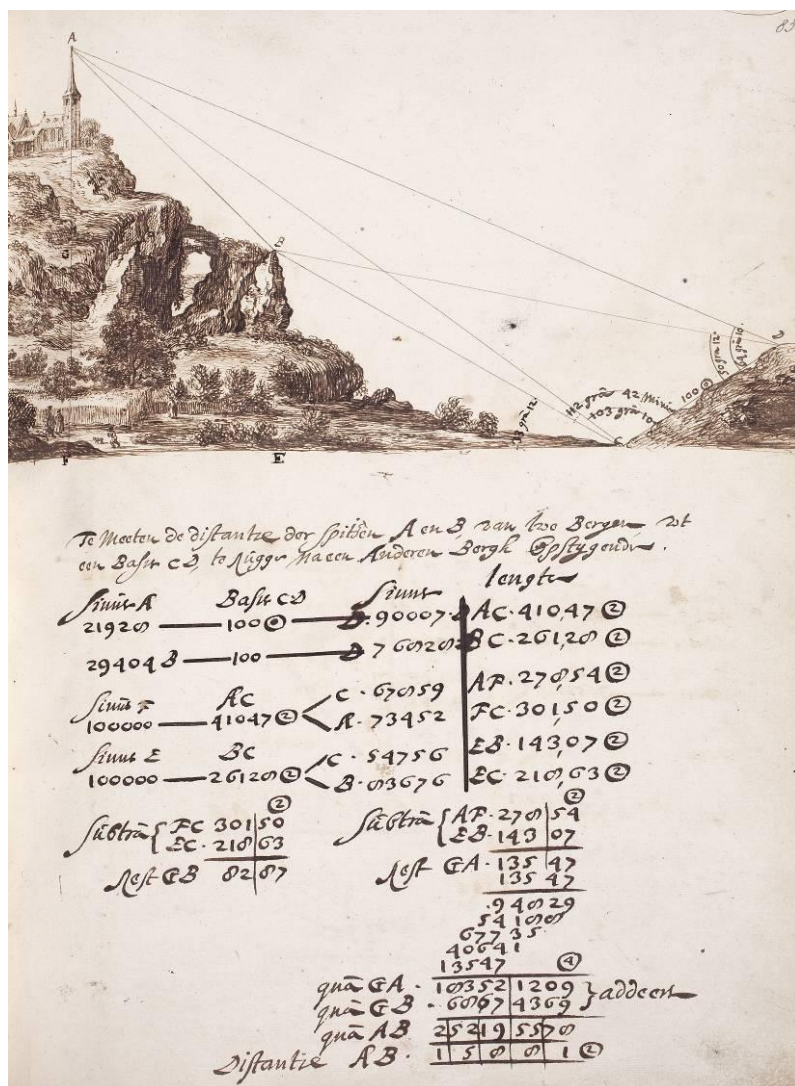


Figure 1 shows one of the exercises from the second half of the manuscript (measuring heights). Looking at the solution presented, there are quite some steps to be thought of and performed by the students in order to get to the required solution. Some virtual points are included (F, E and G). The sine rule is applied repeatedly, in the form of the rule of three. Pythagoras' Theorem occurs once. Van Schooten used an effective notation for decimal numbers, which was not at all common at that time.

When comparing the main content of *Mathematische Wercken* by Van Schooten (table 2) with the *Instruction* by Simon Stevin (table 1) it is obvious that Van Schooten took the *Instruction* as guidance. Both started with arithmetic,

including the use of decimal notation. Stevin, but clearly also Van Schooten, saw the importance of using decimals, e.g. to diminish the length of calculations. Algebraic equations, conic sections, and similar subjects which were at the time important in mathematical research were considered of less use to surveyors and engineers, thus they were not included in the programme. But Van Schooten also omitted subjects which were presumably for his students not necessary, such as the four operations in whole numbers, which were mentioned in the *Instruction*. He added subjects which were helpful to an engineer or surveyor (extraction of square and cube roots, use of trigonometric tables).

Arithmetic: extraction of roots, decimal numbers, calculation of area
Geometry: definitions and axioms, propositions from Euclid, constructions, transformations
The practice of surveying: preparation, measuring distances in accessible land, and calculations
Use of trigonometric tables, measuring in inaccessible lands, making maps, measuring heights (or depths), also measurements without use of tables
Solids: calculations on all kind of shapes and materials, calculating content
Fortification: definitions, plans of fortifications, bastions, calculations

**Table 2: Main content of *Mathematische Wercken* by Frans van Schooten**

The students who came to the lectures of the Duytsche Mathematique were of different background: they might know only the basics of arithmetic or they might be surveyors who wished to improve their knowledge and skills, or craftsmen like carpenters and bricklayers, or university students who wished to study engineering. Van Schooten started with the explanation of square and cube roots, and the explanation of the decimal number system used in surveying, both subjects were linked to geometry (see Figure 2). The manuscript as a whole shows some striking features from the point of view of teaching methods. Strong coherence, elaborate use of visualisation and relevant contexts alternating with rigorous training in mathematical skills without contexts suggest an author who had a good knowledge of the relevant mathematics, who had thought about teaching methods and enjoyed teaching.

Examples of coherence are: consistent use of decimal notation whenever relevant; the use of nearly every element of the part on fundamental geometry (definitions, propositions of Euclid, constructions and transformations of figures) in the part on surveying techniques, in the calculations in three dimensions and in fortification and the repeated use of three main techniques in the notes on surveying (rule of three with sine, fig. 3, the tangent and rule of three with similarity, fig. 4). Throughout the manuscript, illustrations are used with every exercise and every definition, axiom and proposition. The illustrations are always very neatly executed and if necessary labelled with capitals, to facilitate understanding.

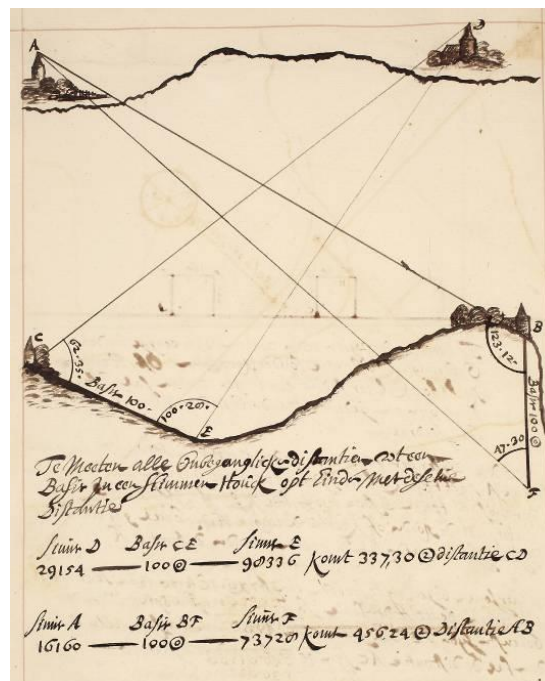
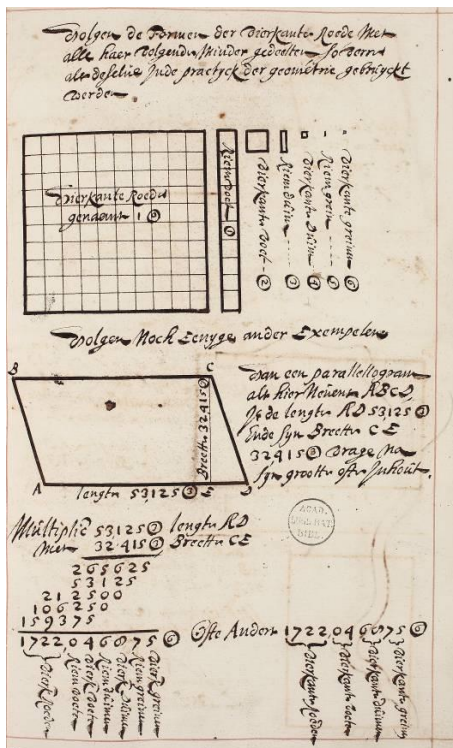
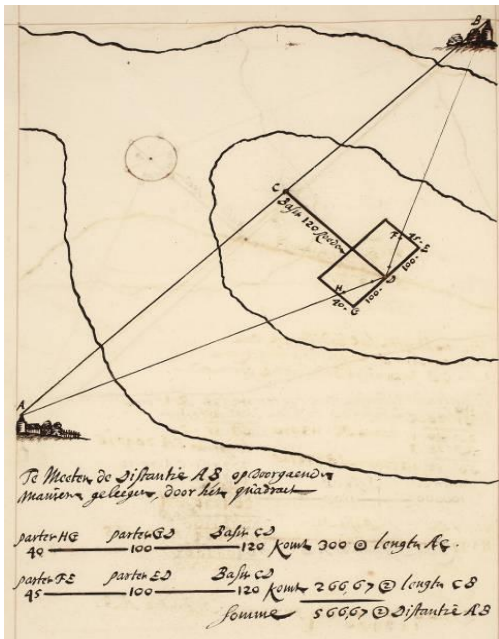
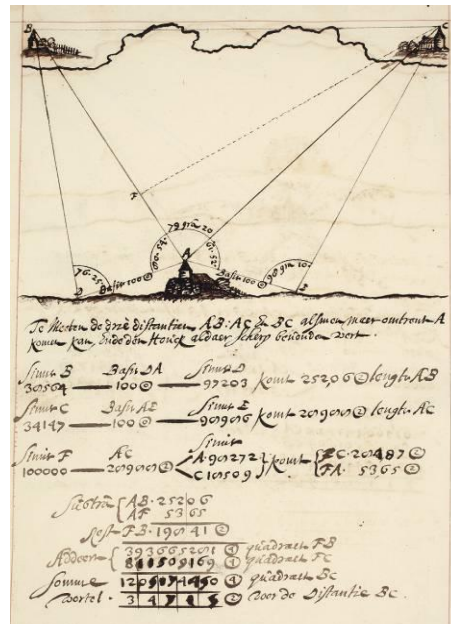


Fig. 2: decimals in surveying, f 45<sup>r</sup>      Fig. 3: rule of three with sine, f 57<sup>v</sup>

Some of the illustrations are of a much higher artistic quality than one usually sees in such texts (see Figure 1). Illustrations may represent a context, combined with the mathematical translation of the problem, for example in the part on surveying (Figures 3 -5). They are also used to facilitate understanding of a new concept (fig. 2).

When van Schooten's lecture notes are compared with publications on the same subjects (Ceulen, 1615; Marolois, 1628; Sems and Dou, 1600; Stevin, 1594; Stevin, 1605) it is clear that none of these authors treated all these topics in one book. Furthermore the books have far more text and less visualisation, for obvious reasons. In his treatment of Euclid, van Schooten is more academic than Sems and Dou in their *Practijck des Lantmetens*. Because of his use of decimal fractions, his calculations are shorter than those of van Ceulen or Marolois. The latter did not use trigonometric tables either. Manuscripts from the seventeenth century on these subjects by other authors are less coherent in the treatment of the subject, the illustrations are of lesser quality, the use of decimal notation is either absent or occasional (Author A?, 1658; Author A? & Cardinael, 17th century; Kechelius, 1655).



**Fig. 4: similarity, f 58<sup>v</sup>**

**Fig. 5: combination of techniques, f61<sup>v</sup>**

The Duytsche Mathematique at Leiden University was successful until about 1668: as is evident from number and background of students (Du Rieu, 1875; Witkam, 1967), repeated mentioning as an example worthwhile to follow and reputation of professors (Krüger, 2010). From about 1668 it went into fairly rapid decline. It was closed in 1679 and resurrected a few years later, but never again became as successful as during the early seventeenth century.

## DISCUSSION AND (PRELIMINARY) CONCLUSIONS

Some factors and actors which influenced the success and decline of this programme are discussed in the remaining part of this paper.

The Engineering School was established in 1600 at the request of an at the time very influential person, prince Maurits, who himself had studied at Leiden University and personally knew the curators. Around 1679 there were no influential people who would take an interest in the Duytsche Mathematique.

The curriculum had a clear aim and was devised by a respected mathematician and engineer, Simon Stevin, who worked for and with Maurits. The content of the curriculum may be considered optimal as opposed to maximal, both in the formal curriculum and in the interpretation of 1622; modern techniques such as up-to-date trigonometric tables and decimal notation, were used if relevant for the aim of the programme. However the mathematical content and structure of the programme hardly changed during sixty years, and so gradually the programme became outdated.

There was a population of prospective students and there were ample employment opportunities for those who had studied Duytsche Mathematique. However during the seventeenth century more institutes offered courses in mathematics for engineers in Dutch language, also during the seventeenth century engineers only got temporary

contracts with the army, which may have diminished the attractiveness of the profession.

Teachers were capable and respected mathematicians and often also professional surveyors. However, the successor of Frans van Schooten, his son, Frans the Younger, was a respected mathematician, but he was not a professional engineer and as far as we know did not show much interest in this profession.

The teaching methods, level of accuracy and contexts were aimed at the requirements of the future professions of students; there was a good combination of theory, practice and fieldwork. After 1645 there is no mention of practical work.

The differences with present-day mathematics education are obvious, but from this case it seems likely that at least some aspects still are important today, both for curriculum design and for mathematics education.

- Agreement on and explicit formulation of realistic aims of a new mathematics curriculum before deciding on the content might well provide a better framework for the whole process of development, provided the formulated aims are kept in view during the process of curriculum design, as was the case with the lecture notes Frans van Schooten wrote around 1622.

More opportunity for teachers to design part of their own curriculum, with strong emphasis on quality, might well improve the teaching of mathematics. Structural links between professionals who use mathematics in their work and schools could help to improve both the teaching of mathematics and motivation of students.

- Even the best curriculum has to change with new developments in society and in technology, or risk becoming outdated.

## NOTES

1. This document and others concerning the Duytsche Mathematique are kept in the archives of the library of the University of Leiden (Arch. Cur. 20, 42); most of them are published in Molhuysen 1913 and later volumes.

## REFERENCES

Manuscripts (LdnUL means the library of Leiden University, Western manuscripts):

[Author, A?] (1658). *Inleidinghe tot de geometria* [Introduction to geometry]. LdnUL BPL 1970.

Author A?. & Cardinael, S.H. (17th century). *Geometria*. LdnUL BPL 2084. Arch. Cur. 20, 42. LdnUL.

Kechelius, S.C. (1655) *Beginnelsen der geometriae* [Introduction to geometry]. LdnUL BPL 1351.

Schooten, F van (1622). *Mathematische Wercken* [Mathematical Works]. LdnUL BPL1013.

Publications:

- Akker, J. van den (2003). Curriculum perspectives: an introduction. In: J. van den Akker, W. Kuiper & U. Hameyer (eds.), *Curriculum landscapes and trends*. (pp. 1-10). Dordrecht: Kluwer.
- Ceulen, L.van (1615). *De arithmetische en geometrische fondamenten* [Fundaments of arithmetic and geometry]. Leyden: Joost van Colster en J.Marcus.
- Du Rieu, W.N. (1875). *Album Studiosorum Academia Lugduno Batavae MDLXXV - MDCCCLXXV*. Den Haag: Nijhoff.
- Gulik-Gulikers, I. van (2005). *Meetkunde opnieuw uitgevonden*. University of Groningen (Dissertation).
- Krüger, J. (2010). Lessons from the early seventeenth century for mathematics curriculum design. *BSHM Bulletin*.  
on line: <http://dx.doi.org/10.1080/17498430903584136>
- Maanen, J.A. van (1987). *Facets of seventeenth century mathematics in the Netherlands*. University of Utrecht (Dissertation).
- Marolois, S.(1628) *Opera Mathematica*. Amsterdam: Jan Janssen,
- Metin, F.(2007). *Adam Fritachs new Fortification*. Prague: workshop ESU5..
- Molhuysen, P.C. (1913). *Bronnen tot de geschiedenis der Leidsche universiteit. Vol.1.* Den Haag Nijhoff.
- Sems, J. & Dou, J.P.(1600). *Practijck des Lantmetens* [Practice of Surveying]. Leyden. Jan Bouwensz.
- Stevin, S. (1585). *De Thiende*. .Leiden: Christoffel Plantijn. Retrieved October 1, 2008, from Technical University of Delft. Retrieved from: <http://www.xs4all.nl/~adcs/stevin/telconst/10e.html#10>.
- Stevin, S. (1594). *De Stercktebouwing*.Leiden: Francoys van Ravelenghie.
- Stevin, Simon (ca. 1605) Meetdaet. In: *Wisconstighe gedachtenissen*. Leiden: Jan Bouwensz.
- Struik, D.J. (1981). *The land of Stevin and Huygens*. Dordrecht: Reidel.
- Struik, D.J. (1995). *Geschiedenis van de wiskunde* [History of mathematics]. Utrecht: Het Spectrum.
- Witkam, H.J. (1967) Jean Gillot (een Leids ingenieur). *Jaarboekje voor geschiedenis en oudheidkunde van Leiden en omstreken*, 1967, 29-47.

# HOW MUCH MEANING CAN WE CONSTRUCT AROUND GEOMETRIC CONSTRUCTIONS?

Snezana Lawrence and Peter Ransom

Bath Spa University, England

*This paper describes a way in which the mathematical heritage can be used to identify potentially 'rich' tasks undertaken by student teachers to deconstruct, and subsequently better understand, the meaning of mathematical concepts they already know and are expected to teach. It is based on a small-scale project undertaken in the South West of England and which is proposed as a pilot for a larger project to map the curriculum against such topics. The project has been generously supported by the National Centre for Excellence in the Teaching of Mathematics (NCETM)<sup>1</sup>.*

Of course, mathematics involves deductivity. Working with the slide rule and the protractor is no mathematics, measuring areas and volumes is no mathematics. But accepting a deduction is no mathematics either, unless you adhere to the interpretation of mathematics as a ready made subject.

Hans Freudenthal, *Geometry between the Devil and the Deep Sea*, 416.

## THE CURRICULUM CHANGES AND THE CULTURAL HERITAGE

Within the past few years a great number of changes have been initiated in the mathematics education in Great Britain<sup>2</sup>. The first of the most dramatic two of those changes was certainly the introduction of the new curriculum which was brought in as the old government prepared to leave the scene and the new was preparing to step in<sup>3</sup>.

One of the most important things for schools to get right is being identified now to be mathematics education<sup>4</sup>. But is it mathematics that we as a society are interested in, or the results of the PISA study<sup>5</sup>, the content of the curriculum, the pedagogy, the level of difficulty of our qualifications, or the power of teachers in classrooms?<sup>6</sup> We will argue in this paper that none of the above gives satisfactory answers to the better teaching and learning of mathematics, and that the disengagement of teachers from the actual mathematical content is the most important reason for disengagement of our students<sup>7</sup>. We will also argue that this is so because the teachers are not inducted in any way in the mathematical culture which they are supposed to transfer and transform through their own practice, either by 'doing maths' or by inducting their own students into that practice. It is our fear that whatever curriculum we have the same will be valid in this regard: when the student teachers engage in doing maths, they follow the 'heritage' path and are rarely aware of the origin of concepts and therefore their meaning. This paper will argue that the awareness of mathematical



cultural heritage, and a set of skills in identifying such heritage, is a necessary component in any preparatory teacher education. This process we identify also as a possible way into introducing the students to the history of mathematics which brings out all the well known benefits to their subsequent practice. (Grattan-Guinness, 2004).

## THE RESEARCH IMPETUS

Grattan-Guinness (2004, 174) describes the type of mathematics education ‘very much guided by heritage.’ But this heritage he identifies thus as one which brings out the

‘...reactions of students – including myself, as I still vividly recall – are often distaste and bewilderment; not particularly that mathematics is very hard to understand and even to learn but mainly that it turns up in “perfect” dried-out forms, so that if there are any mistakes, then necessarily the student made them.’

In this way described, the heritage in secondary school is the tradition of learning Euclid by rote, or in more recent times, following algorithmic learning and teaching of famous examples such as completing the square or the theorem of Pythagoras without the historical accuracy evident in teaching, or the real understanding reported by students. Although not entirely the same, this reminds us also of the Freudenthal’s anti-didactical inversion: in fact we are dealing here with taking the ‘mathematical activities of others... as a starting point for instruction’ (Gravemeijer & Terwel, 2000, 780).

We argue however, that the heritage, and its dried-out mathematics which does not engage students, has a role to play in teacher training if only as a way of exploring a culture of mathematics and a possible initiation route into the historical study of mathematical concepts. In other words, we should use the ‘heritage’ approach of the ready-made mathematics of the prescribed curriculum(s) to identify the concepts which may be rich in their potential to search for ‘meaning’, so that dealing with concepts and doing mathematical problems or exercises which use them, becomes a meaningful activity. The use of curriculum in such a way provides a comfort zone for teachers to structure their exploration in a sustainable way, through gradual learning process rather than sweeping enthusiasm which ends up with not having enough time to engage with the ‘new’ material through research and reading, and subsequently can end as a project never to be completed<sup>8</sup>.

In order to use ‘heritage’ in this way we propose that answers to the questions:

- Why do we call this (concept/process/tool) by this name?
- What does it mean and/or how does it work?

must be possible to answer. For example, Euclidean tool or construction, and a Pythagoras’ theorem would be starting points for an investigation of such type in which heritage would be used to explore the history and gain the meaning about a

mathematical concept.

How else can teachers themselves impart the ‘meaning’ of concepts they are dealing with onto their pupils? This may seem like an obvious fact, but the research presented here will show that imparting the all elusive ‘meaning’ is not a part of everyday practice, and that teachers in education are prone to unquestionably take concepts in a heritage-like, ready-made mathematics way. Should every teacher be aware of why we call certain constructions ‘Euclidean’ (In fact how many teachers actually do so?) and why a certain way of solving equations is done through the method of ‘completing the square’? How can a teacher use these concepts with the secondary age pupils in a meaningful way without herself/himself really being sure what they mean? Not knowing these simple facts by teachers approaches the near-total disengagement with a mathematical culture that the teachers are trying to somehow in turn engage their students with. It may not then come as a surprise to hear that the perception that many young people in Britain have of mathematics is that it is “boring and irrelevant” as a consequence (Smith, 2004, 2).

## **FROM EXPERIENCE TO EXPERIMENT**

While exploring the possible ways in which mathematics ‘subject knowledge’ can be revitalised through the Initial Teacher Education course for Secondary and Middle Years Mathematics Students, we came across the barrier of students wanting to engage with simple exercises and working things out for themselves. Doing mathematics is different to teaching mathematics, but can mathematical concepts be re-examined for the purposes of education without doing the mathematics one already ‘knows’? This was another of the questions that puzzled us as we tried to understand both the wide-spread lack of knowledge about the origin of certain concepts that we teach at secondary level, and the lack of questioning from prospective teachers as they train to use these in their craft, and in their (what it seemed) ultimate goal of covering the prescribed curriculum. In fact, should we be asking the questions: ‘Do they actually care?’ and ‘Does it make a difference to their teaching and students’ learning?’ However those questions are beyond the research of this paper.

Watson (2008, 7) documents the way that we have in fact structured our curriculum through examinations in the recent past:

Questions involving application of theorems can be avoided in UK national tests at 16+ and students still be awarded the highest grades. Theorems and proof of any kind, let alone geometrical contexts, do not play a part in higher school examinations.

The lack of questioning of the premises upon which we should build some mathematical understanding by new graduates in this context does not then seem puzzling. The difference between doing mathematics (seen in all its diverse cultural interactions, a mode of intellectual enquiry but also a mode of intellectual

communication), and doing the ‘school’ mathematics, is described by Watson (2008, 2) in the same paper:

Learners... have a different experience to those taught with a more abstract view, but solving realistic and everyday problems need not lead them to understand the role of mathematics beyond providing ad hoc methods for real problem-solving, or as a service subject which holds tools for moving forward in other domains.

How it is possible that such practice is embedded in the system of education becomes apparent when at the start of such practice it is evident, as will be shown, that the enquiry stops at the gate of the curriculum temple.

The project described here was a pilot project to deal only with the single issue of Euclidean constructions. The question was ‘How much do teacher students understand what a Euclidean construction is and how could they employ such understanding to teach the topic from the curriculum?’ The aim was to:

- find the level of understanding of Euclidean constructions and their historical origins;
- engage the student teachers in exploring the possibilities to do mathematics but also to think about how they would teach it;
- find the ways in which this process could be modelled for other topics from the curriculum.

The pilot will serve as a basis to plan a larger project to examine the issues of

- overcoming the disengagement of teacher students from the meaning of mathematical concepts through
  - doing mathematics first through a heritage-like way
  - identifying the ‘heritage’ elements of mathematics thus ‘done’
  - going onto the exploration of the history of the concept (with all subsequent benefits, but primarily: aiming to understand the way of mathematical thinking associated with the concept, and aiming to systematize the interconnectedness with other mathematical discoveries and concepts)
- devising a system to identify the topics from the curriculum which can be rich to offer such explorations by the teachers in training and
- attempting to define ‘how’ and ‘how much’ history of mathematics should and could be incorporated into the teacher education.

This paper therefore does not give a full and/or comprehensive list of answers to the question of how and why to introduce the teachers in education and training to the history of mathematics. But it does trace a project which shows the increased engagement and motivation giving teachers the confidence to use the syllabus in a

creative way and to explore the concepts they are meant to teach, which would otherwise be made into an empty list to be ticked off as they go into the lessons. It also evidences the students' increased capability to search for cultural and historical roots and construct the meaning around mathematics they are teaching.

### The results of the experiment

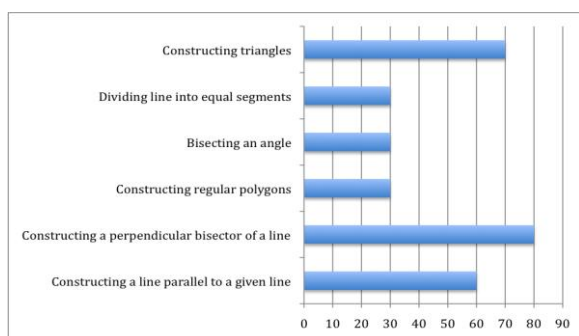
The study concentrated on analysing a group of students at a university in South West England. They were all mathematics specialists: 24 who enrolled on a postgraduate course leading to the Postgraduate Certificate in Education (PGCE) preparing them to teach at secondary level (11-18 year olds) and 12 students preparing to teach in middle-schools (8-14 year olds), but all working on the mathematics related pedagogy for 11-14 year olds.

The students were given a task to complete on geometrical constructions. They needed to:

- research the requirements of the curriculum regarding geometrical constructions
- find as many constructions as possible, appropriate for the curriculum levels and execute them themselves
- devise a learning activity for their prospective pupils.

The first problem that teacher students encountered was the material from which to source their own, and then their students', learning. While many school textbooks deal with geometrical constructions, they do so in a haphazard way without sufficient explanation as regards to the context in which these arise, or were conceived, and often do not provide any underlying conceptual understanding. The history of the concepts is often entirely disregarded, even though one of the most widely available textbooks insists on mentioning 'Euclidean constructions'<sup>9</sup>. This has been identified by authors in other countries as well, such as in recent study of Nicol and Crespo (2006).

At the end of the task, the students were asked to say what constructions they learnt and were able to execute without resource to repeated instruction. The percentages of students being able to complete various constructions are given in table below.



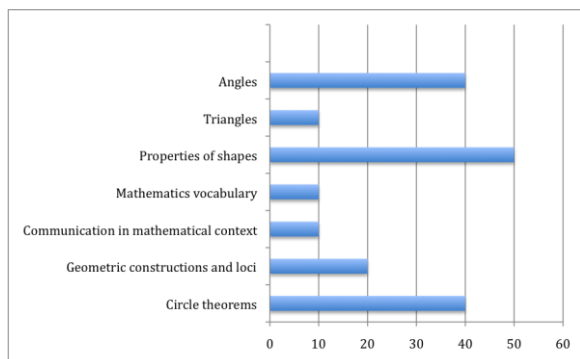
teach topics listed are given below.

However, of more interest was the subsequent students plotting of the constructions against the curriculum for 11-14 olds; the students were asked to say how many topics from the curriculum they could teach (partially or in full) through geometrical constructions. Percentages of student teachers who believed they could

## Working Group 12

In this survey, some interesting results came through:

- 70% of students originally said they could construct triangles, but only 10% declared they could see how geometric construction of the triangles would be relevant to the curriculum topic



- Furthermore, only 20% of students were able to identify or recognise that you could teach geometric constructions and loci (a topic in the curriculum) by actually doing geometric constructions.

When students were asked in a whole-group discussion to make sense of these two

answers, they further clarified:

- They could not see how constructions of triangles aided the conceptualisation of various properties of triangles and/or geometrical reasoning related to this topic
- Whilst they recognised that the geometric constructions and loci were part of the curriculum, they believed this to be an ad-hoc and not important part of actually ‘doing’ mathematics on one hand, and on the other, that engaging with constructions was not going to cover the curriculum topic in its entirety.

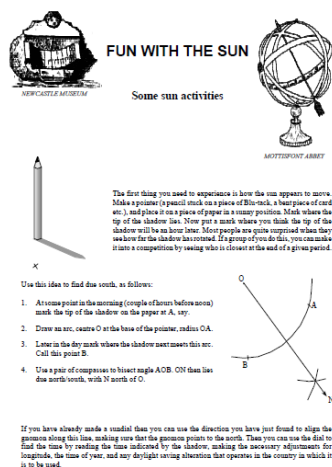
In our opinion the application of geometrical constructions is important to school students. Many texts and exercises consist of the pure geometrical constructions first, before looking at applications of these constructions. Even then the applications are thinly disguised (e.g. boat sailing between two rocks). Our work with 13/14 year old students suggests that by putting constructions into a larger piece of practical work (usually a 2½ hour single session, though this can be spread over three single lessons) more is appreciated of the use of such constructions and there is then time to explain more of the geometrical proof of why the construction works. One example used in the classroom (and shown here) is on how to find true north/south using a vertical pointer. Other examples can be found at Ransom (2004, 22-26) and the student teachers involved in this research project worked through these examples. It was interesting watching them work through the materials since their first reactions to drawing a perpendicular from a point was to use a protractor, just as school students tend to do. This does, however, allow the point to be made that all measurement is approximate (to varying degrees of accuracy), yet the construction method is theoretically accurate. This concept of accuracy does not appear to be important to many.

## Deconstruction and discussion

Without doubt the most important part of the experiment was the further deconstruction of the students' learning process, which they used to improve the design of their learning task. Van Maanen (1992, 223) has claimed that the use of constructions begins only when these are to be designed:

Clearly the crucial point is to find out which construction steps are needed to solve the problem (the analysis stage). When these steps have been discovered the proof of the correctness of the construction (the synthesis stage) is usually easy.

Through this analysis, and subsequent synthesis of the Euclidean constructions, the students first made the discovery that geometry and measurement are not necessarily the same; this led some students to delve further into the history of measures as well as measuring devices on the one hand, and constructions and mathematical instruments on the other.



Whilst the students discussed the two technical meanings of the 'construction' – the construction of the theorem and the construction in a form of a drawing (Martin, 1998, 3), the issues of the 'Euclidean' tools arose (as defined by Martin, 1998, 6, and Holme, 2002, 48). It was this crucial piece of 'meaning', of the difference between the measurement and the theorising through construction with Euclidean tools that led students to a better understanding of the way that

Greek mathematics dealt with magnitudes as tools for understanding the relationships and building upon those which have already been established. At this point in the discussion the student teachers began to be truly engaged in an intellectually active way and see mathematics as a possible way of sharing that intellectual dialogue with their prospective students.

At the beginning of the experiment exactly 50% of the students had never heard of Euclid and 100% didn't know what Euclidean constructions were, although they were happy to include mentioning of both Euclid and his constructions and tools in the learning tasks they devised for their pupils. At the end however, **90%** described that **their main motive for using Euclidean constructions would in future be in order to engage the students in geometrical reasoning and proof.**

## CONCLUSIONS

While small in scope, this experiment showed that the practical way of engaging student teachers through:

- the process of doing (albeit some simple) mathematics;
- discussing the historical context and hence dissecting the meaning attributed to some concepts (Euclidean tools);

- the learning about a topic they are attempting to teach, and a vision of how to transfer that engagement in their classrooms.

It has been noticed, and Gulikers (2001, 224) gives supporting evidence, that there is ‘...growing interest among teachers in the history of mathematics. ...results of two questionnaire surveys... reveal that teachers are interested in the history of mathematics, but at the same time, are not well resourced to actually use such material in their own teaching’. While we agree with this, this small study and the long term engagement with teacher groups from around England<sup>10</sup> also gives cause to believe that

- teachers do not have the **full motivation** and don’t actually see the history as a necessary part of mathematics learning;
- they have not experienced those **crucial insights** that would make the history of mathematics ‘necessary’ to the process of understanding and engaging with a mathematical concept either for themselves or for their pupils.

At the end of the experiment, 75% of student teachers declared the desire to engage more with the history of mathematics and 83% declared that they could see how to do it<sup>11</sup>. So, while it is worthwhile discussing the need to introduce the history of mathematics into mathematics instruction, **the student teachers need to have the experience of how this is useful in their own practice**. They are often bogged down with many daily pressing issues, such as pupil motivation, behaviour management, getting the hang of the mathematics curriculum, understanding the levels which are appropriate for the classes they teach both in terms of age and ability, and finding enough preparation time whilst learning how to plan effective lessons. Only after they have understood a crucial piece of mathematics that they have ‘inherited’ and thus practiced for many years (not very well if we are to judge by the results of the first questionnaire) without questioning, do they begin to see the potential of the history of mathematics. The learning of mathematics includes various other activities that support learning other than doing mathematics. Freudenthal identified those as ‘organising a subject matter... from reality which has to be organised according to mathematical patterns if problems from reality have to be solved. It can also be a mathematical matter, new or old results, of your own or others, which have to be organised according to new ideas, to be better understood, in a broader context, or by an axiomatic approach’ (Freudenthal, 1971). Others, like Lakatos for example, focused on the problem-solving as a way of reconstruction of a pure research mathematical discourse (Lakatos, 1976)<sup>12</sup>. Both of these are valid ways of learning mathematics, but for those who already ‘know’ and whose path to teaching is littered with ready-made mathematics modules and heritage-style pictures of good mathematics like some dusty and beautiful picture of a remote landscape in an old frame, the way of rediscovery can simply be to engage with a question of when did we identify a concept as a concept, why do we call it as we do and what does that mean. Only then does the mathematics history become part of the

## Working Group 12

mathematics education culture, and the pleasure becomes the cultural one<sup>13</sup>. And that is just for teachers, the kids will follow.

### NOTES

1. For similar projects see Teacher Enquiry: Funded Projects <https://www.ncetm.org.uk/enquiry/funded-projects>.
2. These are listed in the pre-university qualifications guide with a timeline of their introductions at [http://www.heacademy.ac.uk/physsci/news/detail/2010/pre\\_he\\_maths\\_guide](http://www.heacademy.ac.uk/physsci/news/detail/2010/pre_he_maths_guide).
3. The new curriculum was made effective from the beginning of academic 2008/9 for secondary subjects.
4. From the speech of David Cameron, now the British Prime Minister, delivered on 2nd February 2009, accessed 20th September 2010 from [http://www.conservatives.com/News/Speeches/2009/02/David\\_Cameron\\_Conservatives\\_Maths\\_Taskforce\\_launch\\_with\\_Carol\\_Vorderman.aspx](http://www.conservatives.com/News/Speeches/2009/02/David_Cameron_Conservatives_Maths_Taskforce_launch_with_Carol_Vorderman.aspx): “When it comes to what those disciplines are, and how they are taught, I believe there few things more important than getting maths in our schools right”.
5. Programme for International Student Assessment, by the Organisation for Economic Co-operation and Development.
6. These are some of the most mentioned topics in the current British media; a separate study has currently been undertaken by the author to identify these over the period of last three years.
7. Report by Adrian Smith, entitled ‘Making Mathematics Count’, was undertaken upon the commission from the Advisory Committee on Mathematics Education, an independent body, based at the Royal Society, London.
8. This has been further expounded upon in Gulikers & Blom, (2001).
9. For the matter of fairness the publisher and the book have not been mentioned.
10. See Lawrence, (2009).
11. As a small aside, around 17% of the students reported that they have acquired the employment while in the course due to their meaningful use of historical context in the interview lessons.
12. Barbin (1996, 1997), who argues “that history of mathematics changes the epistemological concepts of mathematics by emphasising the construction of knowledge out of the activity of problem solving”, gives us a possible way of approaching the solution of how to introduce such activities.
13. Heilbron (2000, 46) describes pleasures while researching geometry: “Finally the pleasure, or my pleasure, has been cultural. Pursuing geometry opens the mind to relationships among learning, its applications, and the societies that support them.”

### REFERENCES

- Barbin, E. (1994). The role of problems in the history and teaching of mathematics. In R. Calinger (ed.), *Vita Mathematica: Historical Research and Integration with Teaching*, MAA, Washington, 17-25.
- Barbin, E. (1997). Sur les relations entre épistémologie, histoire et didactique. In *Repères IREM*, 27, 63-80.



## Working Group 12

- Davis, B. and Simmt, E. (2006). Mathematics-for-Teaching: An Ongoing Investigation of the Mathematics That Teachers (Need to) Know. In *Educational Studies in Mathematics*, 61 (3), 293-319.
- Freudenthal, H. (1971). Geometry between the devil and the deep sea. In *Educational Studies in Mathematics*, 3 (3-4), 413-435.
- Fujita, T and Jones, K. (2008). *The Process of Re-designing the Geometry Curriculum: the case of the Mathematical Association in England in the early 20th Century*. Paper presented to Topic Study Group 38 (TSG38) at the ICME-11 Monterrey, Mexico, 6-13 July 2008.
- Grattan-Guinness, I. (2004). The mathematics of the past: distinguishing its history from our heritage. In *Historia Mathematica*, 31 (2), 163-185.
- Gravemeijer, K. and Terwel, J. (2000). Hans Freudenthal: a mathematician on didactics and curriculum theory. In *Curriculum Studies*, 32 (6), 777-796.
- Gulikers, I. and Blom, K. (2001). "A Historical Angle", A Survey of Recent Literature on the Use and Value of History in Geometrical Education. In *Educational Studies in Mathematics*, 47 (2), 223-258.
- Heilbron, J. L. (2000). *Geometry Civilized, History, Culture and Technique*, Clarendon Press, Oxford.
- Holme, A. (2002). *Geometry, Our Cultural Heritage*. Springer, Berlin, Germany.
- Lakatos, I. (1976). *Proofs and Refutations*. Cambridge University Press, Cambridge, UK.
- Lawrence, S. (2009). *What works in the Classroom – Project on the History of Mathematics and the Collaborative Teaching Practice*. Paper presented at CERME 6, January 28<sup>th</sup> – February 1<sup>st</sup>, 2009, Lyon France.
- Lee, S., Browne, R., Dudzic, S., Stripp, C. (2010). *Understanding the UK Mathematics Curriculum Pre-Higher Education*. The Higher Education Academy, accessed 29<sup>th</sup> September 2010,  
[http://www.heacademy.ac.uk/physsci/news/detail/2010/pre\\_he\\_maths\\_guide](http://www.heacademy.ac.uk/physsci/news/detail/2010/pre_he_maths_guide)
- Maanen, van J. (1992). Seventeenth Century Instruments for Drawing Conic Sections. In *The Mathematical Gazette*, 76 (476), 222-230.
- Martin, G. E. (1998). *Geometric Constructions*. Springer, NY, USA.
- Nicol, C and Crespo, S. (2006). Learning to Teach with Mathematics Textbooks: How Preservice Teachers Interpret and Use Curriculum Materials. In *Educational Studies in Mathematics*, 62 (3), 331-355.
- Ransom, P.(2004). The Maths Busters. In *Mathematics in School*, 33 (2), 22-26
- Smith, A. (2004). *Making Mathematics Count*. Stationary Office, UK.
- Watson, A. (2008) School mathematics as a special kind of mathematics. In *For the Learning of Mathematics*. 28(3) 3-8.

# IDENTITY OF MATHEMATICS EDUCATORS. THE PORTUGUESE CASE (1981 - 1990)

José Manuel Matos<sup>1</sup>

New University of Lisbon

*A historical case study centred on the Portuguese community of mathematics educators intending to understand how its identity was shaped from 1981 to 1990 is presented. Keywords: history of mathematics education; professional identity*

The notion that mathematics education may consist of a world-wide *community* is expressed in the title of the book *Mathematics Education as a research domain: a search for identity* that publishes materials presented at the Study Conference “What is research in mathematics education and what are its results” organized by ICMI in 1994. In the summary, the editors, Anna Sierpinska and Jeremy Kilpatrick, conclude:

In spite of all the differences that divide mathematics education researchers (in terms of theoretical approaches, views on relations between theory and practice, philosophies of mathematics, etc.), they still constitute a community, and it is necessary to search for what constitutes its identity. (1997, Vol. 1, p. ix)

Understanding how such community came about and consequently how such *identity* is shaped, is then a necessary endeavour. An early reflection about the nature of such identity and its diversity can be found in the discussion about a *Theory of Mathematics Education* (Steiner et al., 1984) that took place in ICME 5. Later, Alan Bishop (1992) distinguished three traditions of research according to the goal of enquiry, the role of evidence, and the role of theory. The study of specific traditions — essentially those nationally bounded — has also been the object of analysis<sup>2</sup> centred on paradigms, methods, and results.

But the study of *research* alone does not suffice to understand the *identity* of this community. Claude Dubar (1991) defines identity as the “result simultaneously stable and provisional, individual and collective, subjective and objective, biographic and structural, of the several processes of socialization that, jointly, construct individuals and define institutions” (p. 113) and it is formed through the articulation of two heterogeneous processes, the *identity for others* involving social expectations and acts of attribution, *self-identity* based on self-perspective and acts of belongingness.

By using a historical case study approach, this paper aims at understanding how the *self-identity* of the community of mathematics educators was shaped in Portugal. At the centre of this analysis is the meaning attributed to the term *mathematics educator*. I chose to designate as *mathematics educators* those concerned with the research of problems relating to teaching and learning mathematics. Today most share common practices (teacher education and research, for example), communicate through institutionalized forums, and have professional ties to universities or

colleges. In the Portuguese context, however (and certainly in other countries), boundaries are fuzzy and some primary or secondary teachers engage in research activities. That was not the case in the early 1980s where we can hardly speak of professionals focusing on research in mathematics education in Portugal. This paper will study the actual *emergence* of the community of mathematics educators out of a community of teachers of mathematics. The time frame starts in 1981 — when the first discussions in a democratic context about the state of mathematics teaching and learning occurred — and ends in 1990 — coinciding with the adoption of new curricula. Data collection was centred on articles, books, official documents, and gray literature. Content analysis studied this *corpus* as historical documents (Certeau, 1975) focusing on evidence for the representations and the practices of the participants (Julia, 1995).

### **EARLY 1980s, DISCOMFORT WITH MATHEMATICS PROGRAMMES**

The Portuguese educational system experiences tensions and changes during the 1970s. On the one hand, there is a steady growth in the number of students that is not accompanied by a proportional growth of schools or teachers. On the other hand, there is a shift on the purposes of the system. The separation between schools for students aimed at the universities and professional schools gradually moved from 4<sup>th</sup> grade to 6<sup>th</sup> grade (in 1968), and after 1976 gradually disappeared. Significant changes in some curricula, and especially in school management, occurred as a consequence of the democratic revolution of 1974. The seventies were a period characterized by overpopulated schools, lack of certified teachers, of textbooks, and other materials, and unstable school management and curricula.

At that time, mathematics curricula embodied the Modern Mathematics approach. The ideas of this movement were firstly published in Portugal in 1957, experimented through the 1960s, and adopted in the following decade. Teachers with the responsibility of training other teachers enthusiastically embraced the reform because they saw it as a perspective adequately merging psychological and mathematical approaches (Matos, 2009). However, by the 1970s (and well into the 1980s), a majority of teachers of mathematics were not certified, nor had enough mathematics background. The ways in which the reform changed the daily life of mathematics classrooms has not been researched, but there are studies about its effectiveness suggesting widespread learning difficulties (Matos, 2010).

In the late 1970s documents show discomfort among young teachers of mathematics, some teaching in teacher education courses, but most of them still following courses at universities, about the state of mathematics teaching and learning. Small group discussions over curricular and mathematical issues (Pólya's *How to solve it*, for example) were conducted and in 1981 the participants presented papers at the first professional meeting about teaching and learning of mathematics organized under the new democratic regime, started a newsletter *Inflexão* [*Inflection*], and actively

engaged in a series of meetings planned by the Portuguese Society of Mathematicians to discuss the programmes of upper secondary school. By the end of these meetings, a coherent set of perspectives about mathematics teaching and learning emerged (“Os programas em debate”, 1982). Reacting against what was perceived as “a critical situation”, Modern Mathematics programmes were condemned because they rendered mathematics hermetic, formalized, with a great emphasis on symbols, and foreign to reality and applications. Programmes should change to integrate

- a strong component making use of problems, i. e., great relevance to the role of problems as a means to develop an investigative and discovery spirit (...);

- a strong focus on the practical side [of mathematics], by using calculators (...) [and] computers;

- a special attention to the applications of mathematics and to its relationships to other disciplines, [adopting] a marked interdisciplinary sense;

in sum, an increased relevance to the formative aspect. (“Os programas em debate”, 1982, p. 20)

These three dimensions (problems, technology, and applications) that departed from the official curricular options at the time will integrate the backbone of the idealized mathematics curriculum proposed by the document.

The document approved in 1981, also discusses curriculum development. Criticising previous practice of writing programmes in closed commissions appointed by the Ministry of Education, new curricula, the document stated, should be developed by a Commission composed of elements indicated by the Universities, Secondary Schools, Portuguese Society of Mathematicians, and lastly, the Ministry of Education. This “parliamentary” approach to curricular development, almost marginalizing the governmental body, shows that this emerging community is also seeking the ownership of (or reclaiming its expertise over) specific knowledge dimensions, thereby exerting pressure on other social bodies as a means to acquire power.

In 1982, several papers centred on curricular perspectives were presented at a national meeting. From the distance of time, and by today’s standards, one cannot help to notice their exploratory nature, hardly supported by relevant literature, rudimentary methodologies, and with scant conclusions. But it is also clearly apparent their voluntarism, their strong ties to the life of schools, and the firm desire to change a situation perceived as professionally frustrating.

In brief, during this period, there is a small loose network of young teachers of mathematics and assistant professors, almost all of them still acquiring their scientific or professional certification or their doctorates. Exchanges among the members occurred at small informal gatherings. At the end of the period a newsletter

appeared. Their practice included not only the common practice of mathematics teachers in the country, but also interventions outside regular classes (math clubs), and interventions at meetings requiring changes in mathematics curricula, implicitly seeking public recognition of their expertise on the subject. Many of the members had also intervention in other dimensions of social life (teachers' unions, for example). Their theory was straightforward: school success will come from problem solving, technology and applications of mathematics; school failure should be attributed to repetition, widespread formalization, and premature abstraction, the last two been interpreted as the hallmark of Modern Mathematics.

### **THE CONSTITUTION OF A COMMUNITY (1982 – 1986)**

During the year 1981, Joseph Hooten, Jr., from the Department of Mathematics Education of the University of Georgia, USA, was a visiting professor at the University of Lisbon<sup>3</sup>. Through seminars, visits, small group discussions, and informal conversations, for almost a year he disseminated a coherent perspective about the problems of teaching and learning mathematics.

The concept of *connected history* (Gruzinski, 2003), developed by cultural historians, has been used to understand how communities elaborate their representations and practices in connection with other communities. Going beyond both a mirror perspective — in which one culture tries to emulate another —, and a power perspective —interpreting cultural influences as either acts of emulation or resistance —, the influence of one culture by another is seen as acts of *hybridism* (*métissage*) in which external appropriations are not seen as mere imitations (or distortions) from the original, but as producers of a new originality. In this model, a key role on the circulation of ideas and practices has been attributed to *mediators*, persons that travel among societies and cultures. Posing as mediator, Joe Hooten brought firstly a new pedagogical model. Portuguese Modern Mathematics curricula, inspired essentially on ideas from France and Belgium, were confronted by a distinct perspective centred on problem solving, laboratories, the value of applications of mathematics, and of the use of manipulative materials. Joe Hooten also brought an appreciation for empirical research as a means to consolidate educational knowledge.

One tangible product of this influence is the translation of the book *Agenda for action: recommendations for school mathematics of the 1980s* (NCTM, 1980) that circulated before 1985. Its first proposal, that problem solving be the focus of school mathematics in the 1980s became one of the key elements of a pedagogic alternative of the Portuguese group. In the USA, this emphasis can be seen as a response to the middle 70s “back-to-the-basis” approach thereby valuing problem solving’s allowance for more significant mathematics involving higher-level reasoning, not limited to the memorization of facts and algorithms. In Portugal, however, problem solving was also valued because of its non-abstract and non-formal qualities, which was seen as permitting an alternative to the Modern Mathematics curricula.

By the end of 1986, the small group of young teachers, having completed their graduation and post-graduation, incorporated the first Portuguese association of teachers of mathematics (in 1986), with annual national meetings (since 1985) with a growing participation of teachers, and publishing a journal (since 1987). Three other small non-disjoint groups of teachers integrated this movement. The first was an active group of teachers responsible for in-service teacher education at grades 5<sup>th</sup> and 6<sup>th</sup>. The second were teachers and mathematicians based in Coimbra and promoters of the national Olympiads of mathematics. A third was composed of teachers for schools aiming at forming primary teachers that, as a group, underwent a graduate programme on mathematics education<sup>3</sup> in the University of Boston, thus reinforcing the USA ties and becoming mediators themselves.

All these groups and mediators enlarged the practices of the community and shaped their representations as can be observed in the first national meeting of teachers of mathematics (*ProfMat*) that occurred in 1985 in Lisbon. The proceedings were published in a newly founded journal *ProfMat, Revista teórica e de investigação sobre o ensino da matemática* [*ProfMat, Theoretic and research journal about mathematics teaching*] that, as other scientific journals, had its Editor and Editorial Board<sup>5</sup>. The value attributed to research is expressed in its name, its purpose and is the focus of the first editorial:

[The journal's] purpose is to provide for a broad exchange of perspectives related to the formulation of research problems, methodologies for collecting and analysing data, theoretical foundations, evaluation, and syntheses of results. ("Editorial", p. 3)

The journal includes 11 papers, mostly reporting investigations produced for the completion of graduate studies at American universities. These works show a very distinct style from the ones of the early 80s. Now, there is a clear emphasis on structured empirical research on issues concerning mathematics education, including the definition of a problem, a review of literature, data analysis, and conclusions.

The proceedings also include three texts on computers, calculators, problem solving (originally in English), manipulative materials, and mathematical clubs. The three plenary lectures, given by leading mathematicians, were all devoted to the issue of mathematics and computers.

In 1985, the term "educação matemática" (mathematics education) is used for the first time on the cover of a book presenting a chronology of mathematics teaching (Matos, 1985), and is also included in the name of one the discussion groups at the first ProfMat (on "Theory of Mathematics Education"). In 1986 it is used again on the cover of the second issue of the journal that published the proceedings of the second national meeting. In this issue, the journal itself had a slight change in name replacing "mathematics teaching" for "mathematics education" and the term is discussed in the plenary of the second congress (Ponte, 1986). The consolidation of this denomination in 1985/86, and consequently of the identification of the members

of the community as *mathematics educators*, also suggests the consolidation of the community. It also indicates the departure from other denominations, namely *didactique des mathématiques*.

The foundation of the first Portuguese association of teachers of mathematics (APM) at the second national meeting in 1986 marks the appearance of a national professional organization, specifically concerned with the problems of teaching and learning mathematics and the publication of a new bi-monthly journal *Educação e Matemática*. It also results, internally in the consolidation of the community's membership — a mathematics educator *must* be a member of APM —, and in its external legitimization.

The dominant perspective on the community can be obtained through the first editorial of this second journal, written by Paulo Abrantes (1987). There he complains about the “crisis” in mathematics teaching in recent years and endorses its responsibility to the Modern Mathematics approach, which is “very removed from students’ concrete reality, with considerable importance been given to mathematical structures and their properties.” (p. 3)

He proposes initiatives that may promote change: out-of-class activities centred on applications, problem solving (math week, conferences, problem competitions, and Olympiads), or the use of computers. He notes the importance of initial teacher education and teachers’ meetings. He finishes by stressing the importance of active methodologies, of considering the active and social dimensions into the broad objectives of education, the importance of out-of-school contexts, the new technologies, problem solving, applications of mathematics, and interdisciplinarity.

In summary, by the end of 1986 the community is grouped under the national association for teachers of mathematics that includes virtually all small groups of innovators interested in teaching and learning of mathematics. Almost all of its leading members are now certified teachers, many are working at universities in teacher education programmes, and have some practice in conducting research in the field. There is a functional and complex network and exchanges among their members can occur through national (or regional) meetings, or through the association's journal. For some, a regular presence in international meetings diversified sources for the circulation of ideas. The practice of some members continues to be similar to the common practice of teachers of mathematics, including organizing math clubs. Their theory has become more sophisticated. Research is now seen as a fundamental option to legitimize innovation and knowledge building. But problem solving, technology, and applications of mathematics, together with the importance of the use of manipulative materials continues to be equated with success; repetition, widespread formalization, and premature abstraction, still prevalent in school curricula, continue to be seen as promoting failure.

**CONSOLIDATING IDEAS AND REPRESENTATIONS (1986 — 1990)**

The volumes from the first two years of the teachers' association's journal (1986-7) contain many articles presenting problems, or about problem solving, others relating to applications of mathematics, some discussing the use of technology, or manipulative materials. But, there are almost no accounts of actual interventions in classrooms. The same can be observed as we look at the first books published by the community. Apparently, most of the applications of the theory (valuing problem solving, applications, technology, and manipulative materials) can only be seen outside real classes — in math clubs, for example — in environments that do not confront the national curricula.

In May 1987, intending to adapt the educational system to a democratic society, the Ministry of Education initiated a public discussion of new curricula. The preliminary document raised widespread concerns in the community, as, apparently, it favoured a “back-to-bases” approach. As a reaction, APM convened a meeting for the elaboration of an alternative perspective.

“For four days, morning to night, 25 teachers and researchers discussed some of the essential problems to the renewal of mathematics for basic and secondary grades. (...) The following texts are the product of this work and constitute documents that support an enlarged debate among all members of the Association and, in general, among teachers of mathematics.” (APM, 1988, p. 3)

As a consequence, a collective book was produced (*Renovação do currículo de Matemática* [Renewal of mathematics curriculum], 1988), known as the *Milfontes document*, containing a coherent perspective on learning and teaching mathematics that was also published by the National Commission for the reform. But, perhaps the impact of this document outside the community can be ascertained by the interview given by the responsible for the coordination of the new programmes to the journal of the Association and published shortly after. In this interview, Brigitte Tudichum, overseeing curricula from all areas, agrees on the importance of problem solving, technology, and communication in mathematics teaching (Tudichum & Nunes, 1989). “Mathematics must appear and be explored through problems” (p. 24) she states. The success obtained by this social intervention was new to the Association and reinforced its status. Moreover, the ideas of the *Milfontes paper* prompted some participants in the seminar to test its proposals empirically through a three-year curricular development project in grades 7<sup>th</sup> through 9<sup>th</sup>, Project Mat789 (Abrantes, Leal, Teixeira & Veloso, 1997).

As representations of successful mathematics teaching and learning became more reflective, newly masters' courses on education promoted the development of applied research. In fact, from 1987 teaching experiments centred on the four axes for innovation — technology, problem solving, and in a smaller degree, applications and



manipulatives — were conducted. Moreover, specific research groups appeared, on teacher education, problem solving, and learning.

In brief, the community is now composed of a very large group of teachers and APM is seen nationally as a reference as far as innovation is concerned. There are several working groups in the Association, the annual national conferences experience a growing participation, and throughout the country there is a multitude of regional meetings. At the same time, a differentiation within the community starts to show because, so far, researchers identity can hardly be distinguished from the teachers of mathematics. The emergence of graduate programmes in mathematics education located in Portuguese universities is going to change the representations and the practices of a small number of members and, at the same time, endows them with power over the others. In this period, especially after the publication of the *Milfontes paper*, practice became the application of theory for some members. Theory itself became much more elaborated.

### CONCLUSION

This case study of the emergence of identity in the community of mathematics educators in Portugal has shown how the community itself started from a loose group of beginning teachers, essentially located around Lisbon that gradually incorporated other teachers. It also showed how this community gradually differentiated into a large majority of teachers and a minority of teachers “that teach teachers”.

Identity itself was characterized as an enumeration of practices and theories followed by the community. As for the practices, this study has shown that curricular innovation was an intention from the early beginnings but either the lack of knowledge of adequate comparable practices, or the centralized structure of Portuguese curricula, did not allow for much in-class innovation. Early pedagogical practice is therefore displaced to the outside of the classroom. Only later, essentially after 1987, with new curricula in sight and the generalization of graduate studies, some experiments were performed.

In the beginning, another particular kind of practice emerged, essentially around problem solving. Documents from that time include problems, challenges for solving problems, and examples of the nesting of problems. In some schools teachers prompted students to solve problems in out-of-class activities. Apparently, it was as if the community needed to construct a perspective about what constitutes mathematics distinct from university teachings, which was dominated by Bourbakian approaches.

We have also seen how theories have evolved. From a naïve perspective on mathematics education problems, and with the help of mediators, more elaborate representations emerged, with a strong influence from USA's perspectives. Later, and prompted by external solicitations (a curricular reform), a coherent perspective was collectively elaborated (the *Milfontes* document). The modes and values of

educational research became appreciated, as the empirical scientist tradition (following Bishop's classification) was appropriated.

As we have seen, in the beginning, mathematics educators (today seen as researchers on teaching and learning mathematics) in Portugal were undistinguished from innovative teachers of school mathematics. Differently from other countries where the community was initially composed of trained mathematicians, for example, virtually all the early members began their careers as teachers in secondary schools, and only later a differentiation started to emerge. It may be the case that such origin explains the existence of a strong national association of teachers of mathematics, the numerous presence of school teachers in research meetings (usually from one half to two thirds), the late constitution of an association of researchers in mathematics education, and the affiliation of the Portuguese research journal, *Quadrante*, to the Association of Teachers of Mathematics.

## NOTES

1. This paper was supported by the Project "A Matemática Moderna nas escolas do Brasil e de Portugal: Estudos históricos comparativos" financed by FCT (Portugal) and CAPES (Brazil).
2. For example, Artigue and Douady (1986) discussed the specificity of French research, Arzarello and Bartolini Bussi (1997) examined the Italian tradition, and in Portugal an overview of research on the field was analysed (Ponte, Matos, & Abrantes, 1998).
3. He was supported by a Fulbright grant to work with science education professor Odete Valente.
4. This programme was part of a larger project financed by the World Bank to provide new founded schools for forming primary teachers with teaching staff in all areas. It involved Boston University and the Université de Bordeaux.
5. Domingos Fernandes together with João Ponte and José Matos constituted the Editorial Board.

## REFERENCES

- Abrantes, P. (1987). Associação de Professores de Matemática: esperança e desafio. *Educação e Matemática*, 1, 3-6.
- Abrantes, P., Leal, L. C., Teixeira, P., & Veloso, E. (1997). MAT 789, inovação curricular em Matemática. Lisbon: Fundação Calouste Gulbenkian.
- APM. (1988). *Renovação do currículo de Matemática* [Renewal of mathematics curriculum]. Lisbon: APM.
- Artigue, M., & Douady, R. (1986). La didactique des mathématiques en France. *Revue Française de Pédagogie*, 76, 69-88.
- Arzarello, F., & Bartolini Bussi, M. G. (1997). Italian trends in research in mathematical education: a national case study from an international perspective. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics Education as a research domain: A search for identity* (Vol. 2, pp. 243-262). Dordrecht: Kluwer.

- Bishop, A. J. (1992). International perspectives on research in mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 710-723). New York: Maxwell Macmillan.
- Certeau, M. (1975). *L'écriture de l'histoire*. Paris: Gallimard.
- Dubar, C. (1991). *La socialization. Construction des identités sociales et professionnelles*. Paris: Armand Colin.
- Editorial. (1985). *ProfMat, Revista Teórica e de Investigação sobre o Ensino da Matemática*, 1, 3-4.
- Gruzinski, S. (2003). O historiador, o macaco e a centaura: a “história cultural” no novo milénio. *Estudos Avançados*, 17(49), 321-342.
- Julia, D. (1995). La culture scolaire comme objet historique. *Paedagogica Historica. International Journal of the History of Education, Suppl. Series, I*, 353-382.
- Matos, J. M. (2010). Elementos sobre o ensino e a aprendizagem da Matemática Moderna em Portugal no final dos anos 70. In J. M. Matos & W. R. Valente (Eds.), *A reforma da Matemática Moderna em contextos ibero-americanos*. Lisbon: UIED.
- Matos, J. M. (2009). Changing representations and practices in school mathematics: the case of Modern Math in Portugal. In K. Bjarnadóttir, F. Furinguetti & G. Schubring (Eds.), *"Dig where you stand" Proceedings of a Conference on On-going Research in the History of Mathematics Education, Gardabær, Iceland, June 20-24 2009*. Reykjavik: University of Iceland.
- Matos, J. M. (1985/1989). *Cronologia recente do ensino da Matemática*. Lisbon: Associação de Professores de Matemática.
- NCTM. (1980). *Agenda for action: recommendations for school mathematics of the 1980s*. Reston: NCTM.
- Os programas em debate [Debating programmes]. (1982). *Boletim da Sociedade Portuguesa de Matemática*, 5, 18-22.
- Ponte, J. P. (1986). Investigação, dinamização pedagógica e formação de professores — três tarefas para a renovação da educação matemática. *ProfMat, Revista Teórica e de Investigação em Educação Matemática*, 2, 15-45.
- Ponte, J. P., Matos, J. M., & Abrantes, P. (1998). *Investigação em educação matemática. Implicações curriculares [Research in mathematics education. Curricular implications]*. Lisbon: IIE.
- Sierpiska, A., & Kilpatrick, J. (Eds.). (1997). *Mathematics Education as a research domain: A search for identity*. Dordrecht: Kluwer.
- Steiner, H.-G., Balacheff, N., Mason, J., Steinbring, H., Steffe, L. P., Brousseau, G., et al. (Eds.). (1984). *Theory of Mathematics Education (TME)*. Bielefeld: Institut für Didaktik der Mathematik der Universität Bielefeld.
- Tudichum, B. & Nunes, F. (1989). Da matemática nos novos programas [About mathematics in new programmes]. *Educação e Matemática*, 8, 23-25.

# THE TEACHING OF MATHEMATICS IN PORTUGAL IN THE 18TH CENTURY – THE CREATION OF THE 1ST FACULTY OF MATHEMATICS IN THE WORLD

Catarina Mota<sup>a</sup>, Maria Elfrida Ralha<sup>b</sup>, Maria Fernanda Estrada<sup>c</sup>

<sup>a</sup>CMAT & Didáxis – Cooperativa de Ensino

<sup>b</sup>CMAT & Departamento de Matemática e Aplicações, Universidade do Minho

<sup>c</sup>CMAT – Centro de Matemática, Universidade do Minho

*In 1772 the teaching of Mathematics in Portugal, at a higher level, was transformed with the creation of a Faculty of Mathematics in the University of Coimbra. The newly reformed Portuguese University was ruled by Statutes. In the words of Francisco Gomes Teixeira, these Statutes are a “remarkable dissertation about the teaching of sciences, exquisite both in deepness and form and a monument to healthy pedagogy and high philosophy, written in vernacular and elegant language, where all justifications are clearly explained and justified”<sup>1</sup>. In our article we will analyze some parts of these Statutes, aiming to recover a 250 years old mathematical curriculum in what represents lessons to be used in present mathematics education.*

**Keywords:** Portuguese teaching of Mathematics; Eighteenth century; Statutes; University of Coimbra.

## INTRODUCTION

The university teaching of Mathematics in Portugal suffered a deep reformation in 1772 when the Faculty of Mathematics at University of Coimbra was created. As far as we investigated, it was the first time, in the world, that one University recognized such a high standard of importance to Mathematics that it created a whole Faculty to teach Mathematics (several/different courses for different degrees) to all students of the University (‘Do curso Mathematico’, 1772, p. 141).

As all the other Faculties reformed or created by this reform, the Faculty of Mathematics was ruled by Statutes, written and published under the guidance of king D. José. Those Statutes are undoubtedly a pedagogical document that might be used as an example for those who are responsible for teaching Mathematics nowadays. This work aims to study a 250 years old mathematical curriculum with the objective to retrieve the teachings of this document to the current mathematics teaching and curriculum development in mathematics.

## THE CREATION OF THE FACULTY OF MATHEMATICS AT UNIVERSITY OF COIMBRA

Mathematics was supposed to be taught in Portugal at least since 1559 at the University of Coimbra (Queiró, 1993, p. 4), even though sometimes there was no lecturer available for this course. Mathematics was also taught by the Jesuits who

were installed in Portugal since their very beginnings. Ruling the University of Évora, as well as many other schools/colleges, Jesuits in Portugal became clearly interested in teaching Mathematics and lecturers/teachers were always available.

In the beginning of the eighteenth century Portuguese mathematicians started to publish books on recent mathematics, as *Theorica verdadeira das marés* by Jacob de Castro Sarmiento or *Logica racional, geometrica e analytica* by Manuel de Azevedo Fortes (Queiró, 1993, p. 11).

By 1750, D. José reached the Portuguese throne and called Sebastião José de Carvalho e Melo, later known as Marquês de Pombal, to be his Secretary for Foreign Affairs. Six years later Marquês de Pombal becomes Minister of the Kingdom with larger powers to rule. His powers were in particular used to reform the school system in Portugal, expelling the Jesuits (by 1759) and implementing reforms from the very elementary level to the university level (Teixeira, 1934/2010, p. 223 – 232).

Marquês de Pombal went to Coimbra to formally open in 1772 the newly reformed University. Under this reform the five faculties (Theology, Law, Medicine, Philosophy and Mathematics) included two new ones (Philosophy and Mathematics). The creation of the Faculty of Mathematics was justified right at the beginning of the Statutes:

Mathematics has such an indisputable perfection among all natural knowledge, as well as in the luminous accuracy of its method, and in the sublime and admirable speculation of its doctrines; that it deserves the name of sciences not only in rigor but with property; but it is also the one which has singularly believed in the Man's strength, skills and sagacity. For that reason it is extremely necessary, also for security and improvement of other Faculties, that this science has at the University an adequate establishment of its place, occupied in the General System of Human Knowledge; being clear, that if that same University was dismissed of the Mathematics lights, as it unfortunately was in the past two centuries, it would not be more than a chaos, similar to the Universe, if deprived from the brightness of the Sun. ('Do Curso Mathematico', 1772, p. 141, the translation from Portuguese and the underlines is ours but the punctuation is the original)

## THE STATUTES OF THE FACULTY OF MATHEMATICS

### The Author and his Inspiration

José Monteiro da Rocha (1734 – 1819) was the author of the Statutes for the Faculty of Mathematics. He was born on June 25th, 1734, in small town in the north of Portugal called Canavezes (Amarante). It is also known that in his youth Monteiro da Rocha went to Brazil to study with the Jesuits. By 1752, he was teaching in the Jesuit's "Colégio da Baía". When Marquês de Pombal expelled the Jesuits from Portugal, in 1759, Monteiro da Rocha left the Society of Jesus (Fernandes & Figueiredo, 2006).

Having been a member of the Society of Jesus, Monteiro da Rocha wrote the Statutes naturally influenced by the Jesuit pedagogy. Indeed, at a first glance, the organization of the Faculty Statutes seems very much alike the organization of *Ratio Studiorum*, the pedagogical code for the Jesuits published in 1599, but a comparative study between these two documents remains to be done.

### **The importance of the studies of Mathematics**

We have seen previously the general justification given by Monteiro da Rocha in the Statutes for giving to Mathematics a higher place within the University. But he continues saying that:

[Mathematics] not only goes through a road of lights, from the first Axioms, to the most sublime and recondite Theorems; but it also illuminates the understanding in the study of other Courses: showing them the most perfect example on how to treat a subject according to order, precision, strength, closed chains and even joining some truths to others: inspiring the pleasure and the necessary discernment to distinguish the solid from the frivolous, the real from the apparent; the Demonstration from the Paralogism: and giving them an accuracy, according to the Geometric Spirit; rare and precious quality, without which they cannot go on neither make any progress in the knowledge of Mankind. ('Do Curso Mathematico', 1772, p. 141-142)

The justifications of every single statement are clear, complete and rigorous in all the analyzed documents in the Statutes.

These Statutes also include applications of Mathematics, showing that this science is not only a theoretical one but also a practical: regulation of time; geography; cartography; tactics for army; architecture and engineering are some of the subjects that may be found in the Statutes.

### **Mathematicians: a “special career”**

The goal for teaching Mathematics at the University was to create professional Mathematicians. The career of "Mathematician" and the research on Mathematics were encouraged, as stated in the Statutes ('Do Curso Mathematico', 1772, p. 148-150), by counting the years spent in the Faculty of Mathematics as years spent in war campaigns for the allocation of social benefits, such as social status. Having a degree in Mathematics would, in particular, grant advantages in getting a place in the Army. Furthermore, those who were Doctors in Mathematics would officially be in advantage when applying for jobs.

Awards, called "*Partidos*", were to be given to students with exceptional classifications. These awards would stimulate students to study and encourage them for following a scientific carrier ('Do Curso Mathematico', 1772, p. 216-218).

On the other hand, sanctions were imposed on those who did not recognize the benefits of studying mathematics, for example to those who, when offering jobs, did not give priority to holders of a degree in Mathematics as well as to the students that

did not perform well on the tasks of the courses ('Do Curso Mathematico', 1772, p. 202).

## THE DEGREE OF MATHEMATICS

The degree of Mathematics was composed by four courses distributed in 4 years: *Geometry* in the 1<sup>st</sup> year; *Algebra* in the 2<sup>nd</sup> year; *Phoronomy*<sup>2</sup> in the 3<sup>rd</sup> year and *Astronomy* in the last. There also existed a course in *Design and Architecture* (Civil and Military), for students who showed interest in these courses.

The lessons of the four years of the degree took place every day of the week, with the duration of one and a half hours, according to the years (the students attending 1<sup>st</sup> and 2<sup>nd</sup> year had classes in the morning while the students in 3<sup>rd</sup> and 4<sup>th</sup> years in the afternoon).

The lessons were divided into two different parts. In the first one, the students were asked about the previous lesson and were allowed to ask the lecturer to explain their doubts. The second part of the lesson was for lecturing without interruption from the students and, in this case, the lecturer was advised to follow the "inventor's path", this is, following the historical development of the subjects. The students should then go home to study the lesson and wait for the following day for further clarification of the subject (Do Curso Mathematico', 1772, p. 200-201).

### The lessons of Geometry

The first course in the degree was Geometry and it was taken not only by the mathematics students but also by every other student from all the other faculties. The subject of study was elements of Arithmetic, Geometry and Trigonometry.

The beginning of the course included, each year, the prolegomena where the lecturer would present the subject, its division and generalities such as the method, the utility and the excellence of Mathematics. The lecturer would also present an abstract on the History of Mathematics, including Greek and Christian ages as well as Descartes. This prolegomena aimed to motivate students for Mathematics ('Do Curso Mathematico', 1772, p. 169).

The prolegomena was followed by Arithmetic: the importance of symbols and Arabic notation; the notion of number, unity and the fundamental idea of numeration; the learning of the four fundamental operations and their justification; the quadratic and the cubic numbers; the root extractions; proportions; both arithmetic and geometric progressions; rules of three and false position; logarithms.

Subsequently, Geometry was taught, which was considered vital for the development of mathematical reasoning.

This science [Geometry] requires all possible attentions because it is the basis for the lessons of the following years: and because in that science there must be the use of judgment to feel the evidence of the mathematical reasoning; to search for accuracy and

Geometrical rigor of the Demonstrations; and to think methodically in every subject. ('Do Curso Mathematico', 1772, p. 172)

Here, the lecturer would once again start with history (of Geometry) followed by notions, definitions and fundamental principles, the relationship between theory and practice and finally the stereometry and its application to measurement of tunnels, piles and ships.

In the last part of the course trigonometry was to be studied: preliminary notions, construction of tables for *sines* and *cosines*, use of those tables and analysis of triangles using trigonometry.

### **The lessons on Algebra**

Algebra was the 2<sup>nd</sup> year course because it was understood as being more abstract and containing principles that were more difficult than Arithmetic and Geometry. As a result, the lecturer was due to use all his capacity for making his students understand such an important science.

The lecturer should engage himself with care in the complete and profound instruction of his students in this sublime and important Science, from which it depends the large progress, that can and should be done during the frequency of the degree of Mathematics: because it is the place, where all the spirit of invention, so necessary to this science, is formed; and it is the instrument to everything that can be discovered about quantity. ('Do Curso Mathematico', 1772, p. 175)

The study of Algebra started with a historical abstract and the presentation of the three principal items: to express any and every circumstance, conditions and relationships of quantities in algebraic language; to know how to combine the conditions with each other and to do all operations on them to reach the intended purpose; finally to explain the result of algebraic manipulations.

Then, students would learn literal calculus (the fundamental operations on magnitudes simple, complex, fractional, rational and irrational), equations (general and particular properties, method of preparation and transformation), fundamental rules of analyses, conic sections, differential calculus (definitions, differentiation rules and general theory of curves) and integral calculus (fundamental rules and applications).

One of the most important aspects in Algebra should be its connection with Geometry. To relate these two sciences, the lecturer, after presenting the literal calculus and equations, should present to his students the use of the rules in solving geometrical problems (determined and undetermined) and should practice those exercises until students achieve skill and sagacity. The same should be done for differential and integral calculus ('Do Curso Mathematico', 1772, p. 177-182).



### **The lessons of Phoronomy**

In the lessons of Phoronomy, in the 3<sup>rd</sup> year, the physics of bodies, fluids, light and sound were studied.

The aim to study physics in the Faculty of Mathematics was to give a profound knowledge of the science of movement. The students were meant to attend lectures during the second year, lessons on experimental physics in the Faculty of Philosophy where they were to learn about the physical phenomena. Then, in the lessons of Phoronomy, the lecturer should provide justifications to those phenomena using chains of mathematical reasoning, given by the methods of calculus and geometry. The lecturer was preparing the students to go beyond the obvious, analyzing and generalizing the studied principles ('Do Curso Mathematico', 1772, p. 182-184).

The lessons of Phoronomy started with the study of the physics of bodies, the statics, mechanics, dynamics and ballistics. Following these, the students were to study the physics of fluids: hydraulics, hydrostatics and hydrodynamics. The physics of light contained optics, dioptrics, catoptrics and perspective. Finally, in acoustics, students were to analyze a small number of phenomena such as the vibrations of chords, using the Mathematical calculus.

Although, Architecture was not clearly seen as part of Mathematics, students could also study it in form of mechanical problems.

### **The lessons of Astronomy**

The lessons of Astronomy started, as usual, with a historical abstract: beginning with the Astronomy to Hipparchus, going through Ptolemy, Albategnius, Kepler and Newton. The historical abstract, along with the presentation of the subject and method, aimed to introduce students to the course with pleasure and motivation ('Do Curso Mathematico', 1772, p. 189-190).

The lessons would then continue with a preliminary treatise on spherical trigonometry followed by three different aspects of Astronomy: the knowledge of phenomena deduced by observation; the proof of the physical reasons for the observed phenomena; the establishment, in consequence of these reasons, of rules to determine the studied phenomena at any given time.

The fixed stars, the planetary movements and the eclipses of satellites were also studied. In the last part of the lessons the Chronology and the Calendar were considered without reference to its historical development.

The study of Astronomy was not only theoretical but also practical. An Astronomical Observatory was built and furnished with the best instruments available at the time, so that the students could have practical sessions and be instructed in the observation of astronomical phenomena.

## **The lessons of Design and Architecture**

In the Design and Architecture course, students were able to learn metaphysics of the design (skill, style, accuracy and expression), Civil Architecture, Military Architecture and maps.

As referred previously, these lessons were not compulsory but the students were advised to attend to it during their 3<sup>rd</sup> or 4<sup>th</sup> years.

## **The Exercises**

During the Mathematics degree the students had to make several exercises. Oral, practical and written exercises were proposed daily, weekly and monthly.

The oral exercises aimed to prove the understanding and demonstration of the contents. All students were asked to explain some proposition or a theorem, and those who demonstrated a difficulty in this, would benefit from the help of others, designated as their tutors. This was definitely a proof of real cooperative education ('Do Curso Mathematico', 1772, p. 198-202).

However, the exercises were also practical, since Mathematics was applicable to different uses in daily life ('Do Curso Mathematico', 1772, p. 202).

Written exercises were problems requiring that students meditate about what they had learnt, combining subjects and developing new concepts. Students were encouraged to deliver their solutions as fast as they could, since the lecturer would only receive 5 or 6 good solutions. Those solutions were then published as a method to stimulate students' further learning. There were also monthly exercises, which required some discussion, presented by every student in a brief dissertation. Again, those exercises were used to prepare students for research ('Do Curso Mathematico', 1772, p. 204-205).

## **THE TEXTBOOKS**

The textbooks for which the lecturers should follow their lessons were also considered in the Statutes. Although there was no fixed textbook, at the end of each academic year, the Congregation of Mathematics<sup>3</sup> should choose proper textbooks for the following year, since mathematics was considered to be a science in constant development. Nevertheless, the chosen textbooks were deemed to have to accomplish some conditions:

[The Congregation] should always consider that the Treatises, that have to be explained, are concise and elementary; and contain the most effective and sublime methods that are known; so, those who gain their degree following them, will be entitled to understand without obstacles the deeper scripts that exist in these sciences. ('Do Curso Mathematico', 1772, p. 164)

A case was also considered where no adequate textbook existed for a specific course. In that instance, the lecturer should write his own treatise in order to facilitate the

students' study and to contribute to development and research of Mathematics as well as those sciences that depend on Mathematics. For those treatises that were chosen, no oral commentary was allowed except for the lecturer who, when necessary, had to present all the demonstrations to his students.

The choice of textbooks was therefore a task for the Congregation, but there is one textbook mentioned explicitly in the Statutes: Euclid's *Elements*. The geometrical books of Euclid's *Elements* (from 1<sup>st</sup> to 6<sup>th</sup>) had already been translated into Portuguese, in 1756 by João Ângelo Brunelli, and so this was used for Geometry lessons. The choice of this textbook is also clearly justified in the Statutes:

In it [the Geometry textbook] it is required that not only each geometrical truth is rigorously demonstrated; but also that all together these form a stable, sequenced and continuous chain of matters; there should not exist any lonely proposition; but all interacting with each other. And with these advantages there is no other author to the present date who wrote, with such perfection, as Euclid did in his *Elements*, for the reasons of which the lecturer will follow the lessons. ('Do Curso Mathematico', 1772, p. 164)

Together with Euclid's *Elements* the Statutes also mentioned Proclus' *Commentary on Euclid*, noting that the lecturer should follow it in order to explain to the students the metaphysics of Geometry.

By the time that the Faculty was created, *Elementos de Arithmetica* and *Elementos de trigonometria plana* by Bezout were also chosen for Geometry (along with Euclid's *Elements*). *Elementos de Analisi Mathematica* by Bezout were chosen for Algebra. *Tratado de Mecânica* by Marie, *Tratado de Hidrodinamica* by Bossut and *Optica* by La Caille were the three textbooks for Phoronomy, while *Astronomia* by Lalande was chosen for the 4<sup>th</sup> year of the degree (Freire, 1872, p. 38).

The privileged choice of French authors for the Faculty of Mathematics did not seem casual since Portugal and France had maintained very good relationship in relation to concerns in education and the general development of sciences. Indeed, from the very beginnings of Portugal, we can find Portuguese Mathematicians as well as other scientists attending the prestigious French Universities as students, such is the case of Manuel de Azevedo Fortes or José Soares de Barros e Vasconcelos. Admirably, the cited textbooks were immediately translated into Portuguese facilitating this way the study of these mathematical subjects.

## FINAL REMARKS

The creation of the Faculty of Mathematics at University of Coimbra by 1772 and the principles defended by the Statutes that ruled it conduct to a change in the study and teaching of Mathematics in Portugal. This huge reform of the teaching of Mathematics leads to the development of the teaching of mathematics at other levels. Furthermore, in the Academies which prepared the Portuguese army (Brazil

included) Mathematics was a central subject and many students graduated in the Faculty became lecturers at these Academies.

The foundation of an Academy of Sciences in Lisbon, by 1779/1780, also fomented the study of Mathematics, particularly by one of its founding and most influent partners: Monteiro da Rocha, himself.

Nowadays, in Portugal, as well as in many other countries, the mathematics curriculum follows closely the *Principles and Standards for School Mathematics* published by the National Council of Teachers of Mathematics (NCTM).

These “new” standards for teaching mathematics present, in our opinion, the same structure and the same ideals already presented in the Statutes for the Faculty of Mathematics of University of Coimbra, almost 250 years ago, namely: rigour in teaching a science such as Mathematics; the importance of the study of Mathematics for everyday life; the importance of the History of Mathematics for the contextualization of the content taught and for increasing student motivation; the importance of a systematic and continuous study by students to a better understanding of the content; the cooperative education that allows students to help each other in pursuit of success; the careful choice of textbooks suitable for students and enabling them to acquire mathematical knowledge consistently; and, in addition, the constant appeal to promote excellence and encourage effort in students’ work.

In Portugal, together with this “perfect” historical example, we might/should also speak of the possible cultural advantages of referring/following such an important curriculum in Mathematics.

Simultaneously, teachers and others responsible for the development of mathematics curriculum worldwide might also use these Portuguese Statutes both as a tool in their work, look at the examples of teachings presented in this document in daily classes, and as a reference to reflect upon Principles and Standards of nowadays. In fact, we are in this matter sharing Herodotus’ opinion that by knowing the past we can better understand the present and steadily prepare the future.

## NOTES

1. As cited in Estrada & Ralha, 2008, p. 733.
2. Phoronomy was the name given, in the 18<sup>th</sup> century, to the physics of the movement.
3. The Congregation of Mathematics was a governing body, chaired by the University Rector and formed by all lecturers of the Faculty of Mathematics, with the aim of making the Statutes to be accomplished.

## REFERENCES

“Do Curso Mathematico” [From Mathematics Degree]. (1772). *Estatutos da Universidade de Coimbra [Statutes of University of Coimbra]*, vol. 3, 141-221. Lisboa: Na Regia Officina Typografica.

- ESTRADA, M. F. & RALHA, M. E. (2008). *Reflections Upon a “Method for Studying Maths”, by José Monteiro da Rocha (1734 – 1819)*. In E. Barbin, N. Stehlíková & C. Tzanakis (Ed), *History and Epistemology in Mathematics Education, Proceedings of the 5<sup>th</sup> European Summer University*, (pp. 731 – 739). Plzeň: Vydavatelský servis.
- FERNANDES, J. & FIGUEIREDO, F. (2006). *José Monteiro da Rocha. A Portuguese Astronomer and Mathematician. The work on comets. CIM Bulletin, vol. 21 (1), 26-28*. Retrieved from <http://www.cim.pt/files/publications/b21www.pdf>
- FREIRE, F. C. (1872). *Memória Histórica da Faculdade de Mathematica nos cem anos decorridos desde a reforma da Universidade em 1772 até ao presente [Historical Memoir of the Faculty of Mathematics in the one hundred years since the University's reform in 1772 to present date]*. Coimbra: Imprensa da Universidade.
- MIRANDA, M. (2009). *Código Pedagógico dos Jesuítas. Ratio Studiorum da Companhia de Jesus. Regime escolar e curriculum de estudos. Edição bilingue latim – português [Pedagogical Code of the Jesuits. Ratio Studiorum of the Society of Jesus. Scholar regime and curricula of the studies. Bilingue edition Latin – Portuguese]*. Lisboa: Esfera do Caos.
- National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.
- QUEIRÓ, J. F. (1993). *A Matemática (1537-1771) [The Mathematics (1537-1771)]*. In CORREIA, A. F. et al. (Ed), *História da Universidade em Portugal [History of the University in Portugal]*. Retrieved from <http://www.mat.uc.pt/~jfqueiro/HistUniv.pdf>
- TEIXEIRA, G. (1934/2010). *História das Matemáticas em Portugal [History of Mathematics in Portugal]*. Lisboa: Arquimedes Livros.

# USING STUDENTS' JOURNALS TO EXPLORE THEIR AFFECTIVE ENGAGEMENT IN A MODULE ON THE HISTORY OF MATHEMATICS

Maurice O'Reilly

CASTeL, St Patrick's College Drumcondra, Dublin 9, Ireland

*The author taught a module on the History of Mathematics to a class of nine second and third year students taking a three-year BA in Humanities. Students were required to keep a learning journal for the duration of the module. This paper considers extracts from students' journals that can reasonably be identified as coming from their respective 'participant' persona (Liljedahl, 2007). These extracts have been classified in an effort to gain insight into the affective domain of these students' engagement with the module. Questions of sustainability, contingency and authenticity arise.*

Key words: journaling, affective domain, history of mathematics, persona

## INTRODUCTION

It can take a while for a mathematician to gain confidence to offer a module in the history of his subject. In my own case, eleven years! My first serious question arose about spring 1997, while teaching algebra to computing students at Dundalk Institute of Technology, between Dublin and Belfast. It was this: The civilisation of Baghdad had a flourishing mathematical community in the late 8<sup>th</sup> century (CE). At that time there was a vibrant civilisation in Ireland. What, if any, mathematical activity existed in the latter? I still have not answered this question, yet I know more about it than I did in 1997 (O'Reilly, 2009). I offered a module on the history of mathematics for the first time in the spring semester of 2008. The second time was two years later. In the intervening period, I had the good fortune to attend the HPM satellite meeting at ICME-11 in Monterrey, Mexico. Fourteen months later, Jan van Maanen presented a paper in Dublin (van Maanen, 2009) which stopped me in my tracks. It alerted me to the deeply intricate interactions between History of Mathematics, Mathematics and Mathematics Education, and, in particular, it propelled me to begin to use original sources.

Glaubitz (2010) suggests that using original sources in teaching is a demanding activity, yet the rewards can be very significant. He mentions, *inter alia*, how such sources sharpen students' awareness of representation and provide material for debate on what matters. Indeed, their very strangeness can give rise to useful 'anchor points' for teaching. His appraisal of the use of original sources in the mathematics education is positioned in a landscape of hermeneutics. Kiernan (2010) also acknowledges the challenge to the instructor in designing a History of Mathematics module and, in particular, when the use of original material is considered. These authors echo the work of Jahnke (2002) who recognized the study of original sources

as the most ambitious, yet most rewarding, way to integrate history into teaching mathematics.

Janqvist (2009, 2010) distinguishes between two models of using History of Mathematics in Mathematics Education, one where history is used as a ‘tool’, the other where it is the ‘goal’ of the endeavour. In the planning and implementation of the latter model, ‘meta-issues’ will arise including: how students’ ‘virginal beliefs’ are modified during a history course; how to anchor comments, arguments and discussion; the need to orientate questions in a variety of registers (e.g. historical, sociological or philosophical). He calls for more research in the HPM community to relate HPM to Mathematics Education research. One can expect a healthy symbiosis! Fried (2010) raises a call to arms (or is it peace?):

The tension between the aims of history of mathematics and the other aims in mathematics education must be confronted if one wishes to embrace the history of mathematics not as a tool but as an inquiry important in its own right.

He makes the case for Mathematics Education to be ‘justified’ not just on the basis of utility, but also of culture. This issue was discussed by OReilly (2000), albeit in relation to Mathematics.

This paper considers issues arising from personal experience of offering a one-semester undergraduate module on the history of mathematics in a two-subject BA programme, with mathematics as one of those subjects. An effort is made to situate this experience in a broader setting. Part of this setting is informed by the author’s collaborative work on mathematical identity and narrative (Eaton & OReilly, 2009), although an explicit link to this work is not made here. Instead, students’ learning journals are used as primary data; these are set in a framework modifying that of Liljedahl (2007). The question addressed is: how can evidence from students’ journals be harnessed to explore their affective engagement with mathematics and its history?

## **OVERVIEW OF MODULE DELIVERY**

The module under discussion was a required mathematics module taken by second and third year BA students in the second semester (late January to mid-May) of the 2009-2010 academic year. It was spread over a period of fourteen weeks (excluding a two-week Easter vacation period) and involved 44 contact hours nearly a quarter of which took place in a computer laboratory. Seven second-year and two third-year students took the module. Typically few BA students choose mathematics.

The stated learning outcomes of the module were that students should be able to:

1. Outline in very general terms the timeline of the development of mathematics
2. Describe significant historical periods when key changes in mathematical thought occurred and new areas emerged

## Working Group 12

3. Summarise some important contributions of prominent mathematicians
4. Explain how topics arising in school mathematics developed historically
5. Discuss important examples of cultural factors influencing the development of mathematics
6. Discuss the technical details of specific mathematical problems pertinent to 2-5, above
7. Situate points 2-5, above, in a broader historical context

In retrospect, examining these through the lens of Jankvist's 'tools' and 'goals', it seems appropriate to conceive learning outcomes 1, 2 and 5 as focusing on *the developmental and evolutionary aspects of mathematics as a discipline* (Janqvist, 2009) and hence on 'history as a goal'. On the other hand, learning outcomes 3, 4 and 6 are concerned more with 'history as a tool', that is with *how students learn mathematics*. Outcome 7 serves both of these purposes.

The module was anchored around reading Derbyshire's *Unknown Quantity* (2006). This book was chosen because it is accessible (and affordable!); moreover it covers a broad canvass of the history of algebra as well as touching on aspects of geometry. Anyone acquainted with the book will know that it is not without shortcomings. Some serious reviewers (Grabiner, 2006; Katz, 2006; Segal, 2008) will consider this an understatement, yet the defects are, for the most part, in the detail or in Derbyshire's sometimes arbitrary preferences in his emphasis and his irksomely gratuitous remarks. This author has found the book a helpful bridge for undergraduate students between mathematics and its history. Its shortcomings were compensated for by using additional reliable sources (e.g. Bos 1980; Cardano, 1545; Kwārizmī, 1831; Leibniz, 1675/1920) to expose students to original material in some detail; moreover the book's defects led to interesting discussion on meta-issues relating to the module.

Most of the module dwelt on the history of algebra (from its origins to the solution of polynomial equations to the emergence of abstract algebraic structures) with a three-week insertion of the history of the calculus (with emphasis on Leibniz' contribution). Both the computer algebra system, Maple, and the dynamic geometry system, GeoGebra, were used to support students' exploration of the mathematical detail of the material under consideration (e.g. Kwārizmī's classification and solution of quadratic equations and Leibniz' transmutation rule, following the actual historical conceptions closely). A variety of web (e.g. O'Connor & Robertson, 2010; Lee 2007) and library resources was recommended. Thus there was a significant diversity of register amongst the resources presented to students.

Now let us turn to the focus of this paper, the students' learning journals. As part of assessment for the module, students were required to keep a learning journal (LJ). Students were directed as follows:



## Working Group 12

You should bear three things in mind as you prepare your LJ:

1. It should indicate your engagement with the [module].
2. It should include reflection on this engagement.
3. You should see it as useful for [module] revision when preparing for the final exam.

For 1, you are recommended to include a summary (in your own words) of the main topics/themes encountered, covering both mathematical concepts and historical notes. You should complement each summary by indicating the key technical details of relevant problems.

For 2, you might wish to comment on how you encountered each topic/theme: what you found surprising; what you found easy or hard; to what extent GeoGebra or Maple helped you with your understanding; how your personal understanding of a particular topic/theme has been influenced by studying its historical evolution.

Clearly these directions were not prescriptive, thus allowing each student considerable leeway in choosing how to articulate her/his own learning. All nine students show significant engagement in the module. However some chose to focus exclusively on summarizing mathematical and historical detail (valuable for module revision and exam preparation), rather than reflect on their engagement and reveal aspects of the affective domain.

## DATA FROM STUDENTS' LEARNING JOURNALS

### Students' personas

Liljedahl (2007) considers three personas, *narrator*, *mathematician* and *participant*, that students use in journaling. His goal is to achieve authenticity in representing the processes in which students engage in tackling mathematical problems.

The *narrator* moves the story along. ... The *mathematician* is the persona that provides the reasoning and the rational underpinnings for why the mathematics behind the whole process is not only valid but also worthy of discussion. Finally, the *participant* speaks in the voice of a real-time evolving present. This persona reveals the emotions and thoughts that are occurring ... as he is experiencing the phenomenon. (p. 663)

Let us adapt these personas to the purpose of the title of this paper. The *narrator* chronicles the progression of the module. The *mathematician* documents the mathematical detail (both conceptual and technical). The *historian*, a new persona necessary in this context, documents the historical detail (narrative, timeline, context, etc.). The *participant* reveals what is actually happening for the student as the module progresses. For our purpose, we focus exclusively on the participant persona, for it is precisely this persona who reveals a student's affective engagement in the module. It is important to point out that personas, by their nature, overlap to some extent, and so we choose (as outlined below) to consider a rather broad expression of

the participant persona including situations when this persona draws on the substance of mathematics or history, but with an affective lens.

Of the nine students taking the module, only five of them expressed their participant persona significantly in their journals. There were 161 such expressions amongst these five: Michael (13), Olivia (33), Rachel (37), Seán (30) and Tara (48).

### **Classification of data in the affective domain**

In classifying these 161 instances, nine codes were identified:

- Expressions of *interest* (33)
- Expressions of *enjoyment* (17)
- Expressions of *surprise* (29)
- *Anticipating* (looking forward) (10)
- *Recalling* (linking to already known facts) (5)
- *Reflecting* (articulating general insight) (34)
- *Understanding* (articulating specific insight) (25)
- Identifying what is *easy/hard* (39)
- Identifying what is *helpful* (36)

The distinctions between the first three of these are not always clear-cut. Typically ‘interest’ refers to a fairly neutral expression, ‘enjoyment’ indicates something stronger, while ‘surprise’ identifies situations where a student has shown a significant change of attitude or deepening of insight.

In the context of the data, it makes sense to refine Liljedahl’s participant persona into five sub-personas:

- The *maths participant* who expressed evolution of mathematical understanding (95)
- The *history participant* who expressed evolution of historical understanding (32)
- The *inner participant* who expressed a strong personal conviction indicating attitudinal change (15)
- The *outer participant* who expressed a considered judgement about the world outside (13)
- The *spectator* – the other participant who articulated the affective domain without evidence of particular understanding, conviction or judgement (27)

The frequency of each code and sub-persona is indicated in brackets. Of course the sum of the frequencies, in each case, exceeds 161. The general approach to classification here is well-established grounded theory adapted to the context and scale of the data. Here, the maths participant is to be found where the mathematician

and participant personas overlap; likewise the history participant is located where the historian and the participant meet. The intention is that this refinement of the participant persona will bring out more clearly distinctions within the affective domain as students reflect on mathematics and its history. The use of the nine codes appears to be of much less significance. In the presentation of data and discussion which follow, we will focus on the personas.

## PRESENTATION OF DATA

Let us read the words of each persona (identifying the student by pseudonym and indicating the codes in each case). First, the *maths participant* relating how he feels and thinks about making progress with his understanding of mathematics:

Question 2 [Let  $x = t$  be a root of the cubic  $x^3 = cx + d$ . Find the other two roots in terms of  $c$ ,  $d$  and  $t$ .] was one of the questions I struggled with but when it was explained on the board, the ease with which  $x = t$  was found to be a solution of  $x^3 = cx + d$  was eye opening. It certainly made me feel it could be accomplished easily. (Michael; surprise, easy/hard.)

This was called Leibniz' transmutation. It is developed through the use of the characteristic triangle and yields a transformation of the quadrature of another curve, related to the original curve through a process of taking tangents. I found this lesson useful for developing the idea of how Leibniz developed the transmutation. (Michael; helpful.)

The *outer participant* joins in with the *maths participant* as they make a judgement about the presenter's behaviour yet in the context of mathematical reflection:

In today's lecture we watched Donny Lee from Singapore, work on Leibniz and his formula for  $\pi$ , on You Tube. He links the famous formula ... to Leibniz transmutation rule. He does this with great enthusiasm, so much so that when watching it for the first time I believe he got 'carried away' to a certain extent which resulted in me being completely lost. However when I watched it for a second time I think I learned and understood a little more. I hope this will continue to happen. (Rachel; reflecting, understanding.)

Now we hear the *maths* and *history participants* together as they place mathematical development within an historical narrative:

For the essay I picked the history of notation as I thought it covers so much time that it would help me learn more about the development of Maths. I enjoyed doing it as I was interested in the development [through] the ages and what different people at different times [did]. But I found it very hard to condense such a massive topic. I could have [written] pages and pages. I also [tried] my best to bring in some [GeoGebra] and Maple. (Tara; interest, enjoyment.)

Next the *history participant* alone, reflecting on how the historical discipline helped her understanding:

## Working Group 12

From keeping the Journal and constructing the time line, I know if I had to write a bit about all of the above, I could. I found this really [helpful]. (Tara; helpful.)

The *history* and *inner participants* join forces showing a powerful transformation in perspective, articulated unambiguously:

Since researching my essay on women mathematicians my attitude towards mathematics has been altered. Although of course these male mathematicians achieved great things during the 16<sup>th</sup> & 17<sup>th</sup> century, I can't help but think about women such as Sophie Germain and Maria Agnesi. These women had to fight so much harder than their male counterparts to achieve [and] succeed in this area. They did not receive the same respect or education that they should have been entitled to. (Olivia; surprise, reflecting.)

Now we hear the voice of the *inner participant* alone, clearly asserting his own learning style and suggesting a personal strategy on how to gain even more insight:

Felt in a much better mood about the maths since last class and was raring to go. As a result, dove in today head first into analysing the diagrams given on 23/2 (yesterday) and using the graphs and seeing Geogebra in action helped me visualise. From today's [two] classes I have concluded that I am in fact a very visual learner rather than theoretical [and] algebra side of things. All I need to do now is to make stronger links between them! (somehow) ☺ (Seán; anticipating, reflecting.)

Here is the strong voice of the *outer participant* in the presence of the *history participant*, critical of Derbyshire's treatment of the narrative of early algebra:

I feel that Derbyshire presents a biased attitude towards Al-Khwarizmi's achievements and the Arabic way of thinking in general. He has a very negative opinion of Al-Khwarizmi's way of thinking. In fact, I feel that he has an extremely condescending attitude towards this great Arabic mathematician. On the other however Derbyshire appears to believe that the Hellenic way of thinking is much superior to the Arabic way. He praises Diophantus throughout. (Olivia; reflecting.)

Finally an example of the *spectator*, expressing interest in the preoccupation of the Arabic mathematician with Islamic law:

We learn that the Islamic law of inheritance played an extremely important role behind the development of algebra. I find it extremely interesting that Al-Khwarizmi devoted a chapter in his book 'al-jabr' to the solution to the Islamic law of inheritance using algebra. (Rachel; interest, reflecting.)

## DISCUSSION AND CONCLUSIONS

Here the cohort of students is small – only nine – and the cohort of those who engaged in the affective domain a mere five. Yet it is remarkable how rich the data is and how varied the style of its expression. The classification by code and by (sub-) persona provides a structure underpinning an intricate interaction between different aspects of students' learning processes in the affective domain and the implementation of diverse teaching approaches.

The voices of these participants, and, in particular, that of the *inner participant*, resonate with recent work of Rosas Mendoza and Pardo Mota (2010). These authors report on how students engage in a very personal way with mathematical ideas when working with original sources, such as Newton's *Principia*. Two of the extracts given above (the second and last) demonstrate significant engagement with original sources. There is strong evidence, especially in the voice of the *history participant*, of a humanistic understanding of the development of mathematics (Liu, 2010), and that sustained collective effort with mathematics bears fruitful results. All participants, and especially the *maths participant*, displayed evidence of 'working with doubt' and of the 'struggling mathematician' which/who draws a parallel between the learning of mathematics and the development of mathematics by its first inventors (van Maanen, 2010). We see here that working with an historical lens invigorates learning.

The module described here drew from a variety of sources: YouTube, 'popular' mathematics, fundamental invention (Lee, 2007; Derbyshire, 2006; Leibniz 1675/1920); it evoked diverse affective (and other) engagement from students as evidenced by the personas and sub-personas described above. Liljedahl (2007) suggests that students work with more mathematical awareness when they tune in explicitly on personas while journaling. Such 'specific focus' places demands on students. To what extent are such demands reasonable? It seems that the use of journals is particularly fruitful when used in the context of history of mathematics. Moreover, it is helpful to refine the participant persona as described in this paper since such refinement helps distinguish aspects of students' affective engagement with mathematics and its history. There are, of course, practical questions when we seek to implement such a framework in the teaching situation. How might students tune in to their personas and sub-personas, even when not journaling? How can this work be sustained, be implemented efficiently, be deepened? How can its contingency be nurtured, its authenticity assured?

## REFERENCES

- Bos, H.J.M. (1980). Newton, Leibniz and the Leibnizian tradition. In I. Grattan-Guinness (Ed.) *From the calculus to set theory 1630-1910* (pp. 49-93). Princeton University Press.
- Cardano, G. (1545). *Ars magna*. Retrieved on 14 September 2010 from [http://www.filosofia.unimi.it/cardano/testi/operaomnia/vol\\_4\\_s\\_4.pdf](http://www.filosofia.unimi.it/cardano/testi/operaomnia/vol_4_s_4.pdf)
- Derbyshire, J. (2006). *Unknown quantity: A real and imaginary history of algebra*. Washington DC: John Henry Press.
- Eaton, P. & O'Reilly, M. (2009). Who am I and how did I get here? Exploring the mathematical identity of student teachers. In D. Corcoran, T. Dooley, S. Close & R. Ward (Eds), *Proceedings of Third National Conference on Research in Mathematics Education* (MEI 3), Dublin, 228-237.

- Fried, M.N. (2010). History of mathematics in mathematics education: Problems and prospects. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.
- Glaubitz, M. (2010). The use of original sources in the classroom – Empirical research findings. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.
- Grabiner, J. (2006). A backward look at modern algebra [Review of Derbyshire, op. cit.]. *American Scientist*, 94(5).
- Jahnke, H.N. (2002). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds), *History in Mathematics Education*, New ICME Study Series, 6, 291-328.
- Jankvist, U.T. (2009). A categorization of the “whys” and “hows” of using history in mathematics education. *Educational Studies in Mathematics*, 71(3), 235-261.
- Jankvist, U.T. (2010). An implementation of two historical modules: outcomes and perspectives. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.
- Katz, V. (2006). A faulty survey of algebra’s roots [Review of Derbyshire, op. cit.]. *Science*, 312(5779), 1473-4.
- Kiernan, J.F. (2010). The use of original sources in an undergraduate history of mathematics class. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.
- Kwārizmī, Muhammed b. Mūsā al- (1831). *The algebra of Mohammad ben Musa* (Rosen, F., Trans. & Ed.). Retrieved on 14 September 2010 from <http://www.archive.org/details/algebraofmohamme00khuwuoft>
- Lee, D (2007). Leibniz quest for pi pt1. *YouTube* retrieved on 14 September 2010 from <http://www.youtube.com/watch?v=GAKkTa2XpEc>
- Leibniz, G.W. (1675/1920). *The early mathematical manuscripts of Leibniz* (Child, J.M., Trans. & Ed.). Retrieved on 14 September 2010 from <http://quod.lib.umich.edu/cgi/t/text/text-idx?c=umhistmath;idno=AAS8079.0001.001>
- Liljedahl, P. (2007). Persona-based journaling: Striving for authenticity in representing the problem-solving process. *International Journal of Science and Mathematics Education*, 5, 661-680.
- Liu, P.-H. (2010). Evolution of college students’ epistemological views of mathematics in a history-based class. In *Proceedings 6<sup>th</sup> European Summer*

*University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.

- O'Connor, J.J. & Robertson, E.F. (2010). *The MacTutor history of mathematics archive*. Retrieved on 14 September 2010 from <http://www-gap.dcs.st-and.ac.uk/~history/>
- O'Reilly, M. (2000). Mathematics and culture – a vital symbiosis. Retrieved on 14 September 2010 from [http://staff.spd.dcu.ie/oreillym/mathematics\\_&\\_culture.htm](http://staff.spd.dcu.ie/oreillym/mathematics_&_culture.htm)
- O'Reilly, M. (2009). Mathematics in early medieval Ireland. Retrieved on 15 September 2010 from <http://staff.spd.dcu.ie/oreillym/pages/memi.htm>
- Rosas Mendoza, A.M. & Pardo Mota, L. del R. (2010). Newton's Principia Mathematica in a twenty first century classroom. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.
- Segal, S.L. (2008). Unknown quantity [Review of Derbyshire, op. cit.]. *Notices AMS*, 55(5), 583-5.
- van Maanen, J. (2009). Maths for All. In D. Corcoran, T. Dooley, S. Close & R. Ward (Eds), *Proceedings of Third National Conference on Research in Mathematics Education* (MEI 3), Dublin, 53-66.
- van Maanen, J. (2010). The teacher as a researcher in the history of mathematics. In *Proceedings 6<sup>th</sup> European Summer University on the History and Epistemology in Mathematics Education*. Vienna: Technische Universität Wien. To appear.

# **A CROSS-CURRICULAR APPROACH USING HISTORY IN THE MATHEMATICS CLASSROOM WITH STUDENTS AGED 11-16**

Peter Ransom

Bath Spa University

*Over the past decade I have developed many lessons that use history in the mathematics classroom. These lessons are cross-curricular using events from history to enthuse students in their learning of mathematics. They show that the mathematics used today had, and continues to have, applications to everyday life. As well as being classroom lessons these episodes form mathematics masterclasses done on Saturday mornings around the UK incorporating Texas Instruments handheld technology. The lessons use old and new technology to provide students with an insight into how they are made and used. Recent developments with the STEM (science, technology, engineering and mathematics) initiative in the UK mean these lessons are even more relevant. The two newest episodes are described here.*

## **THE RESEARCH IMPETUS**

I incorporate events from history into mathematics lessons because I find it very interesting to see the practical applications of mathematics set into the period when it was used. My research questions are qualitative and are concerned with whether students find it interesting and if it helps improve their attitudes to mathematics. I chose to write about these two episodes since they have been developed over the past year and are both very rich in mathematical history. The Galileo episode emerges from an anecdotal point of view incorporating cross-curricular work with science, the Brunel one uses examples from his original calculation books.

Using Jankvist and Grattan-Guinness previous works (Jankvist 2009, Grattan-Guinness 2004), Tzanakis & Thomaidis (2011) classify the arguments and methodological schemes for integrating history in mathematics education and my episodes fit into the two-way table mainly as *History-as-a-tool* and *Heritage* though there are overlaps into the *History-as-a-tool* and *History* cell (*op.cit.* section 4, Table 1). The over-riding concept in my work is *History-as-a-tool*.

The reliability of this research in the sense of reproducibility by someone else is impossible to quantify, since teachers use such episodes in different ways with different students and probably not in costume! Every session I do with students is different according to local conditions and the knowledge students bring to the sessions, so results will vary.

## **THE EVOLUATION OF THE EPISODES**

Back in the 1990s I was introduced to the Royal Institution mathematics masterclasses for gifted and talented 13/14 year old students. These took place (and still do) on Saturday mornings and last for 2½ hours. They are organised by local



groups of interested teachers and generally take place at prestigious places covering schools within travelling distance. They put on a programme of 4 to 8 sessions (though most places do 6) inviting staff at local schools to nominate a couple of students who would benefit from these masterclasses. Members of staff are also invited to attend and without these volunteers the sessions would be difficult to run since they attract between 30 and 60 students each session. These volunteers help when students are involved in the tasks since the sessions involve a lot of interactivity. The topics offered vary considerably and cover a wide range of mathematical topics. As the years progressed I felt that there was plenty of material that was suitable for general classroom work and so by developing the masterclasses I do in my own classroom I was able to trial the work with a wide variety of ages and abilities.

In my opinion the work should have relevance to the students, so I always try to include material that covers mathematics primarily and history secondly. However science, engineering and geography also play significant parts and all these are brought together to provide a holistic coherent scenario in which to develop the mathematics. Technology changes daily and in preparing our students for the future we need to use the technology of today as well as that of yesterday so that students become digital natives rather than digital immigrants. In my school and masterclasses I use handheld technology by Texas Instruments since that is what I had in school and it fits the work remarkably well, being portable and wireless.

Sometimes the opportunities arise by chance: a suitable anniversary, an interest or an opportunity can be the spur to spending many hours pursuing threads which gets sewn into a rich tapestry of mathematics. The STEM (science, technology, engineering and mathematics) initiative in the UK is aimed at developing an interest in those subjects since they have shown a drop in numbers at degree level over the years. The STEM programme aims to rationalise and improve the provision of support for students. It puts into practice the recommendations of the STEM Programme Report that was published in October 2006. This report focused on how best to support STEM through school, post-16 education and university and how to streamline the current numerous STEM initiatives and implement them more effectively in every school, college and learning provider. The Government wants to increase students' STEM skills in order to provide employers with the skills they need in their workforce help to maintain the UK's global competitiveness and make the UK a world-leader in science-based research and development. The proposals in the STEM Programme Report build on the Government's strategies for developing a strong supply of scientists, engineers, technologists and mathematicians. By tackling these areas with scenarios that show how they were dealt with in the past I believe it gives students enthusiasm for mathematics and provides them with a rich background of general knowledge. The episodes I mention here are the two latest

ones that have been trialled with teachers and students in the classroom and then improved after feedback from them and my peers in the last year.

## GALILEO AND PENDULA

This was the basis of two one-hour classroom lessons with 15 low-achieving 14 year olds. I wanted to introduce some wireless handheld technology (TI-nspires) so that the class could develop their IT skills, so we set up an experiment that involved just string and a mass to act as a pendulum. The students were asked to work in pairs since there was a need for one to record data and the other to time the pendulum swinging. After logging on to the wireless system in the classroom, all students were sent a file to the nspire handhelds. This consisted of a page about Galileo and a bit about his work with pendula, but not the result he found. The advantage to the student (and teacher) is that the work they are doing is relayed wirelessly to the interactive white board and refreshed every 30 seconds so the teacher can see how the students are progressing with the work. There is also the added advantage that any student's work can be made to fill the board, so misconceptions can be identified and corrected at an early stage, or good work can be brought to the notice of the whole class with the student presenting their work.

The materials provided in the room included rulers, string, sticky tape, a few stopwatches and some small bits of metal with holes in that can be used as the pendulum's mass. They then worked in pairs to measure the time taken for 10 oscillations of a pendulum (using a stopwatch or their iPhone!) and recorded the time and length of the pendulum, after having discussed what factors they thought affected the time of a swing.

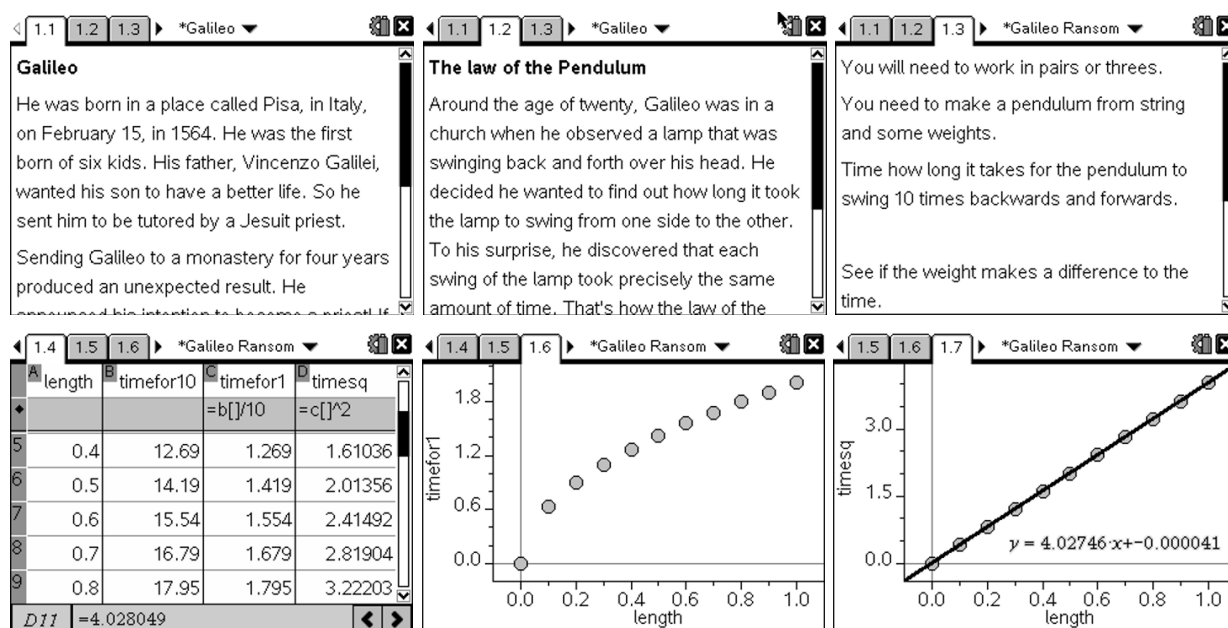


Fig 1: Screenshots from the pendulum experiment and analysis

## Working Group 12

They recorded the data collected in a spreadsheet page, using that to calculate the time of one swing. They then used a statistics page to show the data from the spreadsheet. With the low attainers in this class we stopped at that point though higher attainers can experiment with changing the variables on the axes until they find a straight line. Built in menus allow the student to find the equation of the straight line that links the length of the pendulum with the square of the time, then by considering the gradient an approximation for the value of acceleration due to gravity can be found.

Previously I thought this activity was too difficult to do without the technology since the low attainers did not have the graphing skills necessary and this would have frustrated their learning. Using the technology removed this blockage and allowed them to realise that there was a connection between the length of the pendulum and the time taken for a single oscillation.

The other benefits to this class' learning included the following: working cooperatively; developing their creative skills and recording the lengths in metres when they tended to use centimetres to measure. Watching them talk to each other about how and where to put the pendulum was fascinating. Some students attached the string to the doorframe, some to the backs of chairs which they put on the table so they could work with a pendulum length of about one metre. Others tried small lengths of less than 0.2 metres, but soon realised that there were some timing difficulties.

This STEM based activity links the science of pendula with the mathematics and involves students thinking for themselves about making working pendula. This episode was repeated with 90 15 year old students in a large hall at a London comprehensive school in November 2010. It was less successful than in the classroom due to other mathematical activities taking place at the same time meaning that the whole group could not be engaged in discussion about the findings. Feedback from students in the final stages was positive however – they enjoyed working on group activities more than working as individuals since they had learnt from each other and the new technology had not impeded their learning. (Sometimes it is the teacher who fears that technology they do not understand will interfere with their students' learning, but this is rarely the case.) They mentioned that they saw a reason why they had been taught gradient in the past because it now referred to a physical problem.

Did the historical aspect make a difference? I have no evidence to say 'yes' or 'no' at this point since no question was directed at this aspect. However the experiment lesson was preceded by one, where I asked the students to prepare a PowerPoint presentation about Galileo, his life and his mathematical works. This exceeded my expectations since the students were totally engrossed in the work, demonstrating their ability to use the Internet to obtain facts and create impressive files. (This links

in with the History side of the two-way table mentioned in Tzanakis & Thomaidis (2011).)

When trialling new work with my students I never know what will happen, but with over 30 years' experience of the 11-18 year classroom I think I can judge reasonably well what will be accepted. In the plenary at the end of a session students are asked about what they learnt and it is only the comments that students offer at the end of the lesson that give me an insight into whether these lessons are successful or not. Comments such as 'I learnt loads today about Galileo and some maths'; 'It was good working in pairs' encourage me to continue with such lessons adapting them for different classes. Comments such as 'It was too hard'; 'We didn't have a stopwatch' means I make amendments to the work and try again with another class. This episode survived and will be developed further over the next year.

## BRUNEL: BUILDING BRIDGES

Isambard Kingdom Brunel (1806-1859) was an engineer who designed and built many things thought impossible in his day. His Clifton suspension bridge still spans the Avon gorge near Bristol. Much has been done in the UK to encourage teachers and students to work with this image (and others) using dynamic geometry software to find the equation of the bridge's curve.



### Your investigation

Use a chain and supports to allow a chain to hang freely.  
Stick a paper ruler along the horizontal support.  
Use a ruler to measure the vertical distance of the chain from the horizontal support at intervals from where the paper tape starts.  
Enter your data into a [spreadsheet](#) page on the [nspire](#).  
Use a graph page to show a scatterplot of the data and see if you can find the equation of the chain.

**Fig 2: Isambard Kingdom Brunel yesterday and today and the student investigation**



**Fig 3: The chain and framework**

I decided to make this a more practical activity by investigating the shape of the curve caused by a suspended chain (see above). Initially the work was trialled with prospective teachers at Bath Spa University, then with actual teachers at the Brunel Museum in Rotherhithe, London. The work with 14/15-year-old students was on a large scale with 90 of them in a school hall. The equipment used was a chain (1 metre long) and framework I have previously used in another activity. It is possible to use a cereal box with the front panel cut away to suspend the chain.

They also work in a more practical sense by suspending chains on a framework and measuring the horizontal and vertical distances from an origin. This data they put into a spreadsheet page on a TI-nspire handheld then try to find the equation of best fit using either a graph or statistics page. This helps develop their measuring, algebraic and IT skills.

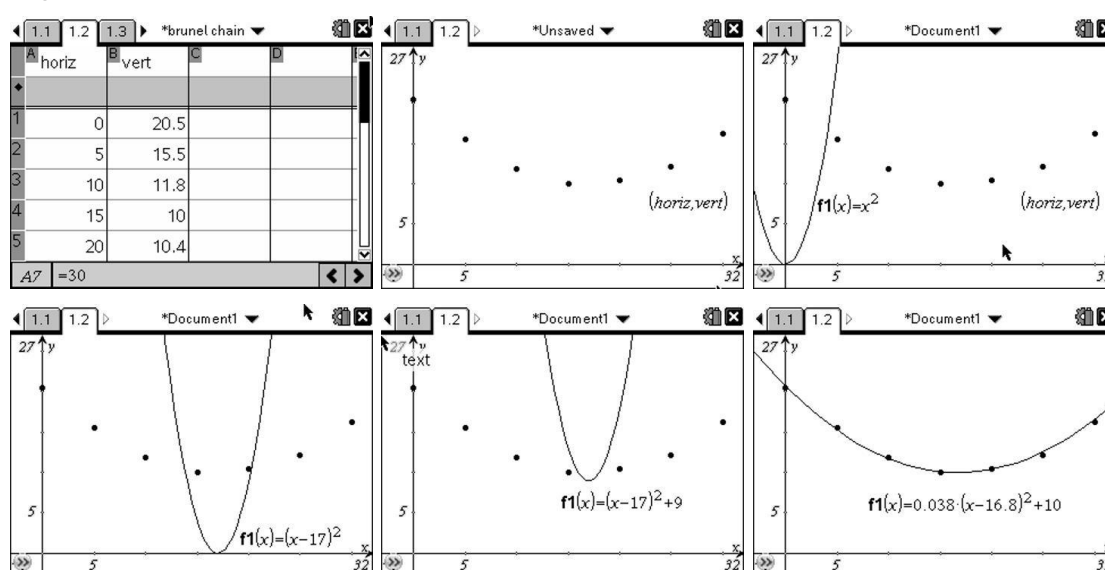


Fig 4: Screenshots of the data and successive transformations of  $y = x^2$

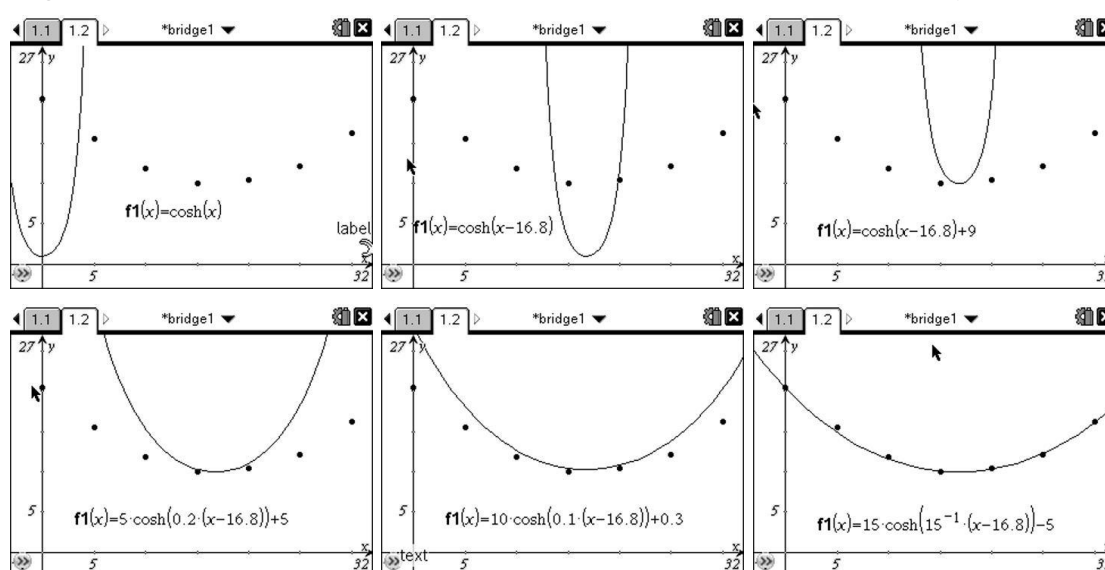
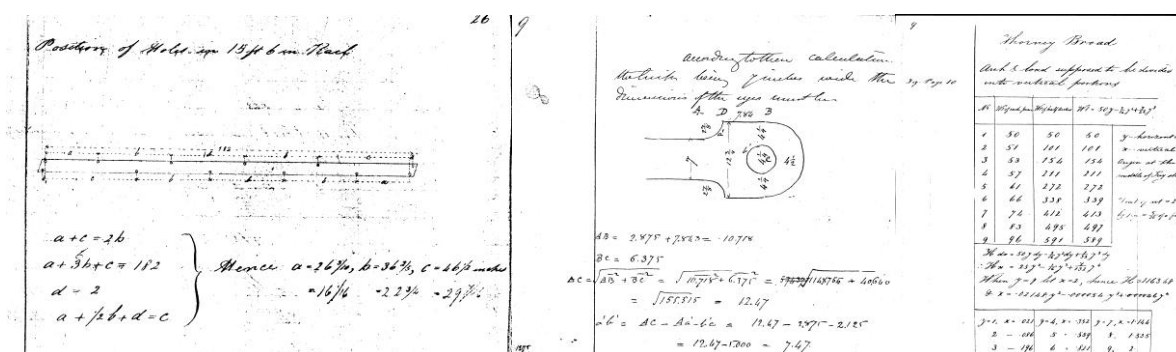


Fig 5: Screenshots of transformations of  $y = \cosh x$

Teachers and students appreciated the practical approach and felt that it strengthened their understanding of transformations. The work with the hyperbolic trigonometric function was not mentioned to students under-16 since that is not part of their syllabus, but the link between the transformations of the quadratic and  $\cosh x$  was appreciated by the teachers, though that aspect has yet to be trialled in the classroom.

It is possible to visit the Brunel collection at Bristol University and I have acquired photocopies of Brunel's actual calculation books (see illustrations below). Some of the mathematics he used is appropriate in today's classroom and worksheets have been developed that have been trialled in various situations.



**Fig 6: Brunel's use of simultaneous equations, Pythagoras and calculus**

Students are asked to check Brunel's solution to the simultaneous equations above, comment on the results and look at the proportion of the two sets of answers. These are used in calculating the regular distances between holes in a metal rod. One student found this work fascinating commenting that it was very interesting to see how simultaneous equations were used in real life.

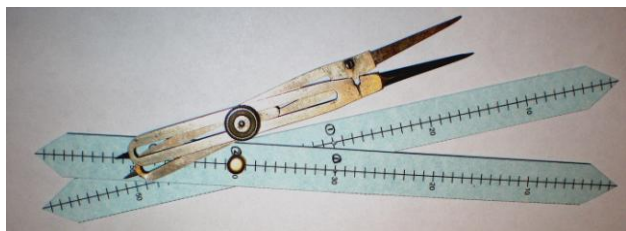
Students under 16 work with the Pythagoras manuscript, those who have experience of calculus also work with the Thorney Broad sheet. I cover up some of the terms and solutions and students have to work these out themselves.



**Fig 7: Brunel's box of mathematical instruments**

Proportional dividers, seen here at the bottom of the box (Isambard Kingdom Brunel's initials, IKB, are engraved on them) are used to enlarge diagrams with given scale factors, or used to divide circumferences of circles into a given number

of parts. Traditionally the sides are labelled with circles (to divide a circumference into 6, 7, 8 ..., 20 parts); lines (to enlarge a given line with a scale factor from 1 to 10); planes (to enlarge a given area with a scale factor from 1 to 10); solids (to enlarge a given volume with a scale factor from 1 to 10). The two legs are equal in length and the central slide allows the pivot to move, thus in effect creating two similar triangles. I have produced replicas in card that students cut out and make, using a paper fastener to act as a pivot. This gives a practical use of ratio and develops an understanding of similar triangles, enlargement and dividing a line in a given ratio. The accuracy is quite astounding!



**Fig 8: Actual and student made proportional dividers**

## SURVEY ANALYSIS

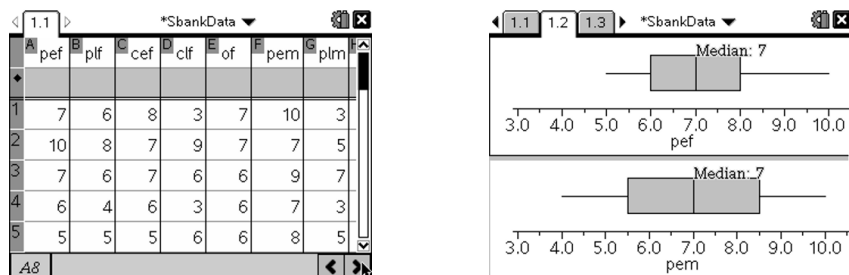
No formal survey was conducted when developing the materials due to time limitations in the classroom. However there was an opportunity in January 2011 to collect some data from a group of 24 students aged 15 and 16 from the Southbank International School who attended a session at the Brunel Museum at Rotherhithe, London. The students were at the museum for just over 3 hours which included a visit to the shaft that Brunel built. The purpose of the visit was to work with previously unused handheld technology to collect data that could be analysed in the classroom when they returned. This was a joint venture between the mathematics and science departments and both a mathematics and a science teacher were present from the school. The students were told to write down the data so that further study could take place when they returned to school. This meant that there was not much time available at the session to actually engage in the theory of the mathematics and science and that probably explains why the students gave it higher marks for enjoyment rather than for what they had learnt!

50 minutes was spent on each of the activities described (plus another 50 minutes on air resistance) and in the final 5 minutes an evaluation sheet was given to each student to gauge some opinion of each activity. They were asked to circle a number out of 10 for each activity and give some opinions. There were three lines available for students to write about what they had learnt and their comments.

The data was analysed using TI nspire Teacher Software and some of the screen shots of the analysis are shown below.



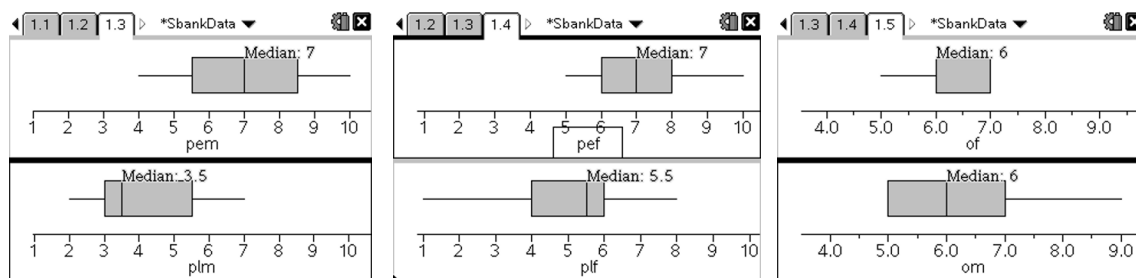
## Working Group 12



**Fig 9: Spreadsheet results with box and whisker plot comparing gender enjoyment**

This shows part of the spreadsheet with the column headings. The following abbreviations were used: p (for the pendulum activity), c (for the chains activity), e (for the enjoyment of the task), l (for the learning that took place), f (for a female response), m (for a male response), o (for the overall enjoyment). Therefore **pef** refers to the number given on the **p**endulum activity for the **e**njoyment by the **f**emales.

Here two box and whisker plots compare the pendulum enjoyment of males and females. We see that the median in each case is 7 and dispersion for males is more than that for females.



**Fig 10: Box and whisker plots of enjoyment and learning for each gender together with the overall enjoyment of each gender**

It is perhaps more interesting to compare what the students felt about their learning with their enjoyment. The screen shot on the left compares the males enjoyment and learning; that on the right compares the females enjoyment and learning for the pendulum activity. It appears that the females felt they learnt more than the males, though the small sample size means that further investigation would be needed before any firm conclusions could be deduced. Since no formal testing of the learning took place I make no claims for the efficacy of the activities apart from the enjoyment factor!

Some of the comments, being an open response, were not very illuminating. Here are some which were more relevant:

‘I learnt the difference between a catenary and a parabola.

It was fun and interactive, made learning easier.

I learnt how to find the equation of a curve.

I learnt how the length of the string affects the time.



I learnt how to use the calculator to adjust the graph.  
Good and interesting.’

It was a pity that I did not have more time with this student group to have more input into the analysis of the data collected with the activities, but the prime purpose of the day was to collect the data for them to analyse in their lessons. The historical aspect combined with the handheld technology provided the enthusiasm to make this an enjoyable experience overall as the third screen shot above shows.

Nobody scored the event below 5 and both the male and female median value was 6. The mean value by both genders was 6.2 (to 1 d.p.).

## CONCLUSION

My experience with using history in the mathematics classroom is in relating the mathematics students do with episodes from history when it was used. I believe students need to see where mathematics has been used to appreciate its importance. These activities were developed for use in the classroom and masterclasses, not as a research project, so the evidence of their success (or failure) is mainly ephemeral through comments received by participants after each session.

## REFERENCES

- Brunel, I.K.B. (1838?). Clifton Suspension Bridge volume 2 [book of manuscript calculations by the author]. Brunel Special Collection (CSB Vol. II: calculations, notes & sketches. 1838?-1850). University of Bristol Library
- Byrne, E. & Gurr, S. (2006). *Isambard Kingdom Brunel, a graphic biography*. Bristol: Brunel200
- Fauvel, J. & van Maanen, J. (eds) (2000). *History in Mathematics Education: The ICMI Study*. Dordrecht: Kluwer
- Grattan-Guinness, I. (2004). “History or Heritage? An important distinction in mathematics for mathematics education”, in G. van Brummelen & M. Kinyon (eds), *Mathematics and the historian’s craft*, Springer, pp.7-21. Also published in *Am. Math. Monthly*, **111**(1), 2004, 1-12.
- Jankvist, U. T. (2009). ‘A categorization of the ‘whys’ and ‘hows’ of using history in mathematics education’. *Educ. Stud. in Math.* **71**(3), 235–261.
- Tzanakis, C. & Thomaidis, T. (2011). Classifying the arguments & methodological schemes for integrating history in mathematics education. CERME7 Proceedings. To Appear.

[http://inventors.about.com/od/gstartinventors/a/Galileo\\_Galilei.htm](http://inventors.about.com/od/gstartinventors/a/Galileo_Galilei.htm)

## **INTRODUCTION TO THE PAPERS OF WG13:**

### **“EARLY YEARS MATHEMATICS”**

Ingvald Erfjord, Ema Mamede, Götz Krummheuer

This Working Group dealt with the research domain of mathematics learning and mathematics education in the early years, age 3 to 7. The working group met for the second time during a CERME Conference. Several members met in CERME VII for the second time. As already found at the first meeting at CERME VI, this research group has to cope with a tremendous variety of educational approaches in preschool and kindergarten in the different countries. The following table gives an overview of the national origins of the presenters of working group 13:

<b>Country</b>	<b>Number of papers</b>	<b>Number of posters</b>	<b>Presenters</b>
Cyprus	1		1
Germany	3		5
Israel	2		2
Italy	1		1
Norway	3	1	6
Portugal	1		2
Spain	1		2
The Netherlands	1		1
United Kingdom		1	1
<b>TOTAL</b>	<b>13</b>	<b>2</b>	<b>21</b>

**Table 1: Number of papers, posters and presenters from different countries**

In these countries, especially Kindergartens serve different tasks. This variety is related to the diverse societal conception of early years education, with major differences in what role direct teaching of mathematics and other academic subject matters play and to the individual diagnosis of learning progress in these domains. Due to these differences, certain research questions and methods which are highly relevant for one country are almost unimaginable to consider in another one. This diversity offers a great challenge and opportunity to establish an European wide leading research group.

### Working Group 13

In congruence to this initial situation, the group was confronted with a wide array of research questions and their related theoretical founding, which is shown in the following table:

Main theoretical orientation	Topics
Educational psycholgy	
• Conceptual development	Concept of half Geometrical knowledge Concept of numbers and operations
• Classroom teaching experiment	Fractions Early algebraic thinking
Educational science	Relationship of playing and learning
Cultural psychology	Co-learning Mathematical and communicative competence
Sociology, socio-linguistics	Mathematics learning Support system Codes and frames Multi-culturalism
Semiotics	Gestures

**Table 2: Different theoretical orientation and topics**

Despite this large range of theoretical interests, the members of working group 13 accomplished a conversation of mutual respect and partly envisioned options of „bridging“ the gap among the different theoretical paradigms with respect to the topic of the development and teaching of early years mathematics.

# ISSUES ON CHILDREN'S IDEAS OF FRACTIONS WHEN QUOTIENT INTERPRETATION IS USED

Ema Mamede

CIFPEC, University of Minho

Manuela Oliveira

EB1 Campo

*This paper focuses on children's understanding of fractions when quotient interpretation is used to introduce them this concept. An intervention program was conducted with a 7-years-old classroom, from a public primary school, in Fafe, Portugal. This intervention program comprised seven sessions in which children learned the representation of fractions and were challenged to solve some problems of ordering and equivalence of fractions. These sessions were organised following the official curricular content (starting by the equal sharing problems) but also according to the children's rhythms and demands. Children's performance and their arguments solving the tasks of ordering and equivalence of fractions are presented here. Issues on their learning process are characterized and discussed.*

## FRAMEWORK

Fractions is one of the most complex concept that children have to learn during the elementary school, but also a necessary one. Literature already provided information about students' difficulties (see Behr et al., 1984; Hart, 1981; Kerslake, 1986) with fractions. More recently, literature has been discussing the issues related to the effects of the interpretations for fractions on children's understanding of this concept (see Mamede, Nunes & Bryant, 2006; Mamede & Nunes, 2008; Nunes, Bryant, Pretzlik, Wade, Evans & Bell, 2004) and on the children's schemes of action (Nunes, 2008; Nunes & Bryant, 2008).

Distinct interpretations of fractions seem to affect differently children's understanding of the ideas of fraction. At the primary school, the children are supposed to understand at least fractions in quotient, part-whole and operator interpretations. But in these interpretations the meaning of the numerator and denominator differ. In part-whole interpretation, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So,  $\frac{2}{4}$  in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient interpretation, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation,  $\frac{2}{4}$  means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situation a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction  $\frac{2}{4}$  can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each

(Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). In operator situations, the denominator indicates the number of equal groups into which a set was divided and the numerator is the number of groups taken (Nunes et al., 2004). In an operator interpretation, if a boy is given  $\frac{2}{4}$  of 12 marbles, means that the 12 marbles are organized into 4 groups (of 3 marbles each) and the boy receives 6 marbles – that is 2 groups of the 4 into which the 12 marbles were organized. Thus number meanings differ across these interpretations. These differences affect children's understanding of fractions when building on their informal knowledge.

Mamede, Nunes and Bryant (2006) conducted a survey on 80 first-grade children, aged 6 and 7 to compare their understanding of ordering and equivalence of fraction presented to them in quotient and part-whole interpretations. These children had received no school instruction about fractions. The results show that children's performance on problems presented in quotient interpretation was much better than in part-whole interpretation. In quotient interpretation the rates of success were 55% for 6-year-olds children and 71% for 7-year-olds children, for ordering problems; and 35% for 6-year-olds and 77% for 7-year-olds children, for equivalence problems. In part-whole interpretation the rates of success were 24% for 6-year-olds children and 20% for 7-year-olds children, for ordering problems; and 9% for 6-year-olds and 10% for 7-year-olds children, for equivalence problems. The children's resolutions were also analysed giving evidence that strategies based on correspondence combined with partitioning were popular among the group of children who solved the problems in quotient interpretation whereas partitioning was the strategy adopted by those who worked in part-whole situations.

More recently Mamede (2008) conducted an intervention program with 37 first graders (ages 6-7) to introduce fractions in distinct interpretations. The children were addressed randomly to work in part-whole, quotient and operator interpretations of fractions. Again the children had received no instruction about fractions. The results showed that those who were introduced to quantities represented by fractions in quotient interpretation could succeed in ordering, equivalence and labeling tasks; those who were introduced to fractions using part-whole and operator situations were able to succeed only on the labeling of fractions, but not on the ordering and equivalence tasks.

Thus, the type of interpretation used to work with fraction in the school interferes with students understanding of fractions. This idea is also supported by Nunes et al. (2004) who describe the results of a survey conducted with 130 students in Year 4 and 5 (8- and 9-year-olds) to analyse the pupil's ability to compare equivalent fractions presented in Quotient and Part-whole situations. In quotient situation item the pupils were asked to compare the fractions  $\frac{1}{4}$  and  $\frac{2}{8}$ ; in part-whole situation they were asked to compare  $\frac{2}{4}$  and  $\frac{4}{8}$ . Results show that the rates of correct responses were 46% for the part-whole item and 77% for the quotient item. Thus, in

spite of considering different fractions in each situation, these results suggest that children perform differently in these two situations.

Research has been giving evidence that quotient situations are more suitable for children to build on their informal knowledge for fractions. The informal ideas about fractional quantities appear much earlier than the formal learning of fractions in school. Research developed with younger children shows that in a division situation, there are some children as young as 6-year-olds who can understand the inverse relation between the divisor and the quotient, when the dividend is the same (Correa, Nunes & Bryant, 1998) when discrete quantities are involved, and when continuous quantities are involved (Empson, 1999; Kornilaki & Nunes, 2005). This understanding of the inverse relation between the divisor and the quotient can be seen as a precursor of understanding of the logic of fractions: the greater the divisor (which would be represented by the denominator in a quotient interpretation), the smaller the quantity.

Streefland (1991, 1997) recommends the use of quotient situations to introduce fractions to children because these situations rely on the idea of fair sharing, which can provide the model for fractions and the part-whole concept related to equivalence and operational relations. The author not only recommends but also provides evidence of success in the use of the quotient interpretation to introduce fractions to children, describing a theory for teaching fractions based on the realistic approach that uses this type of interpretation to introduce fractions to children (see Streefland, 1991). Starting from problems using situations taken from daily life focused on division situations, Streefland produced good improvements on children's understanding of fractions, helping them to perceive the meaning of numerator and denominator as connected to each other, forming a correct mental object for the concept of fraction.

Traditionally, in many European countries, including Portugal, and the U.S. (see Behr, Harel, Post & Lesh, 1992; Behr, Lesh, Post & Silver, 1983; Kerslake, 1986; Mack, 1990; DEB, 1998) children are introduced to fractions at school using the part-whole interpretation and then this work with fractions is extended to include operator situations. In Portugal, in the primary school levels (1<sup>st</sup> to 4<sup>th</sup>-grades) students are introduced to fractions representation using the part-whole interpretation, and in some cases students have their first contact with fractions on the 5<sup>th</sup> grade. Portugal is experiencing a new curriculum for the elementary school levels. This new curriculum refers that fractions should be introduced to children in an informal way, in the second grade, relying in partitioning and equal sharing; and explored in the third and fourth grades in the quotient, part-whole, operator and measure interpretations should be explored. Nevertheless that document gives no other indication for teachers to introduce and explore fractions in the classroom. Literature already provided evidence of success when children are introduced to fractions in quotient interpretation (see Streefland, 1991, 1997; Mamede, 2008).

However, for many Portuguese primary school teachers the concept of fraction only makes sense when the part-whole interpretation is involved. Knowing that quotient interpretation of fractions can help children to build on their informal knowledge with understanding, how can teachers explore this interpretation in the classrooms? This paper tries to give evidence of a well succeeded experience conducted in the classroom in which fractions are introduced to children using quotient interpretation.

The study reported here describes children's understanding of fractions when they experience partitioning and equal sharing activities, and then received instruction on fractions using the quotient interpretation. The teaching experiment follows the Portuguese official curriculum for the 2<sup>nd</sup> grade mathematics, but goes further anticipating children's first contact with fractions to this level. Previous related studies give evidence of success of children's understanding of quantities represented by fractions when quotient interpretation is used but they do not follow the Portuguese curriculum in the classroom.

The part of the study reported here focuses on children's understanding of fractions when they are introduced to them using quotient situations, after a contact with partitioning and equal shared activities. It tries to address two questions: (1) How do children understand ordering of fractions when introduced to this concept using the quotient interpretation? (2) How do children understand the equivalence of fraction in this interpretation?

## **METHODS**

An intervention study was conducted using qualitative methods to describe children's performances and characterize the processes involved in their learning to represent and compare fractions. Children's answers, as well as their arguments and solving strategies were analysed to reach an insight on their ideas of fraction.

### **Participants**

The participants were a class of 8 students from a public primary school from Fafe, in the north of Portugal. The children were all 7 years-old. The teacher of the class is one of the researchers. These children had received no instruction about fractions.

### **Design**

The intervention comprised 7 sessions, of approximately 90 minutes each, in which children were introduced to fractions using quotient situations. In the first two sessions children were challenged to solve problems involving equal sharing; they were also introduced to the symbolic representation of fractions, in which the quotient situation or interpretation was used. The remaining sessions were designed to explore ordering and equivalence of fractions in quotient situations.

There were 6 tasks of ordering of fractions and 4 of equivalence of fractions. The fractions used in these tasks were all less than 1. In the ordering tasks children were

asked to solve a problem such as: “Two girls are going to share fairly a chocolate bar, and there is nothing left; four boys are going to share fairly a chocolate bar and there is nothing left. These chocolate bars are equal. Do you think that each girl is going to eat more chocolate than each boy, each boy is going to eat more chocolate than each girl, each girl and each boy are eating the same amount of chocolate? Can you write the number that represents the amount of chocolate that each child eats?”. They were also asked to compare fraction given only symbolically. Analogous tasks were presented to them involving equivalence of fractions, in a problem such as: “Two girls are going to share fairly a chocolate bar, and there is nothing left; four boys are going to share fairly two chocolate bars and there is nothing left. These chocolate bars are equal. Do you think that each girl is going to eat more chocolate than each boy, each boy is going to eat more chocolate than each girl, each girl and each boy are eating the same amount of chocolate? Can you write the number that represents the amount of chocolate that each child eats?”. In some sessions the ordering and equivalence problems were presented with no pictorial support.

### **Procedure**

In all sessions the tasks were presented to the children with the support of PowerPoint slides. Each child had a worksheet with the same information presented by the teacher, in which they could draw as they wish; and manipulative aid was provided as coloured paper with squared, rectangular and circular shapes were available. In each session the tasks were presented by the teacher to the class orally to ensure children’s understanding of the problem, as they usually do in the math class. Then children were asked to solve the problems presented to them and justify their results. Then the teacher challenged them to write down their arguments and verify their solutions.

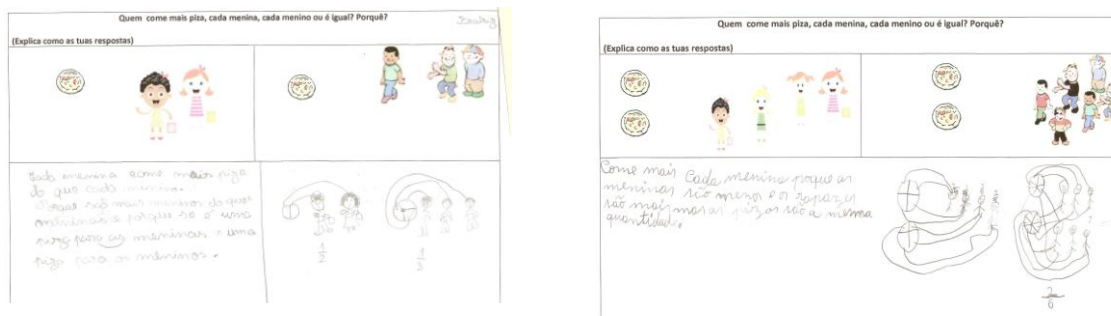
Data collection was carried out with the use of video and audio records, students’ worksheets and field notes taken by the researcher.

### **Results**

In this section it is presented the results concerning children’s performance and arguments solving ordering and equivalence problems of fractions. The children solved the tasks individually and wrote their answers on their worksheet; then they wrote their justification and only after that they were challenged to verify their solution.

When the children were asked to compare  $\frac{1}{2}$  and  $\frac{1}{3}$ , 7 of the 8 children succeeded; these 7 children gave a correct answer and then wrote down the explanation. Majority of children wrote down the explanation and then drew the pictures in their worksheets to verify their reasoning. Children’s performance was even better when they were asked to compare  $\frac{2}{4}$  and  $\frac{2}{6}$ . Figure 1 gives examples of children’s performance on these ordering problems.





*'Each girl eats more pizza than each boy. Because there are more boys than girls and there is one pizza for the girls and one pizza for the boys.'*

*'Each girl ate 1/2. Because there are only two girls. And the boys ate 1/3.'*

**Figure 1: Children's resolutions comparing  $1/2$  and  $1/3$ , and  $2/4$  and  $2/6$ .**

The children's arguments were improving along the sessions. By the fourth session it was possible to hear valid explanations such as "... each girl eats more pizza because there are fewer than boys and there is equal pizzas" referring to the equal number of pizzas; or "Each girl eats more pizza because each girl eats a bigger piece and the boy eats a smaller piece than the girls"; or "Each girl eats more because there are two pizzas for four girls and each boy eats less because there are two pizzas for six boys". The children's arguments were improving along the sessions.

In another episode involving an ordering task, the children were able to recognise and generalize the inverse relation between the divisor and the quotient, when the dividend is the same, as it shows the following transcription of children's discussion.

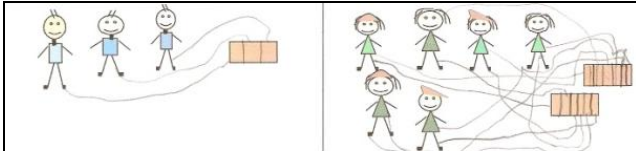
- Tutor: So, if you wish to get the biggest amount of paper which fraction would you choose?
- M: One half... it's more than the others.
- Tutor: How do you know that is more?
- M: One paper for two children is more than one paper for three...
- J: And more than for four and more than three...
- Tutor: If it is so, what sign did you write J?
- J: The 'bigger' sign...
- Tutor: Why did you put the 'bigger' sign?
- J: It's always 1 paper and there is always more children. They were 2, then 3, then 4....
- R: The smallest is one-fifth... I circled that one!

**Transcription 1: Children's explanation of the inverse relation between divisor and quotient for the same dividend.**

## Working Group 13

The children were also able to solve problems of equivalence of fractions (7 out of 8 succeeded in all these tasks). Some episodes in which children revealed difficulties will be presented in the conference.

Figure 2 shows children's resolutions on two equivalence tasks. In these tasks some children found more difficult to explain their arguments in a written mode, in spite of solving the tasks correctly. These examples as well as children's difficulties will be presented in the conference. However, their oral justifications improved after drawing their schemes to verify the solutions, as their pictures were giving them some support in this task.




Quem come mais, cada menino ou cada menina?  
 Quem come mais não é ninguém porque  $\frac{1}{3}$  é igual a  $\frac{2}{6}$ .  
 Eu pus o sinal igual porque  $3+3=6$  meninos e  $1+1=2$  chocolates.

Escolhe o sinal adequado e coloca o entre as frações.

$\frac{1}{3}$	$=$	$\frac{2}{6}$
	$> < =$	

'No one eats more because  $1/3$  is equal to  $2/6$ . I put the equal sign because  $3+3=6$  boys and  $1+1=2$  chocolates.'



Quem come mais, cada menino ou cada menina?  
 Quem come mais não é ninguém porque  $\frac{1}{3}$  é igual a  $\frac{2}{6}$ .  
 Eu pus o sinal igual porque  $3+3=6$ meninos e  $1+1=2$  chocolates.

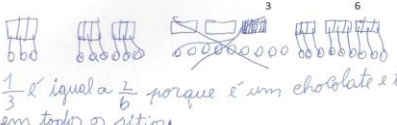
Escolhe o sinal adequado e coloca o entre as frações.

$\frac{1}{3}$	$=$	$\frac{2}{6}$
	$> < =$	

'Eat the same because if we cut two-sixths the pieces are the same of one-third.'

Coloca o sinal adequado entre as frações. ( $>$ ,  $<$ ,  $=$ )  
 Explica a tua resposta.

$\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$



$\frac{1}{3}$  é igual a  $\frac{2}{6}$  porque é um chocolate e três meninos em todos os sítios.

' $1/3$  is the same of  $2/6$  because there is one chocolate and three children in all places.'

### Figure 2: Children's resolutions of distinct equivalence tasks.

In the majority of the tasks presented, children seemed to rely on the use of correspondence to reach the solution. When solving equivalence tasks the use of correspondence is even more evident as in many cases this idea is also supported by their justifications (see Figure 2).

When asked to compare  $1/3$  and  $2/6$  almost all succeeded (7 out of 8 children). Children's resolutions relied mainly on partitioning and correspondence. But a few children seemed to reveal some type of proportional reasoning when presenting their justifications. This is suggested when they try to explain that fact with numbers and expression familiar to them, such as ' $1+1=2$ ' and ' $3+3=6$ ' trying to express the double of quantities involved (see Figure 2).

## DISCUSSION AND CONCLUSIONS

The results of this study allow us to establish some remarks. First, this experience gives evidence that children can understand fractions when introduced to them in quotient situations, in agreement with Streefland (1991, 1997), Mamede, Nunes and Bryant (2006), Mamede and Nunes (2008) who previously studied this issues. In the sessions of this study, the ordering problems seemed to help the children to easily understand the inverse relation between the divisor and the quotient, when the dividend is the same. This relation is essential to understand the meaning of fractions. Second, these children learned easily fractions labels. They were introduced to the representation of fractions in the beginning of the intervention, and soon they master the symbolic representation of fractions, when quotient interpretation was used. In this type of interpretation, the magnitudes involved in the fractions refer to two variables of different nature (Nunes *et al.*, 2004), - numerator refers to the number of items to share, denominator refers to the number of recipients - and this may facilitate children's learning of fractions labels. Third, because in quotient interpretation the numerator and the denominator relate to variables that are different in nature (Nunes *et al.*, 2004), children easily relied on the use of correspondence to solve many of the tasks. This finding was also documented previously by Mamede, Nunes and Bryant (2006) when observing 6-7-year-olds children's strategies solving ordering and equivalence problems, when interviewed individually. Nunes (2008) argues that in a division situation, there are two types of action schemes: partitioning, which involves dividing the whole into equal parts; and correspondence which involves two quantities (a quantity to be shared and a number of recipients of the shares). The development of these action schemes defers. Children of 5 to 6-year-olds can establish correspondence to produce equal shares (see Kornilaki & Nunes, 2005; Mamede, Nunes & Bryant, 2006; Nunes, 2008), but they find more difficult to accomplish partitioning of continuous quantities. These schemes of action (Nunes, 2008) are fundamental for the learning of the mathematical concepts. Fourth, the equivalence problems presented in quotient interpretation gave the children an opportunity to promote their proportional reasoning. When solving the equivalence problems many children establish a proportional relation between the numbers of items to share and the number of recipients in order to reach the solution; some of them could express that relation in a written way, others by drawings. Proportional reasoning was also a strategy identified by Mamede (2007) and Nunes *et al.* (2004) when analysing students' strategies solving equivalence problems presented to them in quotient interpretation of fractions. To conclude, this short intervention program allowed the teacher to understand children's possibilities of success with fractions when they are introduced to the children using quotient situations. We hope that this experiment can contribute to promote a change in the classroom practices, following the Portuguese official curriculum, giving the primary teachers an example of a well succeeded experience. As correspondence seems to have an important role on children's reasoning on

fractions problems, it seems to be relevant to give young children the opportunity to develop sharing experiences based on correspondence (one-to-one and one-to-many) since kindergarten.

More research is needed in order to explore other ways of introducing fractions to children in the classroom, using quotient interpretation of fractions. In this experiment, the children learned the fraction representation in the beginning sessions. For further research in this area it would be interesting to develop a longitudinal research to analyse the influence of interventions based on quotient interpretation of fractions on young children's understanding of other interpretations of fractions.

## REFERENCES

- Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational Number, Ratio, and Proportion. In D.A. Grows (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp.296-333). New York: MacMillan Publishing Company.
- Behr, M., Lesh, R., Post, T. & Silver, E. (1983). Rational-Number Concepts. In R. Lesh and M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 92-127). New York: Academic Press.
- Behr, M., Wachsmuth, I., Post, T. & Lesh, R. (1984). Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. *Journal for Research in Mathematics Education*, 15 (5), 323-341.
- Correa, J., Nunes, T. & Bryant, P. (1998). Young Children's Understanding of Division: The Relationship Between Division Terms in a Noncomputational Task. *Journal of Educational Psychology*, 90(2), 321-329
- DEB (1998). *Organização Curricular e Programas do 1.º Ciclo - Ensino Básico*. Lisboa: Ministério da Educação.
- Empson, S. (1999). Equal Sharing and Shared Meaning: The Development of Fraction Concepts in a First-Grade Classroom. *Cognition and Instruction*, 17(3), 283-342.
- Hart, K. (1981). Fractions. In K. Hart (Ed.), *Children's Understanding of Mathematics: 11-16*, (pp. 66-81). London: John Murray Publishers.
- Kerslake, D. (1986). *Fractions: Children's Strategies and Errors – A Report of the Strategies and Errors in Secondary Mathematics Project*. Berkshire: NFER-NELSON.
- Kornilaki, E. & Nunes, T. (2005). Generalising principles in spite of procedural differences: Children's understanding of division. *Cognitive Development*, 20, 388-406.

- Mack, N. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education*, 32, 267-295.
- Mack, N. (1990). Learning Fractions with Understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16-32.
- Mamede, E. (2008). Focusing on children's early ideas of fractions. In Bozena Maj, Marta Pytlak & Ewa Swoboda (Eds.), *Supporting Independent Thinking Through Mathematical Education*, pp. 61-67. Poland: Rzeszow, WUR.
- Mamede, E. (2007). The Effects of situations on Children's Understanding of Fractions. PhD Thesis (unpublished thesis), Oxford Brookes University. Oxford: OBU.
- Mamede, E. & Nunes, T. (2008). Building on Children's Informal Knowledge on the Teaching of Fractions. In O. Figueras, J. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the Joint Meeting of the Psychology of Mathematics Education 32 and Psychology of Mathematics Education – North America*, Vol. 3, pp. 345-352. Mexico: Morelia.
- Mamede, E., Nunes, T. & Bryant, P. (2006). The Equivalence and Ordering of Fractions in Part-whole and Quotient Situations. In Helen Chick & Jill L. Vincent (Eds.), *Proceedings of the 29th Psychology of Mathematics Education*, Vol. 3, pp. 281-288. Australia: Melbourne.
- Nunes T. (2008). Understanding Rational Numbers. *Proceedings of the Conference of European Association for Research on Learning and Instruction – EARLI*, pp. 23-50. Hungary: Budapest.
- Nunes T. & Bryant, P. (2008). Rational Numbers and Intensive Quantities: Challenges and Insights to Pupils' Implicit Knowledge. *Anales de psicologia*, 24 (2), diciembre, 262-270.
- Nunes, T., Bryant, P., Pretzlik, U., Evans, D., Wade, J. & Bell, D. (2004). Vergnaud's definition of concepts as a framework for research and teaching. Annual Meeting for the Association pour la Recherche sur le Développement des Compétences, Paper presented in Paris : 28-31, January.
- Nunes, T. & Bryant, P. (1996). *Children Doing Mathematics*. Oxford: Blackwell Publishers Ltd.
- Streefland, L. (1991). Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research. Norwell, MA: Kluwer Academic Publishers.
- Streefland, L. (1997). Charming Fractions or Fractions Being Charmed? In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics – An International Perspective* (pp. 347-372). Psychology Press Ltd.: East Sussex.

# FINGER-SYMBOL-SETS AND MULTI-TOUCH FOR A BETTER UNDERSTANDING OF NUMBERS AND OPERATIONS

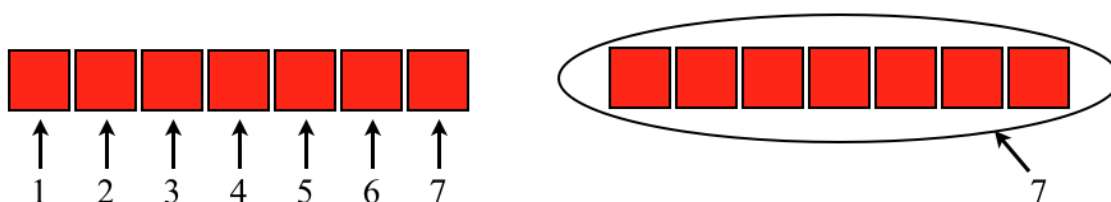
Silke Ladel and Ulrich Kortenkamp

CERMAT, University of Education Karlsruhe, Germany

*Basic concepts of numbers and operations are fundamental for mathematical learning. Suitable materials for developing such basic concepts are hands and fingers. Among other things, this is because of their natural structure of 5 and 10. To support the development of concepts and the process of internalization a linking between different forms of representations by the computer can be helpful. To benefit of both, the advantages of the hands and fingers and the automatically linking, we suggest using multi-touch-technology, i.e. computer input devices that are able to recognize several touch gestures at the same time. Here, children can present numbers with their fingers that produce virtual objects. These objects can be automatically linked with the symbolic form of representation.*

## THE ORDINAL AND CARDINAL CONCEPT OF NUMBERS AND OPERATIONS

“How many things are there?” – For parents as well as for mathematicians, this is a common question to pose, if a child already has knowledge about numbers. For the child, this question is almost always the initiation to start counting verbally by saying the number words in a row (Fuson, 1988). The fundamental principles needed for answering the question are a) the one-one-principle that relates every single object to exactly one numeral (Gelman & Gallistel, 1978), b) the stable-order-principle prescribing the correct order of numbers (Fig. 1, left), and c) the last-word-rule that assigns the last said numeral not the last counted object, but to the quantity as a whole (Fig. 1, right).

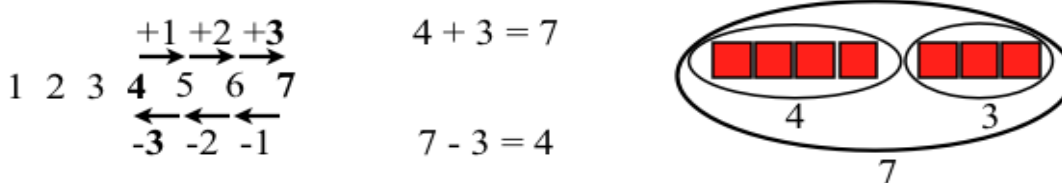


**Figure 1: ordinal (left) and cardinal (right) concept of numbers**

Here, the change from the ordinal concept of numbers, where the numeral is part of the numeral row, to the cardinal concept of numbers, where the numeral identifies a quantity, is necessary. It is not necessary to count a quantity in order to know it, that is, the ordinal concept is not a necessity for the cardinal concept. Resnick (1991) distinguishes the development of mathematical knowledge by two components that are developed independently: *protoquantitative schemata* and the *mental number line*. To build-up a well-developed concept of numbers, these two threads of

development have to be linked. For many children this is a critical problem. Fuson refers to the following example: a child counts a quantity of five cars. As an answer to the question „*How many?*“ the child points to the last-counted car and says: „*This is the five cars.*“ (Fuson, 1992 p. 63).

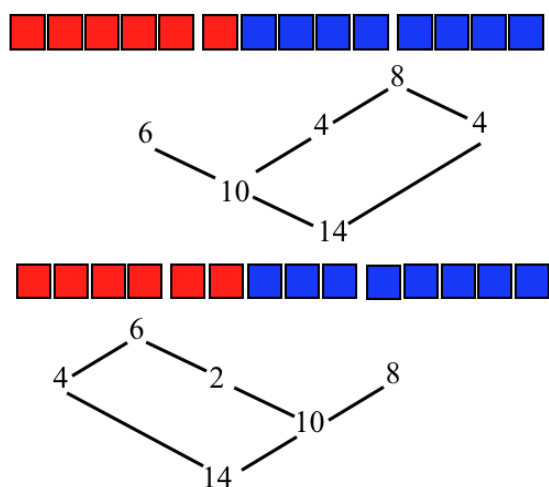
Children who do not have a proper linking between the two concepts can misinterpret addition and subtraction as a demand to count forwards or backwards (fig. 2 left). As long as the children calculate with numbers smaller than 20 they can apply this strategy successfully. But, for instance, when they want to add 55 to 27 and begin to count „28, 29, 30, 31, ...“ there is no chance to come easily and quickly to the correct result. Thus, it is important that children acquire a part-whole schema of numbers as a foundation for addition and subtraction (fig. 2 right).



**Figure 2: Addition and subtraction with the ordinal (left) and the cardinal (right) concept of numbers**

„*The protoquantitative part-whole schema is the foundation for later understanding of binary addition and subtraction and for several fundamental mathematical principles, such as the commutativity and associativity of addition and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.*“ (Resnick, 1991 p. 32).

For example when you want to add 6 and 8 with the use of the part-whole schema you can split and add in lots of ways (e.g. fig. 3).



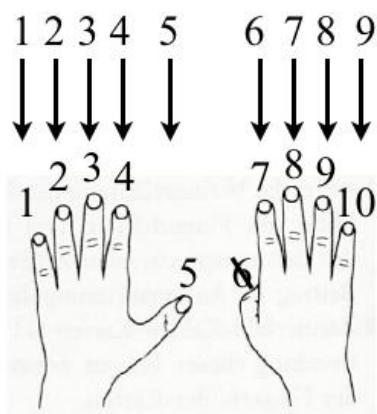
**Figure 3: different ways to add with the part-whole-schema**



Because of our decimal number system and with regard to the decimal analogy the secured understanding of the number range up to 10 is assigned a key position in the development of mathematical competences (Claus & Peter, 2005 p. 11). If you know all the possible decompositions of the numbers up to 10, you are able to add and subtract in bigger number ranges easily, e.g. if  $7 + 8 = 7 + (3 + 5) = (7 + 3) + 5 = 15$ , then  $67 + 8 = 67 + (3 + 5) = (67 + 3) + 5 = 75$ . The missing knowledge of and fluency in decompositions of the numbers up to 10 as much as the lack of capability to recall them quickly and effortlessly is the cause of many subsequent difficulties in mathematical learning.

## FINGER SYMBOL SETS

Calculating with fingers has a very bad reputation in mathematics lessons, as it is usually seen as an indicator for counting. Most children do as they have learned from young days on and count objects by „Counting-Word Tagging to Number“ (Brissiaud, 1992). According to the ordinal concept of numbers each finger is related to exactly one numeral. But if each finger is labelled by a number, counting children are encouraged to stay with their strategy and this consequently leads to further problems. To illustrate this we ask what happens if the sixth finger is buckled? The „name“ of the last finger, that indicated the quantity, was „10“ before, but now the finger has to be renamed into „9“ (Fig. 4).



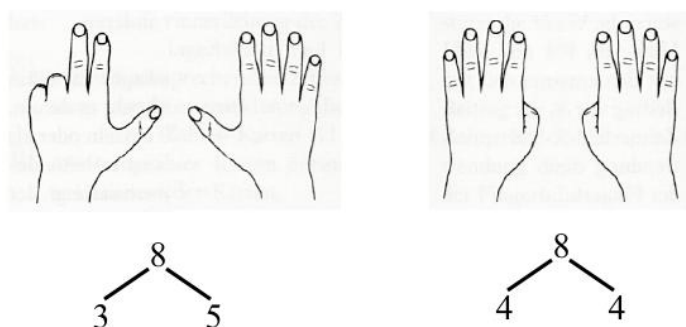
**Figure 4: Order-irrelevance principle**

This procedure can puzzle some children. Therefore it is important not to assign names the fingers – there is no “6-finger.” The child has to know that it is irrelevant which fingers it uses to present a quantity. To present „3“, the thumb, the index finger and the middle finger can be used as well as the little finger, the middle finger and the thumb, or any other combination of three fingers. As we point out below, the cognitive process behind this fact can be experienced and thus supported by the use of multi-touch-technology.

Amongst others, the advantages of fingers and hands are their permanent availability and their natural structure in 10 fingers per child with 5 fingers per hand. The 10 fingers qualify the hands to work out questions about the decimal number system,



e.g. „How many children do we need to see 30 fingers all at once?“ The 3 children stand for 3 tens. Just as well, the different decompositions of all numbers up to 10 can be presented with the hands. The „power of five“ (Krauthausen, 1995) is due to the ability to instantaneously recognize quantities (subitizing) up to 4. Applying this to the hands, the shown quantity of the fingers of one hand can be conceived simultaneously and hence the fingers of both hands can be conceived quasi-simultaneously. Furthermore, one hand gets a special status because children tend to present numbers greater than five sequentially (Brissiaud, 1992 p. 61). For example, to present „7“, they tend to use one full hand and then add two fingers of the other hand. In this way the decomposition of the numbers from 1 to 10 with the power of five can be worked out. But not only these, also all other decompositions are possible (Fig. 5) and can be conceived quasi-simultaneously.



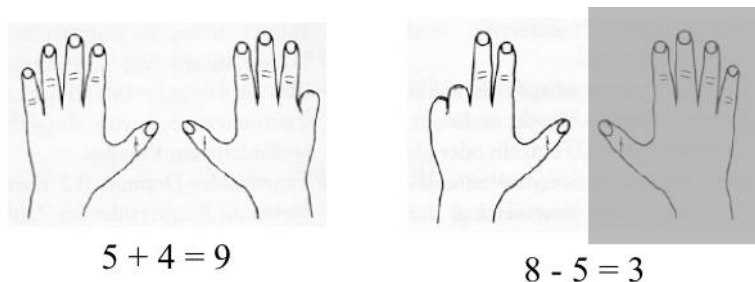
**Figure 5: Decomposition of numbers with finger-symbol-sets**

If the fingers are used like this, in sense of the part-whole schema, they are a qualified working material for a well-developed concept of numbers and operations (cf. Steinweg, 2009). Brissiaud (1992) coined the notion „From Finger Symbol Sets to Number“:

*„Certain children who were not exposed early to the use of finger symbol sets may become counters, whereas children who were encouraged to use finger symbol sets may preferentially choose finger strategies“.*

He could show that this way of gestured representation of quantities by some children is established early and in parallel to the development of the numerical row as an autonomous type of numerical representation.

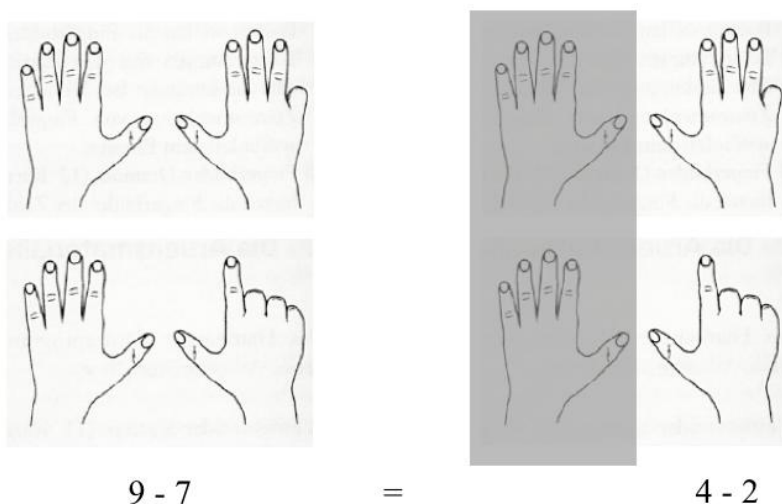
The decompositions of all numbers up to 10 that were acquired in this way can now be utilised in the following process of mathematical learning of addition and subtraction (Fig. 6).



**Figure 6: addition and subtraction with finger symbol sets**

If children have a part-whole schema of numbers the transition to addition and subtraction is easy. It is just another way of nonverbal-symbolic representation of the fact that „two parts make a whole“.

Further strategies like variation in the opposite or in the same direction can than be worked out easily: If one finger is buckled, than another finger must be stretched to keep the same quantity. To get the difference of two quantities, e.g. of 9 and 7, you can vary the numbers in the same direction. For example, a whole hand can be omitted, which corresponds to subtracting five from each quantity. It is evident that the difference of 9 and 7 is the as the difference between 4 and 2 (Fig. 7).



**Figure 7: Strategy of variation in the same direction with finger symbol sets**

Based on such strategies the decadic analogy can be build up.

It is important to pay attention to the fact that the children stretch their fingers *simultaneously* to represent quantities with them. If they show them one-by-one the positive effects of these strategies are lost and the children will still use counting for addition and subtraction.

As well as addition and subtraction, multiplication and division can be presented via hands and fingers. If 5 children all show 6 fingers, this represents five times six. The inverse operation is found when we start with 30 fingers that shall be represented by 5 children.

This introduction can only serve as a small insight into the possible representations of numbers and operations by hands and fingers and their usage in early arithmetic. It is the process of internalization that is of essential importance: How can the children benefit from the mathematical content of these representations and actions and use them in their mental processes?

## **THE PROCESS OF INTERNALIZATION SUPPORTED BY THE USE OF MULTI-TOUCH-TECHNOLOGY**

According to Aebli, the process of early mathematical learning follows four stages, independent of the arithmetical subject (Grissmann & Weber, 2000; Aebli, 1987). Coming from concrete manipulations with different objects (stage 1), the children have to abstract these manipulations and operations to pictorial representations (stage 2). Subsequently they pass over to symbols (stage 3) with the aim to automate their actions (stage 4). For us, stage 2 is of special importance, because there the process of internalization takes place. The child has to comprehend the manipulation of concrete objects as a representation of a quantitative structure and it has to capture the structure and the relations of the concrete manipulation in an intellectual activity (Gerster & Schultz, 2004 p. 47). Lorenz calls this process „focus of attention“. To facilitate this process of focus and abstraction and to develop it, a dialog is essential (Lorenz, 1997): *„In talking about the working material and the relations between numbers and operations that it represents, the concepts in development of the learner are going to be clarified by verbalisation.“* In this sense, Aebli (1987) suggests that the children should review their concrete manipulations and make forecasts about further actions. Doing this, they comment their own manipulations by iconic illustrations till they are able to reproduce the structures and relations of the manipulations in conceptions. To support this process Aebli (1987 p. 238) established the following rule:

*„Every new, more symbolic representation of the operation must be linked as closely as possible with the precedent one.“*

As shown in Figs. 5 to 7, the enactive form of representation with finger symbol sets should be related to the nonverbal-symbolical form of representation (MER<sup>1</sup>) (Ainsworth, 1995; Mayer, 2005). But as studies show some of the children even don't link the different forms of representations when they are designed in form of MERs (Clements, 2002). For them, an automatic linking designed with the computer (MELRs<sup>2</sup>) can help them to experience the relations (Thompson, 1992; Clements, 2002; Ladel, 2009). This experience should be as natural and directly as possible. In this article we suggest to use multi-touch-technology for this experience, where the children can manipulate with their hands and fingers and an automatic linking with all other forms of representation can take place. In the remainder of this article we assume the availability of a multi-touch-enabled table. Such a table consists of a

display surface connected to a computer and some tracking hardware that can recognize several touches on the display simultaneously and report them to the computer software. Similar technology with a different form factor is available in desktop monitors, tablet computers and devices like the Apple iPad, or mobile phones. With the availability of hardware as already imagined by Kay (1972) we now have to answer the question of the educational implications more than ever.

The basic underlying idea for all the activities sketched only briefly in the following is that the computer can track the children's actions on the table and give nonverbal-symbolic representations of either the current situation or the action that lead to it in form of a written protocol.

In a first scenario, the children represent numbers with their hands and fingers as described before. This enactive form of representation shall produce an iconic one on the display. The computer creates quadratic pads on the surface of the multi-touch-table. Already at this stage, the focus of attention of the children can be laid on the fact of bundling the 5 fingers of one hand to a bar of 5 and the 10 fingers of two hands to a bar of 10. Through the contact of the fingers with the multi-touch-interface there is not only a link between the enactive form of representation with other forms of representation but also between the tactile and the visual sense. While representing numbers enactively and thus iconically, there is an automatic link to a nonverbal-symbolic form of representation. This representation can be imagined like a paper tape or sales slip and serves as a kind of protocol for the manipulations the children do. Such a protocol can support the focus of attention and the numerical aspects of a task (Dörfler, 1986).

In this activity it is possible for children to experience that it is of no particular importance which fingers they use to present quantities. Using the thumb and the index finger or using the little finger and the ring finger both yield the number "2". At a table, it is also possible that the children work in teams: Two children can "share the work" to present two fingers if each touches the table with one finger. While this sounds funny for the number two, it is of great importance for partitions of larger numbers. Two partners can try to find all ways to partition numbers up to 20 into two numbers up to 10.

Working in teams or groups the children are also able to present numbers greater than 10, emphasizing the social aspects of learning. Because the protocol immediately reflects the actions of the children their focus of attention is on the mathematical content of their actions automatically, guiding them to abstraction.

We can also use the technology as a diagnostic tool for the number concepts of children. Recording the way the children touch the table with their fingers it is possible to measure the time intervals between the touches of each finger. We can judge whether the children are still counting to put a given number of fingers onto the table, or whether they work already with suitable finger symbol sets.

Just showing the number of fingers touching the surface can also help the children to experience the difference between the ordinal and the cardinal aspect of numbers. The child can touch the table with its fingers in different sequences, different positions or with different fingers – in all cases the protocol just depends on the quantity of fingers.

It is also possible to support the four basic arithmetic operations and their basic concepts in such an environment. Regarding addition, students can develop the basic concept of a union by manipulating the virtual objects (pads) and arrange them close to each other. For example, if the child merges a group of 3 pads and a group of 5 pads the protocol will show the symbolic representation of this action as „ $3 + 5 = 8$ “. Here the focus of attention lies on the fact that this action constitutes a basic concept of addition, together with its nonverbal-symbolic form of representation. In multi-touch-technology there is also the possibility to draw a circle around some pads with the effect that these pads are bundled (a so-called lasso-gesture). This again is a manipulation based on the basic concept of union. Another task in the realm of addition and subtraction may be that 3 pads are shown and the child should create so many pads that in the end there are 7 ( $3 + \_ = 7$ ).

The basic concept of balance can be represented as well. Children can create quantities, remove from them, manipulate them with their fingers, and see the consequences of the manipulation at the same time in the nonverbal-symbolic protocol. Likewise it is possible to give instructions in the nonverbal-symbolic form and to see the output in the iconic forms with the pads.

It is rather easy to imagine that addition and subtraction can be done in such an environment, and we have shown some ways how the action or the state can be linked to a nonverbal-symbolic representation. For multiplication and division it is advisable to take advantage of the time as another dimension. The temporal-successive idea of multiplication that can be traced back to a repeated addition is mapped to a repeated touch action of the same quantity of fingers several times. The protocol may then show, for four touches with five fingers, „ $5 + 5 + 5 + 5 = 20$ “ as well as „ $4 \cdot 5 = 20$ “. Thus the children can see, that there are different ways to protocol their manipulation. If several children are working together they can take advantage of the spatial-simultaneous idea of multiplication, creating the same quantity by several children at the same time. For division, one example activity would be to move pads and build piles of the same amount to divide a given number of pads.

### **FORECAST**

We are currently working on implementing the above scenarios using a multi-touch-table built at CERMAT. A first study that examines the critical point in translating numbers and operations from and in different forms of representation is going to take place in October 2010. At the same time we will conduct a pre-study about the way

children touch with their fingers and present quantities on a table. The programming of the multi-touch-learning-environment is currently in progress using a multi-touch extension of a dynamic geometry software system.

Finally, we aim to answer the research question about the impact of the availability of such multi-touch-learning-environments regarding the diagnosis and the support of acquiring basic concepts of numbers and operations.

## NOTES

1. MER: multiple external representations (Ainsworth, 1999)
2. MELRs: multiple equivalent linked representations (Harrop, 1999)

## REFERENCES

- Aebli, H. (1987). *Zwölf Grundformen des Lehrens*. Stuttgart: Klett-Cotta.
- Ainsworth, S. (1995). The role of multiple representations in promoting understanding of estimation: An empirical investigation. ESRC Centre for Research in Development, Instruction and Training. University of Nottingham. Technical report number 30.
- Brissiaud, R. (1992). A Toll for Number Construction: Finger Symbol Sets. In J. Bideaud, C. Meljac & J.-P. Fischer (ed.) (1992). *Pathways to number. Children's Developing Numerical Abilities*. New Jersey. Lawrence Erlbaum Associates.
- Claus, H. & Peter, J. (2005). *Finger, Bilder, Rechnen. Förderung des Zahlverständnisses im Zahlenraum bis 10*. Vandenhoeck & Ruprecht GmbH & Co. KG, Göttingen.
- Clements, D.H. (2002). Computers in Early Childhood Mathematics. *Contemporary Issues in Early Childhood*, Volume 3, Number 2, 2002.
- Dörfler, W. (1986): Zur Entwicklung mathematischer Operationen und gegenständlicher Handlungen. In: *Beiträge zum Mathematikunterricht*, 1986, S. 88-91. Bad Salzdetfurth: Franzbecker.
- Fuson, K.C. (1988). *Children's Counting and Concepts of Number*. New York: Springer.
- Fuson, K.C. (1992). Research on Learning and Teaching Addition and Subtraction of Whole Numbers. In G. Leinhardt, R. Putnam & R.A. Hattrop (eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale: Lawrence Erlbaum Associates. 53-187.
- Gelman, R. & Gallistel, C.R. (1978). *The child's understanding of number*.

Cambridge, MA: Harvard University Press.

- Gerster, H.-D. & Schultz, R. (2004). Schwierigkeiten beim Erwerb arithmetischer Konzepte im Anfangsunterricht. Bericht zum Forschungsprojekt Rechenschwäche – Erkennen, Beheben, Vorbeugen. Freiburg im Breisgau.
- Grissemann, H. & Weber, A. (2000). Grundlagen und Praxis der Dyskalkulietherapie. Bern: Hans Huber.
- Krauthausen, G. (1995). Die „Kraft der Fünf“ und das denkende Rechnen. In Müller, G. & Wittmann, E.Ch. (ed.). Mit Kindern rechnen. Arbeitskreis Grundschule – Der Grundschulverband – e.V. Frankfurt am Main. 87 – 108.
- Kay, A. (1972). A Personal Computer for Children of All Ages. In: Proceedings of the ACM National Conference. Retrieved from <http://www.mprove.de/diplom/gui/Kay72a.pdf>
- Ladel, S. (2009). Multiple externe Repräsentationen (MERs) und deren Verknüpfung durch Computereinsatz. Zur Bedeutung für das Mathematiklernen im Anfangsunterricht. Didaktik in Forschung und Praxis, Bd. 48. Verlag Dr. Kovac, Hamburg.
- Lorenz, J.-H. (1997). Der gescheiterte Rechenunterricht: Rechenversagen – Ursachen und “Therapie”. In W. Stark, T. Fitzner & Ch. Schubert (ed.) Grundbildung für alle in Schule und Erwachsenenbildung p. 90-97. Stuttgart: Klett.
- Mayer, R. (ed.) (2005). The Cambridge Handbook of Multimedia Learning. Cambridge University Press. New York.
- Resnick, L.B., Bill, V., Lesgold, S. & Leer, M. (1991). Thinking in arithmetic class. In B. Means, C. Chelemer & M.S. Knapp (ed.), Teaching advanced skills to at-risk students: Views from research and practice (pp. 27-53). San Francisco: Jossey-Bass.
- Richter-Gebert, J. & Kortenkamp, U. (2006). The Interactive Geometry Software Cinderella, Version 2.1, URL: <http://cinderella.de>.
- Steinweg, A.-S. (2009). Rechnest du noch mit Fingern? – Aber sicher! In MNU Primar 4 (1), 124 – 128.
- Thompson, P.W. (1992). Notations, conventions and constraints: contributions to effective uses of concrete materials in elementary mathematics. Journal for Research in Mathematics Education, 23 (2), 123-147.

# **SIMILAR BUT DIFFERENT - INVESTIGATING THE USE OF MKT IN A NORWEGIAN KINDERGARTEN SETTING**

Reidar Mosvold, Raymond Bjuland, Janne Fauskanger, and Arne Jakobsen

University of Stavanger [1]

*In the last decades, a lot of research regarding the professional knowledge of mathematics teachers in school has been conducted. Far less research has been done concerning the professional knowledge of kindergarten teachers, and there is little evidence regarding what kind of mathematical knowledge for teaching that is needed by kindergarten teachers. This paper analyzes a case study from a Norwegian kindergarten and discusses how the tasks of teaching [2] that have been observed in this kindergarten setting differ from those of teachers in school. Based on the analyses, we suggest that part of the theoretical framework has to be adjusted before it can be used in a kindergarten setting.*

## **INTRODUCTION**

In the last decades, a lot of research has been conducted in order to learn more about the mathematical knowledge that mathematics teachers need in their teaching practice and the impact of such knowledge (e.g. Ponte & Chapman, 2006). Researchers agree that knowledge is important (e.g. Hiebert, Gallimore, & Stigler, 2002), and some suggest that there is a connection between the teachers' mathematical knowledge and the students' results (Hill, Rowan, & Ball, 2005; Baumert et al., 2010). So far, most of the research has focused on teachers in elementary and middle school.

Based on results from several studies, Ball and her colleagues (e.g. Ball, Thames, & Phelps, 2008) present a framework of mathematical knowledge for teaching (MKT). According to their framework, mathematics teachers need a combination of content-specific knowledge and pedagogical content knowledge in order to facilitate learning of mathematics. Although their framework has been developed from studies of mathematics teachers in elementary and middle school (in the U.S.), it is tempting to suggest that kindergarten teachers also need some kind of MKT in their practice. When reviewing the literature, however, attempts to investigate the kind of mathematical knowledge needed by kindergarten teachers has so far been scarce. Lee (2010) is one of very few examples, but her focus is on pedagogical content knowledge, and she does not mention the MKT framework. Carlsen, Erfjord and Hundeland (2010) have investigated the competence of kindergarten teachers from a sociocultural view (and so have others), but they did not in any way build upon or relate to the MKT framework. In this paper we use a case study from a Norwegian kindergarten as a context for a first attempt to investigate some possibilities and limitations of using the MKT framework as an analytical tool in a kindergarten context. The following research question has been formulated:



What are the similarities and differences between the “tasks of teaching” of mathematics teachers in school and those of kindergarten teachers?

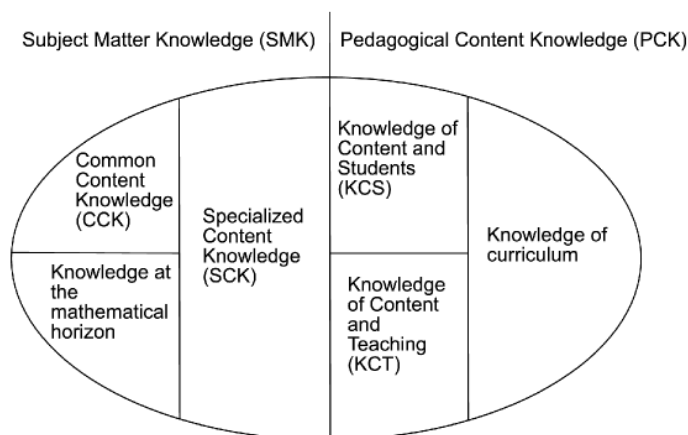
## **THEORETICAL BACKGROUND**

Before going into the MKT framework in more detail, it is important to clarify a few issues related to the Norwegian kindergarten context. As identified in *Starting Strong II* (OECD, 2006, chap. 3), there is often made a distinction internationally between two main traditions in early childhood education. The Nordic countries (including Norway) and some countries in central Europe are often placed within what is referred to as a social pedagogy tradition in early childhood education. Within such a tradition, kindergartens are viewed as institutions where a broad concept of pedagogy is used, and a combination of care, upbringing and learning is emphasized (ibid.). The other main tradition is referred to as the pre-primary tradition, and within this tradition, kindergarten is viewed as an institution with a main focus on preparing the children for school. Within the social pedagogy tradition, free play is emphasized, whereas kindergartens in countries with a pre-primary tradition often have a stronger resemblance with traditional schools (ibid.). Such differences are important to have in mind when discussing the use of the MKT framework in a kindergarten setting.

Another important aspect when discussing Norwegian kindergartens is related to a particular political initiative. In 2009, the Norwegian Government decided (through law regulations) that all children (1-6 year olds) should have the right to go to kindergarten (Kunnskapsdepartementet, 2009). As a result, a number of new kindergartens were built. This massive increase of kindergartens did not lead to a corresponding expansion of kindergarten teachers, and several municipalities still experience a severe lack of educated kindergarten teachers. At the same time, the knowledge and competence of the kindergarten teachers was strongly emphasized by the Government (ibid.). Norwegian kindergartens are thus faced with a huge challenge, and in this connection it is appropriate to approach the question of what kind of mathematical knowledge these kindergarten teachers need.

The focus in research concerning teacher knowledge has shifted from using the number of courses or study points as an indicator (e.g. Begle, 1979) in order to document teachers' apparent lack of understanding about the mathematics that their students are supposed to learn (Cooney, Shealy, & Arvold, 1998), to a focus on the many facets of teachers' mathematical knowledge for teaching (e.g. Ball et al., 2008). A large proportion of the research that includes the latter focus builds upon Shulman's (1986) conceptions of the different aspects of teachers' professional knowledge. His model originally distinguished between seven different kinds of knowledge, whereas three of those had a subject-related content: pedagogical content knowledge, content knowledge, and knowledge of curriculum. Shulman's model has now been developed further by Ball and her colleagues (e.g. Ball et al., 2008), and

these researchers from the University of Michigan present a content specific model for teachers' mathematical knowledge for teaching.



**Figure 1. Aspects of MKT (from Ball et al., 2008, p. 403).**

These researchers have a particular focus on the so-called work of teaching, and as a result of their studies, a list of some distinct and important tasks of teaching was proposed [2]. Although these tasks of teaching have been identified from studies of mathematics teachers in school, it is interesting to investigate whether or not kindergarten teachers are faced with the same tasks in their work. We discuss this in the context of a shopping game in a Norwegian kindergarten, and we focus on a selection of tasks that appeared relevant for this particular context.

## METHODS

The context for the study is a group of 3-year old children in a Norwegian kindergarten. Mary is the main kindergarten teacher in this class, and she does not have an approved education as kindergarten teacher yet. The reason for selecting Mary and her group as a case in this connection, is that we believe that analyses of her teaching practice can be used as a starting point for identifying the MKT that kindergarten teachers need, and which she appears to be in the process of developing. Hiebert, Gallimore and Stigler (2002) argue that practitioner knowledge along with professional public knowledge provide a knowledge base for the teaching profession, and this argument can be used as rationale for our selection of case as well.

In a larger context, Mary and her group had been part of a developmental study for about half a year at the time of this case study. During that year, a number of visits had been made to the kindergarten by the main researcher in order to discuss, plan and evaluate learning activities with the kindergarten teachers.

The data we analyze here is from an activity where Mary decided to stimulate the development of the children's concept of number and counting in the context of a shopping game. In addition to video recordings of the actual activity and field notes,

data were gathered from audio recording of interviews and discussions with Mary in connection with the planning and evaluation of the activity.

The recordings of audio (and video) were transcribed, and these transcripts were analyzed through content analysis (e.g. Kvale, 1996) with MKT as a theoretical lens.

## RESULTS

When the interview starts, Mary and the researcher have a little discussion about their last project. In that project, their focus had been on figures and shapes, and now they want to shift their focus towards numbers and counting. The children had already made experiences with numbers (or numerals) in relation to emergency numbers (somewhat equivalent to 911, only there are three different in Norway). They had talked about these numbers and linked them with photos of an ambulance, a police car and a fire engine respectively. Mary starts elaborating on her ideas and thoughts so far, and a discussion about the new project evolves. Mary does not want to leave their previous focus on shapes and figures completely behind, but she still wants to shift her main focus somewhat. She discusses this with the researcher:

Researcher: Yes. Mmm. Then I think it sounds reasonable that you choose something [an issue] that you really [want to] spend time on.

Mary: Yes, because I observe that, when we have spent so much time on circles and triangles, then they manage it! And then ... then we know that we should spend some more time on other things [issues] as well. And then ... no, I think that the counting [in itself], that [issue] we manage to connect with most [of our activities], both in songs, books, in assembly - well, we simply count all kinds of [objects]. Then we'll be able to manage it.

They keep discussing how the issue of counting appears in different settings, like in games. When playing games, especially board games where dices are used, the children need to count the steps while they are moving pieces along the board. This kind of counting appears to be difficult for the children in Mary's group, she reveals. In their continued discussion, they focus on the connection between counting out loud and making a one-to-one correspondence, which the children have to master when counting objects. Mary and her colleagues have experienced that the children count asynchronously in many occasions. This particular kind of knowledge, which might be defined as knowledge of content and students/children, appears to be of importance to the kindergarten teacher. In order to be able to support the children's learning, a kindergarten teacher needs some knowledge about how children develop number sense, counting skills etc. Mary tells the researcher that the children in her group know the numbers 1-3, and some children can even count further, but something recently came up which surprised her:

Mary: Yes, they do! Well, some even manage to count till 10 in both Norwegian and English. But even though he is counting both in Norwegian and English

### Working Group 13

till 10, then I was actually surprised when we were sitting down reading in that “duck-book” - that counting book - eh ... that counting by pointing, and I didn’t quite manage to connect the dots there. So it became a bit: Oh, [inaudible] that counting by pointing then it went a bit pell mell, so ... and then I was a bit worried based on what I maybe had anticipated, because I expected something [more] from a child who speaks so extremely well! And [he] manages to explain himself incredibly well. But maybe it was ... I hadn’t actually - then I got a bit surprised.

Here, Mary shares her experiences from a situation that emerged when she was reading together with a boy who she believed could count really well. He knew his numbers from one through ten in both English and Norwegian, but he still appeared to have problems using the counting as a tool in order to find the quantity. This illustrates a kind of knowledge of content and students/children, and in this example it was a matter of Mary discovering that there was something she did not know about before she experienced this situation.

Mary decides to facilitate the playing of a shopping game in her group, and she intends to use this as an arena to stimulate the children’s further development of counting skills. The children need to figure out how much a certain good costs, count out the correct amount of money, pay for the goods, and possibly receive the correct amount of change if they have paid more than the costs. Mary’s idea was that all of this would provide a nice context for stimulating the children and provide them with rich experiences in relation to numbers, quantity and counting.

The next time the researcher visits their kindergarten Mary and her colleagues have already started experimenting with the shopping game. They had decided to keep the prices low, and they agreed to represent the prices with a numerical representation along with the correct number of dots. This decision relates to a task of teaching where the kindergarten teacher needs to select the appropriate representation for particular purposes, and the knowledge involved in this process can be labeled as what Ball and colleagues (2008) refer to as knowledge of content and students.

The starting point, based on Mary’s experiences with a particular boy, was a belief that the children would not be so skillful in pointing by counting. When they started playing the shopping game, Mary soon experienced that she had to adjust her goals somewhat:

Mary: No, what has happened is that ... well, before we managed to finish this properly, we were very focused on that counting by pointing. And suddenly then all managed to ... [count] till more than 10. I think there are two - we had a round at an assembly, and then they counted, yes then they counted till more than 10. I think there were two [of the children] who started to shake with the finger at 6.

Researcher: Mmm.

### Working Group 13

Mary: So, they have managed to come very far with this [already]!

Researcher: Yes.

The children were obviously motivated to play the shopping game, and all of a sudden most of them were able to count objects up till 10 and even appeared to understand about the price tags. Then something happened:

Mary: What became a little bit funny was when [laughs a bit] Kari [a colleague] came to film [laughs a bit more]. Then they easily understood [about] the prices. They did, that was not a problem. What became difficult [for them] was to count out the correct amount of money. They didn't manage that ... properly. Then they were ... actually, they were more concerned with those items ... and the shopping part. They were very eager to shop.

The children had rather quickly developed their counting skills, and they were able to count the number of coins they possessed. When they were going to use their counting skills to "count out" the correct amount of money from a larger amount, however, many children struggled.

Mary: But it became difficult when they had - when I had given them some money, and said: "How many [coins] do you have to pick out then?"

Researcher: Yes, so finding the correct amount [of money], that was difficult.

Mary: It was a bit difficult in the beginning. Yes, but I don't think it will take long before they understand that as well.

Researcher: No, because there is a little difference in asking about how many coins there are, and finding ... four coins, in a way.

Mary: Yes, because that was a bit difficult [for them], because they had to pick out that many [coins] when I had given them more, or when I held them [those coins] in my hand and said: "Yes, but now - how many [coins] do you need now in order to pay for that thing?" Then they just wanted to take all the coins. Actually, even though it was 7, and maybe they should only use 4 (...)

Mary's experiences in this situation apparently went beyond the practice based knowledge and experiences that she had made before.

The discussions between Mary and the researcher indicate that counting is a complex activity. To use a context like this in order to facilitate children's learning, a particular kind of knowledge is needed. In the discussions below, we investigate how the tasks of teaching that Mary face are related to some of the tasks of teaching that were identified by Ball and colleagues (2008) in their studies of mathematics teachers in U.S. schools.

### DISCUSSION

In a typical Norwegian kindergarten, there are no mathematics lessons, no mathematics textbooks, no blackboards, and no traditional classroom with desks. The

children's learning takes place in everyday activities and play situations. As a result, the (Norwegian) kindergarten teachers are faced with challenges that are different from those of mathematics teachers in school. When discussing a possible application of the MKT framework in a Norwegian kindergarten setting, such contextual (and cultural) differences are important to be aware of.

The dialogue between Mary and the researcher has indicated that Mary has some kind of MKT about counting that she makes use of in the context of the shopping game. She relates previous experience from shapes and figures to the new topic of counting. The need for some kind of pedagogical content knowledge (including knowledge of content and students/children in particular) appears evident in this situation. The challenge is to try and distinguish what kind of content-related knowledge (specialized content knowledge in particular) that kindergarten teachers like Mary need. A focus on tasks of teaching is a relevant tool to use when approaching such a challenge. Mary is also involved in work related to presenting ideas. Mary's work of teaching is, however, quite different from that of a mathematics teacher in school. In our particular case, Mary introduced an activity that she believed would provide the children with various experiences related to counting. She would not, however, present ideas of counting to a whole group of children in a lecture-like way. Instead, she has to react to challenges as they appear within the context of a play situation. Mason and Spence (1999) propose that the absence of the active knowledge (knowing-to) blocks teachers and students from responding constructively in the moment. In the same way, it is an important feature for a kindergarten teacher to act in the moment and to show awareness in a particular teaching situation. In Mary's situation, we might refer to this as situated knowledge of responding to the children's questions. The question she had to respond to, however, just appeared from the children's acting in the play situation. It was not asked directly by the children.

For a mathematics teacher in school, teaching is often related to a particular textbook or other curriculum material. The teacher presents a mathematical idea to the class, and appropriate examples are used to illustrate or make specific a certain point. A Norwegian kindergarten teacher, like Mary, does not have a mathematics textbook. Instead, she presents a mathematical idea by using examples as they appear naturally in the play situation, and she is trying to facilitate a context in which the children can experience a certain idea. We would suggest that a reformulation is necessary in order for the tasks of teaching to be relevant in a kindergarten context. One suggestion is to merge the two tasks that were originally defined as "presenting mathematical ideas" and "finding an example to make a specific mathematical point" into "facilitating activities that enable children to experience mathematical ideas".

Another task of teaching that we identify in this case is that of connecting a topic taught with topics from prior years. Several issues appear problematic when trying to apply such a description to the tasks involved in Mary's practice. Most importantly,

she is not teaching the children in a traditional sense, and Norwegian kindergarten teachers (and educators) often avoid using the word “teaching” altogether. Different topics are covered through experiences from play situations and informal discovery activities, and it is therefore hard to track previously taught topics in Norwegian kindergartens. She facilitates the children’s learning and exploring in a play context. The aims that she has set for the activity are based on her impressions of where the children are in their mathematical development rather than knowledge of previously taught topics. It becomes a challenge for Mary to figure out what kind of informal knowledge the children have previously built in relation to this topic. Oftentimes, they have made experiences in free play (where Mary has not been present), or their informal knowledge has been built from experiences in everyday situations at home or in kindergarten. Whereas a mathematics teacher in school can go back to curriculum materials (like textbooks), previous tests or worksheets, Norwegian kindergarten teachers do not have this possibility. The task of connecting with previously taught topics thus becomes a different task of teaching when compared with that of teachers in school.

Other tasks of teaching (as presented by Ball et al., 2008) are more relevant for describing the work of teaching of Norwegian kindergarten teachers, but the ones discussed above are somewhat problematic.

## CONCLUSIONS

When Ball and colleagues (e.g. 2008) proposed their theoretical framework of MKT, they also included a list of common tasks of teaching that were intended to describe the work of teaching of mathematics teachers in school. The research that this framework builds upon was conducted in the U.S. only, but the tasks of teaching as well as the theoretical framework as a whole were still supposed to be of a more universal nature. When trying to apply this framework to describe the professional knowledge of a Norwegian kindergarten teacher, some problematic issues appear. Although the model’s distinction between various aspects of teacher knowledge appears to be applicable in a kindergarten setting, some of the proposed tasks of teaching have to be modified. The main reason is that the work of teaching of a Norwegian kindergarten teacher is very different from that of a mathematics teacher in U.S. Schools. Whereas mathematics teachers in school are constantly faced with challenges regarding how to present mathematical ideas, the entire concept of presenting mathematical ideas needs to be described differently in order to apply as a description of the work of teaching that Norwegian kindergarten teachers are involved with. Although it can be described as a similar kind of challenge, the way a kindergarten teacher has to use play situations and everyday activities in order to facilitate children’s informal experiences with mathematical ideas is quite different from when mathematics teachers in school attempt to present mathematical ideas to their pupils. Our suggestion is to merge the two tasks that were originally defined by Ball and colleagues (2008) as “presenting mathematical ideas” and “finding an

example to make a specific mathematical point” into “facilitating and using activities and play situations that enable children to experience mathematical ideas”. Similarly, the task of teaching related to connecting a topic taught with topics from previous years also needs some adaptation. For a Norwegian kindergarten teacher, it becomes more of a challenge related to uncovering and building upon the children’s previous informal knowledge and experiences from everyday activities and play situations.

The Norwegian kindergarten belongs to a social pedagogy tradition (see OECD, 2006), and we suggest that some of the challenges with using the MKT model in a kindergarten context might be relevant for other kindergartens and kindergarten teachers within such a tradition. The work of teaching of kindergarten teachers within the pre-primary tradition (ibid.) is probably more similar to that of mathematics teachers in school, and the MKT model along with the proposed list of tasks of teaching might be more directly transferable in such a context. We believe that further research is needed in order to learn more about the tasks of teaching that kindergarten teachers are faced with. Although the MKT model can be used to describe some relevant aspects of the professional knowledge that is needed in a kindergarten setting, vital differences between the work of teaching as well as the tasks of teaching of kindergarten teachers and those of mathematics teachers in school exist. Further studies are needed in order to investigate such similarities and differences, and we believe that special attention needs to be made in relation to cultural as well as situated aspects.

## NOTES

1. Our research project has been supported by OLF, The Norwegian Oil Industry Association.
2. *Tasks of teaching* refer to a specific list of challenges that have been identified. The complete list includes: Presenting mathematical ideas, Responding to students’ “why” questions, Finding an example to make a specific mathematical point, Recognizing what is involved in using a particular representation, Linking representations to underlying ideas and to other representations, Connecting a topic being taught to topics from prior or future years, Explaining mathematical goals and purposes to parents, Appraising and adapting the mathematical content of textbooks, Modifying tasks to be either easier or harder, Evaluating the plausibility of students’ claims (often quickly), Giving or evaluating mathematical explanations, Choosing and developing useable definitions, Using mathematical notation and language and critiquing its use, Asking productive mathematical questions, Selecting representations for particular purposes, Inspecting equivalencies (Ball et al., 2008, p. 400).

## REFERENCES

- Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y.M. (2010). Teachers’ mathematical



- knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, 47(1), 133-180.
- Begle, E. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Reston, VA: National Council of Teachers of Mathematics.
- Carlsen, M., Erfjord, I., & Hundeland, P.S. (2010). Orchestration of mathematical activities in the kindergarten: The role of questions. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6, Lyon, France)*, (pp. 924-933). Université de Lyon 1. Retrieved on September, 6, 2010 from <http://tinyurl.com/39k8qf2>
- Cooney, T., Shealey, B.E., & Arvola, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, 29(3), 306-333.
- Hiebert, J., Gallimore, R., & Stigler, J.W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher*, 31(5), 3-15.
- Hill, H.C., Rowan, B., & Ball, D.L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371-406.
- Kunnskapsdepartementet (2009). *St.meld. nr. 41 (2008-2009): Kvalitet i barnehagen*. Oslo: Det Kongelige Kunnskapsdepartementet [Ministry of Education and Research].
- Kvale, S. (1996). *Interviews. An introduction to qualitative research interviewing*. London: SAGE Publications.
- Lee, J. (2010). Exploring kindergarten teachers' pedagogical content knowledge of mathematics. *International Journal of Early Childhood*, 42(1), 27-41.
- Mason, J. & Spence, M. (1999). Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment. *Educational Studies in Mathematics*, 38(1-3), 135-161.
- OECD (2006). *Starting Strong II: Early childhood education and care*. Paris: OECD Publishing. Retrieved September 6, 2010, from <http://tinyurl.com/2uqtfa3>.
- Ponte, J.P., & Chapman, O. (2006). Mathematics teachers' knowledge and practice. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education - past, present and future* (pp. 461-494). Rotterdam: Sense Publishers.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

# KINDERGARTNERS' PERSPECTIVE TAKING ABILITIES

Aaltje Berendina Aaten\*, Marja van den Heuvel-Panhuizen\*, Iliada Elia<sup>^</sup>\*

\*FIsme, Utrecht University, the Netherlands

<sup>^</sup>Department of Education, University of Cyprus, Cyprus

*This study investigated kindergartners' imaginary perspective taking (IPT) abilities by examining their ability to imagine whether an object is visible from another viewpoint (IPT Competence 1) and how an object looks from another viewpoint (IPT Competence 2). The participants were 308 4- and 5-year-old kindergartners in the Netherlands. A paper-and-pencil test of various perspective-taking pictorial tasks was developed and administered to the children. The results show that IPT Competence 2 is more difficult than IPT Competence 1, and that both competences develop during the kindergarten years. Also, for both IPT Competences 1 and 2 a positive relationship was found with mathematics performance, while no gender difference was found on either IPT Competence.*

*Key-words: Kindergartners, Spatial reasoning, Perspective taking ability, Gender*

## SPATIAL GEOMETRY FOR YOUNG CHILDREN

Geometry is an indispensable part of contemporary early childhood curricula and educational programs (Sarama & Clements, 2009). It is not confined to plane geometry, but spatial abilities play an important role as well. For example, in the Standards of the National Council for Teachers of Mathematics (NCTM, 2000) in the K-2 grades much attention is paid to *Specifying locations* (which includes interpreting relative positions in space) and *Using visualization* (which includes creating mental images of geometric shapes using spatial memory and spatial visualization and recognizing and representing shapes from different perspectives). Similarly, the TAL teaching/learning trajectory for geometry (Van den Heuvel-Panhuizen & Buys, 2008) includes the sub-domain *Orienting* which focuses on localizing and taking a point of view in both the first and second year of kindergarten (K1 and K2).

This emphasis on a spatial interpretation of geometry for young children is not surprising. It is the natural way in which children encounter geometry. They discover the world around them while they walk, play, and look around. They are in fact investigating their environment all the time; by doing so, they learn to find their way, to determine their own location within the environment, to describe to others their own position or the position of an object such as their teddy bear. Also visualization and spatial reasoning abilities develop through children's activities, such as playing hide-and-seek. Children try to hide in a place in which they will not be visible to the child that is looking for them. As such, they try to imagine or to reason what the other child will and will not be able to see while wandering around.

Although the above seems very plausible now, in the past the teaching of geometry to children started with plane geometry. It was Freudenthal (1973) who argued strongly for changing this and starting with spatial geometry at an early age.

“Geometry is grasping space . . . that space in which the child lives, breathes and moves. The space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it.” (Freudenthal, 1973, p. 403)

Enhancing young children’s spatial abilities is important for several reasons. As described above, spatial abilities help children to understand their environment. Furthermore, the development of spatial abilities is important for the development of mathematics ability in general. According to Clements (2004, p. 267), “[g]eometry and spatial reasoning form the foundation of much learning of mathematics and other subjects.” Spatial reasoning in particular is recognized by mathematicians as a useful strategy in mathematical problem solving, for example through the use of diagrams and drawings to solve problems (Casey, Andrews, Schindler, Kersh, Samper, & Copley, 2008). Moreover, teaching spatial ability enhances children’s mathematical attitude (Casey et al., 2008; Van den Heuvel-Panhuizen & Buys, 2008).

Altogether, spatial abilities are important for young children to learn and therefore, it is worthwhile to gain more insight into how they develop these abilities. In this study, we focused on a specific spatial ability of kindergartners, namely their competence to mentally take a particular point of view.

## IMAGINARY PERSPECTIVE TAKING

Important research on imaginary perspective taking (IPT) has been done by Piaget and Inhelder (1956). One of the tasks that they used to investigate children’s IPT was the “Three Mountain task”. In this task children were positioned on one side of a table and asked to describe how the scene on the table would look from the opposite side. In this way it was found that children up to the age of nine tended to describe the way the scene looked from their own position. They were not able to take a perspective from another position than their own.

Flavell, Everett, Croft, and Flavell (1981) came up with a distinction into two abilities of perspective taking. The so-called Level 1 competence concerns the *visibility of objects*: a child that has achieved this competence is able to deduce which objects are or are not visible from the other viewpoint. The Level 2 competence relates to the *appearance of objects*: a child that has attained this competence is able to indicate how an object looks as it is observed from a different viewpoint. Hughes (1975, as cited by Donaldson, 1980) has independently proposed a very similar model, in which “projective” and “perspective” abilities correspond respectively to Level 1 and Level 2 competence.

In connection to the distinction into two abilities, Flavell et al. (1981) also found that the abilities differ in their rate of development. Children of three years of age performed well on the Level 1 tasks but had difficulties with the Level 2 tasks.

### **Mathematics performance**

Many studies (e.g. Burnett, Lane, & Dratt, 1979; Casey, Nuttall, & Pezaris, 1997; Geary, Saults, Liu, & Hoard, 2000) have found positive correlations between mathematics performance and spatial ability, but most of these studies were done with high school and university students. Only a few studies were carried out with young children. For example, Robinson, Abbot, Berninger, and Busse (1996) found a high correlation between two-dimensional reasoning and quantitative skills in precocious preschoolers and kindergartners. Guay and McDaniel (1977, as cited by Lean & Clements, 1981), who did a study with primary school students, focused on a broader range of spatial abilities not only encompassing two-dimensional visualization tasks, but also tasks involving three-dimensional mental images and mental transformation of these images. The results indicated a positive correlation between mathematics performance and two- and three-dimensional spatial tasks in primary school students.

### **Gender**

The findings with respect to the relation between gender and spatial ability are various. Several studies found a male advantage for three-dimensional mental rotation (Casey et al, 2008), two-dimensional mental rotation as well as translation (Levine, Huttenlocher, Taylor, & Langrock, 1999), and spatial visualization (Tracy, 1987) already in existence at kindergarten age. Horan and Rosser (1984) investigated children aged 4, 6, and 8 years who were offered dimension-transcending tasks in which the questions and the answers were formulated in a different dimension: a three-dimensional object was shown and the child was asked how this object looked to an observer in another position after the object was rotated 90°. The child could answer the question by selecting the correct two-dimensional picture. In these dimension-transcending tasks, boys performed better than girls. However, if the question and the answer were both presented in the form of two-dimensional pictures, girls performed better than boys.

Some studies, though, did not reveal a gender effect for spatial ability. For example, Lachance's and Mazzocco's (2006) longitudinal study on children in lower primary school did not show sustainable gender differences in spatial ability in general. Also, Newcombe and Huttenlocher (1992) did not find gender differences with respect to perspective taking in their study in which they asked young children which object would be in a certain position relative to another observer.

### **RESEARCH QUESTIONS AND HYPOTHESES**

In light of the above, we formulated three research questions to gain more insight into kindergartners' imaginary perspective taking abilities:

1. How able are kindergartners in IPT Competence 1 (determining whether an object is visible from another viewpoint) and IPT Competence 2 (imagining how an object looks from another viewpoint)?
2. How are these IPT competences related?
3. How are these IPT competences related to kindergarten grade, mathematics performance and gender?

Based on previous research we could only formulate two hypotheses:

1. Kindergartners will perform better on tasks requiring IPT Competence 1 than on tasks requiring IPT Competence 2.
2. There will be a positive relation between mathematics performance and the IPT competences.

## METHOD

### Assessment Instruments

*Assessment of IPT.* To answer our research questions, we developed a paper-and-pencil test to measure kindergartners' perspective taking abilities. This test contains two booklets of test items, and was administered in two test sessions. The test items consist of a picture illustrating what each question is about and four pictures depicting the possible answers. Before implementing the test items in our study, we piloted them to ensure children's correct understanding of the test items.

In Figure 1a, an example of a test item is given. This *Basket* item is meant for measuring IPT Competence 1. Figure 1b shows the *Mouse* item, meant for measuring IPT Competence 2.

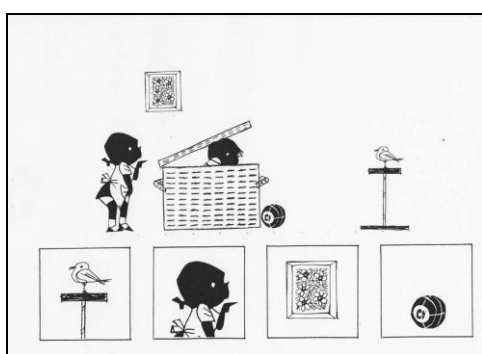


Figure 1a: *Basket* item

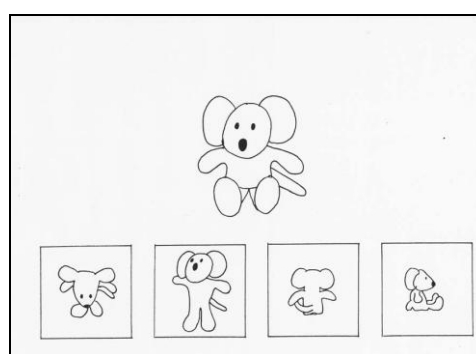


Figure 1b: *Mouse* item

The questions were read to the children by trained test administrators. For the *Mouse* item, the question was: "How do you see Mouse if you look at him from above like a bird?" Children answered the question by underlining the picture that shows the correct answer. Correct responses were coded as 1, and incorrect ones as 0. Six items in the test concerned IPT Competence 1 and five items concerned IPT Competence 2.

*Assessment of mathematics performance.* To assess children's mathematics performance, we made use of a standardized mathematics test for kindergartners, the CITO Test Ordering, which is in nation-wide use. This test has separate versions for the first (K1) and second year of kindergarten (K2); the Cronbach's alphas of both versions are .85 and .81 respectively (Van Kuyk & Kamphuis, 2001).

## Participants

Our study involved 384 Dutch kindergartners. Children who did not do both test booklets were excluded from the analysis, which diminished our sample to 308 children, 146 girls and 162 boys; 109 children attended K1 and 199 children were in K2. The K1 children had an average age of 4 years and 8 months and the K2 children were on average 5 years and 8 months old.

## RESULTS

### Kindergartners' abilities in IPT Competences 1 and 2

The data analysis showed that the kindergartners achieved a significantly higher success rate on test items that require IPT Competence 1 ( $M = .73$ ,  $SD = .20$ ) than on the IPT Competence 2 items ( $M = .34$ ,  $SD = .22$ ) [ $t(307) = 26.42$ ,  $p < .01$ ], see Figure 2.

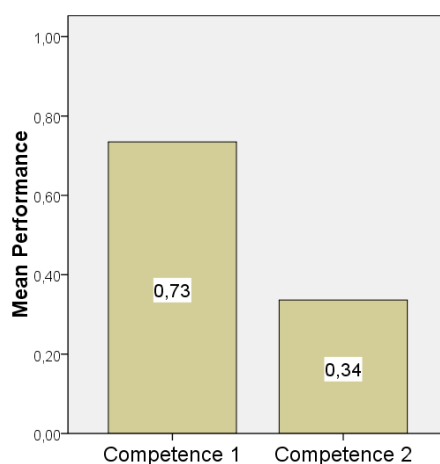


Figure 2: Mean performance on IPT Competence 1 and IPT Competence 2 items

### Relationship between IPT Competences 1 and 2

A statistical implicative analysis (Lahanier-Reuter, 2008) was used to investigate the relationship between the two competences based on the children's responses to the test items. Statistical implicative analysis leads to results such as "if we observe success on item *a* in a subject, then in general we observe success on item *b* in the same subject". We conducted this analysis for both kindergarten grades separately. In Figure 3 the results for K1 are depicted on the left and the results for K2 on the right. The figure shows how the items are related. The item names indicate what competence the item measures. For example, the item *C2cucumber* requires IPT Competence 2. The diagram shows that in K1, success on this item implies success on two IPT Competence 1 items. The probabilities of these implications are 90% for the grey arrows (e.g.

$C2cucumber \rightarrow C1tower$ ) and 95% for the blue arrows (e.g.  $C2cucumber \rightarrow C1hole$ ).

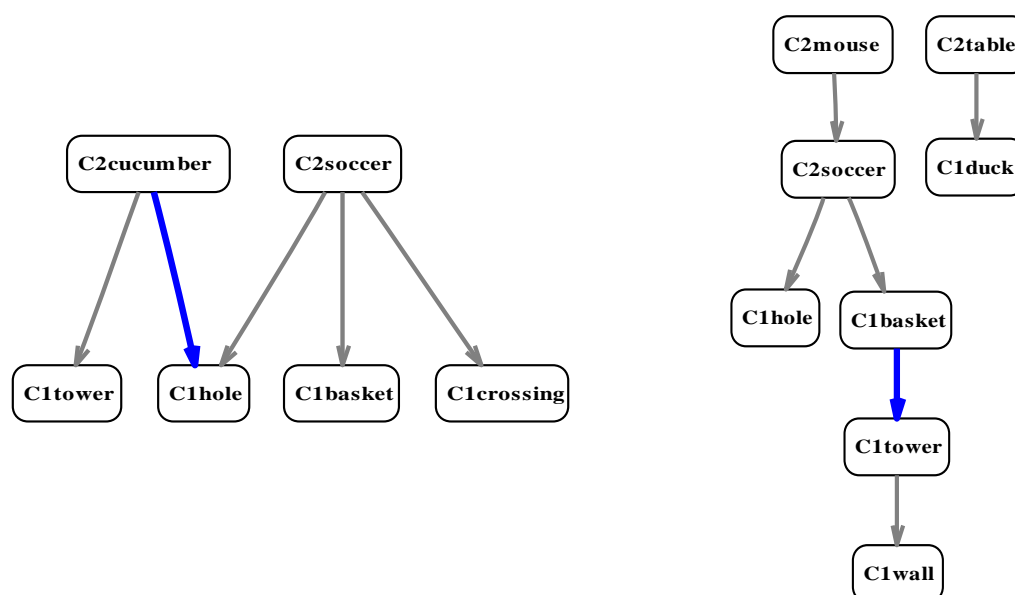


Figure 3: Implicative Relationships between successes on test items by K1 children (left) and K2 children (right)

The implicative diagrams for the K1 and K2 data show a similar pattern. In both diagrams, the items that measure IPT Competence 2 are high up in the diagrams, whereas the IPT Competence 1 items are positioned below them, i.e. success on the IPT Competence 2 items implies success on the IPT Competence 1 items.

### Development of IPT Competence 1 and 2

To gain more insight into the development of IPT Competences 1 and 2, we compared the performance on items measuring these IPT competences between kindergartners in K1 and K2 by carrying out independent samples t-tests.

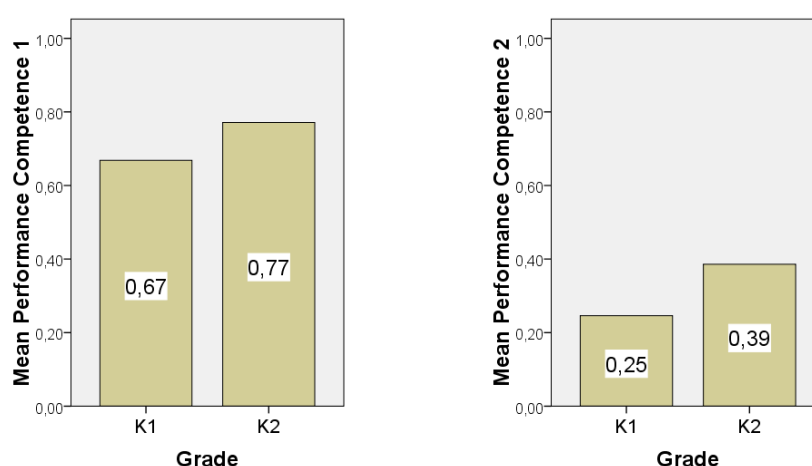


Figure 4: K1 and K2 children on IPT Competence 1 items (left) and IPT Competence 2 items (right)

As illustrated in Figure 4, K2 children's performance on IPT Competence 1 items ( $M=.77$ ,  $SD=.18$ ) was significantly higher than the K1 children's performance ( $M=.67$ ,  $SD=.21$ ) [ $t(306) = -4.44$ ,  $p < .01$ ]. Also on IPT Competence 2 items, K2 children performed better ( $M=.39$ ,  $SD=.22$ ) than K1 children ( $M=.25$ ,  $SD=.19$ ) [ $t(306) = -5.59$ ,  $p < .01$ ].

### Mathematics performance and IPT

To investigate a possible relation between IPT Competences 1 and 2 with mathematics performance, one-way analyses of variance (ANOVA) were carried out with the IPT competences as the dependent variables and children's mathematical achievement level as the independent variable. Children's mathematical achievement level was based on their score on the CITO Mathematics Test. The scores on this test are indicated by a letter. An A-score means that the test score belongs to the highest 25% scores of the complete Dutch population of kindergartners, a B-score stands for the next 25%, a C indicates that the score is in the third quartile, while D concerns the next 15% of scores and E the 10% lowest scores. The results of the analyses are shown in Figure 5.

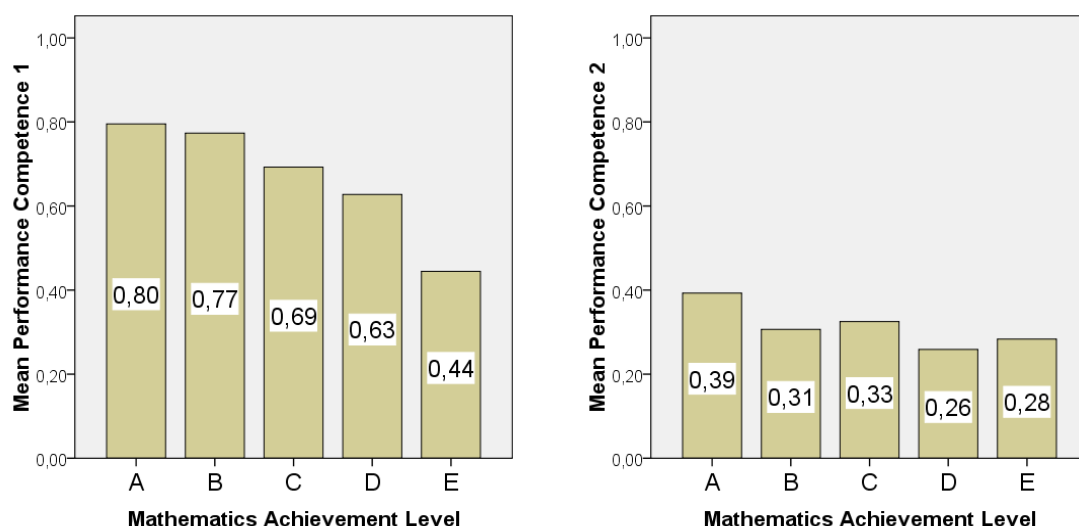


Figure 5: IPT Competence 1 (left) and IPT Competence 2 (right) relative to mathematics performance

For IPT Competence 1 [ $F(4, 298) = 14.56$ ,  $p < .01$ ] as well as for IPT Competence 2 [ $F(4, 298) = 3.12$ ,  $p = .02$ ] we found a significant relationship with the children's mathematics performance level.

### Gender and IPT

Independent-samples t-tests were used to compare boys' and girls' performances on the items measuring the two IPT competences. The results indicated no gender difference on either IPT Competence 1 (Boys:  $M=.73$ ,  $SD=.20$ ; Girls:  $M=.74$ ,  $SD=.21$ ) [ $t(306) = -0.12$ ,  $p = 0.90$ ] or on IPT Competence 2 (Boys:  $M=.34$ ,  $SD=.22$ ; Girls:  $M=.33$ ,  $SD=.22$ ) [ $t(306) = 0.57$ ,  $p = 0.57$ ], as is illustrated in Figure 6.



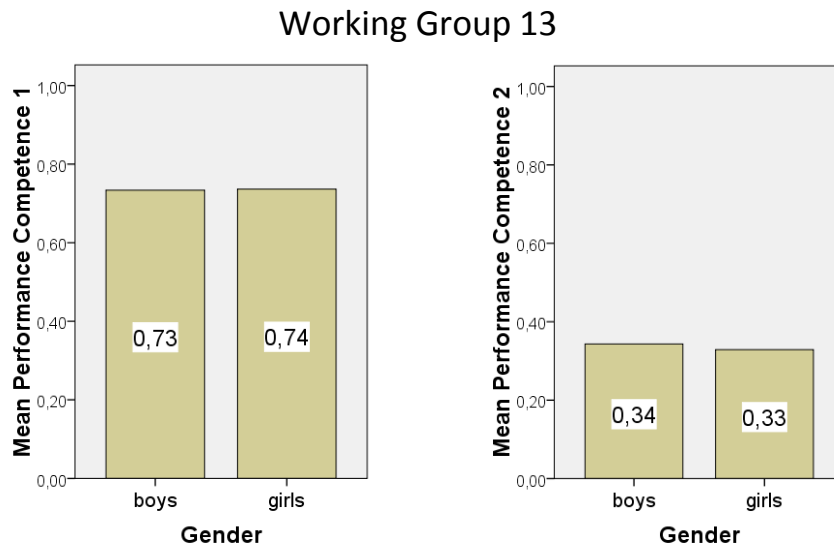


Figure 6: Performance in IPT Competence 1 (left) and IPT Competence 2 (right) by boys and girls

## DISCUSSION

In this study we investigated kindergartners' competences in determining the visibility and appearance of objects as seen from another point of view. In particular, we examined how these abilities are related, how they develop, and how they are related to gender and mathematics performance.

Concerning kindergartners' ability to determine whether an object is visible from a different viewpoint (IPT Competence 1), we found an overall success rate of 73% while the success rate on items that require the ability to determine the appearance of an object from a different point of view (IPT Competence 2) was significantly lower, i.e. 34%. This result indicates that the IPT Competence 2 items are more demanding for the children than the IPT Competence 1 items (Hypothesis 1), which is in line with previous studies (e.g. Flavell et al., 1981).

Statistical implicative analysis showed that success on several of the IPT Competence 2 items implied success on either other IPT Competence 2 items or on IPT Competence 1 items. The other direction, an implicative relation of an IPT Competence 1 item and an IPT Competence 2 item, was not found. This suggests that the development of IPT Competence 1 precedes the development of IPT Competence 2. This conclusion is in agreement with our finding that the children performed better on the IPT Competence 1 items than on the IPT Competence 2 items.

Children at kindergarten age experience major development in many fields. This turned out to be the case for the IPT competences as well. In our study we found that both IPT competences increase significantly from K1 to K2.

With respect to factors that possibly influence IPT we found that the children's mathematics performance level is significantly related to their IPT competences. This is in line with earlier findings on the relation between spatial ability and mathematics

performance. However, in our study the relationship between mathematics performance and IPT was stronger for Competence 1 than for Competence 2. Future research could explore the causes of this difference and whether it changes over time.

Comparison of IPT competences from the view of gender did not reveal significant differences. Both for the IPT Competence 1 items as well as for the IPT Competence 2 items, the results were similar for boys and girls. This finding is in agreement with previous studies on perspective taking, but contrasts with studies into the relationship between gender and spatial ability as assessed, for example, by rotation and translation tasks. Spatial ability clearly has a multi-dimensional structure of which the different aspects might have a different gender profile. Further research is indicated on this topic.

## REFERENCES

- Burnett, S. A., Lane, D. M., & Dratt, L. M. (1979). Spatial Visualization and Sex Difference in Quantitative Ability. *Intelligence*, 3, 345-354.
- Casey, M. B., Nuttall, R. L., & Pezaris, E. (1997). Mediators of Gender Differences in Mathematics College Entrance Test Scores: A Comparison of Spatial Skills With Internalized Beliefs and Anxieties. *Developmental Psychology*, 33(4), 669-680.
- Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The Development of Spatial Skills through Interventions Involving Block Building Activities. *Cognition and Instruction*, 26, 269-309.
- Clements, D. H. (2004). Geometric and Spatial Thinking in Early Childhood Education. In D. H. Clements & J. Sarama (Eds.), *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Donaldson, M. (1980). *Children's Minds*. Glasgow, Scotland: Fontana.
- Flavell, J. H., Everett, B. A., Croft, K., & Flavell, E. R. (1981). Young Children's Knowledge About Visual Perception: Further Evidence for the Level 1-Level2 Distinction. *Developmental Psychology*, 17(1), 99-103.
- Freudenthal, H. (1973). *Mathematics as an Educational Task*. Dordrecht, the Netherlands: D. Reidel Publishing Company.
- Geary, D. C., Saults, S. J., Liu, F., & Hoard, M. K. (2000). Sex Differences in Spatial Cognition, Computational Fluency, and Arithmetical Reasoning. *Journal of Experimental Child Psychology*, 77, 337-353.
- Guay, R. B., & McDaniel, E. D. (1977). The Relationship between Mathematics Achievement and Spatial Abilities among Elementary School Children. *Journal for Research in Mathematics Education*, 7, 211-215.

- Horan, P. F., & Rosser, R. A. (1984). A Multivariable Analysis of Spatial Abilities by Sex. *Developmental Review*, 4, 387-411.
- Hughes, M. (1975). *Egocentrism in pre-school children*. Edinburgh University: unpublished doctoral dissertation. [cited by Donaldson, 1980]
- Lachance, J. A., & Mazzocco, M. M. M. (2006). A longitudinal analysis of sex differences in math and spatial skills in primary school age children. *Learning and Individual Differences*, 16, 195-216.
- Lahanier-Reuter, D. (2008). Didactics of Mathematics and Implicative Statistical Analysis, *Studies in Computational Intelligence*, 127, 277-298.
- Lean, G., & Clements, M. A. (1981). Spatial Ability, Visual Imagery, and Mathematical Performance. *Educational Studies in Mathematics*, 12, 267-299.
- Levine, S. C., Huttenlocher, J., Taylor, A., & Langrock, A. (1999). Early Sex Differences in Spatial Skill. *Developmental Psychology*, 35(4), 940-949.
- National Council of Teachers of Mathematics (NCTM) (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.
- Newcombe, N., & Huttenlocher, J. (1992). Children's Early Ability to Solve Perspective-Taking Problems. *Developmental Psychology*, 28(4), 635-643.
- Piaget, J., & Inhelder, B. (1956). *The child's conception of space*. London, United Kingdom: Routledge & Kegan Paul.
- Robinson, N. M., Abbott, R. D., Berninger, V. W., & Busse, J. (1996). The Structure of Abilities in Math-Precocious Young Children: Gender Similarities and Differences. *Journal of Educational Psychology*, 88(2), 341-352. Tracy, 1987
- Sarama, J., & Clements, D. H. (2009). *Early Childhood Mathematics Education Research. Learning Trajectories for Young Children*. Routledge, New York.
- Tracy, D. M. (1987). Toys, Spatial Ability, and Science and Mathematics Achievement: Are They Related? *Sex Roles*, 17(3/4), 115-138.
- Van den Heuvel-Panhuizen, M., & Buys, K. (Eds.). (2008). *Young Children Learn Measurement and Geometry*. Rotterdam, the Netherlands: Sense Publishers.
- Van Kuyk, J. J., & Kamphuis, F. (2001). *Verantwoording van de toetsen uit pakketten Ruimte en Tijd, Taal voor Kleuters en Ordenen*. Arnhem: Cito.

# THE LINGUISTIC CODING OF MATHEMATICAL SUPPORTS

Anna-Marietha Hümmer

Goethe-University Frankfurt a.M., Germany

*Within the framework of recent research in mathematics education a subject specific performance of children and the learning of mathematics is not only depending on cognitive dispositions, which are mediated by individual performance motivation, but in a large part constructed socially. Mathematical concepts and procedures as well as the description of accepted participation in mathematics thereby are not conveyed explicitly, but transmitted implicitly in terms of so-called “codes”, which have a socio-linguistic dimension. In the micro sociological focus of this paper the question should be addressed, to which extent the acquisition of basal mathematical knowledge in early childhood is regulated by the ability to recognize such linguistic codes and their meaning. In that case sequences from mathematical situations with children age 4 are analysed concerning to (subject specific) socio-linguistic theories according from Bernstein (1996) and Carruthers and Worthington (2006). First theoretical considerations and empirical results will be presented.*

## INTRODUCTION

According to the later work of Basil Bernstein (1996), verbal skills and the ability of “reading” and reproducing linguistic codes in interactions are playing a significant role in learning mathematics and for the performance of mathematics’ discourses (e.g. Schütte 2010; Gellert/Hümmer 2008; Lubienski 2004). “Success” in learning mathematics therefore seems to be dependent to the ability of decoding relevant meanings and rules of participation in mathematical interactions on the verbal and non-verbal plane in order to comprehend the mathematical content and subject specific mathematical demand on presenting a content. With respect to this approach the paper focuses on the question, to which extent the development of mathematical thinking and learning in early childhood is regulated by the ability to recognize such linguistic codes and to decode their meaning. To put it in more concrete words: What kinds of codes can be specified in discourses within the kindergarten context and which impact do they have on the acquisition and the learning of mathematics in the early years.<sup>1</sup>

On the theoretical plane this paper tries to concatenate the socio-constructivistic approach of Bruner (1983) concerning learning theory with the socio-linguistic models of Basil Bernstein (1996) to take in to account the impact of linguistic representations on the process of negotiation of meaning.

From a socio-constructivist point of view the individual ability of learning mathematics and to participate successfully in mathematical discourses develops in the course of interaction with other members of the culture. Bruner (1983) emphasizes that it is impossible for an individual to acquire knowledge without

social interaction. The participants construct in their interactional moves a system of linguistic codes. For this reason, we analyze the mathematical situations of play and exploration that we initiated in day care centres by reconstructing this process of “code-production”. In the following analysis there is chosen a situation with exemplary character to give a short but detailed look towards the research questions and the current work within the project.

## **THEORETICAL PERSPECTIVE**

From a socio-reconstructivistic perspective the acquisition of mathematical knowledge is not only an act of the (inner-psychological) internalization of concepts and procedures, rather it is a process that is conditional on the negotiation within interactions. The cognitive development of the individual itself is seen as constitutive product moderated through the participation in such interactions. The social interaction therefore is a condition for the probability of learning (e.g. Miller 1986, p. 27f; Krummheuer 1989, p.241f). Within these interactions conducive instants can probably emerge.

### **Supports as a condition of mathematical learning**

Concerning to Bruner (1983) the process of learning starts when an adult and a child „create a predictable format of interaction that can serve as a microcosm for communicating and constricting a shared reality“(Bruner 1983, p.14). In this interactional framework „ shared procedures of interpretation and negotiation “ (ibid., S.17) take place. Bruner developed, in the first instance without knowing Vygotsky’s work, a theoretical approach within learning is afforded by a so-called “Supportive system”, which is moderated by a capable other. With regard to the acquisition of mother tongue, which Bruner observed, he considered such a “Language Acquisition Support System“ (LASS) (ibid., p.32) as necessary for the activation of the inborn “Language Acquisition Device” (LAD). On the empirical plane Bruner reconstructs the so-called “format”(ibid., p.33) for which he accounted a supportive or rather a conducive function during the process of learning. The “format” hereby is “a standardized, initially microcosmic interaction pattern between an adult and an infant that contains demarcated roles that eventually becomes reversible“ (ibid., p. 120). For Bruner increasing autonomy within these “formats“ was seen as an indicator for learning progress. In his later approaches Bruner sees a direct conjunction between his perception and the “zone of proximal development” developed by Vygotsky (1978). This way he introduces the concept of “scaffolding” (Bruner 1990). Krummheuer (1989) proves these supportive structures also in mathematical learning processes in discourses in primary school.

### **Codes within mathematical supports**

From a perspective on learning theory the supports that are mentioned before have the function to afford the acquisition of knowledge and transmit the rules of accepted

participation in a specific discourse on the structural plane. This way they establish conditions for the learning process and can be understood as structural and structuring elements of a pedagogical situation. Thereby both the plane of meaning as regards contents and the social plane of accepted participation are structured by the formatting of meanings and attitudes. Though in his research Bruner focuses mainly on the function of supports concerning to the early acquisition. The difficulties given by the form of this structures (supports) and their influence on the development of meanings he leaves untouched. How these difficulties arise on a verbal plane Schütte (2010) describes. He perceives in his work on the linguistic plurality in mathematics classroom that the implicit transmission of knowledge and subject's content are main phenomena in interactions of the mathematics classroom. Therefore he obtains to the concepts of Basil Bernstein who speaks within his theory of codes of an "invisible pedagogy" (e.g. Bernstein 1996). According to Bernstein pedagogical situations and interactions are affected by regulative principles, so called codes. They are determined by implicit rules, which select and integrate relevant meaning, the way of realisation and the generating context (Bernstein 1996, p.111). Bernstein in that case speaks of the classification and the framing of the pedagogical knowledge (Bernstein 1996). The classification he assumes from the sociological concepts of Durkheim. It describes what belongs to a specific discourse and what is not part of it. This way the focus of the description is not the conjunction between different discourses rather the description is focussing on the boundaries between discourses. The classification also affects which content and meanings are valid for a specific discourse. It regulates which kind of content and meaning are acceptable within the interaction. This content and meanings have to be recognized. Bernstein calls this the „recognition-rule“ (Bernstein 1996, p. 30ff.). The second abstract, theoretical concept Bernstein introduces is the concept of "framing". It controls the social accepted behavior and accordingly the participation. It controls thereby the social relationship, the hierarchy and the performance on presented content. In case of the concept of framing Bernstein introduces parallel to the concept of "recognition rules" the approach of "realisation rules". On the one side both concepts deal with the rules of recognizing which symbols or gestures are closely connected with which meanings and on the other side how the accepted realisation of acquired knowledge looks like. In other terms: What is the relevant message of the communication and how a reply has to be communicated. These are two functions that are fundamental to each communicative situation. Schütte (2009) describes that concerning to the implementation of new contents in mathematics these rules should be transmitted explicitly to make the acquisition more easy for a child. This way the process of acquisition would be more successful. To which extent the acquisition of early mathematics also follows this appraisal should be observed and discussed in the following on an empirical and theoretical plane.

Another approach which Bernstein mentioned in his earlier work from the sixties and seventies and which he extended till his death in 2000 are the concepts of the

restricted and the elaborated code (e.g. Bernstein 1970; 1985; 1996). As elaborated the production of texts and contributions are seen if they are independent from every day expressions and generally admitted. As restricted are seen those, that depend on a specific situation and a context and only concern the given symbols and meanings (e.g. Bernstein 1972; Leufer & Sertl 2010; Lubienski 2004).

### Formal and informal speech

The approach of Carruthers and Worthington (2006) concerning the learning of Mathematics of preschoolers can be seen as an amplification of the further concepts of Bernstein. Similar to Bernstein, for them Mathematics takes place in two different languages: the informal every day language and the formal language of mathematics which is determined through specific technical terms and algorithmic diagrams. Thereby the transfer between both “languages”, as Carruthers and Worthington declare, is not distinctive or does not exist. So there is a discrepancy between formal and informal mathematics language. Over the intervening years children develop a so-called multi-competence, which enable them to have more than one ambassador of a linguistic approach simultaneously in mind. A kind of “interims speech” which is determined by rules and structures the children produces by themselves takes the part of connecting informal aspects with the formal ones. The knowledge of the informal every day language therefore is used to develop a formal speech. A main competence of children according to the acquisition in the early years seems to be inferential to develop such a multi-competence und to switch flexibly between both (the formal and the informal) languages and their assignment of meaning.

Together with the further concepts of Basil Bernstein the result for the examination of subject specific codes in Mathematics can be diagrammed in the following Matrix.

<div> <div>socio-linguistic</div> <div>subject specific</div> </div>	restricted	elaborated
	formal	informal

plane of meaning

**Fig. 1: Matrix of the mathematical codes**

Interactions in mathematical Supports therefore can be coded at the same time as well formal mathematical as verbal restricted. This seems to be confusing at the first glance, but it is characteristic for mathematical Codes. One example is the use of the term “if... then” which is characterising the mathematical logic of proof. Concerning to the interactionistic perspective of negotiation and accordingly the situational emerging meanings in interactions the theoretical approaches of Bernstein and

Carruthers and Worthington get a third dimension – the plane of meaning. In the following analysis this dimension should be useful for the exemplary description of the development of mathematical concepts and procedures and give a deeper insight.

## **METHODOLOGY**

### **Process of reconstructive analysis**

Having regard to the theoretical approaches and the attempt to identify support structures and their coding in mathematical interactions, in the following there is conducted an analysis of interaction, which refers to the interactional theory of learning (Cobb & Bauersfeld, 1995). The method was devised by a working group round Bauersfeld in reference to ethnomethodological conversation analysis (Eberle 1997; Sacks 1998; Garfinkel 1967). It focuses on the reconstruction of the meaning and the structure of interactions (Krummheuer, in press). Therefore it is proper to describe and analyse topics with regards to contents and the negotiation of meaning in the course of interactional processes with a supportive character. Thus the verbal coding within the supports and the connected concept of the “formatting” (as a distinguished structural element in the analysis) is focussed in the analysis there is an emphasis on elements of the conversation analysis. Therefore the analysis lays stress on the functional aspects of the interactional process and highlights the rules and methods by which the actors construct the communication. It should be asked, how the functional aspect effects the performance of the process of learning.

### **Short description of the sample**

The analysed sequence is part of the empirical work within the project erStMaL (early Steps in Mathematics Learning) at the Centre on Individual Development and adaptive Education of Children at Risk (IDeA) in Frankfurt a.M. (Germany). The research design contains the observation of 277 children age 3,5 to 6. The children come from 12 day care centres in the area of Frankfurt a.M. in Germany.

## **ANALYSIS - „AND WHICH ARE BELONGING TOGETHER“**

### **Short description of the situation**

The situation of play and exploration, which is described in the following chapter is accomplished by a nursery teacher. She is the attachment figure of the children and their group leader at the day care centre. Additionally she is in charge of the group of the so-called “small researchers” in the kindergarten. The nursery teacher achieves the specification of the mathematical domain for the accomplishment of the situation, which is “Measurement”. The children who are taking part in the situation are: Hannah (3;3 years), Michael (3;7 years), Bettina (4;7 years) und Martha (5;3 years).

The material the nursery teacher uses in the situation is: two green paper circles with different diameters (0,5m and 1,0m), a gunnysack which is filled with ten different objects in each case in two different sizes. During the time of the sequence the



children are sitting on a carpet together with their nursery teacher. To paper circles are lying on this carpet in the way that the children are able to sit in a hemicycle around it. The sequence takes place in the middle of the situation. In the sequence before the nursery teacher asked the children to allocate the object to the two paper circles according to their size. It takes some turns till every object is related to a circle.

### Transcript

- 001 N. : And which are belonging together/  
 002 Bettina: *knuckles down to the paper circles* That one\ *pointing with one finger*  
 003 *at the big wooden cuboid which is lying on the big paper circle* and  
 004 *that one pointing at the smaller wooden cuboid on the smaller paper*  
 005 *circle*  
 006 N. : Take a look Bettina (.) put two things together\ here we make a line  
 007 *pointing with her finger in a line right beside the paper circles*  
 008 *parallel to the edge of the carpet* start right here\ *pointing at one point*  
 009 *near the edge of the carpet*  
 010 Bettina: *takes the smaller pin from the smaller paper circle*  
 011 Michael: laughs  
 012 N. : Two things that belong together\  
 013 Bettina: *takes the bigger pin from the bigger paper circle*  
 014 N. : Okay\ take a look\ one here\ *pointing with her finger to the same point*  
 015 *she marked before* and one here\ *pointing at a place a little bit more*  
 016 *on the left hand side next to the place she marked before*  
 017 Bettina: *placing the bigger pin to the place that is marked second and the*  
 018 *smaller pin to the place that is marked first by the nursery teacher*  
 019 N. : Exactly\ this way\ *adjusts the pins on the carpet the way that they are*  
 020 *lying parallel to the edge of the carper and the heads of the pins are*  
 021 *abreast* Who wants to search for two things that belong together now/

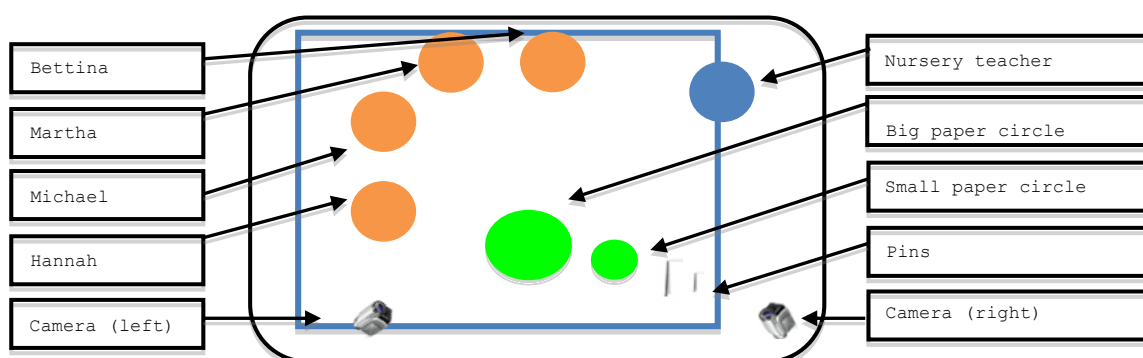


Fig. 2: Position of the children, nursery teacher and the material

Annotation: Spatial dimensions are given from the perspective of the observer (position of the camera).

**Analysis – *Focusing linguistic codes within the supports***

The sequence is opened through the comment of the nursery teacher. Therefore she structured the proceedings by her question, which the children should answer. This way she implicates a call for action. Through the temporarily advice towards the immediate presence of the question which is given on the verbal plane, the nursery teacher confines the sequence from the previous one. The content of this section was the children distribute the objects according to their relative dimension to the two paper circles. But this confining is also denoted implicitly and is not fastened as regards contents (of the previous sequence). On the content plane the expected structuring thereby is not taking place. Thus the reference, which is underlying the given question, can be only understood in mathematical terms by decoding informal verb “belong”. The realisation rule that marked the accepted participation on the thematically plane extensively last open. In that case one can speak of an informal, restricted coding.

Bettina seems to be able to decode. She marks two objects of similar form and different measure. The regulating system, which forms the basis of her action, seems to be formal. She develops a reference, which can be seen as a subset of the implicated system of the teacher. She abridges the system through the addition of the attributes form and measure. Both of these attributes can be seen as formal mathematical issues, which are representations of the mathematical domain geometry. Same counts for the centric dilatation, which is essential to a proof of congruence. The teacher does not amplify, neither positively, nor negatively. She “replies” through a further call for action <006-009>. This way she is structuring another time. The call for action from the beginning of the sequence is now being extended towards system of rules: the objects should not only being marked, they have to be collected and placed in line beside the paper circles and separated from the other ones. However there is no direct reference to the action of Bettina and obviously Bettina could not integrate the extended aspects of the system to her own. Probably the previous action effects a controversial conflict for Bettina and she takes another object. The teacher impairs another time in a structuring way and references implicitly to her system: two objects should be marked <011>. Also this turn lingers informal verbal and restricted in the linguistic code. However Bettina seems to understand and takes a second pin. In the following expression <012> the nursery teacher repeated the second aspect of the call for action from line <006-009> and specifies the placement of the objects as well through her verbalisation as trough her gesture, which can be seen as exemplification. That way she structures the interaction another time. Bettina decodes also this expression as a call for action, which has to be complied. This becomes obvious through her action in line <014-016>. The formal interpretation of the implicit, informal-coded and extended system

of meaning, which is given by the teacher, here has to be speculative. The subsequent positive evaluation by the nursery teacher in line <019> “Exactly\ this way\” can be seen as structural limiting point of the sequence. Her non-verbal action in line <019-021> indeed can be seen as a further add-on of the system, but it may also an exemplarily given advice for the next action. It is underlined by her comment in line <021> through which she invites the other children to take part in the situation.

Concerning to the structure of the support one can assume that the formatting of the nursery teacher is characteristic for supportive structures in learning situations as they are observed in case of child-parents-discourses (e.g. Tiedemann 2010) and also discourses in mathematics classroom. Thereby the content and the accepted participation in the discourse is - in terms of the invisible pedagogy – strongly framed and classified. Both classification and framing is obvious through the so-called „Trichtermusters“ (e.g. Bauersfeld 1978). This pattern structurally partitions the interactional sequence und sequences or rather “portions” the contributions of the child. The strong classification is also observable through the format of content, which is underlined through the gestural exemplification or rather diagrammatics in terms of Peirce and focuses concrete systems of mathematical meanings.

## SUMMARY

In the context of the analysed sequence supports in the acquirement of basal and mathematical concepts and procedures seem to be strongly categorized and regulative formatted, but they are mostly implicit in terms of content. On the verbal plane this is expressed by an informal and mostly restricted codification. However the regulative system, which is underlying the supports in mathematical interactions, has to be seen as formal and elaborated.<sup>2</sup> This has enormous consequences for children concerning their process of learning: Namely there can emerge conducive moments in the interaction through the strong formatting of the course within the supports in mathematical situation<sup>3</sup>, though in terms of content children have to provide an enormous performance to decode the implicit, informal remarks respectively to their implicated, formal mathematical meaning. The presentation of the solution process has also to correspond to this implicit formal regulation system. The decoding becomes more complicated by the fact that the implicit and formal reference of framework is probably subjected to a fast changing development process. Though there is mostly just a little gap between the changing references or framings<sup>4</sup>, which can be seen as exemplifications attending to foster the process of learning, but the implicit coding seems to make it difficult for the children to understand the change in meaning of these exemplifications. The result can be the persistence on the previous reference and framing. With respect to the previous theoretical considerations and the approach referring to Carruther’s and Worthington’s concept of „multi-competence“ (2006) children have to develop the ability to translate not only between informal and formal language, but also between

informal and restricted language and the implicated formal meaning. In what way the so called “invisible pedagogy” (u.a. Schütte 2009; Gellert/Hümmer 2008; Lubienski 2004; Zevenbergen 2001) has to contemplate out of a development process, which is determined by the implication in the insemination of mathematical concepts and procedures, remains open. Hereto the longitudinal study erStMaL analyses primary empirical data, which can convey incipient stages. erStMaL accompanies children in their development from kindergarten to primary school. Another question to be answered empirically is, whether the phenomenon of the implicit insemination of mathematical knowledge can be seen as cross-cultural and equally occurring in all social classes. In the style of Bernstein’s early remarks to specifics in the learning process in the sixties and seventies may be assumed that the social status plays an important role (vgl. Bernstein 1985). Considering the cultural and linguistic plurality (Schütte 2009) future research has to explore to which extend migrant children from the aforementioned settings have advantages or disadvantages in the learning of mathematics and giving good performances in mathematics.

### NOTES

1. Hereby the concept of acquisition (as a uncontrolled and “natural” way of comprehension) is used additionally to the concept of learning (as a controlled, curricular process) in this paper. Both processes (the controlled as well as the uncontrolled one) are taking part and overlapping in mathematical discourses with children and nursery teachers in the kindergarten. They also both have an impact on the development. This way the dichotomic aspects of both concepts will not be focussed any further in this paper.
2. This conclusion corresponds with the findings from the empirical research of Leufer and Sertl (2010), which find out that subject specific linguistic codes in mathematics classroom are always elaborated.
3. This is commensurate to the description of the acquisition of mother tongue by Bruner (1983)
4. Framing here has to be seen in terms of Goffman (1990).

### REFERENCES

- Bernstein, B. (1970a). *Lernen und soziale Struktur, Aufsätze 1965-1970*. Amsterdam, de Munter.
- Bernstein, B. (1970b). *Soziale Struktur, Sozialisation und Sprachverhalten, Aufsätze 1958-1970*. Amsterdam, de Munter.
- Bernstein, B. (1985). *Studien zur sprachlichen Sozialisation*. Berlin , Ullstein.
- Bernstein, B. (1996). *Pedagogy, Symbolic Control, and Identity*. Lanham M.D., Rowman & Littlefield.
- Bruner, J. (1983). *Child’s talk. Learning to use language*. Oxford, Oxford University Press.
- Bruner, J. (1990). *Acts of meaning*. Cambridge: Harvard University Press.
- Carruthers, E. & M. Worthington (2006). *Children's Mathematics. Making marks, making meaning*. Los Angeles usw., Sage.

### Working Group 13

- Cobb, P. & H. Bauersfeld (1995). Introduction: The coordination of psychological and sociological perspectives in mathematics education. In P. Cobb & H. Bauersfeld (Hg.), *The emergence of mathematical meaning: Interaction in classroom cultures* (1-16). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Eberle, T. S. (1997). Ethnomethodologische Konversationsanalyse. Sozialwissenschaftliche Hermeneutik. R. HITZLER & A. HONER. Opladen: Leske + Budrich.
- Garfinkel, H. (1967). *Studies in ethnomethodology*. Englewood Cliffs, Prentice-Hall.
- Gellert, U. & A.-M. Hümmer (2008) Soziale Konstruktion von Leistung im Unterricht. *Zeitschrift für Erziehungswissenschaft* 11 (2), 288-311.
- Goffman, I. (1975). *Frame Analysis: An Essay on the Organization of Experience*. Cambridge, Harvard University Press.
- Krummheuer, G. (1989). Die Veranschaulichung als "formatierte" Argumentation im Mathematikunterricht. *Mathematica Didactica*, 12, 225-243.
- Krummheuer, G. (2010). Die Interaktionsanalyse. in press
- Leufer, N. & Sertl, M. (2010). Kontextwechsel in realitätsbezogenen Mathematikaufgaben. Zur Problematik der alltagsweltlichen Öffnung fachunterrichtlicher Kontexte. - Brake, Anna; Bremer, Helmut (Hg.): *Alltagswelt Schule. Die soziale Herstellungssch*  
nchen: Juventa, S. 111 – 133.
- Lubienski, S. T. (2004). Decoding mathematics instruction. A critical examination of an invisible pedagogy. Reading Bernstein, Researching Bernstein. J.Muller, et.al.: London, New York: 108-121
- Morgan, C. (2003). The linguistic construction of social identities in mathematical communities. *Educational Perspectives on Mathematics as Semiosis: From thinking to interpreting to knowing*. A. Sáenz-Ludlow, S. Zellweger and V. V. Cifarelli. Ottawa, Legas: 109 - 128.
- Pimm, D. (1987). *Speaking mathematically. Communication in mathematics classrooms*. London, New York, Routledge.
- Sacks, H. (1998). *Lectures on conversation*. Malden, Ma, Blackwell, 3. Auflage
- Schütte, M. (2009). *Sprache und Interaktion im Mathematikunterricht der Grundschule*. Münster, New York, München, Berlin, Waxmann.
- Tiedemann, K. (2010). Support in mathematischen Eltern-Kind-Diskursen: funktionale Betrachtung einer Interaktionsroutine. In: B. Brandt, M. Fetzer & M. Schütte (editor): *Auf den Spuren Interpretativer Unterrichtsforschung*. Münster: Waxmann: 177-207.
- Vygotsky, L.S. (1978). *Mind in Society: Development of Higher Psychological Processes*. Cambridge: Harvard University Press.
- Zevenbergen, R. (2001). Language, social class and underachievement in school mathematics. *Socio-cultural aspects of mathematics education: An international perspective*. B. Atweh, H. Forgasz, B. Nebrez. Mahwah, NJ, Lawrence Erlbaum. 1: 201 - 215.

# **ANALYSIS OF MATHEMATICAL SOLUTIONS OF 7 YEAR OLD PUPILS WHEN SOLVING AN ARITHMETIC PROBLEM ON DISTRIBUTION**

Edelmira Badillo\*, Vicenç Font\*\*, Mequè Edo\* & Núria Planas\*

\*Universitat Autònoma de Barcelona, \*\*Universitat de Barcelona

*We analyse mathematical solutions of 7 year old pupils when they individually solve an arithmetic problem. The analysis has used “the configuration of objects”, an instrument provided by the onto-semiotic approach to the mathematical knowledge, combined with the organisation of data into a “systemic network”. Results are illustrated by three cases. The aspects inferred from the overall analysis of the mathematical solutions include the use of iconic representations as a counting instrument, and the demonstrative nature of the arguments developed by the pupils.*

## **INTRODUCTION**

The objective of this research is to analyse the written and verbal mathematical solutions of 7 year old children when solving an arithmetic problem in an individual context of work.

Solutions by children when solving arithmetic problems tend to have a predominance of iconic and symbolic representations and a lack of explicit verbal argumentations (for example, Saundry & Nicol, 2006). For this reason, different research projects have shown an interest in studying the representations produced by children when solving problems (Edo, Planas & Badillo, 2009; Saundry & Nicol, 2006). To analyse the productions made by pupils we take their representations and other elements such as calculation procedures, argumentations, etc. This is why in order to analyse the underlying mathematical activity of pupils we use instruments of the onto-semiotic approach to the mathematical knowledge -OSA (Godino, Batanero & Font, 2007).

We start with a brief review of the literature. We then present the theoretical and methodological instruments that have been applied to our analysis. After that, we describe part of the design of the study and present some of the most relevant data. We end with a discussion of results and final conclusions.

## **REVIEW OF THE LITERATURE**

Many research works have shown that students can solve a different multiplicative problem before the instruction about the multiplication and the division has been given (Mulligan & Mitchelmore, 1997). Carpenter et al. (1993) founded out that even students of early years could learn how to solve multiplicative problems. Such Studies have also been carried out with “diverse” students (Nunes et al., 2009)

Much research has been done on primary age arithmetic problems on distribution in which the task is to share a number of elements that are to be shared out one by one among a variable number of participants; for example, to share out a number of cookies among different children on the basis of questions with multiple solutions (Davis y Hunting, 1990). Research has also been done on problems that work on the idea of the division of units and distribution. Charles and Nason (2000), in a study of the development of the concept of fractions among 8 year old children, proposed a type of problem in which the unit(s) is/are divided into parts.

We examine the particular case in which the context requires the elements to be separated into groups, which involves a distribution in which not everything is a unit (or several units) that has to be divided into parts, but in which everything has to be separated into discrete sets, that can(not) have a different cardinal.

## **THEORETICAL AND METHODOLOGICAL FOUNDATION**

Some studies involving the OSA (Malaspina & Font, 2010), in which mathematical solutions have been analysed, first consider the mathematical practices and then the mathematical objects and processes that are activated in them. In this study we adapt such approach with the pupils' practices being the reading of the text of the arithmetic problem and the production of a written answer. Due to space limitations, we will only analyze the mathematical objects that are activated by said practice.

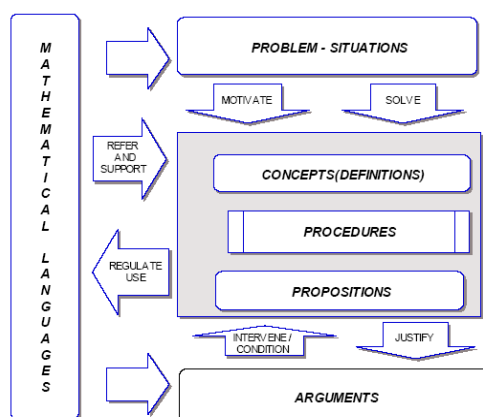
If we consider the mathematical objects activated in undertaking a practice that enables the resolution of a problem situation (e.g. tackling and solving an arithmetic problem), we observe the use of verbal, iconic, symbolic and other representations. These representations are the ostensive part of a series of concepts/definitions, propositions and procedures that intervene in the production of arguments to decide whether the practice is satisfactory. So, when a pupil performs and evaluates a mathematical practice s/he activates a conglomerate formed by problem situations, representations, concepts, propositions, procedures and arguments, which are articulated in the configuration of Figure 1 (Font & Godino, 2006, p. 69).

To move from the individual analysis on pupils' mathematical solutions to a more general analysis on the whole group, we used a systemic network. This is a classical instrument from the organisation and interpretation of qualitative data proposed by Bliss, Monk and Ogborn (1983).

## **DESIGN OF THE STUDY**

The participant sample was made up of 21 primary school pupils (7 years of age) at a school in Barcelona, Spain. The mathematical task presented to the pupils, which was to be solved individually and in writing, was: 1) an arithmetic problem involving distribution in which everything had to be separated into discrete sets of various elements, which could (not) have a different cardinal; 2) an open-ended situation;

and 3) a feasibly resolvable task using the pupils' prior knowledge. The problem was: "If you have 18 wheels, how many toys with wheels can you have?"



**Figure 1. Configuration of objects**

The problem was read aloud and the pupils were expected to solve it with paper and pencil during a one-hour class. When they finished the task, they were individually asked, "what did you do?" Their answers were recorded in audio and transcribed.

## DATA ANALYSIS

Two types of analysis were performed, an analysis of each of the cases and then a global one of all of the mathematical solutions. For the former, an analysis (Figure 1) was made of each pupil's solutions. Table 1 illustrates part of an example.

Following the categories suggested by Malaspina and Font (2010), the data was analysed as indicated in Table 2. Each pupil's mathematical practice was analysed individually, paying attention to 1) representations, 2) concepts, 3) properties, 4) procedures, and 5) arguments. We present the analysis of one of the mathematical practices by one of the pupils, Pupil 15.

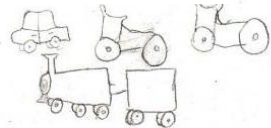
Figure 2 shows the systemic network obtained from the overall analysis. It is organised into categories and aspects (using the terminology by Bliss, Monk and Ogborn, 1983). We use braces ({} to represent inclusive aspects and lines to group exclusive categories (|).

The analysis of the mathematical practices leads to two main categories. First we have pupils that put the emphasis on the cardinal of the set. Here there are three subcategories: 1) those which give a single answer (e.g. Pupil 10 says, "if I had 18 wheels I'd have 6 toys with wheels"); 2) those whose answers suggest more than one answer (e.g. Pupil 15 writes, "I could have four toys with wheels"); and 3) those who give more than one answer (the only case is Pupil 18 who gives four different answers "...you could have 9 motorbikes, you could have 6 tricycles..."). Second we have pupils who point to the set and only refer to it by extension (e.g. Pupil 12 says,



### Working Group 13

“I have made a car, a bike, a car, a scooter and another scooter”) or give the cardinal for the subsets (e.g. Pupil 6 says, “four cars and a bicycle make 18 wheels”).

Written production	Verbal production
<p>Si tens 18 rodes, quantes joguines amb rodes podries tenir?</p>  <p>Un cotxe, dues motos i un tren tenen 18 rodes.</p> $4 + 2 + 2 + 6 + 4 = 18$ <p>Podria tenir 4 joguines amb rodes.</p>	<p><i>I have drawn a car that has four wheels, a motorbike that has two, another motorbike that has two and a train that has ten wheels. And here I have explained what I have drawn and how many wheels they all have. And here I have added them up and this is the answer.</i></p>

**Table 1. Data from Pupil 15**

For each of the five subcategories above, we organised the data on the basis of the mathematical objects following the configuration of objects in Figure 1. We considered each object as an aspect in the systemic network. On this occasion, we have grouped the procedures and properties as a single aspect and we have left the argument aspect for another occasion. Given the richness of the responses, for each aspect we have introduced meanings that have been used as categories; we do not go into the details of all of them. We now illustrate three significant cases.

#### The case of Pupil 15

Pupil 15 solves the problem well by giving the cardinal of one of the possible sets and concluding, “I could have 4 toys with wheels”. We consider that she is suggesting there is more than one answer, as she uses the verb tense “could”. First, we examine the richness of her representations. She starts with an iconic representation of the toys in perspective (Table 1) and then translates this into a symbolic numerical representation ( $4+2+2+6+4=18$ ) and a verbal one (one car, two motorbikes and one train have 18 wheels).

In relation to concepts, this pupil breaks down the set of wheels (18) into parts or subsets (she draws a 4-wheeled car, two 2-wheeled motorbikes and a 10-wheeled train). She is then able to treat each of the subsets as an element (a toy) in a new set (the set of toys). Finally, she implicitly distinguishes between a set and the cardinal of a set, because in her answer she refers to the cardinal of the set of toys.

## Working Group 13

If you have 18 wheels,  
How many toys with  
wheels can you have?

The answer is the cardinal of the set	Single answer	Representations	Makes various translations/ conversions	Symbolic numerical $\rightarrow$ Iconic	(3*)
			Makes translation/ conversion	Iconic	(4*, 10*)
		Procedures / properties	Breakdown into equal addends	Reaches multiplication   Doesn't use commutative property	(3)
			Breakdown into different addends	Breaks into 2 different addends (fours and twos)	(4, 10)
	Concepts / definitions		With multiplication	(3)	
			Without multiplication	(4, 10)	
	More than one answer is suggested	Representations	Makes various translations/ conversions	Iconic $\rightarrow$ symbolic numerical	(15*)
			Makes translation/ conversion	Symbolic numerical $\rightarrow$ Iconic	(17*)
		Procedures / properties	Makes translation/ conversion	Iconic	(12*)
			Breakdown into equal addends	Reaches multiplication   Doesn't use commutative property	(17)
			Breakdown into different addends	Breaks into 2 different addends (fours and twos)	(12)
			Breakdown into different addends	Breaks into 3 different addends (fours, twos and sixes)	(15)
		Concepts / definitions	With multiplication	(17)	
			Without multiplication	(12, 15)	
	More than one answer	Representations	Makes various translations/ conversions	Symbolic-verbal $\rightarrow$ Iconic	(18**)
			Makes translation/ conversion	Iconic	(18)
		Procedures / properties	Breakdown into equal addends (implicit)	Reaches multiplication   Uses commutative property	(18)
			Breakdown into different addends (implicit)	Breaks into 2 different addends (fours and twos)	(18)
The answer is only the set	Set is given by extension	Representations	Makes various translations/ conversions	Iconic + symbolic   Symbolic numerical	(8)
			Makes translation/ conversion	Symbolic numerical $\rightarrow$ Iconic	(19**)
		Procedures / properties	Makes translation/ conversion	Iconic	(21*, 16*, 2*, 9**, 7**)
			Breakdown into equal addends	Reaches multiplication   Doesn't use commutative property	(19)
			Breakdown into different addends	Doesn't reach multiplication	(9, 7)
			Breakdown into different addends	Breaks into 2 different addends (fours and twos)	(8, 16, 2)
			Breakdown into different addends	Breaks into 3 different addends (fours, twos and eights)	(21)
		Concepts / definitions	With multiplication	(19)	
			Without multiplication	(8, 21, 16, 2, 9, 7)	
	Cardinal of the subsets is given	Representations	Makes translation/ conversion	Iconic	(14**, 5*, 13**, 1*, 6**)
			Makes translation/ conversion	Iconic and symbolic	(11**, 20)
		Procedures / properties	Breakdown into different addends	Breaks into 2 different addends (fours and twos)	(11, 5, 1, 6)
			Breakdown into different addends	Breaks into 2 different addends (dosos i deu)	(14)
			Breakdown into different addends	Breaks into 3 different addends (quatre, dosos i sisos)	(13)
		Concepts / definitions	Without multiplication	Breaks into 3 different addends (fours, twos and eights)	(20)
			Without multiplication	(14, 11, 20, 5, 13, 1, 6)	

Figure 2. Systemic network from the overall analysis of the pupils' answers<sup>2</sup>

## Working Group 13

This pupil applies the property that “a number can be broken down into the sum of smaller numbers”, in order to break down 18 (into three different addends) and 10 (into two different addends):  $4+2+2+2+6+4$ . We consider this pupil to be aware of the application of this property because she writes  $(6+4)$  and draws a two-carriage train with 6 and 4 wheels, though in her verbal answer she refers to a train with 10 wheels.

<i>Mathematical Object</i>	<i>Mathematical practice</i>
<b>Problem situation</b>	If you have 18 wheels, how many toys with wheels could you have?
<b>Representation</b>	<ul style="list-style-type: none"> <li>• <b>Iconic</b> with perspective</li> <li>• <b>Symbolic</b> <ul style="list-style-type: none"> <li>- Verbal (one, four, two, ten)</li> <li>- Numbers (4, 2, 6, 18)</li> <li>- Signs (+, =)</li> </ul> </li> </ul>
<b>Concepts</b>	<ul style="list-style-type: none"> <li>• Addition (Previous)</li> <li>• Implicit terms of the addition (addends and results)</li> <li>• Number (Previous)</li> <li>• Subtraction (implicit)</li> <li>• Set</li> <li>• Elements of a set</li> </ul>
<b>Properties</b>	<ul style="list-style-type: none"> <li>• A number can be broken down as the sum of smaller numbers (this is applied to 10 and to 18)</li> </ul>
<b>Procedures</b>	<ul style="list-style-type: none"> <li>• Combination of numbers to obtain 18</li> <li>• Add and subtract (mentally)</li> <li>• Determination of a set by extension</li> </ul>
<b>Arguments</b>	<ul style="list-style-type: none"> <li>• <b>Explicit thesis:</b> I could have 4 toys with wheels (to make 18)</li> <li>• <b>Graphic argument:</b> draws the 4 toys</li> <li>• <b>Verbal argument:</b> describes the elements of the set (a 4-wheeled car, two 2-wheeled motorbikes and a 10-wheeled train)</li> <li>• <b>Numerical-written argument:</b> <math>4 + 2 + 2 + 6 + 4 = 18</math></li> </ul>

**Table 2. Configuration of objects in Pupil 15's answer**

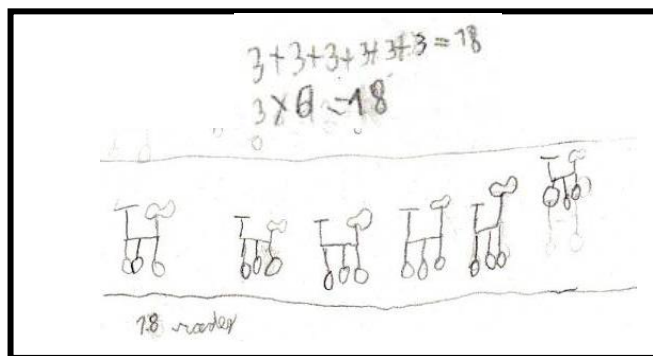
In relation to procedures, she uses the previous property to break down number 18. She seems to take a first number (she draws a 4-wheeled car), then adds another number (she draws a two-wheeled motorbike), and as the result is less than 18, she adds another addend (a two-wheeled motorbike); given that the result is still less than 18, she adds another addend (a ten-wheeled train). She iconically determines the set by extension.

Finally, the explicit thesis of her demonstrative argument (she could have 4 toys with wheels) is justified by the ostensive presentation of the set (iconic representation and verbal description) and by the numerical-written verification ( $4+2+2+2+6+4=18$ ), of which she is aware because she says, “...and here I have added them up...”.

### The case of Pupil 19

Pupil 19 solves the problem implicitly in that he draws 6 tricycles (Figure 3). We consider this to be implicit because the pupil expresses the cardinal of the set of wheels (18), which leads to start the solving process using symbolic representations.

This pupil starts his answer with a symbolic-numerical representation based on the sum ( $3+3+3+3+3+3=18$ ) and translates this to another symbolic expression based on multiplication ( $3 \times 6=18$ , see Figures 2 and 3). Later, he switches to an iconic representation without perspective.




**Figure 3. Representations used by Pupil 19**

In relation to concepts, this pupil comes to the concept of multiplication and seems to be clear of its concept as a repeating addition. We consider this because he uses a mathematical property: “18 can be broken down as the repeated addition of number three”. In relation to procedures, he uses the previous property to break down number 18. He likely takes a first number, 3, then adds another 3, and given that result is still less than 18, he adds another addend (3), and so on successively until he reaches number 18. He iconically determines the set by extension.

Finally, the explicit thesis of his argument (6 tricycles make 18 wheels) is justified by the ostensive presentation of the set (iconic representation). He gives a verbal description of the procedure he used to get to number 18 (I did three plus three...).

### The case of Pupil 20

Pupil 20 solves the problem implicitly, as he draws two cars, a truck and a scooter (Figure 4). He also gives the cardinal of the subsets as a verbal response (2 cars, 1 scooter and 1 truck). We find his type of representation significant, and we have named it in the systemic (Figure 2), iconic and symbolic (Figure 4) networks. The drawings are not in perspective but the pupil represents the total number wheels on each toy using numerical symbols (). The only conversion he makes is to switch from an iconic and symbolic representation of the set of toys to a verbal and written description of the cardinal of the subsets.

In relation to concepts, he breaks down the set of wheels (18) into parts or subsets (he draws 2 cars with 4 wheels, 1 scooter with 2 wheels and a truck with 8 wheels).

After, he gives the cardinal of the subsets (2 cars, 1 scooter and 1 truck). Meanwhile, he implies the mathematical property: “18 can be broken down into the sum of smaller numbers” in order to break down 18 (into four addends, three of which are different): four, four, two and eight.

In relation to concepts, this pupil uses the previous property to break down the eighteen. We consider that he takes a first number (he draws a four-wheeled car), adds another number 3, then adds another (he draws another four-wheeled car), and as the result is less than 18 he adds another addend (and eight-wheeled truck). He iconically determines the set by extension. The explicit thesis of his demonstrative argument (2 cars, 1 scooter and 1 truck) is justified by the ostensive presentation of the set (iconic representation and verbal-written description).

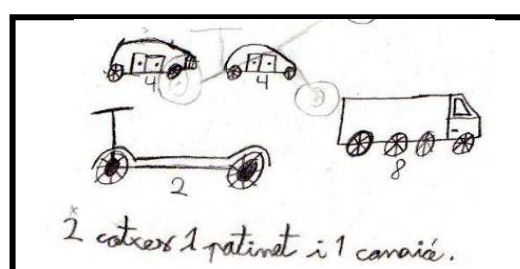


Figure 4. Representations used by Pupil 20.

## CONCLUSION

All of the pupils make an iconic representation of the set of toys. It could be said that this is because of the need at this age to work using contextualised scenarios. However, there is also the need to use drawing as a counting instrument, as has been shown by Saundry and Nicol (2006). In our study, this use of iconic representations as a counting instrument is made clear in the representation that we have called iconic and symbolic (Figure 4). This is a type of representation (used by Pupils 8 and 20), that can be considered an intermediate step between flat representations (used by Pupils 6, 7, 9, 11, 13, 14, 18, 19) and representations in perspective (used by Pupils 1, 2, 3, 4, 5, 8, 10, 12, 15, 16 17, 21).

There are three pupils (3, 17, 19) who separate the set of 18 wheels into discrete sets with an equal cardinal and start solving the problem using written symbolic-numerical representations (they break 18 down into equal addends). In all three cases, they translate this representation into another written symbolic expression in which they use the concept of multiplication (Figure 5), to end with a conversion to an iconic representation of the set of toys. In cases 3 and 17 this is in perspective and in case 19 without perspective (Figure 3).

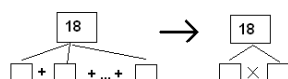


Figure 5. Breakdown of equal addends into multiplication

Pupil 18 uses multiplication in his four different answers (which are given verbally and iconically). But he does not use written symbolic-numerical representations, so we suppose he reached his answer by making mental calculations. From his verbal responses, we infer that he makes an implicit use of the commutative property (“you can have 6 tricycles with three wheels or you can have 3 limousines with 6 wheels and you also get 18”) and that, unlike the three previous pupils, he does not need to explicitly break 18 down into equal addends to break 18 down into the product of two factors. He does not need to add first in order to get to multiplication.

All of the pupils implicitly or explicitly use the property of breaking 18 down into addends (we include the extreme case of Pupil 18 who breaks 18 down using  $4 \times 4 + 2$ ). When the addends are equal, this facilitates the use of the concept of multiplication, and on the other hand, facilitates the process of giving the cardinal for a set of toys of a certain type (e.g. Pupil 17’s answer, “6 tricycles”), which implies that the term “toy”, which is more abstract, is not used. However, the two pupils that explicitly use that term in their answers (e.g. Pupil 15, “I could have 4 toys with wheels”), break number 18 down into different addends.

In this last case we have the close relationship between properties and concepts. The use of a certain mathematical property (a type of breakdown of number 18) conditions the use of certain mathematical concepts (addition or multiplication). Meanwhile, the use of multiplication -a concept that is considered, in curricular terms, to be more difficult than addition- involves, in this case, less abstraction in solving the problem.

We observe two fundamental procedures. One is related with the application of the mathematical property/ies that guarantee the breakdown of number 18 into addends. The pupils mentally apply addition and subtraction, and even multiplication, to reach that breakdown. The other is the determination by extension of the set (via an iconic representation). This latter method, used by all pupils, is the one that enables them to defend, explicitly or implicitly, their answer. These are demonstrative arguments that consist of the ostensive presentation of the set (the drawings of the toys).

We consider the theoretical categories provided by the OSA to facilitate an in-depth analysis of the pupils’ solutions and to reveal the complexity of objects (concepts, representations, properties, etc.) that are activated when solving arithmetic problems.

The systemic network has also been a powerful instrument of organization that has led to construct a taxonomy of the pupils’ responses when they solve a problem in which the whole has to be separated into discrete sets of various elements, which can also have (or not) a different cardinal.

## ACKNOWLEDGEMENTS

The research is part of Projects EDU2009-07113/EDUC and EDU2009-08120/EDUC, from the Spanish Ministry of Science and Innovation.

## NOTES

1. \*The represented drawings are in perspective. \*\* Without in perspective.
2. The pupils that draw an iconic representation and give a verbal description of the set of toys (almost all), are part of the aspect “makes a translation/conversion”. When the pupil also makes another type of translation/conversion, s/he is part of “makes several translations/conversions”.

## REFERENCES

- Bliss, J., Monk, M., & Ogborn, J. (1983). *Qualitative data analysis for educational research*. London, UK: Croom Helm.
- Carpenter, T., Ansell, E., Franke, M., Fennema, E. & Weisbeck, L. (1993). Models of problem solving: a study of kindergarten children’s problem-solving processes. *Journal for Research in Mathematics Education*, 24(5), 428-441.
- Charles, K., & Nason, R. (2000). Young children’s partitioning strategies. *Educational Studies in Mathematics Education*, 43(2), 191-221.
- Davis, G. & Hunting, R. P. (1990). Spontaneous partitioning: Pre-schoolers and discrete items. *Educational Studies in Mathematics*, 21(4), 367-374.
- Edo, M., Planas, N., & Badillo, E. (2009). Mathematical learning in a context of play. *European Early Childhood Education Research Journal*, 17(3), 325-341.
- [Font, V., & Godino, J. D. \(2006\). La noción de configuración epistémica como herramienta de análisis de textos matemáticos: su uso en la formación de profesores. \*Educação Matemática e Pesquisa\*, 8\(1\), 67-98.](#)
- Godino, J. D., Batanero, C. & Font, V. (2007). The onto-semiotic approach to research in mathematics education, *ZDM-The International Journal on Mathematics Education*, 39(1-2), 127-135.
- Malaspina, U., & Font, V. (2010). The role of intuition in the solving of optimization problems. *Educational Studies in Mathematics*, 75(1), 107-130.
- Mulligan, J. & Mitchelmore, M. C. (1997). Young children’s intuitive models of multiplication and division. *Journal for Research in Mathematics Education*, 28(3), 309-330.
- Nunes, T., Bryant, P., Burman, D., Bell, D., Evans, D. & Hallett, D. (2009). Deaf children’s informal knowledge of multiplicative reasoning. *Journal of deaf studies and deaf education*, 14(2), 260-77.
- Saundry, C., & Nicol, C. (2006). Drawing as problem-solving: Young children’s mathematical reasoning through pictures. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (v. 5, pp. 57-63). Prague, Czech Republic: PME.



# KINDERGARTNERS' USE OF GESTURES IN THE GENERATION AND COMMUNICATION OF SPATIAL THINKING

Iliada Elia<sup>1,2</sup>, Athanasios Gagatsis<sup>1</sup>, Paraskevi Michael<sup>1</sup>, Alexia Georgiou<sup>1</sup>, Marja van den Heuvel-Panhuizen<sup>2</sup>

<sup>1</sup>Department of Education, University of Cyprus, Cyprus

<sup>2</sup>Freudenthal Institute for Science and Mathematics Education, Utrecht University, The Netherlands

*Recent studies have advocated that mathematical meaning is mediated by gestures. This study explores how kindergartners use gestures in a semiotic transformation activity involving the description of spatial relationships between objects. The two 5-year-old children that participated in the study used gestures throughout the whole activity, mainly iconic gestures (representing images of objects) and gestures combining iconic and deictic (locating objects in space) properties. A multidimensional linkage between children's gestures and speech, as well as, a significant effect of the researcher's gestures on one child's gestures were found. Findings showed that gestures are essential in the construction and communication of early mathematical meaning and raise important questions for future research.*

*Keywords: gestures, kindergartners, spatial thinking, semiotic transformation*

## INTRODUCTION

The semiotic approach to the learning and teaching of mathematics constitutes a very important trend in the research field of Mathematics Education (e.g. Gagatsis, 2003; Presmeg, 2006). A new trend of this research area concentrates on the examination of bodily movement and particularly on gestures. In the last years gestures and bodily movement have been considered as a source of information and a contributor in mathematical thinking and communication (Edwards, 2009).

The study of gesture is still a young research field within mathematics education and many theoretical and methodological questions remain open (Radford, 2009). Existing research in the mathematical domain has focused on the role of gestures in the generation and the communication of meaning by students in primary and secondary education level. To our knowledge, the gestural activity of preschool and kindergarten children in relation to communicating and thinking about mathematics has not been addressed to a great extent yet. In this paper we analyze gestures as a semiotic resource used by kindergartners into the learning process of mathematics and specifically in a semiotic transformation task of spatial character, involving the conversion of a visual spatial array into verbal description.

## THEORETICAL FRAMEWORK

### **Semiotic representations and gestures in mathematics education**

Mathematics education includes a wealth of ideas and concepts and constitutes an area of human activity and thinking, which is characterized by the use of multiple



representations. Gagatsis, Michaelidou and Shiakalli (2001) support that the advancement of mathematical knowledge is accompanied by the creation and development of new semiotic systems that coexist and operate simultaneously with the first and basic system, that of natural language.

Representations are often concerned as connected with a sign. Saussure defined the sign as a combination of two mental constructs, roughly translated as a “signified” together with its “signifier” (Presmeg, 2006). Within this wide conception of sign Arzarello, Paola, Robutti and Sabena (2009) regard gestures as an important semiotic resource related with the more traditional signs (such as spoken or written language, mathematics symbols, and so on).

Gestures are defined as spontaneous movements of the arms and hands, closely synchronized with the flow of speech (McNeill, 1992). Parrill and Sweetser (2004) define the meaning of a gesture as “the relationship between how the hands move in producing a gesture, and whatever mental representation underlies it, as inferred both from the gesture and the accompanying speech” (p. 197).

McNeill (1992; 2005) proposes five dimensions of gestures with respect to their meaning: 1) *deixis* (locating existing or virtual objects and actions in space with respect to a reference point), 2) *metaphoricity* (presenting an image of an abstract object or idea), 3) *iconicity* (standing for images of concrete entities and actions), 4) *temporal highlighting* (simple repeated gestures used for emphasis) and 5) gestures that modulate *social interactivity*. These dimensions play essential roles in communicating and thinking about mathematics.

### **The role of gestures in mathematical thinking**

Mathematical cognition is not only mediated by written symbols, but is also mediated, by actions, gestures and other types of signs (Radford, 2009). Particularly, children’s semiotic activity in mathematics includes action, gestures and speech. When mathematics is seen as an embodied, socially constructed human product, gestures constitute a particular modality of embodied cognition (Edwards, 2009).

Roth and Thom (2009) claim that gestures are genuine constituents of thinking. Radford (2009) describes gestures as the very texture of thinking and as important sources of abstract thinking. Gestures, along with speech and inscriptions, jointly support the thinking processes of students (Arzarello et al., 2009). When a better understanding occurs, actions become shorter and gestures and language become more relevant (Radford, 2009). McNeill (1992) noted that, “Speech and gesture are elements of a single integrated process of utterance formation in which there is a synthesis of opposite modes of thought” (p. 35).

Gestures can be fundamental for the effectiveness of mathematical communication (Roth, 2001), as well. Along with oral speech, written inscriptions, drawings and graphing, gestures can serve as a window on how learners think and talk about mathematics (Edwards, 2009) and on inner thoughts or as conveyors of ideas that are

already somewhere in the mind awaiting the proper material, namely, verbal expression (Radford, 2009).

### **Spatial thinking in early childhood mathematics**

Geometry and spatial thinking are very important areas of mathematics learning in all educational levels, including early childhood education (Sarama & Clements, 2009). They involve grasping the space in which the children live and move (Freudenthal, in NCTM, 1989). A major spatial competence is spatial orientation, which involves the understanding and operating on relationships between various positions in space (Sarama & Clements, 2009).

Mathematical understanding and, in terms of this study, understanding of spatial concepts and relationships evolves through the transformation of semiotic representations (Duval, 2006). Thus, as children link their spatial knowledge to verbal, analytic knowledge they move beyond visual thinking which is restricted to surface-visual ideas. Connecting spatial representations to language can help children develop the ability to reason and communicate about space and thus gain spatial sense (Sarama & Clements, 2009).

In the present study, a spatial orientation activity is designed and used. This activity requires semiotic transformations, that is, conversions between spatial representations and verbal descriptions.

### **Gestures and spatial thinking**

In activities that entail communication about space, besides words, people frequently use gestures (Ehrlich, Levine & Goldin-Meadow, 2006). A number of studies have revealed that gesture and spatial thinking are connected to one another. Krauss (1998) has found that people use gestures more frequently in defining spatial words than non-spatial words. Gestures occur often in people's descriptions of their navigation in space (Emmorey, Tversky & Taylor, 2000).

Despite the growing evidence about the link between spatial skills and gestures, limited attention has been given on the role of gestures in the development of spatial thinking (Ehrlich et al., 2006). Ehrlich et al. (2006) explored the strategies 5-year-old children used to solve spatial transformation tasks. The findings of the particular study showed that children frequently produced gestures whose meaning was not necessarily expressed in the accompanying speech. Children who referred to spatial information in their gestures but not in their speech were more likely to succeed. These findings suggest that gesture has the potential to improve early spatial skills.

The present study addresses the issue of the role of gestures in young children's ability to mentally manipulate spatial information, by examining the gestures kindergartners produce in a spatial orientation task that requires describing a spatial array of objects the children constructed themselves to another person who could not see it. Specifically, the questions we tackled were the following:

1. What are the dimensions of the gestures kindergartners produce when carrying out conversions from a spatial representation to verbal description?
2. What is the relationship between the kindergartners' gestures and language in carrying out a conversion from a spatial representation to verbal description?
3. What is the effect of the researcher's gestures on the kindergartners' gestures in the conversion activity?

## METHOD

The participants were two five-year-old kindergartners from a private kindergarten in Nicosia, Cyprus. In this study we will refer to the two children as *child 1* and *child 2*.

The spatial orientation activity, in which the children were involved, was designed on the basis of a method of the Didactics of Mathematics that refers to the designation of students' mathematical thinking and focuses on situations of mathematical engagement and communication. According to this method, students solve problems in pairs or in groups, in order to have the chance for oral or written communication between them. This method allows for a precise analysis of students' answers and a specification of their mathematical ideas (Brousseau, 1997).

Thus, the activity, which had the form of a game, required two players, one of which was the researcher. Each player had to use wooden bricks of different shape in order to make a construction. The two players sat opposite each other, having a small wooden wall in the middle, so that each player was not able to see what the other player was constructing. The aim of the activity was the first player to create the same construction with the opposite player, based on the latter's description. The activity involved three parts (rounds) and was carried out separately for each child. In the first and the third part of the activity the child had to make the construction and describe it to the other player (researcher), while in the second part the researcher was the player that had this role.

To examine children's gestures and language, their reactions and utterances during their participation in the activity were video-recorded.

## RESULTS

To identify the dimensions of children's gestures during the activity (Research Question 1), McNeill's (1992; 2005) proposed dimensions of gestures were used. This analysis showed that the meaning of children's gestures was multidimensional. Many gestures used by both children were of *iconic* nature. They used the iconic gestures mainly to present the shape of the various bricks they used. An example concerns the shape of cylinder, for which child 1 moved her finger to make a round line vertically in the air. For the same shape, child 2 used her pointing finger and thumb to form a circle (Figure 1).



**Figure 1: Iconic gestures for the shape of cylinder by (a) child 1 and (b) child 2**

In both children's gestures we also distinguished a great number of gestures that can be considered as *deictic and iconic* at the same time. Children used deictic and iconic gestures when speaking of spatial relationships between bricks. For instance, to explain the placement of four bricks in her construction, child 2 used the expression “one on the top, one at the bottom, one on the left and one on right” moving both her hands to the corresponding positions (deictic property). At the same time she opened her hands to form a flat surface and put them next to one another in a horizontal direction and opposite to one another in a vertical direction, respectively, in order to present the image of the orientation of the parallelepipeds (iconic property) (Figure 2).



**Figure 2: Child's 2 iconic and deictic gesture (a) for the placement of the parallelepiped at the bottom of the construction in a horizontal direction, (b) for the placement of two parallelepipeds on the left and on the right of the construction in a vertical direction**

A small number of gestures were identified as only *deictic* for both children, who used them to present the location of some bricks with respect to bricks that they were already described. For instance, to show the position of a brick, which was on the right side of another brick, child 2 moved her right hand up and down once (Figure 3a).

*Temporal highlighting* gestures were used even less frequently by the children, mainly in trying to emphasize and enforce to the researcher what they were saying. For example, child 2 moved her hands repeatedly one on top of the other tapping them at the same time in order to highlight that three bricks should be put on top of each other (Figure 3b).

Metaphoric gestures were the most rarely produced. Child 1 was found to use a metaphoric gesture for the concept “small”. As she was explaining to the researcher



to “*take two circles (meaning cylinders), but small*”, for the word “*small*”, she moved her hands close to her face and formed fists (Figure 3c).



**Figure 3: (a) Child’s 2 deictic gesture for the position of a brick on the right of another brick, (b) Child’s 2 temporal highlighting gesture for the placement of three bricks on top of each other, (c) Child’s 1 metaphoric gesture for the word “*small*”**

With respect to the connections between language and gestures (Research Question 2), we observed that children were using gestures and language simultaneously. In some cases, the meaning of children’s words and gestures coincided. While the children were trying to explain to the researcher where to place the bricks, they were like holding the brick and placing it in the position they were trying to explain. An example is the verbal expression of child 2 “...*take a roof and place it on top*” which was accompanied by the gesture shown in Figure 4a.

In other cases children replaced verbal expressions with gestures, in order to explain what they were thinking. When the children tried to describe the position of a brick, they frequently used the expressions “*like this (not this)*”, “*place it... emhhh, place it this way...*”, “*place it here*” and simultaneously produced a gesture to illustrate what they mean. For example, child 1 while trying to explain to the researcher how to place the two “long shapes” (parallelepipeds), she was showing with her hands their position, highlighting that they should be placed “*like this*” (apart) and “*not like this*” (not together) (Figures 4b, 4c).



**Figure 4: Child’s 1 gesture accompanying the verbal expressions (a) “...*take a roof and place it on top*”, (b) “*put them like this*”, (c) “*not like this*”**

An aspect of the linkage between language and gestures that was examined was how this connection changed over time during the activity. An interesting example is that in the first part of the activity child 1 opened her hands, one hand to the left and the other to the right side of her body (iconic gesture) to represent the “long shape”,

meaning the parallelepiped (Figure 5a). In the third part of the activity, every time child 1 used the particular term, she produced a different iconic gesture consistently. She stretched out one hand vertically to her body and formed a straight line in the air by moving her hand near her chest (Figure 5b). This is explained by the different position of the parallelepiped between the first and the third part of the activity. In the first part of the activity, the brick was in a horizontal position while in the third part it was in a vertical position. However, the last time she referred to this shape, she did not use any gesture. For the same shape, child 2 used consistently a “smaller” iconic gesture than child 1, during the first part of the activity. When talking about the shape (e.g. “*take 4 lines*”) she moved her pointing finger vertically towards her body to draw a small straight line in the air (Figure 5c).



**Figure 5: (a) Child’s 1 gestures for the parallelepiped in the first part of the activity, (b) Child’s 1 gestures for the parallelepiped in the third part of the activity, (c) Child’s 2 gestures for the parallelepiped in the first part of the activity**

This gesture was not used in the third part of the activity when child 2 used the verbal expression “line”. It is noteworthy that both children, when using words referring to the position of this shape (where to put the parallelepiped), such as “in front of”, “on the right”, “on the left”, produced the same form of (iconic) gesture, consistently, changing only the corresponding place of their hand in space. In particular, they opened their hand(s) keeping the fingers closed to form a flat surface (iconic property) and moved it to the corresponding place and direction (deictic property).

The researcher had an obvious effect in the way child 1 was behaving during the activity (Research Question 3), since she was imitating her expressions and gestures in respective situations. In one case, child 1 even “extended” the researcher’s gesture. Specifically, in the second part of the activity, while the researcher was explaining that two bricks are attached to each other she put the palms of her hands together. In the third part of the activity, when child 1 had to describe the relative position of two bricks and in particular that they should be apart from one another, she used the same gesture and the “opposite” gesture that she observed before from the researcher. In particular, the child first moved her hands away from one another highlighting that they should be placed “*like this*” (apart from each other) (Figure 5b) and then put the palms of her hands together clarifying that they should not be

put in this way (not attached) (Figure 5c). On the contrary, the gesture or language used by child 2 was not influenced by the researcher's behavior.

## CONCLUSIONS

This research examined the gestures two kindergarten children use while communicating in the context of a semiotic transformation activity which involved spatial orientation abilities. Children were found to use gestures throughout the whole activity. The most commonly used gestures were the iconic gestures and the gestures that combined iconic and deictic properties. This finding can be attributed on the one hand to the spatial and geometric nature of the activity which encourages imagistic thinking and on the other hand, to children's developmental stage which is characterized by concrete thinking processes. Metaphoric and temporal highlighting gestures were rarely used by the children. In contrast to producing iconic and/or concrete deictic gestures, in which children only think about the object or spatial relationship the gesture is representing, producing these types of gestures are cognitively complex, involve meta-cognitive abilities and therefore are developed in later years (McNeill, 1992). Further investigation on how the occurrence of abstract gestures varies in activities of spatial context or other mathematical domains among young children and the associations of this gestural behavior with children's mathematical achievement and other child characteristics (e.g. age, gender) could be theoretically and practically important.

A close multidimensional relationship between gestures and language was revealed. This relationship appeared in students' behavior in three distinct ways. Firstly and most frequently, while describing their constructions children's gestures had the same meaning as their verbal expressions. Secondly, in some cases children replaced language with gestures and thirdly, children's gestures complemented and enriched their verbal descriptions. Although previous research (McNeill, 1992) suggests that by the age of five gestures co-occur with speech, the use of gestures only without the corresponding words, by the children of this study, in some cases, could be explained by a lack of flexible knowledge in the construction of sentences for describing spatial relations. Furthermore, the spatial character of the activity, and the fact that the spatial arrangement the children described had been constructed by themselves and was in front of them, probably endorsed the visual elements rather than the analytic elements of their thinking. As a consequence, gestures were stronger than verbal utterances in some parts of their descriptions.

That some dimensions of the children's thought, such as the orientation of the shape, were presented in the gesture and others, such as, the form of the shape, were presented in linguistic form provided evidence for the complementary role of gestures to speech. Each mode of representation made its own contribution to the whole and was essential and valuable in representing children's spatial thinking. In other words, children's thinking for a number of spatial or geometric concepts was characterized by a dialectic of gesture (imagistic thinking) and language (analytic

thinking), that is, a synthesis of language and gesture into one comprehensive presentation of meaning (McNeill, 1992).

A main concern of the study was the role of the researcher's gestures in the communicative process during the activity. In the specific activity the choice of having each child to start the game, the researcher to follow and then the child to play again, gave us the opportunity to identify the effects of the researcher's behavior on the children's behavior. Zooming at the interaction between the researcher and the children during the game, it emerged that only one of the two children was influenced by the researcher's gestures. This child not only emulated the researcher's gestures, but added a contrast to the gesture of the researcher. Specifically, the position of two bricks in the child's construction (apart from each other) was opposed to the position of the two bricks of the researcher's construction (attached) made previously. The child used a gesture that represented the relative position of two separated bricks in her construction, and then a gesture to show how this position differs from the image of two attached bricks (counter-example), which had been previously represented by the researcher's gesture. According to McNeill (1992) adding contrasts is considered as a mechanism by which gestures can affect thought. Thus, this finding provides further evidence for the important role of gestures in young children's development of spatial thinking. Evidence is also provided for the influential role of the teacher in the mathematics classroom, as the teacher is often a model for the students, who tend to be affected by her actions and expressions. However, there was not an effect of the researcher's gestures on the other child's gestures examined here. This inconsistent finding between the children raises an important question regarding the relationship between mathematics teaching and gestures: What factors influence the extent to which teachers' gestures affect children's gestures and their learning outcomes in mathematics instruction? The child characteristics, such as the child's prior knowledge, the complexity of the learning activities and the characteristics of the teacher's behavior could be some of the factors that future research may explore in concern with this issue.

## REFERENCES

- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70, 97-109.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, Boston: Kluwer Academic Publishers.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in learning of mathematics. *Educational Studies in Mathematics*, 61, 103- 131.
- Edwards, L. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics*, 70, 127-141.



- Ehrlich, S., Levine, S., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental Psychology*, 42, 1259-1268.
- Emmorey, K., Tversky, B., & Taylor, H. A. (2000). Using space to describe space: Perspective in speech, sign, and gesture. *Spatial Cognition and Computation*, 2, 157-180.
- Gagatsis, A. (2003). *Comprensione e appredimento in Matematica - Un approccio multidimensionale*. Bologna: Pitagora Editrice Bologna.
- Gagatsis, A., Michaelidou, E., & Shiakalli, M. (2001). *Theories of representations and mathematical learning*. Nicosia: University of Cyprus, ERASMUS IP (in Greek).
- Krauss, R. M. (1998). Why do we gesture when we speak? *Current Directions in Psychological Science*, 7, 54-60.
- McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: The University of Chicago Press.
- McNeill, D. (2005). *Gesture and thought*. Chicago: University of Chicago Press.
- National Council of Teachers of Mathematics (NCTM) (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, Virginia: NCTM.
- Parrill, F., & Sweetser, E. (2004). What we mean by meaning. *Gesture*, 4, 197-219.
- Presmeg, N. (2006). Semiotics and the "connections" standard: significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics*, 61(1-2), 163-182.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111-126.
- Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, 70, 91-95.
- Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research*, 71, 365-392.
- Roth, W.-M. & Thom, J. S. (2009). Bodily experience and mathematical conceptions: from classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics*, 70, 175-189.
- Sarama, J., & Clements, D. H. (2009). *Early Childhood Mathematics Education Research. Learning Trajectories for Young Children*. New York: Routledge.

# HOW DO CHILDREN'S CLASSIFICATION APPEAR IN FREE PLAY? A CASE STUDY

Vigdis Flottorp

Oslo University College

*I analyze an episode from a field work in a multilingual kindergarten in Oslo, using semiotic theory. I examine verbal and non-verbal expressions of two boys playing in a sandpit. A key part of their play is creation of structure. My findings indicate that this structuring become conscious experiences to the children. I would argue that we cannot know about the children's mathematical and communicative competence without knowing the physical context, the play in the sandpit, and the friendship between the boys.*

## BACKGROUND

In Norway we got a new *Framework Plan for the Content and Tasks of Kindergartens* in 2006, which had a separate chapter devoted to mathematics, named 'Numbers, spaces and shapes'. It says:

(...) the staff must listen and pay attention to the mathematical ideas that children express through play, conversation and everyday activities (...) and support the mathematical development of children on the basis of their interests and modes of expression. (Ministry of Education, 2006:42)

The Norwegian Framework stresses that mathematics have an intrinsic value for children, not only a value for the future. This concerns the justification for mathematics in kindergarten. According to some political signals, mathematics is important first of all for school readiness. A white paper brings out:

In school many pupils struggle with doing arithmetic as a basic skill. There is a need for creating more positive attitudes to the subject. The general work in kindergartens and especially with the learning area 'Numbers, spaces and shapes' is important in this context. (Ministry of Education, 2009:77)

In the recent years *learning* has been stressed, focusing on basic skills. In Norway the kindergarten became a part of the educational system with the transition from The Ministry of Children and The Family to Ministry of Education.

This has caused a revived debate concerning the concepts of play and learning. Most people agree that kindergarten shall be a learning arena. The question is *how* this can be realized in mathematics, and what the concept of learning implies.

Much research on mathematics in the early years is based on experimental situations, often with single children. It's harder to find research on mathematics in free play and in informal everyday situations. Some studies though, I have found. Björklund (2007) uses video and analyses toddlers in order to discover how they come to understand different aspects of mathematics like parts and whole, similarities and

differences. Her findings indicate that the children become aware of how things can be seen as a group of items with similar qualities, by first experiencing units as being separate from other units. The children tend to focus on *visual* characteristics of items especially when they *differ* from others. The study is interesting because of her focus on very small children in everyday activities, and because it reveals methodological problems interpreting the actions of small children.

Fauskanger (1998) analyzes an incident where some children are playing pirates for weeks on a farm in the countryside. Measuring, counting and making record of prisoners are central parts of their play. Her study focuses on how the children construct mathematical knowledge in social context meaningful to them. This is similar to my case study, but while the episode I am analyzing is lasting for some minutes, hers is going on for weeks.

Tudge & Doucet (2004) investigate children's engagement in mathematical activities of a complete day, comparing white and black children. The variation among them cannot be explained by ethnicity or class. Seo & Ginsburg (2004) perform a similar study, but include parents' income. These studies reveals challenging methodological issues, concerning how to record mathematical engagements in daily life activities.

The latter study is indicating that enumeration is a relatively small part of the children's mathematical activities compared to shapes and patterns. My interest from the beginning was on geometrical phenomena more than enumeration. The research on small children's number sense is waste, while studies on their geometrical understanding are relatively sparse (Clements, 2003). When findings indicates there is less enumeration than other types of mathematics in kindergarten, but more research on enumeration, then my interest coincides with what seemed important to focus on.

The main aim for my field work was to study the mathematics in children's everyday activities, focusing especially on geometrical concepts and on how children express them. I needed a tool for analysing all kinds of expressions - actions, gestures, body language and verbal utterance, and that's why I chose a semiotic approach.

## **THEORY**

Semiotics is the study of culture as signs, where signs incorporate all kinds of tools used in communication, from linguistic to physical tools. Since these tools are human made, all concepts can be regarded as historically created. This also applies to mathematical concepts.

Ideas and mathematical objects (...) are conceptual forms of historically, socially and culturally embodied reflective, mediated activity. (Radford, 2006:42)

Mathematical concepts are like «lighthouses that orient navigators' sailing boats» (ibid.), but they are not ideas separated from our world. Their abstract and general

aspects are results of human activity, and new constructions can arise in another context. Consequently, knowledge is created and recreated in every situation.

Traditionally, *thinking* is regarded as a mental activity. Radford (2009) advocates a multimodal perspective, where language, gestures and tools are considered as “*genuine constituents*” of cognitive activity. “Thinking does not occur solely in the head, but *in* and *through* language, body and tools. (Radford, 2009:113). Accordingly, mathematics can become manifest in many ways.

Concept formation is closely related to the context. This does not imply that knowledge can be reduced to individual constructions, because we use tools in the concept formation, tools which already have content. The relation between the subjective and the cultural content are like two sides of the same coin (Radford, (2006). One side is the subjective comprehension, intimately related to the person’s experience. On the other side is the cultural content, transferred through the cultural tools in the act of meaning making (ibid., 52). This is why participation is regarded as crucial for learning, rather than acquisition.

Sociocultural psychologists prefer to view learning as becoming a participant in certain distinct activities rather than as becoming a possessor of generalized, context-independent conceptual schemes. (Sfard, 2001, p. 23)

Mathematics can be conceived in many ways. As a subject matter in school, three conceptions can be distinguished (van Oers, 2001). According to the first one, mathematics is synonymous with arithmetical operations. The second conception says that mathematics is about abstract structures applied to concrete situations, and the last one advocates that mathematics is about problem solving with symbolic tools. The second conception presupposes that structures are stable and a priori. This is not consistent with a sociocultural view, whereas the third conception is. According to this view, mathematical activity organizes human experience in a systematic way, also called *mathematising*.

I myself insist on including in this one term the entire organizing activity of the mathematician, whether it affects mathematical content and expression, or more naïve, intuitive, say lived experience, expressed in everyday language. (Freudenthal, 1991:30)

What make Sfards *distinct activities* and Freudenthals *lived experiences* mathematical? According to Mason and Johnston-Wilder (2004) desirable mathematical activity consist of some ways of acting; stressing and ignoring, specializing and generalizing, distinguishing and connecting, imagining and expressing, conjecturing and convincing, organizing and characterizing.

Bishop (1988) argues that *classifying* is a fundamental part of *explaining*, which is one of what he sees as the six universal mathematical activities (explaining, counting, measuring, locating, designing, playing,). Explaining is a central part in all activities, because it is basically about exposing relations between phenomena, and he claims that the most important connection between phenomena has to do with

*similarities*. The quest for explanation is the quest for “unity underlying apparent diversity” (ibid:48). This is what happens in classification.

## METHOD

I visited a kindergarten in Oslo weakly for one year. The kindergarten is a preschool for practicum, located in an area with a high amount of minority speaking people. The section I followed, had 17 children aged 2-6 years. Five of the children had Norwegian as first language.

In order to find out what was going on in the children’s activities, I became participating observer. I could not plan the children’s play or give them instructions, but followed their daily activities. Since I wanted to study what children do alone, I tried not to give them too many suggestions.

I was interested in the social aspects of meaningmaking, and consequently I focused on situations with interaction between children. I was also curious about the role of the verbal language in a multilingual kindergarten. Hence, I concentrated on children with verbal language.

I looked for situations with mathematical potential, like block building, drawing, games and conversation. Quite soon I distinguished some children because of their concentration and creativity. These children seemed to participate in the most interesting mathematical episodes. I do not infer that there is coherence between concentration and mathematics. My data selection does not tell anything about what is common or typical, but they can suggest *how* children express mathematical ideas.

My data consist of notes, photos and videos. I classified the material, using Bishops categories. First I transcribed the videos roughly, dividing them in episodes after distinct activities. Then I chose the most interesting ones and transcribed them more thoroughly. In the interesting episodes some kind of problem arose, and these episodes were rich with regard to meaning and expression. Usually these episodes lasted over some time. Some episodes were interesting because they were surprising. Sometimes it was hard in advance to spot the interesting episodes.

## ANALYZE

The episode lasts for only 3-4 minutes. When I start recording, the boys have placed different toys on the edge of the sandpit. It ends when another child enters the scene.

One of the boys, Mohammed [1], excels with his interest in systems and numbers. The other one, Waqas, is a quiet and concentrated boy. They speak Norwegian with each other as their first languages are different.

The sandpit toys constitute an essential part of the activity. They are artifacts with cultural meaning, which in this case is ambiguous. They are *toys* for children, but at the same time they resemble *articles for daily use*, - some look like kitchen equipment, other like miniature garden utilities. The cones are special since they are

### Working Group 13

neither kitchen nor garden utilities. They are the only items which I have given a mathematical name. The boys never mention any of the toys by name, but their actions reveal what kind of meaning they give to them. They pretend to drink from the cups and stir in the pans. From earlier observations I know that the children often use the toys for making food. Sometimes they expand this activity by making a restaurant, - pretending to serve and sell food.

It looks like the play is about doing the same, because when one of the boys is drinking or stirring, the other one is mimicking. Consequently, it is essential to have the same types and the same number of sandpit toys, which they do not have.

Mohammed	Waqas
cup	cup
cone	cone
2 pans	1 pan
sea star	sieve

**Fig 1: The distribution of the sandpit toys in the beginning**

The problem with Waqas having only one pan, is solved easily. He throws the sieve away and finds a frying pan. This item he bangs in the frame of the sandpit, declaring: “Look what I’ve found!” A frying pan is without holes and can have the same function as a cooking pan. This common feature becomes pronounced, when Waqas suggest that they put sand in “everything”. Mohammed responds that he will put sand “only in two”. Then both put sand in their pans, including the frying pan, showing that they agree on what “everything” shall mean.

The difference between sea star and sieve, is more difficult to solve, since there is no extra sea star around. Mohammed watches the toys, turns around, grasps the sieve and places it on Waqas’ side, second to the edge. At the same time he points at his own sea star, saying: “Look, sea star!” Then he turns the sieve *bottom up*. Then he controls the system by making one-to-one correspondence: He touches Waqas’ toys one by one while he follows his own toys with the eyes, saying “putting it there” for every item. In the end he declares: “Now everything is alright.”



**Fig 2: Mohammed pointing at the cones while he declares:” Now everything is alright.” The sea star is hidden second from the end to the right.**

Mohammed creates a double similarity between the sea star and the sieve. First, he makes up a new criterion for classification – *bottom up* – which distinguishes the sea star and sieve from the other toys. Secondly, he creates a common feature by reflective symmetry, *next further out*.

Waqas					Mohammed				
pan	sieve	pan	cup	cone	cone	pan	pan	sea star	cup

**Fig 2: The categories and the sequence of the toys on the edge of the sandpit.**

The symmetry is not incidental, because short afterwards the spades are placed in the sequence. Mohammed places his spade furthest out to the end with the handle turned to the middle. Waqas asks for help, consequently he must have understood that the placement is special and important. Mohammed puts his spade on his knee, handle turned to the middle, saying: “It’s like this.” Then Waqas places his spade on *his* side, the handle turned to the middle.

The symmetry is not perfect, but Mohammed still declares that everything is alright. The purpose of the symmetry is to create a similarity between the sea star and the sieve. What is right in the play for the boys, is the essential condition, which is to do the same with the same number and kind of toys. This can be interpreted like a token of friendship where the boys mimicking each other.

Friendship is an essential part of socialization. In this episode Mohammed is the one in command, and many would argue that reciprocity and balance are necessary for a friendship. According to Greve (2009) every friendship has its own style and history. What matters, is that the children have experiences which give them a *common we*. It looks like this is the case in this episode. They have a common project that they communicate to each other without misunderstandings.

## DISCUSSION

The play in the sandpit has many similarities with desirable mathematical activity described by Mason and Johnston-Wilder (2004). The boys are creating structure and order of the artifacts by stressing and ignoring, distinguishing and connecting. The toys can be classified in many ways, for example after size, color, form or function. The boys distinguish the pans from the rest by size. The pans and sieve are separated from each other by stressing topological aspects. By looking at form, the boys distinguish the cones from the rest. The sea star and the sieve are connected by placement and symmetry, ignoring all other aspects.

The boys cognitive activity becomes manifest first of all through their *communicative actions*. They show, throw, bang and fill sand in the toys. Sometimes the artifacts themselves are mathematical expressions, for example when the spades are placed symmetric. The boys have few verbal utterances, and most of the utterances are *deictic*: “I will get one like *that*,” says Waqas, searching for an equivalent to a pan. “It is like *this*,” demonstrates Mohamed to his mate demonstrating how to place the spade. These words do only have meaning in the context, which is common to the boys. They talk well Norwegian, but still they communicate without many words because they do not need it. Their actions and deictic signals are sufficient. Accordingly it is irrelevant that they are bilingual. They do not use any mathematical words, yet they have made a structure of classification and symmetry. This can be compared to ethno mathematical studies, for example of children’s geometrical patterns (Gerdes, 2007). These are describes in a mathematical language which is as strange to the children of Angola, as it is to these boys. Neither the patterns, nor the boys’ system are coincidental. While the boys’ structure is ad-hoc, the patterns of the Angolan children are stored knowledge, supported by tradition. The boys are making their own unique genre; they do not repeat a pattern. Most of the studies of children’s mathematical classification concerns geometrical forms, based on features with unambiguous definitions. The studies examine usually to what extent children are able to separate the defining features from the irrelevant ones (Clements, 2003).

While all mathematical concepts have a clear definition, the daily life concepts are often ambiguous. What makes a triangle to a triangle, are the human made definitions which have to be learned. Without them, a form with tree corners and curved lines could be a “triangle”. In the boys’ classification, nothing is defined beforehand. They make up their own criteria, depending on the play and the feature of the toys. Their play reflects the adult’s word, but at the same time it is different because they make the rules. They are in our world and in a make-believe world at the same time.

Structures are just temporarily stabilized ways of approaching a problem. Mathematical activity in school – in order to be realistic – should focus above all on the processes of



structuring instead of the mastery of fixed and prescribed structures. (van Oers, 2001, p. 63)

In this episode the boys solve a problem, but do not use any traditional mathematical concepts or tools. Their actions, gestures and toys are semiotic signs which create a structure.

### CONCLUSION

One single case cannot *prove* anything about the mathematics of small children. The analyses *show* that physical experiences and actions are fundamental in classification. It looks like all the communicative signs are intended. Hence, the children's mathematical experience cannot be unconscious. The children communicate in the most efficient way in the situation. In this case it means very few words. We emphasize the importance of verbalization, but in efficient communication we avoid unnecessary information. It is a challenge for the staff to create situations where verbalization is *necessary* and meaningful to the children. They should be encouraged to verbalize because of the benefits of using word, not to please the teacher.

The study shows how classification is useful to children in their play, and that mathematics can have an intrinsic value for children. Often the structures are less sophisticated than in this case. Anyhow, classification is a structuring process and an informal learning situation, stimulating children's logical ability.

The article started with a quote from the *Framework Plan* which stated that the staff should "support the mathematical development of children on the basis of their interests and modes of expression" (Ministry of Education, 2006). The structuring in this case study is an example of this. There is a need for much more research on how the staff can support and develop learning situations on basis of children's activities.

### NOTES

<sup>1</sup>All the names are anonymous. They have been translated culturally, - for example, children with Urdu names have got usual Pakistani names.

### REFERENCES

- Bishop, A. J. (1988). *Mathematical enculturation: a cultural perspective on mathematics education*. Dordrecht: Kluwer.
- Björklund, C. (2007). *Hållpunkter för lärande. Småbarns möten med matematikk. Critical conditions of learning – toddlers encountering mathematics*. Phd. Åbo: Åbo Akademi University Press.
- Clements, D. H. (2003). Teaching and Learning Geometry. In J. Kilpatrick, W. G. Martin & D. Schifter (Eds.), *Research Companion to Principles and Standards for School Mathematics*. Reston, VA: NCTM.

### Working Group 13

- Clements, D. H., Sarama, J. A., & DiBiase, A.-M. (2004). *Engaging young children in mathematics: standards for early childhood mathematics education*. London: Lawrence Earlbaum.
- Fauskanger, J. (1998). *Lek i seksåringenes matematikkundervisning*. Thesis. Oslo: Universitetet i Oslo.
- Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Boston: Kluwer Academic Publishers.
- Gerdes, P. (2007). *Drawings from Angola: living mathematics*. Maputo, Moxambique: Research Centre for Mathematics, Culture and Education.
- Ginsburg, H. P., Inoue, N., Seo, K. H. (1999). Young Children Doing Mathematics. Observations of Everyday Activities. In J. V. Copley (Ed.), *Mathematics in Early Ears: National Association for the Education of Young Children*, Washington D. C., NCTM, Inc Reston V. A.
- Greve, A. (2009). *Vennskap mellom små barn i barnehagen*. Oslo: Pedagogisk forum.
- Ministry of Education. (2006). *Rammeplan for barnehagens innhold og oppgaver. Framework Plan for Content and Tasks of Kindergartens*. Oslo: Regjeringen.
- Ministry of Education. (2009). *Kvalitet i barnehagen. Quality in Kindergartens*. (Vol. nr. 41 (2008-2009)). Oslo: Regjeringen.
- Mason, J., & Johnston-Wilder, S. (2004). *Designing and using mathematical tasks*. St. Albans: Tarquin.
- Radford, L. (2006). The Anthropology of Meaning. *Educational Studies in Mathematics*, 61(1-2), 39-65.
- Radford, L. (2009). Why Do Gestures Matter? Sensuous Cognition and the Palpability of Mathematical Meanings. *Educational Studies in Mathematics*, 70(2), 111-126.
- Seo, K. H., & Ginsburg, H. P. (2004). What is Developmentally Appropriate in Early Childhood Mathematics Education? Lessons from Research. In D. H. Clements & J. Sarama (Eds.), *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*. London: Lawrence Earlbaum.
- Sfard, A. (2001). There Is More to Discourse Than Meets the Ears: Looking at Thinking as Communicating To Learn More about Mathematical Learning. *Educational Studies in Mathematics*, 46(1-3), 13-57.
- Tudge, J. R. H., & Doucet, F. (2004). Early Mathematical Experiences: Observing Young Black and White Children's Everyday Activities. *Early Childhood Research Quarterly*, 19(1), 21-39.
- van Oers, B. (2001). Educational Forms of Initiation in Mathematical Culture. *Educational Studies in Mathematics*, 46, 59-85.

# **ERSTMAL-FAST (EARLY STEPS IN MATHEMATICS LEARNING - FAMILY STUDY)**

Ergi Acar

Mathematics Education in Goethe University

*Abstract. In this paper social-constructivist approaches of learning-as-participation will be applied. In this study, from a socio cultural perspective, we focus on family situations, which deals with mathematical problems as an everyday affair. It will be primarily presented as a comparison of an identical mathematical play situation of two immigrant Turkish families. The basic research questions: (1) cultural differences while playing, (2) the participation of siblings and (3) code switching during the play.*

*Key words. Early Childhood Mathematics, Family, Code switching*

## **ERSTMAL AND ERSTMAL-FAST**

erStMaL, early Steps in Mathematics Learning, is a research project of the interdisciplinary research center of IDEa, Individual Development and Adaptive Education of Children at Risk, which is extensively interested in the development of children at risk. This project is designed as a longitudinal study and relates to the investigation of mathematical cognitive development in preschool and early elementary school age from a socio-constructivist perspective. The Family Study of the erStMaL Project is named as erStMaL- FaSt, early Steps in Mathematics Learning-Family Study. It is also designed as a longitudinal study, which belongs to the erStMaL Project.

## **THEORETICAL FOCUS OF ERSTMAL-FAST**

This project deals with the impact of the familial socialization for mathematics learning. In empirical level, we know from research projects like CEMELA and MAPPS (Civil 2005), that familial activities in mathematical context are corner stones of children's mathematical abilities and acquisitions. As Mills pointed out, parents are their children's first and continuing educators and nobody knows their own children like themselves. (Mills 2002,p.1) Thus, the family functions as parallel to an ongoing "support system", parallel to preschool, kindergarten and (primary) school for the learning of mathematics. The more children experience mathematical situations in their families, the more learning of mathematics in early years occurs in the different emerging forms of participation in everyday situations in their families.

By the term "support system" it is referred to the idea of any socio- constructivist theory, which means that the cognitive development of an individual is constitutively bound to the participation of this individual in a variety of social interactions. They move on learning -but inevitably- they also support his/her development. With respect of Bruner's concept of a Language Acquisition Support System (LASS) that

exists beside an internal Language Acquisition Device (LAD), we propose a similar concept for the learning of mathematics, which we call analogically the “Mathematics Acquisition Support System” (MASS) (Bruner 1990).

The process of engagement with the adult enabled the children to refine their thinking or their performance to make it more effective. Hence, the development of language and articulation of ideas was central to learning and development. (Atherton 2010) In these sense, familial contexts in multi-ethnic societies-like in Germany-, immigration, multilingualism and multiculturalism also inevitably have roles in early childhood mathematics education. As Suárez-Orozco & Suárez-Orozco write in their study on immigrant children "Immigrants are by definition in the margins of two cultures. Paradoxically, they can never truly belong either 'here' nor 'there.'" (2001, p. 92) Thus, in bilingual immigrant families it is clear that their children at risk.

From Zevenbergen’s perspective (2003), according to social and cultural differences the way in which action and practice are structured highly influences the social and individual construction of identities while learning mathematics. Thus, especially in this study I am interested in mathematical discourse in the context of bi-cultural families.

As authors Suárez-Orozco point out, the identity issues that immigrant children confront in feeling caught between their parents’ culture and the culture in their new country: “Children of immigrants become acutely aware of nuances of behaviors that although ‘normal’ at home, will set them apart as ‘strange’ and ‘foreign’ in public...Immigrant parents walk a tightrope; they encourage their children to develop the competencies necessary to function in the new culture, all the while maintaining the traditions and (in many cases) language of home” (Suárez-Orozco, Suárez-Orozco 2001, pp. 88-89).

As H.Coffey explained, a practice of moving among variations of language in a different context is defined as a “code switching”. He identifies “code switching” as the practice of switching between a primary (L1) and a secondary (L2) language or discourse in an educational context. In P.Auers’s (1998) opinion, code-switching can be related to indicative of group membership in particular types of bilingual speech communities, such that the regularities of the alternating use of two or more languages within one conversation may vary to a considerable degree between speech communities. The conversations in bilingual immigrant families are usually the mixture of two languages in mathematical play situations. Paradoxically, with these multi-culturalism and -lingualism they create a new language, which occurs by both languages.

Li Wei sorts Code Switching in 4 Categories:

Inter-sentential switching occurs outside the sentence or the clause level (i.e. at sentence or clause boundaries).

## Working Group 13

Intra-sentential switching occurs within a sentence or a clause.

Tag-switching is the switching of either a tag phrase or a word, or both, from language-B (L1) to language-A (L2), (common intra-sentential switches).

Intra-word switching occurs within a word, itself, such as at a morpheme boundary.

As mentioned above, tag switching and also intra-word switching can be performed by immigrant bilingual families in everyday affairs. Especially with regard to the young age of the children of erStMaL-FaSt, it can be assumed that these forms include intensive narrative argumentation (Krummheuer 2009, Van Oers). In mathematical play situations with families, explanations and narrative presentations which are strongly linguistic are used. By linguistic matters, code switching can be strongly influenced by the functioning of MASS in bilingual familial context. Through all these aspects, it will be interesting to find out the functioning of MASSs in bilingual families.

### METHODOLOGY OF ERSTMAL-FAST

For the Family Study, 12 children who are about 4 years old are chosen from a larger sample that belongs to the project erStMaL. The criteria are the ethnic background (German or Turkish), duration of the formal education of the parents and sibling situations within the families.

Our research design can be shown in the following table:

		<b>more than one children</b>	<b>only one child</b>
<b>Higher Educational Families</b>	<b>Turkish</b>		
	<b>German</b>		
<b>Lower Educational Families</b>	<b>Turkish</b>		
	<b>German</b>		

**Figure 1:** Table of research design.

As shown above, two comparable families with different nationalities are matched.

For erStMaL-FaSt, four play situations are conceived, which refers to the two mathematical domains: geometry and measurement. They are constructed according to specific design patterns, which will not be explained here.

Data collection comprises of recorded videos and their transcripts. The erStMaL child is recorded with the members of family while they are playing.

The data collection is organized according to following table:

Settings	A)	erStMaL child as single child	B)	erStMaL Child as sibling
erStMaL Child is playing with	A-1)	mother <b>or</b> one member of family (e.g. father)	B-1)	mother <b>or</b> one member of family(e.g. father) <b>and</b> sibling
	A-2)	mother <b>and</b> one member of family (e.g. father)	B-2)	mother, sibling <b>and</b> one member of family (e.g. father)

**Figure 2:** Table of research setting.

In this paper, I want to present two mathematical play situations of two different immigrant Turkish families as in Setting B-1.

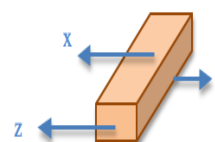
### EPISODE- “BUILD”

The mathematical play “Build” refers to geometry and spatial thinking. From given wooden bricks, the children and their families build three-dimensional bodies whose two-dimensional image is given on the playing cards. Supposedly, they perform the relations between two- and three-dimensional representations.

All playing cards are placed face down on the table. Each card has of difficulty level from 1 to 3. The cards with the number 1 are the easiest ones and the cards with the number 3 are the hardest ones. To be clear in explanation, each dimension of a wooden brick is named also as follows:



**Figure 3:** Examples of playing cards.




**Figure 4:** Dimensions of a wooden brick.

The player chooses one card from the pile of playing cards and tries to build the image on the card with the wooden bricks. In the transcription of two Scenes, to show the code switching, Turkish talks are written in bold; German talks are written in bold and cursive style.

#### Scene 1

In 1st Scene, family Gül performs a “polyadic interaction”. (Krummheuer 2007). Necessary information about Family Gül is given in the table below:

### Working Group 13

Can	Didem	Berk	Family Gül	
	erStMaL Child Berk		4;9 years old	speaks Turkish and German
	Older Brother Can		11 years old	speaks Turkish and German goes secondary school
	Mother Didem		studied 7 years	(Low-Educated)

**Figure 5:** Information about Family Gül.

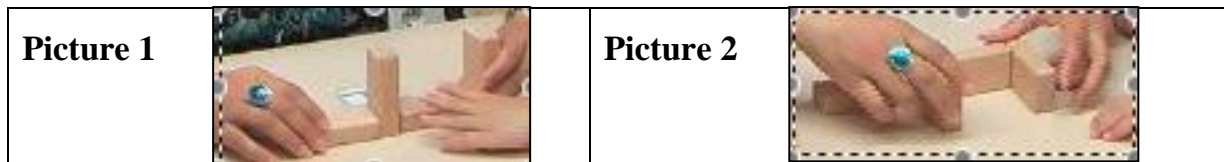
Play round begins with Berk. He takes a card, with number 1, the top of the pile of playing cards and builds it with the support of his mother and brother. After his turn, Can chooses one of the playing cards which is number 3. Although the card is of the hardest degree, he builds it easily without any help. Then mother Didem's turn comes. She chooses a card, with number 1, from the pile of playing cards.



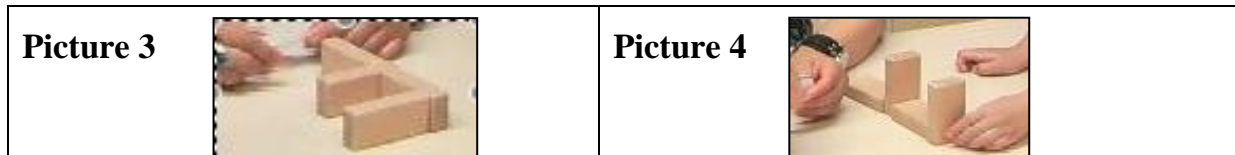
**Figure 6:** Chosen Card-Family Gül.

At the first sight as Didem sees her card, she reacts, as it is too hard to build this figure. But opposed to her react, Can says that it is easy to build that figure, while she is looking obsessed to her card. After a while, Didem starts out to build and lays two wooden bricks on x side with her both hands in front of her. But again Can reacts contrary to his mother and tells her that she is doing it the wrong way. After she takes another wooden brick, she lays all of them on y side. Berk also reacts same as his older brother and he says that she is doing it the false way. While Can backs up Berk, Didem reacts *just a momeeennt*<93>. Up to now, although the three family members speak Turkish, for the first time, Didem changes language, makes a tag switching and moves from Turkish to German.

After a while, Can seems that he cannot be patient and interferes the figure which is builded by Didem, by saying how she should do. But Didem looks this building just a second and changes the direction of whole figure as showed below:



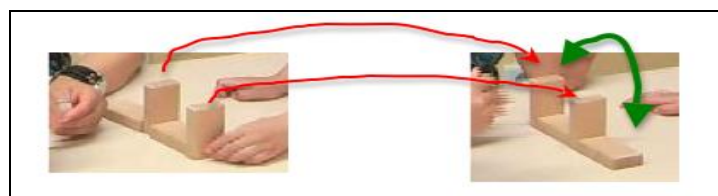
But Can instantly denies and says **No you can't** <98>. He takes a card in front of her, holds it up and shows her **you are looking like that. That's why it is like that** <99>. Then he puts it back in front of her. Didem turns the whole figure 90° as showed on picture 3:



Then she says **it doesn't matter if it lays like that..or like that**. Can denies her again and responds **then it is not difficult it must stand up then it is difficult** <100-101>. We understand here, the word “stand up” refers to the y coordinate axis in a Cartesian coordinate system. It also shows us the differences between home-language and school-language as an inter-sentential switching.

Then Can takes a card from his mother's hand and by looking the card, he changes the position of the bricks again as showed above on the picture 4. At the same time, Berk reacts **off maamm please it is soo. yeess sooo**, and he smoothly regulates the wood bricks <117-118>. Berk's this reaction can be seen as a tag switching again. In the sentence, he moves from Turkish to German and then again from German to Turkish back. With this utterance, he addresses to his mother, and it seems that he got bored from the discussion between his mother and brother, too.

Didem holds up the card, looks it again by showing to Can and says **look I'm looking like this**. In return of Can, he takes the card from her hand and by showing says **that's why it's up on your head** <120-123>. Here, it is seen again inter-sentential switching with the sentence “it's up on your head”. As a logical nexus, this sentence might refer again to y coordinate axis in a Cartesian coordinate system. On the other hand, between Didem and Can a direct participation occurs due to addressing each other. Afterward, Can regulates again the figure and changes places of two wooden bricks as shown as red lines in Figure 7.




**Figure 7:** Putting order of the bricks.



But Didem interrupts him and tells that there is no difference among both figures and in conclusion, both figures are F. By explaining the form of the figure on the card, Didem refers it as an F, which can be named an intra word switching. It also shows us, how an adult can configure the mathematical figures and introduce them to her/his child (ren). After Can responds **you don't understand.it must be like on the card** <137-139>, Didem expresses that she just understands. Hereby Didems' turn ends up.

## Scene 2

In 2.Scene, Family Ak performs and occurs “dyadic interaction”. (Krummheuer 2007). Necessary information about Family Ak is given in the table below:

Leyla	Aleyna	Family Ak	
		erStMaL Child Aleyna	4;8 years old speaks Turkish and German
		No Sibling	
		Mother Leyla	studied 12 years (High-Educated)

**Figure 8:** Information about Family Ak.

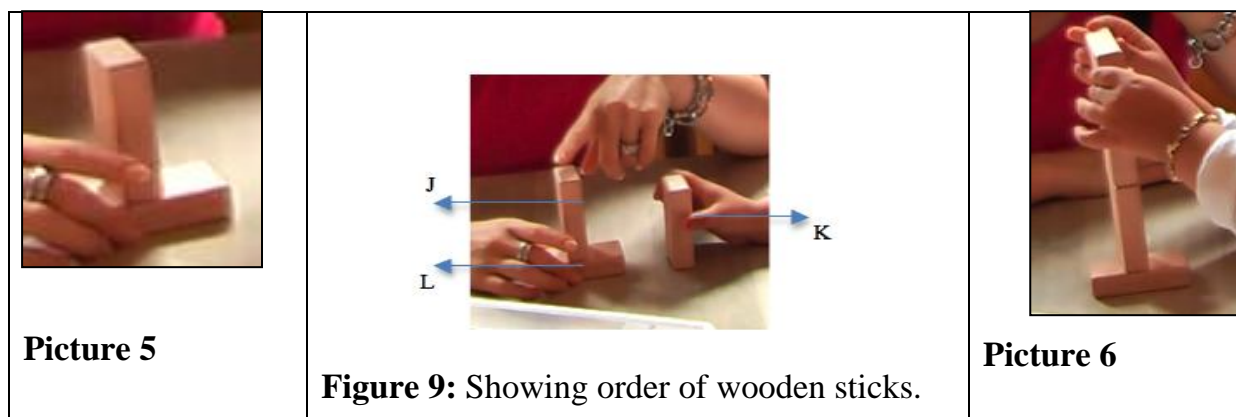
Play round begins with Aleyna. They play 4 turns each. They mostly pick up second-degree cards. But the figures, they built, interestingly are not the same figures on the card. This situation follows at the fifth turn of the mother, which will be exemplified in this part, too.



**Figure 9:** Chosen Card-Family Ak.

Leyla picks up an upper given card, takes 2 wooden bricks from the box and lays them on the x side with her both hands in front of her. Then she regulates them as shown in picture 5. She asks for help to Aleyna while she is going on building the figure. By shaking her head up and down, Aleyna reacts. This reaction can be seen as a positive meaning, just like a confirmation of Aleyna. Right after, Aleyna attempts

to put a wooden brick, like her mother, near the other one. But Leyla denies her and shows with her finger up on the wood stick by saying *up on it. up.* <294>. (see fig.9)



Aleyna puts K on L in the same art, just like how J stands. While Leyla is assisting Aleyna by putting up, she says *you look around the card* <296>. It is not clear enough what she really means, but it can be meant as an opposite side that Aleyna looks the card from the opposite side of her mother. As it is seen although they speak only German, there can be different utterances by using second language (L2/ German) as a first language (L1/ Turkish) because of thinking in Turkish. But it cannot mean that is a Tag switching.

Meanwhile Aleyna takes another wooden brick from the box and puts the new taken wooden brick on y side by centering up on the K as in picture 6, Leyla reacts *no no that must be like round*, shows to Aleyna the card with her finger and adds *look like that look should be an L.* <306-308>. But Aleyna lets the wooden brick stand upon the K as just before. By this explaining the form of the figure on the card, Leyla refers it as an L and makes a type of code-switching. It also shows us again as in Family Gül, how an adult can configure the mathematical figures and can introduce her child. Although it is not so clear, which wooden bricks she means (E or H or I), they built something else as another figure. Leyla takes this brick out and puts the wooden brick in the same coordinate axis on the K but towards the front. Her act shows us what she means by L for a while. However, Aleyna clicks her tongue: *tthh.* <329>. By this react, Tag-switching and typical reaction of Turks are seen, which means no.

Then Aleyna follows by showing a card with her finger and tells that it doesn't look like that. By centering up again the same wooden brick on the K, she expresses that it has to be like that. Surprisingly, her mother acclaims and acknowledges her mind. But Aleyna seems not to understand and asks unintelligible by showing on the card *it must be like a hammer* <403>. Leyla responds by showing it upon the wooden bricks *we had already. it was the hammer like here on the front.* <407-411>. Thereby, they finish the play. By the explaining the form of the figure on the card, Aleyna and Leyla refer it as Hammer and make a type of code-switching. The word Hammer can be implied such as a sum of two functions in a x,y coordinate axis. This utterance is

just like a metaphor and with this sense; it can be named as an “Intra word switching”.

## CONCLUSION

In this paper, a scene from Family Gül and Ak is introduced and reconstructed with the reference to the concept of learning-as-participation. With respect to their learning-as-participating, they cooperate by their discussion. They paraphrase a given figure on card and bring it in a new idea or perspective. Therefore, these forms are new representations of the sensitizing concept of learning-as-participation (Krummheuer, G. 2011). Considering the dynamics of the interactional turn-taking processes and specific relationships among utterances in Family Gül, it can be said that in a learning process a high degree of attentiveness occurs. In the first given scene, clearly learning-as-participation in a family is seen. While Didem comes up with a figurative meaning, Can insists on dealing with the cards according to the play-rules. There are not fixed roles in this family situation. However, the attitudes of the elder brother also affect his brother. Due to Berk's utterances, it can be clearly seen that he perceives his brother as a recipient. At this point, we see the importance of the participation of the sibling. The elder brother initiates an emerging mathematical process in the way he is discussing with his mother.

In this point, a question comes to mind: what would be or happen, when elder brother does not exist in this play? In this sense, Scene 2 can be given as an answer. Because only the mother is an adult and a supportive person, it cannot be seen such an interactional turn-taking processes on the example the Family Gül. There is a high degree of attentiveness but no one remains true for playing card. Thus, Leyla supports her child not in a way, which would be based on the rules of the game. She does not act according to these rules and replies her daughter in the whole episode as if she is right, although the figure they built is completely different from the figure on the card. On the contrary of Family Gül, in Family Ak's situation, there are certain roles. Unfortunately in this familial situation, a defective emerging mathematical process is occurred by false building the figure and not enough or intensive discussing to justify the built figure. Due to Leyla's utterances, it can be said that her understanding of the support can be just to give a positive feedback to her daughter by saying “yes, right or you are doing very good”. Possibly this supportive system lacks an older sibling, as in Family Gül, who would be able to show the right way of the playing game.

While we see often code-switching during the turn taking in an interaction of Family Gül, in an interaction of Famils Ak seldom code-switching occurs.

On this account, it can be shown as a reason: emotional expressiveness. While the family member is disturbed, bored, or stressed, then he/she moves from one language to other language. For that several reasons, weak vocabulary, false grammar knowledge, cut corners, to be integrated etc can be given. When it is taken into

account that this family is an immigrant Turkish family, it will be not a surprise to cause code switching because of their emotions. To show a high emotion or reaction is a clear cultural difference between Turkish and German families. Hence, it is really often that they are seen 'normal' at home, 'strange' and 'foreign' in public. With these multi-culturalism and –lingualism, they create a new emergent mathematical model in the learning process with their different attitudes. It remains still open, if this hangs a positive or negative effect. In both families, the common situation is to describe the figures as a letter of the alphabet. It can be explained as an effort of the adults to decrease to level of comprehension for their children.

In conclusion, one might ask whether this newly created emergent mathematical model by immigrants can be used or improved by the researchers as a model for mathematical learning process in the immigrant families as a new learning process in multi-ethnic countries. More Research is needed in order to declare a model of mathematics learning in everyday situations of immigrant families. These first insights limit at some mathematic theoretical concepts, like zone of proximal development, capable adult etc.

## REFERENCES

- Atherton, J. S. (2010). *Learning and Teaching Constructivism in learning* <http://www.learningandteaching.info/learning/constructivism.htm> Accessed: 28 November 2010 [On-line] UK: Available
- Auer, P. (1998). *Code Switching in conversation, Language, Interaction and Identity*. London: Routledge
- Bruner, J. (1990). *Acts of Meaning*, Cambridge, MA: Harvard University Press.
- Civil, M., Planas, N., & Quintos, B. (2005). Immigrant parents' perspectives on their children's mathematics. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 37 (2), 81-89.
- Coffey, H. (2009). Code-switching. Retrieved from <http://www.learnnc.org/lp/pages/4558?ref=search>
- Krummheuer, G. (2007). Argumentation and Participation in the Primary Mathematics Classroom. Two Episodes and Related Theoretical Abductions, *Journal of Mathematical Behavior*, 26(1), 60 - 82.
- Krummheuer, G. (2009). Inscription, narration and diagrammatically based argumentation. The narrative accounting practices in the primary school mathematics lesson. In W.-M. Roth (Ed.), *Mathematical representation at the interface of the body and culture* (pp. 219 - 243). Charlotte, NC: Information Age Publishing

### Working Group 13

- Krummheuer, G. (2011). Representation of the notion “learning-as-participation” in everyday situations of mathematics classes. *Zentralblatt für Didaktik der Mathematik (ZDM)*, in press.
- Mills, J. (2002). Early numeracy: children’s self-initiated recordings (3-5 years), unpublished PG Diploma Assignment, Swift Masters Programme, College of St Mark and St John. Plymouth.
- Suárez-Orozco, C. & Suárez-Orozco, M. (2001). *Children of immigration*. Cambridge, MA: Harvard University Press.
- Zevenbergen, R. (2003). Ability grouping in mathematics classrooms: a Bourdieui an analysis. *For the Learning of Mathematics*, 23(3), 5-10
- Wei, L. (2000). *The Bilingualism Reader*. London: Routledge.

# CHANGING MATHEMATICAL PRACTICE OF KINDERGARTEN TEACHERS. CO-LEARNING IN A DEVELOPMENTAL RESEARCH PROJECT

**Per Sigurd Hundeland, Martin Carlsen and Ingvald Erfjord**

*University of Agder (UiA), Kristiansand, Norway*

*The aim of this paper is to report results from a three-year project concerning collaboration with kindergarten teachers in implementing mathematics in the kindergarten. In our co-learning with the kindergarten teachers we all participated in a developmental project. The frame for analysing our data was developmental research. The analysis shows that the kindergarten teachers changed their practice throughout and because of their participation in the project. These changes were exemplified through increased mathematical consciousness in orchestrating mathematical activities, emerged emphasis on disseminating thoughts and ideas to colleagues, and changed ways of communicating with the children.*

introduction

During the last few years, mathematics has gained increased focus as a separate subject area as regards the content of Norwegian kindergartens. In particular, this was due to an increased emphasis on mathematics in the present curriculum for the kindergarten. In this curriculum mathematics is explicitly mentioned as an area with which children are supposed to be engaged; an emphasis not made before. This increased focus on mathematics, made by the authorities, put the kindergarten teachers in a challenging situation as to how to implement and orchestrate mathematical activities in the kindergarten.

A professional development project called Learning Better Mathematics (LBM<sup>63</sup>) has been initiated as a joint project with teachers in our university region and didacticians at the University of Agder (UiA). This project has enrolled in close relationship with a research project called Teaching Better Mathematics (TBM<sup>64</sup>) initiated at the University of Agder. In these two projects, teachers and didacticians are collaborating in order to promote mathematics teaching and learning (cf. Carlsen, Erfjord, & Hundeland, 2010). In this paper we report from some of the results of this collaboration, analysing to what extent the kindergarten teachers' practices have changed due to their participation in LBM. The study focuses on the kindergarten teachers' conceptions about their own development in orchestrating mathematical activities. We use the notions of *co-learning* and *co-learning agreement* (Wagner,

---

<sup>63</sup> The LBM project was supported by The Competence Development Fund of Southern Norway, Tekna and Vest Agder County.

<sup>64</sup> The TBM project was supported by the Research Council of Norway (NFR no. 176442/S20) and The Competence Development Fund of Southern Norway.

1997) in order to describe the nature of our collaboration with the kindergarten teachers. According to Wagner (1997), co-learning agreement is described as follows:

In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling (Wagner, 1997, p. 16).

We are aware that the kindergarten teachers (practitioners) and we as didacticians (researchers) bring different kinds of expertise when engaging in collaborative activities. These different resources contribute to one another as to possibly develop the orchestration of mathematical activities in the kindergarten. With this as background it is interesting to scrutinise the outcomes of the kindergarten teachers' participation in LBM. The following research question has therefore been formulated: *In what ways have the kindergarten teachers' mathematical practices changed, due to their participation in the project?*

### Project organisation

The project Learning Better Mathematics encompassed nine schools and four kindergartens. One important element of this project was workshops at UiA. The content of these workshops was devoted to mathematical topics such as geometry and number calculation combined with didactical topics such as communication in mathematics teaching and learning and designing of mathematical tasks. At these workshops there were plenary sessions and group sessions. In the group sessions, the kindergarten teachers were organised in separate groups, discussing, to them, relevant issues of how to engage the children in mathematical inquiry relative to the mathematical focus of the workshop. Thus, they shared thoughts, ideas, and engaged in mathematical activities. Another important element of the project was institutional visits by didacticians in kindergartens. At such visits, kindergarten teachers and didacticians discussed and reflected on how to implement and orchestrate mathematical activities in the kindergarten, and didacticians often observed kindergarten teachers' mathematical activities with children.

### Theoretical framework

A theoretical ground for this study is a sociocultural perspective on learning and development. We view learning as a fundamentally social and situated process of appropriation where individuals, i.e. kindergarten teachers and didacticians, make concepts, tools, and actions their own through their collaboration and communication (Rogoff, 1990; Wertsch, 1998). The reason for situating our study within this theoretical position is our aim of scrutinising and making sense of the institutionalised interaction and learning activities taking place. By studying the

professional development of kindergarten teachers through interviews, we are analysing their processes of appropriating ways of orchestrating mathematical activities.

According to the project's theoretical stance in general (cf. Jaworski, 2007), *inquiry* is a central theoretical notion. The aim and intention of the project has been to collaborate with teachers in order to promote development of mathematics teaching through inquiry (Wells, 1999; Jaworski, 2005). According to Wells (1999), inquiry is a process described as “a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them” (p. 121). Moreover, according to Cochran-Smith and Lytle (1999), the core of inquiry is a position they call an *inquiry stance*. That is, when one as a researcher and/or teacher takes a positively, critical position towards ones own practice, reflecting on this practice and asks critical questions in order to make a way into the deeper substance of the practice. In our case the practice is thus that of the kindergarten teacher, the what, how and why behind the mathematical learning activities orchestrated in the kindergarten. In our study, the notion of inquiry serves as a tool to describe how the kindergarten teachers' practice has changed.

### Developmental research

The methodology of this study is developmental research (Freudenthal, 1991; Gravemeijer, 1994). Didacticians and teachers collaborate in order to change and develop the mathematical practice in the classroom, engaging in these developmental processes, critically reflecting on these practices, and discussing their teaching experience. The basic assumption is hence made, that if there is going to take place any changes in the mathematical practices in schools and kindergartens, it is the teachers that have to bring such changes about.

In developmental research there is a cyclical relationship between research and development. The research guides the development and the development nurtures the research (Goodchild, 2008). In combining this methodology with co-learning (Wagner, 1997), both teachers and didacticians get the opportunity to participate in the research process. The project activities are hence designed within such a frame. By adopting this methodology our aim is to experience “the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience” (Freudenthal, 1991, p. 161). Furthermore, our aim is to make contributions both to the field of mathematics education researcher and to the field of practitioners in their orchestrations of mathematical activities in the kindergarten. Through developmental research we simultaneously study both the promotion of development and the developmental process as such (Bjuland & Jaworski, 2009).

### Analysis and Results



### Working Group 13

The empirical basis for our analyses was three focus group interviews, one in each kindergarten, made by one of the didacticians within the project, as well as observations in these kindergartens. The three kindergartens are approximately equal in size, both as regards number of staff (25) and children (70-80). Eleven kindergarten teachers participated as indicated in Table 1. In Norway, kindergarten teacher education before 1994 did not include any mathematics course. After 1994 only a minor course (6 ECTS) was included. In our study, the majority of the participants was educated before 1994.

Name of kindergarten:	Naturbarnehagen	Pinocchio	Andungen
Name of the of kindergarten teachers in the interview:	Else, Sam, Marit, Ronny	Julie, Maiken, Lotte	Vilde, Line, Unni, Venke

**Table 1: Overview of participating kindergarten teachers in the interviews**

We took a systematic and iterative approach to the analysis of the interviews. We scrutinised one interview each and identified emerging issues. We discussed and compared these issues and agreed to examine them further in a new phase of analysis, based on commonalities and relevance for our research question. This second phase of analysis revealed three findings:

1. Increased consciousness regarding mathematics in the kindergarten.
2. Willingness to disseminate thoughts, ideas, and experience.
3. Improved ways of communicating mathematical ideas with the children.

In our data there are several parallel utterances confirming the same finding. Thus, we will here present and analyse only some of the kindergarten teachers' utterances as regards to three findings.

#### Increased consciousness regarding mathematics in the kindergarten

When the kindergarten teachers were asked to make their own experience from participating in the project explicit, we found three types of comments with respect to increased consciousness:

- An extended view on the nature of mathematics and how it may look like in the kindergarten practice.
- A changed personal relationship to mathematics.
- Development of experience regarding implementation and orchestration of activities with inquiry in mathematics sessions in kindergartens.

As regards the first bullet point, one of the kindergarten teachers, Lotte, claims that she has changed her way of thinking of mathematics in the kindergarten because of her participation in the project: "It is much easier for me to open up for more mathematical aspects to include in the activities and to think about how I can include these aspects. Now I have several mathematical ideas that I will develop further"

(Lotte, Pinocchio, line 292). A common expressed argument by the kindergarten teachers was that they now were in a position where they were more able to exploit and facilitate mathematical activities with the children. Such a view was by one of them expressed in the following way: “We have all developed experience making us more alert and ready to exploit the potential (for mathematics activities) when they appear. Perhaps we observed similar things before too but then we were unconscious of the mathematics” (Unni, Andungen, line 196). Marit from Naturbarnehaugen added to the picture of what mathematics is and may look like in the kindergarten environment. She expressed that playing activities may be a good arena to promote mathematics: “It is wonderful to see that we can play mathematics” (Marit, Naturbarnehaugen, line 114). These utterances we interpret as reflecting the kindergarten teachers increased consciousness with respect to what mathematics may look like in the kindergarten environment and in unfolding mathematical activities in the kindergarten. They have apparently changed their approach to the mathematical practice as well as developed their view and experience as regards orchestrating mathematical activities in the kindergarten.

With respect to the second bullet point, the kindergarten teachers talked about an earlier fear for mathematics which is no longer present. One of them expressed: “I have been one of those who have disliked mathematics, but now I very much appreciate it” (Venke, Andungen, line 73). Ronny at Naturbarnehaugen said: “It has been fun to see that colleagues I have talked with in our kindergarten, which have had problems to see the meaning of including mathematics, now have started to change their view of mathematics”. Marit, one of Ronny’s colleagues, exemplifies this argument: “We see that the project has contributed in making mathematics less fearful for us” (Marit, Naturbarnehaugen, line 15). We interpret these utterances as confirming that they have changed their personal relationship to mathematics.

Concerning the third bullet point increased consciousness and the changed personal relationship to mathematics have lead to new actions with the children. Throughout the project, many of the kindergarten teachers several times expressed that inquiry was well known and familiar to them. This was repeated in the interviews: “The method inquiry is not new to us. It is an old method in kindergartens” (Venke, Pinocchio, line 73). However, it appeared that working with *mathematics* in an inquiry way was new to them. Maiken argues: “We have worked with mathematics before, but not in this way, not so actively (Pinocchio, Maiken, line 26). Related to the third bullet point, several of the kindergarten teachers commented on what it means to interact and communicate with children in an inquiry way; the third main finding of this study.

The kindergarten teachers emphasised how they have progressed in their implementation of mathematical activities. They pointed to the workshops as an important arena to discuss and share ideas. Line reflected on her development: “During the workshops, my brain starts to work and I am thinking on how I can use

this, whether it is too difficult to implement or if I can use parts of it? In the workshop on symmetry (in a plenary, pictures of symmetric leaves and flowers were presented), I thought that this is useful and possible because we are, in our kindergarten, always focusing on outdoor activities.” (Line, Andungen, line 102). This utterance indicates a readiness to implement mathematics in the kindergarten. On a more general level, Lotte said: ”I believe it is important to move beyond the stage ’mathematics is everywhere’ and ’we are counting’. I believe we have to move further than that. Of course, we are counting, but we have to be more thorough, to be more systematic, and justify what we are doing as regards mathematics” (Lotte, Naturbarnehaven, line 274). This was not only at a superficial level, but they were aware that in order to support and facilitate mathematical activities with the children, they needed to be more systematic and comprehensive.

#### Willingness to disseminate thoughts, ideas, and experience

From our analysis of the focus group interviews we found that the kindergarten teachers are eager to disseminate the basic ideas of the project to other kindergarten teachers in their own kindergarten, to kindergarten teachers in the project, to fellow kindergarten teachers in their local city, and to parents.

Only a small number of kindergarten teachers in each kindergarten participated in the project and came to workshops at the University. But kindergarten teachers in the project paid attention to disseminating project ideas to their colleagues, including assistants without pedagogical education. Unni said: “The whole kindergarten participates. Thus, it becomes a part of the kindergarten, not only something a few are familiar with. Everyone becomes part of it.” (Unni, Andungen line 7). Also at Naturbarnehaven and Pinocchio they prioritised dissemination to colleagues. Maiken at Pinocchio talked about the situation at her department of the kindergarten, saying that: ”I think the whole group of staff is more involved now than they used to be” (Maiken, Pinocchio, line 26). Staff meetings were mentioned as an arena for dissemination within the kindergartens: ”In the staff meetings, apart from project information, we have sometimes worked in groups and we have had visits from UiA where inquiry and the project have been focused” (Unni, Andungen, line 14).

A second kind of dissemination took place *between* kindergartens in the project. In all the interviews, kindergarten teachers emphasised the learning benefit of sharing ideas. Julie stated this in the following way: “We are learning from each other” (Julie, Pinocchio, line 119). In fact Unni expressed that she would have liked to share even more with the other project participants: “Sometimes, when we have finished the work in groups (in the workshops at UiA), I think we could have shared more” (Unni, Andungen, line 137). Thus, the kindergarten teachers participating in the project apparently shared their ideas and experience with utilising inquiry in orchestrating mathematical activities.

Dissemination to *other kindergartens* were also taking place, for instance through arrangements for kindergarten staff in the local city. When referring to an upcoming planning day for those people, where Lotte and Julie were going to give a lecture, Lotte said: “We are going to tell how we have worked with mathematics in our kindergarten, and we have planned to give several examples of inquiry in mathematics to the audience, because that is what we believe the audience is interested in” (Lotte, Pinocchio, line 390). One of the authors of this paper collaborated with these kindergarten teachers in giving this lecture. Therefore, we are able to conclude that dissemination of basic ideas indeed took place.

The fourth kind of dissemination was related to involving the children’s *parents* in the ideas of inquiry as an approach to mathematics. The kindergarten teachers reported about concern raised by some parents to implementation of mathematics in the kindergarten. However, this situation has changed: “They were afraid that the children would loose the possibility to play freely and they were unsure about this (the mathematics). But gradually we now only get good feedback, excellent feedback from the parents” (Venke, Andungen, line 47). The kindergarten teachers also reported that they have got feedback indicating that some parents with children in other kindergartens were jealous since their children were not approaching mathematics in the same way: “Other parents in other kindergartens, they are a bit yealous” (Unni, Andungen, line 54). These utterances we interpret as confirming a desire to disseminate project ideas broadly. For instance, dissemination took place through documentation of activities with posters used in staff meetings and meetings with parents. Written documentation gave possibilities for disseminating ideas to assistants and substitutes in order for them to engage with mathematical inquiry. They also had a wish to create a resource bank electronically available for the staff.

#### Improved ways of communicating mathematical ideas with the children

From the interviews we found evidence of improved ways of communicating with children in mathematical activities. One of the kindergarten teachers pointed to inquiry as a tool that increased the awareness of how to communicate: “Inquiry in a way makes us much more conscious about how we talk (Else, Naturbarnehaugen, line 17). Ronny reflected about how inquiry facilitates profound and prolonged discussions with children: “It is fun to sit together and talk with them. It is an excellent method for children to have a long conversations; a good long conversation” (Ronny, Naturbarnehaugen, line 179).

Maiken at Pinocchio emphasised the importance of children answering questions and coming up with new questions themselves: “I have to let the children get the possibility to answer. Now I have to let them get the opportunity to find possible answers and to come up with new questions, and resist putting words in their mouth” (Maiken, Pinocchio, line 22). A similar concern for questions was visible in a dialogue between the kindergarten teachers Marit, Else and Ronny at Naturbarnehaugen:

### Working Group 13

- 31 Marit: I am now much more conscious when I am together with the children,  
what I say to them and I am concerned what they might come up with  
32 Else: They are now more clever to ask and we draw on the opportunities  
arising in daily life and use them actively.  
33 Marit: We now pay more attention to this issue  
34 Ronny: We see it much clearer.

Marit gave an example of how she facilitated mathematical communication. An example is the word “under” in connection with an outdoor track which included hurdles: “I asked him afterwards, how did you move?. He answered that he crawled *under*. It is important that they use their own words; it is not me who should tell them (Marit, Naturbarnehaugen, line 29).

We did not explicitly find utterances concerning new ways of communicating in the interview at Andungen. However, in Carlsen et al. (2010) we reported from a mathematical activity led by Unni at Andungen. The communication was dominated by a variety of question giving the children opportunities to participate and contribute with ideas and arguments in the mathematical activity. In our conversation with Unni after the activity, she claimed that this intensive use of questions was due to her participation in the project.

To summarise, we found that the kindergarten teachers wanted the children to take more active roles than before. This awareness gave children several opportunities to inquire; taking charge of finding solutions, asking their own questions and creating new mathematical challenges. We interpret this as a shift in the relationship between kindergarten teachers and the children.

### Discussion

We set out the following research question for this paper: *In what ways have the kindergarten teachers' mathematical practices changed, due to their participation in the project?* We argue, based on the kindergarten teachers' comments, that they by being participants in the LBM project have changed and developed their view of mathematics. Furthermore, they have developed experience in implementing mathematical activities in kindergarten. They claimed that these changes have resulted in a more focused orchestration and facilitation of mathematical activities characterised by inquiry. We interpret these changes as indicating a changing practice. We will argue that the kindergarten teachers' changed practices are due to their participation in co-learning processes (Wagner, 1997) at different arenas; between didacticians and kindergarten teachers, between kindergarten teachers and between kindergarten teachers and children. When participating in such processes, the kindergarten teachers increased their consciousness as regards mathematics, they disseminated thoughts, ideas and experience, and they improved their ways of communicating and discussing mathematical ideas.

The kindergarten teachers' awareness and valuing of the role of questions and contributions by the children in such activities, illustrate that the teachers have taken an inquiry stance towards mathematical activities in the kindergarten (Cochran-Smith, & Lytle, 1999; Jaworski, 2005, 2007). In accordance with how Wells (1999) describes an inquiry process, the kindergarten teachers critically scrutinised their own practice, which facilitated the children's possibilities to inquire into the mathematics. In their orchestration of mathematical activities, the kindergarten teachers emphasised the social and interactive dimensions of learning. This was also visible in our observations, where the kindergarten teachers organised goal-directed mathematical discussions among themselves and a small group of children. We will argue that these indicators exemplify the kindergarten teachers' processes of appropriating inquiry as a tool and as a way of being as regards mathematics (Rogoff, 1990; Wertsch, 1998).

We would also like to argue that developmental research, as a methodology for conducting the kind of research presented in this paper, has been fruitful when it comes to scrutinising the various aspects of how the kindergarten teachers have changed their own practice of orchestrating mathematical activities. The development of the practice has indeed nurtured our research, and it is our hope that our research may guide possible development in the future (cf. Freudenthal, 1991; Goodchild, 2008; Gravemeijer, 1994).

The kindergarten teachers claimed that they currently think differently about mathematical activities. They reflected more thoroughly in their approach to orchestrating these activities, including careful considerations of how to communicate mathematically with children. Furthermore, they were eager to disseminate their thoughts and ideas at several arenas as regards mathematics in the kindergarten. They also claimed that they have gained from their participation in the project, by making references to ideas and tasks presented and discussed at workshops and staff meetings. These transformations we interpret as exemplifying what Jaworski (2005) means by appropriate inquiry as a way of being in orchestrating mathematical activities in the kindergarten. Through our three-year collaboration with the kindergarten teachers, we have observed their orchestration of mathematical activities, their discussions of mathematical and didactical issues at workshops, and their dissemination of ideas and experience from the project. These observations, together with the analytical findings reported in this paper, contribute to our argument that they have changed their practices. In Norway, in-service training typically has been offered as short term ad hoc courses. Our study implies that professional development is achievable through long term co-learning and fostering of inquiry communities among didacticians and kindergarten teachers.

The three findings that have been identified as regards the research question, all address in what ways the kindergarten teachers' have changed their practice. However, we do not consider this as evidence of successfulness of the project as

such. Further research might reveal constraints and problematic issues as regards implementation of the project's aims and ideas.

## References

- Bjuland, R., & Jaworski, B. (2009). Teachers' perspectives on collaboration with didacticicians to create an inquiry community. *Research in Mathematics Education*, 11, 21-38.
- Carlsen, M., Erfjord, I., & Hundeland, P. S. (2010). Orchestration of mathematical activities in the kindergarten: The role of questions. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education. January 28th - February 1st 2009* (pp. 2567-2576). Lyon: Institut National De Recherche Pédagogique.
- Cochran-Smith, M., & Lytle, S. L. (1999). Relationships of knowledge and practice: Teacher learning in communities. In A. Iran-Nejad & C. D. Pearson (Eds.), *Review of research in education* (pp. 249-305). Washington, DC: The American Educational Research Association.
- Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer Academic Publishers.
- Goodchild, S. (2008). A quest for 'good' research. In B. Jaworski & T. Wood (Eds.), *International handbook on mathematics teacher education, Vol. 4: Individuals, teams, communities and networks* (pp. 201-220). Rotterdam: Sense Publishers.
- Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, 25, 443-471.
- Jaworski, B. (2005). Learning communities in mathematics: Creating an inquiry community between teachers and didacticicians. In R. Barwell & A. Noyes (Eds.), *Research in mathematics education: Papers of the British Society for Research into Learning Mathematics, Vol. 7* (pp. 101-120). London: BSRLM.
- Jaworski, B. (2007). Theoretical perspectives as a basis for research in LCM and ICTML. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild, & B. Grevholm (Eds.), *Læringsfelleskap i matematikk - Learning Communities in Mathematics* (pp. 121-138). Bergen: Caspar Forlag.
- Rogoff, B. (1990). *Apprenticeship in thinking. Cognitive development in social context*. New York: Oxford University Press.
- Wagner, J. (1997). The unavoidable intervention of educational research: A framework for reconsidering research-practitioner cooperation. *Educational Researcher*, 26, 13-22.
- Wells, G. (1999). *Dialogic inquiry: Towards a sociocultural practice and theory of education*. Cambridge, MA: Cambridge University Press.
- Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.

# **“LOOKING FOR TRICKS”: A NATURAL STRATEGY, EARLY FORERUNNER OF ALGEBRAIC THINKING**

Maria Mellone

Dipartimento di Matematica e Applicazioni “R. Caccioppoli” – Università Federico II di Napoli

*Starting from the assumption that very young children exhibit some naïve forms of algebraic skills, in this work we present and discuss an episode occurred within a grade 2 classroom experimental work. This work is centred on an exploratory activity (namely, looking for regularities) in the domain of natural numbers. We observe that this kind of activity, if suitably managed, can activate the search for tricks to help mental calculation, inducing in children a first shift from an arithmetic behaviour, based on search for results, to an algebraic attitude, based on the search for relationships. Moreover, we notice how a guided use of suitable representations can transform children’s natural bent to look for regularities into ‘awareness of, or attention to, structure’, in the sense of (Mason et al., 2009).*

**Key-words:** natural strategy, algebra, structural thinking

## **INTRODUCTION AND THEORETICAL FRAMEWORK**

In the last years an increasing number of researches pointed on recognizing naïve forms of algebraic thinking in very young children (Carraher *et al.*, 2001; Lee and Freiman, 2004; Malara and Navarra, 2008; Radford, 2010). Along these lines, the quite simple view that identifies a one-way “ontogenetic” path from the arithmetic thinking to the algebraic one has been doubted (Radford, 2001, 2010), while a more complex relationship has been hypothesized.

Among many attempts to deeply investigate on children’s abilities in mathematical generalization and abstraction, various research experiences have been developed concerning the study of pattern work at early school years (see for example Stacey, 1989; Lee and Freiman, 2004; Radford, 2010 and Rivera, 2010). The work of Lee and Freiman, carried out also at kindergarten level, “*confirmed [...] the more positive observation, that children engage easily and enthusiastically in pattern work having little or no difficulty in ‘seeing a pattern’*”, but at the same time stressed “*the more negative observation, that the final stage of pattern work, checking the generalization, appears to be absent*” (Lee and Freiman, 2004, p. 2). Before, Stacey in her work (1989) already noticed a similar behaviour in primary school students by observing a certain reluctance to generalize or the presence of only some rough forms of generalization: “*students have demonstrated a willingness to grab at relationships without subjecting them to scrutiny*” (Stacey, 1989, p. 163).

A different view is adopted by Radford (Radford, 2010). His study, besides deeply analyzing some early forms of algebraic thinking, in the conclusions faces the problem of monitoring their development. In other words, the relevant question is



not only to identify what children can or cannot do at an early stage, but also how to favour the development and possibly to improve these embodied, non symbolic forms of algebraic thinking. In this direction, the question arises that is crucial in the present study: what kind of early work leads to the development of algebraic thinking?

Indeed, whilst it is now widely acknowledged that very young children are capable of some forms of algebraic skills, it seems also evident that “*algebraic thinking does not appear in ontogeny by chance, nor does it appear as the necessary consequence of cognitive maturation. To make algebraic thinking appear some pedagogical conditions need to be created*” (Radford, 2010, p. 79), but “*the task of creating appropriate didactical tools for the teaching of algebra in the early grades [...] has barely begun.*” (Lee & Freiman, 2004, p. 1).

In our research group, the methodological and didactic aspects and the related teaching proposals for an early approach to algebra, follow from a basic cognitive assumption, supported also by some neuroscience results, namely that (i) the aptitude to abstraction is typical of human nature and spontaneously exerted from the very early age, and that (ii) the purpose of education is just to refine and to formalize the forms of abstraction and to become conscious of them, what can be obtained in situations of cultural immersion and social interaction, together with a progressive development of linguistic skills (Iannece & Tortora, 2008). Therefore, in the last years we have been particularly interested in finding and analyzing all the *natural strategies* that can be seen as rough forms or precursors of the mathematical procedures, abilities and structures, and in designing suitable didactic interventions with the aim of let them emerge, be exploited and refined (Iannece *et al.*, 2009 and Iannece *et al.*, 2010).

In the case of algebraic thinking, if we start from the assumptions that the algebraic thinking can be viewed as a particular way of expressing abstract relations and processes (for example arithmetical ones) and that children have a natural tendency to abstraction, the purpose is to lead learners to what Mason *et al.* have called the *awareness of, or attention to, structure* (Mason *et al.*, 2009). As they emphasize in their work, learners manifest *awareness of, or attention to, structure* when they begin to focus on what stays and what changes – that is, “*becom[e] accustomed to considering invariance in the midst of change*” (p. 13).

Assuming a vision of thinking as interpersonal communication (Sfard, 2008), but also as intrapersonal communication made tangible by social practice and materialized in the body, in the use of signs and artefacts (Radford, 2010), algebraic thinking can be seen as a kind of “discourse” (inter/intrapersonal) in which we can recognize this *attention to structure*.

In this study we will show how an activity of free exploration in natural numbers domain can promote this attention, in particular when the need of obtaining untrivial

arithmetic results favours the search for tricks to easily perform mental calculations, inducing in children a first shift from an arithmetic behaviour (simple execution of correct operations), to an algebraic attitude, based on *attention to structure*. Furthermore we analyze how this first form of attention to structure can develop into more ‘formal’ knowledge and into sound linguistic formulations, taking advantage from the guided use of suitable representations (objects, drawings, etc.) for the sake of social interaction.

## ARITHMETIC EXPLORATION

In this section we present a second grade classroom experimentation. It is based on an exploratory activity on natural numbers, which starts with the following task:

***Take three consecutive numbers and add them up. Repeat this several times, using other groups of three consecutive numbers. What do you observe?***

It is an open task, that we usually propose to students of different grades, with more or less deep goals, depending on their age. In our case (grade 2), the “simple” calculation of several sums, possibly of big numbers, is in itself a demanding task. This is not made by chance: indeed, it is just the obstacle consisting in children’s still poor calculating skills, that stimulates a sort of tension between arithmetic and algebraic behaviours. As we will better see in the sequel, also due to teacher’s mediation, children are forced to look for tricks that enable them to perform quite difficult calculations. In this way their attention naturally shifts from the simple search for the result of a calculation to the detection of a general structure.

In our opinion, our experience corroborates Radford’s view (Radford, 2001, 2010) that the ontogenetic relation between arithmetic and algebra is quite complex. In fact, the described activity has been naturally intertwined with the usual arithmetic curriculum of grade 2, in particular with the normal acquisition of calculation skills. But the insertion of this kind of activities can help to move from the diacronic feature of arithmetic (where the mental processes of calculations are linearly distributed in time, and the results are obtained from the action of an individual) toward the time-free and impersonal features of the structural world of algebra.

As suggested by Mason *et al.* “*any sensible work approach to teaching combines work on understanding concepts with work on mastering procedures, and combines tasks designed to stimulate learners to express their own thinking using technical terms with task designed to highlight the use of important routines*” (Mason *et al.*, 2009, p. 26). We think that the activities of exploration with natural numbers, suitably managed, can stimulate a structural vision of procedures. In this direction we have found very effective as an interpretative tool the distinction proposed in (Mason *et al.*, 2009) between *empirical counting* and *structural generalization*. In the forthcoming analysis we will try to distinguish between an action performed by a child in order to obtain a particular result and an action performed in the attempt to clarify the underlying structure.

## Research Method

The episode that we present here took place in Winter, 2010, in a class of 19 Grade 2 students. The class teacher is in service for many years, with her own more or less unconscious pedagogical beliefs, and her own educational experience. During the 2010 the teacher was involved in a professional development course, held by our research group, arranged around several topics, among which the early approach to algebra<sup>65</sup>. The activity of exploration on natural numbers above described, is formerly faced in collaborative way during this professional course.

During the experimentation in her class, the teacher alternated work in small groups (5 groups composed by 3-4 children) and work within the whole class group. A member of our research team recorded the class discussions and took some notes regards the gestures performed by students. The collected data consist in the class discussions transcripts, in representations produced by children during the small group activities and in the observing researcher's notes.

According with the vision of thinking as interpersonal and intrapersonal discourse I conducted a multi-semiotic analysis of young students' behaviours (representations use, artefacts use and linguistic expressions) in order to follow the development of linguistic skills. Here I choose to analyses some excerpts of the activities in order to show how the guided use of suitable representations supports the development process of linguistic abilities.

## Results of the analysis

At the beginning of the first lesson the teacher presents the task and divides the children into groups. Going around, she observes that in this first phase the pupils' natural behaviour is to arrange the numbers in the ordered sequence and then to obtain the various terns grouping all numbers by three. So, since all the groups choose their various terns not by chance but arranged in regular sequences, it seems that even though the task doesn't deal with patterns, the children autonomously build some patterns: 1-2-3; 4-5-6; 7-8-9; etc. Then, all the groups concentrate on the search of relationships among the sums of consecutive terns. Similar behaviours have been already observed in pattern work studies, for example Lee and Freiman say: *"What surprised us was how strong that urge to see a pattern is – strong enough to compel the student to impose a pattern by modifying or ignoring some elements in a given configuration"* (Lee & Freiman, 2004, p. 5). Here our hypothesis is a bit more refined: namely, a task like the above one, where the attention has to be directed toward a generic element of an infinite set, requires a harder abstraction jump in comparison with a typical pattern work which deals with an already ordered infinite

---

<sup>65</sup> An analysis of possible changes in teacher's practice and beliefs should be interesting, but it is not the focus of this work.

set, asking to determine the  $n^{\text{th}}$  element of a given succession. But children, looking for patterns, perhaps exhibit the ability to split this jump in easier steps.

Almost one hour later the groups are invited to communicate their findings to the rest of the class. The regularities recognized by children, expressed in natural language, are: ‘the alternation of odd and even numbers in the sums’, ‘the possibility to obtain any successive sum by adding 9 to the previous one’ and, finally, the only one destined to survive, ‘all sums are multiple of three’.

The teacher, respecting the children’s need to work with terns arranged in a regular way, suggests a slight variation, that is to work with different sequences, but ‘ordered’ (as the children call them) as before:

0-1-2; 1-2-3; 2-3-4; 3-4-5; ...

0-1-2; 2-3-4; 4-5-6; 6-7-8; ...

Here, the teacher’s management of the activity utilizes, more or less consciously, children’s predisposition to look for “what stays and what changes”, that is one of the central points of Mason’s ‘variation theory’: “*human beings naturally detect similarity through becoming aware of variation*” (Mason *et al.*, 2009, p. 12). Indeed, the children realize very quickly that in the new sequences the difference between two consecutive sums is still a constant but not 9 anymore, and moreover, working with the last sequence, they also see that the regularity of ‘alternation of odd and even numbers’ doesn’t work. Finally, only the regularity ‘all the sums are multiple of three’ works. The teacher helps the children to synthesize this first experience on a poster, to record their findings (Fig. 1)

0-1-2 3 + 6 = 9	2-3-4 6 + 6 = 12	4-5-6 9 + 6 = 15	6-7-8 12 + 6 = 18	8-9-10 15 + 6 = 21
10-11-12 33 + 6 = 39	12-13-14 36 + 6 = 42	14-15-16 39 + 6 = 45	16-17-18 42 + 6 = 48	18-19-20 45 + 6 = 51
20-21-22 63 + 6 = 69	22-23-24 66 + 6 = 72	24-25-26 69 + 6 = 75	26-27-28 72 + 6 = 78	28-29-30 75 + 6 = 81

IN UNA SEQUENZA ORDINATA LA DIFFERENZA  
TRA UNA SOMMA E L'ALTRA E' UN NUMERO COSTANTE,  
IN QUESTO CASO E' +6, E SEMPRE  
UN MULTIPLO DI 3.

Fig. 1

During the second lesson the teacher decides to work with the whole class group and proposes to choose a sequence of terns not ‘ordered’ as before, but to take groups of three consecutive numbers at random. In a natural way the children start taking into consideration scattered terns: Gabriele, for example, takes 1-2-3; Giuseppe 12-13-14; Angelo 39-40-41, Ivan 83-84-85 and so on, with even larger numbers. In this way the children soon face the problem of calculating quite large sums, not easily manageable by mental calculation. Some children spontaneously use the additive algorithm which can be interpreted as *empirical counting*. Clearly, not all the young children are experts with the additive algorithm when the sums become so big.

Pointing on this difficulty, the teacher asks to carefully observe the three consecutive numbers and to try to find an easier way, a trick, to calculate their sum. As Mason *et al.* observe, “*inviting children to find ‘quick way’ to do arithmetic calculations [...] can be an entry into appreciating structure.*” (Mason *et al.*, 2009, p. 14).

The children work in groups for a little time and then, after several attempts, Ivan observes: “Teacher, I understand! It’s enough to do 84 times 3”. He seems to become suddenly aware of an action that he has repeated many times. In this way Ivan seems to generalize not a figural pattern, but an ‘action’ pattern.

Not all the other children seem to understand Ivan’s observation, therefore the teacher invites Ivan to explain his finding, also because the rest of the class is not working on this particular tern. “If we take away one unit from 85 and give it to 83 than they become three 84, and so if we do 84 times three, we do it faster.”<sup>66</sup>. We can recognize in Ivan’s statement an attention to structure in which his *empirical counting* performed on the particular tern moves toward a *structural generalization*. Here an able teacher’s intervention is needed in order to stimulate a more structural vision. Indeed, the teacher invites Ivan to use some objects like the rulers, or to draw a picture, to better explain his thinking. After a few time Ivan comes to a graphic representation, so clear and convincing that the whole class decides, following teacher’s advise, to put it on a poster (Fig. 2).

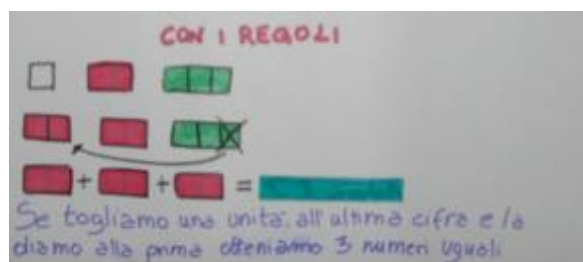


Fig. 2

It is interesting to note how Ivan’s drawing, based on the rulers, though obviously concerning a single, and even very simple numerical case, bears a true generalization, much as in the use of geometrical figures. In fact, even if Ivan uses the “little square” to represent two different things, namely the generic number and the unit, we can recognize in his representation a naïve form of notational symbolism. Quoting Mason again, “*there is a basic awareness based on physical manipulation of objects which tells the people the answer without having to do particular cases*” (Mason *et al.*, 2009, p. 15). Moreover, it is the need to communicate his hypothesis to the rest of the class that allows Ivan to go a step farther and to produce a graph with some algebraic feature, thereby to reach a form

<sup>66</sup> Unfortunately, we cannot report here about the interesting gestures by which Ivan accompanied his words, since we don’t have any video recording of the episode, but the observing researcher describes the gestures as a kind of forerunner of the representation in Fig. 2.

of algebraic thinking, sophisticated enough. That the use of such representation strongly induces a clearer structural vision is proved by the fact that Ivan, after his drawing, is able to articulate a refined explanation in natural language: “If we take away one unit from the last number and we give it to the first one, the three numbers become equal, and if we multiply this number by three, we are done!”

The search for regularities continues also in the following lessons, but what is remarkable here is that all the children begin to use Ivan’s finding to calculate large sums, in other words the children like the “trick” so much, to transform it into a patrimony of the class. Furthermore, in the following lesson the children use the discovery to perform the inverse task, which is given a number multiple of three to identify the consecutive numbers tern of which this number is the sum. This kind of task is suitably chosen by the teacher to allow the children to grasp a more clear structural vision of the relationships involved in the problem. Through a collective discussion mediated by the teacher the children easily understand that they can find middle number of the tern by dividing the number for three.

### **SOME CONCLUSIVE REMARKS**

We have tried to show how some guided activities of exploration and search for regularities in the domain of natural numbers can promote algebraic thinking in very young children. In particular, the activity presented here, profiting from young children’s poor calculation skills, stimulates the invention of tricks to succeed in performing calculations. In our opinion, this invention can be seen as a first kind of ‘attention to structure’; in other words, ‘looking for tricks’ can act as a very early forerunner of the abstraction process, typical of the algebraic thinking. Therefore to induce in children the need for simplifying difficult tasks can be a good way to start them off towards the algebraic thinking.

Many questions still remain to be faced. For instance, since it happens that some children, like Ivan, are more ready to develop algebraic thinking than others, how to let the intuitions of an individual be acquired by others; and, even more crucial, which kind of didactic mediation can help to transform a possibly deep but unstable intuition of a child into a more formal knowledge. The questions are not easy, and in our opinion cannot be solved in a simple or unique way: in this sense we agree with Tall’s perspective, according to which, “*the study of ‘early algebra’ needs to be seen not only as an activity in itself but also as part of a longer-term development.*” (Tall, 2001, p. 152).

Anyway, a small contribution to both questions can come from the use of suitable representations and from a good context of social interaction, as we hope to have shown before. Indeed, teacher’s invitation to use representations seems to help Ivan to reach more sophisticated forms of structural thinking and, from the other side, the proposal of exposing Ivan’s discovery in a poster contributes to transform his trick into a patrimony of the whole class.

## REFERENCES

- Carraher, D., Schielmann, A. D., & Brizuela, B. M. (2001). Can young children operate on unknowns? In van den Heuvel-Panhuizen, M. (Ed.), *Proceedings of the 25<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 130-140). Utrecht, The Netherlands: PME.
- Caspi, S & Sfard, A. (2010). Spontaneous meta-arithmetic as a first step toward school algebra. In Pinto, M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 249-256). Belo Horizonte, Brazil: PME.
- Iannece, D., Mellone, M., & Tortora, R. (2009). Counting vs. measuring: reflections on number roots between epistemology and neuroscience. In Tzekaki, M., Kaldrimidou, M. & Sakonidis, H. (Eds.), *Proceedings of the 33<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 209-216). Thessaloniki, Greece: PME.
- Iannece, D., Mellone, M., & Tortora, R. (2010). Early multiplicative thought: a kindergarten path. In Pinto, M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 121-128). Belo Horizonte, Brazil: PME.
- Iannece, D., & Tortora, R. (2008). Resonance: a key word in mathematics teaching-research. In Czarnocha, B. (Ed.), *Handbook of Mathematics Teaching Research: Teaching Experiment - A Tool for Teacher-Researchers* (pp. 59- 70). Rzeszów, Poland: Rzeszów University Press.
- Lee, L., & Freiman, V. (2004). Tracking primary students' understanding of patterns. In D. McDougall & J. Ross (Eds.), *Proceedings of the 26<sup>th</sup> PMENA* (Vol. 2, pp. 245-251). Toronto, Canada.
- Malara, N.A., & Navarra, G. (2008). An Early Algebra Glossary and its role in teacher education. In Czarnocha, B. (Ed.), *Handbook of Mathematics Teaching Research: Teaching Experiment - A Tool for Teacher-Researchers* (pp. 193-207). Rzeszów, Poland: Rzeszów University Press.
- Mason, J.; Stephens, M. & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, 21 (2), 10-32.
- Radford, L. (2001). Of course they can! A Reaction to Carraher et al.'s paper "Can young students operate on unknowns?". In van den Heuvel-Panhuizen, M. (Ed.), *Proceedings of the 25<sup>th</sup> Conference of the Int. Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 145-148). Utrecht, The Netherlands: PME.
- Radford, L. (2010). Elementary form of algebraic thinking in young students. In Pinto, M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34<sup>th</sup> Conf. of the Int.*

### Working Group 13

*Group for the Psychology of Mathematics Education* (Vol. 4, pp. 73-80). Belo Horizonte, Brazil: PME.

Rivera, F. D. (2010). Second grade students' preinstructional competence in patterning activity. In Pinto, M. F. & Kawasaki, T. F. (Eds.). *Proceedings of the 34<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 81-88). Belo Horizonte, Brazil: PME.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, mathematizing*. Cambridge, UK: Cambridge University Press

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics*, 20, 147-164.

Tall, D. (2001). Reflections on early algebra. In van den Heuvel-Panhuizen, M. (ed.), *Proceedings of the 25<sup>th</sup> Conference of the Int. Group for the Psychology of Mathematics Education*, (Vol. 1, pp. 149-152). Utrecht, The Netherlands: PME.



# **CAN YOU TAKE HALF?**

## **KINDERGARTEN CHILDREN'S RESPONSES**

Dina Tirosh, Pessia Tsamir, Michal Tabach, Esther Levenson, and Ruthi Barkai

Tel Aviv University

*The notion of 'half' is familiar to children from their everyday experiences before formal schooling begins. This study explores kindergarten children's ability to take half of an increasingly large amount of discrete items and the strategies they employ while doing so. It also explores the difference between taking half of a set of discrete items and half of a set of continuous items where each item may itself be halved. Results indicate that kindergarten children know that taking half implies two sets and that they employ different strategies when halving different amounts.*

### **BACKGROUND**

The importance of engaging children with mathematics during their early years has recently come to the fore. This is reflected in both position papers and national standards which often mention number and operations as a key domain to be developed during these years. The notion *half* is included in the domain of number concepts. For example, in their guidelines for grades Pre-K through grade 2, the Principles and Standards for School Mathematics states that "All students should... understand and represent commonly used fractions, such as  $\frac{1}{4}$ ,  $\frac{1}{3}$  and  $\frac{1}{2}$ ." (NCTM, 2000, p. 78). The Israel National Mathematics Preschool Curriculum (INMPC, 2008) states that "a child should be familiar with the concept of half and its practical use in everyday life" (p. 17). In addition it states that, "the concept of half relates to half of one unit, such as half a pita bread, as well as half of a quantity, such as half of the cookies in a box...In kindergarten, it is unnecessary to introduce the graphic symbol of half." (p. 45). In Australia, the New South Wales K-6 curriculum (2002) states that for Early Stage 1, students should learn about "using the term 'half' in everyday situations", and "use fraction language in everyday situations e.g. 'one-half of a cake has been eaten'" (p. 60).

Previous studies investigated young children's notion of half within the context of fair-sharing. Fair-sharing contexts relate to situations where the total resources available are divided equally among the users. Thus, if an amount is shared equally between two people, we may say that each person has received half of the amount. Regarding the sharing of a continuous item, Hunting and Sharpely (1988) investigated the behaviours of preschool children requested to share a blanket and a skipping rope between two dolls. They found that while most children understood that a single partition was necessary, most lacked an anticipatory or a verification strategy for assessing the resulting equality. Regarding discrete items, it was found that young children were able to divide 12 crackers between two dolls by dealing one cracker at a time to the dolls (Hunting & Davin, 1989 in Davis & Hunting, 1990).

Yet, in an unstructured interview, when it came to sharing 12 jelly beans between two children, the same children did not share equally (Davis & Hunting, 1990). Davis (1989) found that students learning division may not take into consideration the reality of a problem. When asked to share five balloons between two children, one student suggested cutting the fifth balloon in half.

While the previous studies investigated children's notion of half within a fair-sharing context, this study investigates the notion of half when it is not explicitly related to fair-sharing. Such situations arise when a child eats half an apple or takes out half of the blocks from a container. These situations are very different from fair-sharing contexts. For example, Parrat-Dayana and Vonech (1992) found that giving half of six apples to one doll is more difficult than carrying out this sharing task with two dolls. "Asking a child for half of six apples forces him or her to think of the six apples as a totality, such that the union of the parts forms a whole, where the two parts are disjuncted." (Parrat-Dayana & Voneche, 1992, p.74).

Our current study continues and expands upon previous investigations in two ways. First, it investigates young children's notion of half when fair-sharing is not explicit, in situations when it is possible to take half and in situations when it is not possible to take half. In addition, previous studies did not focus explicitly on children's strategies for taking half. This study focuses also on the strategies.

Our research is guided by the following questions: (1) To what extent can kindergarten children take half of a set of homogenous discrete objects before they are explicitly taught the concept of half and what strategies do they employ when doing so? (2) Given two sets of objects, one where the objects themselves are continuous and may each be halved and one where each of the objects is discrete and cannot be halved, is there a difference between kindergarten children's success in taking half of the set?

## METHOD

### Participants

Four preschool classes in low-socioeconomic neighbourhoods participated in this study. Each class consisted of approximately 30 pre-kindergarten and kindergarten children between the ages of four and six years old. In this paper we focus on all 64 kindergarten children (between the ages of five and six years old) who were expected to enter first grade in the upcoming school year. The kindergarten teachers were participating in a two-year professional development program, *Starting Right: Mathematics in Kindergarten*<sup>67</sup>. The children worked with their teachers on tasks from various mathematical domains, including geometry, measurement, and number and operations. Counting was part of the children's daily routine. Other

---

<sup>67</sup> This research was conducted in collaboration with the Rashi Foundation.

mathematical activities included composing and decomposing numbers and dividing objects into two, three, or more sets, equal as well as unequal sets (e.g. Tsamir, Tirosh, Tabach, & Levenson, 2010). Fractions and the concept of half were not explicitly taught.

## Tools

Individual interviews were conducted in a quiet corner of the kindergarten during the last month of the school year. The interviewer followed a set protocol and was responsible for writing down each child's actions as well as utterances. Each interview consisted of eight tasks. Table 1 presents the tasks and accompanying questions as well as the settings of each task, in the order in which they were implemented. The tasks were implemented in the same order for each child.

	The setting	The tasks
1.	Six bottle caps, all the same size, but not necessarily the same color, are placed on the table in a bunch.	Please take half of the caps. How do you know you took half of the caps?
2.	Six bottle caps are placed on the table in a bunch. The interviewer tells the first part of a story and then removes four caps from the table. The interviewer continues with the questions.	Yesterday I was in a different kindergarten and asked a girl named Dina to take half of the caps. Here is what Dina did. (The interviewer removes four caps from the table.) Do you think Dina was correct? Explain why.
3.	Five bottle caps are placed on the table in a bunch.	Please take half of the caps. How do you know you took half of the caps?
4.	Five congruent pieces of note paper are placed on the table in a pile.	Please take half of the papers. How do you know you took half of the papers?
5-8.	Four (eight, ten, and 14) bottle caps are placed on the table in a bunch.	Please take half of the caps. How do you know you took half of the caps?

**Table 1: The interview settings and questions**

The interview began with the placement of six bottle caps on the table. We began with this amount because we considered it non-trivial and yet not too large for five year olds to handle. The second task of the interview presented a situation where the child's role was reversed. Instead of asking the child to solve a problem, the child was presented with an incorrect solution given by a fictitious child and then requested to evaluate that child's actions. We were interested in investigating if the child would be swayed by a countersuggestion (Kamii, 1982).

For the next pair of tasks, five objects were used – first five caps and then five pieces of note papers. We began with the caps and not with the note papers because the first two tasks had used caps and we did not want to change both the amount and type of

object at once. A bottle cap cannot be halved. On the other hand, the pieces of paper used in the study were such that they could be torn in two without the use of scissors, resulting in two halves of a piece of paper. Thus, taking half of five bottle caps cannot be executed whereas one could take half of five papers by taking two and a half pieces of paper. These tasks had two aims. First, would children accept and acknowledge that not every task may be executed? Second, would children differentiate between taking half of five bottle caps and taking half of five pieces of paper?

The last part of the interview consisted of a sequence of four similar tasks, taking half of an even amount of bottle caps, in which the amount of caps gradually increased from four, which was thought to be quite simple, to eight, ten, and finally fourteen caps. This allowed us to investigate if the children would be able to take half of a growing number of elements. It also allowed us to investigate the children's strategies for taking half of a different number of caps.

### Methods for data analysis

Data analysis consisted of three components. The first component related to the correctness of the child's final result or response. For example, did the child take four caps when requested to take half of eight caps? Did the child correctly respond that he could not take half of five caps? In the case of taking half of five pieces of paper, we considered a correct solution to be the tearing of one piece of paper into approximately two equal halves, thus producing two and half pieces of paper.

The second component related to analyzing the methods employed by the child when he or she attempted to take half of the presented objects. Methods were recorded regardless of the final outcome and were derived from the actions performed. Some children immediately took half of the objects without any observable strategy. Others employed one of three identified strategies: (1) Children who first counted all of the caps in the given set, concluded how many would be considered half, and took this amount employed the *count all* strategy. (2) *Divide and adjust*, was displayed by children who either arbitrarily or perhaps by visual estimation, divided the given set into two subsets, counted the number of caps in each subset, and then adjusted the two subsets by moving a cap or two to equate the number of caps in each subset. (3) Children who built two subsets simultaneously, putting one cap at a time in each of two subsets, until all the caps were used up were said to *build stepwise*. Some children using this strategy put two caps at a time in each new set. Finally, some children took actions that did not fall into any category.

The third component in the analysis related to children's explanations. As expected from young children, the explanations were brief phrases and not necessarily full sentences. Qualitative analysis of explanations added a dimension to our understanding of why children acted as they did and the possible reasons for their responses. They also provided some insight into the children's conceptions of half.

## RESULTS

In this section we begin by presenting results of the first two tasks – halving six and responding to an incorrect suggestion for halving six. We follow with the next two tasks, halving five caps and five pieces of note paper. Finally we report on responses to the sequence of halving an increasing even amount of caps.

### Half of six

Most children (90%) correctly took three caps out of the set of six caps. Out of those who explained their actions, some pointed to each pile of three caps, claiming "half and half" or "both are equal". This type of explanation could possibly be referring to the child's attempt at fairness. Some children merely reiterated what they had done. For example, one child commented, "Three, half is three". Some children decomposed the number six stating, "Three and three is six". Of the children who did not take three caps, five children took two caps explaining, "a half is a little bit". This is in line with studies which have shown that children refer to any part of the whole, but preferable the smaller part, as a half (Parrat-Dayana & Voneche, 1992). One child did not take any caps at all nor did he respond verbally. Children's methods for taking half of six are discussed along with the results of the last four tasks where children were repeatedly requested to take half of an increasing amount of caps.

Regarding the second question, 81% of the children correctly rejected Dina's suggestion. Similar to children who in the first task merely reiterated what they had done, here, some children pointed out the result of what Dina had done saying, "Here there are two [caps] and here there are four." Others pointed out, "It should be equal." Others told the interviewer what Dina should have done, "She (referring to Dina) did not take three." Among the 19% of children who incorrectly accepted Dina's suggestions, the explanations varied. One child claimed that Dina was correct because "she left some for the others as well". Another child seemed to sympathize with Dina claiming that "she wanted to play with the caps". Finally, just as children decomposed six into three and three, one child claimed that Dina was correct because "four and two is six". Table 2 presents the results of the first two tasks.

	Took 3 caps	Did not take 3 caps	Total
Accepted Dina's suggestion	12	7	19
Rejected Dina's suggestion	78*	3	81
Total	90	10	

\*Correct response to both tasks

**Table 2: Frequency (%) of responses to Tasks one and two.**

We first note that while only 10% of the children did not answer the first question correctly, 19% did not answer the second question correctly. Perhaps, countering a

proposed suggestion is more difficult than proposing one's own solution. Regarding consistency, most of the children (85%) were consistent while answering the two questions - 7% gave incorrect responses in both cases while 78% gave correct responses for both tasks. Regarding children who were not consistent, we note that 12% of the children correctly took three out of six caps but then accepted Dina's incorrect suggestion. One such child explained his correct solution to the first task by claiming that "three and another three is six." His explanation to his incorrect response on the second task was "four and two is six." In other words, this child used the same type of explanation for both tasks, decomposing the number six.

### Half of five

Tasks three and four related to taking half of five, first five caps and then five pieces of paper. The same child, who did not respond to the first two tasks, did not respond to these tasks either. Results, summarized in Table 3, indicated that almost 80% of the children responded that it is not possible to take half of five caps. Of those that claimed it was possible, one child took all five caps for himself. Six children took three caps and six children took two caps. All of the accompanying explanations reverted to describing what was just done. As one child claimed, "I took two."

Response	Cannot take half of 5 caps	Can take half of 5 caps	Total
Cannot take half of 5 papers	70	2	72
Can take half of 5 papers			
Tore one piece of paper	3*	2	5
Did not tear one piece of paper	5	16	21
Total	78	20	

\*Correct response to both tasks

**Table 3: Frequency (%) of responses to tasks three and four, taking half of five items.**

Regarding the 78% of children who correctly claimed that half of five caps could not be taken, many tried to take half, moving the caps back and forth in an attempt to take half or to create two sets with an equal amount of caps. Accompanying explanations included, "This is two and this is three" or "You have half and I don't". Others suggested how the situation may be remedied, claiming either, "One cap should be added" or "One cap should be removed." Others simply stated, "It is not possible".

Most of the children (70%) claimed that one cannot take half of five, regardless of the object at hand. It could be that not being able to take half of five caps influenced children's performance on the paper task. In fact, children's explanations for why they could not halve five pieces of paper were similar to those they gave for the bottle caps. Only 5% of the children suggested halving one of the papers. The

children who did so first asked permission from the interviewer, asking if they could tear one of the papers. When granted permission, each of the children tore one of the papers into two approximately equal parts, gave two and a half papers to the interviewer and took two and a half papers for themselves. They then claimed, "Three papers for me and three for you." Perhaps children were relating to the torn piece as a whole piece. Regarding the 21% who claimed that they could take half of five papers but did not tear any of the papers, one child took four pieces of paper, 10% of the children took 2 pieces of paper and 10% took three pieces of paper.

### Halving an even number of bottle caps

Recall that the last four tasks requested of the children to repeatedly take half of an increasing amount of bottle caps. Interestingly, the one child who did not respond to the first four tasks did complete the last four tasks. It is possible that it took some time for this child to warm up to the interviewer. Table 4 summarizes children's responses to these tasks. (C denotes the number of caps presented.)

Responses	C=4	C=8	C=10	C=14
Correct response	98	92	85	70
Took less than $C/2$	--	3	6	8
Took more than $C/2$	--	5	3	2
Did not complete the task	2	--	6	20

**Table 4: Frequencies (%) of children's responses to Tasks 5-8, taking half of an even number of items**

First, we note a difference between children's responses when taking half of four and ten caps, and again between ten and 14 caps. Among incorrect responses, approximately a third of the responses consisted of taking less than half while approximately a fifth of the responses consisted of taking more than half. Among them were two children who correctly took half of four and eight caps (and also of six caps in the previous task) but then took four caps out of ten and again four caps out of fourteen. These children, as did most children who completed the task incorrectly, did not explain their actions.

When the number of caps was 14, seven (11%) children did not even attempt to complete the task. A fifth of the group attempted the task but did not complete it. As one girl commented, "It's too many. I won't succeed." Another girl claimed, "I can't. It's hard for me." Others acknowledged that taking half of 14 might be difficult but persisted. Faced with 14 caps, one child who eventually correctly completed the task, commented, "Wow, this will take time". On the other hand, almost all of the children immediately halved four caps. We also note that one child consistently took two caps and one child consistently took all of the caps, regardless of the amount of caps presented. Neither of the children explained their actions.

Children employed a variety of methods when attempting to take half of the caps. As noted in the method section, three strategies were identified: *count all*, *divide and adjust*, and *build stepwise*. Table 5 presents the percentage of children who employed each strategy and gave a correct solution. Strategies are presented in order from the most to the least frequently used strategy. Regarding incorrect solutions, when the number of caps was less than 10, no specific strategy was noticeable. When there were 10 caps, 5% of the children attempted to use the build stepwise strategy and the rest (10%) did not have a noticeable strategy. When there were 14 caps, 11% of the children did not even attempt the task, 5% of the children attempted to use the divide and adjust strategy and the rest (14%) did not have a noticeable strategy.

Method	M=4	M=6	M=8	M=10	M=14
Immediate	96	88	39	--	--
Strategy					
Build stepwise	2	2	28	27	38
Divide and adjust	--	--	14	27	19
Count all	--	--	11	23	3
Other	--	--	--	8	10
Total	98	90	92	85	70

**Table 5: Frequencies (%) of children's methods accompanying correct halving**

We first note a difference between the amount and types of strategies employed by the children when dealing with four and six caps versus the larger sets. Approximately 90% of the children immediately and correctly took half of four and six caps while none of the children even tried to do so when there were 10 or 14 caps. One reason for this difference may be children's ability to subitize – "the direct and rapid perceptual apprehension of the numerosity of a group" (Kaufman, Lord, Reese, & Volkmann, 1949, in Clements & Sarama, 2007) that allows children to easily handle sets with a small number of caps. Perhaps being able to rapidly identify two and three allowed the children to easily take half of four and six without employing more complex strategies.

The build stepwise strategy was the strategy most frequently used. The count all strategy was the strategy least employed. In fact, only two children even attempted to use this strategy when presented with 14 caps. On the other hand, it consistently led to a correct solution. This was not the case with the divide and adjust strategy. Some children, who attempted to use this strategy when halving 14, divided the caps into two not necessarily equal subsets and then found it difficult to adjust them. That is, they could not figure out how many caps should be moved from one set to the other in order for both to have the same amount of caps.



Not being able to complete the task also occurred when using the build stepwise strategy. Three children, who used this strategy for taking half of ten, started off by placing two in one set, two in another set and again two in the first set and then two in the second. They then ended up with two caps and concluded that the task could not be completed. Perhaps because they used two caps as a building block, it did not occur to them that this block could also be divided. When it came to taking half of 14 caps, two of the children adjusted the strategy. One child began as before, building stepwise with a block of two caps. When again faced with two caps he paused and said, "This is hard." He then thought a little more and separated the two caps, adding one to each set, successfully completing the task. The second child adjusted the strategy by building stepwise using one cap instead of the two she had previously used. Thus, she had no difficulty in halving 14.

## DISCUSSION

This study set out to investigate children's concept of taking half of a set of items. Being that the sample was relatively small and came from one country, we cannot make any generalizations. However, for the most part, children in this study succeeded in taking half of the objects presented to them, carefully constructing two sets with an equal number of elements and then taking one of these sets. There are two equally important components of this understanding. The concept half implies two sets. The concept half implies equal sets. While both of these are necessary components of half, it seems that the necessity of two sets may be more dominant or rooted than the necessity for the sets to be equal. None of the children thought to divide the caps into more than two sets and only one child consistently took all the caps for himself. Children who completed the tasks incorrectly almost always divided the caps or papers into two sets that were not equal. These children may have related the halving task to an everyday sharing task, where sometimes, it is important to share but maybe less important to share equally. This is similar to the study mentioned in the background of children's unequal sharing of 12 jelly beans (Davis & Hunting, 1990). It is important to note also that some children knew that their answer was incorrect or incomplete, but could not fix it. It could be that these children had not yet developed the skills to use a strategy efficiently and flexibility.

Halving five pieces of paper turned out to be a complex task. This complexity may be related to the items themselves. On the one hand, the pieces of paper are discrete items. On the other hand, each piece is continuous and may be halved. It could be that just as very young children may be reluctant to cut a cake in half (Piaget & Szemirska, 1952), some children may be reluctant to tear paper in half. Perhaps a more authentic problem, such as taking half of five chocolate bars, would have prompted the children to break one bar in half. Or it could be that children were responding to a social norm whereby tearing paper is unacceptable. Recall also that 20% of the children claimed that they could take half of five caps. It could be that these, as well as other children were responding to a didactical contract (Brousseau,

1997) and specifically the implicit rule that all problems must have a solution. Regarding children's incorrect response to Dina, young children may be more naïve than older children and believe that if Dina took four caps, she must have had a good reason to do so. In other words, children's knowledge of half is fragile.

In addition to increasing our understanding of children's conception of half, this study also points to the importance of implementing a variety of tasks when trying to assess children's knowledge as well as requesting children, even young children, to explain their actions. Had we not presented the children with Dina's incorrect solution, we may have missed how their empathies may affect their mathematics. Had we not gradually increased the amount of caps to be halved, we may have missed the opportunity to investigate children's strategies for taking half. Asking children to explain each action also proved insightful. We may have thought it sufficient and correct when a child explained why three is half of six by decomposing the number six into three and three. However, when the same child explained his incorrect acceptance of Dina's actions by decomposing six into four and two, we realized that our previous assessment may have been hasty and that this child may not yet relate equality of sets to the notion of half.

Finally, we note that little difference was found between this study, which investigated the children's conception of "taking half", and that of studies which investigated the notion of half within a fair-sharing context. Will taking a third be similar to fair-sharing between three? Will taking a fourth be similar to fair-sharing between four? We suggest more research into children's conceptions of half, as well as other fractions, which will offer insight into the different tasks we might implement with kindergarten children, as well as the different items we may employ in these tasks, in order to promote young children's initial fraction knowledge.

## REFERENCES

- Board of Studies New South Wales (2002). *Mathematics K-6*. Sydney, Australia.
- Brousseau G. (1997) Theory of didactical situations in mathematics. Dordrecht: Kluwer Academic Publishers
- Clements, H. D., & Sarama, J. (2007). Early childhood mathematics learning. In Frank K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, (pp. 461-556). Charlotte, NC: Information Age Publishing.
- Davis, R. B. (1989). The culture of mathematics and the culture of schools. *Journal of Mathematical Behavior*, 8, 143-160.
- Davis, G. & Hunting, R. (1990). Spontaneous partitioning: Preschoolers and discrete items. *Educational Studies in Mathematics*, 21, 367-374.
- Hunting, R. & Sharpely, C. (1988). Fraction knowledge in preschool children. *Journal for Research in Mathematics Education*, 19,(2), 175-180.

### Working Group 13

- Israeli National Mathematics Preschool Curriculum (2008). *Mathematics Curriculum for Pre-Elementary School*. (in Hebrew).
- Kamii, C. (1982). *Number in Preschool & Kindergarten*. National Association for the Education of Young Children, Washington, D. C.
- National Council of Teachers of Mathematics [NCTM] (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Parrat-Dayana, S., & Vonech, J. (1992). Conservation and the notion of "half". In J. Bideaud, C. Meljac, J. P. Fischer (Eds.), *Pathways to Number*, Hillsdale, (pp. 67 – 82). NJ: Lawrence Erlbaum Associates.
- Piaget & Szemirska, A. (1952). *The child's conception of number*. London: Routledge & Kegan Paul.
- Tsamir, P., Tirosh, D., Tabach, M. & Levenson, E. (2010). Multiple solution methods and multiple outcomes – Is it a task for kindergarten children? *Educational Studies in Mathematics*, 73(3), 217- 231.

# INVESTIGATING GEOMETRIC KNOWLEDGE AND SELF-EFFICACY AMONG ABUSED AND NEGLECTED KINDERGARTEN CHILDREN<sup>68</sup>

Pessia Tsamir, Dina Tirosh, Esther Levenson, Michal Tabach, and Ruthi Barkai

Tel Aviv University

*This study investigates kindergarten children's identifications of examples and nonexamples of triangles, pentagons, and circles and their self-efficacy related to these tasks. The participants of this study included children who had been abused and neglected. When comparing the group of abused and neglected children with other children, results indicated that both groups of kindergarten children had high self-efficacy beliefs which were not significantly related to knowledge. Significant differences in knowledge were found between the two groups.*

Early knowledge of mathematics is often seen as a predictor of later school success (Jimerson, Egelstad, & Teo, 1999). Abused and neglected children<sup>69</sup> are especially at risk, as these children lag behind their peers in cognitive development (Gowen, 1993). During the elementary school years, the mathematics achievement scores of abused and neglected students are significantly lower than their peers, even when controlling for socioeconomic status (Kendall-Tackett & Eckenrode, 1996). One of the key mathematical domains during the preschool years mentioned by many national guidelines is geometry. During these years, children are developing and refining their spatial and geometric thinking. The first aim of this study is to investigate the geometrical knowledge of kindergarten children, including abused and neglected kindergarten children. Are differences in geometrical knowledge already noticeable in kindergarten?

Abuse and neglect during the preschool years can have a significant, as well as lasting impact on an individual's self-perception (Waldinger, Toth, & Gerber, 2001). One aspect of self-perception related to the promotion of knowledge is self-efficacy (Bandura, 1986). Bandura (1986) defined self-efficacy as "people's judgments of their capabilities to organize and execute a course of action required to attain designated types of performances" (p. 391) and claimed that, "...beliefs of personal efficacy make an important contribution to the acquisition of the knowledge structures on which skills are founded" (Bandura, 1997, p. 35). Primary caregivers, as they provide feedback of children's performances, play a significant role in developing children's self-efficacy (Bandura, 1993). Thus, abusive parents may contribute to negative self-efficacy. On the other hand, an inflated self-efficacy belief

---

<sup>68</sup> This study was supported by the Haruv Institute (R.A.) of Israel.

<sup>69</sup> Throughout the paper, the term "abused and neglected children" refers to children who have either been abused or neglected or both.

may result as a form of self-protection in the face of parental abuse and neglect. In such cases, a high self-efficacy gives the child a false sense of self. The second aim of this study is to investigate kindergarten children's geometric self-efficacy beliefs, that is, beliefs related to performing geometrical tasks. Is there a difference between the geometry related self-efficacy of abused and neglected children and other children?

When investigating children's knowledge it is important to consider both real achievement and perceived achievement in tandem. One study of elementary school children found that maltreated children, more so than nonmaltreated children, tend to overestimate their level of competence, particularly for arithmetic (Kinard, 2000). The third aim of this study is to investigate the relationship between children's geometric knowledge and self-efficacy beliefs. We investigate this relationship among kindergarten children, including abused and neglected kindergarten children.

### **Theoretical background**

Two main issues are at the heart of this study: young children's geometric knowledge and young children's self-efficacy beliefs. This section begins by describing previous studies related to young children's geometric knowledge and then reviews studies related to mathematics self-efficacy.

Young children learn about and develop concepts, including geometrical concepts, before they begin kindergarten. At this age, young children begin to perceive attributes but need guidance in order to assess which attributes are critical for identifying a figure and which are not (van Hiele, 1958). For example, studies have found that when a triangle is not oriented with a horizontal base, children may not identify it as a triangle (e.g. Burger & Shaughnessy, 1986). Children may also accept curved sides, either concave or convex, when identifying triangles (Clements, Swaminathan, Hannibal, & Sarama, 1999). Within the domain of geometry, the Early Years Foundation Stage Statutory Framework in England (2008) and the mandatory Israel National Preschool Mathematics Curriculum (2008) specifically require that by the end of kindergarten children use mathematical language to describe two-dimensional figures. This study focuses on identifying triangles, pentagons, and circles.

Few studies have investigated preschool children's self-efficacy. This may be due to children's difficulty in differentiating between what is real and what they desire to be real (Stipek, Roberts, & Sanborn, 1984). Research findings are mixed. Some studies have found that young children may have overly high self-efficacy beliefs (Stipek, Roberts, & Sanborn, 1984) while others have found that young children are able to understand the process of self-evaluation and may fairly judge their own competence (Anderson & Adams, 1985). "Mathematics self-efficacy...is a situational or problem-specific assessment of an individual's confidence in her or his ability to successfully perform or accomplish a particular task or problem" (Hackett & Betz, 1989, p. 262).

Research related to self-efficacy and mathematics has shown that regardless of mathematical ability, students with a higher self-efficacy tend to expend more effort on mathematics tasks than students with lower self-efficacy (Collins, 1982). Such students are willing to rework problems, discarding faulty strategies in favor of trying new ones, and in general display a more positive attitude towards mathematics than students' with a lower self-efficacy. Studies have also shown that students' self-efficacy beliefs predict mathematics performance (Bandura, 1986; Pajares, 1996) and do so to a greater degree than mathematics anxiety (Pajares & Miller, 1995). Among first and second graders, academic self-efficacy was found to be related to mathematics achievement (Liew, McTigue, Barrois, & Hughes, 2008). It is important to note that self-efficacy beliefs may be domain specific or general. Most studies related to mathematics self-efficacy measured a very general belief in mathematics self-efficacy which did not necessarily relate to specific mathematics topics (i.e. Usher, 2009). This study will focus on the child's self-efficacy while engaging in geometrical tasks and will investigate the relationship between kindergarten's children's geometric knowledge and their geometric self-efficacy.

## METHODOLOGY


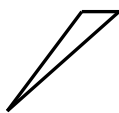


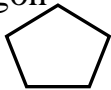


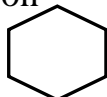
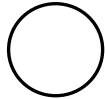


The participants of this study included 141 kindergarten children, ages 5-6 years old, living in low socio-economic neighbourhoods. All of the children were scheduled to enter first grade during the following school year. Of the 141 children, 69 children were labelled as abused and neglected by the social welfare department of their municipality. All of the children attended municipal kindergartens in their local neighborhood in the morning. While most children go home after school is over, the 69 abused and neglected children were bussed after school to day-care centres run by their municipality where they received hot meals and enrichment.

The research took place in the last three months of the school year. A structured interview was developed for this study interweaving questions related to geometric self-efficacy with questions related to geometric knowledge. Children who were identified by the social welfare department of the city as being abused and neglected were interviewed individually in a quiet corner of the day-care center which they attended in the afternoons. The other children were interviewed individually in a quiet corner of their kindergartens in the morning.

The focus of this study was on identifying and reasoning with triangles, pentagons, and circles and associated self-efficacy beliefs. The interview began with the following self-efficacy questions: If I show you a picture of a shape, will you be able to tell me if the shape is a triangle? Are you very sure or only a little bit sure? These self-efficacy related questions were based on the *Pictorial Scale of Perceived Competence and Social Acceptance for Young Children* (Harter & Pike, 1984). In that study, children were shown pictures of two children engaging in some task, one successful and one not successful. The interviewer asked the child to point to the child he or she identified with. After the child pointed to the appropriate picture, the

child was asked if he or she was a lot like the child in the picture or a little bit like the child in the picture. Thus, a four point scale was created. Likewise, in the current study, the first two questions, taken together, created a scale of 1-4 describing children's belief in their ability to identify triangles. For example, if a child answered "yes" to the first question and "a little bit" to the second question, his self-efficacy was graded at 3. If he answered "no" to the first question and "very sure" to the second question, his self-efficacy was graded at 1.

Children were then presented one at a time with four figures, each figure drawn on a separate card, and asked, "Is this a triangle"? Why? The entire set of questions, including the first two self-efficacy related questions, was then repeated for a pentagon and a circle with a different set of figures presented for each shape. Figure 1 displays the figures presented for each set of questions. Figures were presented in the order shown in each row.

Is this a...	Intuitive example	Non-intuitive example	Non-intuitive non-examples	
triangle?	Equilateral triangle 	Scalene triangle 	Rounded-corner "triangle" 	Pizza 
pentagon?	Regular pentagon 	Concave pentagon 	Curved-sides "pentagon" 	Hexagon 
circle?	Circle 		Spiral 	Decagon 

**Figure 1: The set of figures presented in this study**

In choosing the figures, both mathematical and psycho-didactical dimensions were considered. When considering triangles, the equilateral triangle may be considered a prototypical triangle and thus intuitively recognized as a triangle, accepted immediately without the feeling that justification is required (Hershkowitz, 1990; Tsamir, Tirosh, & Levenson, 2008a). The scalene triangle may be considered a non-intuitive example because of its "skinniness". Whereas a circle may be considered an intuitive non-example of a triangle, the pizza-like "triangle" may be considered a non-intuitive nonexample because of visual similarity to a prototypical triangle (Tsamir, Tirosh, & Levenson, 2008a). Similarly, the regular pentagon was thought to be easily recognized by children who had been introduced to pentagons whereas studies have shown that even among children who had been introduced to pentagons, the concave pentagon is more difficult to identify (Tsamir, Tirosh, & Levenson, 2008b). Triangles and pentagons may vary in the degree of their angles providing a

wide variety of examples. In contrast, the circle's symmetry limits the variability of its characteristic features. Thus, only one example of a circle was given. The nonexamples of each shape were also chosen in order to negate different critical attributes. Due to the young age of the children, we chose to limit the amount of figures presented to each child and thus did not include in this study intuitive nonexamples. Finally, we hypothesized that, in general, the triangle and circle would be figures known to the children from their surroundings whereas the pentagon is a figure less known but part of the preschool mathematics curriculum.

## RESULTS

This section begins by describing children's self-efficacy beliefs related to identifying the different shapes. It then describes the results related to children's actual identification of the figures. Finally, it analyzes the relationship between self-efficacy and knowledge.

### Self-efficacy beliefs

Recall that a scale of 1-4 was used to grade children's self-efficacy, 4 being very high and 1 being very low. Results, presented in Table 1, indicated that, in general, the children had a high self-efficacy related to identifying the different shapes. In addition, no significant difference between the self-efficacy of the two groups of children was found for any of the shapes.

Children	Abused and neglected children		Other children	
	M	SD	M	SD
Triangle	3.7	.62	3.7	.67
Pentagon	3.1	1.1	3.5	.75
Circle	3.7	.75	3.9	.46

**Table 1: Children's geometric self-efficacy per shape per group**

Cronbach's alpha was used to investigate internal consistency between the self-efficacy scores for each shape per group of children. A weak internal consistency was found in both groups,  $\alpha=.50$  for the group of abused and neglected children and  $\alpha=.34$  for the other children. These results indicate that, for these children, self-efficacy may be task or shape specific. This is consistent with previous self-efficacy studies which claimed that mathematics self-efficacy may be problem-specific (Hackett & Betz, 1989). On the other hand, when considering that the self-efficacy questions all related to identifying shapes mentioned in the preschool curriculum, we allowed that a general geometric self-efficacy grade may still be calculated for each group. Results indicated no significant difference between the geometric self-efficacy of the abused and neglected children ( $M=3.7$ ,  $SD=.43$ ) and the geometric self-efficacy of the other children ( $M=3.5$ ,  $SD=0.61$ ).

### Geometric knowledge



We begin by describing children's identifications of the individual figures presented to them. Due to limited space, this paper does not analyze the children's explanations which accompanied identifications but provides sample illustrations where relevant. Results, summarized in Table 2 indicated that all of the children correctly identified the equilateral triangle. This coincides with studies which have found that the equilateral triangle with a horizontal base may be considered a prototypical triangle and is thus intuitively identified as such (e.g. Tsamir, Tirosh, & Levenson, 2008a).

Figure name	Abused and neglected children	Other children
Equilateral triangle	100	100
Scalene triangle	20	35
Rounded-corner "triangle"	19	22
Pizza	46	56
Regular pentagon	71	71
Concave pentagon	29	24
Curved-sides "pentagon"	57	70
Hexagon	26	32
Circle	100	100
Spiral	51	61
Decagon	83	85

**Table 2: Frequencies (in %) of correct identifications per figure per group**

The rounded-corner "triangle" was the most frequently misidentified figure. As one child claimed, "It has the shape of a triangle". Interestingly, the equilateral pentagon was identified correctly by less than three-quarters of the children in both groups, though learning to identify pentagons is part of the kindergarten curriculum. As expected, few children in both groups identified correctly the concave pentagon. One child explained, "It looks like a bridge and has only four points." Regarding the circle, although all of the children correctly identified the circle, approximately half of the children incorrectly claimed that the spiral was a circle. Perhaps, the children focused on the roundness of the spiral and the absence of sides. One child claimed it was a circle and added "it continues to roll." Finally, although few children correctly identified the scalene triangle, when comparing the groups of children, this was the only figure for which a significant difference was found  $\chi^2(1, N=138)=4.33, p<.05$ .

After reviewing the results of children's responses to the individual figures, we grouped together the figures according to the shape they were intended to investigate. For each shape, triangles, pentagons, and circles, the mean score was configured resulting in a grade for each child ranging from 0-100% for each shape.

Results, presented in Table 3, indicated that abused and neglected children had a significantly lower triangle grade than other children,  $p < .05$ . No significant differences were found between the two groups of children for the other shapes. Finally, averaging all 11 figures and creating a general geometric knowledge grade, we noted that the neglected and abused children scored significantly lower ( $M = .57$ ,  $SD = .11$ ) than the other children ( $M = .62$ ,  $SD = .14$ ),  $t(117) = 241$ ,  $p < .05$ ).

Children	Abused and neglected children		Other children	
	M	SD	M	SD
Triangle	.46	.20	.54	.20
Pentagon	.46	.18	.50	.19
circle	.78	.24	.82	.24

**Table 3: Children's geometric knowledge per shape per group**

### **Relating geometric knowledge and geometric self-efficacy**

The third aim of the study was to investigate if children's geometric knowledge was related to their geometric self-efficacy. Nonparametric correlations were configured for each geometric shape per group of students. Results for both groups of children indicated that no significant relationship was found between children's ability to identify triangles, pentagons, and circles and their respective self-efficacy beliefs. Finally, when considering general geometric knowledge and general geometric self-efficacy, no significant relationships were found in either group.

## **SUMMARY AND DISCUSSION**

This paper describes an investigation of geometric knowledge and geometric self-efficacy among kindergarten children, including children who were abused and neglected. We begin by discussing similarities in self-efficacy and then similarities in knowledge. We then discuss differences, which for this group, arose when comparing geometric knowledge.

When asked to assess their abilities to identify triangles, pentagons, and circles, children in both groups reported a high self-efficacy, believing greatly in their ability to identify each of the mentioned shapes. There are several possible reasons for this response. First, it could be that children have a tendency to reply in a positive manner or to the high end of any question or scale posed to them. Thus, asked if they can or cannot do something, they respond almost automatically in the positive. Perhaps a naïve belief in one's own capabilities is indicative of all children who are young. While these, and possibly other reasons, may explain the high self-efficacy rating children exhibited, it remains that for the participants of this study, there were no significant differences between the self-efficacy of abused and neglected children and other children. In addition, children's high self-efficacy did not correlate with their knowledge. As mentioned in the background, young children may find it

difficult to differentiate between what is real and what they desire to be real (Stipek, Roberts, & Sanborn, 1984). And yet, among first and second graders, academic self-efficacy was found to be related to mathematics achievement (Liew, McTigue, Barrois, & Hughes, 2008). Perhaps, between kindergarten and first grade, there is a leap in the development of children's sense of self. Perhaps differences between the two groups of children regarding geometric related self-efficacy, and possibly other academic related self-efficacy, may come to the fore at a later age.

Regarding children's identifications of geometric shapes, if we focus on the first example of each shape presented to the children, the equilateral triangle, the equilateral pentagon, and the circle, we note that the frequencies of correct identifications were exactly the same for each group of children. That is, figures which are symmetrical and possibly prototypical of their shape in general, may be easily identified by all kindergarten children regardless of their home backgrounds. In addition, there were no significant differences between the two groups of children in their general knowledge of pentagons and circles. Regarding pentagons, this finding may not be surprising. The pentagon is less common in children's everyday experiences and is usually first introduced in kindergarten. On the other hand, knowledge of circles was also similar between the two groups. It was thought that knowledge which might stem from the child's everyday experiences might produce different results for the different groups.

When looking at the differences between the two groups, less correct identifications were noted among the abused and neglected children than among the other children for the non-intuitive scalene triangle, as well as for each of the nonexamples of triangles, and a significant difference between the two groups of children was found in their general knowledge of triangles. Finally, when the results of the other shapes were also taken into consideration, abused and neglected children exhibited significantly less knowledge than other children. These findings indicate that even before children begin first grade, differences are detectable between the two groups of children. Knowledge of geometric shapes most often begins before formal presentation in school. As such, these differences may possibly stem from the home environment.

Abused and neglected children learn in the same kindergartens as other children. Thus, in order to plan lessons and interventions, it is important to note both the similarities and differences among these children. A high self-efficacy which is not realistic is an issue common to both groups of children and needs to be addressed. In addition, the non-intuitive examples of triangles and pentagons were incorrectly identified by most of the children in both groups. Thus, it is important to actively promote this knowledge among all kindergarten children. And yet, differences do exist. Equity is not only about giving a fair chance to children from different socio-economic backgrounds or minority students. It is about providing "high expectations and strong support for all students" (NCTM, 2000, p. 12). Children who have been

abused and neglected have special needs. Schmid (2007), in his report on children at risk in Israel, suggested that identification of risk factors in early childhood may prevent or minimize problems which develop later on. This study is a first step in considering the mathematics educational needs of children at risk. Additional research is needed to address possible interventions which take into consideration both similarities and differences in knowledge, self-efficacy, and possibly affective issues when promoting mathematics for all children, including children at risk.

## REFERENCES

- Anderson, P. & Adams, P. (1985). The relationship of five-year-olds' academic readiness and perceptions of competence and acceptance. *Journal of Educational Research*, 79, 114-118.
- Bandura, A. (1986). *Social Foundations of Thought and Action: A Social Cognitive*. Englewood Cliffs, NJ: Prentice Hall.
- Bandura, A. (1993). Perceived self-efficacy in cognitive development and functioning. *Educational Psychologist*, 28, 117-148.
- Bandura, A. (1997). *Self-efficacy: The exercise of control*. New York: W. H. Freeman and Company.
- Burger, W. & Shaughnessy, J. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education*, 17(1), 31-48.
- Clements, D., Swaminathan, S., Hannibal, M., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, 30(2), 192-212.
- Collins, J. (1982). Self-efficacy and ability in achievement behavior. Paper presented at the *Annual Meeting of the American Educational Research Association*, New York.
- Statutory Framework for the Early Years Foundation Stage* (2008). Retrieved August 17, 2010 from <http://nationalstrategies.standards.dcsf.gov.uk/node/151379>.
- Gowen, J. (1993). *Effects of neglect on the early development of children: final report, National Clearinghouse on Child Abuse and Neglect, Administration for Children & Families*, Washington, DC.
- Hackett, G., & Betz, N. (1989). *An exploration of the mathematics self-efficacy/mathematics performance correspondence*. *Journal for Research in Mathematics Education*, 20(3), 261-273.
- Harter, S. & Pike, R. (1984). *The pictorial scale of perceived competence and social acceptance for young children*. *Child Development*, 55(6), 1969-1982.
- Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70-95). Cambridge, UK: Cambridge University Press.
- Israel national mathematics preschool curriculum (INMPC)* (2008). Retrieved April 7, 2010, from [http://meyda.education.gov.il/files/Tochniyot\\_Limudim/KdamYesodi/Math1.pdf](http://meyda.education.gov.il/files/Tochniyot_Limudim/KdamYesodi/Math1.pdf)

### Working Group 13

- Jimerson, S., Egelstad, B., & Teo, A. (1999). A longitudinal study of achievement trajectories: Factors associated with change. *Journal for Educational Psychology*, 91, 116-126.
- Kendall-Tackett, K. & Eckenrode J. (1996). The effects of neglect on academic achievement and disciplinary problems: A developmental perspective. *Child Abuse & Neglect*, 20, 161-169.
- Kinard, E. (2000). Perceived and actual academic performance in maltreated children. *Child Abuse and Neglect*, 25, 33-45.
- Liew, J., McTigue, E. M., Barrois, L., & Hughes, J. N. (2008). Adaptive and effortful control and academic self-efficacy beliefs on achievement: A longitudinal study of 1st through 3rd graders. *Early Childhood Research Quarterly*, 23(4), 515-526.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Pajares, F. (1996). Self efficacy beliefs in academic settings. *Review of Educational Research*, 66(4), 543-578.
- Pajares, F., & Miller, (1995). Mathematics self efficacy and mathematics performance: The need for specificity for assessment. *Journal of Counseling Psychology*, 42(2), 190-198.
- Schmid, H. (2007). Children and youth at risk in Israel: Findings and recommendations to improve their well-being. *Children and Youth Services Review*, 29(8), 1114-1128.
- Stipek, D. J., Roberts, M. T., & Sanborn, M. E. (1984). Pre-school age children's performance expectations for themselves and another child as a function of the incentive value of success and the salience of past performance. *Child Development*, 55, 1983-1985.
- Tsamir, P., Tirosh, D., & Levenson, E. (2008a). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics*, 69(2), 81-95.
- Tsamir, P., Tirosh, D., & Levenson, E. (2008b). Exploring the relationship between justification and monitoring among kindergarten children. *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education*. January 28 – February 2, 2009, Lyon, France.
- Usher, E. L. (2009). Sources of middle school students' self-efficacy in mathematics: A qualitative investigation. *American Educational Research Journal*, 46(1), 275-314.
- Waldinger, R., Toth, S., & Gerber, A. (2001). Maltreatment and internal representations of relationships: Core relationship themes in the narratives of abused and neglected preschoolers. *Social Development*, 10(1), 41-58.
- van Hiele, P. M., & van Hiele, D. (1958). A method of initiation into geometry. In H. Freudenthal (Ed.), *Report on methods of initiation into geometry*. Groningen: Walters.

# PLAYING AND LEARNING IN EARLY MATHEMATICS EDUCATION – MODELLING A COMPLEX RELATIONSHIP

Stephanie Schuler

University of Education Freiburg

*Playing and learning – in early childhood often used synonymously, later on antithetically – are both key concepts in educational contexts. Considering theoretical models and empirical research on the relationship of playing and learning there remain no objections on learning while playing. But learning is not without requirements and depends on situational conditions. In the empirical study children are videotaped using selected materials and games in every day kindergarten contexts. The research results in descriptions of mathematical learning opportunities and their situational conditions.*

Keywords: play, games, arithmetic skills, early childhood education, video study

## INTRODUCTION

Concerning early mathematics education there are often brought up two important questions: *What* and *how* should children learn in the early years (3 to 6 years old)? Answers to the first question are mostly connected to fundamental mathematical ideas like quantitative and spatial thinking, patterns and relationships, contents that are not only relevant for a preschool curriculum. In distinction from formal school education there is consensus that young children acquire these skills primarily in a playful way (cf. Moyles, 2010, Pramling-Samuelsson & Fleer, 2009). Kindergarten, the place of early childhood education, is shaped by free play, open offers and informal learning opportunities. Thus *play* and the *relationship of playing and learning* have to be explored more closely when talking about mathematics for the early years.

*Play* is probably one of the most dubious notions. It withdraws defining, because it comprises very different phenomenon.<sup>70</sup> However there are many attempts to describe play with different characteristics. Depending on the *theoretical approach*, phenomenological (cf. Scheuerl, 1990), action-theoretical (cf. Oerter, 1993), cultural-historical (cf. Vygotskij 1933/80), these characteristics differ in quantity and label. Following an *empirical approach* play cannot be uniquely defined but just case-by-case explicated. Hence different types of play and the same concrete activity can be more or less play (cf. Einsiedler 1999). This conclusion matches with the synopsis of a literature review: “In essence, play could be viewed in its broadest

---

<sup>70</sup> Above there exists a linguistic and a cultural problem: E.g. in German “Spiel” includes “play”, “game”, “match” and “gambling”. Beliefs about play and learning vary in different countries (cf. Pramling-Samuelsson & Fleer, 2009).

sense as describing almost all the activities that young children engage in” (cf. Fleer, 2009, 2).

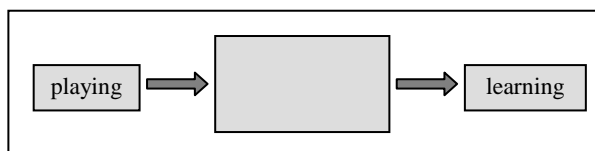
In a *historical perspective* the *role of play* during childhood is described manifoldly. Play is attributed an emotional (cf. Oerter, 1993), a social (cf. Parten, 1932) and a cognitive role (cf. Piaget, 1969). During infancy and toddlerhood play is associated with learning mainly in the context of training and rehearsal. In early childhood, the so called play age, play becomes in terms of Vygotskij (1933/80) not the predominant but the leading activity. It creates the zone of proximal development and allows the child to rise above itself while following challenging rules. Exponents of play pedagogy remark that the utilisation of play is not possible, because children create their own learning contexts in play broadly independent from adults’ learning intentions (cf. Flitner 2002).

Summarising play in early childhood is the motor of development and hence associated with learning. Consequently the underlying question seems not to be “Can children learn while playing?” but rather “How can learning while playing be modelled?” and “Can children learn mathematics while playing?”.

## THEORETICAL MODELS ON THE RELATIONSHIP OF PLAYING AND LEARNING IN CHILDHOOD

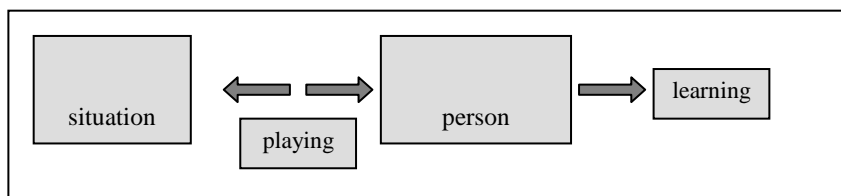
Einsiedler (1982, 5) compares two basic models referred to the relationship of playing and learning:

- Every play implies learning.



**Figure 1: Simple Input-Output-Model**

- Learning while playing is an interactive relationship between personal and situational conditions.



**Figure 2: Person-Situation-Model**

The former model is a historical position against the supposition that playing and learning are incompatible. The latter model points out that learning while playing is possible but depends on personal and situational conditions. Einsiedler (1989, 297) refers to this dependence as the problem of contingency of learning while playing.

Given constructivist assumptions it is disputable if the contingency of learning in play must be considered as a special problem. Even intentional learning, the increasingly leading activity during primary school, is not utterly controllable by teaching. Knowledge is constructed individually linked to former experiences and mental structures. Constructivist, social-constructivist but also cultural-historical theories highlight the organisation of *situational conditions of learning* (cf. Gerstenmaier & Mandl, 1995).

## **EMPIRICAL STATEMENTS ON THE RELATIONSHIP OF PLAYING AND LEARNING**

Mainly there are two different types of empirical studies in the context of playing and learning. First there are short or long term *intervention studies* which are interested in *learning effects* when inserting games (commercial or didactical) in kindergarten or first/second grade (cf. Ramani & Siegler 2008; Kamii 2004; Peters 1998; Floer & Schipper 1975). Second there are *observational studies* which are interested in the context of playing in kindergarten settings and in the *conditions for learning* in these settings (cf. van Oers 2010, Ginsburg 2009).

Studies of the first type with a control group design could show that playing with particularly chosen games (e.g. games with mathematical potential for number concept) are similar successful as teaching. In kindergarten games can foster mathematical abilities of disadvantaged children.

Observational studies draw the conclusion that children in kindergarten need adult guidance or more knowledgeable others in the context of play to promote their mathematical thinking. Ginsburg (2009, 413f) states that developing mathematical thinking depends on the environment, play and the teachable moment. Quite similar van Oers (2010, 34) summarises his studies: „The emergence of mathematical thinking in young children is a culturally guided process, wherein mathematical meaning can be assigned to actions of the child. These actions can be further developed through collaborative problem solving with more knowledgeable others in the context of activities that make sense to the children.”

In essence empirical research on playing and learning in early childhood underlines the central role of the educator and the quality of materials, games and activities.

## **RESEARCH QUESTIONS**

The study generally follows the question if early mathematics education can be organised by playing games. The goal is shortly outlined as follows<sup>71</sup>:

*How can learning opportunities develop in informal contexts like free play and open offers with educational materials and games to acquire the number concept?*

---

<sup>71</sup> Further information about research questions and research results can be reread in Schuler & Wittmann 2009.



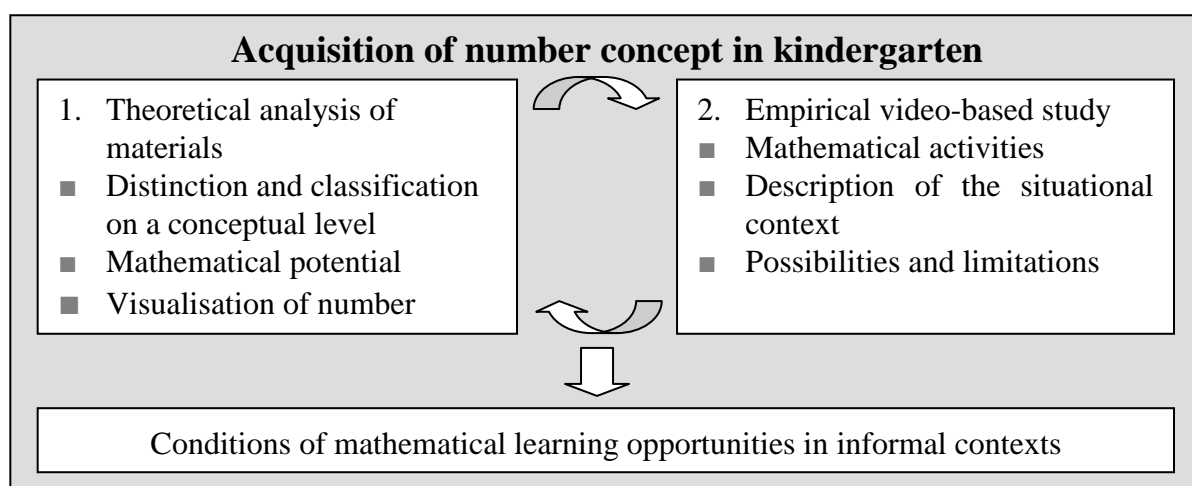
In detail we ask the following research questions:

1. What (theoretical) potential for children's construction of number concept do these materials and games have in principle?
2. Under what conditions can learning opportunities occur while playing these games or with these materials?

## RESEARCH METHODS AND METHODOLOGY

The study is rooted in the methodological frame of *Grounded Theory*, one type of theory generating qualitative research (cf. Strauss & Corbin, 1996). According to the research questions it is a two-phase design (cf. figure 3) that will lead to a (grounded) theory about the conditions for learning opportunities in informal contexts:

- The first phase is a *theoretical analysis of games and educational materials* (cf. Schuler & Wittmann, 2009).
- The second phase is an *empirical evaluation of selected, theoretically proved games and educational materials*.



**Figure 3: Two-phase research design**

According to the *methodology of Grounded Theory*, there are several phases of video propped data inquiry. Children are recorded while playing with selected materials and games during free play and open offers.

Basis of the data analysis are transcripts of video sequences. These are analysed by the following tools of *videography* to deduce and reduce the complexity of video data (cf. Dinkelaker & Herrle, 2008):

- The *segmental analysis* surveys the sequential process and the phases of the whole scene.
- The *configurational analysis* surveys the spatial organisation of the whole scene.
- The *sequential analysis* enables the reconstruction of the sequential meaning.

The following tools of *Grounded Theory* data analysis support the development of theoretical concepts and their relationship and result in a local, data grounded theory (cf. Strauss & Corbin, 1996; Kelle & Kluge, 1999; Mey & Mruck, 2007):

- *Theoretical sampling* means the ongoing change and interplay between action (data inquiry) and reflection (data analysis and theory construction) (cf. Mey & Mruck 2007, 13). Hypotheses are continually phrased and conduct further data inquiry. Video sequences are purposefully chosen for theoretical saturation.
- *Theoretical coding* is not only a description and classification of phenomena but the creation of theoretical concepts that explain the phenomena.
- *Comparative analysis* contrasts phenomena and contexts in order to develop theoretical concepts.

### Mathematical learning opportunities while playing games in kindergarten

For brevity of the article it is not possible to show the whole process of data analysis. Referring to one transcript I illustrate some of the tools mentioned above and coincidentally highlight learning opportunities that may occur while playing the Quips game (cf. figure 4) during the informal context of free play.

#### Game: Quips (Ravensburger)

*Player:* (1) 2 to 4

*Material:* 4 boards, 90 chips in 6 colours, 1 die with colours, 1 die with sets one to three

*Rule:* Each player gets a board and plays the two dice. The cast of dice determines the colour and the quantity of chips that each player can take and put into his board. Surplus chips have to be put back or given away to another player.

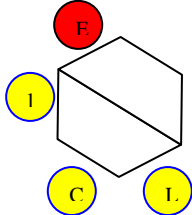
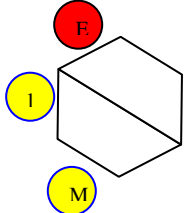


**Figure 4: Rules of the game “Quips”**

The segmental and configurational analysis (cf. table 1) survey the sequential process and the spatial organisation of the whole scene. The scene starts with the open offer of the Quips game during free play and closes with the end of the game. The scene is structured in four segments.


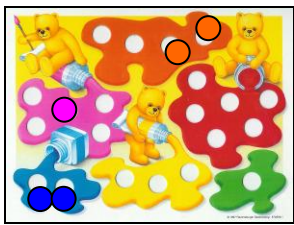
During free play the educator creates liability by proposing a game. Leni takes up the suggestion. Christian and Leon join the two. Their choice reveals social affordance that can be observed quite often in informal contexts. Segment 2 lasts approximately 15 minutes. Each of the players dices four times. Segment 3 lasts 12 minutes and each player dices fourteen times. Christian and Leon primarily want to put chips into the holes on the board (affordance of the material). They have difficulties following the rules: waiting to have a turn, recognizing and naming the colours, counting objects, comparing sets. They leave the game after four rounds. Moritz changes from the status of a spectator to the one of a player (affordance of the game). The game

accelerates and the three players finish the game (liability through rules and the play group).

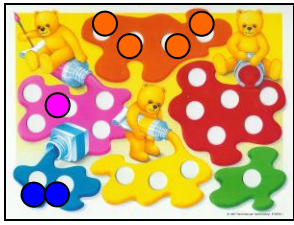
00:00	<b>1. Choice of game, formation of the play group</b> The Educator puts the game „Quips“ during free play on the table. Leni (2;11) takes a seat at the table and opens the box. The educator sits down as well. Christian (3;1) and Leon (3;0) enter the room and come to the table. They want to play too.	
03:04	<b>2. Game with four players (four dice rounds)</b> The rules of the game are clarified while playing the game. Moritz joins the game as a spectator.	
18:40	<b>3. Game with three players (fourteen dice rounds)</b> Leon and Christian leave the game and the room. Moritz (4;10) takes Christian's seat and game board.	
30:53	<b>4. End of game, putting away the game</b>	

**Table 1: Short segmental and configurational analysis**

The following transcript (cf. table 2) is located in segment 2. Its Leni's turn and her fourth cast of dice. Moritz follows the games as a spectator. Leon has already announced that he wants to leave the game. The right column contains figures of dice and game boards and above open codes.

Time	Speaker	Transcript	Materials and codes
16:35	E	LENI THROWS DICE How many have you got Leni?	 question(ing)
	Leni	One two. LENI LOOKS ON HER BOARD	 counting objects
	E	You need two and how many have you diced?	comment(ing) question(ing)
	Leni	Two. LENI POINTS TO THE DIE WITH COLOURS	subitizing
	E	How many are there? How many dots? EDUCATOR POINTS TO THE DIE WITH	question(ing)

### Working Group 13

		SETS	
	Moritz	Three. LOW VOICE	subitizing
	Leni	One two three. LENI POINTS SYNCHRONOUSLY ON THE DIE WITH SETS	counting objects
	E	Exactly, then you can take three orange chips. LENI TAKES TWO ORANGE CHIPS Have you already got three orange chips?	request(ing)
	Leni	No you must two LENI SHOWS THE EDUCATOR THE TWO ORANGE CHIPS IN HER HAND	arguing
	E	But-	arguing
	Leni	Because I have two. LENI PUTS THE TWO ORANGE CHIPS INTO THE HOLES ON HER BOARD	 reasoning
	E	But you have diced three, you can give away one.	arguing, requesting comparing exactly
17:30	Leni	Yes. LENI TAKES ONE ORANGE CHIP AND PUTS IT IN CHRISTIANS BOARD	following request

**Table 2: Transcript from segment 2 with codes**

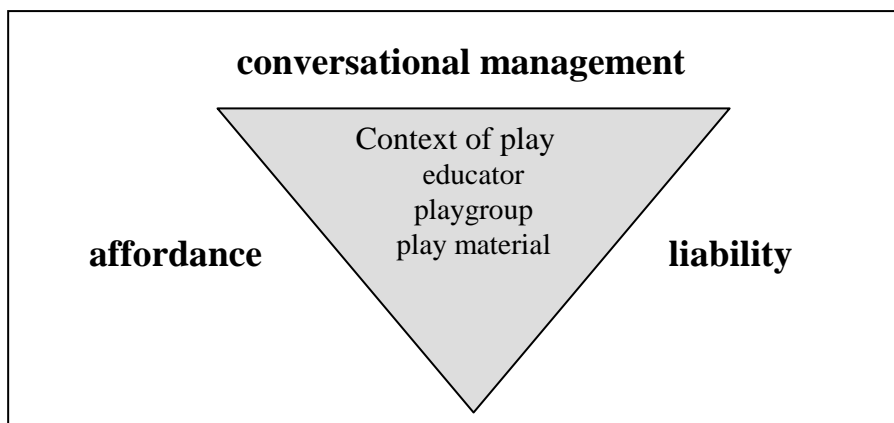
Mainly one mathematical activity can be observed in this transcript: counting objects. Leni counts holes on her board, dots on the die and she takes chips out of the box. The other observation is an argument between the educator and Leni that is based on a different reading of the situation. Leni reads her cast of dice from her board: she needs two more orange chips. She does include the die with sets probably just in a vague but not in an exact manner. Beyond that the educator considers the exact set on the die and compares it with all other boards. The rule to give away surplus chips requires this complex reading. After the argument Leni follows the educator's request that she can give away one. The scene offers *learning opportunities* related to the content idea *quantitative thinking* and to *process ideas*:

- counting objects, subitizing
- comparing sets and units of sets
- reasoning, arguing

The learning opportunities are embedded in the educator's questioning, commenting, requesting and arguing and are based on the game's rules.

## RESEARCH RESULTS AND FURTHER CONCLUSIONS

The focus of our research is on the situational conditions of learning (cf. figure 5). They are necessary requirements for learning but they do not guarantee it. The design of our study does not allow any statements on what single children have learned while playing the game, but just what opportunities to learn may occur (see previous section). Furthermore we can describe more closely situational conditions that support these learning opportunities (cf. Einsiedler 1982). So far we could emerge three main blocks of situational conditions: Affordance, liability and conversational management.



**Figure 5: Situational conditions of learning while playing**

The context of play is generally contingent on three variables: the educator, the playgroup and the play material. Materials and games introduced in kindergarten should offer a *mathematical potential* concerning central content ideas. This potential can be analysed in advance (cf. Schuler & Wittmann, 2009). Abridged affordance and liability encourage children to start, to maintain and to repeat a game. The conversational management is crucial for mathematical learning opportunities.

### Affordance and liability

Each *material* possesses an *intuitive affordance* (cf. Lewin following Heckhausen 2006, 31, 105ff) that supports the involvement with the material, but which does not necessarily lead to mathematical activities. Boards and chips of the Quips-game for example invite children to put the chips into the holes on the board or to pile them up. As we have seen in the transcript above, *rules* can offer mathematical activities beyond a material's intuitive affordance and thus *create liability*. Intuitive affordance of materials is replaced in games by (*the affordance of*) keeping the rules and winning *the game*. In the informal context of free play we can observe that children leave or change a material after a short time and others stay and play the game until the end. No rules or overcharging rules promote the abandoning of materials and games. Individually challenging rules lead to a longer lasting and repeated involvement. Thus the educator has to be sensitive to individual competencies and possible variation of rules. This seems to be a particular challenge in informal

contexts where children of different age and competence play together. *Liability* is also produced *by temporal, spatial and social standards*: e.g. announcing special periods for playing games, playing games at a separate table, presence of the educator. Not only materials are equipped with affordance but also social contexts. Playing with other children and playing with the educator exerts a *social affordance*.

### **Conversational management**

As we have seen in the transcript above the *conversational management* influences the mathematical potential that develops in interaction. Besides content ideas like comparing sets, counting objects there occur process ideas like arguing and reasoning. These general mathematical skills are not material inherent, cannot be analysed in advance and depend on the educator's conversational competence (cf. Schuler & Wittmann 2009).

Mathematical potential develops through the educator's comments on the game's course, through questions that stimulate explanations, reflections on actions and thoughts, and reasoning. She has to communicate individually challenging rules through stimuli, comments, questions and requests what requires a sensitivity for possibilities and variations in the games course.

### **Further conclusions**

Potentially suitable materials and games need a competent educator with regard to didactical and conversational aspects. The educator has to analyse, assess, choose and present materials and games. She has to discern the child's or children's individual approach to the material and has to consider the mathematically productive aspects. Through temporal and spatial organisation she creates liability and she utilises the case of social affordance to involve children in playing. Considering these situational conditions learning opportunities occur during play regarding central mathematical content and process ideas.

To organise early mathematics education in informal contexts like free play with potentially suitable materials and games educators need a mathematical, didactical, pedagogical and conversational competence that cannot be taken for granted. The qualification of present and future educators needs to take into account these aspects.

### **REFERENCES**

- Dinkelaker, J. & Herrle, M. (2009). *Erziehungswissenschaftliche Videographie. Eine Einführung*. Wiesbaden: Verlag für Sozialwissenschaften.
- Einsiedler, W. (1999). *Das Spiel der Kinder. Zur Pädagogik und Psychologie des Kinderspiels*. Bad Heilbrunn: Klinkhardt.
- Einsiedler, W. (1989). Zum Verhältnis von Lernen im Spiel und intentionalen Lehr-Lern-Prozessen. *Unterrichtswissenschaft*, 17, 291–308.

- Einsiedler, W. (1982). Neuere Befunde zum Verhältnis von Spielen und Lernen im Kindesalter. *Spielmittel*, 5, 2–9.
- Fleer, M. (2009): A Cultural-Historical Perspective on Play: Play as a Leading Activity Across Cultural Communities. In I. Pramling-Samuelsson & M. Fleer (Eds.) *Play and Learning in Early Childhood Settings. International Perspectives* (pp.1–17). Springer.
- Flitner, A. (2002). *Spielen - Lernen. Praxis und Deutung des Kinderspiels*. Weinheim: Beltz.
- Floer, J. & Schipper, W. (1975). Kann man spielend lernen? Eine Untersuchung mit Vor- und Grundschulkindern zur Entwicklung des Zahlverständnisses. *Sachunterricht und Mathematik in der Grundschule*, 3, 241–252.
- Gerstenmaier, J. & Mandl, H. (1995). Wissenserwerb unter konstruktivistischer Perspektive. *Zeitschrift für Pädagogik*, 6, 867–888.
- Ginsburg, H. P. (2009). Early Mathematics Education and How to Do It. In O. A. Barbarin, & B. H. Wasik (Eds.) *Handbook of Child Development and Early Education. Research to Practice* (pp. 403–427). New York, London: The Guilford Press.
- Heckhausen, J. & Heckhausen, H. (Eds.) (<sup>3</sup>2006). *Motivation und Handeln*. Heidelberg: Springer.
- Kamii, C. (2004). *Young Children Continue To Reinvent Arithmetic. 2<sup>nd</sup> Grade. Implications of Piaget's Theory*. New York, Teachers College Press.
- Kelle, U. & Kluge, S. (1999). *Vom Einzelfall zum Typus*. Opladen: Leske + Budrich.
- Mey, G. & Mruck, K. (Eds.) (2007). *Grounded Theory Reader*. Köln: Zentrum für Historische Sozialforschung.
- Moyles, J. (Ed.) (2010). *The Excellence of Play*. Buckingham: Open University Press.
- Oerter, R. (1993). *Psychologie des Spiels: ein handlungstheoretischer Ansatz*. München: Quintessenz.
- Parten, M. (1932). Social participation among preschool children. *Journal of Abnormal & Social Psychology*, 27, 243–269.
- Peters, S. (1999). Playing games and learning mathematics: The results of two intervention studies. *International Journal of Early Years Education* 1, 49–58.
- Piaget, J. (1969). *Nachahmung, Spiel und Traum*. Stuttgart: Klett.
- Pramling-Samuelsson, I. & Fleer, M. (Eds.) (2009). *Play and Learning in Early Childhood Settings. International Perspectives*. Springer.

### Working Group 13

- Ramani, G. B. & Siegler, R. S. (2008). Promoting broad and stable improvements in low-income children's numerical knowledge through playing number board games. *Child development* 2, 375–394.
- Scheuerl, H. (1990). *Das Spiel. Untersuchungen über sein Wesen, seine pädagogischen Möglichkeiten und Grenzen*. Bd. 1 Weinheim, Basel: Beltz.
- Schuler, S. & Wittmann, G. (2009). How can games contribute to early mathematics education? – A video-based study. In *Proceedings of CERME 6*, Lyon. 2009. <http://www.inrp.fr/publications/edition-electronique/cerme6/wg14-12-schuler.pdf>
- Strauss, A. & Corbin, J. (1996). *Grounded Theory. Grundlagen Qualitativer Sozialforschung*. Weinheim: Beltz.
- Vygotskij, L. S. (1933/80). Das Spiel und seine Bedeutung in der psychischen Entwicklung des Kindes. In D. Elkonin *Psychologie des Spiels. Studien zur kritischen Psychologie*. (pp. 441–465) Köln: Pahl-Rugenstein Verlag.
- Van Oers, B. (2010). Emergent mathematical thinking in the context of play. *Educational Studies in Mathematics* 1, 23-37.



# INTRODUCTION TO THE PAPERS OF WG14: UNIVERSITY MATHEMATICS EDUCATION

Elena Nardi<sup>1</sup>

Alejandro S. González-Martin<sup>2</sup>, Ghislaine Gueudet<sup>3</sup>,

Paola Iannone<sup>1</sup>, Carl Winsløw<sup>4</sup>

<sup>1</sup>University of East Anglia (UK), <sup>2</sup>University of Montreal (Canada), <sup>3</sup>University of Rennes-1 (France), <sup>4</sup>University of Copenhagen (Denmark)

Research on university level mathematics education is a relatively young field, which embraces an increasingly wider range of *theoretical approaches* (e.g. cognitive/developmental, socio-cultural, anthropological and discursive) and *methods/methodologies* (e.g. quantitative, qualitative and narrative). Variation also characterises research in this area with regard to at least two further issues: the *role of the participants*, students and university teachers, in the research – from ‘just’ subjects of the research to fully-fledged co-researchers – and the *degree of intervention* involved in the research – from external, non-interventionist research, to developmental/action research in which researchers identify problems and devise, implement and evaluate reforms of practice (Artigue et al, 2007).

2011 marks the 20<sup>th</sup> anniversary of the publication of *Advanced Mathematical Thinking* (Tall, 1991), a volume that is often heralded as a first signal of the emergence of this new area of research. A few years later a second signal was given by the 1998 ICMI study that resulted in *The teaching and learning of mathematics at university level* (Holton, 2001). In the meantime Advanced Mathematical Thinking (AMT) groups ran both in previous CERME and PME conferences; sessions exclusively on university mathematics education are part of the EMF conferences since 2006 ; the RUME, UMT and Delta conferences emerged in the USA, the UK and South Africa respectively; the International Conferences on the Teaching of Mathematics at University Level were launched in 1998; etc.

WG14 (*University Mathematics Education*, hereafter UME) emerged out of the above developments and out of the realisation that this is a distinct area of mathematics education research. Its distinctiveness can be attributed to several characteristics. Firstly, the classic distinction between ‘teacher’ and ‘researcher’ does not always apply in UME as researchers in mathematics education in this area are often university-level teachers of mathematics themselves. In particular there is a growing group of mathematicians specializing in research on mathematics education at university level, where advanced mathematical knowledge and experience is really an asset (if not a necessity). Secondly, mathematics education theories and research methods find new uses, and adaptations, at the university level. These adaptations are often quite radical as the post-compulsory educational context is different in many ways – the voluntary presence of students, the important role of mathematics as a

service subject, the predominance of lecturing to large numbers of students, the absence of national programmes for university education, to mention but a few. In this sense UME is a distinct area of mathematics education research, not merely a mirror of mathematics education research at a more advanced educational level. Finally, in recent years, research in this area has been growing in different parts of the world. WG14 is one opportunity to collate evidence of this growing research activity from Europe and beyond.

The WG14 Call for Papers invited contributions from as wide a range of research topics as possible. These included: concept formation, mathematical reasoning and proof in university mathematics; teaching at university level (including the perspectives of university mathematicians); ICT in the teaching and learning of university mathematics (including curriculum design); socio-cultural, anthropological and discursive approaches in university mathematics education; affective and social aspects of the learning and teaching of university mathematics; the transition from secondary to tertiary mathematics; novel approaches to teaching mainstream (e.g. calculus and linear algebra) as well as more advanced topics, both to students of mathematics and other areas (e.g. engineering, sciences, finance); theoretical approaches to the study of teaching and learning mathematics at university (including a focus on specific approaches and on contrasting or combining approaches).

This report draws on the presentations, reactions-to and discussions of the 21 accepted papers that met these terms. The number and quality of these papers marks the recent surge in the quality of research outputs and a move away from the earlier days of perhaps more naïve, less rigorous research in this area – brought about partly by the university sector's increasing urge to adapt teaching to changing student cohorts and by a growing, and wider, tendency towards an in-depth probing into traditional teaching practices in higher education.

Across the WG14 discussions certain themes and questions emerged as crucial. These included: exploring whether UME needs to generate new theories or adapt already existing ones; attending to issues of both theory and practice; acknowledging that research on teaching and learning in higher education develops also outside mathematics education, and benefiting from these developments; working towards the generation of new theories while valuing already accumulated knowledge in the field; etc. Colleagues observed that, beyond staple references to classic constructs from the *AMT* era (such as concept image – concept definition; APOS theory, process – object duality etc.), several works presented in WG14 employ (often in tandem with the above) approaches such as the Anthropological Theory of Didactics (ATD: Chevallard, 1985), and discursive approaches (e.g. Sfard, 2008).

Generally speaking papers seem to be classified into those with a focus on the teaching and learning of particular mathematical topics (calculus continues to attract more attention than other topics) or on wider, cross-topical issues such as the

transition to university mathematics, use of IT, language, motivation, teacher knowledge and development, curricular, pedagogical and institutional issues, etc.. Furthermore an area of growth is of studies that examine the different role of mathematics in courses towards a mathematics degree, courses for pre-service teachers, as a 'service' subject (physics, biology, economics etc.). While a substantial number of papers remains in the increasingly well-trodden area of students' perceptions of specific mathematical concepts (again calculus prevails in these), a focus on university teachers and teaching is also emerging, if often a little timidly, and diplomatically, resulting in descriptive, openly non-judgemental studies. In conjunction with those studies a genre of collaborative studies, with mathematicians engaged as co-researchers, also seems to be on the rise. In the nutshell descriptions of the WG14 papers that follow the order of presentation is loosely structured around some of the themes mentioned above.

*Xhonneux & Henry* is one of the papers that employs the ATD framework to distinguish between *mathematical* and *didactic* praxeologies in the context of teaching and learning of Lagrange's Theorem in calculus courses to mathematics and economics students. *Gyöngyösi, Solovej & Winslow* is another: in it a part of a transitional course in Analysis was taught with a combination of Maple and paper-based techniques and resulted in mixed reception and performance by students. A third is *Barquero, Bosch & Gascón*: from its analyses 'applicationism' emerges as the prevailing epistemology of mathematics in science departments that potentially hinders the teaching of mathematical modelling to science students.

Another set of approaches that was employed by a number of WG14 papers were discursive. *Jaworski & Matthews* employed such approaches to trace university mathematicians' pedagogical discourse and suggest links of this discourse to their ontological and epistemological perspectives. *Biza & Giraldo* described how computational inscriptions – in this case differentiability – have potentialities and limitations that can be helpful in students' exploration of newly introduced mathematical concepts. Three papers made use of Sfard's commognitive framework. *Viirman* employed this framework to trace the variation in the pedagogical discourses of mathematics lecturers in the course of their introducing the concept of function. *Stadler* described students' experience of the transition from school to university mathematics as an often perplexing re-visiting of content and ways of working that seem simultaneously familiar and novel (for example in the case of solving equations). *Nardi* outlined interviewed mathematicians' perspectives on their newly arriving students' verbalisation skills; and, observed that discourse on verbalisation in mathematics tends to be risk-averse and not as explicit in teaching as necessary. Several papers focused on the transition from school to university mathematics (including *Gyöngyösi, Solovej & Winslow* and *Stadler* mentioned above). *Biehler, Fischer, Hochmuth & Wassong* proposed that blending traditional course attendance with systematic e-learning study can facilitate the bridging of

school and university mathematics. *Faulkner, Hannigan & Gill* noted the shifting profile of students who take service mathematics courses (in the context of an Irish institution): many more are diagnosed as at risk, fewer have an advanced mathematics secondary qualification and the percentage of non-standard (e.g. mature) students has grown. *Zimmermann, Bescherer & Spannagel* described MaSE-T, a mathematics self-efficacy test designed to measure the impact that self-efficacy perceptions have on choice of studies. *Vandebrouck* noted that in the transition from school to university, mathematics students need to reconceptualise the concept of function in terms of its multiple representations and its process-object duality. Finally, *De Vleeschouwer & Gueudet* observed that students can learn to appreciate the duality in linear forms (described here in micro-macro terms) if given an appropriate set of tasks that require them to engage with these concepts at both levels.

Many of the papers mentioned above had a clear focus on a specific mathematical concept or issue. In addition to these, *Iannone & Inglis* discussed a range of weaknesses in Year 1 mathematics students' production of deductive arguments (rather than in the oft-reported perception that a deductive argument was expected of them). *Juter* reported the highly individual and not easily classifiable character of pre-service secondary mathematics teachers' concept images of elementary Calculus concepts. And *Souto-Rubio & Gómez-Chacón* mapped out students' difficulty with developing visualisation skills in the context of the Riemann integral.

Some papers focused on particular curricular and pedagogical aspects of university mathematics. *Agathokleous* argued how teaching Abstract Algebra to pre-service primary teachers can facilitate students' appreciation for the connectedness across mathematical domains. *Jukic & Dahl*, through data collected in the Croatian and Danish context, aimed to illustrate that students taught in different styles are likely to perform differently. *Bergsten* presented evidence that students tend to find lectures useful and attractive, despite their bad press in some education quarters.

Finally, a few papers addressed theoretical issues directly. *Barton* outlined efforts to combine the three-fold activity of research, development and theory building into LATUM, a model for learning and teaching university mathematics that is proposed as a model for designing alternative university mathematics delivery. And, *Pettersson* proposed 'threshold concepts', a theoretical construct from the general education literature, as a means for gaining insight into student learning and engaging teachers in pedagogical discussion.

## REFERENCES

- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics Thinking and Learning at Post-secondary Level. In F. K. Lester (Ed.), *The Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1011-1049). Charlotte: IAP.
- Chevallard, Y. (1985). *La transposition didactique: du savoir savant au savoir*

#### Working Group 14

*enseigné*. Grenoble: La Pensée Sauvage Éditions.

Holton, D. A. (Ed.). (2001). *The teaching and learning of mathematics at university level: An ICMI study*. Dordrecht,: Kluwer Academic Publishers.

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourse, and mathematizing*. New York: Cambridge University Press.

Tall, D. (Ed.). (1991). *Advanced mathematical thinking*. Dordrecht: Kluwer Academic Publishers.

# WHY ABSTRACT ALGEBRA FOR PRE-SERVICE PRIMARY SCHOOL TEACHERS

**Eleni Agathocleous**

Cyprus Ministry of Education and Culture

*There has been a rapid increase in the volume of research papers focusing on mathematics teacher education and in particular on the characteristics that should describe the desired mathematical knowledge for teachers. In this paper, it is suggested through theoretical discussion on the character of Abstract Algebra and via Peirce's semiotics, that a one-semester course in Abstract Algebra, specially designed for pre-service primary school teachers, can enhance the development of the basic, multifaceted characteristic of connectedness. Some concrete examples are given in the end of the paper, taken from in-class observations and reflections on a similar course taught by the author, for two consecutive semesters.*

Key words: Teachers' mathematical knowledge, Connectedness, Abstract Algebra, Semiotics.

## INTRODUCTION

In this paper I argue that a course in basic notions of Abstract Algebra, specially designed for pre-service primary school teachers, can help teachers regroup and enrich their existing mathematical knowledge, as well as achieve connectedness across various mathematical domains as well as across time, as a mathematical idea develops and extends. These basic notions of Abstract Algebra include the notions of group, ring and field, accompanied by a very basic introduction to functions, modular arithmetic and complex numbers. Unlike a standard introductory course in Abstract Algebra intended for mathematicians, the aim of this course would not be the teaching of group theory, quotient rings or maximal ideals for example. But instead, the introduction of groups, rings and fields as a means to put existing mathematical knowledge (such as the number systems  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ) or new mathematical knowledge (such as the number systems  $\mathbb{C}$  and  $\mathbb{Z}_n$ ) into new forms, in order to reveal more information on the structure of these mathematical systems as well as present new ways of understanding old knowledge. The theoretical discussion in this paper focuses on the unifying character of the basic notions of group, ring and field, and I employ Peirce's Semiotics to show how the use of these notions is on the highest level of the hierarchy, helping the learner make the desired connections suggested by the literature on teachers' mathematical knowledge.

## TEACHERS' MATHEMATICAL KNOWLEDGE

What mathematical knowledge does it take to teach primary mathematics well? Is it a matter of quantity? Of depth? Or is it a matter of different quality? There has been a rapid increase in the volume of research papers focusing on the relatively new area of

mathematics teacher education (Adler, Ball, Krainer, Lin, & Novotna, 2005) and investigating the “unsolved” (Ball, Lubienski, & Mewborn, 2001) problem of teachers’ mathematical knowledge (see, e.g., Davis & Simmt, 2006; Ball & Bass, 2000; 2003; Ball *et al.*, 2001). It is a shared belief that if a teacher does not know mathematics, then he/she cannot teach mathematics (Ball & Bass, 2003; Murphy, 2006). However, one of the first attempts to investigate the relevance between teachers’ knowledge and students’ achievement showed that the number of advanced mathematics courses taken by teachers, was not relevant to students’ achievement (Begle, 1979). Even though Begle’s findings as well as views concerning research in mathematics education, were later doubted by critics (see, e.g., Elerton & Clements, 1998; Howson, 1980), many others since then have argued in favour of this view, i.e. that advanced mathematics courses offered by mathematicians for mathematicians might be of no value to teachers and can even have negative effects on their pedagogical approaches (Cooney & Wiegel, 2003; Davis & Simmt, 2006; Murphy, 2006). Relevant studies by Askew, Brown, Rhodes, Wiliam, and Johnson (1997), Ball and Bass (2000; 2003), Ball *et al.* (2001) and Davis and Simmt (2006), point out a need for different mathematics for teachers. Davis and Simmt (2006) suggest that teachers’ mathematical knowledge is not a matter of “more of” or “beyond”, but it is of a different quality. This ‘new’ mathematical knowledge should foster teachers’ understanding of mathematics in a broader and unifying sense, so as to enable them to make connections across mathematical domains and help students build a coherent mathematical knowledge (Askew *et al.*, 1997; Ball & Bass, 2000; 2003; Ball *et al.*, 2001). Furthermore, teachers are to anticipate the way mathematical ideas change and grow; hence, they should also be able to make connections across time, as mathematical ideas develop and extend (Ball & Bass, 2003).

Even though there is no general agreement as to the level or the type of mathematical knowledge teachers should have, we see from the above discussion that the multifaceted characteristic of connectedness - across various mathematical domains, conceptual aspects of one same notion, or even across time as a mathematical idea develops - is widely present in the literature. Another example is the study of Ma (1999), in which she compares Chinese and U.S. elementary teachers’ mathematical knowledge. Ma uses the term “profound understanding of fundamental mathematics” to describe teachers’ mathematical knowledge, which is later described by Ball and Bass (2003) as “a kind of connected ... and longitudinally coherent knowledge of core mathematical ideas” (p.4). Ball and Bass (2000) explain that “depth” for Ma is the connecting of ideas with the larger and more powerful ideas of the domain, “breadth” is related to the connecting of ideas of similar conceptual power and “thoroughness” is what groups everything together into a coherent whole.

The need for a connected mathematical knowledge for teachers is also suggested by studies that focus on the teaching and learning of specific mathematical ideas. One such example is a study by Lamon (1996) of children’s partitioning strategies and the

development of their unitizing process. In her section *Implications for Instruction* she writes

The many personalities of subconstructs of rational number that children must conceptually coordinate may all be understood as compositions and recompositions of units. Because the rational numbers are a quotient field, partitioning itself is an operation that plays a role in generating each of those subconstructs ... Students need extensive presymbolic experiences involving these conceptual and graphical mechanisms in order to develop a flexible concept of unit and a firm foundation for quantification, to develop the language and imagery needed for multiplicative reasoning, and to conceptually coordinate the additive and multiplicative aspects of rational numbers. (p.192)

All of the above concerning the multiple faces of a rational number, the multiplicative versus the additive structure, the development of flexible concepts, etc. are characteristics of the students' desired knowledge. One can only begin to understand how complex, rich, versatile and flexible teachers' content knowledge should be, so as to be able to correspond to such high and demanding expectations.

## **WHY ABSTRACT ALGEBRA**

Abstract Algebra is an advanced mathematics course usually meant for mathematics students, and it is no surprise that the literature concerning the pedagogy of Abstract Algebra, focuses on the teaching of the course to university students majoring in mathematics, or to high-school mathematics teachers (see, e.g., Burn, 1996; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Leron & Dubinsky, 1995; Simpson & Stehlíková, 2006). The subject is hard enough, even when taught to mathematics majors and we have the, apparently famous, quote: "The teaching of Abstract Algebra is a disaster, and this remains true almost independently of the quality of the lectures" (Leron & Dubinsky, 1995, p. 227; Simpson & Stehlíková, 2006, p.347). So why should one attempt to teach notions of Abstract Algebra to pre-service primary school teachers?

## **History, Character and Basic Definitions**

Until the end of the eighteenth century, Algebra was concerned with mainly the study of polynomial equations and was regarded as a generalization of Arithmetic. The nineteenth century was for Algebra the period of transition, of complete reform. Some of the basic characteristics of the mathematics of the nineteenth century were a turn toward rigor and the need for axiomatization, and the emergence of abstraction. Following Geometry, Algebra became another branch of Mathematics that mathematicians tried to axiomatize. Furthermore, the attention was now turned to the study of mathematical objects, such as vectors, matrices, transformations, etc, and various operations acting on them, which expanded the role of Algebra to the study of form and structure, giving birth to Abstract Algebra. It is in this century that we have the explicit formulation of the fundamental concepts of a group, ring and field, which grew out of the work of many mathematicians such as Galois (1811-1832),



Cayley (1821-1895) and Dedekind (1831-1916). These three concepts constitute the focal point of the Abstract Algebra course that we propose, since they have by definition a unifying nature. In the preface of his book, Herstein (1999) mentions that one aspect of the role of Abstract Algebra is "... that of a unifying link between disparate parts of mathematics ..." (Herstein, p.xi); and Robert (1987) refers to concepts of the theory of groups as "unifying and generalizing concepts" (in Dorier, 1995, p.175). But what is it exactly that gives the concept of a group such unifying powers? Let us first recall the definition:

**Definition1:** A non-empty set  $G$  is said to be a group if we define an operation  $*$  in  $G$  such that: (1) If  $a, b \in G$  then  $a*b \in G$ , and we say that  $G$  is closed under  $*$ , (2) Given  $a, b, c \in G$  then  $a*(b*c) = (a*b)*c$ , and we say that the associative law holds in  $G$ , (3) There exists a special element  $e \in G$  such that  $\forall a \in G, a*e = e*a = a$ . This element  $e$  is called the identity or unit element of  $G$ , (4)  $\forall a \in G$  there exists an element  $b \in G$  such that  $a*b = b*a = e$ . We write this element  $b$  as  $a^{-1}$  and we call it the inverse of  $a$  in  $G$ . If in addition, we have that (5)  $\forall a, b \in G, a*b = b*a$ , i.e. the commutative law holds, then we say that  $G$  is an abelian group.

From the definition alone, one can recognize the shift of attention from specific objects and operations, to the interrelationships between objects produced under the action of some operation. Seeing for example that  $(\mathbb{Z}, +)$  and  $(\mathbb{Q}^\times, \cdot)$  are both groups, implies, among other things, the realization that  $0$  in the first example and  $1$  in the second play exactly the same role, and therefore the 'identification' of these two seemingly unrelated objects. Similarly,  $z$  and  $-z$  connect together in the same way that  $q$  and  $q^{-1}$  in the second example do, where  $z$  is any integer and  $q$  any rational. Another very useful example involves the even and odd numbers and the realization that  $(2\mathbb{Z}, +)$  forms a group whereas  $(2\mathbb{Z}+1, +)$  does not since closure fails to hold in the second case. Proving this requires seeing the even and odd numbers in their general form as  $2k$  and  $2k+1$  respectively, where  $k$  any integer. And this in turn, requires the learning as well as the use of the definitions of these two kinds of numbers, which is a type of knowledge that is expected from the teacher (see Ball & Bass, 2000, the Introduction).

**Definition2:** A non-empty set  $R$  is said to be a ring if there are two operations  $+$  and  $\cdot$  such that: (1)  $(R, +)$  is an abelian group, (2) if  $a, b \in R$  then  $a \cdot b \in R$ , (3)  $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ , for  $a, b, c \in R$ , (4) (i)  $a \cdot (b+c) = a \cdot b + a \cdot c$  and (ii)  $(b+c) \cdot a = b \cdot a + c \cdot a$ . If in addition we have that (5)  $\forall a, b \in R, a \cdot b = b \cdot a$ , then we say that  $R$  is a commutative ring and the axiom 4(ii) is unnecessary.

The axioms for a ring look familiar since they are a generalization of what happens to the integers. The object  $(\mathbb{Z}, +, \cdot)$  is indeed a commutative ring with unit. One can see that what rings are 'missing' from behaving like the rationals or the reals are the multiplicative inverses and of course the need for the multiplicative identity element,

equivalently called unit, which we denote by  $1$ . What we are about to define as a field, has exactly these two extra axioms. Therefore

**Definition3:** A non-empty set  $F$  is said to be a field if there are two operations  $+$  and  $\cdot$  such that: (1)  $(F, +, \cdot)$  is a commutative ring with unit, (2) for every non-zero  $a \in F$  there is an element  $a^{-1} \in F$  such that  $a \cdot a^{-1} = 1$ .

And so a teacher can now say that  $(Q, +, \cdot)$  is a field. What have we exactly achieved by that? This is the type of questions that will be answered in the next two sections.

### **Groups, Rings, Fields - A Semiotic Perspective**

In this section, through Peirce's semiotic hierarchy I expect to show how the learning and the use of the notions of a group, ring and field lie on the highest level of the hierarchy, thus leading to the making of the desired connections suggested by research on teachers' mathematical knowledge.

Semiotics is the study of signs, especially as elements of a system, and as such, semiotics serves as a natural theoretical framework for the learning and teaching of mathematics, since mathematics requires certain sign systems "to keep a record of and code the knowledge" (Steinbring, 2006) and mathematization "means representing problems or facts by means of symbols, indices and relational representations" (Hoffmann, 2006, p.279).

For Peirce, a sign is anything that "stands for something (its object)" in such a way as to generate meaning (called its interpretant) (Otte, 2006, pg.23). The signs are of three types. On the first level we have the icons, which are, as the word says, icons – pictures – likenesses of what they represent. A number is an example of an icon; i.e. " $1$ " stands for the number or the idea of "one". If now we choose to write " $m$ " to imply any integer, then " $m$ " becomes a name, an index, of something existing. An index is on the second level of the hierarchy. Other examples of indices are the ideas of "unit", "additive inverse" and "identity element", when these are considered as names or categories for the appropriate icons. One way to think of what an index does is that it organizes icons "in higher order relationships" (Davis & McGowen, 2001, p.10). However, if we make the realizations (i.e. prove) that for every integer  $m$ ,  $-m$  is the "additive inverse" of  $m$ ,  $1 \cdot m = m$ ,  $0 + m = m$ , etc. then we are on the third level of the hierarchy, the symbolic level, as the aforesaid realizations are of a law-like nature and they are generalizations regarding the behaviour of the integers. In Peirce's words, the symbol, which is on the third and highest level of the hierarchy, refers to its object "by virtue of law" (in Radford, 2000, p.252). But we see that it is through an index that we can say that the integers have a certain quality. It is the index "additive inverse" for example, that allows us to say that all integers have an additive inverse. In the words of Peirce again

A symbol is a conventional sign which being attached to an object signifies that object has certain characters. But a symbol, in itself, is a mere dream; it does not

show what it is talking about. It needs to be connected with its object. For that purpose, an index is indispensable. No other kind of sign will answer the purpose (in Otte, 2006, p.30).

If we now consider all the indices that appear in the definition of a ring and the way these organize the icons of the integers, such as the numbers themselves, the signs “=”, “+”, etc., then we have, according to Peirce, the connection between the symbol, if I may denote it by “the ring  $(Z, +, \cdot)$ ”, and the integers. And as we attach to the integers the symbol “the ring  $(Z, +, \cdot)$ ” we automatically ‘see’ in the integers all those properties that we all learn from our early years until university. Deacon (1977) refers to the shift to the symbolic level of the hierarchy as

... a way of off-loading redundant details from working memory, by recognizing a higher order regularity in the mess of associations, a trick that can accomplish the same task without having to hold all details in mind. (in Davis & McGowen, 2001, p.10)

The notions of group, ring and field behave exactly like that: they allow which ever object we are investigating, i.e. the integers, the complex numbers, the  $2 \times 2$  invertible matrices, to be “apprehended, pretty much all at once” (Davis & McGowen, 2001, p.10), i.e. through the symbol “the ring  $(Z, +, \cdot)$ ”, “the group  $(Z, +)$ ”, “the field  $(C, +, \cdot)$ ”, “the group of  $2 \times 2$  invertible matrices  $GL(2, R)$ ”, etc., without having to hold all details in mind. As such, the notions of group, ring and field are systems of symbols and they form a part of a connected symbolic system, the symbolic associations of which are being understood from the “myriad connections” between the indexes (Davis & McGowen, 2001, p.10) organized by this system.

From the above discussion we see that by moving up on the hierarchy, we achieve a shift in understanding: a shift from the specific and isolated, i.e. the number 1, 0, 2, -2,  $\frac{1}{2}$ , to ideas such as “unit”, “additive or multiplicative inverse”, “the ring  $(Z, +, \cdot)$ ”, etc., which connect together seemingly different parts of our knowledge, putting them into a coherent whole.

## IN-CLASS OBSERVATIONS AND REFLECTIONS

This last section aims to provide the reader with some concrete examples from, and suggestions, if you will, regarding the teaching of basic notions of Abstract Algebra to pre-service primary school teachers.

While being a Lecturer in Mathematics Education at a university in Cyprus, I taught a course for pre-service primary school teachers, called General Topics in Mathematics. This was the second and last part of a series of mathematics courses for primary school teachers, and it was optional. Only the first course was obligatory. I taught this course for two consecutive semesters. The spring semester course ran over a period of thirteen weeks and my students and I met once a week, for three hours. The summer semester course was seven weeks long and we met twice a week, for three hours each time. I kept the syllabus for both courses the same. I chose a

variety of topics from traditional Algebra, Logic, Set Theory, etc. and during the last three lectures I gave a brief introduction to the complex numbers and the notions of group, ring and field, including examples from areas they had already seen before, as well as some basic examples from modular arithmetic. In class observations and discussions revealed similar reactions and results from both classes.

The following are some examples from in-class observations as well as from the students' answers in the final exam: (1) The students seemed to have appreciated more the importance of the distributive law, as they saw how it connected together the abelian groups  $(Q, +)$  and  $(Q^\times, \cdot)$  for example, in order to give the well-known field  $(Q, +, \cdot)$ . This is a concrete example of the symbolic (in Peirce's language) character of the notions of a group, ring and field in the sense that it shows how the groups  $(Q, +)$  and  $(Q^\times, \cdot)$  and the field  $(Q, +, \cdot)$  are part of a symbolic system and their association is being understood from the connection between the indexes, via the distributive law (see also the quotes from Davis & McGowen, 2001, in p.6 above). (2) It appeared that the notion of closure was being brought to their attention for the first time. They did not understand the necessity of this axiom in the definition of a group. Their reactions were pretty much along the line "if you add a number you will get a number and that is pretty much obvious, so why do we need this statement?" I asked them: what happens if you add 0 and 4? -2 and 100? 1000 and 44? So we concluded that when you add two even numbers you seem to get an even number. What followed was the realization that this does not happen with the odd numbers. The next step was to prove that  $\text{even} + \text{even} = \text{even}$  and  $\text{odd} + \text{odd} \neq \text{odd}$ . The notion of "closure" in this example, intended to force the students to focus their attention on specific sets of numbers and to help them see a 'different' distinction between them. 'Different' distinction in the sense that when examining whether even and odd numbers are groups, these sets stop being just the sets of icons (the lowest level of Peirce's hierarchy):  $\{0, 2, 4, \dots\}$  and  $\{1, 3, 5, \dots\}$  but instead, they become systems of connections in the highest level of the hierarchy, since statements of the form " $(2Z, +)$  is closed" and " $(2Z+1, +)$  is not", lie on the symbolic level. (3) One of the earlier topics that were presented in this course was the notion of function. The concept of a function came up again in one of the examples, where they had to prove that the following set  $G$  of functions is a group:  $G = \{T_{a,b} : R \rightarrow R \mid T_{a,b}(r) = ar + b, a, b \in R, a \neq 0\}$ . At first, there was an immense reaction (expected) from the students about the impossibility of this exercise due to its highly symbolic (not symbolic in Peirce's language) character, or in their words, to the appearance of "too many letters". When I explained that the function  $T_{a,b}$  is merely a generalization of functions like the ones they saw before, for example like the function  $f(x) = 3x+2$ , etc., then the complaints seized. It was very interesting and rewarding to see their surprised faces when they realized that (a) the identity element is the function  $T_{1,0}$  and that this object behaves in exactly the same manner as  $0$  in  $(Z, +)$  for example, or  $1$  in  $(Q^\times, \cdot)$ , (b) even if this problem is so different from everything they saw before,

the procedure to find the inverse element of  $G$  was exactly the same as the steps they followed earlier in the course, in order to find the inverse of a specific function. In other words, they were very surprised to see that the same method could work in such different, for them, situations. As was already discussed in the section above, recognizing different kinds of inverses, i.e. 1, 0 and  $T_{1,0}$  in this example, imply the making of connections that lie on the highest level of Peirce's hierarchy. Furthermore, these are examples of connections that help teachers build connections across different mathematical domains, since in this case we were comparing number systems with groups of functions. The students presented difficulty however, which was also shown in the final exam, in showing that the inverse  $T_{a,b}^{-1}$  is still a member of  $G$ ; i.e. even though almost everyone showed that  $T_{a,b}^{-1}(r) = \frac{1}{a}r - \frac{b}{a}$ , not everyone could see why we are not done until we write  $T_{a,b}^{-1} = T_{1/a, -b/a}$ ,  $1/a \neq 0$ . (4) Difficulty for students also presented in the case of modular arithmetic. Since there was not enough time to really digest this new kind of “numbers”, even though they were pretty quickly able to find the additive and multiplicative inverses of  $Z_5$  for example and they appeared to understand why not every non-zero element of  $Z_6$  has a multiplicative inverse, in the exam when they were asked to explain why  $(Z_4^\times, \cdot)$  is not a group, the majority did not succeed. However the majority succeeded in the two more familiar examples where they had to explain why  $(2Z+1, +)$  and  $(Z^\times, \cdot)$  are not groups. (5) The two new topics, modular arithmetic and complex numbers, created some problems for the students because of time shortage. However, the introduction to complex numbers that the students had, gave them some sort of continuity regarding the number system and it helped them realize and understand the, up until then unknown to them, relationship  $N \subset Z \subset Q \subset R \subset C$ . The last two examples emphasize the need for a one-semester course on these topics, as these new ideas seem to take time to get through to the students.

## CONCLUSION

As it is pointed out by Dorier (1995, p.1), concepts such as the concept of a group, were invented not only to solve new problems but “mainly to find general methods to solve different problems with the same tools” (Dorier, 1995, p.1). All of the above different examples from the area of functions, modular arithmetic, as well as the various number systems, were solved by using the same ‘tools’, i.e. the concept of group, ring and field. This ‘proves’ in a way how the unifying character of these ‘tools’ can be used in the training of future teachers, in order to help them achieve the desired connections. However, a remark regarding the cost of such a course should be added here, since its potential difficulty can cause problems if the course is taught in universities where teachers do not have to take any kind of actual mathematics courses, but only courses in mathematics education instead.

As it is pointed out throughout this paper, this is a theoretical attempt to show how the concepts of a group, ring and field can help to regroup and enrich teachers’

mathematical knowledge in a way agreeable with the literature on teachers' mathematical knowledge. What can follow this paper and produce useful and insightful information, is research that investigates which part of this knowledge and in which form, teachers actually transfer into their own classroom, and which also examines the actual effect of such a course on teachers' mathematical knowledge.

## REFERENCES

- Adler, J., Ball, D., Krainer, K., Lin, F., & Novotna, J. (2005). Reflections on an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics*, 60(3), 359-381.
- Ball, D.L. & Bass, H. (2000). Interweaving content and pedagogy in teaching and *learning* to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (83-104). Westport, CT: Ablex.
- Ball, D.L. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt and B. Davis (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'Étude en Didactique des Mathématiques*, CMESG/GCEDM (3-14). Edmonton, AB.
- Ball, D. L., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4<sup>th</sup> ed.) (433-456). New York: Macmillan.
- Begle, E.G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Washington, DC: Mathematical Association of America and national Council of Teachers of Mathematics.
- Burn, B. (1996). What are the fundamental concepts of group theory?. *Educational Studies in Mathematics*, 31(4), 371-377.
- Cooney, T. & Wiegel, H. (2003). Examining the mathematics in mathematics teacher education. In A. Bishop *et al.* (Eds.), *Second international handbook of mathematics education* (Vol.2) (795-828). Dordrecht: Kluwer.
- Davis, G.E. & McGowen, M. A. (2001) *Embodied objects and the signs of mathematics*. Paper prepared for the PME 25 discussion group "Symbolic Cognition in Advanced Mathematics". Utrecht, The Netherlands: Freudenthal Institute, University of Utrecht. July, 2001.
- Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, 61(3), 293-319.
- Dorier, J. (1995). Meta level in the teaching of unifying and generalizing concepts in mathematics. *Educational Studies in Mathematics*, 29 (2), 175-197.
- Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, 27 (3), 267-305.

## Working Group 14

- Elerton, F & Clements, M. (1998). Transforming the international mathematics education research agenda. In A. Sierpiska and J. Kilpatrick (Eds.), *Mathematics education as a research domain, An ICMI Study* (153-176). Dordrecht: Kluwer Academic Publishers.
- Hernstein, I. N. (1999). *Abstract Algebra* (3<sup>rd</sup> ed.). USA: John Wiley & Sons, Inc.
- Hoffmann, M. (2006). What is a "Semiotic Perspective", and what could it be? Some comments on the contributions to this special issue. *Educational Studies in Mathematics*, 61 (1/2), 279-291.
- Howson, G. (1980). Untitled [Review of the book *Critical Variables in Mathematics Education: Findings from a Survey of the Empirical Literature* by E. G. Begle], *The Mathematical Gazette*, 64(427), 64-68.
- Lamon, S. (1996). The development of unitizing: Its role in childrens' partitioning strategies. *Journal for Research in Mathematics Education*, 27 (2), 170-193.
- Leron, U. & Dubinsky, E. (1995). An abstract algebra story. *The American Mathematical Monthly*, 102 (3), 227-242.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers; understanding of fundamental mathematics in China and the United States*. Hillsdale, NJ: Lawrence Erlbaum Associates.
- Murphy, C. (2006). 'Why do we have to do this?' Primary trainee teachers' view of a subject knowledge audit in mathematics. *British Educational Research Journal*, 32 (2), 227-250.
- Otte, M. (2006). Mathematical epistemology from a Peircean semiotic point of view. *Educational Studies in Mathematics*, 61 (1/2), 11-38.
- Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, 42 (3), 237-268.
- Robert, A. (1987). De quelques spécificités de l'enseignement des mathématiques dans l'enseignement post-obligatoire. *Cahier de didactique des mathématiques*, 47, IREM de Paris VII.
- Simpson, A. & Stehlíková, N. (2006). Apprehending mathematical structure: A case study of coming to understand a commutative ring. *Educational Studies in Mathematics*, 61 (3), 347-371.
- Steinbring, H. (2006). What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics*, 61 (1/2), 133-162.

# **‘APPLICATIONISM’ AS THE DOMINANT EPISTEMOLOGY AT UNIVERSITY LEVEL**

Berta Barquero\*, Marianna Bosch\*\* & Josep Gascón\*\*\*

\*Universitat Pompeu Fabra (Barcelona), \*\*Universitat Ramon Llull (Barcelona)

\*\*\*Universitat Autònoma de Barcelona

*Recent research on the teaching of modelling highlights the existence of strong institutional restrictions on the widespread and long-term diffusion of mathematical modelling practices in current educational systems. We present a research project that aims at studying these restrictions in first year Sciences University degrees, especially those arising from the ‘prevailing epistemology’ concerning the role attributed to mathematics in Natural Sciences. We characterise this epistemology as ‘applicationism’ and describe its main features. The analysis of teaching materials and a survey and interview to science university teachers bring concrete evidence to the prevalence of such ‘applicationism’ and the way it may hinder the teaching of mathematics as a modelling tool.*

## **THE ‘ECOLOGY’ OF MATHEMATICAL MODELLING**

This paper focuses on studying the teaching of mathematics in Natural Sciences in universities degrees. It focuses more specifically on the didactic problem of studying the ‘ecology’ of mathematical modelling in these institutions. By ‘ecology’ we mean the necessary conditions for an activity to ‘live’ in an institutional environment and the restrictions hindering the evolution of implementing this activity. The origin of this ecological problem, which was first applied to mathematical objects and practices before being enlarged to a wider institutional perspective, can be located in the study of the process of *didactic transposition* and its related phenomena (Chevallard 1985, see also Bosch & Gascón 2006). More recently, to study the ‘ecology’ of the mathematical practices that exist (or could exist) in a teaching institution and the possible ways of constructing them (the didactic organisations), Chevallard (2002, p. 42) introduced a hierarchy of ‘levels of didactic co-determination’ that consists in the following sequence:

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔ Domain ↔ Sector ↔ Theme ↔ Subject

As we indicated in Barquero *et al.* (2010b), this hierarchy goes from the most generic level –Civilization– to the most concrete one – the subject or questions that are to be studied by a group of persons. The lower levels going from the *discipline* to the *subject* are considered as the *mathematical levels* if the considered discipline is *mathematics*. The didactic transposition process (Chevallard 1985) tends to provide a structure of taught disciplines into different domains, sectors and themes that usually comes from the scholar discipline but is modified according to the education



institution considered. For instance, in Spanish universities, a first year course of mathematics for science students is usually structured into three domains: calculus, linear algebra and differential equations, divided into classical sectors such as ‘real-valued functions’, ‘limits’, ‘derivatives’, ‘integration’, etc. containing different themes, to which every subject or question to study belongs. At secondary school level, other domains appear depending on the curricular reform in force. These low levels are the ‘specific’ ones and are used to analyse the constraints coming from the specific way of organising teaching contents at school: from the division into disciplines and blocks of contents, until the low-level concatenation of subjects.

The upper levels of determination refer to the more general constraints coming from the way Society, through School, organises the study of disciplines (pedagogical level). They concern the status and functions traditionally assigned to educational contents and the general way of organising teaching and learning activities at school. These levels offer important conditions – and also set a lot of constraints – that concretely affect what the teacher and students can do in the classroom. For instance, the amount of hours and sessions assigned to the teaching of a concrete discipline, the possibilities for disciplines to interact more or less easily, the way students are grouped (by age, by level, by gender, etc.), the organisation of the school space, etc. All those conditions and constraints belong to the *school level*, while the *pedagogical level* refers to those only affecting the teaching and learning of ‘disciplines’. The way disciplines are grouped, valued, linked, diffused belongs to this level, as for instance the choice of an interdisciplinary way of studying questions or the way of presenting disciplines as independent. The *society and civilization levels* concern the way the rationale, functions, aims of school instruction is considered and valued.

We focus here on the general problem of integrating mathematical modelling, considered a central aspect of mathematics (see Barquero *et al.* 2008 and 2010b), into current educational systems. Our objective is to study, analyze and describe some institutional constraints that hinder the implementation of these modelling activities. From our perspective, the study of these constraints and the way new teaching proposals can overcome them appear as a necessary step for the dissemination of mathematical modelling activities at all school levels.

Taking into account the different levels of didactic co-determination is not only useful to reach these variety of constraints acting on the classroom activities, but also to know better at what level it is necessary to act in order to set up suitable conditions that make the development of this mathematical modelling activities possible<sup>72</sup>. In this paper, we focus on analyzing some restrictions which come from

---

<sup>72</sup> Within the framework of the ATD, most of the research related to mathematical modelling and teaching practices (Artaud 2007, Bolea *et al.* 2004, Barquero *et al.* 2008, Barbé *et al.* 2005) takes into account the problem of the ‘ecology’ of didactic organisations, which has to be located at the core of all research aiming to integrate mathematical

the most generic co-determination level, especially all those related to what we call the ‘dominant epistemology’ in university institutions. By ‘dominant epistemology’ we understand the way in which the university as an institution and, more specifically, the community of university teachers (and students) consider what mathematics is and how it is related to the natural sciences. This situation can be summarized in the formulation of the following didactic problem:

In which way does the university community understand the role of mathematical activity and, especially of mathematical modelling in the development of Natural Sciences? What restrictions are derived from this ‘prevailing epistemology’ in relation to the ‘life’ of mathematical modelling in university settings? What conditions, at what level of co-determination, can help overcome these restrictions?

## GENERAL CONSTRAINTS ON THE TEACHING OF MODELLING ACTIVITIES

### 2.1. A characterization of ‘applicationism’

At the levels of *society* and *school*, we focus on investigating the *dominant epistemology* in the teaching institution considered (the university), and its effect on the different mathematics teaching practices. Our first hypothesis is that the widespread understanding of mathematics and its relation to natural sciences is what we can call ‘applicationism’ and this can be depicted in the following way: first mathematical tools are built within the field of mathematics and then they are ‘applied’ to solve problematic questions from other disciplines, but this application does not cause any relevant change, neither in mathematics nor in the rest of disciplines where the questions to study appeared. ‘Applicationism’ thus assumes a strict separation between mathematics and other disciplines. For example, in the majority of the Spanish university courses we have examined, the study of population dynamics is a subject located in the sector of differential equations under the label of ‘application’, as if some dynamic laws could exist without any mathematical tool to describe them and, in the same way, as if differential equations could independently exist without any extra-mathematical problem to solve. In this context, mathematical modelling is understood as a mere ‘application’ of previously constructed mathematical knowledge or, in the extreme, as a simple ‘exemplification’ of mathematical tools in some extra-mathematical contexts artificially built in advance to fit these tools.

To be more specific, we propose to characterise ‘applicationism’ using the following indicators (Barquero *et al.*, 2010a):

## Working Group 14

*I<sub>1</sub>: Mathematics is independent of other disciplines* ('epistemological purification'): mathematical tools are seen as independent of extra-mathematical systems and they are applied in the same way, independently of the nature of the system considered.<sup>73</sup>

*I<sub>2</sub>: Basic mathematical tools are common to all scientists*: all students can follow the same introductory course in mathematics, without considering any kind of specificity depending on their speciality (biology, geology, etc.).

*I<sub>3</sub>: The organisation of mathematics contents follows the logic of the models* instead of being built up from considering modelling problems that arise in the different disciplines. All occurs as if there were a unique way of organising mathematical contents and different ways of applying them.

*I<sub>4</sub>: Applications always come after the basic mathematical training*: the result is then a proliferation of isolated questions that have their origin in the different systems. The first thing is to learn how to manipulate the mathematical concepts and later learn about their use. The models are built from concepts, properties and theorems of each theme independently of any extra-mathematical system.

*I<sub>5</sub>: Extra-mathematical systems could be taught without any reference to mathematical models*, that is, there exists the belief that natural science can be taught without any mathematics.<sup>74</sup>

To empirically contrast to what degree applicationism prevails in university institutions, we decided to use these indicators to analyse teaching materials (syllabi and textbook prefaces) and to interview and survey some geology and biology teachers of a science faculty in Catalonia.

### 2.2. Applicationism in syllabi and book prefaces

The analysis of about 30 syllabi of mathematics for Natural Sciences university courses of 10 different Spanish universities constitutes a good indicator of the mathematics proposed to students. Here we consider the particular case of a first course of mathematics in the degrees of biology, geology, chemistry and environmental sciences. The written description of different mathematical courses of Spanish first year university scientific programmes states that the teaching of mathematics follows a double objective: on the one hand, they strive to give students basic mathematical training; on the other hand, they try to introduce students to mathematical modelling [*I<sub>1</sub>*, *I<sub>2</sub>*]. For instance, in the case of a geology degree (2006/07) at the UAB university, the following is proposed (our translation):

---

<sup>73</sup> This indicator is more general than the others, as it refers to a characteristic of mathematics as a discipline, and not to the way it is taught.

<sup>74</sup> This is an extreme indicator of the independence between mathematics and natural sciences (especially in the case of biology and geology) that is surprisingly widely shared to the point that, in most cases, people state that scientific systems could be studied without any mathematical tool.

## Working Group 14

This program claims a double objective. The first and most important one is to give students the basic mathematical training focused on linear algebra and on one-variable differential calculus, which will allow them to understand the language of Science. The second aim is to introduce them to the field of Geology, that is, to mathematical modelling, using simple examples that could be analyzed by previously introduced mathematical tools.

In a more detailed analysis of the mathematical contents of these courses, we notice they are organized in ‘topics’, ‘themes’ or ‘modules’ centred on a main concept (limits, derivatives, integration, linear applications, diagonalisation, ordinary differential equations, etc.) each including a number of definitions, properties, theorems, proofs, various techniques and types of problems. At the end of the study process, problems tend to turn into ‘applications’ such as giving a ‘rationale’ to the contents and showing their functionality [I<sub>5</sub>]. The corresponding study programme is generally structured in three main areas: linear algebra, one-variable differential and integral calculus, and ordinary differential equations [I<sub>3</sub>]. Always left to the end, the teaching of modelling problems is basically absent from Spanish ‘real’ university curricula.

It stands out that the ‘traditional’ organization is not structured around modelling problems or ‘applications’, but rather follows a very standard list of topics or blocks of themes. We also observe that this organization places the technological and theoretical block at the origin of mathematical activity. As a consequence, it tends to propose types of problems, which are particularly limited and isolated, to obtain examples of all the notions and properties of each topic [I<sub>4</sub>, I<sub>5</sub>]. This situation leads us to think that these mathematical tools only provide ‘qualitative’ information on the study of scientific facts. It suggests the possibility that natural sciences could be taught without mathematics [I<sub>5</sub>].

Similarly, some of the prefaces and content organizations in most of the recommended books for these courses helped us to uncover the presence of applicationism. Good examples of this are Salas & Hille (1995) and Anton (2003). The main aim of these books is to introduce students to a ‘basic common language’ for all scientists and in a completely independent way of natural sciences disciplines. [I<sub>1</sub>, I<sub>2</sub>]. Salas & Hille (1995) explain it in the following terms (our translation):

In this edition, you will find some easier applications to physics and, as extra chapters, some more difficult applications. [...] Despite the incorporation of more applications, this book is still a mathematics book, not a science book or an engineering book. It is about calculus and its main basic ideas are limits, derivatives and integrals. The rest is secondary; the rest could be left out. Salas & Hille (1995, p. 7)

The organization of these books is again and again structured around several blocks of themes, following a purely deductive logic. The common structure of most textbooks contains the following blocks of themes: limits and continuity,

differentiation, mean-value theorem and applications, integration, applications of integrals, transcendent functions, integration techniques, conic sections, sequences, infinite series [I<sub>3</sub>]. The references to applications usually appear at the end of each chapter as an example of how to use the mathematical tools introduced. Often, these applications are included in a separate section labelled ‘additional exercises’, which offers an expansion of the field of problems that can be solved with the mathematical techniques introduced in that chapter [I<sub>4</sub>].

### 2.3. Applicationism as described by teachers and researchers in Natural Sciences

The next step of our research was to look into the discourses of university teachers and researchers when they are asked about the criteria used to design and manage mathematics teaching in Natural Sciences degrees. For this purpose, we first conducted a survey on the indicators that characterize applicationism. The survey was distributed and answered by 30 teachers and researchers in the departments of biology, geology and environmental sciences at the Autonomous University of Barcelona (UAB). In a second phase of the research, we completed the findings with interviews to four teachers that had participated in the survey, with the objective of explaining and expanding their answers. The results obtained in the survey were, for most questions, quite clear. We next summarize some of the main findings and complement them with the explanations provided in the interviews.

In the first question, more than 50% of the respondents did not agree on the fact that the mathematics taught in their respective scientific disciplines were any different from the ones taught in the rest of Natural Sciences. One can hence confirm that the mathematics taught in one particular science is not specific to it [I<sub>2</sub>]. Respondent C added:

“I think that the differences between programs need not be very significant. The mathematical language is unique, and the attraction for mathematics can come from different scientific perspectives but is always the same. Nevertheless I think that details matter, the type of examples, the adaptations, being able to visualize that those equations can be translated into specific phenomena [...]”

1. “The mathematics of the first year Biology degree is different from the rest of Natural Science degrees.”		
1. Strongly disagree	8	27%
2. Disagree	8	27%
3. Neither agree nor disagree	4	13%
4. Agree	3	10%
5. Strongly agree	1	3%
6. I do not know	6	20%
<b>TOTAL</b>	<b>30</b>	<b>100%</b>

2. “Mathematics is generally introduced independently of the biological systems that can be mathematically modelled.”		
1.. Neither agree nor	2	7%
2. Agree	15	50%
3. Strongly agree	13	43%
<b>TOTAL</b>	<b>30</b>	<b>100%</b>

Where people agreed most was in the second question: 93% agreed (43% strongly agreed) that mathematics is introduced with independence from scientific systems [I<sub>1</sub>]. Regarding that idea, respondent A agrees with this separation:

## Working Group 14

“I do not think that mathematics should be introduced linked to geological systems. How could one link it with such basic mathematics? I do not find it convenient or pertinent because we do not have to explain why mathematics is needed in each one of our courses. Mathematics must be there, for their own sake.”

90% of the respondents agreed with the third statement (37% strongly agreed), that the mathematics that is taught is structured around problems of a mathematical nature, and not of a scientific nature [I<sub>3</sub>]. On this point, respondent C explains:

“I think that (the mathematics taught) are still centred on mathematical problems, but I do not know whether this is bad, I find it natural. I said earlier that programs did not have to be completely different and the differences should appear in the examples. What I meant is that I am glad that the mathematics programs for physicists and for bio-scientists do not contain large differences and share a common body, which cannot be physics, chemistry or biology: it has to be mathematics [...]”.

3. “Mathematics is structured around mathematical problems and not around biology problems.”		
1. Strongly disagree	0	0%
2. Disagree	3	10%
3. Neither agree nor disagree	0	0%
4. Agree	16	53%
5. Strongly agree	11	37%
<b>TOTAL</b>	<b>30</b>	<b>100%</b>

It is also quite clear that the respondents disagreed with the fourth statement (47% strongly disagreed, 40% disagreed), that mathematics are only taught after the need for it arises from the study of scientific phenomena [I<sub>5</sub>]. On the other hand, 90% disagreed (40% strongly disagreed) with the idea that extra-

4. “Mathematics is taught only after the study of biological systems requires it.”		
1. Strongly disagree	14	47%
2. Disagree	12	40%
3. Neither agree nor disagree	3	10%
4. Agree	1	3%
<b>TOTAL</b>	<b>30</b>	<b>100%</b>

5. “Biological situations where mathematics is applied are linked to a general biological problem that is relatively unified.”		
1. Strongly disagree	12	40%
2. Disagree	15	50%
3. Neither agree nor disagree	3	10%
<b>TOTAL</b>	<b>30</b>	<b>100%</b>

mathematical situations where mathematics are used appear linked to a general Natural Science problem that is relatively unified rather than a set of independent questions. This idea, introduced by I<sub>4</sub>, supports the progressive disappearance of a general scientific problem. Instead, there is a proliferation of isolated problems (arising from

different systems). The interviews suggest a similar trend:

“A set of isolated and separated questions appears, and, I must add, it is a shame. But this is what is done. It is not only that there is no relationship with the instrumental courses, but also that there is no relationship between the descriptive courses! They are all satellites on their own isolated orbits [...] We could look for very serious reasons, but one always ends up finding the usual suspects, inertia and lack of unity.” (Respondent C)

## Working Group 14

“Each one seems to reign over his/her little kingdom, and explains the topics separately. The student will never have a global view, but I think that we, the teachers, do not have it either. There is no tradition of working in community, of meeting and agreeing on the content of each subject [...] I do not see how it can be possible to avoid this tradition in the future, to escape from this inertia around us.” (Respondent A).

Regarding the last statement in one of the most extreme indicators of applicationism, we find divided opinions, only 10% remaining neutral. Around 47% disagree that it would be possible to teach a Natural Science degree by reserving mathematics only for the quantitative aspects. But what we find surprising and to some extent worrying is that approximately 44% agree with the statement. This suggests that it would be possible to achieve a relatively complete and deep study of scientific phenomena without using mathematics.

6. “An entire Biology degree can be taught almost without using mathematics”		
1. Strongly disagree	5	17%
2. Disagree	9	30%
3. Neither agree nor disagree	3	10%
4. Agree	10	33%
5. Strongly agree	3	10%

The interviews also reflect this divergence of opinions. Those who agree with that possibility add:

“Yes, it would be possible. I have known several students that have obtained their degree without passing the mathematics subjects, and even worse, without even trying.” (Respondent A). “One could do it, but the degree would be incomplete, it would be merely descriptive [...] But as soon as one would need to study deeper or establish relationships between phenomena, then mathematics are needed.” (Respondent B).

Respondent C is clearly opposed to this point of view:

“Absolutely not. In fact, one cannot live without it at any degree. One should have the capacity of not being afraid of an abstract formulation of any discipline, and even more so for sciences. It would be madness that someone suggested removing mathematics or said it is not necessary. Anyone that tries to do forecasting or modelling will need it.”

Lastly, respondent D insists on the ‘error’ of thinking of mathematics as being unique:

“I would say that if the mathematics taught in secondary schools were well understood by a large percentage of students, one would not need much more mathematics, especially in the more general subjects. The problem is that this is not the case. It is a different matter when one needs some type of specialization that requires some specific mathematical tool. But it would be a mistake to think that all kinds of biologists require the same kind of mathematics.”

### 3. CONCLUSIONS

One of the main characteristics of ‘applicationism’, which represents one of the strongest restrictions on the normal life of mathematical modelling, is that it establishes a *clear distinction between mathematics and the rest of natural sciences*.

It is furthermore supposed that both ‘worlds’ evolve with independent logic and without many interactions. This fact, which partly results from the first three indicators of applicationism, leads to greatly reduce the possible ways of teaching *mathematics as a modelling tool* for the study of scientific systems. In general the mathematics that are taught at first year sciences degrees present a highly stereotyped and crystallized structure that does not mingle with the systems that are modelled and, besides, the taught mathematics are never ‘modified’ as a consequence of being applied in the construction of such models.

This radical separation between mathematics and natural sciences makes it very difficult for students to consider *mathematics as a constituting tool* of natural sciences and, thus, to value the necessity of its learning. This is one of the most generic restrictions that hinder mathematics teaching at this level, a restriction that appears at the general levels of *society* and *school* at the scale of didactic codetermination levels.

Our research on the implementation of modelling activities into the teaching system through ‘Study and Research Courses’ (Barquero *et al.* 2008 and 2010b) shows, on the contrary, how problematic scientific questions can be successfully situated at the origin and core of the mathematical activity that is taught in natural sciences degrees. It has been seen how the study of these questions starts a process of mathematical modelling that ends up with the construction of an entire mathematical organisation. This ‘first’ process generates new questions of an extra-mathematical nature that must not be abandoned or ignored but must generate new modelling processes. In these dynamics it no longer makes sense to think of two kinds of different and independent logic that rule both ‘worlds’, the mathematical one and the world of the rest of the scientific disciplines. It even becomes necessary to think of a single world, the world of scientific activity of which mathematical modelling is an integral part.

Another restriction of ‘applicationism’ on the life of mathematical modelling shows up in the usual organisation of mathematical study programmes taught at natural science degrees. In fact, neither the structure given to the contents nor the way of developing them in class make it possible to carry out mathematical modelling work based on the study of problematic questions that arise in scientific fields which are close to the speciality chosen by the students. The *raison d’être* (mathematical or extra-mathematical) of the contents, which are part of this basic mathematical training the students need to acquire, is not part of the study programme. The ‘modelling activity’ is restricted and limited to the simple illustration or occasional and anecdotal exemplification of certain pre-established models to systems fitted out with pre-established problems. This set of restrictions, which greatly limits the ‘nature’ and ‘structure’ of the possible mathematical activities taught in natural sciences degrees, may first of all be situated at the levels of *pedagogy* and *discipline*. However, these restrictions have a great impact on the more specific levels, that is,



on the specific way how mathematics taught are organized in *areas, sectors, themes* and *questions*.

The characterization of ‘applicationism’ appears as a useful tool to analyse the *restrictions* on the life of mathematical modelling which are derived from the current interpretation of mathematical activity and its role in natural sciences. It seems clear that research cannot ignore the proper level of the scale of levels of co-determination in which these restrictions appear, as general a level as it could be. Only by knowing these restrictions well, will we be able to put forward the necessary conditions to achieve a ‘real’ integration of mathematical modelling, conditions that have to take into account the necessary evolution of the prevailing epistemology and the establishment of a ‘new epistemology’ as, for instance, the one proclaimed by Chevallard (2006). These are the questions that lead our current research project.

## REFERENCES

- Anton, H. (2003). *Introducción al álgebra lineal*. Editorial Limusa. Grupo Noriega Editores.
- Artaud, M. (2007). Some Conditions for Modelling to Exist in Mathematics Classrooms. In W. Blum, P. L. Galbraith, H-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 371-377). New York: Springer.
- Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher’s practice. The case of limits of functions in Spanish High Schools. *Educational Studies in Mathematics*, 59, 235-268.
- Barquero, B., Bosch, M., & Gascón, J. (2008). Using Research and Study Courses for Teaching Mathematical Modelling at University Level. In D. Pitta-Pantazi & G. Pilippou (Ed.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 2050-2059). Larnaca: University of Cyprus.
- Barquero, B., Bosch, M., & Gascón, J. (2010a). Ecología de la modelización matemática: Restricciones transpositivas en las instituciones universitarias. In A. Bronner, M. Languier, M. Artaud, M. Bosch, Y. Chevallard, G. Cirade & C. Ladage (Éds), *Diffuser les mathématiques (et les autres savoirs) comme outils de connaissance et d’action*. Montpellier: Université de Montpellier.
- Barquero, B., Bosch, M., & Gascón, J. (2010b). The ‘ecology’ of mathematical modelling: Constraints to its teaching at University level. *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 2146-2155). Lyon: Université de Lyon.

#### Working Group 14

- Bolea P., Bosch M. & Gascón J. (2004). Why is modelling not included in the teaching of algebra at secondary school? *Quaderni di Ricerca in Didattica*, 14, 125-133.
- Bosch, M. & Gascón, J. (2006). Twenty-Five Years of Didactic Transposition. *ICMI Bulletin*, 58, 51-64.
- Chevallard, Y. (1985). *La transposition didactique, du savoir savant au savoir enseigné*. Grenoble, France: La Pensée Sauvage (2d edition 1991).
- Chevallard, Y. (2002). Organiser l'étude : 3. Ecologie & régulation. *XIe école d'été de didactique des mathématiques (Corps, 21-30 de agosto de 2001)* (pp. 41-56). Grenoble: La Pensée Sauvage.
- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the 4<sup>th</sup> Conference of the European Society for Research in Mathematics Education (CERME 4)*. (pp. 21-30). Barcelona: FUNDEMI-IQS.
- García, F.J., Gascón, J., Ruiz Higuera, L. & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics, *ZDM: The International Journal on Mathematics Education*, 38(3), 226-246.
- Salas, S. & Hille, E. (1995). *Calculus*. Barcelona: Editorial Reverté.

# DESIGNING ALTERNATIVE UNDERGRADUATE DELIVERY: OIL & MASSAGE

Bill Barton

The University of Auckland

*As we reconsider undergraduate mathematics, we need to pay attention to theory, practice, research, and development. I draw on a theory-building project, an intimate examination of practice, a radical re-examination of undergraduate delivery, and diverse research literature, to identify some key features of undergraduate mathematics education. The interplay of constructs labelled 'mathematical essence' and 'learning culture' emerges as a useful focus of attention for my understanding of the undergraduate teaching and learning. Not only does it encompass the recurring themes of the three activities, but, connected with the concept of 'curricular contribution', it helps me design new developments and research questions.*

Key words: undergraduate mathematics, research framework, mathematical knowledge, learning culture.

Sheer pleasure is any academic's reaction to fortuitous conjunctions of their teaching practice, current research, and aspirations for development. It should be no surprise that validated insight and significant advance in understanding also result. This year I have experienced such a feeling as I engaged with colleagues in three activities concerned with the undergraduate experience. My aim has been to find a way to explain, or even just talk analytically about, some phenomena of the undergraduate mathematical environment. This paper creates a framework that will help.

Two features of mathematics learning emerge, Phoenix-like, from recurring discussions in all three activities. The Phoenix refers to the resurrection of ideas that can be found in much mathematics education literature, but they are burnished anew by a focus on undergraduate mathematics, and the intimate examination of practice. I identify these constructs as 'mathematical essence' (the real oil) and 'learning culture' (the pedagogical environment that massages our actions).

But let us start by describing the three activities, and follow with three recurring themes within those activities.

## THREE ACTIVITIES

### Research: Videoing Lecturers

In 2009 a group of researchers began a three-pronged investigation of undergraduate mathematics. The Lecturer Perspective component, based on Alan Schoenfeld's 'Knowledge Orientation Goal' framework (2008), involves a group of lecturers videoing their lectures and discussing excerpts as a professional community. As one of the lecturers involved, this intensely personal experience has given me insights

into both my own practice, but also my own practice *vis a vis* that of my colleagues (Barton, Oates, Paterson, & Schoenfeld, 2009). The Student Perspective component builds on the work of Rodd and Bartholomew (2006) characterising the mathematical identity of university students (both mathematical majors and others). The Interactions component focusses on the way questions can be used in lectures and tutorials to promote deeper mathematical engagement.

### **A Public Lecture: The Pleasure Principle**

In April, 2010, I argued that undergraduate mathematics was founded more on tradition and habit than on any consideration of educational aims or research. The US National Research Council report *Revitalizing Undergraduate Mathematics* (1991) endorses and gives detail to such a view. The challenge to my department to consider a radical revision of how we present mathematics to undergraduates has been taken up. A group are preparing a proposal for a course in mathematics for those not intending to major in the mathematical sciences. Given the economies of scale of the existing course (that is, existing student/staff ratios and workloads), how might we rationally design the programme for improved learning? Can we design a course that will give non-majoring students a sensation of the intense pleasure in mathematics experienced by researchers (see related discussion in Madsen & Winsløw, 2008)?

### **Theory: Building a framework**

During 2010 an opportunity arose for researchers in England and New Zealand to meet three times to discuss a framework for describing undergraduate mathematics. Previous frameworks (e.g. APOS theory of Dubinsky and colleagues, see Asiala et al., 1997) focus on learning theory more than the whole environment. The framework is intended to help us formulate research questions that will explain the phenomena we observe in mathematical science teaching and learning. The meetings were driven by an awareness that much of the existing mathematics education research literature is based on school assumptions. In what ways is the undergraduate experience different and similar? What research conclusions are likely or unlikely to be valid? What frameworks are in use in undergraduate research, and what important issues are being overlooked? Opportunities for a wider-ranging approach exist in the recent emergence of university centres dedicated to undergraduate learning and teaching at The University of Auckland (CULMS <<http://www.math.auckland.ac.nz/CULMS/>>) and Loughborough University (MEC <<http://mec.lboro.ac.uk/>>).

## **RECURRING THEMES**

### **Recurring Theme #1: Responsibility for learning**

The most explicit recurring theme in the three activities is student responsibility for their own learning. It was quickly identified as a significant point of difference between school and undergraduate environments in the theoretical framework

discussions. In schools, teachers assume (or have thrust upon them) responsibility for getting the students to complete homework, engage in tasks in class, and reach mathematical understanding. At university, voluntary lecture attendance may be ameliorated by terms requirements of tutorial participation or assignment submission, but failure is firmly a student failure, not a lecturer failure. In our developmental discussions of course delivery mechanisms, it is no surprise that university conventions around lecture and tutorial attendance are also major issues. However the complexity of learning expectation and responsibility was driven home in the research activity when examining a video-tape of my own lecturing.

With my senses honed by the accompanying critical eyes of my colleagues, I saw myself on video act under unwarranted assumptions about what students would do to support their own learning, and also I undermined their learning independence. For example, several times I adjusted my expectations of what the students would have done with respect to revision of the previous lecture, thereby eventually removing all consequences of a student not having taken the responsibility for getting themselves prepared. Alerted to the phenomenon, I saw repeated versions of it in subsequent videos of both myself and my colleagues. The unintended changes in outcome are reminiscent of the Topaze effect of Brousseau's theory of didactic situations (1987, p25). Recent analysis of this effect in undergraduate mathematics has been done by Grønbaek, Misfeldt and Winsløw (2009).

The combined effect of these experiences is an awareness of the interaction between the assumptions and the actions of lecturers and students. We claim increased student independence at university, but as lecturers we may rarely address the issue explicitly and our actions may counteract any statements of expectation. The didactic contract (to again use Brousseau, 1987) is complex, and our resulting actions can be contradictory, in my own case at least.

### **Recurring Theme #2: The Discipline of Mathematics**

A second theme to emerge from all three activities is an abiding concern for mathematics as a discipline—and this word is intended in both meanings. Mathematics is a field of enquiry, and mathematics is a mode of being. I include the connotations of strictures and suffering. Our theoretical framework discussions repeatedly harked back to the role of research mathematicians as pedagogues and how that defined some aspects of university education both positively (direct contact with the discipline and authentic mathematical “modes of enquiry” (Watson & Barton, 2010)) and negatively (lack of attention to pedagogical theory). Our course development discussions very rarely descended into arguments about content or sequencing, but often reached swift agreement on the need for ‘mathematical habits of mind’ and extended debates about what, exactly, these constituted. The group contained several mathematicians, and personal accounts of wrestling with mathematical problems at the research level exposed a common core of experiences and emotional attachments that would be beneficial for students, even those not

entering mathematical careers. The ultimate aim of non-continuing mathematics courses were expressed as the ability to recognise the mathematical nature of a problem and not fear the need to address it, possibly by learning new mathematics for oneself. Much discussion concerned the nature of a mathematical mind.

Attention to the ‘mathematicality’ of the undergraduate experience emerged also in the videos of lectures. We noted a difference between lecturers who were mathematics educators and lecturers who were mathematicians. The former, who came from school teaching backgrounds, expressed concern about the correctness of their mathematics and wanted to discuss the mathematical implications of the videoed excerpt. The latter, who have mathematics research credentials, were more worried about the pedagogical consequences of their actions. Judy Paterson referred to this as our teacher selves and mathematical selves, focusing attention on the way our conceptions of ourselves affects the teaching, and hence learning, experience.

In both the development and research environments the correctness of the mathematics is also regularly discussed. On the surface the discussion is about how much an introduction to a mathematical concept can be limited in the interests of simplicity, and what are the downstream effects of the resulting misconception. For example, what are the consequences of introducing all vectors as free vectors for later definitions of vector spaces? Differing perspectives of the same concept is related to what Mike Thomas calls versatile thinking (Thomas, 2008) and developing students’ ability to “see” many perspectives is non-trivial.

But correctness in the sense that there is a right way to present material also implies a certain attitude towards mathematics, or at least towards undergraduate mathematics education. A focus on correctness, especially when the teaching mode is a transmission one, conveys the assumption (intended or not) that mathematics is a given—it is like this, learn it and use it in this way. Any other way is wrong. That such a view is, in fact, held by many mathematicians emerges from Leone Burton’s (2001) study of seventy mathematicians. Such an attitude contradicted our other discussions about the essential nature of mathematics being a rational exploration of the consequences of possible constructs, and the idea (also found amongst mathematicians in Burton’s study) that mathematics is a social construct.

In all three activities the changing and contemporary nature of mathematics was also discussed (Lóvasz, 2006). Compared with the average ten-year period for major school curricular revisions, university courses are minimally revised individually rather regularly, but the whole programme is typically extremely stable over long periods. Personal and factional investments in each part of the programme make changes to keep pace with the evolution of the subject very difficult.

### **Recurring Theme #3: The Tyranny of Examples**

Viewed cold, I was surprised at how much my lectures focused on doing examples and exercises rather than teasing out problems, asking open questions, or exploring

deep understanding. Compared with my goals (which the research project required me to write down prior to the lecture and reflect on afterwards), I devoted a much larger proportion of the time available to short-term technicalities of standard mathematical exercises.

The mathematical paucity of practice like mine has been noted before many times.

Just as any sensitive human being can be brought to appreciate beauty in art, music or literature, so [they] can be educated to recognize the beauty in a piece of mathematics. The rarity of that recognition is not due to the “fact” that most people are not mathematically gifted but to the crassly utilitarian manner of teaching mathematics and of deciding syllabi and curricula, in which tedious, routine calculations, learned as a skill, are emphasised at the expense of genuinely mathematical ideas, and in which students spend almost all their time answering someone else’s questions rather than asking their own. (English mathematician and Bletchley Park code-breaker, Peter Hilton, 1998)

Nevertheless the dominance of exercise mode, and the perception amongst students (enhanced by assessment practices) that successful mathematics means solutions to exercises, is an enduring phenomenon in schools and universities. Changing students’ focus on exercises emerged as a problem to be solved in both the theoretical and developmental discussions.

## **EMERGING CONSTRUCTS**

These three (and other) recurring themes speak to both the mathematics and the learning culture. The parallels with Bernstein’s concepts of classification and framing are noted, and his strong and weak characterisations can be applied (Bernstein, 1990).

What I wish to call the “mathematical essence” refers to the focus on mathematics not just as a syllabus list of content, but as a research and practical discipline. Mathematics in all its manifestations is the “real oil” that makes learning at this level different. At university level, practitioners in the field interact with learners for the first time. How they act shapes the perception of mathematics, its nature, its uses, and what it is like to act mathematically. Mathematical essence has several components which we can, artificially, categorise into three groups: content and ‘horizon’ knowledge (Ball, 2003); processes, habits of mind (Cuoco, Goldenberg, & Mark, 1996) or modes of enquiry (Watson & Barton, 2010); and attitudes and VPRO’s (an amalgam of values, philosophy, roles and orientations, (see Barton, 2009)). We know sufficiently well from the research literature in each of these areas that all aspects of the mathematical essence need to be taken into account in any framework that purports to characterise undergraduate mathematics teaching and learning.

On the educational side, rather than approach teaching and learning from a pedagogical perspective, I argue (from the experiences above) that it is more

productive to consider the students' and lecturers' personal experience of pedagogy, that is, take a cultural point of view. Hence I will talk about a learning culture.

The concept of a learning culture is common in business (for example, Conner & Clawson, 2004) where it refers to the atmosphere in a corporation that either promotes or hinders development (in particular of management) in such matters as client relations or employee participation. Culture massages us in particular ways. In the undergraduate mathematics environment I use the term to refer to the expectations, intentions and actions of both students and lecturers with respect to pedagogical structures (many of which will be university conventions). For example, what do lecturers and students expect of, and how do they respond to, grading requirements, the physical layout of lecture theatres and tutorial rooms, enrolment processes and prerequisites, course guidelines and notes, opportunities to interact, and a myriad of other components of university life?

Another way of understanding learning culture is through the concept of the hidden curriculum (Snyder, 1970). He discusses the inferred tasks a student experiences "that are rooted in the professors' assumptions and values, the students' expectations, and the social context in which both teacher and taught find themselves" (Chpt.1.) and notes that neither professors nor students desire the resulting study habits.

Perceiving the educational situation in a personal way enables me to examine actual events rather than abstract constructs: I am forced to think about how I write an examination (and what that means) rather than examinations themselves as objects; I must worry about the advice I give to students for their next courses (and how they could respond) rather than focus on calendar entries of pre-requisites; I pay attention to the use I make of my lecture time rather than to an abstract concept of a lecture.

So far, however, all I am proposing is a re-categorisation of mathematics and pedagogy, or, perhaps, a re-viewing from a more elaborate and personal perspective respectively. But my experiences this year have made me realise that my interest lies not in each of these separately, but in the interplay between the mathematical essence and the learning culture (see Madsen & Winsløw, 2008, for a detailed study of this relationship). Teaching is located in the tensions between them, and the teaching task is to make them work together to create better learning. Student learning is the background against which the effectiveness of teaching is judged.

Yet now I wish to introduce a further dimension, adapted from the work of French mathematics educator Michèle Artigue (2002). In discussing the role of technology in mathematics education, Artigue distinguishes between three "values". The pragmatic value refers to the productivity of the technology, how does it help us in the mathematical action we are currently undertaking, how efficient is it, how useful? For example, a calculator allows us to multiply real numbers efficiently without pencil and paper algorithms. The epistemic value refers to how technology helps students understand the mathematical objects they are dealing with. For example, the



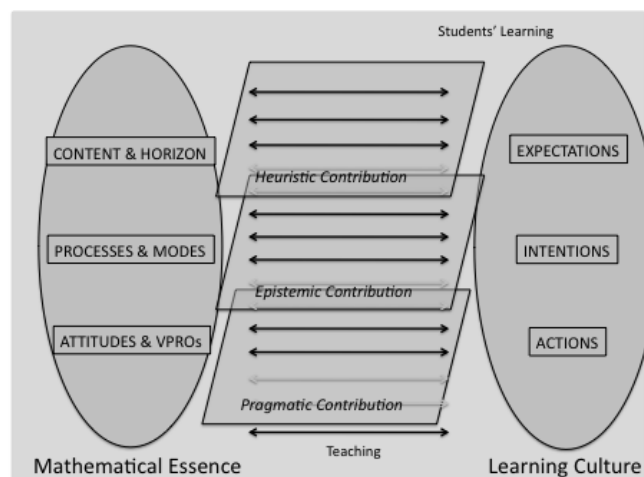
ability of a graphics calculator to flip between tabular, algebraic and graphical representations of a function promotes a versatile understanding of function in general. The heuristic value refers to how technology contributes to understanding future concepts, or how technology prepares students for more advanced concepts. For example, the ability of Matlab to draw first a surface, and then its contours on the domain, paves the way for a visual understanding of the grad of a function as a vector field.

Let us apply these three values to undergraduate mathematics. Let us call them curricular contributions rather than values. The pragmatic contribution refers to the way that an educational expectation, intention or action is directly addressed to the learning of some aspect of the mathematical essence. For example, what sort of assignment will contribute to a students' use of a partial derivative. Does it expand their ability to use them into other contexts, does it help the student evaluate them?

The epistemic contribution refers to the way some aspect of mathematics is more deeply understood. For example, can I write an examination that will contribute to a students' understanding of a partial derivative? Does it expand the contexts for partial derivatives, does it help a student see partial derivatives as elements of a matrix?

The heuristic contribution refers to future understanding, or future mathematical pathways. How can a lecturer increase the range of situations in which students will consider partial derivatives? What attitudes can I impart that will encourage students to learn about partial derivatives beyond that presented in my course?

Layering these contributions onto the interactions between the mathematical essence and the learning culture, I end up with a framework as in Fig 1.



**Fig 1: A model of Learning And Teaching in Undergraduate Mathematics (LATUM)**

The LATUM model enables me to talk about many phenomena observed in researching, practising, and theorising about undergraduate mathematics. For

example, from Recurring Theme #1, a students' acceptance for responsibility for learning is dependent on the value they put on the mathematics, whether they see it as content or a mode that is being learned, and which of the contributions they perceive to be at play (for example, a pragmatic "I need to understand this so that I can find eigenvalues from matrices presented to me in tests and examinations", or an epistemic "doing these exercises will show me what eigenvalues are all about"). We can see immediately that responsibility based on a perceived heuristic contribution ("eigenvalues are a stepping stone to advanced methods in stochastics") will only occur if a lecturer has actually pointed to such a possibility. Do we do this routinely?

Describing responsibility in this way immediately generates researchable questions of direct practical value. What are the contributions perceived to be at play by students who do take responsibility for learning compared with those who do not? Do students who take responsibility for learning focus more on modes and processes (rather than content) than their less responsible companions? What values, philosophies, roles and orientations are communicated by lecturers who best elicit student responsibility?

From Recurring Theme #2, what of the lecturers with a school background who are worried about the way they present as mathematicians? The LATUM model presents undergraduate mathematics as an interplay between learning culture and mathematical essence. A lecturer's role, therefore, is to keep the two aspects in balance, ensuring that the learning environment is not dominated by either pedagogy nor mathematics. School teachers are adept at managing a learning culture (they do not survive otherwise), and it has become second nature to most. We should not be surprised, therefore, that their attention should be on the mathematical essence, and particularly on those parts of it where they lack confidence, in order to be able to maintain the required balance. And *vice versa* for research mathematicians.

The research question arising from this is to observe the way the balance is held in a lecture or tutorial, and investigate whether bringing this to the attention of the lecturer will result in better balances. Can we treat "lecture balance" as a variable in student learning? Does better balance result in better learning?

The tyranny of examples (Recurring Theme #3) fits naturally into the LATUM model. My observation of university mathematics leads me to postulate that, in the mathematical essence, content is dominant; in the learning culture, assessment is dominant; and the curricular contribution most at play is the pragmatic contribution. Both students and lecturers buy into these preferences. The tyranny of examples is the result. Note that, in this construction, the tyranny of examples is an output of the undergraduate mathematics environment, not a cause of learning outcomes.

Hence, an intervention style research project that upsets the dominance of one (or a combination of) content, assessment, and pragmatism in a systematic way over a long period of time is strongly indicated. What will replace the tyranny of examples?

Ultimately, of course, we want to be able to manipulate the environment to produce a set of prior agreed learning outcomes. This is didactic engineering (Brousseau, 1987) on a macro scale, and has previously been theorised by Winsløw (2006).

## CONCLUSION

As we teach in undergraduate mathematics courses, as we reflect and theorise about what happens, as we research questions of interest, we find that certain themes recur. These are not new insights—we can find them talked about amongst our older colleagues, and read about them in the earliest writings on undergraduate teaching. Who has sat in a common room and not heard complaints about the lack of preparation of students? Who has walked university corridors and not met enthusiastic students wanting to share their latest insight? Who has walked into a lecture theatre and never had the experience of a student coming up afterwards and asking a question that showed they lacked the prior knowledge you had assumed? Who has leaned, pen in hand, over a tutorial group and never learned something new from a student who saw a problem in an unexpected way? Who has marked finals examinations and not had moments of despair about their own ineffectualness?

The complexity of the environment requires a framework that will help us to explain, or just talk analytically about, some of these phenomena. We also need ways to ask relevant questions. I find the framework presented above helps me with some, (not all), of my wonderings as I seek alternative answers to some of my own practices.

## ACKNOWLEDGEMENTS

The research project is funded by the NZ Council for Educational Research's TLRI fund. I am indebted to the work of my colleagues at The University of Auckland. The development project relies on the work of a group of colleagues from The University of Auckland's Department of Mathematics and Centre for Academic Development.

The theoretical framework project is an ongoing collaboration between Barbara Jaworski (Loughborough University), Elena Nardi (University of East Anglia), Tim Rowlands (Cambridge University), and Mike Thomas (The University of Auckland).

## REFERENCES

- Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *Intl. Jour. Computers for Mathematical Learning* 7(3), 245–274.
- Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D, & Thomas, K. (1997). A Framework for Research and curriculum development in undergraduate mathematics education. *Research in Collegiate Mathematical Education II*, 6, 1–32.

- Ball, D.L. (2003). What mathematical knowledge is needed for teaching mathematics? *Secretary's Summit on Mathematics*, U.S. Department of Education, February 6, 2003, Washington, D.C.
- Barton, B. (2009). Being mathematical, holding mathematics: Further steps in mathematical knowledge for teaching. In R. Hunter, B. Bicknell, & T. Burgess (Eds.) *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group Australasia*, (Vol.1.), pp 3-10, Palmerston North: Mathematics Education Research Group Australasia
- Barton, B., Oates, G., Paterson, J. Schoenfeld, A. (2009). Have six colleagues view your lecture: Just why did I agree to that? Presentation at the Sixth Southern Hemisphere Symposium on Undergraduate Mathematics and Statistics Teaching and Learning, Southern Right Delta, Gordons Bay, South Africa, 29 Nov – 4 Dec.
- Bernstein, B. (1990). *The Structuring of Pedagogic Discourse* (Vol. 4). London: Routledge & Keegan Paul.
- Brousseau, G. (1987). *The theory of didactical situations in mathematics 1970-1990*. (Translated). Dordrecht: Kluwer Academic Publishers
- Burton, L. (2001). Research mathematicians as learners—and what mathematics education can learn from them. *British Education Research Journal*, 27(5), 589-99.
- Conner, M. & Clawson, J. (2004). *Creating a learning culture: Strategy, technology and practice*. Cambridge, UK: Cambridge University Press
- Cuoco, A. Goldenberg, P. & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula, *Jour. Mathematical Behaviour* 15, 375-402.
- Grønbaek, N., Misfeldt, M., & Winsløw, C. (2009). Assessment and contract-like relationships in undergraduate mathematics education. In O. Skovsmose et al. (Eds), *University science and mathematics education. Challenges and possibilities*, pp. 85-108. New York: Springer Science
- Hilton, P. (1998). Review of: The pleasures of counting by T W Körner, *American Mathematical Monthly* 105 (5), 481-485.
- Lóvasz, L. (2006). Trends in mathematics: How they could change education? Paper delivered at the Future of Mathematics Education in Europe conference, Lisbon.
- Madsen, L. & Winsløw, C. (2008). Relations between teaching and research in physical geography and mathematics at research-intensive universities. *Intl. Jour. of Science and Mathematics Education*, 7, 741-763.
- National Research Council (1991). *Moving beyond myths: Revitalizing undergraduate mathematics*. Washington, DC: National Academy Press

#### Working Group 14

- Rodd, M., & Bartholomew, H. (2006). Invisible and special: Young women's experiences as undergraduate mathematics students. *Gender and Education*, 18(1), 35-50.
- Schoenfeld, A. H. (Ed.) (2008). *A study of teaching: Multiple lenses, multiple views*. A volume in the *Jour. for Research in Mathematics Education* monograph series. Reston, VA: National Council of Teachers of Mathematics.
- Snyder, B. (1970). *Hidden curriculum*. New York: Alfred A. Knopf
- Thomas, M. O. J. (2008). Developing versatility in mathematical thinking. *Mediterranean Journal for Research in Mathematics Education*, 7(2), 67-87.
- Watson, A. & Barton, B. (2010). Teaching mathematics as the contextual application of mathematical modes of enquiry. In K. Ruthven & T. Rowlands (Eds.) *Mathematical Knowledge in Teaching*. Cambridge University Press.
- Winslow, C. (2006). Research and development of university level teaching: the interaction of didactical and mathematical organisations. *Proceedings of CERME4*, pp. 1821-1830. Accessed 02.12.2010 at <  
<http://ermeweb.free.fr/CERME4/>>

# WHY DO STUDENTS GO TO LECTURES?

Christer Bergsten

Linköping University

*This paper investigates, from a student perspective, the fact that, despite arguments put forward against the educational value of large group lectures, this teaching format prevails and attracts students in undergraduate mathematics education.*

Keywords: mathematics lectures, students' attendance, case study.

## INTRODUCTION

Doubts about the value of large group lectures as a teaching format at universities have been raised frequently (Bligh, 1972; Fritze & Nordkvelle, 2003; Holton, 2001). The lecture is an efficient model of teaching as an outcome of the massification of higher education. In case of an eminent scholar who presents a new theory or view, a lecture is certainly appropriate. Given that undergraduate mathematics courses are highly standardised (and so the teacher could be viewed as “exchangeable”) and often based on textbooks, the purpose of the lecture format is questionable. Students might rely on texts and might want to spend more time in study groups, where they could ask questions after studying the texts. Consequently it would be of interest to investigate how students view lectures. One can assume that they go to lectures because they get something out of it. In this paper it will be discussed what this “something” may refer to. Reasons may not only be related to learning mathematics as disciplinary content, but also to issues such as becoming informed about examination specifics or social issues such as participation in the academic community.

## RESEARCH ON MATHEMATICS LECTURES

Fritze and Nordkvelle (2003) identify, from a systems theory perspective, three different functions of a lecture. The lecture reflects both scientific truth, by ways of argumentation and reflection, and educational decisions in order to make this scientific content accessible to students. Thirdly, since students regard lectures in a course as “a part of a socialization scheme”, the lecture takes on a third function of an “organization activity” (ibid., p. 232). The case study reported in this paper, aimed at further differentiating the first two of these functions.

Much of the literature on tertiary mathematics education focuses on the quality of teaching and learning of subject content matter (e.g. Bradley, Sampson, & Royal, 2006). In the general context of tertiary education, Biggs (2003, p. 75) lists four aspects of teaching/learning that from the literature seem to support quality learning of content matter: (i) a well-structured curriculum; (ii) an appropriate motivational context; (iii) learner activity, including interaction with others; and (iv) self-monitoring. The minimal extent to which the third of these categories applies to large group lectures has been given as an argument for the low learning potential of

lectures (Fritze & Nordkvelle, 2003; Bligh, 1972). However, what the “learner activity” consists of remains unspecified. The educational value of large group lectures has been questioned for several additional reasons:

- students’ attention cannot be maintained during a whole lecture (Bligh, 1972);
- lectures are often not understood by the students (Rodd, 2003, p. 15);
- lectures are most often linearly well ordered outlines of a ready made mathematical theory, not offering a view of mathematics as a human social activity, coloured by creativity, struggles, and other emotional aspects involved in mathematical activity (Alsina, 2001; Weber, 2004), thus showing only what sometimes is called the “front” of mathematics and hiding the “back”;
- students do not learn much from lectures (Leron & Dubinsky, 1995);
- lectures are not effective in stimulating higher-order thinking (Bligh, 1972);
- lectures do not provide feedback and social interaction (Bligh, 1972).

However, despite such research results on lecturing, “the lecture survives, probably because it serves many functions not so well observed in the present research” (Fritze & Nordkvelle, 2003, p. 328). Rodd (2003) argues that “university mathematics departments recognise the potential of lectures, not as information-delivery venues, but as a place where the ‘awe and wonder’ of mathematics can be experienced” (p. 20), and claims that ‘active participation’ and ‘identity and community’ can also be experienced as a ‘witness’, similar to the act of experiencing a theatre play. Effects of inspiration may thus be an essential outcome from a good lecture. Related to this, the lecturer as a person, and humour, have been seen as critical for how lectures are appreciated (Fritze & Nordkvelle, 2003). The notion of *teacher immediacy* (Frymier, 1994) thus refers to issues of closeness in classroom student-teacher interaction. Arguments for the importance of *personalisation* in a mathematics context can be found in several studies (Anthony, 1997; Anthony, Hubbard, & Swedosh, 2000; Forgasz & Swedosh, 1997), and the lecturer can be seen to provide a personalisation of the formal mathematical discourse (Anthony et al.; 2000; Wood & Smith, 2004). Inspiration is also emphasised by Alsina (2001, pp. 3-6). Weber (2004) identified three teaching styles used in undergraduate mathematics (small group) lecturing. In the *logico-structural* style a strictly formal way of working was used. While the *procedural* lecture style had the main focus on the technical work, the *semantic* style emphasized the intuitive meanings of the concepts. In large group lectures these styles can be mixed (e.g. Bergsten, 2007), as well as the linguistic modes used. Wood and Smith (2004) noted that “[l]ecturing is a mixed mode activity” (p. 3), using verbal and non-verbal means to organise students’ attention to “written language, mathematical notations, visual diagrams” (p. 3). The authors also noted that “in the spoken text... the lecturer makes use of a range of words like *actually*, *fairly*, *obviously* to personalise and introduce values and judgments into the presentation” (p. 7). Wood and Smith conclude that these

differences of modes and representational forms require a lot from the students. The study by Anthony (1997) showed that the importance given to lectures was higher by students than by lecturers, both for success and failure, including “boring presentations of lectures” and “non attendance of lectures.” Students also “placed more importance than lecturers on active learning and note-taking” during lectures. Successful students found “the availability of worked examples in lectures and tutorials” and “clear presentation of lectures” more important than did non-successful students (pp. 60-61). Moreover, the study by Hubbard (1997) showed that students value the information about what is “important” provided by lectures but are often dissatisfied with the format of a lecture as well as lecturers’ ability to teach. This may well be due to a discrepancy in beliefs and perceptions about the role of lectures between lecturers and students (Anthony et al., 2000, p. 250).

In the case study by Bergsten (2007) of a mathematics lecture in a large group of first year engineering students, a key observation was how the complexity and richness of different educational aspects of a lecture in undergraduate mathematics can come into play during a time of only 90 minutes. Within a TPA format (theorem-proof-application), the observed lecture/lecturer was content-driven and rich in information; used a mixed mode of semantic-procedural teaching styles; exhibited a formal separation of theoretical and practical knowledge; showed an overall strong coherence, higher in rigour than the aims stated; displayed a dominance of an algebraic mode over an imagistic; was rich in gestures and informal language but at the same time establishing mathematical norms for the course and for Mathematics; created a relaxed atmosphere of doing mathematics as it seems together; and used various semiotic means to objectify the target knowledge, as he conceived it, for the students. With an aim to structure this complexity, Bergsten (2007, pp. 69-70) outlined a systemic triangular model for critical characteristics of a mathematics lecture, consisting of mathematical exposition, teacher immediacy, and general quality criteria for mathematics teaching. Mathematical exposition involves the dynamic interplay of mathematical content, mathematical process, and institutionalisation.

## **A CASE STUDY**

### **Method**

Lectures make up a substantial component of beginning calculus courses in many university engineering and science programmes. These lectures are directed towards large student populations and thus have a major impact on mathematics education at the tertiary level. It was therefore an obvious choice for this study to choose lectures within this context. The case studied, two experienced lecturers running a calculus course with engineering students, one group of 141 engineering students specializing in industrial economy, and one group of 132 students studying physics and electric engineering, thus represents a common situation for the problematique in focus.



## Working Group 14

All data were collected by the author, who videotaped the lectures (which took place in an early part of the course), distributed and collected a questionnaire immediately after the lectures, and interviewed a sample of the students attending the lecture (after the lectures). In this paper only data from the questionnaire will be reported.

Bergsten (2007) identified critical aspects that could influence students' appreciation and attendance of lectures. These aspects provided the basis for the questionnaire used in the present study. To enhance validity, the questions in Part 2 (see Table 1) were discussed with a research colleague and tried out in a small pilot study before the final questionnaire was used. It employed four level Likert items and was structured in three main parts; Part 1: four items concerning attendance, reading and note taking (see Table 2); Part 2: 16 items on views about lectures in general; Part 3: 18 items on views about the particular lecture just attended. The answers could be commented and a final open question was asked. In this paper the analysis will focus only on Part 1 and Part 2 of the questionnaire. The items in Part 2 were designed to address the overall question *Why do students go to lectures?* more or less explicitly (item group A), more implicitly by asking on some specific issues that should be provided by lectures (item group B), or by asking if some aspects count as important for mathematics lectures (item group C). Finally, two questions were asked (item group D) to investigate whether students find that lectures generally employ too much formal mathematics. On the following statements in Part 2 of the questionnaire the students were asked to mark one of the categories Fully Agree, Agree, Partly Agree and Disagree. The items from the different groups were mixed in the questionnaire according to their order numbers given in Table 1:

<i>Group A items</i>	<p>1. It is easier to understand the course material by attending a lecture than reading the textbook only</p> <p>2. It is from the lectures that one understands what is important for the examination</p> <p>7. It is by way of the lectures that I learn to understand what proof is and why proof is needed</p>
<i>Group B items</i>	<p>4. A mathematics lecture should link the pure mathematical results to applications outside of mathematics</p> <p>6. A mathematics lecture should present formal definitions and proofs for those mathematical concepts and results that are being treated</p> <p>12. Lecturers in mathematics should sometimes use everyday language, gestures and diagrams to illustrate the mathematical concepts and methods used</p> <p>14. A lecture should provide extra inspiration to continue working with mathematics</p> <p>15. A lecture should make the strength and beauty of mathematics visible</p> <p>16. A lecture should include some humour</p>
<i>Group C items</i>	<p>3. It is better when the lectures cover as much as possible of the content of the course, instead of going into some selected topics more deeply</p>

### Working Group 14

	<p>5. In lectures in mathematics it is important to show methods for solving theoretical tasks and tasks where mathematical definitions and theorems must be used</p> <p>8. It is more important that a lecturer in an intuitive way explains the meaning of the mathematical concepts and results than providing formal definitions and proofs</p> <p>10. It is better to highlight the important ideas in a proof than doing each step of the proof in detail</p> <p>11. In a lecture it is more important with examples that show how to solve tasks than to prove the theorems that are used</p>
<i>Group D items</i>	<p>9. The lectures in mathematics normally employ too much algebraic calculations</p> <p>13. The lectures in mathematics normally employ too much proof</p>

**Table 1: Part 2 of the questionnaire with items ordered by groups.**

In addition to answer frequencies, a simple correlation analysis was performed to base the interpretation of the data. The answers were also analysed for each of the two lecture groups, showing no significant difference between the groups concerning the frequency distributions, or pair wise item correlations for Part 2. This homogeneity can also be interpreted as an indication of reliability, as the groups are comparable in terms of previous mathematics studies.

### Questionnaire results

In this section the results from the questionnaire will be presented before discussing them in the section that follows. Table 2 shows that more or less all students who attended the observed lectures always go to the calculus lectures, while the attendance to the classes (study groups) is lower but still high. A great majority of the students also take extensive notes during lectures. This strong focus on the importance of lectures is also witnessed by the high percentage of students who do not study the corresponding sections in the textbook in parallel.

<b>Statement \ answer category</b>	<b>always</b>	<b>often</b>	<b>sometimes</b>	<b>rarely</b>	<b>N</b>
I attend the lectures in this course	92	7	0	1	273
I attend the classes in this course	58	33	6	3	273
	<b>almost everything</b>	<b>a lot</b>	<b>a little</b>	<b>almost nothing</b>	
During lectures I normally take notes on	85	11	2	2	271
	<b>before</b>	<b>after</b>	<b>both</b>	<b>neither</b>	
I normally read the relevant chapter in the textbook before or after the lecture	8	35	12	44	266

**Table 2: Part 1 of the questionnaire: Statements and frequencies (%).**

## Working Group 14

In Part 2, the answers to item group A (see Table 3) point to the common view among the students that by attending lectures it is easier to understand the content than only by textbook study; one understands what is important for the exam; and one learns to understand what proof is and why it is needed.

On group B items the students agreed or strongly agreed that a lecture should present formal definitions and proofs for those mathematical concepts and results that are being treated; sometimes use everyday language, gestures and diagrams to illustrate the mathematical concepts and methods used; provide extra inspiration to continue working with mathematics; make the strength and beauty of mathematics visible; and include some humour. The students agreed or partly agreed (on item 4) that a lecture should link the pure mathematical results to applications outside of mathematics.

Item	Fully Agree	Agree	Partly Agree	Disagree	Mean	Median	N
1	81	15	4	1	1.3	FA	273
2	25	51	22	2	2.0	A	271
7	23	40	31	6	2.2	A	269
4	16	41	36	7	2.4	A	269
6	28	42	25	5	2.1	A	269
12	56	31	10	3	1.6	FA	273
14	41	40	16	3	1.8	A	272
15	34	41	20	5	1.9	A	269
16	59	29	8	3	1.6	FA	273
3	33	42	21	4	1.9	A	271
5	62	30	7	1	1.5	FA	270
8	45	40	13	2	1.7	A	269
10	40	34	19	7	1.9	A	267
11	43	31	22	4	1.9	A	273
9	1	12	51	36	3.2	PA	263
13	11	29	42	18	2.7	PA	272

**Table 3: Frequencies (%), means and medians for Part 2 of the questionnaire (ordered by item groups; A: 1 2 7, B: 4 6 12 14 15 16, C: 3 5 8 10 11, D: 9 13).**

Within item group C the students agreed or strongly agreed that it is better when the lectures cover as much as possible of the content of the course, instead of going more deep into some selected topics, indicating that students want lectures to give a full overview of the course; it is important to show methods for solving theoretical tasks and tasks where mathematical definitions and theorems must be used; it is more important that a lecturer in an intuitive way explains the meaning of the mathematical concepts and results than providing formal definitions and proofs; it is better to highlight the important ideas in a proof than doing each step of the proof in detail, and it is more important with examples that show how to solve tasks than to prove the theorems that are used.

The answers to item group D showed that the students generally do not find that lectures employ too much algebraic manipulations or proof. Individual correlations

## Working Group 14

between the items 1-16 are shown in Table 4, which is structured by item groups A-D (within group correlations are marked in bold). The item groups A, B, C and D are analytically defined and not empirically observed categories. Therefore correlations between items from different groups are also to be expected, due to their content in other aspects than those defining the groups. Nevertheless, it is evident from Table 4 that items within the item groups are positively correlated (to different degrees), most strongly in group B with items related to the teacher immediacy dimension. One exception here is item 6. This item correlates naturally with item 7 and that items 6 and 7 both have a negative correlation with item 13 makes sense. However, the negative correlation between item 6 and both items 8 and 11 opens up for the interpretation that the students want the lectures to present formal aspects of mathematics but that intuitive and exemplary ways of explaining are seen as more important ingredients of a calculus lecture. The strong correlations between the three items 8, 10 and 11 within group C point to a preference among the students for a more intuitive presentation of mathematics than giving full formal proofs, an interpretation supported also by the positive correlations between item 13 and items 8-12. Item 12 correlates naturally with several items (on intuition, ideas and examples), thus indicating how those presentations are preferred.

2	27														
7	<b>10</b>	<b>9</b>													
4	0	14	2												
6	8	4	36	<b>7</b>											
12	20	16	5	<b>17</b>	<b>-8</b>										
14	12	3	6	<b>11</b>	<b>2</b>	<b>26</b>									
15	-4	-3	14	<b>11</b>	<b>14</b>	<b>15</b>	<b>46</b>								
16	23	5	6	<b>9</b>	<b>-3</b>	<b>27</b>	<b>35</b>	<b>39</b>							
3	11	16	1	14	-2	9	2	7	-1						
5	19	6	-2	14	0	15	6	-2	9	<b>4</b>					
8	14	12	-19	11	-27	24	9	1	11	<b>14</b>	<b>16</b>				
10	15	13	-3	0	-18	21	2	-6	5	<b>12</b>	<b>13</b>	<b>20</b>			
11	8	5	-14	14	-35	35	9	-7	11	<b>9</b>	<b>14</b>	<b>33</b>	<b>31</b>		
9	-14	-4	-1	12	-13	13	2	2	6	6	-8	3	3	20	
13	-3	9	-21	14	-42	27	6	-5	5	2	10	29	18	42	<b>29</b>
item	1	2	7	4	6	12	14	15	16	3	5	8	10	11	9

**Table 4: Pearson correlations between individual items of Part 2 of the questionnaire.**

To the final open question: *What is it that makes a good lecture?*, there was a great variety among the 200 answers delivered. A first categorization of around 50 different aspects was made, where the most common were the following (with the number of comments found in brackets along with some illustrative student quotes):

- The lecturer as a person (100; e.g. inspiration 38, humour 26, engagement 17, pedagogical 9, using easy and clear language 5). Quotes: *The charisma of the lecturer!* ; *Humour and contact with the audience*; *That it is easy to listen to the lecturer, not boring or monotonous*; *The lecturer speaks clearly and uses*

*an everyday language. The lecturer should also be funny so that I do not fall asleep during the lecture.*

- Demonstration of examples (40; how to do it, usefulness for exams). Quotes: *Examples, examples, examples, how to solve the problems. Tricks and methods to approach the tasks; Many examples, applications so that one understands how to use theorems/proofs.*
- Good explanations, to support understanding (35; one can follow, what it is for, why it works). Quotes: *When it says click and you feel you understand a problem and the structure of its solution, then mathematics all of a sudden is fun!; Illustrative diagrams and the engagement of the lecturer.*
- Clarity (33; not skipping linking details, structure). Quotes: *A good structure, with a mix of theorems, proofs, examples and humour; Structure and good writing on the whiteboard; That you understand what the lecturer is saying.*
- A comfortable pace (20; not too quick, not too slow, calm). Quote: *Pace combined with understandability.*
- Coherence (15; read thread). Quote: *A red thread that is interesting to follow.*
- Good mix between theory and methods (11). Quote: *A good mix of examples and theory, and an inspiring lecturer.*

Some students gave only one or two aspects, while others combined several to a whole. Further comments pointed to the importance of a good balance of level of difficulty and pace, also to be able to take notes. Interesting mathematics and applications were asked for, as well as providing a good complement to the textbook and aha-experiences, and be forced to be active.

## DISCUSSION

Students in this study go to lectures. In the previous study (Bergsten, 2007), the interviewed lecturer also stated that his students normally go to the lectures:

I don't know why. It is an easy way to get something done, they think they can use things from the lecture, collect materials, thinking the lecturer will say something that is useful for the exam. (p. 63)

The data presented here show that students do not only value such usefulness. Explicit answers to the main question *Why do students go to lectures?* point to the view that it is easier to understand the mathematical content of the course by attending lectures than only by studying the textbook. Another reason is that one understands what is important for the examination, as found also by Hubbard (1997), and what mathematical proofs are and why they are needed. There is a strong emphasis on the value of both, intuitive explanations (diagrams, metaphors, gestures) and formal presentations (definitions, proof, algebraic calculations), as well as examples, but less strong need of applications from outside mathematics. The data indicate that the semantic mode of presentation is a stronger factor in attracting students to lectures than the procedural or logico-structural style (in the sense of

Weber, 2004), even if clarity and coherence were emphasised. It was a strong agreement among the students about the importance of teacher immediacy, that the mathematics lecturer should inspire to study mathematics and use humour, make the strength and beauty of mathematics visible, and use everyday language, gestures and diagrams to illustrate the mathematical concepts and methods used. The dissatisfaction with the lecture format and the lecturers' teaching ability, as found by for example Hubbard (1997), was completely absent in the questionnaire responses reported here. The high attendance as well as the closed form items and the open comments indicate, rather, that the opposite view was dominating. In relation to Bergsten's (2007) triangular model for a quality lecture, the importance of the dimensions teacher immediacy and mathematical exposition find strong support in the data presented here, while general criteria for quality teaching (Biggs, 2003), not directly addressed in the questionnaire, are more implicit in aspects such as coherence, clarity, applications, pace, and student activity in terms of note taking.

That lectures have impact on the students at a diversity of aspects and levels was evidenced by the great variety of descriptions on what characterizes a good lecture. This points to the complexity and "richness" of the lecture format, as discussed above with reference to Bergsten (2007). It is clear from the students' open comments in the present study that one main reason for the "success" of a lecture is given to the lecturer as a person, being able to engage and inspire the students. This emphasis on teacher immediacy has been pointed to by several authors and can be seen as part of the game of lecturing, as features of acting. Illustrating also adds non-symbolic and non-discursive elements to the semiotic objectification of knowledge, which can function as critical elements in the meaning-making processes of the students. Wood, Joyce, Petocz and Rodd (2007) found that despite availability of textbooks and online material, for their learning students value lectures higher.

Other aspects mentioned by many students in the questionnaire, such as clarity and coherence, a mix of (many) examples and theory, an optimal pace, clear language and writing, can be related more to lecturers' pedagogical awareness (Nardi, Jaworski, & Hegedus, 2005) than personal *charisma*. An aspect less discussed in the literature is the importance of the opportunity for students to take notes. From the data in this study, including student interviews not reported here, it is evident that this is a student activity component of the lecture format that is essential for what the students' get out from attending a mathematics lecture.

The results presented here will be critically discussed in the full study, where all questionnaire data will be analysed in relation to student interviews and an analysis of the video recorded lectures as well as interviews with the lecturers.

## ACKNOWLEDGEMENT

The author wishes to thank Eva Jablonka for sharing her knowledge in the field through critical comments during the writing of this paper.

## REFERENCES

- Alsina, C. (2001). Why the professor must be a stimulating teacher. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: an ICMI Study* (pp. 3-12). Dordrecht: Kluwer.
- Anthony, G. (1997). Students' and lecturers' perceptions of factors influencing students' success in first-year undergraduate mathematics courses. In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (Proceedings of the 20<sup>th</sup> annual conference of MERGA, pp. 58-64). Rotoura, NZ: MERGA.
- Anthony, G., Hubbard, R., & Swedosh, P. (2000). Undergraduate mathematics teaching and learning. In K. Owens & J. Mousley (Eds.), *Mathematics education research in Australasia 1996-1999* (pp. 243-269). Ascot Vale, VIC: MERGA.
- Bergsten, C. (2007). Investigating quality of undergraduate mathematics lectures. *Mathematics Education Research Journal*, 19(3), 48-72.
- Biggs, J. (2003). *Teaching for quality learning at university*. (Second edition) Berkshire: Open University Press.
- Bligh, D. (1972). *What's the use of lectures?* Harmondsworth, Middlesex: Penguin.
- Bradley, K., Sampson, S., & Royal, K. (2006). Applying the Rasch rating scale model to gain insights into students' conceptualisation of quality mathematics instruction. *Mathematics Education Research Journal*, 18(2), 11-26.
- Forgasz, H. & Swedosh, P. (1997). The University of Melbourne's mathematics students: Who are they? Why are they? Will they stay? In F. Biddulph & K. Carr (Eds.), *People in mathematics education* (Proceedings of the 20th annual conference of MERGA, pp. 170-176). Rotoura, NZ: MERGA.
- Fritze, Y., & Nordkvelle, Y.T. (2003). Comparing lectures: effects of the technological context of a studio. *Education and Information Technologies*, 8(4), 327-343.
- Frymier, A.B. (1994). A model of immediacy in the classroom. *Communication Quarterly*, 42, 133-144.
- Holton, D. (Ed.). (2001). *The teaching and learning of mathematics at university level: an ICMI Study*. Dordrecht: Kluwer.
- Hubbard, R. (1997). *Why do students attend lectures and what do they learn?* Paper presented at the Delta'97 Symposium of Undergraduate Mathematics, Brisbane.

#### Working Group 14

- Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly*, 102(3), 227-242.
- Nardi, E., Jaworski, B., & Hegedus, S. (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From “tricks” to “techniques”. *Journal for Research in Mathematics Education*, 36(4), 284-316.
- Rodd, M. (2003). Witness as participation: the lecture theatre as a site for mathematical awe and wonder. *For the Learning of Mathematics*, 23(1), 15-21.
- Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor’s lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23, 115-133.
- Wood, L., & Smith, G. (2004). *Language of university mathematics*. Paper presented at Topic Study Group 25 (Language and communication in mathematics education) at ICME10, Copenhagen 2004. See <<http://www.icme10.dk>>
- Wood, L., Joyce, S., Petocz, P., & Rodd, M. (2007). Learning in lectures: multiple representations. *International Journal of Mathematics Education in Science and Technology*, 38(7), 907-915.



# DESIGNING AND EVALUATING BLENDED LEARNING BRIDGING COURSES IN MATHEMATICS

Biehler, R.\*; Fischer, P.R.\*\*; Hochmuth, R.\*\*; Wassong, Th.\*

\* University of Paderborn, Germany, \*\* University of Kassel, Germany

*Varying mathematical skills, rising dropout rates and growing numbers of first year students confront the universities with major organizational and pedagogical problems. This paper describes an innovative way of teaching and learning that claims to improve this situation by specific bridging courses particularly including self-diagnostic e-assessment and supporting self-regulated learning. In order to give an overview of our whole bridging-course programme we will discuss our material with regard to content-related and pedagogical aspects as well its integration in various course scenarios. Focusing on selected results of an accompanying evaluation study we will finally substantiate the acceptance and success of our courses and highlight some interesting findings regarding our learners.*

Keywords: bridging courses, eLearning, blended learning environments.

## INTRODUCTION

The transition between school and university studies is a difficult one. The gap between school and university seems to be larger in mathematics than in other subjects (cf. Gueudet, 2008, Bescherer, 2003, de Guzman, 1998, Holton, 2001, Tall, 1991).

In 2003, the project VEMA – “**V**irtuelles **E**ingangstutorium **M**athematik (Virtual Entrance Tutorial for Mathematics) (<http://www.mathematik.uni-kassel.de/vorkurs>) started the development of multimedia resources primarily for supporting the pre-term bridging courses, which are intended to bridge this gap. VEMA was initiated at the University of Kassel and was extended to the Universities of Darmstadt and Paderborn later on. During the years the project extended its concern and redesigned the whole pre-term courses by new course scenarios that better integrate the multimedia learning material into the course. The material as well as the course scenarios have been continuously improved taking into account our yearly experiences and evaluations.

## THE INTERACTIVE MATERIAL OF VEMA

### The content of the VEMA-material

In order to support students to individually recapitulate certain topics we decided to structure the content into small learning units called “modules”. Each module essentially concentrates on one mathematical topic. In its latest version the learning material contains six chapters: “Arithmetic”, “Powers”, “Functions”, “Higher Functions”, “Analysis” and “Vectors”. On average each chapter has about 10

modules. The clear module-structure supports self-regulated learning and helps teachers in choosing content suitable for the field of study of their students.

### Structure of a learning unit

We chose a well-defined and consistent structure for all modules, i.e. each module consists of identical types of *knowledge units*. This structure helps learners in their navigation through the material, which is further supported by the layout of the interactive book: There are two navigation frames. One enables learners to choose the modules to which they like to switch and the other enables selecting the different units of a chosen module by clicking on the corresponding icons.

The structure of a module mainly consists of the units: *overview*, *introduction to the domain*, *info*, *Info/Interpretation/Explanation (IIE)*, *application*, *typical mistakes*, and *exercises*. Before and after learning with a module, learners can perform a *diagnostic test* to assess their knowledge of the domain.

- i. The *diagnostic pre-test* contains 4 to 5 exercises and gives the opportunity to the students to test their pre-knowledge concerning the content of the module. After a student has performed all tasks, the system automatically corrects his answers and provides feedback in form of a score for each exercise and for the test as a whole, a model solution, an individual feedback on his mathematical competencies and provides learning advice for further working on the module. With this individual feedback the students are supported in structuring their learning.
- ii. Then the modules start with the *overview* unit, which essentially consists of a list of the major topics and learning goals.
- iii. The second unit is called *introduction to the domain*. It uses discovery-based, inductive and exemplary approaches to familiarize the learner with the content. We also support the knowledge construction process by interactive exercises: learners have the opportunity to make mistakes, to withdraw them and to recapitulate the task until finding a correct solution. The content is presented to learners on a concrete level, with visualizations and references made to their assumed previous knowledge.
- iv. The third *info* unit lists the definitions, theorems and algorithms of the module. These are the central concepts of the module. The *info unit* presents the content on an abstract mathematical level, the pure definitions, theorems and algorithms are presented without examples or exercises.
- v. The fourth *IIE* unit (*Info / Interpretation / Explanation*) repeats the central definitions, theorems and algorithms of the *info* unit. A network to other concepts is built. Illustrations, concrete examples and explanations are added. In case of theorems one can find plausible arguments and/or proofs for their correctness. The learners also find interactive exercises, flash-films and animations they can interact with and which help them to developing a deeper

understanding of the concepts. Since the learners can look at the concepts from various perspectives the memorisation of knowledge is supported, too.

- vi. The fifth unit is called the *application* unit: Here such applications inside and outside mathematics are presented that show the connection of the actual domain to other mathematical and non-mathematical domains. This unit may contain examples e.g. from engineering contexts that are relevant for the engineering students but may be also relevant for other students to see the practical relevance of mathematics. The inner-mathematical applications are used to connect the definitions, theorems and algorithms within mathematics.
- vii. The sixth unit is called the *typical-mistakes* unit: In this unit erroneous argumentations or solutions are presented to the learner, who is invited to find the mistakes, to correct them and to explain possible reasons for them. These exercises are provided to train the learners' diagnostic competencies and to depict misconceptions in order to avoid them in the future. The learners can check their answer by comparing to a correct argumentation or solution. For future mathematics teachers this is important for training their diagnostic competence (cf. Wittmann 2007).
- viii. The last unit is the *exercises* unit: This unit is important for the learners for checking their understanding of the topic and to give opportunities for practicing the concepts. For each exercise a model solution is available to compare own solutions with. These model solutions can also be used as hints for getting an initial idea or for helping when the solution process gets stuck.
- ix. The *diagnostic post-test* has the same structure as the diagnostic pre-test. Its idea is to give the students the opportunity to check their performance after having worked on a module. The diagnostic pre- and post-tests also aim at the elaboration of the students' abilities in self-regulation and self-evaluation, which are major factors for successful learning (Ibabe & Jauregizar, 2010).

## **TYPES OF BLENDED LEARNING SCENARIOS**

For our bridging courses we combined self-directed and externally-regulated learning types of instructional formats (cf. Niegemann et al., 2008, p.66). Both formats have their justification in the specific case of bridging courses: on one hand learners are new at the university, so they have to acclimatise themselves with the new learning environment. Here attendance phases can help them to familiarize before the terms start. On other hand learners at university level have to be more self-directed in their learning than at school. Here eLearning phases can help to adapt their learning behaviour (cf. Mandl & Kopp, 2006). For our bridging courses we developed two different blended-learning course scenarios: a course scenario with an extensive attendance part (P-course) and a course scenario with an extensive eLearning part (E-course). When registering to the bridging courses each learner can freely choose between these types according to individual preferences.

### **The P-course**

This course scenario is structured and led by the teacher while the learner has fewer opportunities for self-regulated learning. The course lasts 4 weeks; each week consists of three days with attendance at university with three hours of lectures and two hours of practice-session each. The remaining days are free for individual learning and homework. This homework consists of two parts: one part has exercises on the topics that were taught in the lectures and another part has specific tasks for individual working on the modules, aiming at recapitulating or preparing content for the next attendance day. Some of the diagnostic tests are available and recommended to the students.

### **The E-course**

This course covers 4 weeks with 6 days attendance at the university. The remaining time is to be spent for online learning. The first week starts with one or two orientation days, where the learners are introduced to the learning system and course material and get advice how to learn with the material. The first modules are presented by means of lectures. Later in the course there is only one attendance day at the end of every week. The learners have the opportunity to ask questions about the content in the first part of the morning session and can pre-select the topics for the lectures in the second part of the morning. The afternoon is devoted to small group learning with exercises related to the content of the morning lectures. The small group work is supported by a tutor.

The rest of the learning time is free for learning with the online resources. Questions that come up in this process can either be asked and discussed on the next attendance day, posted in the forum of the learning platform or posed to the human online tutor, who is available during all normal working hours, including opportunities for online chatting. Moodle supports the learners in choosing their learning paths: The diagnostic tests with the individual feedback support students in structuring their learning, and a list of recommended modules for every study programme helps to identify the most important topics. Besides, we provide a text that explains the use of the material, the diagnostic tests and the role of the days at the university

### **THE EVALUATION-STUDY**

In context of his PhD-project the second author of this paper extensively investigated the 2008 bridging courses in Kassel. His PhD project aims at designing, evaluating and refining the bridging course scenarios as described above. Major questions of the study were the identification of the reasons for the students' choice of the course variants, the description of the participants concerning personal aspects, the investigation of the course effects on the learners' performance and attitudes, the analysis of the acceptance and the rating of both, courses and learning material, and the investigation of the students' use of the learning material (cf. Fischer 2008).

For data collection, three questionnaires, one at the beginning, one in the middle and one at the end of the course, and two assessment tests were used. The questionnaires were anonymous online-forms requiring a personal key that enables us tracing the students' answers while keeping the students anonymous to us. Part of the items were adapted from different studies (Prenzel et al., 2002, Baumert et al., 2008, Bescherer, 2003) and items from the general course evaluation questionnaire of Kassel University. Thus we composed a new instrument for an investigation of blended learning scenarios for mathematical bridging courses. An electronic pre- and post-test was administered under exam conditions in a computer room for measuring students' mathematical proficiency levels. While the pre-test included exercises from school-mathematics, the post-test focussed on the bridging courses' content. In the following we can discuss only a few selected results of the study.

### The courses from the learners' perspective

For investigating the acceptance of our bridging courses in general as well as the two course scenarios in specific, the students had to answer to three questions: 1. *"In general I was satisfied with the bridging course"*, 2. *"The participation in the bridging courses is absolutely recommendable"* and 3. *"I would decide for the E-/P-course of the bridging course again"*. A Likert type scale with four answering categories was used here: (1) "is not true", (2) "is rather not true", (3) "is rather true" and (4) "is true".

Question	P-course			E-course		
	M	SD	N	M	SD	N
1: <i>"In general..."</i>	3.57	0.62	254	3.64	0.53	96
2: <i>"The participation in..."</i>	3.69	0.62	254	3.69	0.56	96
3: <i>"I would decide for ..."</i>	3.67	0.68	254	3.48	0.79	96

**Table 1: Results for questions concerning the acceptance of the courses.**

Table 1 reveals very high scores for the courses in general and similar results for the two course types. Hence we can state that the learners were very satisfied with the bridging course type they had chosen. This proves the success of our course design decisions from this point of view.

### Results from pre- and post-test assessments

The pre-test showed very similar results for both course types, but for the post-test, the results in the E-course are even better than in the P-course (see Table 2).

An analysis of variance for the results of the post-test considering the course variant as dependent variable and the results of the pre-test as covariant showed that the difference in the results of the course variants is highly statistically significant.

Test	Results for P-course			Results for E-course		
	M	SD	N	M	SD	N
<b>Pre-test 2008</b> Maximum: <b>19 points</b>	8.52	3.14	226	8.52	3.64	146
<b>Post-test 2008</b> Maximum: <b>20 points</b>	9.21	3.13	131	10.93	4.02	72

**Table 2: Assessment results.**

Since we only hoped to achieve at least comparable results for both course types in order to disprove the argument that the E-course may be a popular scenario for some students but it will not improve the students' performance as much as by traditional scenarios, we were happy to have such positive results.

### **Students' reasons for choosing a course variant**

The students had to indicate which factor of a given list was relevant for their decision for a course variant and how important the respective factor was (Likert type scale). For each factor we calculated the mean in order to identify reasons with a high impact and reasons with a low impact.

For the **E-course** we found that the mean scores for extrinsic factors such as job-related restrictions, living situation, being on vacation or other external reasons had low values between 1.24 and 2.4. In contrast, the questions for intrinsic reasons revealed high mean scores from 2.73 to 3.52. Therefore we can interpret them as main factors for the decision: this includes reasons concerning the opportunity for a more self-regulated learning within the E-course, the possibility of individual timing as well as a personal interest in eLearning as a learning method. It is not surprising that the reduced numbers of days with compulsory attendance was a further important reason for the students' choice ( $M = 2.7$ ).

The results for the **P-course** showed again that extrinsic reasons like the availability of a computer, the internet or an internet-flat rate had very low mean scores from 1.06 to 1.32. Aversions to learning with the computer ( $M = 2.13$ ) or bad experiences in eLearning ( $M = 1.33$ ) were also reasons with a low impact. Instead the opportunities of personal contact with other students ( $M = 3.4$ ) and with the teacher ( $M = 3.64$ ) as well as the opportunity of experiencing typical lectures were reasons with high mean scores (between 3.4 and 3.64) and can therefore be interpreted as main factors. We also asked for doubts in one's ability of self-regulated learning ( $M = 2.6$ ) and doubts concerning the method eLearning itself ( $M = 2.61$ ) but these results show that these are not strong factors for or against the choice of the course variant.

At the beginning of the study we assumed that especially those students would decide for the E-course who either have an affinity to working with the computer or who already have made (positive) experiences in learning with the PC. That's why

we asked for these aspects in both course-scenarios and surprisingly found no substantial differences between the answers of the P- and E-course participants:

Question	Results for P-course			Results for E-course		
	M	SD	N	M	SD	N
<i>"I have already experiences in eLearning"</i>	1.97	0.4	376	1.95	0.33	209
<i>"I like to learn with the PC"</i>	3.23	0.74	376	3.4	0.7	209
<i>"In the last year in school I have already learnt with a PC"</i>	3.32	1.58	376	3.39	1.55	209

**Table 3: Results for questions on learning with computer. Answering categories for the third question: (1) almost every day, (2) 2-5 times a week, (3) about once a week, (4) 1-2 times per month, (5) less often, (6) never.**

### Usage of the learning material within the E-course

Within the E-course the students were asked questions concerning their use of the diagnostic tests and of the modules. The participants had to indicate how often they had used the diagnostic tests. The results can be found in Table 4:

Test	(1) practically all	(2) most of them	(3) some of them	(4) barely none	M	SD	N
<b>Pre-tests</b>	28.5%	33.8%	22.5%	15.2%	2.25	1.03	151
<b>Post-tests</b>	19.9%	30.5%	23.8%	25.8%	2.56	1.08	151

**Table 4: Use of the diagnostic tests.**

The results show a slightly higher average usage of the diagnostic pre-tests, which is also supported by the user data that were collected in moodle: The number of pre-test-users is always higher than the respective number for the post-tests. The variability in the test usage is fairly high.

We also asked the participants to indicate how helpful the diagnostic tests were for them. Those students who didn't use the pre-tests (10.6% of the interviewees) or the post-tests (19.9%) could indicate it separately and were filtered out. The following table shows very positive results for both test types.

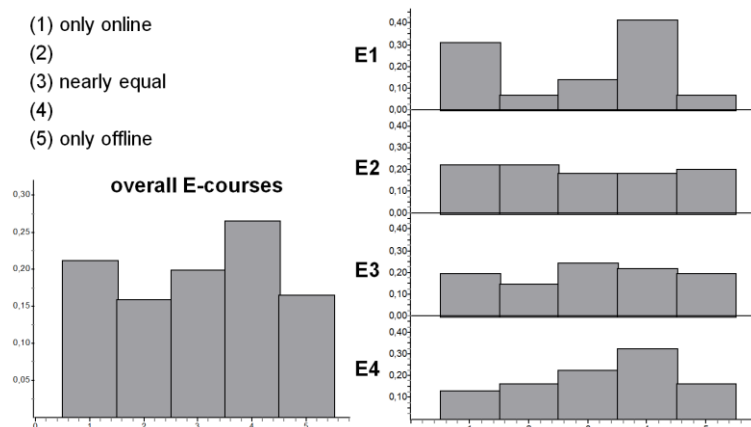
Tests	(1)	(2)	(3)	(4)	(5)	(6)	M	SD	N
<b>Pre-tests</b>	30.4%	42.2%	19.3%	6.7%	0.7%	0.7%	2.07	0.97	135
<b>Post-tests</b>	21.5%	48.8%	23.1%	4.1%	2.5%	0%	2.17	0.9	121

**Table 5: Acceptance of the diagnostic test by those you used them. Answering categories: (1) "helpful" ... (6) "not helpful at all".**

## Working Group 14

Since the students had the opportunity to use a CD offline instead of learning online with moodle, we asked them “*Did you learn online within the learning platform or offline with the CD?*” For this we used a scale with options from (1) “only online” over (3) “nearly equal” up to (5) “only offline”. The quite high spread of  $SD \approx 1.4$  however revealed quite varying student opinions, so we decided to have a more detailed look at the results.

Hence we analyzed the results differentiating four different groups in view of the fields of study: E1 (electrical engineering & computer science), E2 (construction engineering & mechanical engineering), E3 (bachelor of mathematics or science & mathematics teachers for grammar schools) and E4 (teachers for primary and lower secondary schools):



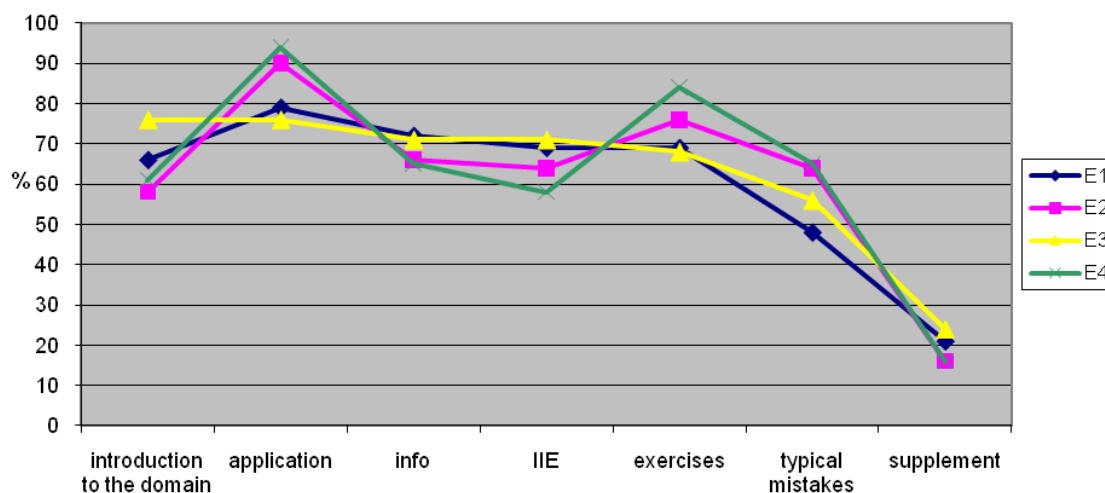
**Figure 1: “Did you learn online within the learning platform or offline with the CD?”**

It is noticeable that the groups E2 and E3 answered very similarly, while group E4 tends to learn offline. For group E1 we can identify two subgroups: One that is only learning online and another one that is learning almost only offline. Further data analyses showed that this split into subgroups can neither be explained by gender nor by the field of study (construction engineering & mechanical engineering).

Obviously there seem to be typical learning approaches that depend on the field of study, while others are independent of it. This assumption is further emphasized by an analysis of the students’ use of the modules. We asked the students to indicate for each module unit within the first three chapters how intensively they have typically used them. We calculated the percentage of all users that indicated an intensive usage and visualized the results for the groups E1-E4 in figure 2. The y-axis of this diagram displays the percentage of “intensive users” of the respective module unit that can be found on the x-axis. For comparing the profiles of the different groups, we sorted the units on the x-axis with respect to the results of group E3 (Bachelor of mathematics and science, mathematics teacher for grammar schools) in decreasing order.



## Working Group 14



**Figure 2: Percentage of intensive users for the units of a module.**

We see that the learning profiles of the groups E1 and E3 are very similar as well as the profiles of the groups E2 and E4. This result was not expected since the fields of study of the groups would rather imply a different pairing. This suggests that it is not only sensitive to evaluate the courses with regard to the variants and the fields of study but also to classify different types of learners and to explore them.

## PERSPECTIVES

The second author of this paper is currently working on different aspects of the evaluation study in the context of his PhD project. We have collected data on learners attributes e.g. personality, motivation, attitude towards mathematics and abilities in self-estimation and self-regulation. A classification of different types of learners and of typical learning strategies will be related to the learning behavior in the course and the effects of the courses on students' mathematical knowledge and attitudes. The data on students' rating of different elements of the courses and the learning material will be used for identifying aspects for improvement of the course design. We also revise our diagnostic tests, develop new content and design a new course structure in moodle for a better integration of interactive material. We expect that the instruments that we have developed will also be useful for the evaluation of blended learning bridging courses in general.

## NOTES

This paper is partly elaborated in Biehler, R., Fischer, P. R., Hochmuth, R., & Wassong, Th. (in press). Self-regulated learning and self assessment in online mathematics bridging courses. In A.A. Juan, M.A. Huertas, S. Trenholm, & C. Steegmann (Eds.), Teaching Mathematics Online – Emergent Technologies and Methodologies. Hershey, PA: IGI Global.

## REFERENCES

- Baumert, J. et al. (2008). *Professionswissen von Lehrkräften, kognitiv aktivierender Mathematikunterricht und die Entwicklung von mathematischer Kompetenz (COACTIV): Dokumentation der Erhebungsinstrumente (Materialien aus der Bildungsforschung, Nr. 83)*. Berlin: MPI.
- Bescherer, C. (2003). *Selbsteinschätzung der mathematischen Studierfähigkeit von Studienanfängerinnen und -anfängern* (Doctoral thesis). Accessed September 15th, 2010, from <http://opus.bsz-bw.de/phlb/volltexte/2004/1626/>
- De Guzmán, M., Hodgson, B.R., Robert, A., & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. In *Proceedings of the International Congress of Mathematicians*, Volume III (pp. 747 – 762).
- Fischer, P.R. (2007). E-Learning als effizienteres Mittel für den Brückenschlag zwischen Schule und Universität? In: *Beiträge zum Mathematikunterricht 2007*. Münster: WTM-Verlag.
- Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67 (3), 237-254.
- Holton, D. (Ed.). (2001). *The Teaching and Learning of Mathematics at University Level. An ICMI Study*. Dordrecht: Kluwer.
- Ibabe, I. & Jauregizar, J. (2010). Online self-assessment with feedback and metacognitive knowledge. *Higher education*, 59 (2), 243-258.
- Mandl, H. & Kopp, B. (2006). *Blended Learning: Forschungsfragen und Perspektiven*. (Forschungsbericht Nr. 182) München: Ludwig-Maximilians-Universität.
- Niegemann, H.M., Domagk, S., Hessel, S., Hein, A., Hupfer, M., & Zobel, A. (2008). *Kompendium multimediales Lernen*. Heidelberg: Springer.
- Prenzel, M., Senkbeil, M., Ehmke, T. & Bleschke, M. (Hrsg.). (2002). *Konzeption, Evaluationsinstrumente und Unterrichtsmaterialien des SEMIK-Projekts "Didaktisch optimierter Einsatz Neuer Medien im naturwissenschaftlichen Unterricht"*. (IPN-Materialien) Kiel: IPN.
- Tall, D. (1991): *Advanced Mathematical Thinking*. Dordrecht: Kluwer.
- Wittmann, G. (2007). Von Fehleranalysen zur Fehlerkorrektur. In *Beiträge zum Mathematikunterricht 2007*. Münster: WTM-Verlag.

# EMPLOYING POTENTIALITIES AND LIMITATIONS OF ELECTRONIC ENVIRONMENTS: THE CASE OF DERIVATIVE

Irene Biza\* and Victor Giraldo\*\*

\*Loughborough University (UK), \*\*Federal University of Rio de Janeiro (Brazil)

*Abstract. The aim of this paper is to discuss potentialities and limitations (and the suitable balance of these) of electronic environments in the development of students' understanding of the derivative. We re-analyse results of two studies in which the property of local straightness was used. In the first (Greek) study sufficiently accurate computer inscriptions are used to build students' understanding on the derivative and tangent line. In the second (Brazilian) study explicitly inaccurate inscriptions shake a student's faith in the computer's reliability, towards more solid theoretical understanding.*

Keywords: teaching and learning of Analysis, derivative, tangent line, local straightness, electronic environments.

## INTRODUCTION AND THEORETICAL BACKGROUND

In this paper we address the role of dynamic electronic environments in students' understanding of advanced mathematical notions and critical mathematical thinking. Particularly, we discuss how inscriptions of mathematical objects in certain electronic environments and their pedagogical use may interact with students' understanding of these objects. As an inscription, we consider any representation of mathematical objects (e.g. graphs, figures etc.) on the computer screen. Our focus is on the pedagogical role of the potentialities of these representations, as well as their limitations associated with the electronic and software technical constraints.

Especially, we address the case of *local straightness* (Tall, 1989) as a property of a curve with twofold particular interest. Firstly, this property can express the differentiability (or non-differentiability, through its lack) of a function at a point. In this sense it can be used towards introducing the definition of derivative and the general definition of tangent line. Secondly, the examination of this property on *critical examples* of functions offers the pedagogical opportunity to contrast the infinite mathematical processes (e.g. limit) with the limitations of the electronic inscriptions. In this sense it can be used towards facilitating students' more sophisticated mathematical understanding (Giraldo, Carvalho & Tall, 2003).

To this aim we report and highlight connections between results from two studies. In the first one (Greek study), students are introduced to the concept of derivative with the mediation of an electronic environment in which *sufficiently accurate inscriptions* of local straightness are offered (Biza, 2008). In the second study (Brazilian study), the students have been previously introduced to the notion of derivative through the property of local straightness in an electronic environment,

and now are asked to investigate cases of functions in which the inscription is *explicitly inaccurate* (Giraldo, 2004). By *sufficiently accurate inscriptions* we refer to the inscriptions that are designed to be as faithful as possible to the mathematical theory (e.g. the graph is not deformed when it is zoomed on); and by *explicitly inaccurate inscriptions* to the ones that are purposely conceived as in conflict with the mathematical properties (e.g. when a continuous graph seems discontinuous). Given the space limitation, we provide only the necessary methodological and theoretical information for each study and we confine ourselves to some of the results in order to discuss the interaction between students' perspectives with the inscriptions in the electronic environments.

Research highlights that, for the purpose of conceptual understanding in Analysis, instruction should be focused not only on the use of algebraic representations but additionally should take into account the geometric and intuitive representations of the corresponding mathematical objects as well as the interactions among these representations (Kaput, 1994). Such modes of instruction have been gaining strength also due to the advances in computer technology (Habre & Abboud, 2006). The introduction of technology in the teaching of Analysis in recent years proved very "important to facilitate students' work with numerous epistemological discontinuities such as discrete/continuum, finite/infinite, determinate/indeterminate" (Ferrara, Pratt & Robutti, 2006, p. 257). Using technology for the introduction of limit and derivative demands an approach that can balance and bring together intuitive and formal perspectives (ibid). One such approach is suggested by Tall's (1989) dynamic and visual idea of *local straightness*. This notion refers to the fact that, if we focus close enough to a point on a function graph, where the function is differentiable, then this graph looks like a straight line. This *straight line* estimates the tangent line of the graph at this point. The introduction to derivative through the local straightness can be supported by zooming tools in appropriately designed software (Tall, 1989).

Nevertheless, the visual representation of analytic concepts – in which mainly the limit is involved (e.g. continuity, derivative) – often contrasts with the formal status of these concepts. Giaquinto (2007) claims that there is a mismatch between these concepts and their visualisation and in order "to employ visual means to arrive at analytic beliefs we have to import assumptions that link the perceptual with the analytic" (p. 179). However, he acknowledges that "visual thinking can increase a student's understanding" (p. 184). Furthermore, the use of technology in the teaching of analytic concepts may be influenced by the limitations of the electronic environments due to their finite structure (e.g. finite step calculation, graph sketching through polygonal estimation, etc.). Giraldo et al. (2003) introduced the term *theoretical-computational conflict* to describe "situations in which a computational representation for a mathematical concept is (at least potentially) contradictory to the associated theoretical formulation" (p. 63). If these conflicts are exploited within a suitable pedagogical approach – rather than avoided – they may facilitate the

enrichment of students' understanding (Giraldo & Carvalho, 2006). Taking into account the above discussion, we consider electronic environments as artefacts that offer special means for students in their interaction with mathematical objects.

the Greek study

The context of the study

The participants of this study (Biza, 2008) were fifteen Year 12 students (aged 17-18 years) of a Greek secondary school. The students were selected because of their availability and their previous experience in the use of Dynamic Geometry (DG) environments in their school mathematics. This group of students aimed for tertiary studies related to mathematics, science, engineering or medicine and, according to the Greek curriculum, they had to attend an introductory course to Analysis. By the time the research took place, the students had been taught functions, limits and continuity and were about to be introduced to the derivative. In addition, in their previous studies, the students had encountered the tangent line to the circle in Euclidean Geometry (Year 10) and to other conic sections (parabola, hyperbola and ellipse) in Analytical Geometry (Year 11) courses. This previous experience was regarded as crucial on the design of the experiment, which was based on previous research results about students' understanding of tangents (e.g. Biza, Christou & Zachariades, 2008; Vinner, 1991, Castela, 1995). These results have highlighted the strong influence of the circle tangent on the students' general perspectives on tangents. For example (Biza et al., 2008), students who had met the tangent line in different mathematical contexts (Geometry and Analysis) demonstrated *Intermediate* perspectives on tangents between the *Analytical Local* perspective – the tangent is defined through the slope or the derivative applied locally at the tangency point – and the *Geometrical Global* perspective – the tangent preserves geometric properties applied globally on the entire curve. These *Intermediate* perspectives are a fusion of geometric and analytic properties applied either globally on the entire graph or locally at a neighbourhood of the graph. These perspectives may prove to be inappropriate when non-trivial cases are dealt with, such as cases in which the tangent line has more than one common point or when it coincides with the curve, as well as in cases of *inflection* and *edge* points.

The aim of the experiment was the reconstruction of previous restricted perspectives about tangents; the introduction to the local straightness as a property of a curve that has tangent line; and, the creation, through the introduction of the derivative, of a more general understanding of tangency. To this end the experiment intended to deploy the dynamic visual graphics in the electronic environment – especially the magnified image of a part of the graph in comparison with the entire graph – and the symbolic expressions – especially the limit of rate of change and the derivative – as *mediators* in students' understanding about tangency. According to the above aim the inscription of the magnified graph in the electronic environment and the selected examples had to be accurate enough in order to be in accordance with the

mathematical properties. For the needs of the experiment the mathematical classroom was transformed into a laboratory with five portable computers and a sixth linked to an overhead projector. The students were working in groups of three in each of the five computers and they kept notes on worksheets prepared for the needs of the instruction. The experiment was run by the researcher (first author) who was operating the sixth computer, with the support of the classroom teacher. Data was collected through post- and pre- teaching experiment questionnaires, audio recording of the experiment, students' responses on the worksheets and notes.

This experiment was developed in the context of the European Union funded project called *CalGeo* (<http://www.math.uoa.gr/calgeo/en/>) and utilised a DG software named EucliDraw (<http://www.euclidraw.com>). In addition to DG facilities, this software offers a function editor and function graph sketch environment as well as some tools appropriate for Analysis instruction. Indicatively, I refer to the magnification tool that can magnify a specific region of any point on the screen in a separate window. This magnification can be repeated as many times as the user specifies through a magnification factor. The graph and its magnification are presented at the same time on the screen. Technically, the highly magnified graph is produced by recalculation and redesign of the graph. Especially for the functions used in the experiment the magnified image was accurate enough and not restricted by the size of the pixels. Thus, the produced curves were not deformed.

In the analysis we used the Vygotskian term of semiotic mediation (1978) in order to describe students' interaction with the DG environment and their construction of the meaning of differentiability and tangent line. The visible objects (e.g. mathematical formulae, graphs, drawings, and diagrams) mediate students' understanding and the analysis focused on the evolution of students' attribution of meaning and the role of the DG environment, the examples and the researcher as teacher in this evolution.

Interpretation of the image towards students' introduction to tangency

In the first stage of the experiment, the students had constructed the tangent line and a secant line to a point  $A$  of a circle in the DG environment and they had been introduced to the local straightness as a property of a circle. They also had used the magnification tool in order to see how the two lines and the circle's curve were represented in the magnified image of a region around point  $A$ . The aim of this introductory activity was to connect the existence of a tangent line at a point of a curve with the property of this curve to look straight in the magnification window, if we focus on this point close enough. Just after this point the classroom was invited to investigate this property in the case of the semicircle in order to generalise it later for any function graph. The students worked on the semicircle, initially with pencil and paper in their worksheets and then in the electronic environment (Figure 1), in which the tangent line and the secant line  $AB$  had different colours. The point  $B$  could be moved along the curve and the region close to the point could be magnified as many times as the students wanted.

At this point the students were invited to comment on the magnified image and a student, Alexandros, responded: “[The tangent] seems to coincide [with the semicircle] because if the magnification number is big, we cannot see the difference. That is because of the low resolution of the screen”.

For this student the inscription of the magnified graph was connected with the inherent technical limitations of the electronic environment although that was not the case. Given that students at this age are computationally literate the above comment was not unlikely. However, the previous discussion on the local straightness and its connection with the existence of a tangent line led us to expect responses based on the mathematical meaning of the magnified image. Thus, from the perspective of the instructional design, Alexandros’ response was welcome but unexpected and led to a slight – but crucial – change of the initial lesson plan.

Firstly, we discussed the differences between the images of the tangent line and the secant line  $AB$  (Figure 1). The secant line  $AB$  did not visually coincide with the curve whereas the tangent did. We argued that if the screen resolution was an issue, both lines and circle should match. Despite this comparison the students could not connect this image with the curve’s properties. Another example of curve was needed, maybe a case in which the curve does not have a tangent at a point. For that reason the class was invited to examine the same situation if the point  $A$  is a vertex of a parallelogram. In this case the line could not coincide with the curve regardless of how big the magnification factor was (Figure 2). During the comparison of these two cases and trying to explain the differences between these two inscriptions the same Alexandros exclaimed: “this happened because the line is the tangent of the circle”.



**Figure 1: Magnification on a circle point    Figure 2: Magnification on a vertex**

This incident indicated the conflict between the meaning the instructor (first author) had given to the image in the magnification window and the meaning given by the student(s). The inscription in the electronic environment had been constructed in order to be as accurate as possible and mediate students’ understanding about tangency. However, the students’ brief experience of the discussion on the locally straight image of the circle (stage one) was not enough to raise their awareness of the mathematical meaning of the magnified image. As a result another explanation was put forward by a student who blamed a perceived inaccuracy of the electronic environment. As long as there was a tangent and the curve became straight in the magnification window the image acted as a physical illustration without any connection with the curve’s properties. The connection started to be made only when



we considered a case in which the property could not be applied (Figure 2). Using this case moved the students away from the distracting perception of the computational inscription as inaccurate and helped them focus solidly on the intended properties. Let us now consider a somewhat different case where a knowingly inaccurate computational inscription is used for a similar purpose.

the brazilian study

The context of the study

Participants were six Year 1 students, attending a mathematics undergraduate course for prospective mathematics teachers in a major mathematics department in Brazil. Participants were selected among the students who volunteered to the experiment and had not had contact with the limit and derivative concepts previously at school (Calculus is not included in the Brazilian secondary school regular syllabus). At the time of the experiment, they were in the end of the first term of the academic year, attending the *Calculus I* course. During this course, they had regular weekly lectures in the computer lab, where they used Maple V (Waterloo Maple Inc., 1997) software to perform tasks such as: sketching and analysing graphs; interpreting a graph's behaviour under the change of graphic windows (e.g. by zooming in/out); and, determining derivatives and tangent lines (both numerically and symbolically). The study aimed to analyse the influence of *theoretical-computational conflicts* (Giraldo et al., 2003) in the participants' understanding of derivatives and limits.

The empirical study was organized in structured (G0, G1, G2) and semi-structured, task-based (T1, T2, T3, T4, T5, T6) individual interview sessions, which run alternately (G0, followed by T1/T2/T3 set, G1, T4/T5/T6 set, and G2). The whole interviewing process took approximately one month. All the interviews were audio-recorded and fully transcribed. The structured interviews included questions about the participants' conceptions of limits, derivatives and the use of the computer to study Analysis, and aimed for mapping out how these conceptions evolved through the experiment. In the semi-structured interviews, participants were given tasks intentionally designed to engender theoretical-computational conflict situations. They were asked to interpret results produced by the computer, which would apparently clash against mathematical theory due to the computational algorithms' intrinsic limitations. These results included, for example, graphs looking polygonal when they should be smooth or continuous when they should be discontinuous. Thus, a confrontation between the finite structure of the computer algorithms and infinite mathematical processes was likely to emerge.

Data analysis was based on the concept image/concept definition theoretical perspective (Tall & Vinner, 1981). Thus, we aimed to trace out the influence of theoretical-computational conflict situations on the participants' concept images. Data analysis was organised in two steps. Firstly, conflict episodes were identified in each participant's task-based interviews and then these episodes were compared to



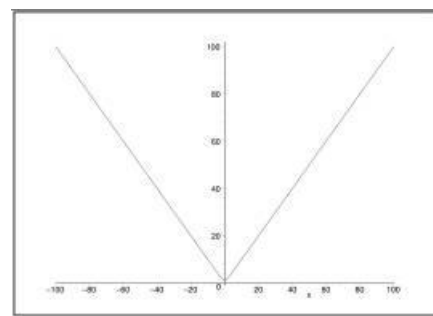
their answers in the structured interviews. This allowed us to outline a map of the development of each participant's understandings throughout the experiment. Secondly, we investigated these maps for the effect the conflict episodes had on the participant's conceptions. These effects were classified into five categories: *confirmation*, *reformulation*, *reconstruction*, *inclusion* and *reversion* (Giraldo, 2004). Our aim was not to draw up a general taxonomy for the effects of theoretical computational conflicts. Rather, we aimed to identify and understand the effects that emerged from our data. In the next section we follow and discuss the development of Júlio's (one of the participants) perceptions and attitudes. Júlio had good grades in the courses and was generally regarded by lecturers as a student above average.

Interpretation of the image towards the elaboration of students' understanding

In the beginning of the study, Júlio had expressed his belief that computer results can be used to check mathematical correctness. In the first structured interview (G0) Júlio talked about the role of the computer in the verification of the symbolic calculation without expressing any doubts on the reliability of the image:

Júlio: We calculate the derivative and then we can put the function on the computer to see if it is really right. (Giraldo, 2004, p. 102)

Later, in interview T1, Júlio was given the algebraic formula of the function  $h: \mathbb{R} \rightarrow \mathbb{R}$ ,  $h(x) = \sqrt{x^2 + 1}$  and its function graph sketched in the graphic window  $-100, 100 \times 0, 100$  of the software (Figure 3). Based on the above he was asked to decide whether the function was differentiable at  $x_0 = 0$  or not. Note that, even though this function is differentiable at  $x_0 = 0$ , the image in Figure 3 suggests the opposite. Júlio, relying on this image, and despite of his familiarity with elementary functions like this, claimed that the function is not differentiable at  $x_0 = 0$  and justified his claim as follows:



**Figure 3. The graph of  $h(x) = \sqrt{x^2 + 1}$  on the graphic window  $-100, 100 \times 0, 100$ .**

Júlio: It was a second degree thing, then we took the square root and the degree became 1. It is going to be a modulus.

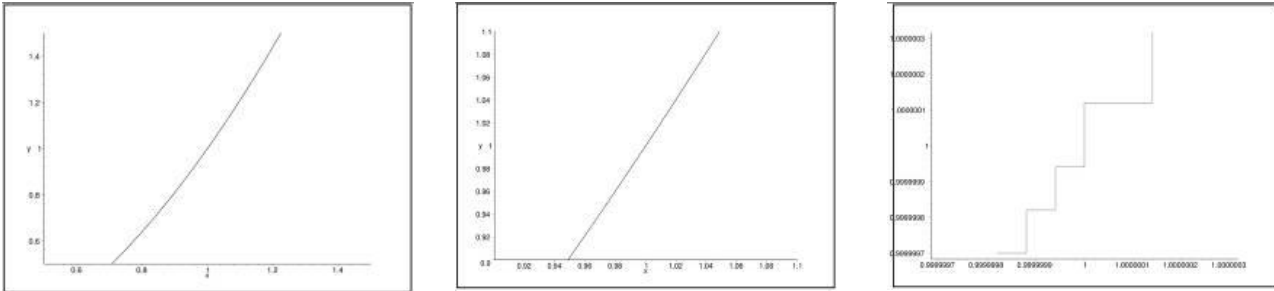
Interviewer: But do you think it's differentiable?

Júlio: Because I'll take out the square root, when I take the square root it's going to be the modulus of the thing that is coming out. Then, the modulus function has a corner [...] that's why this one is going to have one too. (Giraldo, 2004, p. 102)

Following his claim above, he insistently tried to eliminate the square root with algebraic manipulations, without success and no attempts or references to the

necessity of changing the graphic window. During the interview, he didn't touch the computer and considered as correct the image of the graph in the screen. Finally he attributed his failure to express the function formula with modulus to his algebraic weaknesses.

Later, in interview T2, Júlio was asked to explain the local magnification process of the curve  $y = x^2$  as it is represented in Figure 4.



**Figure 4. A curve that is differentiable, but seems not to be.**

After thinking for a while, Júlio explained:

Júlio: That's weird. [...] I think that's because the computer cannot recognise it. The approximation is too small. It no longer distinguishes the curve.

[after zooming in a bit more]

Júlio: Even more. [...] It just links the points and the approximation is way too small. The sketch is coming out deformed. This is not what it's supposed to be! (Giraldo, 2004, p. 168)

Júlio's use of the images in the electronic environment in interviews T1 and T2 is remarkably different. In the former, he trusted the image in the computer and he tried to find an algebraic explanation for it; whilst in the second, his familiarity with the graph of the function generated a resilient and confident doubt about the image. In the first case, not having any knowledge about the function graph, he did not experience any conflict and relied on the graph with somewhat blind faith. This behaviour is consistent with the belief he had expressed in the first interview (G0). Whereas in the second this belief had started to be shaken and Júlio reconsidered the correctness of the computer's image and juxtaposed it with his knowledge about the function. Júlio's attitudes in the following interviews remained similar to the ones he had in interview T2. Later in the final structured interview (G2), he showed that his perspective had considerably changed:

Júlio: What I find cool is that [...] sometimes, one function you have no idea what's going to happen, you put it on the computer, you can make experiences, you can turn it inside out, and upside down. Then, you can figure out what is going on, at least for most of the functions. So, the computer helps to understand things, it gives us a more global comprehension.

[and later]

Júlio: It is cool, but I only rely on it [...] how am I to say [...] I only trust on the computer using the theory I know. [...] Because the computer flaws too, like in many, many times we have seen here. [...] So, we have to use the computer, you know, to make experiences, but always together with the theory (Giraldo, 2004, p. 166).

In the last interview, Júlio kept regarding the computer as an important mean for investigation. However, he no longer considered the computer as a warrant for mathematical truth. Rather, he claimed that the computer outcomes need to be checked through mathematical theory.

## CONCLUDING REMARKS

In the first (Greek) study the students are introduced to local straightness through the visual representation of this property in the electronic environment. To this aim the chosen examples and the inscriptions were designed to be as *faithful* to the mathematical theory as possible. However, for Alexandros, the image of the *straight* curve had no mathematical meaning and thus he attributed inaccuracies to a sufficiently accurate inscription. Actually, the episode evidences a conflict between the teacher's and the student's meanings. The conflict is resolved through the use of a counterexample illustrated in the same electronic environment.

The example from the second (Brazilian) study follows almost a reverse path. The student has been introduced to the property of the local straightness and the concept of derivative and now he discusses some cases in the context of a knowingly (to the teacher) inaccurate inscription in the electronic environment. The aim was to shake the student's somewhat passive faith in the environment. Indeed, Júlio was initially carried away by the image on the graphic window. Then the engagement with familiar examples shook his faith in the computer and, at the end, he stated very clearly that computational inscriptions can contain inaccuracies and that they have to be verified through mathematical theory.

The presented examples are complementary. In both cases the electronic environment mediates the students' understanding: in the first one through sufficiently accurate representations and in the second through explicitly inaccurate representations. In the first we ask the students to trust the electronic environment and in the second to mistrust it. We claim that both of these approaches should be welcome in the teaching and learning process. We need strong and dynamic inscriptions to visualise mathematical objects and properties and at the same time we need to be aware of the limitations of some of these inscriptions. In both cases, the confrontation between the students' previous conceptions and the features of the inscriptions (potentialities and limitations), within suitably designed pedagogical approaches, played a crucial role to change, and possibly deepen, the students' understanding.

## ACKNOWLEDGMENT

We would like to thank Elena Nardi for her helpful comments on earlier drafts of this paper. The re-analysis of the data from the two studies we draw on in this paper was made possible through a Visiting Professor's grant of the postgraduate mathematics education programme in the institution of the second author.

## REFERENCES

- Biza, I. (2008). [*Students' intuitive perspectives on the tangent line in the context of Analysis*] (Unpublished Doctoral Thesis). University of Athens, Greece.
- Biza, I., Christou, C., & Zachariades, T. (2008). Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education*, 10(1), 53-70.
- Castela, C. (1995). Apprendre avec et contre ses connaissances antérieures: Un exemple concret, celui de la tangente. *Recherches en Didactique des mathématiques*, 15(1), 7-47.
- Ferrara, F., Pratt, D., & Robutti, O. (2006). The role and uses of technologies for the teaching of Algebra and Calculus. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future* (pp. 237-273). Rotterdam, The Netherlands: Sense Publishers.
- Giaquinto, M. (2007). *Visual thinking in mathematics*. New York: OUP.
- Giraldo, V. (2004). *Descrições e conflitos computacionais: O caso da derivada* (Unpublished Doctoral Thesis). Universidade Federal do Rio de Janeiro, Brazil.
- Giraldo, V., & Carvalho, L. M. (2006). Generic Organizer for the Enrichment of the Concept Image of Derivative. In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds.), *Proceedings of PME30* (Vol. 3, pp. 185-192).
- Giraldo, V., Carvalho, L. M., & Tall, D. O. (2003). Using theoretical-computational conflicts to enrich the concept image of derivative. *Research in Mathematics Education*, 5, 63-78.
- Habre, S., & Abboud, M. (2006). Students' conceptual understanding of a function and its derivative in an experimental calculus course. *The Journal of Mathematical Behavior*, 25(1), 57-72.
- Kaput, J. (1994). Democratizing access to calculus: New roots to old roots. In A. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 77-156). Hillsdale, NJ: Lawrence Erlbaum.
- Tall, D. O. (1989). Concept images, generic organizers, computers and curriculum change. *For the Learning of Mathematics*, 9(3), 37-42.
- Tall, D. O. & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies*

## Working Group 14

*in Mathematics*, 12, pp. 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of Mathematics. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer.

Vygotsky, L. S. (1978). *Mind and society: The development of higher mental processes*. Cambridge, MA: Harvard University Press.

Waterloo Maple Inc. (1997). Maple V Release 5. <http://www.maplesoft.com/>

# THE CHANGING PROFILE OF THIRD LEVEL SERVICE MATHEMATICS IN IRELAND (1997-2010)

Fiona Faulkner<sup>1</sup>, Ailish Hannigan<sup>1</sup> and Olivia Gill<sup>2</sup>

<sup>1</sup>Mathematics and Statistics Department, University of Limerick, Ireland

<sup>2</sup>NCE-MSTL Department, University of Limerick, Ireland

*In 1997 diagnostic testing was introduced in the University of Limerick (UL) to identify and notify students who were likely to require help to complete first year service mathematics courses successfully and proceed through to second year. A database of diagnostic test and end of semester examination results was initiated in 1997 and is updated annually by the authors. It has been observed by the authors that the student profile and their needs have changed in the 13 years since the initiation of the database. In this paper the authors describe the profile of the current student cohort in first year service mathematics courses in UL and how it has changed in the last 13 years. Furthermore the implications of such changes are outlined<sup>1</sup>.*

Keywords: Student profile, at risk, diagnostic testing.

## BACKGROUND AND METHODOLOGY

### Diagnostic testing in the University of Limerick

The University of Limerick (UL) has used diagnostic testing since 1997 to help identify students who may be at risk of failing service mathematics examinations (O'Donoghue, 1999; Gill, 2006). Diagnostic testing is now commonly used in third level education, both in Ireland and abroad, to identify weaknesses in basic mathematical skills (Abou Halloun & Hestenes 1985, Edward 1996; Malcolm & McCoy, 2007). The diagnostic test has aided the identification of the changing student profile in UL both in terms of students' mathematical competency levels and the educational backgrounds of the students. Such changes have been echoed in documentation from other Universities in Ireland such as Dublin City University (Ni Fhloinn 2009). O'Donoghue, a professor of mathematics education in the University of Limerick, developed the UL paper based test in 1997. A number of quality controls were used in the design of the test to ensure it fulfilled its function. The <sup>1</sup>Ordinary Level Leaving Certificate mathematics syllabus, the SEFI core level zero syllabus for engineers (Barry and Steele, 1993) and an extensive literature review were all used to inform its content and structure. After the test was piloted in a number of second level schools and feedback on the prototype from six mathematics lecturers in UL was received, the test was finalised. Each test is marked by hand so as to enable closer inspection of a script if required (O'Donoghue, 1999).

The two largest service mathematics groups in UL, Science Mathematics and Technological Mathematics, are tested annually. 3,389 students have taken Science Mathematics within the period 1998-2010 and 4,439 have taken Technological Mathematics. The diagnostic test consists of 40 questions covering nine topics, the majority of which are aimed at Ordinary Level Leaving Certificate Mathematics<sup>2</sup> standard or below. The diagnostic test has not changed since it was first developed in 1997. The test is distributed to students in their first mathematics lecture of the academic year without prior warning. If students receive 19 out of 40 or below in the test they are encouraged to avail of the support services provided by the Mathematics Learning Centre (MLC). These students are categorised as being ‘at risk’ (O’Donoghue, 1999). 19 was chosen as the cut off point as these students were considered to be seriously deficient in the basic mathematical knowledge necessary for third level mathematics and therefore were categorised as being at risk and in need of support. It should be noted however that there is “a strong argument for extending the ‘at risk’ category to students with higher scores” (O’Donoghue, 1999, p. 15). Research on the changing student profile in the University of Limerick was carried out in 2008. This paper aims to update those findings to the year 2010.

### **Research Methodology**

Descriptive analysis is used throughout this research paper. The analyses is carried out using the statistical software package SPSS (Statistical Package for the Social Sciences Version 16). The data has been described using summary measures, such as means and standard deviations, and graphs, such as box plots, are used to represent cohort changes over time. The documentation of trends over time amongst UL students was informed by the work of O’Donoghue (1999) and Gill et al (2010) and guided by the work of Hunt and Lawson (1996). The investigation into the implications of the changing profile of students in terms of changes in support services needed over the time period 1998-2010 was carried out by detailing the chronological order in which new mathematics support services were introduced to UL. The pilot study in 1997 examined data on the Technology mathematics students only and so the changing profile of both the Science and Technology service mathematics students is most effectively analysed between the years 1998-2010.

### **MAIN FINDINGS-THE CHANGING STUDENT PROFILE (1998-2010)**

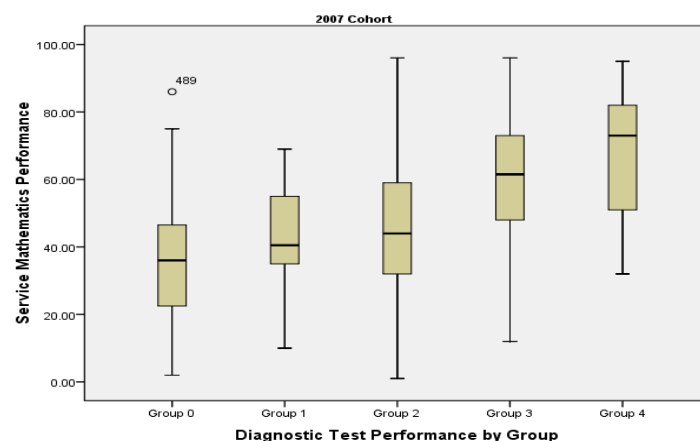
The student profile of Science and Technological Mathematics courses in UL has changed greatly between the years 1998 and 2010 (see Table 1). Changing student profiles of service mathematics students such as this, which have been shown to impact on the teaching and learning of mathematics in third level education, have been documented internationally (Kitchen,1999).

## Degree Programmes

The number of degree programmes required to take Technological and Science Mathematics has increased between 1998 and 2010. Within Technological Mathematics there had been an increase from 8 to 9 degree programmes. Science Mathematics has seen a large increase from 8 degree programmes in 1998 to 15 in 2010 with courses such as Health and Safety and Psychology now being required to take Science Mathematics.

## Percentage of ‘no-show’ and ‘at risk’ students

In 1998 100% of students registered for Technological and Science mathematics sat the diagnostic test i.e. they were present in their first lecture of term. The number of Technology mathematics students sitting the diagnostic test has declined to 82.6% in 2010, which represents a 21.1% decline from the 1998 baseline. A slightly larger decline to 76.5% occurred for Science mathematics students, which represents a 30.7% decline from the 1998 baseline (see Table 1). The reasons for these students not turning up to their first lecture to take the test has not yet been examined however upon analyses of the ‘no-show’ students’ end of semester results from a previous year (2007) the vast majority of ‘no-show’ students performed below average when compared to the rest of the cohort (see Figure 1). Figure 1 highlights that students who do not turn up to take the diagnostic test may be at risk of failing their end of term examinations as well as those who perform poorly in the diagnostic test. There has been an increase in the percentage of ‘at risk’ students, i.e. students who receive 19 out of 40 or below in the diagnostic test. The number of ‘at risk’ students in the Technological mathematics group increased by 25.2% in 2010, which represents a 77% increase from the 1998 baseline. An even larger increase of 24.9% occurred for Science students in 2010, which represents a 117% increase from the 1998 baseline (see Table 1).



(Note- 0= did not sit the test, 1= 1-10 in test, 2=11-20 in test, 3=21-30 in test, 4= 31-40)

**Figure 1 ‘No-show’ students’ performance in end of term test against the performance of students who sat the diagnostic test.**



## Working Group 14

A decrease in mathematical standards such as this may be due to, as suggested by Hunt and Lawson (1996), a shift in emphasis of second level mathematics towards topics not covered in the diagnostic test. Although the suggestions of Hunt and Lawson (1996) may hold in an Irish context, it is thought that the increase in at risk students may also be explained by the decrease in students taking Higher Level Leaving Certificate Mathematics, a decrease of 12.7% and 16.4% occurred within Technological and Science Mathematics cohorts respectively in 2010, which represent a 31% and 30% decline from the 1998 baseline. From Table 1 it is evident that within the Science and Technological Mathematics student cohorts the percentage of students taking Higher Level Leaving Certificate Mathematics has declined over the period (1998-2010). This is an important finding as performance in mathematics in third level education has been shown to be better when students have higher level mathematics as pre-requisite knowledge (Barry & Chapman, 2007).

The increase in the number of non-standard students is also contributing to the decline in diagnostic test performance over time. Non-standard students consist of mature students (i.e. those over the age of 23), non-national students and those who have completed previous degree/diplomas/certificates and have used these as an entry qualification to UL. A 12.4% and 8.6% increase of non-standard students occurred in Technological and Science Mathematics respectively in 2010, which represents a 4133.3% and 600% increase from the 1998 baseline (see Table 1). A change in student intake such as this has been shown to influence overall mathematical performance in UL as well as in other institutions such as Coventry University (Lawson, 2003). Second Level mathematics can no longer be assumed as the previous knowledge of students. Lecturers of these modules are therefore faced with new challenges which call for adaptations in teaching styles to cater for the current cohort and not that which sat in front of mathematics lecturers twelve years previous (Hourigan & O'Donoghue, 2007). The measures put in place to cope with these changes are discussed in section 4.

Year	1998	2010		1998	2010
<b>Technological Mathematics</b>			<b>Science Mathematics</b>		
<b>% of total taking test</b>	100%	82.6%		100%	76.5%
<b>% 'at risk'</b>	32.8%	58.0%		21.3%	46.2%
<b>% doing HL</b>	41%	28.3%		55.4%	39.0%
<b>% doing OL</b>	58.7%	69.8%		43.1%	59.4%
<b>Non-standard students</b>	0.3%	12.7%		1.5%	10.1%

**Table 1: Profile of students in Science and Technological Mathematics**

## PROFILE OF NON-STANDARD STUDENTS IN 2009

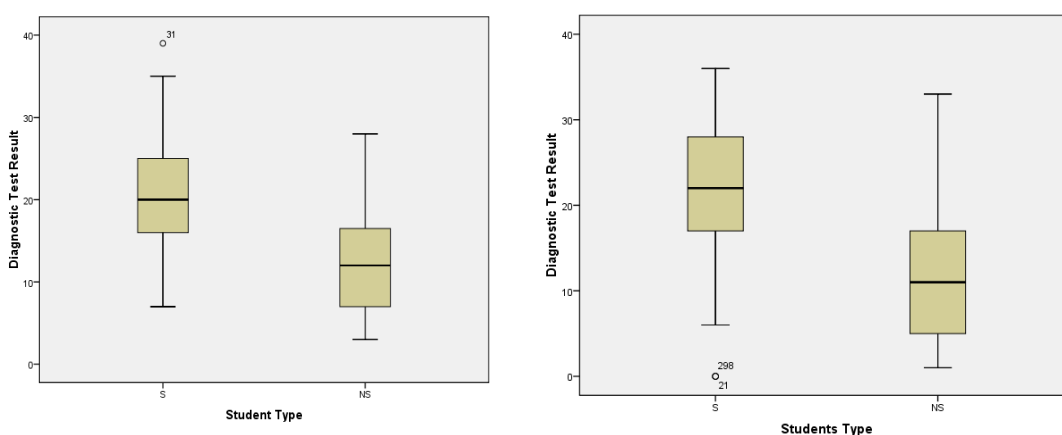
Due to the large increase in the prevalence of non-standard students undertaking degree programs which involve service mathematics it is useful to examine the performance of these students and how it may impact on mathematics education within the University of Limerick. This examination focuses on non-standard students' performance in 2009 as at the time of writing the current non-standard students have not yet sat their end of term examination.

### Performance of non-standard students in the diagnostic test

Mature students make up the majority of the non-standard cohort in UL, therefore the majority of non-standard students have not engaged in mathematics for a number of years. This is apparent in their diagnostic test results (see Figure 2). Non-standard students have mean diagnostic test scores below that of the standard students who have come directly from Leaving Certificate. According to their performance in the diagnostic test, the majority of non-standard students are at risk of failing their end of semester examinations. Figure 2 highlights that non-standard students are mathematically less prepared entering UL than standard students are.

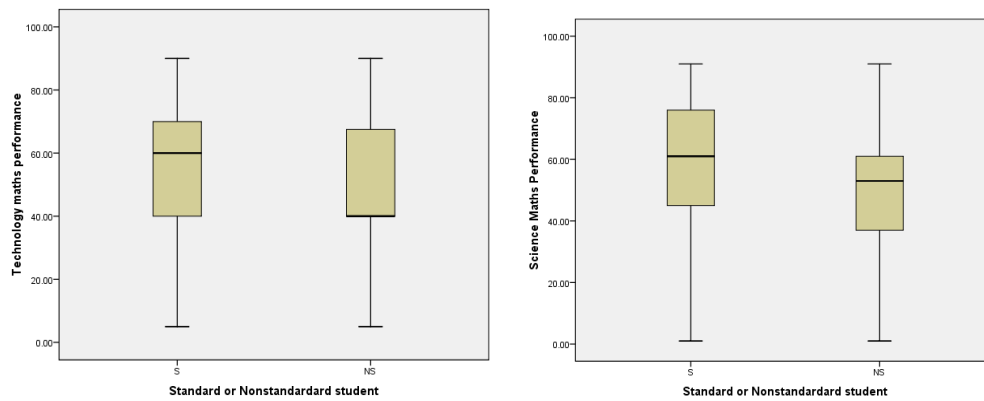
### Performance of non-standard students in end of semester examination

Non-standard students' performance improves in their end of semester examinations, for both Science and Technological cohorts, when compared to their diagnostic test performance. When Figures 2 and 3 are compared it is clear to see that the gap between standard and non-standard students is less obvious in terms of their examination performance. There is still a gap between the performance of standard and non-standard students in end of term service mathematics. However the majority of non-standard students pass their semester 1 examination (the pass mark being 40%) despite the fact that the majority of them were considered to be at risk of failing as highlighted by the diagnostic test (see Table 2 and 3 and Figure 3). Probable reasons for this improvement in performance are discussed below.



**Figure 2: Comparison of diagnostic test performance for Standard and Non-standard students- Technological (left) & Science (right) Mathematics (2009)**

## Working Group 14



**Figure 3: Comparison of end of semester 1 examination performance for Standard and Non-standard students-Technological (left) Science (right) Mathematics (2009)**

Technological Students		Science Students	
Diagnostic Test	Mean value out of 40(SD)	Diagnostic Test	Mean value out of 40(SD)
Standard students	51.5 (16.3) n=260	Standard students	55.5 (17.0) n= 290
Non-standard students	31.8 (16.5) n=44	Non-standard students	30.2(20.0) n= 49
Semester 1 examination	Mean value out of 100(SD)	Semester 1 examination	Mean value out of 100(SD)
Standard students	54.8 (6.0) n=257	Standard students	60.4 (19.8) n=254
Non-standard students	49.4 (23.9) n=43	Non-standard	47.1(24.0) n=46

**Table 2 Technological and Science Mathematics students' diagnostic tests (expressed as a percentage of correct answers out of 40 questions) and end of semester results in 2009**

### Comparison of attendance at support tutorials of standard and non-standard students

Tables 2 and 3 and Figures 2 and 3 highlight that the non-standard students improve in their mathematical performance between the beginning of the semester and the end. Why might this be? The percentage of non-standard students attending support-tutorials, run by the Mathematics Learning Centre (MLC), is much higher than that of the standard students (support tutorials will be outlined in greater detail in section 4). Technological and Science Mathematics students are offered a weekly support tutorial in addition to their regular weekly tutorial. The findings shown in Figures 2

and 3 and Tables 2 and 3 reveal positive indicators for engaging in support services such as support tutorials and one-to-one consultations in the Mathematics Learning Centre. These findings could also be accredited to the fact that non-standard students may be more motivated than standard students (Hirst, 1999).

## **IMPLICATIONS OF THE CHANGING STUDENT PROFILE-INCREASE IN SUPPORT SERVICES**

The provision and uptake of support structures in place in UL reflect the changing student profile which has been documented thus far.

Support services were initiated in 1997 when O'Donoghue carried out his pilot study to measure the extent of the errors and gaps in students' mathematical knowledge. 'Front-end' tutorials were set up in the first two weeks (one each week) of the first term to enable students to revise fundamental mathematics skills in arithmetic and algebra. In addition, students were strongly encouraged to attend a 'support tutorial' on a weekly basis throughout the entire term. Support tutorials run alongside regular tutorials and cover the same material at a slower pace and in a smaller group. The Department of Mathematics and Statistics acknowledged that further support was warranted so in 2001, the Mathematics Learning Centre (MLC) opened. This fully supervised drop-in centre is a place where students can study and/or receive 1-1 attention. This service is provided for 20 hours a week. In the first year of operation, 1516 visits to the drop-in centre were recorded. In 2009/10, a total of 1,129 individual students availed of the drop-in centre making 4527 contacts/attendances over the academic year. Significant increases in the number of support tutorials has also occurred since 1997, when two support tutorials (one for traditional students and one for mature students in Technology Mathematics) ran on a weekly basis. In 2009/10, a total of 228 support tutorials in 21 modules were provided by the MLC with a total of 3,835 contacts/attendances recorded at these classes. This was an increase of over 1,400 contacts on the previous academic year. Between the drop-in centre, examination revision and support tutorials, 8,946 contacts were made with the MLC.

Online support was set up by the MLC with the view to increasing contact time with students who needed mathematics support. Fact sheets tailored specifically for UL service mathematics courses were designed and peer reviewed by colleagues from the Department of Mathematics and Statistics, and made available online for all students. Past examination papers and sample solutions are made available on the site as are links to useful websites such as the University of Loughborough's Engineering Mathematics website (<http://www.ul.ie/~mlc/support/Loughborough%20website/book.html>).

One further development made by the MLC to cope with growing numbers of non-standard students was the introduction of a one week refresher mathematics course, entitled 'Head Start Mathematics'. This course was designed to help students who

had been away from formal mathematics education for a long period catch up on essential mathematics skills they were likely to need for their service mathematics courses. The course, which is run in August before students start in UL, is free of charge and covers topics such as number systems, algebra and graphing etc. In August 2010 2 weeks of this course were run to accommodate the increasing numbers who wished to receive the free tuition. The feedback from participants thus far has been very positive (Gill 2010).

The mathematics learning centre is currently putting together screen casts of areas of mathematics which students often have difficulty with in order to try to alleviate some of the pressures put on the centre by the growing number of students. This additional means of mathematics support is currently being developed in collaboration with Swinburne University in Australia.

### **DISCUSSION- IMPLICATIONS OF THE CHANGING STUDENT PROFILE**

Gill et al (2010) highlighted, through the use of diagnostic testing, the declining mathematical standards of third level students entering Technological and Science Mathematics courses in UL. This paper offers a discussion on a probable contributor to this decline i.e. the changing profile of students within these service mathematics courses. The main changes in student profile between 1998 and 2010 can be summarised as follows:

- An increase in the number of degree programmes within the service mathematics courses
- An increase of 77% of Technological Mathematics students being labelled at risk and an increase of 117% of Science Mathematics students being labelled at risk from the 1998 baseline
- The proportion of students entering UL with Higher Level Mathematics has declined along with a corresponding increase in the proportion of students entering with Ordinary Level Leaving Certificate maths
- The percentage of non-standard students has gone from approximately 1% of the entire cohort to almost 12% for Science and Technological Mathematics combined
- The ‘no-show’ students, i.e. those who did not sit the diagnostic test in their first mathematics lecture, perform poorest in their end of semester examination. These students along with those who perform poorly in the diagnostic test are currently being targeted for an intervention, the results of which are to be reported in a future paper.

An investigation of the non-standard students’ performance in 2009 revealed that they perform below average in the diagnostic test when compared to the standard students but improve by the time they sit their end of term examination. Non-

standard students are more likely to avail of the support services put in place by the Mathematics Learning Centre. A major implication of the changing student profile was the need for increased mathematics support services which were more tailored to the specific/changing needs of the UL students within the period 1997 and 2010.

The increase in support services is but one of the changes that needs to occur in mathematics education in order to attempt to cater for the varying backgrounds of new student cohorts. Lecturers and tutors of mathematics need to take note of the changes which have occurred over time and plan their teaching content and style accordingly. A lack of willingness to change to suit the needs of the current student profile would be likely to lead to “deterioration in the effectiveness of the learning” in UL and indeed in other third level institutions which are experiencing similar changes in student profile (Hunt & Lawson, 1996, p. 171). The need for yearly assessment of the student profile is imperative if student-appropriate teaching is to take place.

### NOTES

<sup>1</sup> This paper revisits the UL database which was formally reported on up to 2008.

<sup>2</sup> Leaving Certificate Mathematics can be taken at three levels in Ireland; Higher, Ordinary and Foundation Level. The Higher Level curriculum is the most advanced level. It covers more topics and to a more sophisticated standard than the other levels. The minimum mathematics entry requirement for direct entry to third level education in Ireland is a grade C or higher in Ordinary Level Leaving Certificate Mathematics.

### ACKNOWLEDGMENT

This work reported in this paper was supported by MACSI - the Mathematics Applications Consortium for Science and Industry ([www.macsi.ie](http://www.macsi.ie)), centred at the University of Limerick.

### REFERENCES:

- Abou Halloun, I. & Hestenes, D. (1985). The initial state of college physics students, *American Journal of Physics*. 53(11), 1043-1055.
- Barry, S.I., & Chapman, J. (2007). Predicting university performance. *ANZIAM J*, 49, 36-50.
- Burgess, R.G. (1989). *The Ethics of Educational Research*. London: Routledge Falmer.
- Edward, P. (1996). Implementing Diagnostic testing for non-Specialist Mathematics Courses. *Open Learning Foundation*, London.
- Gill, O. (2006). What Counts as Service Mathematics? An Investigation into the ‘Mathematics Problem’ in Ireland. unpublished thesis (PhD). Department of Mathematics and Statistics, University of Limerick.

- Gill, O. 2010. Evaluating the Impact of a Refresher Course in Mathematics on Adult Learners. Paper presented at *Adults Learning Mathematics 17 (ALM 17)*, Oslo, June 28-30.
- Gill, O., O'Donoghue, J., Hannigan, A., & Faulkner, F. (2010). Trends in Performance of Science and Technology Students (1997-2008) in Ireland'. *International Journal of Mathematical Education in Science and Technology*, 41(3), 323-339.
- Hirst, K. (1999). Mature students studying mathematics. *International Journal of Mathematical Education in Science and Technology*, 30(2), 207-213.
- Hourigan, M. & O'Donoghue, J. (2007) Mathematical under-preparedness: the influence of the pre-tertiary mathematics experience on students' ability to make a successful transition to tertiary level mathematics courses in Ireland. *International Journal of Mathematical Education in Science and Technology*, 38, 461-476.
- Hunt, D.N., & Lawson, D.A. (1996). Trends in the Mathematical Competency of A Level Students on Entry to University. *Teaching mathematics and its applications*, 15, 167-173.
- Kitchen, A.(1999). The Changing Profile of Entrants to Mathematics at A Level and to Mathematical Subjects in Higher Education. *British Educational Research Journal*, 25(1),57-74.
- Lawson, D. (1997). What can we expect from A-Level students? *Teaching Mathematics and its Applications*, 16(4),151-156.
- Lawson, D. (2003). Changes in student entry competencies 1991-2001. *Teaching mathematics and its applications*, 22(4), 171-175.
- Malcolm,R.K., & McCoy,P. (2007). Evaluation of numeracy skills in first year pharmacy undergraduates 1999-2005. *Pharmacy Education: An International Journal for Pharmaceutical Education*, 7,53-59.
- Ni Fhlionn,E. (2009). Diagnostic Testing in DCU-A five-year review, "Mathematics for all"-Extending Mathematical Capacity. *Proceedings of ME13*, St.Patrick's College Dublin, Ireland, 376.
- O'Donoghue, J. (1999). *An intervention to assist at risk students in service mathematics courses at the University of Limerick: University of Limerick teaching fellowship scheme*. University of Limerick, Limerick.

# USING CAS BASED WORK TO EASE THE TRANSITION FROM CALCULUS TO REAL ANALYSIS

Erika Gyöngyösi, Jan Philip Solovej, Carl Winsløw

EG<sup>75</sup> and CW: Department of Science Education, University of Copenhagen

JPS: Department of Mathematical Sciences, University of Copenhagen

*We investigate how the integration of a computer algebra system (Maple 13) in parts of students' work may help students relate theoretical and practical aspects of convergence of sequences and series of real numbers and functions.*

## TRANSITIONS IN INTRODUCTORY ANALYSIS: THE PROBLEM

Many universities are aware of the difficulties students experience during their first months of study, and they therefore introduce a variety of measures to ease the institutional transition. In particular, university mathematics programmes adapt the first courses to the students' background from upper secondary school. In Denmark this means for instance that basic differential and integral calculus are revisited and extended progressively to functions of several variables, along with other standard topics from the field of calculus. Following the style of North American calculus text books, these first courses focus on students' practice with concrete calculations and analysis of functions, while the theoretical part (precise definitions, theorems and proofs) is left in the background or it is entirely omitted. If the basic courses are shared with students from other specialties like physics or engineering, this may further motivate such a choice.

However, the problem will not go away. Mathematics students soon encounter real analysis where a rigorous approach to topics such as continuity and convergence is inevitable not only as something talked about in lectures, but also as something worked on by students. As pointed out by Winsløw (2008) the students are in fact presented with two types of challenges or transitions, often in rapid succession:

- (1) they will have to do autonomous work on theoretical aspects of practices they previously encountered in calculus (calculations with “ordinary functions”)
- (2) they will also encounter new practices and theories which are based on these theoretical elements (eg. function spaces, abstract metric spaces, norms,...).

Our research addresses mainly the first transition. More concretely, we investigate – theoretically and also empirically – concrete designs for integration of a computer algebra system (a CAS, in this study *Maple 13*) in parts of students' work, aiming to help students relate theoretical and practical aspects of *convergence of sequences and series of real numbers and functions*. In particular we are interested in designing new types of tasks which may realise the *lever-potential* and the *materialization*

---

<sup>75</sup> Permanent address, EG: University of Miskolc, Hungary. EG was supported by Eotvos Hungarian State scholarship.



*potential* (cf. Winsløw, 2003), and we are particularly interested in the effects, for “weak students”, of such *reasoned instrumentation* (cf. Elbaz-Vincent, 2005, 63).

## INSTRUMENTED TECHNIQUES AND INSTRUMENTATION

We now outline the theoretical framework for this study. Our basic model for the mathematical activity of students comes from the anthropological theory of didactics (Chevallard 1999; Barbé et al., 2005, sec. 2). That is, we consider the work of students (and any mathematical activity) as organised by types of tasks  $T$ , carried out using a technique  $\tau$ ; the discourse about the practice block  $(T, \tau)$  is organised in a *technology*  $\theta$ , which may pertain to a whole family of practice blocks; and several technologies are structured and justified within a wider *theory*  $\Theta$ , which in our study is essentially the theory of convergence of series of functions and numbers within the mathematical domain of real analysis (involving several definitions, theorems and so on). This way, a mathematical activity is always modelled with a *praxeology*  $(T, \tau, \theta, \Theta)$ . Of course, the main point regarding the use of a CAS is that it provides an inventory of *instrumented techniques* (where by “instrumented” we indicate that a more or less powerful tool is involved in the technique; cf. Artigue, 2002). These may at first present themselves simply as alternative techniques for solving an existing type of task, such as determining the limit of a sequence of real numbers. But in practice, the new techniques rarely correspond exactly to existing types of tasks, and at the same time they often need to be combined with non-instrumented techniques. Moreover the theory blocks may need to be adjusted – not least at the level of technology – to accommodate such combinations.

Elbaz-Vincent (2005) studied examples and principles of how *Maple* computes limits of functions and series in symbolic mode (these routines are essentially based on power series expansions); it disposes of a range of other techniques for numerical computations. As any techniques, instrumented techniques will only provide meaningful or correct answers for a restrained type of tasks, which could in principle be determined theoretically. However, most users – including students – will have to relate to the results in a more pragmatic way. Even in situations where all that matters is to solve a given task, it then becomes important to develop practical means to evaluate the result (or absence thereof) from applying an instrumented technique. This also means that, most of the time, a complex interplay of instrumented and non-instrumented techniques will have to be constructed, made explicit and reasoned within theory blocks which become, as a result, different from what they are in the absence of instrumented techniques. In this paper, we will give a number of examples of such modifications of praxeologies to be developed by students and teachers.

These modifications will have to be understood relatively to the affordances of the involved techniques, as defined by Artigue (2002, 248): their *pragmatic value*, i.e. the efficiency for solving tasks; and their *epistemic value*, i.e. the insight they

provide into the mathematical objects and theories to be studied. These values may be experienced to differ greatly, especially by teachers.

Such differences may be studied in more detail from the point of view of *semiotic representations* of mathematical objects (cf. Winsløw, 2003). An instrumented technique provides, within its scope, access to transformations of representations (eg. symbolic or graphical) whose epistemic value depends both on the users' means to access or perform intermediate steps (the absence of which is often referred to as a *black box effect*). It also depends on the users' means to relate them to a didactically targeted theoretical block. The possibility of using instrumented techniques leading to a stronger focus on, and access to, theoretical objects and principles, is referred to as the *lever potential* and the *materialization potential*, respectively.

To achieve this we need to construct entirely new organisations of tasks for students, involving instrumented techniques in a way that develop their *epistemic* value for students (cf. also Artigue, 2010, 467f). In the present project, this means that students' work with the designed instrumented practices should be related to, and support, their work with theory blocks, and so facilitate the transition (1) mentioned in the introduction. This way instrumented techniques should also become a more integrated element of the students' developing praxeological equipment, instead of a set of alternative techniques which are at best tolerated because of their pragmatic value. While this process of *instrumental genesis* may be studied in an individual perspective, it is clear that in didactic contexts it can be also heavily dependent on institutional conditions and norms, as the teaching institution takes more or less responsibility to organise and evaluate the use of instrumented techniques. This aspect of teaching is often referred to as *instrumental orchestration* (Trouche, 2005).

To establish and study efforts towards reasoned instrumentation at this institutional level, taking into account in particular the potential effects of instrumented semiotic representations, is indeed the overall purpose of this study. We now proceed to outline the concrete mathematical and institutional context in which it is situated.

## **SERIES AND SEQUENCES – INSTRUMENTED TECHNIQUES**

At the University of Copenhagen, the bases of mathematical analysis are taught in three courses during the first year: an introductory course involving mainly practical blocks of calculus in one and several variables; a follow-up course (*Analysis 0*) in which theoretical blocks, in particular evolving around “ $\epsilon\delta$ -arguments”, are also worked on by students; and finally a more advanced course (*Analysis 1*) in which notions related to convergence, series (including Taylor and Fourier expansions), and metric spaces are to be studied, with a strong focus on theoretical blocks.

In the introductory course as well as in a corresponding course on linear algebra, instrumental orchestrations based on *Maple* have become an integrated and relatively stable part since the first developmental project around the year 2000 (data from that

project were analysed in Winsløw, 2003). In particular, students become familiar with basic use of *Maple* techniques in common tasks in these areas. But until now, the more advanced courses have not made systematic efforts in this direction, probably because of the more theoretical focus.

At the same time, while the introductory course has a very high passing rate, *Analysis 0* and in particular *Analysis 1* seem to filter out a number of students (up to 25% in recent years), who then drop out of the programme – a state of affairs of concern to the institution and to Danish society, in which mathematicians are sorely needed, not least for teaching positions in upper secondary schools.

The hypothesis that these problems reflect a transition problem of type (1) has been further elaborated by Winsløw (2008), and it is likely to mirror the situation in many similar institutions. On the other hand, there seems to be little research done on instrumental orchestration in this context. This motivated us to explore the potentials of resuming *Maple*-based instrumental orchestrations in the course *Analysis 1*, through the design of tasks which explicitly appeal to instrumented techniques. The aim was to realise materialization and lever potentials through the facilitation of students' autonomous work with technology and theory, facilitated by the use of instrumented techniques to solve at least part of the problems presented to them. Before going into the theoretical and empirical analysis of concrete problems, we first map out the target praxeologies appearing in the first weeks of *Analysis 1* and considered in this study, with a special regard to the design of tasks.

### Target praxeologies

The praxeologies focused on here are all concerned with deciding on the following questions for a given *sequence* or *series* of *numbers* or *functions*: does it converge? How? To what? And why? This gives, as a first “map” of the praxeologies concerned, the “grand tasks” shown in Table 1.

	SEQUENCES	SERIES
NUMBERS Given $(a_n)$	$T_{11}$ : is $(a_n)$ convergent? $T_{12}$ : if so, what is the limit? $T_{1k}^*$ ( $k=1,2$ ): prove answer	$T_{21}$ : is $\sum a_n$ convergent? $T_{22}$ : if so, how (absolutely,... ?) $T_{23}$ : ...and what is the limit? $T_{2k}^*$ ( $k=1,2,3$ ): prove answer
FUNCTIONSG iven $(f_n)$ , e.g. $f_n(x) = a_n x^n$	$T_{31}$ : is $(f_n)$ convergent? $T_{32}$ : if so, how (uniformly ?) $T_{33}$ : ...where (eg. interval(s))? $T_{34}$ : ... what is the limit? $T_{3k}^*$ ( $k=1,2,3,4$ ): prove answer	$T_{41}$ : is $\sum f_n$ convergent? $T_{42}$ : if so, how (uniformly ?) $T_{43}$ : ...where (eg. interval(s))? $T_{44}$ : ...what is the limit? $T_{4k}^*$ ( $k=1,2,3,4$ ): prove answer
“REVERSE”	$T_{5c}$ : Given number or function, find approximating sequence or series according to some criteria $c$ (eg. a power series...)	

**Table 1: Grand types of tasks, further subdivided according to techniques**

According to the techniques employed, the tasks listed in Table 1 are further subdivided. For example,  $T_{21}$  is associated with a number of *particular techniques*, such as the ratio test  $\tau_{21,RT}$  whose domain of application correspond to a type of task  $T_{21,RT}$  for which the limit of  $|a_{n+1}/a_n|$  can be found using a technique for  $T_{11}$  (and the result is either less than 1, in which case the result is yes, or greater than 1, in which case the answer is no). We notice that in this case the technique involves solving a task of type  $T_{11}$ , and in fact several techniques corresponding to the “grand tasks”  $T_{ik}$  listed in Table 1 involves solving another task  $T_{i'k'}$  (often but not always with  $i' < i$ ), which is why the didactical process will often address the tasks more or less in the order of the table. It is also clear that the non-instrumented techniques for types of tasks included in  $T_{ik}$  are typically closely associated with theory (theorems and definitions) which, when it comes to solve  $T_{ik}^*$ , are to be used explicitly and with attention to precise conditions.

The instrumented techniques (in fact common to all  $T_{2k}$ ) offered by Maple consist in symbolic ( $\tau_{2,S}$ ) or numeric ( $\tau_{2,N}$ ) evaluation of sum ( $a_n$ ,  $n=1..infinity$ ); while their domain of validity may be less clear, it nevertheless corresponds to types of tasks  $T_{2,S}$  and  $T_{2,N}$ . The technology pertaining to these techniques involves further explanations related to the syntax of *Maple*, variations that can be tried, etc. While the pragmatic value of these techniques is high (for tasks belonging to these types) their epistemic value is “low” in the sense they are not associated with satisfactory techniques for  $T_{ik}^*$ . In fact, the *theory* according to which technologies are to be justified and related is still that of convergence of functions and series, involving definitions and theorems pertaining to various special cases (closely related to the non-instrumented techniques, which will then have to be mobilized eventually, for the solution of  $T_{ik}^*$ ).

The types of tasks covered by  $T_{5c}$  include, in particular, determining a power series that approximate a given function. It is inverse to the task  $T_{44,PS}$  of determining, for a power series that converge on some interval, the corresponding function.

### Designing tasks for instrumented work

We now give some examples of how instrumented techniques were integrated into the work of students, all coming from the first two of the four “take home exams” based on which the students were graded in this course.

*Example 1.* The students were given four series, including

$$(a) \sum_{n=1}^{\infty} (3n^2 + 7n + 1)^{-n} \quad \text{and} \quad (c) \sum_{n=1}^{\infty} (-1)^{-n} \frac{n^2 + n - 1}{3n^2 + n}$$

The instruction is to use *Maple* to compute five partial sums, or alternatively do a point plot, in order to get an idea of whether the series converge or not; and also to see if *Maple* can decide this directly. Finally the students should give an answer with proof to the question of whether the series are divergent, conditionally convergent or absolutely convergent (tasks included in  $T_{21}$ ,  $T_{22}$ ,  $T_{21}^*$ ,  $T_{22}^*$ ). As regards  $T_{21}$ , a

## Working Group 14

general challenge consists in determining the *type* of the given task and hence that will solve it. Having an idea of whether the series converges or not is indeed helpful, and this is where the instrumented techniques come in. The most immediate,  $\tau_{2,S}$ , does not work for the given tasks (so they do not belong to  $T_{2,S}$ ), while  $\tau_{2,N}$  apparently works for both (as it returns finite numbers):

```
> A:=sum((3*n^2+7*n+1)^(-n), n=1..infinity);
```

$$A := \sum_{n=1}^{\infty} (3n^2 + 7n + 1)^{(-n)}$$

```
> C:=sum((-1)^n*(n^2+n-1)/(3*n^2+n), n=1..infinity);
```

$$C := \sum_{n=1}^{\infty} \frac{(-1)^n (n^2 + n - 1)}{3n^2 + n}$$

```
> evalf(A), evalf(C);
```

0.09228936139, -0.07614037580

However the partial sums ( $\tau_{2,PaS}$ ) and even more the point plot method ( $\tau_{2,PoP}$ ) reveal that the series (c) alternates between two values (about 0.09 and -0.24) and the value given by  $\tau_{2,N}$  is the average of these. This means that the series (c) does not give a task of type  $T_{21,N}$ , while students may in fact be confused by the apparent contradiction of the results achieved with  $\tau_{2,N}$  and  $\tau_{2,PaS}$  or  $\tau_{2,PoP}$ . The task (c) thus demonstrates the necessity of combining instrumented techniques with other ones – an important point brought out here by careful instrumental orchestration. At the same time, the materialization potential realised by  $\tau_{2,PaS}$  or  $\tau_{2,PoP}$  helps to choose non-instrumented techniques for  $T_{21}$ ,  $T_{22}$ ,  $T_{21}^*$ , and  $T_{22}^*$  (for (a), root criterion, for (c): divergence criterion related to a task of type  $T_{11}$  or  $T_{12}$  to show that  $|a_n| \not\rightarrow 0$ ).

*Example 2.* For the power series  $\sum x^n/(n(n+1))$  the students were asked to find the sum function ( $T_{44}$ ) using *Maple*, then prove the result ( $T_{44}^*$ ). Here, the symbolic technique  $\tau_{44,S}$  gives:

```
> sum(x^n/(n*(n+1)), n=1..infinity);
```

$$-\ln(-x + 1) + \frac{\ln(-x + 1)}{x} + 1$$

which the students might not easily find by other methods. However, using the Taylor series of  $\ln$  and the above formula, it is easy to solve  $T_{44}^*$ . This use of  $\tau_{44,S}$  also helps in reasoning about the convergence of the series at 0 and  $\pm 1$ , which are in play in later parts of the exercise, and thus to a realization of the lever potential.

*Example 3.* Students were asked to use *Maple* to compute the exact value of  $s = \sum_{k=1}^{\infty} k^{-4}$  using *Maple* (the result  $s = \pi^4/90$  is immediate, using  $\tau_{2,S}$ ) and then to investigate which of the sequences  $a_n = \sum_{k=1}^n k^{-4}$  and  $b_n = \frac{1}{3}n^{-3} + \sum_{k=1}^n k^{-4}$  gives the best approximation to  $s$ . This task belongs to  $T_{5c}$  (approximate numbers by sequences or series). Here, using numeric computations or point plot one sees that

$a_n \leq s \leq b_n$  for all  $n$ , and that in fact  $b_n$  is closer to  $s$  than  $a_n$  for  $n \geq 3$ . The students are finally asked to prove their hypothesis. This is achieved using non-instrumented techniques, taken from the proof of the integral criterion, but the experiment with numeric values of the sequences can support this transition to the theory block when materialization of the two sequences has already suggested a hypothesis. A further epistemic value of the instrumented technique in this task resides in the fact that students may (in fact should) develop themselves the criterion  $c$  for the “best” approximation, by looking at the numeric values of  $s-a_n$  and  $s-b_n$ .

*Example 4.* The students were asked to produce a plot of the functions  $f_n(x) = 1/[1 + \exp(n(2-x))]$  for five values of  $n$ , in order to “illustrate the convergence properties of the sequence”. This in fact involves  $T_{3k}$  for all  $k$ . Just as in the previous example, the materialization potential can be realized to generate strong hypotheses for later work (on  $T_{3k}^*$ ) if the instrumented techniques related to simultaneous plotting are used well (this includes the choices for  $n$  and the interval on which the functions are plotted).

## SOME FIRST RESULTS

The instrumental orchestrations exemplified above were first used in the spring 2010 edition of *Analysis 1*, with about 150 students (divided in 6 classes for exercise sessions). Although all students have easy access to computers with *Maple*, the course had previously not involved instrumented techniques. For this and other reasons, a “modest” approach was adopted, with *Maple* being involved in just a few classroom exercises per week, and in about 20% of the questions in the take home exams (these were handed in by students in two paper copies, including print-outs of *Maple* sheets). The main purpose of this intervention was to get basic information about the effects of the tasks and the feasibility of more ambitious designs in this and similar courses. The goal of these elements of instrumental orchestration was of course to support students’ access to tasks that require more autonomous work with theoretical blocks, but we did not expect that the limited experiment would result in major documented impact of this kind. Instead we analysed, more locally, the performance of students on tasks with instrumented techniques, mainly in their written productions in the two first take home exams. We present the results for the four examples given in the previous section below. Also, secondary and more informal evidence was collected from teachers and students. In particular, the standard course evaluation questionnaire filled by students at the end of the course was extended with items about their experience with *Maple*-use in the course. Some of this evidence is also presented below.

### Performance on exam tasks: instrumented techniques vs. overall performance

In Tables 2, we present the overall results of students’ work on the four examples of exercises that were explained in the previous section. The tables show the number of

## Working Group 14

students in groups defined by the following rough but objective criteria (so that, for instance, the sum of the first row gives the number of students in group  $H$ ):

H: overall high achievers, with at least 75% of points in the take home exam

L: overall low achievers, with less than 75% of points in the take home exam (still plenty of room to pass the course - the limit is an average of 50% in the four exams).

+I: correctly used instrumented techniques in the exercise(s) described in the ex.

–I: did *not* (or *not correctly*) use instrumented techniques in the exercise(s) described

The first three tables (corresponding to example 1, 3, 4) pertain to the same set of exercises, so the groups H/L are the same.

Ex.1	+I	–I	Ex.3	+I	–I	Ex.4	+I	–I	Ex.2	+I	–I
$H$	73	9	$H$	71	10	$H$	77	4	$H$	73	4
$L$	42	17	$L$	40	21	$L$	47	13	$L$	54	1

**Tables 2: Performance of students on exam tasks involving instrumented techniques**

We notice the similarity of results for Ex.1 and 3, probably because they require students to work with very similar combinations of instrumented techniques for computing series and sums. While the *Maple* commands to be used are in themselves quite basic, most mistakes in group  $L$  are in fact also quite basic, like using or plotting the first five partial sums to conclude that a series is not convergent (while in fact it is). Similar errors are found in Ex. 3 where the first terms may also confuse. Also, error messages like “Float(undefined)” are misinterpreted as “divergent” by a few students. In Ex. 4, the difficulties with instrumented techniques appear slightly smaller for both groups, probably because only one (plotting) is required and works (unlike in Ex. 1 and 3 where one must carefully select and combine techniques). The difficulty is the choice of plotting interval – some students, who choose a wrong one, just get a bunch of parallel lines.

We finally notice that virtually all students have used the required instrumented technique correctly in Ex. 2 (the four students in  $-I \cap H$  simply abstained from Maple use, a phenomenon further discussed later). This can be easily explained by both the ease and the pragmatic value of using the instrumented technique ( $\tau_{44,S}$ ) here. Moreover this occurred at a later stage of the course where the students had seen many more complicated uses of *Maple*, including Ex. 1 and 3. The absence of challenge does not mean that this type of use of *Maple* is useless (it works, in this case, as an intelligent “table of series”), just that the lever potential achieved is of a more pragmatic value than in the cases where strong theoretical perspectives are activated in the choice of an instrumented technique.

### Student questionnaire : diverging ‘voices’

It is not surprising that students with an overall lower performance also commit more errors in using instrumented techniques. Our purpose was not to make easy tasks but to include practical (and instrumented) elements in the tasks that would ease the transition to more theoretical parts, at least for low achieving students. To know about this would require a rather intensive study of these students' work during the course, which we did not manage to do. The weaker but easier alternative, structured interviews based on select tasks, had to be given up because we did not get a sufficient number of volunteers (exam period was followed immediately by summer holiday).

Nevertheless, interesting evidence is found in the course evaluation, done via the course website towards the end of the course and with student replies being anonymous. Three questions concerned the use of *Maple* in the course. In the two first, students are asked to indicate their level of agreement with the propositions "I think that the use of *Maple* in the home exercises contributed to my understanding of them", and "I think the use of *Maple* in the home exercises improved my solutions". Here, the 80 responding students are divided rather equally on agreeing and disagreeing. The third question invites for "comments and suggestions regarding the use of *Maple* in this course" and is of course with open field response. As many as 32 students have responded to this field; they can not be considered "representative" but they do represent strong opinions on the matter. These are also divided roughly in two halves, students who think the use of *Maple* is superfluous or tedious, and students who found it useful and supportive. Here are some examples of these comments (translated from Danish):

*I think it was good that the Maple part was relatively simple, and without it certain exercises would have been difficult to get into.*

*Maple is not necessary in a course like An1. The pure mathematical methods totally suffice to understand and solve all the exercises (...) without ugly tricks like numerical evaluation of expression. In all my home exams I did everything without Maple, and in the end I did the Maple part without being surprised of any result.*

*I am pretty sure I didn't get more understanding from the use of Maple, and I have understood from older students that we will not need Maple in the future [authors: more advanced courses in the mathematics programme]. It's irritating.*

*The extent now [authors: of Maple use in the course] is fine, but it shouldn't be more (maybe a bit less). But it clearly helps your understanding.*

## CONCLUDING REMARKS

Combined with our analysis of take home exams and observations in a few classes where *Maple* was used (often computers were missing), we think that two minorities of students may be identified relatively to the categories of Table 2:



$-I \cap H$  (“the proud purists”): Relatively successful students who don’t think *Maple* has its place in a “real” mathematics course and who sometimes choose to deliberately not do the *Maple* part (according to the course policy, declared in the first weekly, they are not penalised for this if they solve the non-*Maple* parts correctly).

$+I \cap L$  (“the challenged but helped”): students who have troubles with several of the exercises, particularly the more abstract and technically demanding ones, but who appreciate and succeed with the parts where instrumented techniques can be used. This is in part because these parts are relatively easy (according to one instructor, they even take some *L*-students above the “passing line” of 50%) but also, at least for (and according to) some students, because these parts help their “understanding”.

Making the optional character of *Maple*-use more apparent could probably satisfy the first group. The needs and work of students in the latter group has to be studied more intensively than we were able to in this study.

## REFERENCES

- Artigue, M. (2002). Learning of mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning* 7, 245–274.
- Artigue, M. (2010). The future of teaching and learning with digital technologies. In C. Hoyles and J. B. Lagrange (Eds.), *Mathematics education and technology – rethinking the terrain*, 463-475. New York: Springer.
- Barbé, J., Bosch, M., Espinoza, L. & Gascon, J. (2005). Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools. *Educational Studies in Mathematics*, 59(1-3), 235-268.
- Chevallard, Y. (1999). L’analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, 19(2), 221-266.
- Elbaz-Vincent, P. (2005). A CAS as an assistant to reasoned instrumentation. In : D. Guin, K. Ruthwen and L. Trouche (Eds), *The didactical challenge of symbolic calculators : turning a computational device into a mathematical instrument*, pp. 41-66. New York: Springer.
- Trouche, L. (2005). Instrumental genesis, individual and social aspects. In : D. Guin, K. Ruthwen and L. Trouche (Eds), *The didactical challenge of symbolic calculators : turning a computational device into a mathematical instrument*, pp. 197-230. New York: Springer.
- Winsløw, C. (2003). Semiotic and discursive variables in CAS-based didactical engineering. *Educational Studies in Mathematics* 52 (3), 271-288.
- Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l’abstrait en analyse réelle. In A. Rouchier & I. Bloch (Eds) *Actes de la XIIIème école d’été en didactique des mathématiques* (cd). Grenoble : La Pensée Sauvage.

# UNDERGRADUATE STUDENTS' USE OF DEDUCTIVE ARGUMENTS TO SOLVE “PROVE THAT...” TASKS

Paola Iannone<sup>1</sup> and Matthew Inglis<sup>2</sup>

<sup>1</sup> School of Education and Lifelong Learning, University of East Anglia, UK

<sup>2</sup> Mathematics Education Centre, Loughborough University, UK

*In this paper we report findings from an investigation of 222 proof attempts produced by 74 year-one undergraduate mathematics students at a university in the UK. We classify the proofs according to an extended classification originally used by Stylianides and Stylianides (2009). We found that already at the beginning of Year 1 most undergraduate students in our sample associate the request for a proof of a statement to the production of a deductive argument. Moreover, when students failed to produce a correct proof this was mostly because of difficulties in producing deductive arguments. We suggest that more attention should be given to the process of producing deductive arguments rather than to the types of non-deductive arguments that some students produce as proofs.*

## INTRODUCTION

Undergraduate students' difficulties with proof production at university level have been widely documented in the literature (Moore, 1994; Harel & Sowder, 1998, 2007; Weber, 2001; Selden & Selden, 2003; just to cite a few examples). Indeed in 2009 the ICMI Study 19 was dedicated to “Proof and proving in mathematics education” with one study group solely focusing on proof and proving at tertiary level. Much of the attention of researchers in this field has been devoted to the types of arguments that undergraduate students produce when asked to solve “prove that...” tasks. One of the most popular theoretical frameworks for analysing students' proofs (not only at undergraduate level) has been proposed by Harel and Sowder (1998, 2007). This framework offers a comprehensive taxonomy of students' Proof Schemes, where

...a person's proof scheme consists of what constitute ascertaining and persuading for that person. (Harel & Sowder, 1998, p. 244)

The taxonomy offered by Harel and Sowder (1998) is based on extensive empirical work at all levels of school and at university. The authors describe three categories of proof schemes (each one comprising several sub-categories) which they call *External Conviction proof scheme*, *Empirical proof scheme* and *Analytical (deductive) proof scheme* (ibid. p. 245). An *External Conviction proof scheme* is found when students

... merely follow formulas to solve problems, they learn that memorisation of prescriptions, rather than creativity and discovery, guarantees success. And when the teacher is the sole source of knowledge, students are unlikely to gain confidence in their ability to create mathematics. (ibid. p. 245)

An *Empirical proof scheme* is one in which

... conjectures are validated, impugned, or subverted by appeals to empirical facts or sensory experiences. (ibid. p. 252)

and an *Analytic Proof Scheme* is

... one that validates conjectures by means of logical deduction. (ibid. p. 258)

Harel and Sowder (1998) also indicate that the latter is the proof scheme that students should aspire to and the one generally held by the community of mathematicians. Because of the prominence that Harel and Sowder (1998) give to deductive arguments, and the statement that this is indeed the proof scheme shared by professional mathematicians, we will take deductive arguments to be the desired outcome of “prove that...” tasks. In fact Harel and Sowder (2007) refine their definition of Analytical proof scheme as to include two proof scheme classes: Transformational proof schemes and Axiomatic proof schemes and state that:

All the transformational proof schemes share three essential characteristics: generality, operational thought, and logical inference. (ibid. p. 7)

In the studies cited by Harel and Sowder (2007) a picture emerges of university students still uncertain about what the role and place of proof at university level is, with students gaining conviction about mathematical statements through various types of arguments, including many students relying solely on empirical arguments. To cite just one more example of research in this direction, Recio and Godino (2001) reported that only very few of the university students in their sample were successful in proof tasks and that 40% of the students in the sample relied on empirical arguments as proof. Whilst proof schemes offer a “truly comprehensive” (Harel & Sowder, 2007) perspective on learning and teaching proof, it can be argued that much of the focus of this framework is on the *types* of arguments that students find convincing when asked to solve proof tasks rather than the *process* they use to produce such arguments. In this paper we examine what types of arguments year-one mathematics students at a high-ranking university in the UK produce when asked to solve proof tasks and we argue that, at least in this case, students are aware of the requirement of proof but their difficulties lie in the process of producing correct deductive arguments.

## THE STUDY

The main aim of the study reported in this paper was to find out whether self-efficacy is an accurate predictor of academic performance, and in particular of proof production. For the scope of this study we have defined self-efficacy, following Bandura (1977), as the judgment students make of their own capability of performing a given task (in our study the task is proof production). The results of this part of the study were presented in Iannone and Inglis (2010). For the scope of his paper we have analysed the proofs that the students produced in the second part of the

questionnaire they were asked to fill in. Seventy-six first year students in mathematics (or on a joint degree with a substantial mathematics component such as computer science or natural sciences) at a high-ranking university in the UK took part in this study. Data collection took place in week 8 of the first semester during one of the Linear Algebra lectures. A booklet of questions was given to each participant, and the students were asked to work through the questions at their own pace. The first section of the booklet consisted of 28 statements; ten consisted of our Proof Self-Efficacy Scale (Iannone & Inglis, 2010), designed following Bandura's (2006) guidelines. A further ten statements were formed from the items of the General Self-Efficacy Scale (Schwarzer & Jerusalem, 1995). Eight extra items were also introduced to readdress the balance between forward- and reverse-scored items (taken from the Need for Cognition scale, Cacioppo & Petty, 1982). The participants were asked to read and decide the extent to which they believed the statements were characteristic of them, using a five-point Likert Scale (from "extremely uncharacteristic" through to "extremely characteristic"). The order in which the statements appeared was randomised for each participant. The second part of the booklet consisted of four proof construction tasks novel to the students and designed in collaboration with a mathematics lecturer so that they would represent appropriate tasks for this cohort, in the sense that they were tasks similar to ones that the students had encountered during the lectures. The order in which the proof tasks appeared was again randomised for each participant. The participants were given 20 minutes to work on the proof tasks.

The proof tasks included in the booklet were:

- A. Prove that the sum of two odd numbers is even.
- B. Prove that the sum of the first  $n$  natural numbers is equal to  $\frac{1}{2}n(n+1)$ .
- C. Let  $d$ ,  $a$  and  $b$  be integers. Prove that if  $d \mid a$  and  $d \mid b$  then  $d^2 \mid (a^2 + b^2)$ .
- D. Prove that if the sum of the digits of a natural number is divisible by 3 then the number itself is divisible by 3.

The proof Task D was introduced even if after the meeting with the mathematics lecturer it was deemed to be too demanding for this cohort. As we will see in the analysis of the data this was indeed the case. A marking scheme was devised to evaluate the proofs. Each proof was marked out of five. In Appendix we report the questions in the questionnaire, the marking scheme and the model solutions to the tasks. We analysed the proofs and classified them following the refinement of a proof classification originally developed by Stylianides and Stylianides (2009) in the context of a study with primary prospective elementary teachers. Whilst categories M1, M4 and M5 of the original classification served our aims, categories M2 ("valid general argument but not a proof", *ibid.* p. 246) and M3 ("unsuccessful attempt for a valid general argument", *ibid.* p. 246) were too broad and we decided to divide them

further in categories P1 to P5. So we obtained the following classification of proof tasks:

Type	Description
<b>M1</b>	Correct proof
<b>P1</b>	The last part of the proof is missing
<b>P2</b>	The hypothesis are expressed correctly but the proof stops after the statement of the hypothesis
<b>P3</b>	The solution does not represent the most general case
<b>P4</b>	The solution resembles the correct proof but there are not enough details to see whether this is correct or not
<b>P5</b>	The solution follows a correct deductive argument but some mistakes in the calculations occur
<b>M4</b>	The solution consists of an empirical argument
<b>M5</b>	Some mathematical statement is presented but this is unrelated to the proof requested
<b>0</b>	Task left blank

**Table 1: Responses categories**

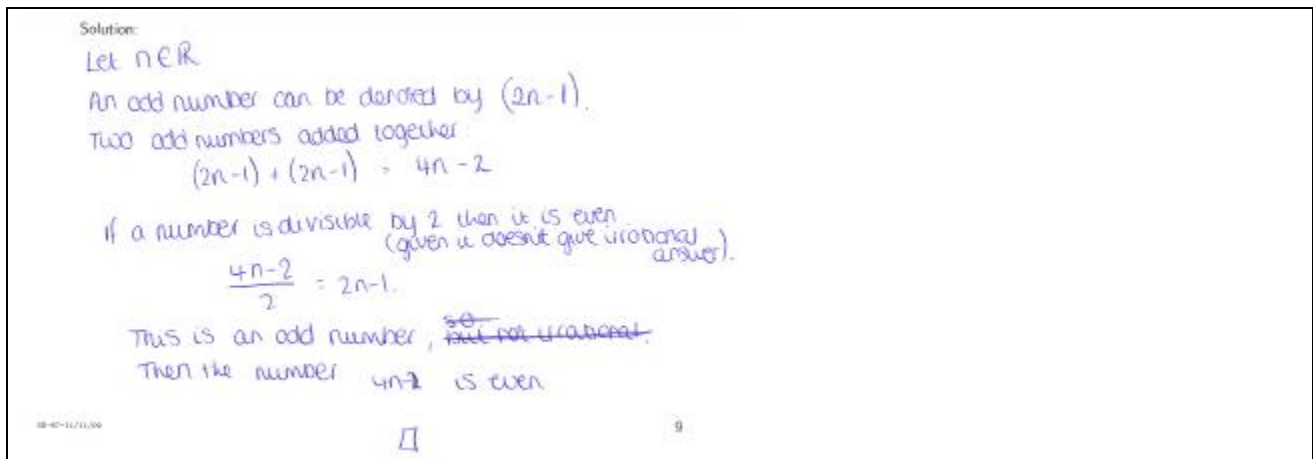
Before we discuss the classification of the proof tasks we give here some examples to clarify the categories mentioned above. We assume that the categories M1 and 0 are self-explanatory so we will very briefly discuss the other categories giving some examples from the students' work.

#### **P1 – The last part of the proof is missing**

Solutions in this category comprise deductive arguments without conclusions. Typical examples here are proofs by induction (Task B) which stop at the end of the inductive step.

#### **P2 - The hypothesis are expressed correctly but the proof stops after the statement of the hypothesis**

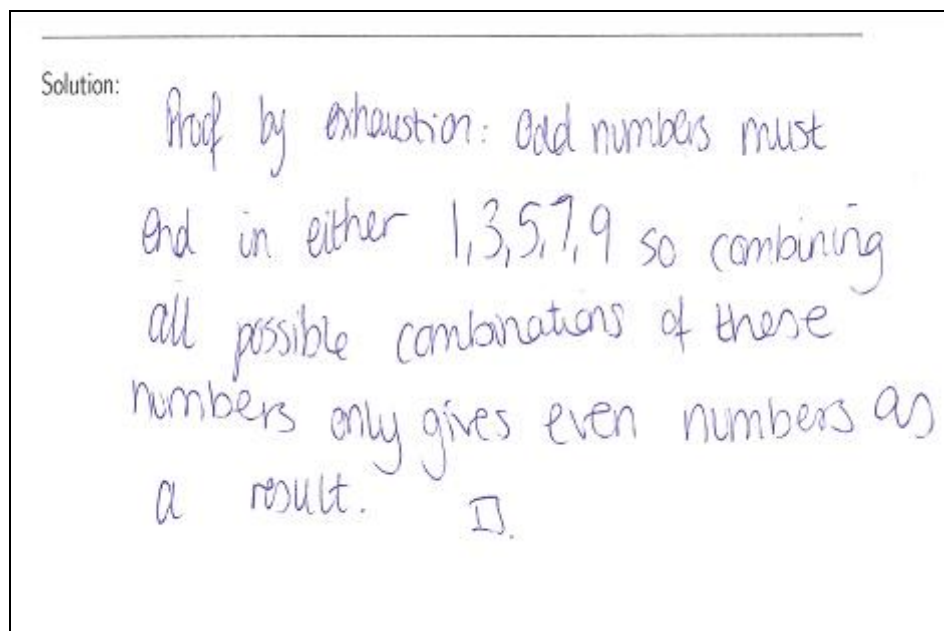
In several of the solutions for the proofs Tasks A and C students wrote the hypothesis correctly (for example stated correctly that  $d \mid a$  means  $a = kd$  with  $k$  an integer number) but were unable to proceed. Attempts of proof of this kind resonate with Moore's (1994) findings who observed that students can sometimes state the hypothesis in the theorem to prove, but are unable to start the proof.

**P3 - The solution does not represent the most general case****Fig. 1: Example of category P3**

This category was mostly found in the solution of Task A. An example is reported in Fig. 1 above. In this case the student failed to express the most general case, proving instead the statement: the sum of two equal odd numbers (i.e. twice an odd number) is even.

**P4 - The solution resembles the correct proof but there are not enough details to see whether this is correct or not**

An example in this category is the following:

**Fig. 2: Example of category P4**

In this example the student has tried to give a verbal argument for the proof. However lack of clarity and details mean that it is not possible to ascertain whether the argument is correct or not.

**P5 - The solution follows a correct deductive argument but some mistakes in the calculations occur**

Typically this occurred in the proofs by induction where the students made a mistake in the calculations for the inductive step.

**M4 - The solution consists of an empirical argument**

In this category we have grouped examples of “proof by example” where students gave a numerical example as a proof. This is clear in the following example:

Working:

$$\begin{array}{l}
 d/a \quad d/b \\
 d=3 \quad a=9 \quad b=27 \\
 d^2=9 \quad a^2=81 \quad b^2=729 \\
 9^2/(81+729) \\
 9/810 \quad \checkmark
 \end{array}$$

**Fig. 3: Example of category M4**

Here the student gave one numerical example and terminated the answer with a tick – perhaps signaling that s/he was satisfied with the answer or was satisfied with having found a confirming example.

**M5: Some mathematical statement is presented but this is unrelated to the proof requested**

Responses in this category included all sorts of unrelated mathematical facts, but no recognisable deductive arguments.

**ANALYSIS AND FINDINGS**

Two participants were removed from the analysis of the data as they attempted none of the proof tasks. This left us with 74 responses.

A classification of the proof tasks excluding Task D, according to the categories outlined above yield to the following table:

	A	B	C	Total
<b>M1</b>	28	34	21	83
<b>P1</b>	0	11	5	16
<b>P2</b>	3	8	8	19
<b>P3</b>	30	0	1	31

### Working Group 14

<b>P4</b>	3	2	2	7
<b>P5</b>	0	3	6	9
<b>M4</b>	0	0	2	2
<b>M5</b>	7	4	6	17
<b>0</b>	3	12	23	38
<b>Total</b>	74	74	74	222

**Table 2: Classification of the proof tasks excluding Task D**

We excluded Task D as it was solved correctly only two times across the sample, with only five students accruing marks for it.

The distribution of the marks over the three remaining proof tasks had mean 7.76 and standard deviation 4.35, suggesting that the three tasks were suitable for this cohort of students (i.e. not at ceiling or floor levels). As for the classification of the proof tasks Table 2 shows that 37% of the total number of tasks attempted by the students were correct, with the task solved correctly the most times being task B (solved correctly 34 times) followed by Task A (solved correctly 28 times) and Task C (solved correctly 21 times). Considering categories P1, P5 and P3 as (attempts at producing) deductive arguments and category P2 as manifestation of the inability to produce a deductive argument (we infer this by the formal statement of the hypothesis but the lack of any other writing to follow, see also Moore, 1994) the distribution gives the following frequencies:

<b>Deductive arguments (M1, P1, P2, P3, P5)</b>	158
<b>P4</b>	7
<b>M4</b>	2
<b>M5</b>	17
<b>0</b>	38
<b>Tot</b>	222

**Table 3: Summary of the types of arguments used across the tasks**

Table 3 shows that more than two thirds of the arguments that the students produced (or attempted to produce) were in fact deductive arguments. Only two arguments produced out of 222 were empirical arguments.

## CONCLUSIONS

In this paper we have classified 222 solutions to proof tasks collected during a study involving year-one mathematics students in a university in the UK. Our classification of proof tasks aimed at investigating what type of arguments students produce when



asked to give a proof a mathematical statement. We found that the students in our sample, at a very early stage in their university degree, know largely what the result of a proof construction task ought to be (at least in algebra), namely a deductive argument. Their difficulties were mostly related to failure to produce a correct deductive argument rather than failure to recognise that a proof involves some kind of deductive argument. Remarkably, in our sample, we found only 2 “empirical proofs” pointing to the fact that our students on the whole did not regard investigating one or more examples of a conjecture to be sufficient proof. Our findings resonate with other findings in the research literature (e.g. Weber, 2001; Weber & Alcock, 2004) where researchers have repeatedly pointed out to the difficulties students have in producing correct deductive arguments. Our findings also seem to be in contrast with other research findings on proof production (for example Recio & Godino, 2001) where researchers have pointed out the reliance of university students on empirical arguments to solve “prove that...” tasks. One possible explanation for this is that the students in some of these research projects are not always mathematics students. For example in the paper by Recio and Godino (2001) we have cited previously, participants were “students who took a mathematics subject in different faculties and polytechnic” (p. 84). Although the authors do not give more details about their sample, we argue that this could include students from many departments and not necessarily sciences. The participants in our sample were mostly mathematics students, with some students coming from other disciplines with a strong mathematics component (e.g. computer science, natural sciences, mathematics with economics). Perhaps it is the background of the students that accounts for such discrepancies in the findings. The data we have presented seem to indicate that the students in our sample (e.g. students with a strong mathematics background) already operate with an analytical proof scheme, and therefore the goal of instruction in this case should be to help the students become versed in the production of deductive arguments rather than the elaboration of their proof schemes. If we confront this with Harel and Sowder (2007) where they write:

We emphasize again that despite this subjective definition the goal of instruction must be unambiguous—namely, to gradually refine current students’ proof schemes toward the proof scheme shared and practiced by contemporary mathematicians.

(Harel & Sowder, 2007, p. 7)

we would argue that perhaps more attention (in research and in teaching) should be given to the *process* by which students produce evidence to gain conviction about statements and write deductive arguments to produce proofs rather than to what are the *types* of evidence that students at university level offer as proofs.

## REFERENCES

- Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavior change. *Psychological Review*, 84, 191-215.

- Bandura, A. (2006). *Self-Efficacy Beliefs of Adolescents*. Information Age Publishing.
- Cacioppo, J.T., & Petty, R.E. (1982). The need for cognition. *Journal of Personality and Social Psychology*, 42, 116-131.
- Harel, G., & Sowder, L. (1998). Students proof schemes. *Research in collegiate mathematics education*, 3, 234-282.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (p. 805-842). Charlotte, NC: Information Age Publishing.
- Iannone, P., & Inglis, M. (2010). Self-efficacy and mathematical proof: are undergraduate students good at assessing their own proof production ability? *Proceedings of the 13th Conference on Research in Undergraduate Mathematics Education*. Raleigh, North Carolina. Available at <http://sigmaa.maa.org/rume/crume2010/Abstracts2010.htm>
- Moore, R. (1994). Making the transition to formal proof. *Educational Studies in Mathematics*, 27, 249-266.
- Recio, A. M., & Godino, J. D. (2001). Institutional and personal meanings of mathematics proof. *Educational Studies in Mathematics*, 48, 83-99.
- Schwarzer, R., & Jerusalem, M. (1995). Generalized self-efficacy scale. In J. Weinman, S. Wright, & M. Johnston (Eds.), *Measures in health psychology: A users portfolio. Causal and control beliefs* (p. 35-37). Windsor, UK: NFER-Nelson.
- Selden, A., & Selden, J. (2003). Validations of proofs written as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4-36.
- Stylianides A.J., & Stylianides G.J. (2009). Proof constructions and evaluations. *Educational Studies in Mathematics*, 72(2), 237-253.
- Weber, K. (2001). Students' difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, 48(1), 101-119.
- Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, 56, 209-234.

## APPENDIX

In this appendix we report the questions included in the questionnaire we gave to the students (labelled here Task X), a solution designed according to what the students would have seen in the lectures and the marking scheme (the last two items not included in the questionnaire).

**Task A:** Prove that the sum of two odd numbers is even.

### Working Group 14

**Proof:** Let  $n = 2k_1 - 1$  and  $m = 2k_2 - 1$  be odd numbers. We have

$$m + n = 2k_1 - 1 + 2k_2 - 1 = 2(k_1 + k_2) - 2 \text{ even.}$$

2 points for expression of odd number; 2 for expression of sum; 1 for conclusion

**Task B:** Prove that the sum of the first  $n$  natural numbers is equal to  $\frac{1}{2}n(n+1)$ .

**Proof:** For  $n = 1$ :  $\frac{1}{2}n(n+1) = 1$ .

Suppose this is true for  $n - 1$  and show this is true for  $n$ . We have

$$S_{n-1} = \frac{1}{2}(n-1)(n-1+1) = \frac{1}{2}(n-1)(n).$$

$$\text{Add } n \text{ and obtain } S_n = \frac{1}{2}(n-1)(n) + n = \frac{1}{2}n(n+1).$$

Hence this is true for all  $n$  in  $N$ .

2 points for base step; 2 for inductive step; 1 for conclusion.

**Task C:** Let  $d$ ,  $a$  and  $b$  be integers. Prove that if  $d \mid a$  and  $d \mid b$  then  $d^2 \mid (a^2 + b^2)$ .

**Proof:**  $d \mid a \rightarrow a = kd$  and  $d \mid b \rightarrow b = md$ , hence

$$(a^2 + b^2) = (kd)^2 + (md)^2 = k^2d^2 + m^2d^2 = d^2(m^2 + k^2) \rightarrow d^2 \mid (a^2 + b^2).$$

2 points for expression of divisibility; 2 for expression of sum of squares; 1 for conclusion.

**Task D:** Prove that if the sum of the digits of a natural number is divisible by 3 then the number itself is divisible by 3.

**Proof:** Let  $d$  be a natural number. Let  $d = d_k d_{k-1} \dots d_0$  its expression in digits. If we expand this expression in base 10 we have

$$D = 10^k d_k + 10^{k-1} d_{k-1} + \dots + d_0 \text{ we can write this as}$$

$$d = (10^k - 1)d_k + (10^{k-1} - 1)d_{k-1} + \dots + 9d_1 + d_k + d_{k-1} \dots + d_1 + d_0$$

Note that all the items in the form  $10^x - 1$  are divisible by 3 (they are in fact a string of 9s). So we can write  $d = 3A + d_k + d_{k-1} \dots + d_1 + d_0$ . Hence  $d$  is divisible by 3 if and only if  $d_k + d_{k-1} \dots + d_1 + d_0$  is divisible by 3.

2 points for expression base 10; 2 for changing into  $(10^{k-1} - 1)$  etc; 1 for conclusion.

# HOW WE TEACH MATHEMATICS: DISCOURSES ON/IN UNIVERSITY TEACHING

Barbara Jaworski and Janette Matthews,

Loughborough University, UK

*A series of seminars called “How we Teach” has focused on the teaching of mathematics at university level. Participants are mathematics educators and mathematicians in a university School of Mathematics. Video recordings of the seminars have been analysed to uncover a teaching discourse in the mathematical community involved. Analysis reveals the preferred approaches to teaching (within institutional boundaries), the resources teachers use and the ways teachers think about students’ learning of mathematics. Issues arising are illustrated by key quotations and the paper offers a first attempt at characterising the discourse and suggesting implications for teaching development and the education of new teachers.*

Keywords: university mathematics teaching, discourse, community, development.

## INTRODUCTION

This paper results from analysis of data from a series of seminars presented by university mathematics teachers on the topic “How we Teach”. They take place within a *School of Mathematics* which includes a *Mathematics Education Centre* and focus on research into university mathematics learning and teaching. Over two and a half years, with 20 seminars to date, most have been video-recorded and 10 seminars have been analysed. We have chosen to analyse seminars regarded as being most representative of an established university discourse. They were each presented by a mathematician or a mathematics educator, to a mixed audience, all of whom teach mathematics to mathematics or engineering students.

Our intention is not to reveal how mathematics teaching *is*, but rather how those teaching mathematics *talk about* their teaching within their institutional setting. We have recorded and analysed what teachers *say they do*, how and why, and the related discussion between seminar participants, covering approaches to teaching, reasons behind these approaches, and issues arising for the presenter, or in discussion.

## WHAT DO WE KNOW ABOUT TEACHING AT UNIVERSITY LEVEL?

Various members of the international research community suggest that we do not, as yet, know very much about how mathematics is taught at university level. For example, from New Zealand and the US respectively, we have:

It is apparent on inspection of the literature that ... there is relatively little research into teaching and learning of mathematical sciences at the undergraduate level and very few academic journals that focus on publishing such research. (Barton & Thomas, 2010, p. 1).

... while some mathematicians have written *about* their teaching,, others have analysed aspects of their teaching and students' learning in innovative collegiate courses, ... very little research has focused directly on teaching practice – what teachers do and think daily, in class and out, as they perform their teaching work (Speer, Smith & Horvath 2010, p. 99). Speer *et al.* (2010), point to five (rare) examples of published research on “collegiate teaching” in the US which provide

... empirical analyses that describe teaching at a sufficiently fine level of detail that teachers and other researchers can inspect and learn from the instructional choices and learning of others.

Some current research in Europe is also looking at university teaching with such level of detail in trying to characterise teaching practices, inform our community and open up a wider awareness of what we are doing to allow us to address how we might do it better (e.g., Jaworski, Treffert-Thomas and Bartsch, 2009; Ioannou & Nardi, 2009; Petropoulou, Potari, & Zachariades, 2010). Clearly we need more of this kind of research as well as outlets for publishing it.

The research reported in this paper aims to open up the field of study of university teaching for better communication and developing awareness. However, its focus, in seeking *a teaching discourse*, is not on the practice of teaching per se, but on how the mathematical community expresses its thinking about teaching and the design of its teaching. Concomitantly, this research, and the actual practice of the research (in this case, the motivation for and holding of the seminars), can draw teachers' attention to alternatives to common practice, encourage critical approaches to thinking about teaching, and foster teaching development.

Two words that have appeared in the above text and perhaps need further remarks are *discourse* and *community*; although space restricts what it is possible to say. Rom Harré and Grant Gillet (1994) write: “Actions and acts they accomplish make up discursive practice” which is “the repeated and orderly use of some sign system, where these uses are intentional, that is, directed at or to something” (p. 28). Also, “... a discursive practice is the use of a sign system, for which there are norms of right or wrong use ...”(p. 29). Derek Edwards (1997) writes that a focus on discourse shifts notions of human actions as rule governed and based on cognitive representations “towards a concern with norms and descriptions”, and that “norms do not govern actions, but ... actions are done and described in ways that display their status with regard to some rule or expectation (p. 7). Harré and Gillet write further, “The crucial insight that allows us to explain psychological phenomena as patterns of discursive acts is that norms and rules emerging in historical and cultural circumstances operate to structure the things people do” (p.33). We would claim that, in the teaching of mathematics, we are dealing with multiple *sign systems*, each with their own “norms and rules”. Mathematics itself offers symbolic sign systems, as well as conventions for writing mathematics, for example, proof structures. Teaching mathematics uses a complex set of sign systems, which include the

mathematical: these would encompass ways of describing teaching and learning with all the ontological and epistemological associations on which such descriptions rest.

We work within a community of practitioners in mathematics, including teachers and researchers and various areas of research in both mathematics and mathematics education. We can talk here of a “community of practice” in which

*The concept of practice* connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do. In this sense practice is always social practice (Wenger, 1998, p.47).

Etienne Wenger describes three ‘dimensions of practice’ which he explains are ‘the property of a community’; these are ‘joint enterprise’, ‘mutual engagement’ and ‘shared repertoire’ (p. 73). Briefly, they involve common purpose and activity and a shared language. Ways of interpreting or describing such a community can be seen as a *discourse* as set out briefly above. So, in our study reported in this paper, we seek to identify aspects of both community and its discourse about teaching mathematics, as seen through analysis of the words uttered by practising mathematics teachers in *How we Teach* seminars.

## METHODOLOGY

One intention in creating the seminars was to provide a forum for such a discourse to become overt and consequently open to challenge. We wished to reveal common practices, taken for granted assumptions, and differences in ways of seeing established practices, and, through making all of these more visible, give our community a chance to look critically at *what* we do, and *how* and *why* we do it. In one sense we wished to flesh out our ‘joint enterprise’, ‘mutual engagement’ and ‘shared repertoire’, or recognise issues which challenge such concepts. One essential recognition was the institutional norms and expectations within which modes of practice were taken for granted. For example, programmes of study are modularised and the main medium of teaching is the lecture addressed to a given cohort of students (up to 250 in some mathematics modules). Our data shows an overwhelming focus on teaching through *lectures*: those who teach are most commonly called lecturers. Although seminar participants address tutorials and other media for learning, discussion of the activity of lecturing by lecturers predominates. This makes it harder to see alternative modes of teaching except in the context of how they would fit into a lecture mode. In addition, the two constituencies, of mathematicians and mathematics educators, may actually see things differently. So we have needed to look critically at *who* is making the observations we note.

We have analysed data from 10 seminars. Our choice has been to analyse those which have seemed to contribute most centrally to an established university discourse at this university – that is those dealing with the mainstream teaching of mathematics to students studying mathematics as part of a mathematics or engineering degree. Thus, we have four from established mathematicians (who do

## Working Group 14

research in mathematics), two from former mathematicians (who now do research in mathematics education), three from mathematics educators (who do research in mathematics education) and one from a university teacher (who does no research). The excluded seminars are not so centrally focused and are seen as more peripheral to our main aims for analysis.

Analysis has involved several stages:

1. To ground perspectives, two researchers viewed together the video-recording of one seminar and discussed their interpretations of what the data reveals.
2. Two researchers independently analysed one further seminar, noting key episodes and identifying their content in a linear pass through the recording.
3. Discussion in (2) suggested a style of analysis for the other seminars as follows:
  - a. One researcher produced a data reduction of the recording with a timeline noting factual details of episodes and highlighting significant ideas.
  - b. The same researcher produced a synoptic, descriptive account of the seminar, including the main themes and issues.
  - c. Both researchers discussed the analysis in (a) and (b), seeking justification for interpretations, and starting to identify key ideas and issues in the data. This led to identifying categories in the data.
4. Periodically, two researchers reviewed analysis to date and mapped out an interpretative account of the seminars, seeking a trustworthy description of the discourse overall, a characterization supported by significant quotations, and further analysis of these quotations as indicated below.

## INITIAL CATEGORISATION

Broad categorisation has suggested the following areas of focus:

*What is included in teaching?* This includes institutional practice, students groupings, a VLE (LEARN), and the various (other) resources that are used in teaching.

*Strategies/Approaches to teaching:* These include use of questions, tests, examples, group work, animations, video, weblinks, Electronic Voting Systems (EVS), and Computer Assisted Assessment.

*Provision of resources for students:* This includes provision of notes, problem sheets, solutions, and a series workbooks (HELM)<sup>1</sup> provided free to students.

*Approaches to/relationships with students:* This focuses on different ways in which lecturers and students meet each other and kinds of interactions that result.

*Links to research:* This includes references to research either that undertaken at this university or research in the public domain.

## ISSUES EMERGING DURING CATEGORISATION

Categorisation, as summarized above, may seem unsurprising. What was more interesting (for us as we worked on the data) were the issues that emerged from discussion around the various approaches to teaching, use of resources and interaction with students. We have grouped these issues within further categories or themes. We give examples below and provide key quotations, essential in highlighting the differing voices that make up the discourse. We refer to particular seminar contributors as M or ME to denote Mathematician or Mathematics Educator respectively. So, M2 or ME5 is always the same person.

Some lecturers brought mathematical examples to engage other participants and provide a basis for discussion. However, in the main, the discourse was meta-mathematical, assuming the basis of mathematics but talking about processes and practices rather than focusing *in* the mathematics. There were identifiable *levels* within this discourse, that is levels of reflective pedagogic perception of the teaching process (see Nardi, Jaworski & Hegedus, 2005 for a spectrum of pedagogic discourse). So far we have categorised these as follows: A) *What we do, how we do it and why* – this is offered with a certainty of action and purpose; B) *What the issues are that we are trying to address* – this points to questions relating to students and the uncertainties inherent in trying to promote desired outcomes for students; C) *What we don't know, but that it would be useful to know* – genuine questions to which it would be valuable to know some answers, but which we cannot address with any certainty at the moment. In addition we see remarks which suggest some underpinning theory, even if implicit. It seems clear that lecturers' words often carry with them something of the lecturer's beliefs about learning and how teaching relates to learning. While it is not our intention to associate individual lecturers with theoretical positions, it does seem worth recognising possible areas of theory on which discourse can be seen to be based. We start with 3 such areas, each dependent on perceptions of knowledge.

1. I need to convey knowledge to students (e.g., absolutist/Platonist theory).
2. Students construct their own knowledge (e.g. constructivist theory).
3. We are bringing students into our mathematical culture (e.g. sociocultural theory).

As a starting point for further consideration we have annotated the quotations below as to whether they can be seen to fit into one or more of these levels and areas.

### **Why should students come to a lecture?**

There were differences in viewpoint, among seminar participants. as to whether students should attend lectures. Some lecturers design modules to encourage students to attend, others leave it to students to make their own decisions as to how to study and, correspondingly, make resources fully available to all.



## Working Group 14

[ME2] says, “I don’t care if they don’t come to lectures because it is all on LEARN. They have got all the workbooks. All the learning resources are there”. I think, when you say that, you do yourself a huge disservice. Because you are missing all the little things that you do [in a lecture] that are of benefit to the students, [ME3: B/2/3].

There is recognition that students do not always choose wisely.

I don’t care [about attendance] if they can manage; but it is clear from exams there are students who would benefit from coming who are not. [M1: B/1]

Others see poor attendance as a reflection on their teaching.

I see poor attendance as a challenge. Some say it doesn’t matter if they turn up as long as they are learning ... poor attendance is a sign I am not doing something right. [ME4: B/1]

It therefore seems important to consider what aspects of lectures would make students find the experience valuable enough to wish to come.

We certainly don’t ever define what the place of a lecture is. And we all look at this very differently ... and have different expectations of students. And that might be confusing for students. [ME2: B/C/1]

I think they [lectures] can be fantastic for providing inspiration and structure to students about how a topic should be thought about. [ME5: A/1]

What I would like is for students to come to lectures and be able to do something at the end that they could not do when they came in. [ME3: A/3]

### **What is the role and nature of lecture notes, and who produced them?**

It was an implicit assumption that students need ‘good notes’. There was considerable discussion on the nature and value of good notes, and disagreement on who should provide them, how and why. Some thought students need to learn how to make good notes for themselves. Some lecturers provide a full set on LEARN, and work through these in a lecture. Others produce “notes with gaps” where spaces are left for students themselves to fill in solutions to examples. If they want to be given the solutions, they have to attend the lecture.

I think they need to ... read my lecture notes in advance, come to lectures ... listen to my exposition, make their own notes about the key points, certainly get extra examples down. Then they can go away and use the workbooks productively. [ME8: A/2]

They think they don’t need to attend the lectures. If they have the notes, that is all they need ... . So they had to turn up to get a full set of examples. I don’t know if that is a good way of doing it. [M7: C/3]

So what I have evolved is ... having gappy notes ... There are bits of writing and then there are big gaps where the students do more bits of writing. What this allows me to do ... is put in quite a lot of commentary about ‘this is what we are doing’ and ‘this is what we are supposed to be thinking about’. So it is not really like getting a text book where it says definition, theorem, proof and so on ... all that stuff that I want to say ..., can already be there for the students but they still have this sort of interaction with the notes

## Working Group 14

where they are writing a lot of the mathematics. It allows for the notes to become something that you have but that you ... physically do something with during lecture time. [ME5: A/2]

I think you should treat students like adults. You should put in front of them all they need to do well. You must not restrict their knowledge in any way. [ME2: A/2]

### **How do students come to understand mathematics?**

Possibly related to these different viewpoints on what a lecture can or should provide, there were differing views on what students can gain from a lecture.

Should we teach the students that in lots of situations they shouldn't expect to learn during a lecture but they should be expected to do something after the lecture? [ME3: C/2]

... you give them the materials, make it tolerably understandable but they don't understand a subject until they have done the tutorial sheets, worked their way through it, gone back to lecture notes and then they understand. [M2: B/2]

Consideration of the issue of understanding suggested that perhaps we had differing perspectives on what we mean by "understanding" and how understanding emerges. Some seemed to suggest that understanding will emerge through lots of focused practice, perhaps working through many examples. Perhaps students also believe this.

Maths is not a spectator sport. You can only learn maths by doing it yourself. [M3: A/2]

You know the technique. Once you have done a few hundred [*examples* e.g., differential equations, integration by parts], you do begin to understand. ... The understanding comes in when you have done enough. [M2: B/3]

In an ideal world, the anticipation is that they will be going away and practising and they have a great resource to do it." [HELM workbooks]. [ME8: A/2]

Many of the students want to know the procedure for getting through the problem and if they have that they are less concerned about the conceptual underpinning ... What I would really like to do is get them thinking about the concepts as well as be able to answer the question that is being set. [ME7: B/C/3]

### **What kinds of things can we do to promote understanding?**

Approaches that were discussed include use of examples, discussion of errors and use of tests or quizzes. Examples are seen as important to illustrate/explain mathematical concepts and procedures.

A certain amount of time is taken up with basic definitions of theorems and proofs and the rest should be examples. Maybe what they don't appreciate is that I design the examples to illustrate the concepts of what I am teaching but not necessarily to give examples that are problems they will meet in the exam. [M1: A/1]

I don't worry if they don't understand everything at start. I don't think they will until they have worked their way through some examples. [M2: A/2]

## Working Group 14

Asking students to come up with their own examples requires them to engage with the mathematics rather than memorise.

If you know a test like this is coming ... you can memorise a bunch of definitions. If you are asked to give examples, that requires you to do something...to really think about what does it takes to satisfy two different properties at once. [ME5: B/2]

But perhaps examples alone are not enough.

[it is] important to discuss the conceptual background of the material because examples are much easier for most students. ... . If I do only that [examples], students miss the most important part. [M4: A/B/3]

Some lecturers use short tests at start of lecture, or quizzes to be taken weekly online.

Everyone knows something they did not know a few minutes ago about their own knowledge ... are they up to date with what we have studied in the last few weeks and can they apply it to examples. [ME5: B/2]

By frequently testing and making them go back over past work, I am hoping they will have an overview of all the techniques to a certain level, all the time. [ME8: B/3]

A focus on errors can offer valuable feedback.

Giving students information on the errors that they have made is I think a very valuable way of giving feedback for bright students.... I collected in the errors they made on their course work .... and asked them if they can spot where the errors are. [ME3: A/3]

There was discussion of ways in which our teaching differs according to our perception of students' mathematical strength, or whether students are main stream mathematicians or in some other programme such as engineering.

[Teaching in Context] is motivating for [engineering] students [who] tend to give [it] very good feedback. They start to see the relevance of mathematics. [ME4: A/3]

For the very weakest engineers...I would give them some graphs. I would ask them to describe in their own words some of the properties of some of these functions ... a way of introducing students to some of the terminology. [ME3: A/3]

[I] direct material to an averagely competent student. In narration, I give simplified explanations for those below averagely competent. [M1: A/1]

I feel I have to give something to the good students who are going to get high firsts and who may do research. Perhaps it is just a handful of students in the room but they pay tuition fees like everyone else. [M1: A/2]

### **Use of technology in lecture presentation**

The medium of the lecture is very important for some lecturers, especially those who favour a chalk or white board. For some, use of a chalk board is what *used to* happen when they were students, but has now been superseded by current technology – e.g., overhead projector (OHP). For others, there is value still in a large chalk board:

## Working Group 14

Students need to see the whole calculation. They need to keep looking back to where they started from, see the key points [M2: A/2]

A problem with, for example, OHP or PowerPoint is that only the most recent part of the mathematical argument is visible at any time. Some mathematics lecturers still prefer to use a chalk board to present material in a lecture because the large expanse of board allows (potentially) a view of the whole argument at one time. The university gives credence to this argument by maintaining a number of lecture rooms for this purpose, despite the fact that chalk dust is detrimental to electronic equipment which needs also to be provided. Those using boards also suggest that students like notes to be written in real time – a PowerPoint presentation is too fast to take notes.

At the other end of the technological spectrum a few lecturers are using electronic voting systems (EVS) in lectures, and see many advantages in doing so, both for the students who gain insight into the thinking of their peers and for lecturers who can see the spread of response in their students as a whole.

Students like to see others getting it wrong – know they are not on their own. [They are] getting an experience they don't get by just reading at home on their own. [ME4: B/3]

### **Resources for teaching and learning.**

Use of the LEARN environment for supporting teaching and learning is highly emphasised in the university. However, could the placing of resources on LEARN be a panacea? Should we consider more overtly whether students use these materials?

The advantage for me ... I am more comfortable lecturing knowing there is this material available [e.g., HELM workbooks]. If you don't like what I am doing in lectures that's fine, just read the book. [ME2: A2]

I suspect we don't teach students how to use books [ME3: C/3]

And many students looking at a page of mathematics look at the formula and ignore the words [ME5: B/1/3]

Some lecturers questioned the role of the student and how students learn to learn.

What do you need to know in order to be a successful student who can go to a lecture, take appropriate notes and then go away and do something with those notes that will be productive in terms of increasing your understanding? [ME5: C/2]

### **DISCUSSION**

The characterisation offered above recognises certain aspects of the discourse; it also offers opportunity for discussion of the underlying issues and how they relate to teaching and its development. The discourse has been highly *meta*-mathematical. By this we mean talking *about* teaching mathematics, talking about students, our goals *for* and perspectives *on* students, talking about the resources we use and our reasons for doing so – often in terms of our intentions for students. Where mathematical examples have been used, they have been designed to stimulate a focus on teaching

approaches. So far, we have not looked explicitly at how lecturers treat particular mathematical content – despite the fact that we have a project looking explicitly at the teaching of linear algebra (Jaworski *et al.*, 2009). We have plans to do this more explicitly in the near future. Recently, the seminars have become a part of the University’s professional development programme for new mathematics lecturers; here, the characterisation above can contribute to clearer visions of how the ways we talk about teaching relate to the actual teaching and its promotion of learning.

A final point brings us back to the nature of our community in the School of Mathematics, consisting as it does of both mathematicians and mathematics educators. Readers might discern, from the A, B, C categorisations above, positions of particular individuals with respect to their views on teaching. With regard to our community as a whole, where our *teaching* is concerned, we can justify the terms “mutual engagement” and “shared repertoire” although we have demonstrated small differences in the latter. A “joint enterprise” is harder to justify, and this possibly relates to our categorisation. At level A, teachers operate within their own certainty of what it means to teach and why, and we see here differing kinds of certainty. However, in areas B and C, we see recognition of the complexity of decisions and related issues for teachers, a willingness to question and consider other possibilities, and a growth of common understandings, or at least respect for alternative points of view. These seem to relate to what Nardi *et al.* (2005) call “reflective and analytic” and ‘confident and articulate’ positions in their *spectrum of pedagogic awareness*. In our development of teaching and the education of new lecturers, increased pedagogic awareness, leading to possibilities for changing and improving teaching, seems essential. Categories relating to theoretical positions are much more tentative at this stage and we offer them as starting points in thinking about how theories and beliefs can influence the ways we teach. We welcome discussion on this with other researchers who address such connections.

<sup>1</sup> HELM: *Helping Engineers Learn Mathematics* is a series of booklets developed specifically for engineering students, focusing on basic mathematical concepts and used with various groups of students in mathematics and engineering

## REFERENCES

- Barton, B. & Thomas, M. (2010). Editorial. *CULMS Newsletter*, no 1.
- Edwards, D. (1997). *Discourse and Cognition*. London: Sage.
- Harré, R & Gillet, G (1994). *The discursive mind*. London: Sage.
- Ioannou, M & Nardi, E. (2009). Engagement, abstraction and visualization: Cognitive and emotional aspects of Year 2 mathematics undergraduates’ learning experience in abstract algebra. In Joubert, M. (Ed.) *Proceedings of the British Society for Research into the Learning of Mathematics*, 29(2) pp. 35-40.

#### Working Group 14

- Jaworski, B., Treffert-Thomas, S., & Bartsch, T. (2009). Characterising the teaching of university mathematics: A case of linear algebra. In *Proceedings of the 33<sup>rd</sup> PME Conference*, Vol 3 pp. 249-256, Thessaloniki, Greece: PME.
- Nardi, E., Jaworski, B. & Hegedus, S. (2005). A spectrum of pedagogic awareness for undergraduate mathematics: From ‘tricks’ to ‘techniques’. *Journal for Research in Mathematics Education*, 36(4), 284-316.
- Petropoulou, G., Potari, D. & Zachariades, T. (2010). Inquiring the teaching of mathematical proof at the university level. In *Proceedings of the 34th PME Conference* Vol.2, p. 84. Belo Horizonte, Brazil: PME
- Speer, N., Smith, J., & Horvath, A. (2010). *Collegiate Mathematics Teaching*. Journal of Mathematical Behaviour, 29, 99-114.
- Wenger, E. (1998). *Communities of Practice*. Cambridge: CUP.

# **WHAT AFFECTS RETENTION OF CORE CALCULUS CONCEPTS AMONG UNIVERSITY STUDENTS?**

## **A STUDY OF DIFFERENT TEACHING APPROACHES IN CROATIA AND DENMARK**

Ljerka Jukić

University of Osijek, Croatia

Bettina Dahl (Soendergaard)

Aarhus University, Denmark

*This paper reports a parallel study of two university calculus courses in Croatia and Denmark using different teaching approaches. Both have lectures to a large group of students but they use different types of exercises. In Denmark, the exercises are student-centred, while the Croatian university uses a teacher-centred approach. The content of the courses are similar regarding the concepts we study in this paper. The students' retention was tested two months after the course exam on these concepts. Our statistical data analysis shows that the Danish students of our sample performed significantly better than the Croatian students of our sample on the conceptual questions, and vice versa for the procedural ones.*

### **INTRODUCTION**

The teaching of basic calculus concepts at the undergraduate level is wide and many students who study calculus are not in mathematics study programmes. Calculus at university level is usually taught by professional mathematicians who do not all seem to realize that there may be problems of communication between them and the students who study in non-mathematics study programmes (Maull & Berry, 2000; Guzman *et al*, 1998). When compared to mathematics students, engineering students seem to change their understanding of mathematical concepts as they progress through their studies (Maull & Berry, 2000). In order to gain more insight into the calculus knowledge of non-mathematics students, we investigated the level of retained knowledge in students from technical and natural sciences studies programmes. Our previous survey (Jukić & Dahl, 2010) showed that the students taking part in our experimentation had forgotten a large portion of notions regarding the derivative concept in differential calculus, and furthermore the surveyed students with the lowest course passing grades outperformed the students with high passing grades two months later in our questionnaire. The study reported in the present paper examines the retention of core calculus knowledge at two different non-mathematics student populations.

### **THEORETICAL BACKGROUND**

Conceptual knowledge describes knowledge of the principles and relations between pieces of information in a certain domain and procedural knowledge is knowledge of the ways in which to solve problems quickly and efficiently (Hiebert & Lefevre, 1986). Haapasalo and Kadijevich (2000) redefined conceptual knowledge,

highlighting its dynamic nature; it concerns the ability to browse through networks consisting of concepts, rules, algorithms, procedures and even to solve problems in various representation forms. Grundmeier *et al* (2006) showed that students generally choose a procedural over a conceptual way of dealing with problems in integral calculus. Pettersson and Scheja (2008) discovered that students developed their knowledge in integrals in an algorithmic way, not because of misconceptions, but because it was more suitable for them and enabled them to deal functionally and successfully with the presented tasks. Mahir (2009) investigated conceptual and procedural performance in integration in a group of undergraduate students who successfully completed a calculus course. She found that the students did not have satisfactory conceptual knowledge of integration, but those who had some conceptual knowledge, also showed some good procedural performance.

Teaching strategies can roughly be divided into student-centred and teacher-centred teaching (Killen, 2006). In the teacher-centred model, the teacher has direct control over what is taught and how the learners are presented the information they should learn. In the student-centred model, the learner is put at the focus of the teaching/learning process, instead of the teacher. The teacher has less direct control over how and what the students learn. An example of such approach is the use of small group work or cooperative learning. Studies showed that teaching strategies employed in the class can influence the development of one type of knowledge more than another; teacher-centred methods would favour the development of procedural knowledge and student-centred methods would favour the development of conceptual knowledge (e.g. Garner & Garner, 2001; Allen *et al*, 2005).

We examine what calculus knowledge is retained by students from two different mathematical populations two months after the course instruction and examination have taken place. Since these two populations are not completely comparable, we regard this as a parallel study, so caution is needed when making statements comparing the two populations.

## **THE TWO POPULATIONS: INSTITUTIONAL SETTINGS**

In this section we will describe the institutional settings in the two universities where our survey was conducted: the University of Osijek in Croatia and Aarhus University in Denmark. In order to examine the calculus courses and their contexts, lectures and exercises were observed at the universities. Furthermore the teaching materials, exams and curricula were examined and interviews with lecturers, department heads, and teaching assistants were conducted at both universities to gain insight into the similarities and differences of both study programmes.

### **The Croatian University**

The calculus course consists of lecture lessons and exercise lessons where the teaching approach is teacher-oriented. Lectures are given in a traditional form to a large group of students, and exercises are based on direct instructions, used in groups



of 30 students where a problem-solving or performance procedure is shown to the students. Conceptual ideas are taught in the context of procedural methods. A first year calculus course is divided in two one-semester courses, entitled Calculus 1 and Calculus 2. Differential calculus is part of Calculus 1 and integral calculus is part of Calculus 2. Part of Calculus 1 is oriented on repetition of high school A-level, using formal mathematical theory, what makes it different from high school mathematics. Also, the majority of the calculus courses are focused on functions in one variable. Every science study programme has its own calculus courses, but these courses have 70% of the content in common. The courses differ not just in course content, but also in the number of teaching hours. They may vary between 60 and 105 hours per semester, altogether for lectures and exercises. The process of examining the students' knowledge begins during the calculus course. Students have several written partial exams with open-ended questions during the semester as a substitution for the final exam at the end of semester. Students have to pass all partial exams and their grade is determined after the last partial exams. Those who fail any of the partial exams during the semester have to take the final exam to pass the course. Students' knowledge in formal mathematical theory in theorem-proof style is also examined. Students get the final grade for both calculus courses separately.

### **The Danish university**

The calculus course is a joint course for all mathematics and science study programmes. The course is organized into traditional lecture lessons and exercise lessons. Lecture lessons are given to a large group of students, but exercise lessons use small group work, based on problem solving where the teaching approach is more student-oriented. A first year calculus course is divided in two courses where functions of one variable and several variables are connected to differential and integral topics. Topics investigated in the questionnaire belong to Calculus 1. Both calculus courses take place during a seven-week half-semester (quarter) period with 63 hours, altogether for lectures and exercises. The process of evaluating students' knowledge starts after Calculus 1, where students take a multiple choice test, which determines whether or not the student can take the final written exam after Calculus 2. The grade obtained in the final exam is a joint grade for Calculus 1 and 2.

### **About comparing the two universities**

The calculus content investigated in this paper belonged to the core of all programmes. One of the major differences between the populations was the teaching methods, but that is not the only difference that might explain how the students answer the questions in our survey. This means that pointing to one single factor causing the difference is not possible, therefore caution is needed and we cannot identify a single cause to the differences in the results of both populations.

## METHODOLOGY

We conducted a survey examining a selected number of core concepts in differential and integral calculus through questionnaires given to first year non-mathematics students. The survey took place in the spring of 2009 at University of Osijek and in the autumn of 2009 at Aarhus University.

The Croatian students were given two questionnaires. The first examined their knowledge of derivatives, from Calculus 1, and the second examined their knowledge of integrals, from Calculus 2. The participants were students from the following study programmes: electrical engineering, civil engineering, food technology, physics, and chemistry. 227 students participated in the first questionnaire and were surveyed two months after the exam in differential calculus. 225 students participated in the second questionnaire and were surveyed two months after the exam in integral calculus. More than 94% of the students answered all questions in the first questionnaire and more than 97% of them answered all questions in the second questionnaire.

The Danish students were given one questionnaire combining the questions from the Croatian questionnaires since those concepts are covered in Calculus 1. The students belonged to the following study programmes: biology, chemistry, chemistry & technology, computer science, geology, geo-technology, information technology, molecular biology, medical chemistry, molecular medicine, and nano-science. 147 students participated in the questionnaire. More than 94% of the surveyed students answered all the questions.

The Danish university does not have engineering programmes and the Croatian university does not have all the study programmes surveyed in the Danish university. Since the aim of our parallel study was to examine knowledge retention in non-mathematics students, we do not consider these differences as significant. We wanted to get some insight into the knowledge of non-mathematics students from two different populations, and not in students belonging to a particular study programme. Even though the Danish and Croatian students have met calculus concepts in high school, the university courses provide different approaches to calculus (building calculus conceptions using formal theory) and build relationships between calculus objects (e.g. connecting them with functions of several variables). This diversity in teaching styles between high schools and universities has also been noted by various researchers (e.g. Guzman *et al*, 1998). We wanted to examine the retention of knowledge related to core calculus concepts after university calculus in students coming from different programs, contexts and teaching methods.

### Questionnaire design

We designed the questionnaires with multiple choice questions where the wrong options represented typical misunderstandings and errors. Before being given to the students, professional mathematicians and the lecturers of the courses were consulted

about the relevance of the questions, formulation, and appropriateness of the options of answers as offering typical misunderstandings. We wanted to examine the students' retention of concepts about derivatives and integrals in a short period of time, since the questionnaire had to be filled out by the students while they attended their class/lecture (the permission to pass the questionnaire during lecture time is easier to get, so multiple choice questions seemed to be a very convenient way to assess the students in a short period of time).

There were four questions about derivatives. The question *Tangent* deals with the geometric interpretation of the derivative of a function at a given point. In *Quotient* our intention was to test the students' knowledge about the quotient differentiation rule. *Composite* examines how the students deal with the derivation of a composite function. *Slope* incorporates several key concepts from differential calculus: slope of tangent line as the derivative of the function  $f$  at the given point and the process of differentiation. There were also four questions about integrals. *Area* deals with the geometric interpretation of the integral. It can be argued that among the offered answers "none of the above" would be the correct one, since the answer "the area between the curve  $y = f(x)$  and the  $x$ -axis for  $x$  between  $a$  and  $b$ " is correct only in the special case where  $f(x) \geq 0$  for  $x \in [a, b]$  and  $f(x)$  is bounded. However, as we had conjectured that the students were likely to overlook or ignore the subtlety of the case of non-positive or non-bounded functions, we did not consider "none of the above" as the right option (and we are aware this can be a potentially contentious choice). *Antiderivative* asked what the anti-derivative of a function is. Depending on the approach that was used in teaching, two of the offered answers could be considered as correct. Therefore, in the data analysis we labelled both possibilities as correct. *Method* asked for the most appropriate/easy method for solving a particular indefinite integral. The integration by parts is considered as the only correct option. The use of substitution for this example is "non-standard", and students would need a table to recall the integral of the logarithm. *Basic integrals* consisted of two indefinite integrals that are usually given in the tables of basic integrals and two possible solutions for each integral. The number of offered options for this question was inspired by the number of possible misunderstandings we considered for each integral. All the questions can be seen in the Appendix. The questions can be grouped as mainly involving either procedural or conceptual knowledge. The conceptual category consists of the questions *Tangent*, *Slope*, *Area* and *Antiderivative*, whereas *Quotient*, *Composite*, *Method* and *Basic integrals* are classified as procedural. However, this categorisation in conceptual or procedural questions is not absolute. Some questions could be placed in both groups, since they involve both kinds of approaches. For instance several differentiation rules have to be connected in *Composition*, and this, at least in some cases, can be considered as conceptual knowledge. On the other hand, it is possible that some students had experienced tasks like *Slope*, and thus their solution could be based only on recalling the method without any conceptual knowledge. This is the reason why the question

is formulated in a little tricky way, so that if the students apply a procedure without carefully thinking, they will fail to answer it correctly. Also, in our case, the students were more exposed to the chain rule of differentiation, unlike the question *Slope*.

## RESULTS OF THE QUESTIONNAIRE

Table 1 below shows the distribution of correct answers for all questions in the two populations. No question was answered correctly by all students.

Type	Topic	Question	Croatia		Denmark		Fisher's p-value
			#/total	%	#/total	%	
Conceptual	Differential calculus	Tangent	102/214	46	93/140	66	0.0007
		Slope	34/217	16	51/142	36	<0.0001
	Integral calculus	Area	160/224	71	141/143	99	<0.0001
		Antideriv.	(170+37)/223	93	(109+30)/142	98	0.0504
Procedural	Differential calculus	Quotient	167/221	76	90/139	65	0.0312
		Composite	144/218	66	94/141	67	1.000
	Integral calculus	Method	142/224	63	64/140	46	0.0011
		Integral a	99/223	46	22/141	16	<0.0001
		Integral b	145/220	65	81/142	57	0.0963

**Table 1: Distribution of correct answers. P-values here indicate the size of the differences among the populations.**

There was a significant difference in how the Croatian and Danish students answered six of the nine questions, eight if we accept an alpha of 0.10. The Danish students significantly outperformed the Croatian students in almost all the conceptual questions, but in the procedural questions, the Croatian students significantly outperformed the Danish students in four of the five questions. The fifth question (*Composite*) had an almost identical rate of correct answers.

Table 2 below shows how well each of the two populations solved each of the conceptual questions compared to each of the procedural questions.

The results of Table 2, and data from Table 1, show that for the Croatian students there was a significantly different performance in 16 of the 20 comparisons of the two groups of questions. Of the 16 comparisons which showed a significant difference (alpha of 0.10), nine times the procedural question was answered the best, while seven times, the conceptual question had the best answer rate. Hence it appears that there is almost no difference in how the Croatian students answer the conceptual and procedural questions, just a small preference for the procedural ones. Among the

procedural questions, the Croatian students achieved better results in the derivative questions than in the integral questions. In the conceptual group, their results were better in the integral questions than in the derivative question.

Croatia		Conceptual				Denmark		Conceptual			
		Dif		Int				Dif		Int	
Procedural		Ta	Sl	Ar	An	Procedural		Ta	Sl	Ar	An
Dif	Qu	* p	* p	3356	* c	Dif	Qu	8017	* p	* c	* c
	Co	0001 p	* p	2588	* c		Co	1.000	* p	* c	* c
Int	Me	0011 p	* p	0864 c	* c	Int	Me	0007 c	1151	* c	* c
	Ia	5031	* p	* c	* c		Ia	* c	0001 c	* c	* c
	Ib	0002 p	* p	2207	* c		Ib	1126	0005 p	* c	* c

**Table 2: Fisher's p-values comparing answers to the procedural and conceptual questions by population. \* denotes  $p < 0.0001$ . P-values are noted without 0. The letters p (procedural) and c (conceptual) denotes which question had the best answer rate.**

For the Danish students there was also a significantly different performance in 16 of the 20 comparisons of the two groups of questions. Of the 16 comparisons which showed a significant difference (alpha of 0.10), three times the procedural question was answered the best, while 13 times, the conceptual question had the best answer rate. Hence, it appears that the Danish students of our sample perform much better at the conceptual questions than at the procedural ones. In the procedural group of questions, the Danish students achieved better results in the integral questions than in the derivative questions. In the conceptual group, their results were better in the integral questions than in the derivative questions.

## DISCUSSION AND CONCLUSION

Having in mind that the questionnaires took place only two months after the examination, and that the questions were multiple-choice, we regard the obtained overall results as weak. There was only one question where both populations had a correct answer rate above 80% (*Antiderivative*). The lowest Croatian result is seen in the question *Slope* (16%) and the highest in the question *Antiderivative* (93%). The Danish students achieved the lowest result in the question *Integral a* (16%) and the highest result in the question *Area* (99%).

Both student populations were taught procedural and conceptual knowledge. In terms of long-term retention, procedural knowledge is quite fragile, meaning that procedures are often forgotten quickly or remembered inappropriately (e.g. Allen *et al*, 2005). This is perhaps reflected by the fact that Table 2 shows that 12 times a procedural question did better in comparison with a conceptual question, 20 times the opposite. Also Table 1 shows that no procedural question had a correct answer

rate above 76%, while three of eight times, the correct answer rate to a conceptual question was above 90%. Hence, our data lead us to think that the Danish students retained more conceptual knowledge than procedural knowledge, while the Croatian students were almost equally strong/weak in the conceptual and procedural questions. In terms of long-term retention, conceptual knowledge is stable, but possessing conceptual knowledge without procedural fluency is considered to be ineffective (Bosse & Bahr, 2008).

The results of our study can be connected with a long dispute on which type of knowledge is more important and in which order they should be learnt (Rittle-Johnson *et al*, 2001; Haapasalo, 2003). Today, we regard both types of knowledge as important and complementary, thus universities should focus on attaining balance between conceptual and procedural knowledge. Learning new concepts and practicing the skills associated with those concepts are strongly interconnected, therefore, a balance of learning concepts and procedures with explicit connections to those concepts will enhance the long term retention of both (Schoenfeld, 1988).

If we have a look at the results of our two populations, the Croatian students showed significantly better performance in the procedural questions, and the Danish students were significantly better in the conceptual group of questions. The teaching approach at the Croatian university is teacher-centred while it is more student-centred at the Danish university. One may wonder if these results are connected with the teaching approaches. Some studies showed that the teaching strategies employed in class can influence the development of one type of knowledge over the other; teacher-centred on procedural knowledge and student-oriented on conceptual knowledge. Garner and Garner (2001) found similar results in the case of applied calculus examining the retention of students' knowledge after eight months, but Allen *et al* (2005) found significant differences only regarding conceptual knowledge, and no difference in procedural knowledge between students exposed to different teaching strategies in differential equations, examining them after one year. Schumacher and Kennedy (2008), who examined calculus knowledge in students exposed to teacher-centred and student-centred teaching approach, found no statistical significance in success between the two groups of students. The studies that we refer to here had investigated students' retention in courses that only differed in the teaching approach and in the number of course hours. Students in our study also had some further differences in terms of previous training, of course content and of examination styles. Therefore, caution is needed when trying to point to *one* factor explaining the difference. This will be the topic of future research.

## APPENDIX

### *Derivatives questions surveyed with given options for answers*

1. Question *Tangent*: What is the geometric interpretation of the derivative of the function

## Working Group 14

$f : R \rightarrow R$  at the point  $x_0$ ? Offered answers: maximum/minimum of the function  $f$  at  $x_0$ ; slope of tangent line to the curve  $y = f(x)$  at  $x_0$ ; continuity of the function  $f$  in the given point; none of the above.

2. Question *Quotient*: Differentiate the function  $f(x) = \frac{x^2 + 2}{x^3}$ . Offered answers:  $\frac{x^3(2x) - (x^2 + 2)(3x^2)}{(x^3)^2}$ ;  $\frac{x^3(2x) - (x^2 + 2)(3x^2)}{x^3}$ ;  $\frac{x^3(2x) - x^2(3x^2)}{(x^3)^2}$ .
3. Question *Composite*: Differentiate the function  $f(x) = \sin^2 6x$ . Offered answers:  $2\sin(6x)$ ;  $12\sin(6x)$ ;  $12\sin(6x)\cos(6x)$ .
4. Question *Slope*: Calculate the slope of the tangent line to the curve  $y = (3x)^2$  at the point  $x = 1$ . Offered answers: 9; 18; 6.

### *Integral questions surveyed with given options for answers*

1. Question *Area*: What is the geometric interpretation of the definite integral  $\int_b^a f(x)dx$ ?  
Offered answers: The area between the curve  $y = f(x)$  and the  $x$ -axis for  $x$  between  $a$  and  $b$ ; The arc length of the curve  $y = f(x)$  on the interval  $[a, b]$ ; continuity of the function  $f$  on interval  $[a, b]$ ; none of the above.
2. Question *Antiderivative*: What is an antiderivative of a function  $f$ ? Offered answers:  $\int f(x)dx$ ; every function  $F$  such that  $F'(x) = f(x)$  holds; The set of elementary functions; none of the above.
3. Question *Method*: Which method should be used for computing the integral  $\int xe^x dx$ ?  
Offered answers: substitution  $t = e^x$ ; integration by parts; trigonometric substitution; none of the above.
4. Question *Basic integrals*:
  - a.  $\int \frac{dx}{1+x^2} = ?$  Offered answers: (a)  $\ln(1+x^2) + C$  and (b)  $\arctan x + C$ .
  - b.  $\int \frac{dx}{x^3} = ?$  Offered answers: (a)  $-\frac{1}{2}x^{-2} + C$  and (b)  $\ln(x^3) + C$ .

## REFERENCES

- Allen, K., Kwon, O. N. & Rasmussen, C. (2005). Students' retention of mathematical knowledge and skills in differential equations. *School Science and Mathematics*, 105(5), 227-239.
- Bosse, M. J., & Bahr, D. L. (2008). The state of balance between procedural knowledge and conceptual understanding in mathematics teacher education. *International Journal of Mathematics Teaching and Learning*, 25(11), 1-28.
- Garner, B. & Garner, L. (2001). Retention of concepts and skills in traditional and reformed applied calculus. *Mathematics Education Research Journal*, 13(3), 165-184.
- Grundmeier, T.A., Hansen, J. & Sousa, E. (2006). An Exploration of definition and procedural

## Working Group 14

fluency in integral calculus. *PRIMUS*, 16(2), 178-191.

Guzman, M., Hodgson, B., Robert, A. & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. *Documenta Mathematica, Extra Volume ICMI 1998*, 3, 747-762.

Haapasalo, L. & Kadijevich, D. (2000). Two types of mathematical knowledge and their relation. *Journal für Mathematik-Didaktik*, 21(2), 139-157.

Haapasalo, L. (2003). The conflict between conceptual and procedural knowledge: Should we need to understand in order to be able to do, or vice versa? In: L. Haapasalo, K. Sormunen (Eds.), *Towards Meaningful Mathematics and Science Education*, Proceedings on the 19th Symposium of the Finnish Mathematics and Science Education Research Association, Joensuu: University of Joensuu, 1-20.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In: Hiebert J. (Ed.), *Conceptual and Procedural Knowledge: The Case of Mathematics*, Lawrence Erlbaum Associates, Hillsdale, NJ, 1-27.

Jukić, L. & Dahl Soendergaard, B. (2010). The retention of key derivative concepts by university students on calculus courses at a Croatian and Danish university. *Proc. of PME 34*, 3 (pp. 137-144), Brasil, Belo Horizonte: PME.

Killen, R. (2006). *Effective teaching strategies: lessons from research and practice*. Thomson Social Science Press

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, 40(2), 201-211.

Maull, W. & Berry, J. (2000). A questionnaire to elicit concept images of engineering students. *International Journal of Mathematical Education in Science and Technology*, 31(6), 899-917.

Pettersson, K. & Scheja, M. (2008). Algorithmic contexts and learning potentiality: a case study of students' understanding of calculus. *International Journal of Mathematical Education in Science and Technology*, 39(6), 767-784.

Rittle-Johnson, B., Siegler, R. & Alibali, M. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346-362.

Schoenfeld, A.H. (1988). When good teaching leads to bad results: The disasters of well taught mathematics classes. *Educational psychologist*, 23, 145-166.

Schumacher, P. & Kennedy, K.T. (2008). Lessons Learned Concerning a Student Centered Teaching Style by University Mathematics Professors from Secondary School Educators. *Education*, 129(1), 102-109.



# UNIVERSITY STUDENTS LINKING LIMITS, DERIVATIVES, INTEGRALS AND CONTINUITY

Kristina Juter

Kristianstad University and Linnaeus University, Sweden

*Mathematical concepts are mentally represented differently depending on individual, context and existing conceptions of related concepts among other things. The present paper reports on a study of students' representations in analysis with an emphasis on the types of representations and the links they have between their representations. The data collection was designed to evoke different parts of the students' concept images and also to return to the concepts several times over time at every data collection session. The results show that formal and intuitive representations in combination are rare. The number of links between concepts is not in itself a measure of the quality of the concept image, as there is a vast number of erroneous links misleading the students to think they understand the concepts.*

## INTRODUCTION

Students' experiences of understanding a mathematical concept have a range from being able to explain all aspects of the concept in relation to other concepts (as defined by for example Hiebert and Lefevre, 1986) to just having heard of the concept, depending on how understanding is assessed and personal definitions of understanding. Prior research reveals university students' unjustifiably strong self confidence about their own mathematical abilities of understanding limits of functions (Juter, 2006) despite their inability to explain core features of the concept. The capability to solve routine tasks gives a sense of mastery of the concept that is not changed from an episode of failure in a special case, like the interview sessions in the study. A sense of understanding, false or otherwise, prevents further learning in the particular topic area, which in turn may lead to a weak mathematical ground for new learning. This is serious for future mathematics teachers who are going to provide opportune learning situations for their students. If their concept images (Tall & Vinner, 1981) are weak, or in worse cases wrong, there is no room for the flexibility and deep conceptual discussions necessary for appropriate teaching. This paper reports part of a study of pre-service teachers' understanding of limits, derivatives, integrals and continuity and links between the concepts. The study also concerns teacher identity from a social as well as a cognitive perspective (see for example Juter, 2010). The following research questions were raised: How do the students connect limits, derivatives, integrals and continuity to other concepts? How do the students represent (graphically, formally, through examples or other descriptions) the concepts for themselves? How do their representations with connections work as a base for analysing graphs with respect to the four concepts examined?

Data analysis resulted in a classification system (table 1) useful for categorising students' traces of connections between concepts in their concept images in form of mathematics actions such as problem solving, proving and explaining solutions or methods.

## UNDERSTANDING AND REPRESENTING CONCEPTS

Skemp (1976) distinguished understanding a concept from its core features, *relational understanding*, which enables implementation of the new concept to existing concept images (which is how Hiebert and Lefevre, 1986, defined conceptual knowledge, p 3), from understanding by just being able to perform a particular operation in what he denoted *instrumental understanding*. Either way to understand a new concept requires mathematical development of existing representations. Tall (2004) introduced a model describing development in three different modes, *the conceptual-embodied world* with an emphasis on exploring activities, *the proceptual-symbolic world* focusing the dual features of concepts as objects and processes expressed in symbols or *procepts* (Gray & Tall, 1994), and *the formal world* where mathematical properties are deduced from the formal language of mathematics in definitions and theorems. Students' concept images develop through the worlds with different emphasis on the three modes allowing them to understand concepts differently. Based on Pinto's and Tall's (2001) definitions of *formal learner* and *natural learner* together with Tall's three worlds and Skemp's definitions of understanding, I created a set of categories to classify students' links between concepts presented in table 1 (see Juter, 2009, 2010, for further details). Examples of classifications of students' links from the present study are provided in the table.

The last four types of links are not desirable for the students, who often are unaware of the quality of the links, particularly if irrelevant or invalid links are mixed with valid ones. Links are formed in different situations, e.g. at lectures, with peers or in solitude. Textbooks, lecturers' selections and general interests of the group of students frame the learning environment and therefore affect the representations students are using. Students' abilities, ambitions and confidence also influence their representations. Representations used when learning a certain topic may become vague if they are not endurable enough, e.g. not sturdily linked to other concepts. If a person learns a new mathematical topic in the embodied world and his/her abilities then develop to symbolic treatment he/she has changed the way of thinking to a proceptual-symbolic mode (Tall, 2008). If the learning phase in the conceptual-embodied world has been too short or otherwise inadequate, parts of the concept image may become disjoint or vague, rendering the person unable to explain core features of the concept. In the present study all students are future mathematics teachers, but they are taught at different universities, by different teachers and under

various circumstances which gives them a range of various learning environments to develop mathematical representations from.

<i>Type of link</i>	<i>Definition</i>
Valid link, procedural (vp)	True relevant link with focus on calculations or applications, ex: <i>The derivative of velocity gives acceleration</i>
Valid link, naturally conceptual (vn)	True relevant link revealing a core feature of the concept, not formal, ex: <i>Derivative is the slope of the tangent at a point</i>
Valid link, formally conceptual (vf)	True relevant link formally revealing a core feature of the concept, ex: <i>If the limit <math>\lim_{x \rightarrow a} f(x) = f(a)</math> exists in every point then <math>f(x)</math> is continuous</i>
Irrelevant link, no reason (ir)	No actual motivation for the link is provided
Irrelevant link, no substance (is)	Peripheral true link without substance relevant for the concept, ex: <i>You can add derivatives</i>
Invalid link, misconception (im)	Untrue link due to a misconception of the concept, ex: <i>Continuous means the same change everywhere</i>
Invalid link, counter perception (ic)	Untrue statement contradicting prior statements ex: <i><math>\sin x</math> is continuous and continuous means linear</i>

**Table 1. Definitions of links between concepts. Examples in italics.**

## THE STUDY

Students from four groups, two from each of two different universities in Sweden, were part of the study. All students, a total of 42, were pre-service teachers in mathematics who were studying to teach grades 7 to 9 and upper secondary school. The study started with one group, group 1, two years before the remaining three groups were added to the project as table 2 indicates. All students in the four groups were enrolled in the study.

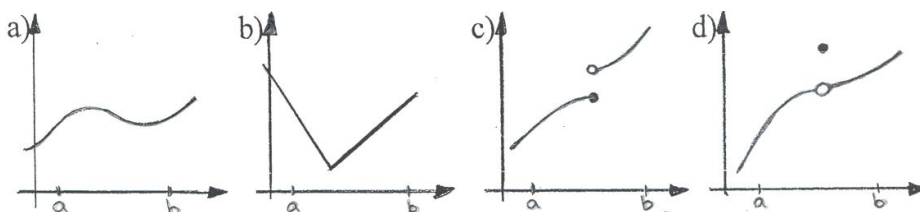
<i>Group</i>	<i>Autumn 1</i>	<i>Autumn 2</i>	<i>Autumn 3</i>	<i>Spring 4</i>	<i>Autumn 4-5</i>
1	Questionnaire Tasks	Interview 1	Interview 2	Interview 3 observation	
2-4			Questionnaire Interview 1	Interview 2	Interview 3 observation

**Table 2: Data collection times for the groups in years 1 to 5**

The first data collection in three of the groups (1, 2 and 4) was at the beginning of the students' analysis course where they filled out a questionnaire aimed at revealing their pre-knowledge of the concepts investigated. The questions were openly formulated for the students to be able to answer without influence from other formulations. Questions about the concepts were "Describe the concept of limit of a function/derivative/integral", "What do you use limits/derivatives/integrals for?" and "What does it mean for a function to be continuous?" There were some questions about the course and a mathematics teacher's main qualities as well. The students'

responses to the questionnaires were used to determine which students to interview. The selection was done to reflect the variety of representations, e.g. formal or intuitive, in the descriptions of concepts among the entire group of students. The purpose of the questionnaire in group 3 was the same but it was somewhat differently designed since the students had completed their analysis course at the time of data collection. The questions about what to use the concepts for was replaced by some of the tasks the other groups got after the course. The questions and tasks in the questionnaires and interviews were designed to make the students respond to the four concepts in various contexts and to come back to the same concept repeatedly. This way the responses were confirmed and clarified and different contexts evoked different parts of the students' concept images allowing a more nuanced picture from the data. All interviews were individually conducted and audio recorded. Data was collected from a teacher identity point of view as well, and methodology and results from that part are presented in Juter (2010). Focus in what follows is on the cognitive representations of the students.

The aim with the first interview in groups 1 and 3 was to investigate the students' representations of the concepts as traces of their concept images, and links among the concepts. The questions and tasks were quite open at first to let the students choose their own formulations of the concepts. Then the instruments used were more directed to different aspects of the concepts. One instrument was a table of 29 words and phrases used with the purpose to work as stimuli for the students to recall parts of their concept images. The words were for example tangent, border, sum, slope, rate of change, infinity and interval (see Juter, 2009 for details). The students were to describe the evoked parts or say if there was no recollection linked to a particular concept (results in table 3). This matrix was used in all groups about half year to a whole year after their analysis course to let the students' conceptions stabilize after the course. Four graphs (figure 1) with different characteristics linked to the four concepts studied were also given to the students to determine whether or not the represented functions have limits, are differentiable, integrable and continuous in every point (results in table 4).



**Figure 1. Graphs of functions for the students to analyse**

For each of the four concepts, a set of four descriptions was presented to the students on separate cards. The aim was to let the students respond to the different ways of representing the concepts. First they got a formal definition and were asked if they could see which concept it was. Then they got a picture describing the definition which could help them determine which definition it was. After that they got a

sentence intuitively explaining the concept and then an example that included calculations related to the concept. The students were asked to choose their preferred representation and also to explain how they would use different representations in their own teaching. They got the concepts in the order limit, derivative, integral and continuity and therefore the last concept could easily be guessed. This was obvious in a few cases when the students were asked to explain the link between the picture and the definition (results in table 5). The first interview in groups 2 and 4 did not include the matrix of recalled links since their analysis courses took place during the same semester. The matrix was presented to those students in the second interview about six months after the analysis courses. The open questions used in the questionnaire were used again in the interviews to reveal how the concepts had developed in the students' concept images. Conceptual issues were addressed again in the second interview in various forms. The first and second interview comprised the same components together for all students, but differently disposed depending on the time of their analysis courses.

## RESULTS

The results are divided in three sections according to the data collection with the matrix of words to evoke the students' concept images, the four graphs in figure 1 for the students to analyse, and the cards with various sorts of representations. The analysis was done to reveal the students' perceptions of the concepts, including links between concepts, as traces of their concept images from a range of various mathematical settings. In table 3 the interviewed students' links from the word matrix are classified in the categories defined in table 1. The numbers after the categories determine the number of links of each kind. The students were divided into three groups, I, II and III, according to their conceptions. The three students in the first group, group I, had few valid links of which none were formally conceptual (vf). Several links were irrelevant (ir, is) or invalid (im, ic). The four students in group II had more valid links than the students in group I, but also no formally conceptual ones. The students had few irrelevant or invalid links. Group III comprised three students with many valid links, including formally conceptual links and very few irrelevant or invalid ones. The labels by the students' names in table 3 indicate which group they belong to. The students are listed in order of strength of their concept images with the weakest first.

<i>Student</i>	<i>Limit</i>	<i>Limit</i>	<i>Derivative</i>	<i>Derivative</i>	<i>Integral</i>	<i>Integral</i>	<i>Continuity</i>	<i>Continuity</i>
Andy	vp 1	ir 3	vp 5	ir 3	vp 5	ir 3	vp 0	ir 3
(I)	vn 2	is 0	vn 0	is 8	vn 0	is 8	vn 0	is 6
	vf 0	mc 2	vf 0	mc 1	vf 0	mc 1	vf 0	mc 8
Betty	vp 0	ir 2	vp 1	ir 1	vp 1	ir 0	vp 0	ir 0
(I)	vn 1	is 2	vn 2	is 2	vn 0	is 0	vn 1	is 0
	vf 0	mc 3	vf 0	mc 1	vf 0	mc 1	vf 0	mc 3
Chris	vp 1	ir 5	vp 4	ir 2	vp 5	ir 2	vp 0	ir 1

# Working Group 14

(I)	vn 1	is 1	vn 0	is 0	vn 0	is 1	vn 3	is 0
	vf 0	mc 0	vf 0	mc 3	vf 0	mc 5	vf 0	mc 2
Diana	vp 4	ir 0	vp 10	ir 1	vp 5	ir 0	vp 0	ir 1
(II)	vn 0	is 1	vn 0	is 0	vn 1	is 0	vn 1	is 1
	vf 0	mc 3	vf 0	mc 2	vf 0	mc 2	vf 0	mc 1
Elise	vp 2	ir 0	vp 2	ir 3	vp 4	ir 1	vp 0	ir 0
(II)	vn 0	is 0	vn 5	is 0	vn 3	is 2	vn 1	is 2
	vf 0	mc 1	vf 0	mc 0	vf 0	mc 0	vf 0	mc 1
Frank	vp 6	ir 0	vp 7	ir 0	vp 3	ir 1	vp 2	ir 2
(II)	vn 1	is 2	vn 2	is 0	vn 3	is 1	vn 1	is 0
	vf 0	mc 1	vf 0	mc 0	vf 0	mc 0	vf 0	mc 3
Glenn	vp 7	ir 2	vp 7	ir 0	vp 5	ir 0	vp 4	ir 0
(II)	vn 2	is 1	vn 4	is 3	vn 0	is 2	vn 1	is 4
	vf 0	mc 5	vf 0	mc 0	vf 0	mc 0	vf 0	mc 2
Harry	vp 7	ir 0	vp 11	ir 0	vp 10	ir 0	vp 3	ir 0
(III)	vn 0	is 1	vn 1	is 2	vn 0	is 1	vn 2	is 2
	vf 1	mc 2	vf 0	mc 0	vf 1	mc 0	vf 1	mc 1
Ivan	vp 8	ir 3	vp 6	ir 5	vp 6	ir 3	vp 2	ir 2
(III)	vn 1	is 2	vn 7	is 0	vn 4	is 3	vn 4	is 2
	vf 2	mc 1	vf 1	mc 0	vf 1	mc 0	vf 0	mc 0
John	vp 5	ir 2	vp 6	ir 1	vp 7	ir 0	vp 3	ir 3
(III)	vn 5	is 1	vn 1	is 1	vn 5	is 1	vn 1	is 1
	vf 3	mc 0	vf 6	mc 0	vf 1	mc 0	vf 0	mc 0

**Table 3. Links between concepts categorised according to table 1. The categories Invalid link, misconception (im) and counter perception (ic) are merged in mc**

Some students readily talk about their conceptions and views while others are not so forward in an interview situation. It is therefore important to take the total number of links for each student into account when reading table 3. Andy had a large number of irrelevant or invalid links but few valid ones, particularly for continuity where he had no relevant links. He had a lot to say about the concept, for example that a continuous function has to be linear, has no peaks in its graph and changes the same way everywhere, but nothing substantially valid. Andy had a total of 59 links of which 34 were irrelevant and 12 invalid. He showed traces of a stronger concept image for limits than for the other concepts. An example of a valid naturally conceptual link (vn) is his explanation of limits as intervals: “a limit is a form of interval which is shortened to a great extent”, and he explained further by drawing a figure of a graph with an interval on the y-axis, i.e. the function values, around a point and saying “it [the graph] closes in on the point from both ways and you press together like this [the endpoints of the interval at the y-axis]”. Other parts of the study confirmed his naturally conceptual understanding of limits, i.e. he was able to discern and explain vital aspects of the concept. Elise had 27 links in all of which 17 were valid, i.e. a considerably larger part of the total number of links than Andy had (13 out of 59 valid). She showed a high level of natural conceptual understanding of the concepts of derivative and integral. John had 53 links. He had 10 formally

conceptual links of his total of 43 valid links, leaving 10 irrelevant links. John showed an overall strong and rich concept image of the concepts studied through the data collection. Continuity was the concept with the least amount of links for John, but also for the other students, which may come from the fact that derivatives and integrals are a major part of the required upper secondary school courses and limits are therefore to some extent dealt with, but not necessarily continuity. The students have consequently not had the same time to implement continuity to their concept images as the other concepts.

<b>Graph a</b>	<b>Yes</b>	<b>No</b>
Limit	ABCEFHIJ	DG
Continuous	ABCDEFGHJIJ	
Differentiable	ABCDEFGHJIJ	
Integrable	ABCDEFGHJIJ	
<b>Graph b</b>	<b>Yes</b>	<b>No</b>
Limit	ABCEFHIJ	DG
Continuous	BDEFHIJ	ACG
Differentiable	ACDFGHI	BEJ
Integrable	ADEFHIJ	BCG
<b>Graph c</b>	<b>Yes</b>	<b>No</b>
Limit	CFGH	ABDEIJ
Continuous	C	ABDEFGHIJ
Differentiable	DEFIJ	ABCGH
Integrable	CEFI	ABDGHJ
<b>Graph d</b>	<b>Yes</b>	<b>No</b>
Limit	ABEGI	CDFHJ
Continuous		ABCDEFGHJIJ
Differentiable	DEFI	ABCGHJ
Integrable	CEFI	ABDGHJ

**Table 4. Students' answers to whether the four graphs have limits, are differentiable, integrable and continuous in the indicated intervals in the graphs in figure 1**

Graph *a* was not a problem to most of the students. On the other hand, it was harder for the students to determine whether Graph *b* is differentiable or not. The peak made them confused and seven students answered incorrectly. Andy, Betty, Elise and Ivan responded correctly to all the limit parts of the tasks and in the last two tasks. Elise, Frank and Ivan were all wrong concerning differentiability. The traces of the concept images in table 3 of these students were very different and in this part of the data collection there is no pattern showing who is high achieving and who is not, or who has a strong concept image or not. Betty, Elise and Ivan had the highest number of correct answers in table 4, but table 3 indicates that they belong to different groups (I, II and III respectively). This type of task is not typically what the students had

seen in their courses so it was new to all of them. Chris, Diana and Glenn had the highest error rates. Diana and Glenn were both in group II in table 3 implying a rather strong concept image for the concepts except for limits. Both students had three of the four limit tasks wrong in table 4. Andy and John, in each end of table 3, had the same score in table 4.

Table 5 shows how the students represented the concepts from the cards with four different representations, formal, picture, calculated example and sentence, for each of the four concepts.

<i>Students</i>	<i>Limit</i>	<i>Derivative</i>	<i>Integral</i>	<i>Continuity</i>
Andy	<i>P, DC</i>	<i>F, P, D<sub>p</sub>C</i>	<i>P, DC</i>	<i>P, D<sub>u</sub>C</i>
Betty	<i>P, D<sub>p</sub>C</i>	<i>S, DI</i>	<i>P, E, S, DI</i>	<i>P, S, DI</i>
Chris	<i>P, S, DI</i>	<i>F, P, E, S, D<sub>p</sub>C</i>	<i>P, S, DC</i>	<i>E, DI</i>
Diana	<i>E, DI</i>	<i>F, P, D<sub>p</sub>C</i>	<i>P, S, DI</i>	<i>P, DI</i>
Elise	<i>E, DI</i>	<i>F, P, E, S, D<sub>p</sub>C</i>	<i>P, S, DC</i>	<i>S, DI</i>
Frank	<i>S, DC</i>	<i>E, S, D<sub>p</sub>C</i>	<i>S, DC</i>	<i>S, D<sub>u</sub>C</i>
Glenn	<i>S, DI</i>	<i>P, E, DC</i>	<i>P, DI</i>	<i>S, D<sub>u</sub>C</i>
Harry	<i>F, P, DC</i>	<i>P, E, DC</i>	<i>P, E, DC</i>	<i>E, S, DC</i>
Ivan	<i>F, P, DC</i>	<i>F, P, DC</i>	<i>F, P, DC</i>	<i>S, D<sub>p</sub>C</i>
John	<i>P, DC</i>	<i>F, DC</i>	<i>P, DC</i>	<i>S, DC</i>

**Table 5. Students' preferred representations of the concepts (*F*: Formal, *P*: Picture, *E*: Example, *S*: Sentence) and their abilities to recognise the formal definitions (*DC*: Definition correct, *D<sub>p</sub>C*: Definition + picture correct, *D<sub>u</sub>C*: Definition correct but unable to explain, *DI*: Definition incorrect)**

Table 5 reveals that continuity was the concept hardest to recognise from the definition, followed by limit. The fact that continuity was the last concept addressed helped the students understand which one it had to be. This became apparent when they were asked to explain the formal representation in relation to the picture (those cases are marked *D<sub>u</sub>C* in the table). All students except Betty recognised the definition of derivative and a majority could identify the definition of integral from the card. Derivative was the concept the students had most kinds of representations for. Chris and Elise both used all four representations intertwined. Students who could recognise the formal definitions (*DC* in table 5) often used pictures to represent the concepts. Pictures in combination with formal representations were also common in the cases where the definition was recognised from the formal representations. Students who were unable to determine which formal definition a card represented, but could determine it from the added picture (marked *D<sub>p</sub>C* in the table) often preferred pictures as a representation for the concept. Betty, Frank and Glenn did not use any formal representations in this context. The students are listed in the same order as in table 3 where the first students have weaker concept images and the last ones have stronger concept images. Andy, who is in the first group, is



different from the others with his ability to recognise the concepts from their definitions despite his results reported in table 3. Among the students who could not recognise the definitions (*DI* in table 5) sentences were the preferred alternative, and never formal which could be expected. The sentences were intuitively stated and typically chosen as alternatives to formal representations. There were only two cases of formal and sentence representations together, and in both cases all four representations were used (Chris and Elise).

## DISCUSSION

The students' links between topics were very diverse and the data collection reported in table 3 gives a picture of continuity not being implemented in their concept images with very few links of all sorts. The other concepts were more elaborated with more links but the links do not only imply understanding. A large number of links may be misleading the students to believe they understand, like in Andy's case, since they are able to talk about the concept. If counter perceptions, which for example Andy and Chris showed evidence of, can be evoked simultaneously the students are more likely to see their flaws than if their misrepresentations are just invalid without a contradiction.

Andy and Betty represented the concepts mainly in pictures (table 5) and that is reflecting how they read the graphs in table 4 as well. Andy and Betty were among the students with the highest rates of correct answers when analysing the graphs (table 4) and the ones with the weakest concept images in the matrix of words (table 3) which is a surprising result. They were often unable to explain their claims correctly, which is a problem especially for future teachers, but they had a sense of the characteristics of the concepts. Ivan, one of the students with the strongest concept images in table 3, had three errors concerning differentiability in the analysis of the graphs (table 4). Ivan's representations in table 5 are mainly formal and by pictures, moreover table 3 reveals that the derivative is in fact a concept he can handle procedurally, conceptually and formally, yet table 4 shows that he is uncertain of the core features of the concept. This type of overarching non-routine tasks requires the students to know more than just definitions in isolated mathematical contexts. They need to be able to relate the characteristics of the concepts to the graphs with their different properties and to use their representations in their analysis. A strong concept image is not enough for students to discern the necessary features. Comparing with Andy and Betty, the similarities are pictures as preferred representations. Dissimilarities are that Andy and Betty do not use any formal representations in either of the contexts reported in table 3 and 5 except one in Andy's case, whereas Ivan does in both. Ivan has an overall richer representation of all concepts studied and uses and refers to formal formulations, but it appears as if he uses his formal representations without having a clear and flexible understanding of them linked to his other representations of the concepts in situations new to him.

Table 3 and table 5 match each other when it comes to the students' use of formal representations and they indicate that a strong concept image means formal representations as opposed to a weaker concept image. Table 4 reveals a complete lack of trends in relation to tables 3 and 5 and the students' representations reported therein. The traces from their concept images depicted in these three tables imply complex relations between abilities to make assumptions about graphs from the various sorts of representations investigated. Rich concept images, including formal representations, do not imply higher abilities to distinguish the essence of the four concepts from the type of tasks used (figure 1).

## REFERENCES

- Gray, E. & Tall, D. (1994). Duality, ambiguity and flexibility: A procedural view of simple arithmetic. *The Journal for Research in Mathematics Education*, 26(2), 115-141.
- Hiebert, J. & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics*. London: Lawrence Erlbaum Associates publishers.
- Juter, K. (2006). Limits of functions as they developed through time and as students learn them today. *Mathematical Thinking and Learning*, 8(4), 407-431.
- Juter, K. (2009). Development of students' concept images in analysis. *Nordic Studies in Mathematics Education*, 14(4), 65-87.
- Juter, K. (2010). Prospective teachers' conceptions of analysis and beliefs about their impending profession (12 pages). *Proceedings of the 16th Conference on Mathematical Views, MAVI16 2010*.
- Pinto, M. & Tall, D. (2001). Following students' development in a traditional university classroom. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (vol 4, pp. 57-64). Utrecht: The Netherlands.
- Skemp, R.R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20-26.
- Tall, D. (2004). Thinking through three worlds of mathematics. In M. Johnsen Høines & A.B. Fugelstad (Eds.), *Proceedings of the 28<sup>th</sup> Conference of the International Group for the Psychology of Mathematics Education* (vol 4, pp. 281-288). Norway: Bergen University College.
- Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, 20(2), 5-24.
- Tall, D. & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

# **‘DRIVING NOTICING’ YET ‘RISKING PRECISION’: UNIVERSITY MATHEMATICIANS’ PEDAGOGICAL PERSPECTIVES ON VERBALISATION IN MATHEMATICS**

Elena Nardi

University of East Anglia (UK)

**Abstract.** *In this paper I draw on interviews with university mathematicians in order to discuss their comments on Year 1 students’ verbalisation skills. Analysis was conducted in the spirit of enculturative and discursive theoretical perspectives and here focuses on: the role of verbal expression to drive noticing; the importance of good command of ordinary language; the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression; and, the precision proviso for the use of ordinary language in mathematics. I conclude with the observation that discourse on verbalisation in mathematics tends to be risk-averse and that more explicit, and less potentially contradicting, pedagogical action is necessary in order to facilitate students’ acquisition of verbalisation skills.*

**KEY WORDS:** university mathematicians, verbalisation, enculturative and discursive perspectives, student learning.

Being able to use ordinary language in order to construct and convey mathematical meaning is an indispensable tool of the mathematical thinker. The mathematics community has always revered eloquent mathematical exposition and, at least until a collectively accepted and used symbolic mathematical language became increasingly dominant from early 19<sup>th</sup> century onwards, substantial ideas in mathematics (by luminaries such as Cauchy and Lagrange) were often conveyed in ordinary language.

Yet research on how mathematicians and their students acquire verbalisation skills is scarce or subsumed in broader studies of mathematical expression. In the early 1990s research on students’ handling of the verbal and symbolic elements of mathematics language often focused on students’ comprehension and response to mathematical texts, rather than students’ own generated verbal utterances. For example, Furinghetti and Paola (1991) discussed students’ understanding of mathematical texts in the context of difficulties with formal proof. Dee-Lucas & Larkin (1991) found that: proofs written in ordinary language resulted in better student performance than equation-based proofs on problems related to both equation and non-equational proof content; the presence of equations cause students to shift attention away from non-equational content; and, learners have more difficulty processing equations than verbal statements of the same content. With regard to students’ own writing some studies (for example, MacGregor, 1990) have suggested that writing sentences helps students write correct symbolic expressions.

In this paper I explore some of the characteristics that make verbalisation skills an important component of mathematical learning. I also discuss some provisos for an appropriate and effective use of verbal expression in mathematics. To this purpose I draw on data collected for a broader study (Nardi, 2008) which aimed to elicit university mathematicians' perspectives on the teaching and learning of mathematics at university level. In the following I introduce briefly the study and the theoretical underpinnings of the discussion I present in this paper.

## A STUDY OF MATHEMATICIANS' PEDAGOGICAL PERSPECTIVES

The study I draw on in this paper is in an area that has been gaining increasing interest in university-level mathematics education research in recent years: the study of university teachers and teaching, with a focus on the pedagogical and epistemological perspectives of university mathematicians (e.g. Jaworski, 2002). The study has its theoretical origins in several traditions of educational research such as: *clinical partnerships* between researchers and practitioners (Wagner, 1997); *communities of practice* (Wenger, 1998); Schön's (1987) *reflective practice*; and, Chevallard's (1985) notion of *transposition didactique*. From these traditions the study has acquired the following characteristics – see (Nardi, 2008, p6-9) for an elaborate account. It is: *collaborative* (namely, it brings together mathematicians and researchers in mathematics education in a collective discussion of pedagogical issues); *context-specific* and *data-grounded* (namely, this discussion is conducted in the specific context of, and with data from, the participating mathematicians' teaching experiences); *non-deficit* (namely, the discussion, while encouraging *self-reflection* and critique, does not primarily aim at the identification of problematic aspects of mathematicians' teaching by the mathematics educators); and, *non-prescriptive* (namely, the discussion, while it encourages the identification of preferred and recommended practice, does not lead to explicit pedagogical prescription). Of course given above characteristics a study of this type clearly serves the dual purpose of research and university teacher development.

The data collected for the purpose of the study consist of *focused group interviews* (Wilson, 1997) with twenty-one mathematicians of varying experience and backgrounds from across the UK. In acknowledgement of the loose sense in which the term 'community' (Wenger, 1998) may have among mathematicians, these backgrounds included pure and applied mathematics, as well as statistics. Eleven interviews with groups of three to five mathematicians were conducted, lasting between two and four hours. Discussion in the interviews was triggered by *Student Data Samples*, namely samples of students' written work, interview transcripts and observation protocols collected in the course of earlier studies of (overall typical in the UK) Year 1 introductory courses in Analysis / Calculus, Linear Algebra and Group Theory. See (Nardi, *ibid*, p9-14) for summaries of these studies. The *Samples* were sent to the interviewees at least a week prior to the interview with a request to read and reflect on them in preparation for the interview.

In accordance with data-grounded theory techniques (Glaser and Strauss, 1967) the approximately 250,000 words of interview transcript were initially arranged in clusters of episodes on the following six themes: four focused on *student learning* (students' mathematical reasoning; in particular their conceptualisation of the necessity for proof and their enactment of various proving techniques (1); students' mathematical expression and their attempts to mediate mathematical meaning through words, symbols and diagrams (2); students' encounter with fundamental concepts of advanced mathematics such as Functions (3) – across Analysis, Linear Algebra and Group Theory – and Limits (4)); one focused on *university-level mathematics pedagogy* (5); and, one focused on *the often fragile relationship between the communities of mathematics and mathematics education* (6). There were 80 episodes. The data I discuss in this paper concern the issue of students' verbalisation skills and collate evidence interspersed in 7 out of the 80 episodes.

The discussion here is in the light of *enculturative* and *discursive* perspectives on mathematical learning. In terms of the former (e.g., Sierpinska, 1994; Wenger, *ibid*), students are seen as incoming participants to the practices of the mathematics community; learning occurs communication and practice; and, in this process the main role of their university teachers is to introduce them to these practices.

In terms of the latter (e.g. Sfard, 2007), the learning of mathematics involves a change of discourse, where discourse is meant as a distinct form of communication that a community engages with. A discourse is made distinct by the community's *word use*, *visual mediators*, *endorsed narratives* and *routines*. *Words* include mathematical terminology. *Visual mediators* include diagrammatic and symbolic representations of mathematical meaning. *Endorsed narratives* include definitions and theorems. And *routines* include practices such as conjecturing, proving, estimating etc.. Sfard (*ibid*) describes the changes of discourse involved in mathematical learning in terms of two levels: *object-level* (namely adding endorsed narratives, e.g. accumulating knowledge of new definitions and theorems) and *meta-level* (namely adding new objects, changing rules of the discursive game, changing word use etc.). Below I outline the analysis of the episodes – overall but particularly those clustered under theme (2) above – in the terms of these two perspectives.

Students' mathematical expression – whether verbal, visual or symbolic – is expected to undergo a substantial shift, particularly in the early parts of their university studies. At least in the UK where the study was conducted, mathematical discourses in school and at university are markedly distinct. Brief examples of the differences between the two discourses involve *routines* such as proving (in school students are rarely, if at all, expected to provide a formal proof of a claim that they make) or the employment of certain *visual mediators* (in school students are not generally expected to make extensive use of formal mathematical language, symbols such as quantifiers etc.).

The interviewed mathematicians paid particular attention to the tension that they see

their students experiencing while undergoing this discursive shift. Below I list five characteristics that, according to the participating mathematicians, typify this tension:

- *Inconsistent Symbolisation*: students' attempts at producing 'acceptable' mathematical writing result in inconsistent use of symbolic language;
- *Ambivalent Visualisation*: students' appreciation of the role of visualisation in gaining and presenting mathematical insight is ambivalent. Their use of it is lacking in confidence or lacking altogether.
- *Undervalued Verbalisation*: students undervalue, and often avoid entirely, expressing their mathematical thoughts verbally.
- *Premature Compression*: students' mathematical writing is typically prematurely compressed, namely ridden with gaps, leaps of logic and omissions.
- *Appearances*: students often enact their perception of the need to *be* mathematical (use the discursive norms of mathematical reasoning such as providing justification or proof etc.) as a need to *appear* mathematical (appear to make extensive use of mathematical symbols, terminology or expressions).

The last two characteristics can be seen as combined repercussions of the first three. Data substantiation of each of the five can be found in (Nardi, 2008, mainly Chapter 4). Further elaboration on relationships across all five is part of a longer paper that is currently in preparation. Here I draw on the developing text of that paper in order to provide a sampler of the data analysis concerning these five characteristics: to this purpose I focus on data from one of the five, *Undervalued Verbalisation*.

the MaTHEMATICIANS' case and provisos for verbalisation

Across the seven episodes the interviewed mathematicians reported extensively the students' lack of ability in and appreciation for verbal expression in mathematics. At the heart of this reporting was the concern that students' inadequate appreciation for verbal expression was an indication of the students' difficulty with – and lack of awareness of their obligation for – making their thinking as transparent as possible. The mathematicians also appear concerned about the students' lack of awareness of the benefits that come with the mastering and employment of verbalisation skills in mathematics. In what follows the discussion of the data is structured around four key issues: *the role of verbal expression to drive noticing and emphasise*; *the role of good command of ordinary language*; *the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression*; and, *provisos for (and issues emerging from) the use of ordinary language in mathematical expression*.

The interviewees stressed that the mere presence of symbols in a mathematical sentence is not sufficient for driving students' attention to the key mathematical idea

in the sentence. As a helpful and efficient *routine*, that the students currently lack but need to adopt the interviewees propose the assistance on this matter from verbal adjuncts to the symbolic expression. They offer the definition of convergence as an example of a statement where such assistance can be potent. The premise for their discussion of this example is the following mathematical problem:

Write down a careful proof of the following useful lemma sketched in the lectures. If  $\{b_n\}$  is a positive sequence (for each  $n$ ,  $b_n > 0$ ) that converges to a number  $s > 0$ , then the sequence is bounded away from 0: there exists a number  $r > 0$  such that  $b_n > r$  for all  $n$ . (Hint on how to start: Since  $s > 0$ , you might take  $\frac{1}{2}s = \epsilon > 0$  in the definition of convergence.)

#### Notes on Solutions

Let  $\epsilon = \frac{s}{2}$  in the definition of convergence. Then there is an  $N$  such that  $n > N \Rightarrow |b_n - s| < \frac{s}{2} \Rightarrow b_n > \frac{s}{2}$ . Then, for any  $n$ ,  $b_n \geq r = \min\{b_1, \dots, b_N, \frac{s}{2}\}$  which is the minimum of finitely many positive quantities, hence is positive.

Fig.1 A Year 1 Calculus question requiring emphasis on the meaning of quantifiers  
Typically students responded with omitting a small but significant number of terms:

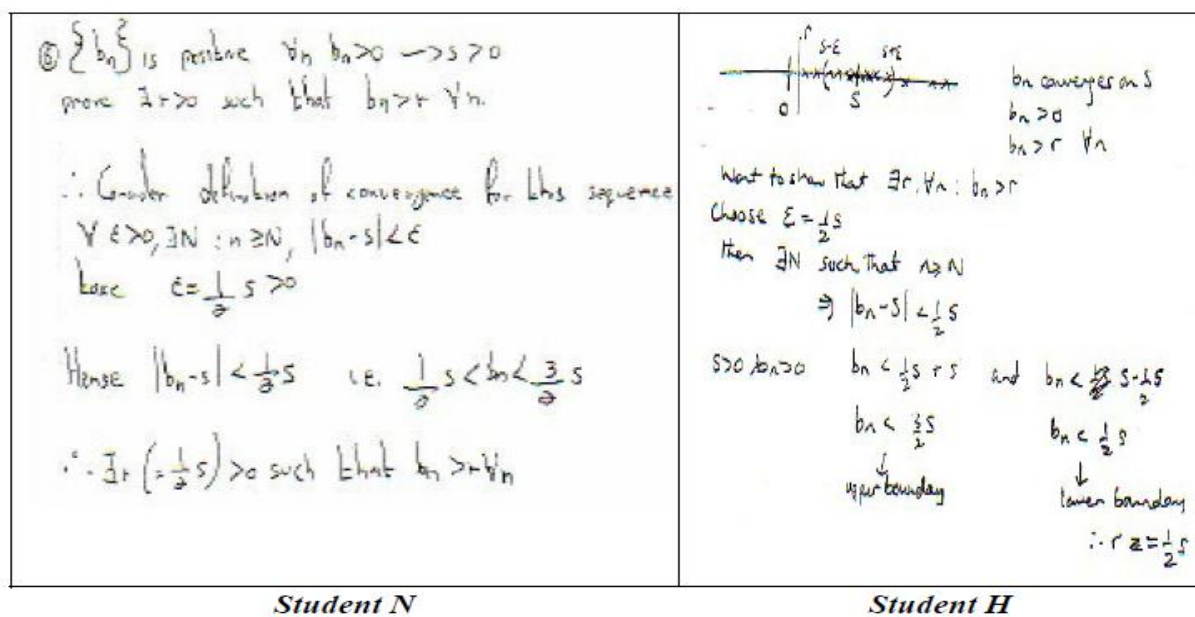


Fig. 2 Two typical responses to the question with missing emphasis on quantifiers  
Discussing what could possibly trigger students' noticing the need to cover all terms of the sequence, the interviewees highlight the importance of full sentences:

'It seems that after all the **presence of the quantifiers** themselves in the text of the question is **not emphatic enough** to suggest universality or existence to the students. And **words, sentences, those creatures ever-absent from students' writing** exist exactly for this purpose: of **emphasis**, of **clarification**, of **explanation**, of **unpacking the information within the symbols**.' p151

## Working Group 14

The more skilled students are in producing such sentences, the better the cognitive support they will gain from verbalising their mathematical thought. Hence the interviewees' discussion of *the importance of good command of ordinary language*. The following mathematical problem is the premise for their statements on this:

Suppose  $A$  in an  $n \times n$  matrix which satisfied  $A^2=0$  (the  $n \times n$  zero matrix).

1. Show that  $A$  is not invertible.
2. Show that  $I_n+A$  has inverse  $I_n-A$ .
3. Give an example of a non-zero  $2 \times 2$  matrix  $A$  with  $A^2=0$ .

Fig.3. A Year 1 Linear Algebra question requiring clarification on implication order

In part (ii) of this question the students need to show that, because the product of  $I_n+A$  and  $I_n-A$  is  $I$ ,  $I_n-A$  is the inverse of  $I_n+A$ . The interviewees notice that one student (Fig.4) starts her response with a different, and incorrect, statement of her intentions:

(ii) Show  $I_n + A$  has inverse  $I_n - A$   
 If this is true, then the product of  $I_n + A$  and  $I_n - A$  will be  $I_n$ .  

$$\begin{aligned} & (I_n + A)(I_n - A) \\ &= I_n^2 + AI_n - AI_n - A^2 \\ &= I_n^2 + A - A - 0 \\ &= I_n^2 \\ &= I_n \end{aligned}$$
  
 $\therefore (I_n - A)$  must be the inverse of  $(I_n + A)$

*Student LD, part (ii)*

Fig. 4 A typical response to the question with apparently muddled implication order

This student script was commented upon as follows:

'I am not very keen on *if this is true, then the product of  $I-A$  and  $I+A$  will be  $I$* , even though she is doing the absolutely right thing, starting from the product and ending up with  $I$ . You know why? Because it's getting so close to **appearing as if she is assuming what she is supposed to be proving**. What she wants to be saying is really *this is true because...* There is a subtlety missing there regarding the converse statement and their **Grammar is not up to scratch** to help them see the difference.'

p59

Unlike the previous example – in which an emphasis on the need to offer a universal coverage of the sequence's terms could have been provided by a verbal accompaniment to the symbolic expression – here the student's inaccurate verbal expression does not lead her astray. However the interviewees take the opportunity provided by such inaccuracies in students' writing in order to stress the strong link between command of ordinary language and 'good mathematical writing':

'It should be made clear to the students that this type of **command of the language** [for example, people being unable to distinguish between a main and a subordinate clause...] is **not irrelevant to good mathematical writing**. And that applies all the way through to completion of their studies. I sometimes see final year students and I wonder whether they deserve marks for a response that I could only detect as correct amidst grammatically incorrect statements. **When you put things on paper with**



**such ambiguity and inconsistency**, such as sentences without verb or subject etc., **maybe you should expect a lesser reward too.**' p151

In the above, the interviewees state that mathematical writing that is characterised, for example, by grammatical ambiguity and inconsistency deserves lesser rewards. They also wonder whether students' awareness about the significance of grammatical and syntactical correctness needs to be emphatically raised. In the concluding remarks in this paper I return to this suggestion in order to discuss briefly whether this appreciation of the role, and employment, of ordinary language in mathematics is perhaps more disproportionately expected from the students than their exposure to, and systematic practising of, this valuable *routine* reasonably allows for. But first, I cite the participants' statements on what they appear to see as the most valuable aspect of verbalisation skills in mathematics: *the role of verbalisation as a semiotic mediator between symbolic and visual mathematical expression*.

The interviewees cite the definition of convergence as a case illustrating the value of the connection between symbolic, verbal and visual forms of mathematical expression. Students' first encounters with the hefty symbolisation employed in this definition are some of the first occasions in which students realise that this mode of expression is the *discursive norm* in mathematical writing and a norm they are expected to accustom to quite quickly. The interviewees stress that verbalisation is a meaningful way to help students face 'what they see as madness' at this stage and steer them away from construing the strings of symbols in the definition as little other than 'formalistic nonsense'. The example of the student in the following is telling:

'...a student who wrote down a neat response to a convergence question – **applying the definition impeccably – and then asked why does this prove the convergence?!** What an excellent question! I tried to explain that this is the *definition* of convergence but students **don't quite understand the relation between this expression and what convergence ought to mean exactly.**' p187

Elaborating on the sources of difficulty with the definition, the following link across visual, verbal and symbolic accounts of the definition was offered:

'... the **difficulty** lies with the **successive appearance of quantifiers in the definition** whereas the primary notion for the students ought to be that *no matter what I specify the  $\varepsilon$  region about the A, from a certain point onwards everything fits inside this box*. **Making this link between this image and the formalisation** behind this is utterly important. Otherwise the definition is **nothing other than formalistic nonsense.**' p188

Verbalization can then invest a symbolic account of the definition with meaning:

'... this is exactly the meaning into **words such as eventually and arbitrarily**, which I constantly put in my writing in my attempt to convey the idea underlying the formal expression of the definition. Most students however simply **ignore them as**

**irrelevant waffle** and copy the definitions only in their notes. I even try grouping the various parts of the definition with **different colours of chalk!**' p188

But then, if a verbal account offers an opportunity to face difficulty with understanding a complex definition such that of convergence, why, we may wonder, do students typically bypass this opportunity? One explanation is offered below:

'But using words is risky: I have seen verbal explanations of the definition which are in fact wrong! Like many textbooks are! Which is not embarrassing given the long debates about acceptable verbalisations of the definition. **Verbalising, geometrising it etc. is fine as long as we stay this side of correctness!**' p189

What the above seem to suggest is that resorting to ordinary language is seen as potentially containing some inherent risk, a jeopardy for mathematical precision. Is it therefore conceivable that the students, in their avoidance of, or trepidation towards, verbalisation are simply adopting the risk-averse discourse of the community towards 'verbalising' and 'geometricising'? The following seems to suggest so:

'... sometimes **steering clear of intuitions and pictures** etc, yes, working through strings of quantifiers, even though one may not be so sure of what is going on, **can be seen as less messy, less risky**. I think some students may in fact see it this way and be happy to work this way and **just do what they are told**. You can view this as the recipe, you can do this, you do this and you do this... You just follow the steps. And in some ways they are **safer this way** because they will not make mistakes as long as they are technically doing the right steps.' p189

Wordless discourse seems to exert some allure on mathematicians (see two examples in Fig.5) perhaps because of its capacity to convey meaning with curt elegance. But does this trepidation towards verbalization contrast with – even contradict... – the expectation that students will appreciate and employ verbalisation in mathematics? I conclude with a few thoughts on this potential contradiction.

### **A contrast between expectations and practice?**

In the above the interviewed mathematicians make a strong case for verbalisation in mathematics: it can drive our noticing of key mathematical ideas and can act as a crucial semiotic mediator between symbolic and visual mathematical expression. They also stress that, for verbal skills to deliver on this potential, good command of ordinary language is important. They observe that students avoid verbalising their mathematical thoughts and they describe this student tendency as missing an opportunity to overcome difficulty with understanding certain complex ideas in mathematics. In discussing certain provisos for what makes verbalisation an acceptable part of mathematical discourse, they cite attention to precision as one such proviso and express their weariness with the potential risk of ambiguity in verbal expression. Their discourse seems to be quite risk-averse and they recognise that students' avoidance of verbalisation may be underlain by a similar aversion to risk.

What is conspicuously absent in the discussion above is the acknowledgement that, if verbalisation skills are an important part of students' learning at this stage (with 'learning' meant as a 'change of discourse' (Sfard, *ibid*)), then **explicit and systematic pedagogical practice has a role to play in facilitating this discursive shift**. At least in these interviews the pedagogical strategies employed towards this facilitation appeared mostly implicit in the interviewees' statements – that they strive for eloquence in their exposition in lectures and that they aim to set a good example, for instance, through their own writing on the board. Given the rather severe absence of mathematical eloquence from most of the student scripts we examined in the course of these interviews, it seems that a more explicit and systematic approach to developing students' verbalisation skills in mathematics is necessary.

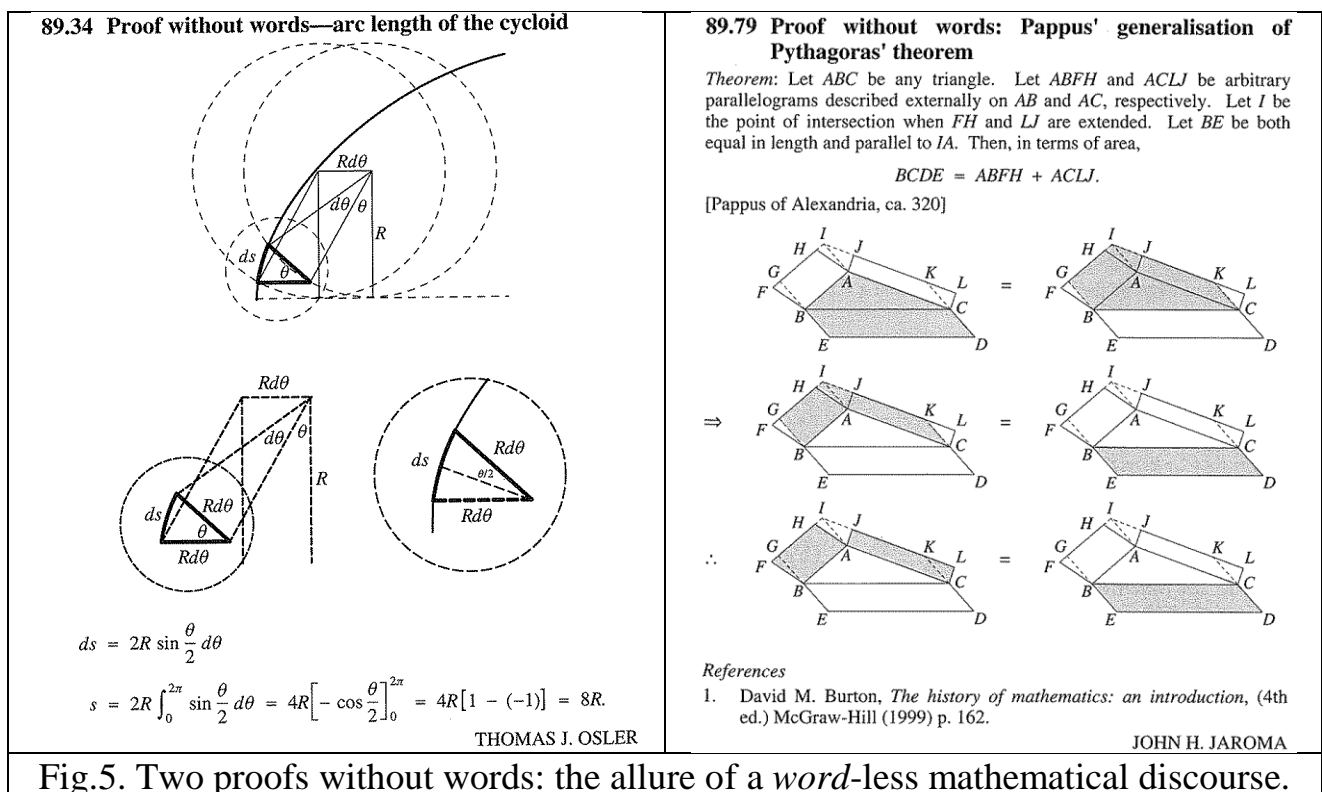


Fig.5. Two proofs without words: the allure of a *word-less* mathematical discourse.

## ENDNOTES

<sup>1</sup>The interview data sampled here are presented in the format of a *re-storied narrative*. The narrative approach of *re-storying* (Clandinin and Connelly, 2000) adopted in this work involves reading the raw transcripts, identifying and highlighting experiences to be told across this raw material and then constructing a new narrative that represents these experiences. So, while fictional, the new narrative is entirely data-grounded. In this sense the interviewee utterances quoted in this paper were constructed entirely out of the raw transcripts of the interviews with the mathematicians. A quotation typifies and condenses the views expressed by a substantial number of participants. (For an example of the re-storying construction process as well as other influences on the data analysis see p27-28 in (Nardi, 2008)).

## ACKNOWLEDGEMENT

This paper draws on studies supported by small grants by the Nuffield Foundation, the UK's Learning and Teaching Support Network (Mathematics, Statistics and Operational Research branch) and the UK's Higher Education Academy. I thank my colleague Paola Iannone for her valuable work with me on these studies.

### REFERENCES

- Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.
- Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience in story in qualitative research*. San Francisco: Jossey-Bass.
- Dee-Lucas, D., & Larkin, J. H. (1991). Equations in scientific proofs: Effects on comprehension. *American Educational Research Journal*, 28(3), 661-682.
- Furinghetti, F., & Paola, D. (1991). On some obstacles in understanding mathematical texts. In F. Furinghetti (Ed.), *Proceedings of the PME 15*, (Vol. 2, pp. 56-63). Assisi, Italy.
- Glaser, B. G., & Strauss, A. L. (1967). *The discovery of Grounded Theory: strategies for qualitative research*. New York: Aldine de Gruyter.
- Jaroma, J. H. (2005). Proof without words - Pappus' generalisation of Pythagoras' Theorem. *The Mathematical Gazette*, 89(516), 493.
- Jaworski, B. (2002). Sensitivity and challenge in university mathematics tutorial teaching. *Educational Studies in Mathematics*, 51, 71-94.
- MacGregor, M. (1990). Writing in natural language helps students construct algebraic equations. *Mathematics Education Research Journal*, 2(2), 1-11.
- Nardi, E. (2008). *Amongst Mathematicians: Teaching and learning mathematics at university level*. USA: Springer.
- Osler, T. J. (2005). Proof without words - arc length of the cycloid. *The Mathematical Gazette*, 89(515), 250.
- Schön, D. (1987). *Educating the reflective practitioner*. San Francisco: Jossey-Bass.
- Sfard, A. (2007) *Thinking as communicating: Human development, the growth of discourses, and mathematics*. UK: Cambridge University Press.
- Sierpinska, A. (1994). *Understanding in mathematics*. London/Washington.: Falmer.
- Wagner, J. (1997). The unavoidable intervention of educational research: A framework of reconsidering researcher-practitioner cooperation. *Educational Researcher*, 26(7), 13-22.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge, UK: Cambridge University Press.
- Wilson, V. (1997). Focus Group: A useful qualitative method for educational research. *British Educational Research Journal*, 23(2), 209-225.

# THRESHOLD CONCEPTS: A FRAMEWORK FOR RESEARCH IN UNIVERSITY MATHEMATICS EDUCATION

Kerstin Pettersson

Department of Mathematics and Science Education, Stockholm University

*A threshold concept is a 'portal' or a 'conceptual gateway' that leads to a previously inaccessible, and initially troublesome, way of thinking about something. A new way of understanding may thus emerge – a transformed view of the subject. The framework of threshold concepts has been used for some years in research of teaching and learning in higher education in several subjects but there are only few articles in mathematics education using this framework. The aim of this paper is to introduce threshold concepts into mathematics education. The result of searching papers in mathematics education using threshold concepts is presented. The need for more research using this framework to create a meeting point for mathematicians and educationalists and to improve students' learning is pointed out.*

**Keywords:** Mathematics education, students' conceptions, threshold concepts, troublesome knowledge, university student learning.

## INTRODUCTION

The aim of this paper is to introduce threshold concepts (Meyer & Land, 2003) as a framework for research in mathematics education. This framework has for some years been used in research of higher education in several subjects but is nearly missing in the mathematics education research. Using the framework of threshold concepts can be very powerful for improving student learning, by providing a language for discussing learning and a meeting point for educational researchers and mathematicians. Threshold concepts have been used successfully to engage subject specialists into pedagogical discussions (Cousin, 2009). My own experience is that the notion of threshold concepts works well also in mathematics (Pettersson, 2008).

Students who are confronted with a mathematical concept sometimes find learning troublesome. There is a large body of research on students' difficulties in shaping their conceptions in accordance with the expectations of the learning environment. For example, the research on students' conceptions in calculus has shown that the learning of several concepts is problematic for the students (for an overview, see e.g. Artigue, Batanero, & Kent, 2007). The concept of limit of functions is one of the problematic instances in the learning of calculus. The concept of limit is a central part of calculus, not only for its own sake, but also because it is used in the definitions of derivative and continuity. Also to get good conceptions of integrals there is a need of a good conception of limit, even though not all definitions explicitly use the concept of limit. 'Limit' is one example of a concept that could be classified as a threshold concept (Meyer & Land, 2003).

The notion of threshold concepts has for some years been used in research in higher education in several subjects. However, in searching for articles in mathematics education using the notion ‘threshold concept’, nearly no findings will appear. In this paper I will first present the theory of threshold concepts and then present some cases using threshold concepts as a framework in economics, engineering and computer science education. I will also present the result of a search for articles using threshold concepts within mathematics education research. Finally I will offer some arguments for using this framework in research on mathematics education. My contribution through this paper is to start a discussion of how focusing threshold concepts can support also research in mathematics education.

## **WHAT IS A THRESHOLD CONCEPT?**

Meyer and Land (2003) introduced the notion *threshold concept*. In students’ efforts to develop their understanding of a subject some concepts may be more crucial than others. A threshold concept can be seen as a ‘portal’ or a ‘conceptual gateway’ that leads to a previously inaccessible, and initially troublesome, way of thinking about something. A new way of understanding may thus emerge – a transformed view of the subject. Threshold concepts are concepts that bind a subject together and the acquisition of such concepts is important to grasping the ways of thinking and practising in the subject (Land, Cousin, Meyer, & Davies, 2005). Threshold concepts are concepts that university teachers typically describe as ‘core concepts’, but it should also be noted that threshold concepts should be regarded first and foremost from a student learning perspective (Meyer & Land, 2003).

A mathematical concept is given by a definition but the definition must be interpreted by a learner. Tall and Vinner (1981) use the notion ‘concept image’ to talk about a student’s interpretation of a concept including everything a student connect to the concept; processes, pictures, examples and also the definition and an interpretation of the definition. Threshold concepts are concepts for which the building of a well connected concept image is troublesome. Meyer and Land (2003) presented the mathematical concepts ‘limit’ and ‘complex number’ as examples of threshold concepts. There are also several examples from other subject areas such as ‘opportunity cost’ in Economics and ‘gravity’ in Physics (Meyer & Land, 2006).

## **CHARACTERISTICS OF THRESHOLD CONCEPTS**

Threshold concepts are characterised as initially troublesome, transformative, integrative, and irreversible (Meyer & Land, 2006). They also tend to serve as boundary markers and may get the students into a liminal space where the students’ understandings are unstable. To identify threshold concepts in a subject area we can search ‘core concepts’ and then use previous research about troublesome learning and students’ conceptions of these concepts. This will give us candidates for

threshold concepts for which we through new research can check the characteristics of threshold concepts.

### **Troublesome knowledge**

The learning of a threshold concept is likely to be troublesome for the students. Perkins (1999) defines ‘troublesome knowledge’ as a kind of knowledge which appears counter-intuitive, alien or seemingly incoherent. A way for students to avoid this troublesome knowledge is to remain in a common sense or intuitive understanding of the concept. Pushing students to change their understanding is not unproblematic as it can involve an uncomfortable and emotional repositioning (Cousin, 2009).

### **Transformative**

Understanding a threshold concept will bring about a significant shift in students’ perception of a subject or a part thereof (Meyer & Land, 2003). It is not just a new understanding of the concept; it involves transformation of the understanding of the whole subject area where the concept is located. It includes an ontological as well as a conceptual shift. The shift in perspective may lead to a reconstruction of subjectivity. It is likely to involve an affective component. The transformation may be sudden, but it is mostly stretched over a long period.

### **Integrative**

Threshold concepts are concepts that bind a subject together. Understanding a threshold concept will expose previously hidden relations between concepts in the subject area (Meyer & Land, 2003). Mastery of threshold concepts helps the student to overcome a fragmented view of the subject, things fall into place (Cousin, 2009). Integrating prior understandings is also part of the transformation of understanding of the whole subject area (Davis & Mangan, 2007).

### **Irreversible**

The change in perspective brought about in the course of developing an understanding of a threshold concept is unlikely to be forgotten or will be unlearned only by considerable effort (Meyer & Land, 2003). This does not exclude modification or rejection for a more refined understanding. One problem in this irreversibility is for teachers, who have transformed their understandings, to look back across thresholds to be aware of the kind of understandings the students are likely to have before transformation has occurred (Cousin, 2009).

### **Boundary markers**

Threshold concepts tend to be bounded in that they “serve as boundary markers for the conceptual spaces that constitute disciplinary terrain” (Land, Meyer, & Smith, 2008, p. x). A student entering a new conceptual space by grasping a threshold concept will find that this new conceptual space will be surrounded by other

threshold concepts. These threshold concepts form a frontier to new conceptual areas.

### **Liminal spaces**

During the process of mastery of a threshold concept the students may enter a liminal space (Meyer & Land, 2006). The notion is taken from the word *limen*, a Latin word meaning boundary or threshold. This space could be compared with the period of adolescence; not yet being an adult, not quite a child. In this unstable state the learner may oscillate between old and new understandings just as adolescents move between acting as a child and as an adult. These liminal spaces are spaces of transformations. Most of the students in this state will oscillate between grasping the concept and then losing that grasp. A learner engaged with the project of mastering a threshold concept would enter the liminal space and hopefully the learner will come out from this space with a transformed understanding of the concept. However, the liminal space can also become a 'stuck place' for the student. Reaching a transformed perspective can be blocked for some students by the 'epistemological obstacles' inherent in the threshold concept (Mayer & Land, 2005). These students will then fake understanding through the practice of mimicry, learning by rote or learning how to solve typical problems (Cousin, 2009).

### **THRESHOLD CONCEPTS IN SEVERAL DISCIPLINES**

Over the past few years, research using the framework of threshold concepts has been carried out in different subjects, such as economics, engineering and computer science (Land, Meyer, & Smith, 2008). Starting with the ETL-project, Enhancing Teaching-Learning Environments in Undergraduate Courses, in UK where several institutions took part, threshold concepts is "now moving from a position of being a leading edge new perspective to one which is catching the interests of academics educational researchers in a growing number of countries" (Meyer & Land, 2006, p.xii).

In a project aimed at empirically identifying threshold concepts in computer science Zander, Boustedt, Eckerdahl, McCartney, Moström, Ratcliffe, and Sanders (2008) used interviews and questionnaires to obtain data from teachers and students. 'Object-oriented programming' and 'memory/pointers' emerged as candidates for threshold concepts. The project also investigated how students understand these concepts with the aim to improve teaching and learning.

Shanahan and Meyer (2006) studied a threshold concept in Economics; 'opportunity cost'. The results points out that there are important implications for the manner in which students are introduced to threshold concepts. When learning threshold concepts 'first impressions matters'. Efforts to make the concept easier by simplifying students' first impressions may set students on a path of ritualised



knowledge that creates barriers to a deeper understanding and prevents students from crossing the threshold.

In electrical engineering Carstensen and Bernhard (2008) considered the troublesome concepts 'frequency response'. They propose that certain concepts can function as a 'key' that opens up the portal of understanding. Teaching such 'key' concepts do not just open up for understanding of that concept, but also the learning of other concepts related to it. Using the tool 'Bode Plots' to illustrate frequency response was found to function as such a 'key'. The results from focusing the teaching on these 'key' concepts opening up for understanding the threshold concepts showed an improvement in the scores achieved by students on a test at the end of the course.

Just a few examples are given within this paper; many more findings have been published (e.g. Land, Meyer & Smith, 2008). Recently a new book on threshold concepts was published (Meyer, Land, & Baillie, 2010). However, there is a lot more to do, especially in relation to mathematics education.

## **THRESHOLD CONCEPTS IN MATHEMATICS EDUCATION**

The notion of threshold concept is established in the research of higher education in general but searching for articles in mathematics education using the notion threshold concepts gives just a very few findings. Using the database ERIC and the combination 'mathematics' AND 'threshold concept' searching in all text produced five hits when searching in the period of 2003-2010 (access 2010-09-06). Two of these five articles are about 'threshold values' of variables used in physics and resources on the Internet. The remaining three articles are published in Educational studies in mathematics (Williams, 2009), European journal of engineering education (Masouros & Alpay, 2010) and Higher education (Scheja & Pettersson, 2010).

Using the database Academic Search Premier and the combination 'mathematics' AND 'threshold concept' searching Boolean/Phrase in all text produced 32 hits of peer reviewed articles when searching in the period of 2003-2010 (access 2010-09-06). However, scanning the list of these articles only seven are in the area of mathematics education; that is reporting research about teaching and learning in mathematics courses. These seven articles include the three articles found in ERIC. The other four are published in Educational studies in mathematics (Bramby, Harries, Higgins, & Suggate, 2009), European journal of engineering education (McCartney, Boustedt, Eckerdahl, Moström, Sandres, Thomas, & Zander, 2009; Booth, 2008) and International journal of mathematical education in science and technology (Pettersson & Scheja, 2008).

Threshold concepts are in focus in two of the articles. Scheja and Pettersson (2010) discuss the transformative aspect of threshold concepts suggesting that the transformation involves a transformation of the students' conceptions as these develop through shifting contextualisations of the concepts. McCartney and

colleagues (McCartney et. al., 2009) also focuses threshold concepts, but mostly related to software engineering. The article summarizes findings concerning how computer science students experience the liminal space and discuss how this might affect teaching.

In the other articles threshold concepts are more or less just mentioned. Masouros and Alpay (2010) focuses on the design of a computer-based mathematics resource. When discussing the mathematics content of the resource they emphasize that special effort should be given to troublesome topics that lead to a transformation in understanding, mentioning threshold concepts and referring to Meyer and Land (2003).

Williams (2009) puts forward the notion ‘threshold moment’ where seeing and grasping at the nexus of two or more activities often seem to be critical to breakthroughs in learning. In a footnote Williams makes a note on literature on ‘threshold concepts’ and also points out that not very much has been published about threshold concepts in mathematics.

Barmby, Harries, Higgins, and Suggate (2009) discuss children’s understanding and reasoning in multiplication. When the authors talk about key representations for a concept they relate this to the notion of ‘key development understandings’ defined by Simon (2006). They make a quotation from Simon and in this quote parallels with the notion of ‘threshold concepts’ are mentioned. Barmby and colleagues do not take this any further.

Booth (2008) addresses the issue of teaching and learning engineering mathematics based on a form of understanding that goes beyond facts, theorems and algorithms. She points out that the mathematicians as mathematics teachers in the engineering education mostly are interested in the learning objects, in what to learn, also mentioning that educational researchers have much knowledge about the ways students might be comprehending the learning objects, “in particular when it comes to ‘threshold concepts and troublesome knowledge’ ” (p. 383) referring to Meyer and Land (2003).

Pettersson and Scheja (2008) explore the nature of students’ understanding of the concepts limit and integral. As a reason to study these concepts it is argued that these concepts are threshold concepts giving references to Meyer and Land (2005, 2006) and to previous research about students’ conceptions in calculus.

Looking at these articles found by searching databases it could be seen that the notion threshold concepts until now is just used in really few research articles in mathematics education. In the next paragraph I point out that this framework is very useful also in mathematics education.

**TO BE USED IN MATHEMATICS EDUCATION**

Using the framework of threshold concepts is potentially even more successful in mathematics than in other subjects. In mathematics the concepts are an important part of the curriculum and the concepts are explicitly defined in a way that is uncommon in other subjects such as economics and engineering. The concepts and the relation between them build the core of mathematics (Davis & Hersh, 1990).

In my own research (Pettersson, 2008; Pettersson & Scheja, 2008; Scheja & Pettersson, 2010) I have studied students' conceptions of function, limit, derivative and integral. Looking through previous research about students' learning of calculus (e.g. Artigue et. al., 2007) these concepts are candidates as threshold concepts. The characteristics of threshold concepts, as integrative and transformative in the learning process, points out the importance of research about conceptions of these concepts. Pettersson and Scheja (2008) points out the students' algorithmic way of interpreting the concepts of limit and integral. Looking at these concepts as threshold concepts and focusing on the transformative aspect of the threshold showed that the students started to pass the threshold (Scheja & Pettersson, 2010). Challenging questions in the interviews showed to be important to start this transformation. Threshold concepts can in this way be used to focus on powerful transformation points in the students' learning; they are 'jewels in the curriculum' (Land, Cousin, Meyer, & Davies, 2006, p. 198). They give the students opportunities to develop fundamental conceptual understanding. Using threshold concepts in teaching a subject offers an opportunity to focus on the points that are really useful in mastering the subject (cf. Carsetensen & Bernhard, 2008). Focusing the most important concepts of the subject will also be a way of avoiding an overstuffed curriculum (Cousin, 2009).

Threshold concepts are characterized as potentially troublesome. The students are likely to encounter troublesome knowledge and experience conceptual difficulty. These concepts are potentially 'stuck points'. The students get into the liminal space with uncertainty and oscillation in understanding. This unsecure phase of learning may be necessary and unavoidable but we do not want the students to stay for a long time in the liminal space. Scheja and Pettersson (2010) argue that the students' shifting from an algorithmic contextualization to a contextualization inviting reflection on conceptual dimensions of limit and integral is a way to move on from the liminal space. Threshold concepts are rarely mastered at a specific point of time, an 'aha' moment, mastery might take years to complete (McCartney et. al., 2009). However, teachers listening to the students' uncertainties give possibilities for helping the students through the liminal space. Knowledge about these potentially stuck places is important in the teaching. Shanahan and Meyer (2006) pointed out the importance of the first impression; to simplify the threshold concepts do not help the students.

One of the big advantages of research on threshold concepts is that it animates interests and discussions among academics. This kind of educational research gives centrality to the subject specialists and gives a platform for partnership between educational researchers and subject specialists. Such a focus on threshold concepts is also a good way to involve both subject specialists and educational researchers in a discussion on student learning, curriculum design and teaching the subject. Getting academics together to discuss and identify threshold concepts in their subject area has proved to be very fruitful (Cousin, 2009). My own experience of discussing teaching and learning mathematics at university with mathematicians is that threshold concepts appear to fit very well in mathematics. Presenting the idea of threshold concepts to mathematicians always bring up discussions about what threshold concepts there can be in calculus, in linear algebra and other subject areas. These discussions usually spark ideas about students' conceptions and how to improve student learning. The metaphor of a threshold to pass seems to be easy to take in and to be a good starting point for educational discussions.

Research using the framework of threshold concepts has the possibility to improve the teaching and learning of mathematics. The framework will be powerful in research of university mathematics education but it can also be used in research on mathematics teaching and learning at other levels such as primary and secondary school.

## CONCLUSIONS

The framework of threshold concepts has successfully been used in educational research into several subjects. However, there are still few publications related to mathematics education. Using this framework in mathematics education will contribute to mathematics education in several ways. There are several concepts in the subject of mathematics that are troublesome for the learners. Knowledge about concepts which are transformative and integrative will help us to improve teaching and learning. To focus on threshold concepts is a way to avoid an overstuffed curriculum (Cousin, 2009). Research has also pointed out that the students scored higher when the teaching was focusing threshold concepts (Carstensen & Bernhard, 2008). The notion of threshold concepts also creates a meeting point for educationalists and mathematicians. The notion provides a language for discussions about how to improve teaching and learning. There is a growing body of results about threshold concepts in several subjects, but there is a lot more to do in relation to mathematics.

## REFERENCES

Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In K. L. Frank (Ed.), *Second handbook of research on*

- mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (p. 1011-1049). Charlotte, NC: Information Age Publisher.
- Booth, S. (2008). Learning and teaching engineering mathematics for the knowledge society. *European Journal of Engineering Education*, 33(3), 381-389.
- Bramby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, 70(3), 217-241.
- Carstensen, A.-K., & Bernhard, J. (2008). Threshold concepts and keys to the portal of understanding. In R. Land, J. H. F. Meyer, & J. Smith (Eds.), *Threshold concepts within the disciplines* (p. 143-154). Rotterdam: Sense Publishers.
- Cousin, G. (2009). *Researching learning in higher education*. New York and London: Routledge.
- Davis, P. J., & Hersh, R. (1990). *The mathematical experience*. London: Penguin Books.
- Davis, P., & Mangan, J. (2007). Threshold concepts and the integration of understanding in economics. *Studies in Higher Education*, 32(6), 711-726.
- Land, R., Cousin, G., Meyer, J. H. F., & Davies, P. (2005). Threshold concepts and troublesome knowledge (3): implications for course design and evaluation. In C. Rust (Ed.), *Improving student learning diversity and inclusivity* (p. 53-64). Oxford: Oxford Centre for Staff and Learning Development.
- Land, R., Cousin, G., Meyer, J. H. F., & Davies, P. (2006). Conclusion. Implications of threshold concepts for course design and evaluation. In J. H. F. Meyer, & R. Land (Eds.), *Overcoming barriers to student understanding: Threshold concepts and troublesome knowledge* (p. 195-206). London: Routledge.
- Land, R., Meyer, J. H. F., & Smith, J. (Eds.). (2008). *Threshold concepts within the disciplines*. Rotterdam: Sense Publishers.
- Masouros, S., & Alpay, E. (2010). Mathematics and online learning experiences: a gateway site for engineering students. *European Journal of Engineering Education*, 35(1), 59-78.
- McCartney, R., Boustedt, J., Eckerdal, A., Moström, J. E., Sanders, K., Thomas, L., & Zander, C. (2009). Liminal spaces and learning computing. *European Journal of Engineering Education*, 34(4), 383-391.
- Meyer, J. H. F., & Land, R. (2003). Threshold concepts and troublesome knowledge: Linkages to ways of thinking and practising within the disciplines. In C. Rust (Ed.), *Improving student learning: Improving student learning theory and practice - Ten years on* (p. 421-424). Oxford: Oxford Centre for Staff and Learning Development.

- Meyer, J. H. F., & Land, R. (2005). Threshold concepts and troublesome knowledge (2): Epistemological considerations and a conceptual framework for teaching and learning. *Higher Education*, 49, 373–388.
- Meyer, J. H. F., & Land, R. (Eds.). (2006). *Overcoming barriers to student understanding: Threshold concepts and troublesome knowledge*. London: Routledge.
- Meyer, J. H. F., Land, R., & Baillie, C. (2010). *Threshold concepts and transformational learning*. Rotterdam: Sense publisher.
- Perkins, D. (1999). The many faces of constructivism. *Educational Leadership*, 57(3), 6-11.
- Pettersson, K. (2008). *Algoritmiska, intuitiva och formella aspekter av matematiken i dynamiskt samspel: En studie av hur studenter nyttjar sina begreppsuppfattningar inom matematisk analys* [Algorithmic, intuitive and formal aspects of mathematics in dynamic interplay: A study of students' use of their conceptions in calculus]. Ph.D. thesis, Department of Mathematical Sciences, Göteborg University.
- Pettersson, K., & Scheja, M. (2008). Algorithmic contexts and learning potentiality: A case study of students' understanding of calculus. *International Journal of Mathematical Education in Science and Technology*, 39(6), 767-784.
- Scheja, M., & Pettersson, K. (2010). Transformation and contextualisation: conceptualising students' conceptual understandings of threshold concepts in calculus. *Higher Education*, 59(2), 221-241.
- Shanahan, M., & Meyer, J. H. F. (2006). The troublesome nature of a threshold concept in economics. In J. H. F. Meyer, & R. Land (Eds.), *Overcoming barriers to student understanding: Threshold concepts and troublesome knowledge* (p. 100-114). London: Routledge.
- Simon, M. A. (2006). Key developmental understandings in mathematics: a direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359–371.
- Tall, D. & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Williams, J. (2009). Embodied multi-modal communication from the perspective of activity theory. *Educational Studies in Mathematics*, 70, 201–210.
- Zander, C., Boustedt, J., Eckerdahl, A., McCartney, R., Moström, J.E., Ratcliffe, M., & Sanders, K. (2008). Threshold concepts in computer-science: A multi-national empirical investigation. In R. Land, J. H. F. Meyer, & J. Smith (Eds.), *Threshold concepts within the disciplines* (p. 143-154). Rotterdam: Sense Publishers.

# CHALLENGES WITH VISUALIZATION: THE CONCEPT OF INTEGRAL WITH UNDERGRADUATE STUDENTS

Blanca Souto Rubio and Inés M<sup>a</sup> Gómez-Chacón

Universidad Complutense de Madrid

*In this paper, some components of a conceptual framework for the study of visualization processes from a cognitive point of view are presented along with a discussion of examples of empirical data relating to the concept of integral: (1) the coordination of registers of representation for explaining some of the students difficulties in the understanding and learning of mathematical concepts (the integral); (2) characteristics of visualization (in Calculus): it is related to the use of the graphic register in coordination with other representations and accompanied by a global apprehension; (3) the use of the graphic register (non-visual, mixed and visual methods are identified) and the higher cognitive difficulty of visual methods.*

**Key- words:** Visualization, Representations, University level, Integral.

## INTRODUCTION

This paper grew out of a study conducted in 2008/2009 with a group of first year students at the Universidad Complutense de Madrid (UCM). The main aim of this study was to improve the teaching of mathematical analysis by emphasising visualization processes (Souto, 2009; Souto & Gómez- Chacón, 2009). From the beginning, a big challenge was: how to characterize visualization processes in order to be able to observe them in our particular context? The literature review related to this topic highlighted a large diversity of terms and theories around the notion of visualization; most of them referred to primary and secondary levels (Duval, 1995, 1999; Arcavi, 2003; de Guzmán, 2002; Presmeg 1985, 2006; Eisenberg & Dreyfus, 1991). Therefore, we noticed a lack of empirical studies on visualization among undergraduate students and of a corresponding adaptation of some of these theoretical elements to this level of teaching.

For the sake of brevity, in this paper we focus on cognitive aspects of visualization. We mostly use the theory of registers of semiotic representation (Duval, 1995, 1999). Firstly, some theoretical ideas from this framework are outlined and reviewed. Secondly, some examples of empirical data are analyzed. It is not our aim in this paper to provide an exhaustive analysis of the results obtained in the previous study (Souto & Gómez- Chacón, 2009), but just to analyze examples of data chosen in order to provide insight into some of the specific knowledge on individual students' reasoning in relation to visualization processes, obtained in this study.

The concept of integral has been chosen because it offers an opportunity to discuss key issues concerning visualization. Research on the concept of integral (Mundy, 1987; González-Martín & Camacho, 2004) emphasize that during the first year of

university, students use the concept of integral in a very mechanical way due to the lack of coordination of the concept of area and of integral, among other reasons. Furthermore, in an attempt to improve the comprehension of the concept, other authors have recommended explicit attention to visualization (de Guzmán, 2002; González-Martín & Camacho, 2004).

## **ABOUT THE CONCEPTUAL FRAMEWORK**

Within the cognitive approach, the theory of the registers of semiotic representation (Duval, 1995, 1999) was useful in order to describe and analyze students' difficulties in the learning of mathematical concepts. In order to explore visualization processes, it has been found fruitful to combine it with some results from research on the role of visualization in mathematical reasoning (de Guzmán, 2002; Arcavi, 2003), on individual differences in the preference to visualize (Presmeg 1985, 2006) or on reasons for a reluctance to visualize (Eisenberg & Dreyfus, 1991), since that allows us making choices of important aspects for visualization inside Duval's theory.

### **Understanding and learning of mathematical concepts**

We agree with Duval (1995, 1999) that the only possible access to mathematical objects is through their representations in the different semiotic registers. From this perspective, the *understanding of a concept* is built through tasks that imply the use of different registers and promote the flexible coordination of representations. Therefore, *learning mathematics* implies "the construction of a cognitive structure by which the students can recognize the same object through different representations" (Duval, 1999: 12).

In this context, *improving learning* implies, in particular, to minimise difficulties, misunderstandings and mental blockings that could appear in different actions related to a register: representation, treatment and conversion (Duval, 1995). As we noted in the introduction, in our specific case - the understanding of the concept of integral – research conducted with the semiotic approach highlights as a source of difficulties the lack of coordination between both the graphic and algebraic registers, and the predominance of the latter in the students' answers. This leads us to pay special attention to the use of the graphical register and to visualization.

### **Visualization in mathematics education**

According to Duval (Duval, 1999: 15), visualization can be produced in any register of representation as it refers to processes linked to the visual perception and then to vision. For the aim of the present study, this notion is too broad; although we take into account some other characteristics of visualization pointed out by Duval. We find more useful Arcavi's definition, which is limited to the use of figures, images and diagrams.

"Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with



technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings” (Arcavi, 2003: 217).

Therefore, in the frame of this research we identify visualization with the use of the graphic register. However, some remarks related to this definition are needed.

In the characterization of visualization in the context of problem solving, we find very useful the difference between *visual* and *non-visual methods* established by Presmeg in her research about preference to visualize (Presmeg, 1985). However, we have to be cautious when combining Presmeg’s and Duval’s approaches. For example, the following equivalence cannot be established: visual method (Presmeg) - use of graphic register (Duval). We must be cautious for two reasons. Firstly, when Presmeg (2006) talks about *visual images*, she includes mental images that belong to the world of mental representations, which are different from Duval’s semiotic representations (Duval: 1995: 14). We adopt this sense of mental images in relation to visualization. Secondly, the use of the graphic register does not imply that the method is visual. Duval (1999: 14) distinguishes two types of functions for the images: the *iconic* and the *heuristic*. The latter involves a global apprehension and it is related to visualization (Duval, 1999: 14). If there is use of the graphic register but there is not global apprehension or the image performs an iconic function, we will not therefore use the term *visualization*. Thus, the relevant connection is between visual methods (Presmeg) and this heuristic function of images (Duval).

Finally, Eisenberg and Dreyfus (1991) identify three reasons to explain the reluctance of some students to visualize: “*a cognitive one (visual is more difficult), a sociological one (visual is harder to teach) and one related to beliefs about the nature of mathematics (visual is not mathematical)*” (1991:30). With the help of Duval’s approach, it will be possible to describe this particular cognitive difficulty of visualization.

## PARTICIPANTS AND DATA COLLECTION

The study was conducted with a first year group of 29 mathematics students at Universidad Complutense de Madrid, 15 female and 14 male. In this first year, the students followed a course called Real Variable Analysis, in which the formal definition of the concept of integral is introduced. However, they were supposed to have learned the basic rules for integration by using primitives as well as its relation to the calculation of some areas under curves already in secondary school.

For the data collection, the instruments used were a questionnaire with problems and semi-structured interviews. The questionnaire was composed of 10 non routine problems in mathematical analysis, some taken from other studies (Mundy, 1987; works quoted in Eisenberg & Dreyfus, 1991). Most of the problems are posed in the algebraic register but they also allow a visual interpretation (Eisenberg & Dreyfus, 1991). Thus these problems allow the analysis of students’ performance with regard

to the coordination of registers, and particularly the use of the graphic register. The results obtained from the questionnaire required deeper investigation into affective, cognitive and sociocultural aspects of individual students. In order to do this, 6 semi-structured interviews were conducted. These were divided into several parts: individual background, tasks about beliefs and preference of visualization, questions on questionnaire's answers. In this paper, attention is paid only to the cognitive aspects of the data.

For the data analysis, we privileged the use of systemic networks for the questionnaire and transcriptions for the interviews. Systemic networks (Figure 8) allow looking simultaneously at all the students' answers to the problems. Both, students and their answers, were labelled with a number from 1 to 29 included between parentheses. In particular, systemic networks favour the observation of the following elements: strategies and kinds of representation used by each student; frequency of use and difficulties of each register; and students' conceptions.

## ANALYSIS AND DISCUSSION OF RESULTS

### Students' difficulties analyzed through the theory of registers of representation

Students' answers to the questionnaire were analyzed using Duval's theory of registers of representation. The results described below are based on the analysis of the systemic network (Figure 8) associated with the following problem (Figure 7), but they are representative of what happens with other problems in the questionnaire.

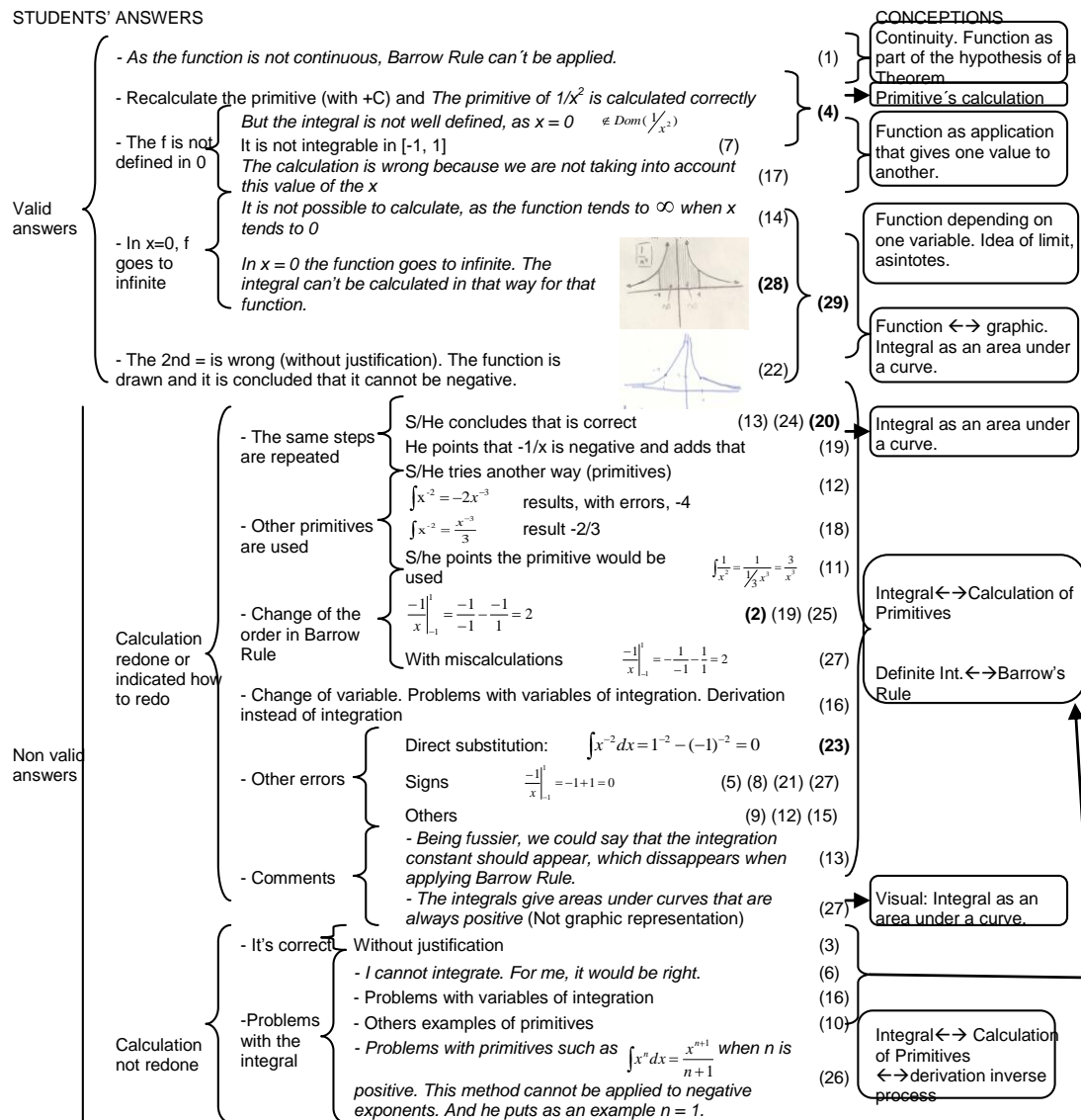
What's wrong in the following calculation of the integral?

$$\int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^1 x^{-2} dx = \frac{x^{-1}}{-1} \Big|_{-1}^1 = \frac{-1}{x} \Big|_{-1}^1 = \frac{-1}{1} - \frac{-1}{-1} = -2$$

Figure 7 Statement of the first problem

Firstly, the choice of representation and register is very important for solving the problem successfully. These decisions are directly related to the conception used for the integral concept. All the students who gave valid answers, placed on the top of Figure 8, were either focused on the function (global properties as continuity or asymptotes; or local at  $x=0$ ) or they contemplated the interpretation of the integral as the area under a curve. This led two of the students to the use of the graphic register. However, most of them (22 students) interpreted the integral as a process (calculation of primitives and Barrow Rule) which leads all of them to the use of the algebraic register. In this case, the students were not able to correctly answer the question. They repeated the same calculation or made some errors (using a different primitive, interchanging the signs when applying the Barrow Rule, considering the constant of integration, miscalculations).

## Working Group 14



**Figure 8: Systemic network associated to the problem**

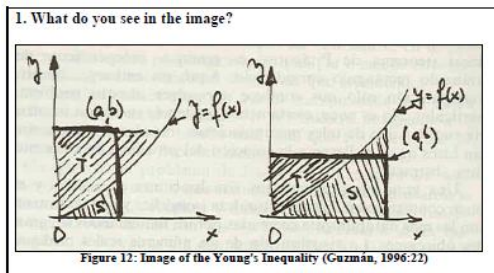
Secondly, we examine the way in which representations are used. The data show that the initial kind of representation chosen does not determine completely the success of the resolution. For example, student 4 was the only one who focused at the beginning on the integral as the calculation of primitives, but answered successfully. This was possible because of the flexible combination of this calculation with another argument about the domain of definition of the function. Moreover, the analysis of the answers highlights how the coordination of registers led to a better understanding of the problem (see in Figure 8, answers 28 and 29). However, there is a risk of making some errors (Figure 8, answers 19, 27) if the mobilisation of both registers is not accompanied by further reflection.

Thus, our results are coherent with previous research (Mundy, 1987; González-Martín & Camacho, 2004) described in the introduction. From a didactic point of view, the use of different kinds of representations and registers and their

coordination seem to be essential. But then, how to promote their flexible coordination when teaching?

### Essential for visualization: the global apprehension

Global apprehension of images is required together with the coordination of registers. However, some students do not go further than having a local apprehension and cannot see the relevant global organization (Duval, 1999: 14). Our analyses adopt these ideas as we try to show with the description of the following episode from the interviews.



2. Do you find some relation with the following heading?

**Theorem 3.2 (Young's Inequality).** *If  $f$  has a continuous and positive derivative on  $[0, c]$  ( $c > 0$ ) and  $f(0) = 0$ , then for  $a \in [0, c]$ ,  $b \in [0, f(c)]$ , we have*

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(x) dx. \quad (3.3)$$

*The equality holds if and only if  $b = f(a)$ .*

Figure 13: Heading of Young's Inequality

### Figure 9: Statement of the task based on Young's Inequality

The episode concerns Young's Inequality (Figure 9) and the interpretation of a graphic representation when asked for the connection with the theorem. Silvia is the name of the student chosen for the interview. She was selected because her responses to the questionnaire showed some preference for the graphic register, while she did not answer satisfactory any problem. The interview enabled us to go deeper into her difficulties with visualization.

At first, only the image was shown. Silvia detected isolated elements and even made some references to the integral as an area. Later, we showed her the statement of the inequality. She assumed the relation to the image, but it did not seem to be clear for her. She frequently requested help by asking questions. Afterwards the following conversation took place. In order to be able to continue with the interview, support is given to help her to identify correctly all the elements in the image with those in the statement. Thus, in spite of the fact that Silvia seemed to be able to coordinate the two registers, there was not any moment of clear understanding.

Interviewer: OK, what kind of explanations would you need with the drawing? Have you understood it completely, the drawing?

Silvia: Um... Well... [...]

Interviewer: OK, this  $[ab, \text{the rectangle}]$  is equal or less than this integral, the one which is in the drawing?

Silvia: Well, it'll be this, from 0 to  $a$  (she points with the finger to an interval over the  $x$ -axis). This one, the  $S$ 's.

Interviewer: OK, and the other?

Silvia: Well,  $T$ 's. (Silence, she seems pensive)

Interviewer: This is a little more difficult for you to see, isn't it?

Silvia: Yes.[...] Well, to understand it [the theorem], with the drawing I wouldn't understand it.

Silvia could not go beyond the mere identification of the represented units. For her, the image was only an illustration, an iconic representation that does not work as a means of visualizing the statement of Young's Inequality. Therefore, the main conjecture for Silvia's difficulties with visualization in this case is the lack of global apprehension. From a didactic point of view, the following challenges emerge: Is it possible to teach how to apprehend an image globally? If so, how can it be done?

### The high cognitive requirement of visual methods

During another task in Silvia's interview, she explained why she chooses "*the way they give [in class], the definition*" as follows: "*I don't know. It's like everything is more mechanical. In the other way [visual] you have to relate, to think. [...] It isn't that I prefer it [algebraic], but it's easier. So, instinctively, I do it*". This excerpt of the interview concerns the cognitive rationale pointed out by Eisenberg and Dreyfus (1991) for the reluctance to visualize. In order to go deeper into this issue, the students' use of the graphic register in the answers to the questionnaire was analyzed. Taking into account the distinction between iconic and heuristic functions performed by images (Duval, 1999), and its relation with non-visual and visual methods (Presmeg, 1985), different kinds of techniques for solving a problem using the graphic register have been detected: *non-visual*, *mixed* and *visual*. The data collected from the following problem (Figure 10) of the questionnaire allow us to illustrate some characteristics of each kind of technique.

If  $f$  is an odd function on  $[-a, a]$  calculate  $\int_{-a}^a (b + (f(x)))dx$

Figure 10: Statement of the second problem

This problem was answered by 20 students, and only 8 used the graphic register. The first kind of resolution (Figure 11) appeared with higher frequency (5 out of 8 students). The images appear together with the algebraic register, in which the main reasoning takes place. The images were employed either to try to remember the definition of odd functions, or to deduce some other properties. Therefore, the image was unnecessary and it performs an iconic function. The method of resolution was considered to be *non-visual*. In fact, the example shown (Figure 11) is accompanied by an image that the student interpreted by giving an incorrect definition of odd functions. In spite of this, this misunderstanding did not affect the algebraic reasoning, which is valid.

In the other two resolutions, images were interpreted as performing a heuristic function. However, they differ according to the number of conversions made between the algebraic and the graphic registers. The second resolution (Figure 13)



was given by two students. It has been called *mixed* as a first step is needed in the algebraic register, in which the additive property of integrals is applied, before converting to two graphic representations, one for each integral. As an informal conversation with the student who gives the answer in Figure 13 clarified, this conversion allowed him calculating the value of the integrals, without performing treatments in the algebraic register, and coming back afterwards to it in order to finish the evaluation of the integral. Thus, two conversions were made (algebraic-graphic- algebraic).

$$\int_{-a}^a (b + f(x)) dx = \int_{-a}^a b dx + \int_{-a}^a f(x) dx =$$

$$= (bx) \Big|_{-a}^a + \int_{-a}^a f(x) dx$$

The integral of  $f(x)$  in  $a$  is going to be equal that the integral of  $f(x)$  en  $-a$ , thus the second summand will be nil and the final result will be:

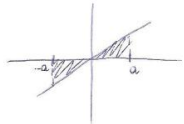
$$(bx) \Big|_{-a}^a = b \cdot a - (b \cdot (-a)) =$$

$$= b \cdot a - (-b \cdot a) =$$

$$= b \cdot a + b \cdot a = 2ba \quad f(a) = -f(a)$$

Figure 11: Non visual method

odd  $\rightarrow$  symmetric respect to  $(0,0)$



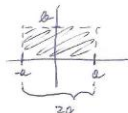
$$\int_{-a}^a (b + f(x)) dx = \int_{-a}^a b dx + \int_{-a}^a f(x) dx = \int_{-a}^a b dx = b \cdot 2a$$


Figure 13: Mixed method

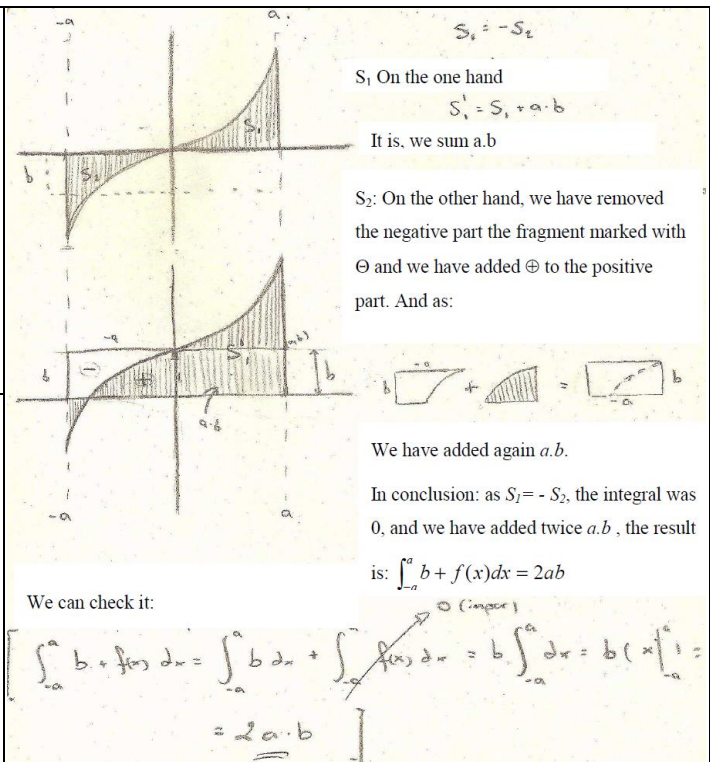


Figure 12: Visual method

The third resolution (Figure 12) is completely *visual* since it includes, at the beginning, just one conversion to the graphic register, in which the main argument is developed. In order to give this visual answer, more concepts and relations than in the non-visual ones (including the pure algebraic one) must be considered simultaneously: a visual interpretation of the integral as an area, and the odd functions as those symmetric around the origin; the recognition that adding a constant quantity to a function means a translation of its graph along the y-axis (this notion is not necessary in the algebraic reasoning); finally some image treatment of that sort of “cutting” and “gluing” areas, taking into account their signs. Therefore, this original answer given by only one student provides the opportunity to show how the cognitive theory of registers of semiotic representation serves to explain and to go deeper into the cognitive difficulty of visualization noted by Eisenberg and Dreyfus (1991). There is also at the end an algebraic answer used for checking. It follows a linear process consisting in the succession of several algebraic treatments, made without errors, giving as result  $2ab$ . Obviously, in both arguments, the result is

the same. However, each kind of argument leads us to see the problem in a very different way.

Therefore, from a didactic point of view, the combination of visual and non-visual arguments when teaching seems to be advisable, since it provides complementary kinds of understanding. However, as was argued in the conceptual framework and as the empirical data have shown, this should not be misinterpreted as just using the graphic register. The following challenges for teaching emerge: how to combine visual and non-visual methods in class in order to improve the understanding of the students? How can the higher difficulty of visual arguments be handled in the class?

## CONCLUSION

In this paper, we aim to: (1) present some theoretical ideas found to be relevant in a conceptual framework for a cognitive perspective of visualization; (2) show the analysis of some examples of empirical data in order to provide insight into two levels, the theoretical ideas presented and the individual students' reasoning.

The theoretical framework of the cognitive theories of registers of semiotic representation (Duval, 1995, 1999) was useful in order to: (1) describe some difficulties of the students in the understanding and learning of mathematical concepts, in this case, the integral; (2) explore some conditions for visualization (in Calculus), related to: the explicit or implicit use of the graphic register, to the coordination with other representations (in the same or different registers) and, as Silvia's episode showed, to the necessity of a global apprehension of the image; (3) examine the students' use of the graphic register and the higher cognitive difficulty of visualization argued by Eisenberg & Dreyfus (1991). Moreover, visualization is related to the *heuristic function of images* (Duval, 1999) which has been identified with the *visual methods* (Presmeg, 1985). This connection led us to distinguish three different kinds of methods for solving a problem using the graphic register: *non-visual*, *mixed* and *visual*. Although this possibility has not been fully exploited in this paper, it enables us to shift our attention to individual differences in the preference to visualize.

From a didactic point of view, some challenges around a specific teaching of visualization emerge: how to use different kind of representations and registers in order to promote a flexible coordination between them? Is it possible to teach how to apprehend an image globally? How could it be done? How to combine visual, mixed and non-visual methods in class in order to improve the understanding of the students? How to manage the higher difficulty of visual methods?

## NOTES

This work has been partially supported by grant AP2007- 00866.

## REFERENCES

- Arcavi, A. (2003). The role of visual representations in the learning of mathematics, *Educational Studies in Mathematics*, 52(3), 215-24.
- Duval, R (1995). *Sémiosis et Pensée Humaine. Registres sémiotiques et apprentissages intellectuels*. Neuchatel: Peter Lang.
- Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In F. Hitt y M. Santos (Eds.), *Proceedings of the 21st North American PME Conference*, 1, 3-26.
- Eisenberg, T. & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmernann, W. & Cunningham (Eds.) *Visualization in Teaching and Learning Mathematics*, MAA Notes n. 19, 25-37. Washington, D.C
- González- Martín, A. S. & Camacho, M. (2004). Legitimization of the graphic register in problem solving at the undergraduate level. The case of the improper integral. *Proceedings of the 28<sup>th</sup> Conference of the PME*, 2, 479-486.
- Guzmán, M. de (2002). The role of visualization in the teaching and learning of Mathematical Analysis. *Proceedings of the 2<sup>nd</sup> International Conference on the Teaching of Mathematics (at the undergraduate level)*. Univ. of Crete, Greece.
- Mundy, J. (1987). Analysis of Errors of First Year Calculus Students, en Theory, Research and Practice in Mathematics Education, Bell, A., Low, B. y Kilpatrick, J. (Eds.), *Proceedings ICME 5*, 170-172 .
- Presmeg, N. C. (1985). The role of visually mediated processes in high school mathematics: A classroom investigation. Unpublished Ph.D. dissertation, Cambridge University, England.
- Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics, *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future. PME 1976-2006*. Sense Publishers, 205-235
- Souto, B. (2009). Visualización en matemáticas. Un estudio exploratorio con estudiantes del primer curso de Matemáticas. Master Dissertation, UCM. (Retrieved from <http://www.mat.ucm.es/vdrmat/TI-08-09/trabajo-master-curso-2008-09-blanca-souto.pdf>)
- Souto, B. & Gómez-Chacón, I. (2009) How to prepare students for a successful first year at university: an experience in visualization, En Tzekaki, M., Kaldrimidou, M. y Sakonidis, H (Eds.) *Proceedings of the 33rd PME*, Vol 5, 495. Thessaloniki, Greece: PME.



# **THE SECONDARY-TERTIARY TRANSITION: A CLASH BETWEEN TWO MATHEMATICAL DISCOURSES**

Erika Stadler

Linnaeus University

*The aim of this paper is to illustrate some aspects of the transition between secondary and tertiary studies in mathematics, based on an analysis of a critical case of two students trying to solve a system of linear equations with help from their teacher. In their conversation, different aspects of the transition appear, which both can be assigned to changes in the mathematical content and differences in the way mathematics should be regarded and communicated. Students draw on a school mathematical discourse, while the teacher answers them within a scientific mathematical discourse. The notion of functional understanding is used to describe students' attempts to build bridges between these discourses.*

Keywords: transition, secondary, tertiary, school mathematics, mathematics teacher

This paper reports from a case study of the secondary-tertiary transition. The study is based on an episode with two novice teacher students, working with a university textbook exercise in linear algebra with help from their teacher. The aim of the paper is to examine the transition in terms of an encounter between two mathematical discourses; school mathematics and mathematics as a scientific discipline. After a short overview of some earlier research concerning the transition, presentation of data is alternated with analyses of the two students working on the exercise with help from their teacher. The paper ends with conclusions and final comments about their transition between secondary and tertiary mathematics.

## **EARLIER RESEARCH ON THE SECONDARY-TERTIARY TRANSITION**

Nowadays, the secondary-tertiary transition is a well-researched area. The transition has been studied from various theoretical perspectives that explicitly or implicitly point at different crucial aspects of students' learning of mathematics in a new learning environment (de Abreu, Bishop & Presmeg, 2002; Gueudet, 2008; de Guzmán, Hodgson, Robert & Villani, 1998). The transition has also been closely connected with the transition from elementary to advanced mathematics as well as from elementary to advanced mathematical thinking (Artigue, Batanero & Kent, 2007), specifically focusing on students' encounters with mathematical abstraction (Nardi, 2000). Lithner (2003) offers a more specific study of university students' work with textbook exercises. He developed different categories of students' reasoning according to what extent the reasoning is based on more superficial features of the exercise or relies on more intrinsic mathematical qualities in the task. In relation to the transition, the new institutional context where the studying takes place is crucial, as are students' actions and statements in relation to new demands on learning mathematics. This complex relation between institution and subject has

been studied within the anthropological theory of didactics where the transition can be regarded as a movement between different praxeologies (Alveres Dias, Artigue, Jahn & Campos, 2010).

Some common features that can be identified in these previous studies about the transition are changes in the way the mathematical content is treated, an increasing level of abstraction and new demands on students as learners. In this paper, I have chosen to take a very detailed perspective on these issues, zooming in on a specific episode where these features of the transition can be understood in terms of the clash between two different mathematical discourses as one additional perspective on the changes that the transition can bring about. The rationale for this approach lies in the fact that the work with textbook exercises and students' interaction with a university teacher are two main activities in university studies in mathematics, and it is reasonable to think that aspects of the transition are visible in these kinds of situations.

## **SCHOOL MATHEMATICS AND MATHEMATICS AS A SCIENTIFIC SUBJECT AS TWO DISTINCT MATHEMATICAL DISCOURSES**

As an attempt to capture the tensions between secondary and tertiary mathematics, I have chosen to view school mathematics and mathematics as a scientific discipline as two distinct communities of discourses (Sfard, 2007) in order to describe differences between treatment of the mathematical content and the learning situation at a social institution. Because a discourse contains specific rules of how to communicate, it can both be inclusive and excluding. Mathematics can be regarded as a specific kind of discourse. According to Sfard (2007) a mathematical discourse consists of mathematical words, visual mediators, routines and narratives. Interlocutors make use of *mathematical words*, such as variables, unknowns, equations and cases, when communicating about mathematics. *Visual mediators*, such as symbolic artefacts and manipulatives, can be used as a cursor on the objects of communication. *Narratives* are descriptions or accounts of objects. These can be any written or spoken text that is used within the discourse and can be subject to endorsement, i.e. narratives can be judged as true or false. *Routines* refer to repetitive patterns in the interlocutors' actions. Routines can be due to properties of mathematical objects but can also be about the rules of the discourse itself. It is plausible to suppose that the mathematical discourse differs between secondary and tertiary mathematics education. I have chosen to give an account for these differences in terms of school mathematics and mathematics as a scientific discipline as two different mathematical discourses.

Even though the notion of "school mathematics" is frequently used, it seems to lack a scientifically grounded definition, but is used in a more common sense and everyday manner. In this paper, I will associate school mathematics with teaching methods that are commonly used in Swedish secondary school mathematics classrooms. Mathematics lessons starts with the teacher giving a short demonstration of some

standard examples. For the rest of the lesson, the pupils work individually with textbook exercises that are similar to the ones that have been demonstrated at the blackboard. A main focus is to learn what to do, i.e. an emphasis on algorithms and procedures. Mathematics as a scientific discipline concerns mathematics as it is considered within an academic and scientific discourse, for example university courses in mathematics (Robert & Schwarzenberger, 1991). The use of exact definitions, theorems and proofs is significant, which is often communicated through generic examples. Thus, a dichotomy between mathematics studies at secondary and tertiary level can be identified. From a students' perspective, the transition can be regarded as a process of enculturation into the university mathematical discourse, based on their previous knowledge and understanding. The notion of functional understanding is used to describe students' attempts to participate in the new mathematical discourse that they meet at university (Stadler, 2009), i.e. to tell new narratives and undertake new routines within a mathematical discourse that is more in accordance with mathematics as a scientific discipline, even though the students are not yet in all respects ready for it.

## METHOD

The data, presented in this paper, is selected as a critical case from a more comprehensive study about the transition (Stadler, 2009). In the larger study five teacher students in mathematics were studied during their first semester of mathematics studies in Calculus 1 and Algebra during a time period of ten weeks, and Calculus 2 and Linear Algebra during another ten-week period. The students were frequently interviewed and observed during lectures, tutorials and their individual work with textbook exercises. The episode reported in this paper was audio recorded and transcribed in full. Analysis of the transcriptions was inspired by grounded theory (Charmaz, 2006) and used constant comparisons, memos, sorting and categorisation. However, instead of generating own categories, the analysis focused on empirical instances of discursive elements of school mathematics and mathematics as a scientific discipline.

The episode took place during the first week of the linear algebra course. The reason for the choice of this episode was that it constitutes a critical case of the secondary-tertiary transition, understood as a clash between two mathematical discourses. The notion of a critical case comes from Flyvbjerg (2006) and can be defined as “*having strategic importance in relation to the general problem*” (p. 229). The episode revolves around a textbook exercise that requires both routines and the use of new words and narratives and an interaction with the mathematics teacher at university. The dialogue about this exercise between the students and the teacher can be regarded as capturing many aspects of the transition in a single conversation and several empirical instances of an encounter between students' previous experiences of a school mathematical discourse and mathematics as a scientific discourse, offered by the teacher.

## RESULTS AND ANALYSIS

Linear algebra has been regarded as a mathematical domain that seems to be cognitively difficult for students to learn (Dorier & Sierpinska, 2001). Vector spaces and their origins is one area that can be particularly problematic for novice university students. However in the first course in linear algebra at the university where the study took place, the course mainly treats systems of linear equations, matrices, determinants, dot and cross products, vectors, changes of basis, eigenvalues and linear transformations in two and three dimensions.

It is against this background that the task, which Jenny and Ellen are working with, should be regarded. The exercise is number 14 in the first chapter of their textbook in linear algebra. The exercise is formulated as:

*Solve the system of equations for all values of the constants  $a$  and  $b$ .*

$$\begin{cases} x + y + z = 5 \\ x - y + az = 3 \\ 2x + y + z = b \end{cases}$$

The students attempt to solve the task by successive elimination.

The third equation is reduced to:  $-\frac{a+1}{2}z = -9 + b$

To proceed, the students would have to analyse different cases depending on the values of the constants. In particular, whether the constant  $a$  equals minus one or not will generate different types of solutions of the system of linear equations. However, Jenny and Ellen get stuck and ask the teacher for help.

Ellen: We want to find the values of these constants. We understand that if  $a$  equals minus one, then the brackets equals zero, and then  $b$  will equal nine.

Teacher: Yes, well, yes. Well, I think we should start with the basic problem here, because you expressed things slightly incorrect. You said that you should determine these constants,  $a$  and  $b$ . But that is not what you should do. You should, for all values on  $a$  and  $b$ , find the solutions of the system of equations.

The task that Ellen and Jenny are working with puts to the fore some interesting aspects of the transition in relation to previous research. There is an increasing level of abstraction (cf. Nardi, 2000) in comparison to prior tasks in the chapter. It requires the students to simultaneously handle variables, constants and parameters. These unknowns have different meanings and functions for the equations and consequently for the solution of the exercises and should be treated in different ways. From a school mathematical context, students seem used to that unknowns, represented by a letter, should be decided or determined by finding their values. Accordingly, Ellen's initial statement that they want to find the values of the constants can be interpreted as a misunderstanding of the aim with the exercise, but also as an everyday and an operational way to describe how they have worked with the task. New requirements are put on students' reasoning with the textbook exercise. The students must relate to intrinsic features of the mathematical content in the task, rather than just using a

## Working Group 14

standardised algorithm for solving systems of linear equations (cf. Lithner, 2003). The teacher's emphasis on finding solutions of the system for all values on  $a$  and  $b$  can be interpreted as an attempt to highlight this distinction and inviting the students to participate in a partly new mathematical discourse.

Instead of giving the students a more direct answer to their questions, the teacher gives a more comprehensive explanation, which can be interpreted as making an effort to highlight the generic character in the exercise concerning role and function of the constants. Ellen and Jenny ask for a routine for how they should proceed to determine the values of  $a$  and  $b$ . The teacher gives them an account of the role of the constants in the system of linear equations. These different approaches can be interpreted as an expression of the transition, where the teacher focuses on the general character of the mathematical content, while the students are primarily interested in a specific routine for finding a solution of the task. Thus, there is an obvious mismatch between the students' and the teacher's mathematical discourses.

Further work with the task results in the expression:  $z = \frac{18 - 2b}{1 + a}$

Jenny: If you divide with  $a$ , it should not be...

Teacher: Yes, exactly! Go on!

Jenny: Minus one.

Teacher: Yes, that's right. If we want an expression for  $z$ , we must divide by one plus  $a$  on both sides. And then, we have to be careful when, as you said, Jenny...

Jenny: ...when  $a$  equals minus one.

Teacher: Yes. Exactly. So you must start looking at... But, then you must not say that  $a$  is or must not equal minus one. You should examine both cases. If  $a$  does not equal minus one, then one plus  $a$  does not equal zero and then there is no problem with division and we can get an expression for  $z$ . But you must not stop there, you must also examine the case when  $a$  equals minus one.

That division by zero is not an acceptable mathematical operation seems familiar to Jenny, but it is a considerable abstraction from knowing this for natural numbers to apply it on a rational algebraic expression. Jenny's use of knowledge in relation to a very local mathematical context, namely the algebraic expression at hand, can be regarded as an empirical instance of functional understanding. At the same time, the teacher attempts to involve the more encompassing ideas of solving systems of linear equations and how a division with zero should be interpreted in that context.

The students and the teacher re-write the system of equations for the case when  $a = -1$  and solve it.

Teacher: Okay, and now  $a$  equals minus one. Let me write a little bit here. That equals minus two  $z$  there [refers to the second reduced equation]. And in the last equation, well, if  $a$  equals minus one, the left hand side will equal zero.

## Working Group 14

And then we have: zero equals minus nine plus  $b$  [ $0 = -9 + b$ ]. So, re-write the system of equations once again for the case when  $a$  equals minus one, and use that as a point of departure for thinking. Which situation do we have now according to the last equation?

Jenny: That one?

Teacher: Yes. What conclusions can you draw?

Jenny: That if  $a$  equals minus one, then  $b$  equals nine.

Teacher: Well, we agreed that  $a$  and  $b$  should be free. You must not say that  $b$  must equal nine. If I ask you the question, if  $b$  equals ten?

Ellen: Then this is not valid. Then we don't have any solutions.

Teacher: No. Exactly. Thus, if  $b$  equals ten, then the last equation says that zero equals one. And the conclusion must be that no solution exists for  $b$  equals ten.

Ellen: Mm.

Jenny: But how do we write this in our solution? Solutions only exist if  $b$  equals nine or what?

Teacher: Yes, but we shall push things one step further and analyse which solutions we get, because we have to organize things in different cases, partial solutions.

Ellen: Oh gosh!

Teacher: But didn't you follow my example?

Ellen: Yes!

The students discuss with the teacher about how they should move on. They consider the case when  $b = 9$  and the system of equations has infinite many solutions.

Teacher: So, an interesting case is when  $b$  equals nine, because then we have  $x$  plus  $y$  plus  $z$  equals five. We have minus two  $y$  minus two  $z$  equals minus two. And finally, the last condition, zero equals zero. So, what about the last equation?

Ellen: We don't need to care about that one.

Teacher: You have to have a parameter there. You'll get infinite many solutions in that case. So, if I say that  $z$  equals four, then  $y$  becomes something and we put it up there and  $x$  becomes something. And if  $z$  equals seven, well, then we'll get a  $y$ -value and then an  $x$ -value.

Regarding the last equation,  $0 = -9 + b$ , the teacher asks the students to analyse and draw conclusions. Jenny's and Ellen's various responses to the teacher can be interpreted as that they are working in a school mathematical context with familiar mathematical routines such as solving equations and writing down complete solutions. Once again, the teacher emphasises that the constants  $a$  and  $b$  do not have

## Working Group 14

any specific values but should be regarded as “free”. As mathematical words within a discourse, these constants get a new meaning. Even though Jenny gets a remark from the teacher not to fix the values  $a$  and  $b$ , her statement “if  $a$  equals minus one, then  $b$  equals nine” could be interpreted as a first sign of understanding the new way of dealing with different cases and treating unknown constants, even if it is just a conclusion of what she thinks the teacher wants her to say. Her question about how the solution should be written indicates a focus on routines that differs from the teacher’s focus but also serves as an attempt to take part in a new way of using narratives and solving a mathematical textbook exercise. At the same time, Ellen manages to draw a correct conclusion about what happens if the last equation ends up in a contradiction or both sides equal zero, but seems to be put under pressure when the teacher wants to push the discussion a little bit further. The teacher constantly continues to push the students towards a more scientific mathematical context, where the solution is not an aim in itself but rather a tool for drawing more general mathematical conclusions.

The students and the teacher discuss the case when  $b$  does not equal nine. The last equation  $0 = -9 + b$  becomes a contradiction, and in this case the system will have no solution. Then the case when  $a$  does not equal minus one is discussed.

Ellen: How do we do with this case when  $a$  does not equal minus one? Shouldn’t we do some kind of “partial solutions”?

Teacher: In general, it depends, but in this case, I can tell, it won’t be needed. Because there you have the equation when  $a$  doesn’t equal minus one. Then you can divide by one plus  $a$  on both sides to get  $z$  and then there won’t be any problems.

Ellen: Sorry, what did you say about... What were we supposed to do here, you said? If... if  $a$  does not equal minus one?

Jenny: Then we can get  $z$  and put it in the...

Ellen: But then we can have a  $z$  that does equal anything and contains both  $a$  and  $b$ ?

Teacher: Yes, and that is not an unnatural thing in any way. If you think about, well, of course the whole system depends on the parameters  $a$  and  $b$ . So it is self-evident that the solutions do so too. Or, well, it isn’t self-evident but it isn’t unnatural either, so to say.

Ellen: Mm...

The teacher leaves the students, who try to finish their work with the exercise.

Ellen: But, should we do something more with this case?

Jenny: I don’t think so.

Ellen: We shouldn’t change it with a  $t$  or something like that?

At first glance, Ellen's final statements can be interpreted as self-deprecatingly ironic, directed towards their mathematical ignorance and inability. However, her comments might also be interpreted as a first attempt to use narratives within a more scientific mathematical discourse. Ellen seems to have grasped that just solving a system of linear equations may sometimes not be enough. Instead, several cases and options may occur during the work. Ellen may not have fully grasped how to deal with different cases, but using the word "case" and trying to fit it in a suitable statement can be regarded as an initial step towards taking part in a more scientifically oriented mathematical discourse.

A main feature in the episode is the discrepancy between what the students regard as their main problem while working with the exercise and the help and explanations that are offered by the teacher. The students seem to work and understand things according to a mathematical discourse, which may be described as school mathematics, based on their previous experiences from upper secondary school. For example, one difficulty for the students is to deal with three different kinds of unknowns; variables, constants and parameters. The students seem to be used to that an expression with unknowns should be solved or treated as an equation. The unknown values should be found, preferably as numerical values. To handle the system of equations by finding  $x$ ,  $y$  and  $z$  and simultaneously taking all possible values of  $a$  and  $b$  into consideration can be a difficult thing to do. A simpler way to think about the different cases is to solve the system for specific values of  $a$  and  $b$ . However, one should not conclude that this is due to students' misconception. Rather, it can be regarded as a pragmatic way to handle a complex situation. What seems to be an immature way to think and act is simply the way that they manage to think and act, because this is what they are familiar with.

### **SUMMARY AND CONCLUDING REMARKS**

In this paper, a critical case was chosen to illustrate the transition as an encounter between a school mathematical discourse and a discourse of mathematics as a scientific discipline. The episode shows several empirical instances of discrepancies between the students' and the teacher's use of different mathematical discursive elements. The students ask for routines that can be used to solve the exercise at hand. The teacher constantly tries to steer focus to more general aspects in the exercise by using words and narratives accordingly. Instead of meeting the students' demands of routines, the teacher offers narratives about the role of the unknown constants.

The mathematical exercise that the students are working with contains an inherent abstraction. On the surface, the exercise seems standardised and ordinary, but demands mixing an algorithmic solution with an extensive analysis of different cases and a new way of talking about the solutions. The switching between different kinds of unknowns also makes the exercise complex, which puts additional demands on students' reasoning (Lithner, 2003). The students encounter situations that they



interpret according to previous experiences and apply well-known routines, when these situations should be regarded and treated in a partly new way.

In the episode, it is obvious that the students and the teacher participate in different mathematical discourses, but it also shows how these students are trying to adapt their reasoning in accordance with how they understand the situation at hand. The students ask for instructions of “what they should do”, but the teacher provides explanations of “how things are” and treats the exercise as a generic example. The teacher puts new expectations and demands on the students when using new discursive elements.

Functional understanding has been defined as students’ attempts to participate in a new mathematical discourse (Stadler, 2009). In the episode, the students’ functional understanding can be identified in empirical instances where the students are trying to use words and narratives in the new discourse even though they are not yet familiar with them. Initially, the students attempt to use routines from a school mathematical discourse, but gradually change their orientations towards new words and narratives, which they are trying to operationalize into new routines within a more scientific mathematical discourse. It can be concluded that the transition from secondary to tertiary mathematics puts new demands on students’ use of mathematical words, their communication with narratives and ways of applying routines. To cope with the new learning environment that the university constitutes and the new mathematical content that they encounter, students both have to learn a partly new kind of mathematics but also learn how to learn mathematics.

The results in this study also contribute to our further understanding of the school to university transition. A recurrent theme in research concerning the transition is the changeover from elementary to advanced mathematical thinking and analysis of students’ learning of mathematics in a new learning environment (Artigue, Batanero & Kent, 2007; de Guzmán, Hodgson, Robert & Villani, 1998), characterising the transition in terms of an increasing level of abstraction (Nardi, 2000) or as a movement between different praxeologies (Alveres Dias, Artigue, Jahn & Campos, 2010). In contrast to these directions of research, where individual, sociocultural and institutional aspects are considered on a more general level, the current study does not attempt to give a comprehensive account of the transition. Instead, by making use of Sfard’s discursive framework (Sfard, 2007), it provides a fine-grained analysis of a critical case for the transition as an encounter between a school mathematical discourse and a discourse of mathematics as a scientific discipline. Specifically, the current study gives an account for difficulties that may be involved in such an encounter and what enculturation can be for novice students in a specific learning situation – namely, their attempts to participate in a new learning environment and handle partly new demands and conditions.

## REFERENCES

- de Abreu, G., Bishop, A. & Presmeg, N. (2002). Mathematics learners in transition. In G. de Abreu, A. J. Bishop & N. Presmeg (Eds.), *Transitions between contexts of mathematical practices* (pp. 7–21). Dordrecht: Kluwer Academic Publishers.
- Alves Dias, M., Artigue, M., Jahn, A. P. & Campos, T. (2010). A comparative study of the secondary-tertiary transition. In M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the PME 34*, (Vol. 2, pp. 129-136). Belo Horizonte, Brazil.
- Artigue, M., Batanero, C. & Kent, P. (2007). Mathematics Thinking and Learning at Post-Secondary Level. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the national council of teachers of mathematics*, (pp. 1011-1050). Charlotte, NC: Information Age Publishing Inc.
- Charmaz, K. (2006). *Constructing grounded theory – a practical guide through qualitative analysis*. London: SAGE Publications.
- Dorier, J.-L. & Sierpiska, A. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 255–273). Dordrecht: Kluwer Academic Publishers.
- Flyvbjerg, B. (2006). Five misunderstandings about case-study research. *Qualitative Inquiry* 12(2), 219-245.
- Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67, 237-254.
- de Guzmán, M., Hodgson, B.R., Robert, A. & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. *Documenta Mathematica. Extra volume ICM 89, volume III*, 747-762.
- Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, 52, 29–55.
- Nardi, E. (2000). Mathematics undergraduates' responses to semantic abbreviations, 'geometric' images and multi-level abstractions in group theory. *Educational Studies in Mathematics*, 43, 169–189.
- Robert, A. & Schwarzenberger, R. (1991). Research in teaching and learning mathematics at an advanced level. In: D. Tall, *Advanced mathematical thinking* (pp. 127–139). Dordrecht: Kluwer Academic Publishers.
- Sfard, A. (2007). When the rules of discourse changes, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Science*, 16(4), 565-613.
- Stadler, E. (2009). *Stadieövergången mellan gymnasiet och universitetet: matematik och lärande ur ett studerandeperspektiv*. (The transition secondary school to university: Mathematics education as seen from a student perspective). Acta Wexionensia nr 195/2009. Växjö: Växjö University Press.

# **STUDENTS' CONCEPTIONS OF FUNCTIONS AT THE TRANSITION BETWEEN SECONDARY SCHOOL AND UNIVERSITY**

Fabrice Vandebrouck

Université Paris-Diderot, Laboratoire de Didactique André Revuz

*In this paper, I would like to introduce the concept of perspective and to stress the importance of dealing with perspectives on functions for students entering university . This theoretical concept is meant as complementary to the process/object duality, enriching thinking in the passage from the conceptual embodied world to the formal axiomatic one. Drawing on a typical task of the transition between secondary school and university, I point out the difficulties for students to solve tasks when algebraic techniques are not sufficient.*

Key words: mathematics, functions, university students, concept image, perspective

In this article I want to investigate one problem that arises in the transition between secondary school and university concerning the concept of function. This area has been extensively studied for several decades but the context of the teaching situation, always changing, justifies that the interest is continuously renewed.

## **SPECIFICITIES OF THE TRANSITION BETWEEN SECONDARY SCHOOL AND UNIVERSITY**

Many studies have already indicated the characteristics of this transition (Artigue 1991, 2007; Guedet, 2008). From an institutional perspective, many macro-ruptures can be identified: a shift from a course with one teacher to lectures and tutorials; acceleration of teaching time with a rapid turnover of content and lessons with faster assimilation; shorter presence of teachers; lack of familiar problems from secondary school; and, wider range of tasks that make their internalization much more difficult than in secondary school, the latter being delegated to personal work of students, who must therefore be more autonomous in their learning.

Robert (1998) also noted a distribution between the types of mathematical notions which is different from secondary school to university, especially the emergence at university level of new mathematical notions carrying a high level of formalism and generalization. She also pointed out the differences in the level and the nature of tasks (necessity of available knowledge, necessity of flexibility in this knowledge, for instance use of different settings and representation systems, new requirements in term of proofs at the university level, etc.).

## **THE COMPLEX NOTION OF FUNCTION**

The notion of function is at the intersection of several mathematical fields (real numbers, limits, algebra, etc...), appears in many frames (Douady 1986) and requires

the consideration of several representation systems (graphical, algebraic, symbolic etc.) (Duval 1991). Functions are therefore complex objects which are still being learnt when students enter university.

The teaching and learning of functions has been studied through many different theories: Tall and Vinner (1981) introduced the distinction between concept image and concept definition, the concept image being generally different from the concept definition, especially for functions. Following Bachelard (1938), Sierpinska (1992) used the notion of epistemological obstacles regarding some properties of functions and especially the concept of limit. Another approach is based on the processes / object duality (Dubinsky 1991): the conceptualization begins with actions on previously constructed mental or physical objects. Then actions are interiorized to form processes which are then encapsulated to form objects. Sfard (1991) also claims that the abstract concepts can be conceived as two different forms: structurally, as objects, and operationally, as processes, the two views being complementary. Tall (1996) also adds the perspective of procept, an amalgam of two components: a process which produces a mathematical object and a symbol that represents at the same time the process and the object. With respect to functions, algebraic or graphical representations are procepts which can both be handled as processes and as objects. Finally, of relevance to these works are Tall's (2004) three worlds of mathematics: in this perspective growth of mathematical thinking occurs as a transition from a conceptual-embodied world to a proceptual-symbolic world and then to a formal-axiomatic world.

With regard to the teaching of functions, I claim that the transition between secondary school and university can be interpreted, in some sense, as a way to move from the conceptual-embodied world to the formal axiomatic one, through embedding a higher level of conceptualization of the notions related to the domain of analysis: indeed, the beginning of the teaching at the university level corresponds to a displacement from functional thinking to set-theoretical thinking, from utilizations of most functions as processes to utilizations as objects, with a higher degree of formalisation and a balancing in the utilization of procepts, especially the use of graphical representations moves from an object's role to a tool for supporting formalizations and proofs.

On the other hand, Balacheff and Gaudin (2002) find two types of concept image (they speak about conceptions) among pairs of students who finish secondary school: a "curve – algebraic" concept image for which functions are primarily special cases of plane curves, those having a specific algebraic form; and, an "algebraic – graphic" concept image for which a function is first an algebraic formula, the associated graph coming after. It seems that the concept image of average students is closer to the latter. They are unable to manipulate functions which are not given in algebraic forms. Moreover, these students cannot easily shift from the algebraic representation system to another. As Coppé *et al.* (2007) stress, that the current practice of teaching

in secondary schools (in France) seems to reinforce the idea that a function is only an object belonging to the algebraic frame.

### THREE PERSPECTIVES ON THE OBJECT OF FUNCTION

In addition to the above works on students' concept images of functions, I consider the notion of perspectives (Rogalski, 2008). More precisely, I claim that different perspectives can be adopted on the object of function: a point-wise perspective, a global perspective and a local perspective. This distinction enriches the different levels of conceptualization introduced above.

Indeed, the balance from conceptual embodied world to the formal axiomatic one is accompanied by the development of local properties about functions: limit, continuity, differentiability, equivalent expressions, Taylor's local expansions near some points which are the basic notions of calculus. I claim that working at university level on functions implies that students can adopt a local perspective on functions whereas only point-wise and global perspectives are constructed at the secondary school. In this paper, I would like to explain some difficulties of novice students at university through associating them with their difficulties to adopt point-wise and global perspectives on functions. These difficulties appear when they are asked to solve tasks where techniques of the algebraic frame are not sufficient. Let me outline what I understand with this notion of perspective.

In the first perspective, functions are considered as correspondences between two sets of numbers, an element of the first set being associated with a unique element of the second set. This point-wise perspective is in accordance with the definition of functions given in textbooks at grade 9 in France, four years before the beginning of university. At this level, functions are represented by numerical formulas that operate as a program for calculation, such as calculus programming. A table of values is also a good representation of a function from this perspective, especially for pupils who are aware only of integers on the real line.

The second perspective is the global one, necessary to understand the notion of variation and to interpret properties such as parity or periodicity. As pointed out by Coppé *et al.* (2007), the table of variation is a good representation of a function from this global perspective. The graphical representation of a function can be manipulated from a point-wise perspective as well as a global perspective but the algebraic expression can represent a function from a global perspective only for experts. Indeed, for students, interpreting an algebraic formula as a function from a global perspective seems relatively natural only for classical functions  $x^2$ ,  $\sqrt{x}$  whose global properties are well known. For more complex algebraic formulas, the most natural perspective is the point-wise one. So, the use of graphical representations should allow easy connections between perspectives. However, a large algebraisation of tasks at the end of the secondary school tends to limit which perspective can be adopted on functions.

The third perspective is the local one. Even though the notions of limit, continuity and differentiability of functions are introduced in secondary school, the local perspective seems necessary only at university level. Again, problems of continuity or differentiability are introduced in an algebraic way and they consist mainly in calculating limits by algebraic rules. So, the algebraisation of tasks erases the point-wise and global perspective and moreover it doesn't allow reaching out to the local perspective.

Finally, I assume that reaching an object level understanding of function is not the only challenge of the secondary school. In fact, I assume that students must be able to articulate different perspectives on functions and especially have to adopt the global perspective on functions to overcome all the obstacles which come with local notions at the beginning of the university: structure of the real line, notion of equality between two real numbers, etc..

### A TEST TO TRACK ABILITIES FOR ADOPTING PERSPECTIVES

In order to diagnose the difficulties that students have with adopting the point-wise and global perspectives on functions, I designed a task for which techniques of the algebraic frame were not sufficient. More precisely, this task was dealing with the function  $G$  below, defined by an integral:

$$G(x) = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt.$$

In secondary school, as well as in the beginning of the university, integrals are defined as areas under curves of functions, and are approached as definite integrals. However, in the two institutions, the link between integrals of continuous and primitive functions is made quickly and students work mostly in the algebraic frame. For this link, the so called fundamental theorem of calculus is usually proven by the teacher, when the function to integrate is continuous, positive and strictly increasing.

I decided to investigate this function  $G$  because in the two institutions its study is very close to well known tasks and its level of difficulty is accessible for both secondary school students and students university. Indeed, even if they are not familiar with this kind of function  $G$ , students of both institutions had already met tasks concerning indefinite integrals over intervals of the form  $[a, x]$  or  $[a, \beta(x)]$  ( $\beta$  being a linear function) at the time of the experimentation. Moreover, questions about global and point-wise properties of  $G$  according to properties of  $f$  are more interesting in the context of this kind of integral (between  $x-1$  and  $x+1$ ), as we will see below.

The experimentation was dealing with one group of students from secondary school (15 students) and one group of students from university level (109 students from University Paris Diderot). The precise statements proposed to students were chosen

## Working Group 14

by their teachers (from one side in a secondary school and on the other side in the university) with instructions for treating questions concerning global or/and point-wise properties of  $G$ .

The beginning of the test given by the teacher at the university level was the following one (a fifth question concerned a local property of limit):

Let  $f$  be a continuous function over  $\mathbf{R}$  and  $G$  the function defined over  $\mathbf{R}$  by

$$G(x) = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt.$$

- 1) Show that if  $f$  is a constant function,  $G$  is also a constant function.
- 2) Show that if  $f$  is even (respectively odd),  $G$  is even (respectively even).
- 3) Show that  $G$  is differentiable and compute  $G'$ .
- 4) Compute  $G$  when  $f$  is defined by  $f(t)=|t|$ .

$f$  is assumed to be continuous. Students have to show global properties of  $G$ . They must also prove that  $G$  is differentiable and compute  $G'$ . Questions 1) and 3) can be treated only in the algebraic frame without any perspective on  $f$  and  $G$ . Problems with perspectives can appear with question 2) – if  $f$  even then also  $G$  is even – and for question 4) – find  $G$  when  $f$  is the absolute value. We will see this below with examples of student responses.

The test given by the teacher at the secondary school level was the following:

In the problem,  $\mathbf{D}$  means the set of differentiable functions on  $\mathbf{R}$ . For each  $f$  in  $\mathbf{D}$ , we define  $G$  such that for all  $x$  in  $\mathbf{R}$ ,

$$G(x) = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt.$$

- 1a) Show that for each primitive  $F$  of  $f$  over  $\mathbf{R}$ ,  $G(x) = \frac{1}{2} [F(x+1) - F(x-1)]$ .
- 1b) Compute  $G$  when  $f$  is defined by  $f(t)=t^n$ ,  $n$  integer greater than 1. Show that if  $f$  is a polynomial function, then  $G$  is a polynomial function with the same degree.
- 1c) Compute  $G$  when  $f$  is defined by  $f(t)=\cos(\pi t)$ .
- 2a) Show that for all  $f$  in  $\mathbf{D}$ ,  $G$  is also differentiable over  $\mathbf{R}$ , and that for all  $x$  in  $\mathbf{R}$ ,  $G'(x) = \frac{1}{2} [f(x+1) - f(x-1)]$ .
- 2b) Show that the following properties are equivalent: (1)  $G$  is constant and (2)  $f$  is periodic with period 2.
- 3a) Suppose  $f$  is increasing over  $\mathbf{R}$ . Show that  $G$  is increasing and that for all  $x$  in  $\mathbf{R}$ ,  $f(x-1) \leq G(x) \leq f(x+1)$ .
- 3b) Suppose  $f$  is defined by  $f(t) = 4 \exp(t) / [t^2 + 4]$ . Study the variations of  $f$  over  $\mathbf{R}$ . Deduce the variations of  $G$  over  $\mathbf{R}$ .

In this exercise, students can work inside the algebraic frame from questions 1) to question 2a) and also for question 3b). It means that the conception of functions as objects belonging to a functional frame (with all its complexity) seems irrelevant. The issues I want to study can appear with question 2b) –  $G$  is constant if and only if



$f$  is 2 periodic - and 3a) – if  $f$  increases then  $G$  increases and for all  $x$ ,  $f(x-1) \leq G(x) \leq f(x+1)$ . We will also see this below.

## RESULTS AND EXAMPLES OF STUDENT RESPONSES

As I was interested in the transition between secondary school and university level, I chose to analyze only responses of secondary school students who were expected to enter university. Only five responses were analyzed. On the other hand, as I was interested only in qualitative results, the analysis of students' responses at university level was done in a separate round, in order to find the characteristics which have been identified in the five responses of the school students.

As expected, most of secondary school students' difficulties were about question 2b) and question 3a). Only one student succeeded in these two questions whereas, except for some minor errors and the second part of the question 1b) (which is more difficult), all of them succeeded in the other questions. I suppose that algebraic techniques are not sufficient to succeed in these tasks. There is a necessity to surpass the algebraic frame and to adopt global and point-wise perspectives on  $f$  and on  $G$ .

For instance, in question 2b), students have to establish global properties on  $f$  and  $G$  –  $f$  is 2 periodic and  $G$  is constant - through considering a point-wise property for  $f$  – for all  $x$ ,  $f(x-1) = f(x+1)$ . Here is a typical response:

<p>2b.</p> <p>(1) <math>f</math> est une fonction constante équivaut à <math>f'(x)=0</math>  (2) <math>f</math> est périodique de période 2 équivaut à <math>f(x)=f(x+2)</math></p> <p>On part donc d'un membre pour arriver à l'autre</p> <p><math>f'(x) = \frac{1}{2}(f(x+1)-f(x-1))</math> et <math>f'(x)=0</math></p> <p>donc <math>\frac{1}{2}(f(x+1)-f(x-1))=0</math> (1)</p> <p><math>\Leftrightarrow f(x+1)-f(x-1)=0</math></p> <p><math>\Leftrightarrow f(x+1)=f(x-1)</math></p> <p><math>\Leftrightarrow f(x+1)+f(1)=f(x-1)+f(1)</math></p> <p><math>\Leftrightarrow f(x+2)=f(x)</math> (2)</p> <p>donc (1) <math>\Leftrightarrow</math> (2).</p>	<p>(1) <math>G</math> is constant <math>\Leftrightarrow G'(x)=0</math>  (2) <math>f</math> is 2 periodic <math>\Leftrightarrow f(x)=f(x+2)</math></p> <p>We start from one side to go to the other:</p> <p><math>G'(x) = \frac{1}{2}[f(x+1)-f(x-1)]</math> and <math>G'(x)=0</math>  ...  <math>\Leftrightarrow f(x+1)=f(x-1)</math>  <math>\Leftrightarrow f(x+1)+f(1)=f(x-1)+f(1)</math>  <math>\Leftrightarrow f(x+2)=f(x)</math> (2) so (1) <math>\Leftrightarrow</math> (2).</p>
---	---

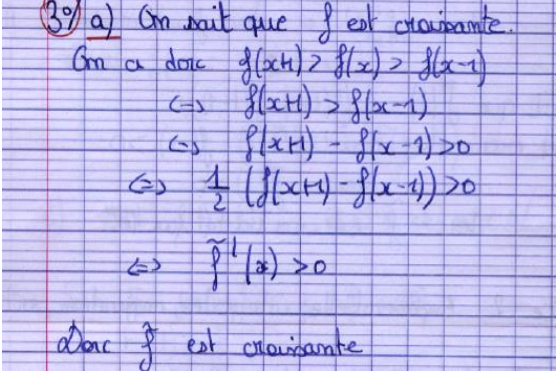
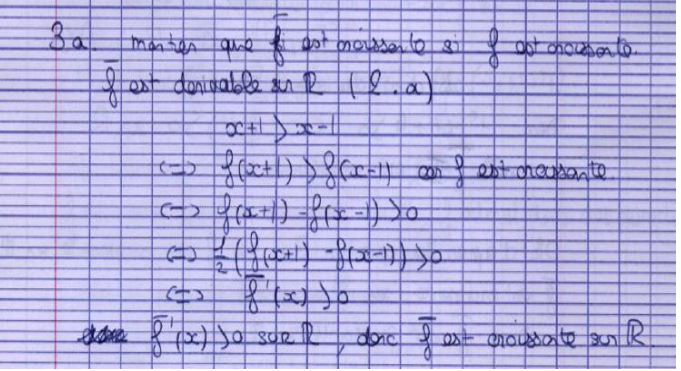
**Figure 1: example of response for question 2b) – secondary school student**

This student explained his procedure as follows: « on part d'un membre pour arriver à l'autre membre » (« we start from one side to go to the other »). There is no quantification, useful to translate the global properties for  $f$  and  $G$  in the formal language. Moreover, equivalences are wrong. The student can not recognize the property of periodicity with the statement  $f(x-1) = f(x+1)$ . It is necessary for him to formulate  $f(x) = f(x+2)$ . Then, all algebraic techniques seem to be good: here he adds  $+f(1)$  to each side. I suppose that this student is unable to reach a global perspective on  $f$  and  $G$ . His reasoning seems to be in the algebraic frame only.



## Working Group 14

On the first part of question 3a), students have to establish a global property on  $G$  –  $G$  is increasing – from a global property on  $f$  –  $f$  is increasing – through point-wise properties – for all  $x$ ,  $f(x-1) < f(x+1)$  and for all  $x$ ,  $G'(x) > 0$ . But, in four of the five students' responses, these interpretations are again done formally without any quantification, in an algebraic way. Students seem unable to see the necessity of two variables  $x$  and  $y$  in order to write the property of growth. They use equivalences which are wrong. Again, I think that the reasoning is only at an algebraic level, not at all in the functional frame.

 <p>3a) On sait que <math>f</math> est croissante.  On a donc <math>f(x+1) &gt; f(x) &gt; f(x-1)</math>  <math>\Leftrightarrow f(x+1) &gt; f(x-1)</math>  <math>\Leftrightarrow f(x+1) - f(x-1) &gt; 0</math>  <math>\Leftrightarrow \frac{1}{2} (f(x+1) - f(x-1)) &gt; 0</math>  <math>\Leftrightarrow \tilde{f}'(x) &gt; 0</math>  donc <math>\tilde{f}</math> est croissante</p>	 <p>3a) montrer que <math>\tilde{f}</math> est croissante si <math>f</math> est croissante.  <math>\tilde{f}</math> est dérivable sur <math>\mathbb{R}</math> (l.a.)  <math>x+1 &gt; x-1</math>  <math>\Leftrightarrow f(x+1) &gt; f(x-1)</math> car <math>f</math> est croissante  <math>\Leftrightarrow f(x+1) - f(x-1) &gt; 0</math>  <math>\Leftrightarrow \frac{1}{2} (f(x+1) - f(x-1)) &gt; 0</math>  <math>\Leftrightarrow \tilde{f}'(x) &gt; 0</math>  donc <math>\tilde{f}'(x) &gt; 0</math> sur <math>\mathbb{R}</math>, donc <math>\tilde{f}</math> est croissante sur <math>\mathbb{R}</math>.</p>
<p>3a) We know that <math>f</math> is increasing.  So <math>f(x+1) &gt; f(x) &gt; f(x-1)</math>  <math>\Leftrightarrow f(x+1) &gt; f(x-1)</math>  <math>\Leftrightarrow f(x+1) - f(x-1) &gt; 0</math>  <math>\Leftrightarrow \frac{1}{2} [f(x+1) - f(x-1)] &gt; 0</math>  <math>\Leftrightarrow G'(x) &gt; 0</math>  So <math>G</math> is increasing.</p>	<p>3a) <math>G</math> is differentiable over <math>\mathbb{R}</math> (2a)  <math>x+1 &gt; x-1</math>  <math>\Leftrightarrow f(x+1) &gt; f(x-1)</math> because <math>f</math> is increasing  <math>\Leftrightarrow f(x+1) - f(x-1) &gt; 0</math>  <math>\Leftrightarrow \frac{1}{2} [f(x+1) - f(x-1)] &gt; 0</math>  <math>\Leftrightarrow G'(x) &gt; 0</math>  <math>G'(x) &gt; 0</math> over <math>\mathbb{R}</math>, so <math>G</math> is increasing over <math>\mathbb{R}</math>.</p>

**Figure 3: examples of responses for question 3a) – secondary school students**

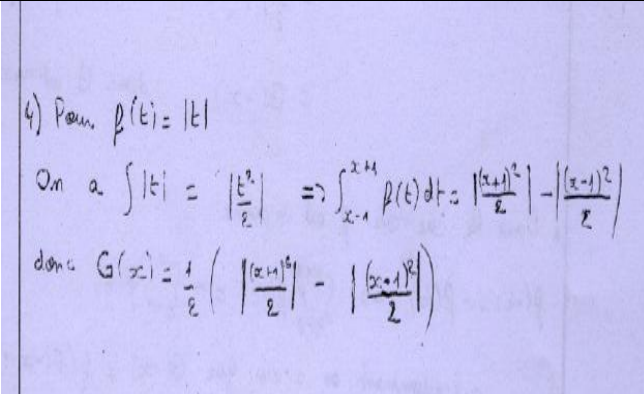
The second part of the question 3a) – to prove  $f(x-1) \leq G(x) \leq f(x+1)$  – is the most difficult task. No student really succeeded in this question. The difficulty seems to be linked to the necessity to adopt a point-wise perspective on  $G$  – the computation of  $G(x)$  for a fixed  $x$  – together with a global perspective on  $f$  – for all  $t$  in  $[x-1, x+1]$ ,  $f(x-1) < f(t) < f(x+1)$ .

This difficulty with perspective appears also in many responses of university students, concerning questions 2) and question 4) of the university test, as it was expected.

In question 4) (university test), there is a necessity to treat several cases according to the fact that 0 belongs or not to  $[x-1, x+1]$ , that is to say  $x < -1$ ,  $x$  in  $[-1, 1]$  or  $x > 1$ . Students must adopt a point-wise perspective on  $G$  – computation of  $G(x)$  for  $x$  fixed – and a global perspective on  $f$  over  $[x-1, x+1]$ . However, many students treat the task at an algebraic level, thinking for instance that the absolute

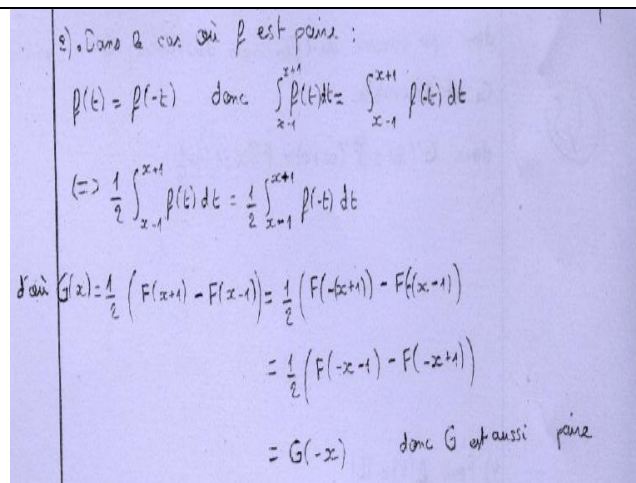
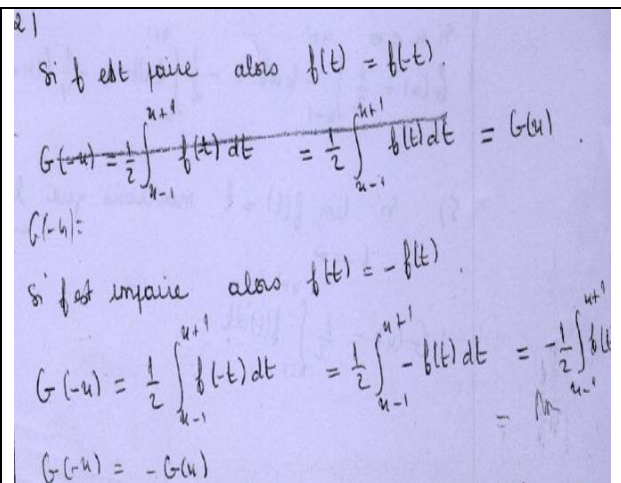
## Working Group 14

value can be integrated without adopting these perspectives. Figure 4 represents a typical example of this mode of reasoning in the algebraic frame:

 <p>4) Pour <math>f(t) =  t </math>  On a <math>\int  t  = \left  \frac{t^2}{2} \right  \Rightarrow \int_{x-1}^{x+1} f(t) dt = \left  \frac{(x+1)^2}{2} \right  - \left  \frac{(x-1)^2}{2} \right </math>  donc <math>G(x) = \frac{1}{2} \left( \left  \frac{(x+1)^2}{2} \right  - \left  \frac{(x-1)^2}{2} \right  \right)</math></p>	<p>4) For <math>f(t) =  t </math></p> <p>We have <math>\int  t  =  t^2 / 2  \Rightarrow</math>  <math>\int_{x-1}^{x+1} f(t) dt =  (x+1)^2 / 2  -  (x-1)^2 / 2 </math>  So <math>G(x) = \frac{1}{2} [  (x+1)^2 / 2  -  (x-1)^2 / 2  ]</math></p>
---	---

**Figure 4: examples of response for question 4) – university student**

In students' responses for question 2) (university test), the same kind of observations can be made. Most of students translate the global property –  $f$  even – without the quantification –  $f(t) = f(-t)$ . Again, the reasoning seems to be in the algebraic/formal frame in many responses as in figure 5:

 <p>2) Dans le cas où <math>f</math> est paire :  <math>f(t) = f(-t)</math> donc <math>\int_{x-1}^{x+1} f(t) dt = \int_{x-1}^{x+1} f(t) dt</math>  <math>(\Rightarrow) \frac{1}{2} \int_{x-1}^{x+1} f(t) dt = \frac{1}{2} \int_{x-1}^{x+1} f(-t) dt</math>  donc <math>G(x) = \frac{1}{2} (F(x+1) - F(x-1)) = \frac{1}{2} (F(-x-1) - F(-x+1))</math>  <math>= \frac{1}{2} (F(-x-1) - F(-x+1))</math>  <math>= G(-x)</math> donc <math>G</math> est aussi paire</p>	 <p>2)  Si <math>f</math> est paire alors <math>f(t) = f(-t)</math>  <math>G(-x) = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt = G(x)</math>  <math>G(-x) =</math>  Si <math>f</math> est impaire alors <math>f(t) = -f(-t)</math>  <math>G(-x) = \frac{1}{2} \int_{x-1}^{x+1} f(t) dt = \frac{1}{2} \int_{x-1}^{x+1} -f(-t) dt = -\frac{1}{2} \int_{x-1}^{x+1} f(t) dt</math>  <math>G(-x) = -G(x)</math></p>
<p>In the case <math>f</math> even:  <math>f(t) = f(-t)</math> so <math>\int_{x-1}^{x+1} f(t) dt = \int_{x-1}^{x+1} f(-t) dt</math>  <math>\Leftrightarrow \frac{1}{2} \int_{x-1}^{x+1} f(t) dt = \frac{1}{2} \int_{x-1}^{x+1} f(-t) dt</math>  so <math>G(x) = \frac{1}{2} [F(x+1) - F(x-1)] = \frac{1}{2} [F(-(x+1)) - F(-(x-1))]</math>  <math>= \frac{1}{2} [F(-x-1) - F(-x+1)] = G(-x)</math>  So <math>G</math> is also even.</p>	<p>If <math>f</math> even then <math>f(t) = f(-t)</math>  <math>G(-x) =</math>  If <math>f</math> odd then <math>f(-t) = -f(t)</math>  <math>G(-x) = \frac{1}{2} \int_{x-1}^{x+1} f(-t) dt</math>  <math>= \frac{1}{2} \int_{x-1}^{x+1} -f(t) dt = -\frac{1}{2} \int_{x-1}^{x+1} f(t) dt</math>  <math>G(-x) = -G(x)</math></p>

**Figure 5: examples of responses for question 2) – university students**

Few responses (about 25%) show the ability for students to adopt a point-wise as well as a global perspective on the manipulated objects. Because of the brevity of this paper, it is impossible to report about them.

## CONCLUSION

In this paper, I wanted to stress the importance of dealing with point-wise and global perspectives on functions for students entering university. I have claimed that this distinction enriches the process / object duality and the students' way of thinking in the passage from the conceptual embodied world to the formal axiomatic one. Through a typical task of the transition between secondary school and university – the study of the function  $G$  – I have pointed at the difficulties for students to solve tasks when algebraic techniques are not sufficient. On one hand, I think that these difficulties are linked to the students' inability to consider functions as complex objects with point-wise as well as global properties. On the other hand, I can think that these difficulties are increased by the current practice of teaching in secondary schools in France, which reinforces tasks belonging to the algebraic frame only (computations of limits, derivative, tracing graphs as objects, not as tool for reflections on tasks...) and which erases the perspective which can be adopted on these objects. In particular, tasks belonging to the graphical register are reduced while they could enrich students' perspectives on functions.

Moreover, I have claimed in this paper that these difficulties with point-wise and global perspectives on functions can be related with difficulties for students to enter on one hand in the formal axiomatic world and on the other hand to develop the local abilities which are necessary at the beginning of the university. I will continue to investigate these ideas in the future by designing a questionnaire for university students which will explore the students' perceptions of local and formal properties (the utilization of the formal definition of limit for instance). On the other hand, I will introduce tasks at the end of secondary school using new technologies to focus on the graphical register of functions, not only the algebraic one, to investigate local problems such as continuity and differentiability problems on functions.

## REFERENCES

- Artigue, M. (1991). Analysis. In D. Tall (Ed.) *Advanced mathematical thinking* (pp. 167-198). Dordrecht: Kluwer Academic Press.
- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1011-1049). Greenwich, Connecticut: Information Age Publishing, Inc.
- Bachelard, G. (Ed.) 1938. *La formation de l'esprit scientifique* Paris: Vrin.

- Balacheff, N., & Gaudin, N. (2002). *Students' conceptions: An introduction to a formal characterization* Cahier Liebnitz (No. 65). Grenoble: Université Joseph Fourier.
- Coppé, S., Dorier, J-L., & Yavuz, I. (2007). De l'usage des tableaux de valeurs et des tableaux de variations dans l'enseignement de la notion de fonction en France en seconde. *Recherche en Didactique des Mathématiques*, 27 (2), 151-186.
- Douady, R. (1986). Jeux de cadre et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7 (2), 5-31.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking Trans.). In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Dordrecht: Kluwer Academic Press.
- Duval, R. (1991). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de didactique et de sciences cognitives*, 5, 37-65.
- Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67, 237-254.
- Robert, A. (1998). Outil d'analyse des contenus mathématiques à enseigner au lycée et à l'université. *Recherches en Didactique des Mathématiques*, 18 (2), 139-190.
- Rogalski, M. (2008). Les rapports entre local et global : Mathématiques, rôle en physique élémentaire, questions didactiques. In L. Viennot (Ed.), *Didactique, épistémologie et histoire des sciences* (pp. 61-87). Paris: PUF.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: On processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy*, Mathematical Association of America Notes, volume 25.
- Tall, D. (1996). Functions and calculus. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 289-325). Dordrecht: Kluwer Academic Publishers.
- Tall, D. (2004). *Thinking through three worlds of mathematics*. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281-288). Bergen, Norway.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.

# DISCOURSES OF FUNCTIONS – UNIVERSITY MATHEMATICS TEACHING THROUGH A COMMUNICATIVE LENS

Olov Viirman

University of Gävle & Karlstad University, Sweden

*This paper reports on an ongoing study focusing on the teaching of functions in undergraduate courses in mathematics at three Swedish universities. In this paper excerpts from the lectures of three teachers at one university are analysed, using communicative theory. Characteristic features of the teachers' discourses about functions are presented. Definitions are found to be the central type of narrative, while theorems and proofs are largely absent, despite the fact that the teaching is of a traditional type, often connected to the "definition-theorem-proof" format. Three main categories of routines are found: substantiation, construction and motivation routines. It is also seen that the teachers are more concerned with questions of "why" to do things than "when" to do them.*

## INTRODUCTION

Students' conceptions of the function concept have been extensively studied (see e.g. Harel & Dubinsky, 1992; Schwarz & Hershkowitz, 1999; Vinner & Dreyfus, 1989). In an earlier study (Viirman, Attorps & Tossavainen, 2011) we looked at a small group of university mathematics students, investigating their concept images of the function concept, and making comparisons with the historical development of the concept. We worked within a theoretical framework building on the idea of concept image, as developed by Tall and Vinner (1981), and also on Sfard's (1991) theories of the process/object duality of mathematical concepts and of the three stages of concept formation. Out of this work grew a desire on my part to continue this research, but with a different focus, leading to the ongoing work of which a first report is given in the present paper. My main interest is now in the teaching, investigating how mathematics teachers at the university level work with the function concept, and what they do to promote the learning of this concept in their students. I believe that there is a need for studies of the actual practice of mathematics teaching at Swedish universities, hopefully gaining insight that might at a later stage be used to help improve university mathematics teaching.

## THEORETICAL FRAMEWORK

Over the last decade or so Sfard has written extensively of acquisition and participation as basic metaphors underlying theories of learning (cf. Sfard, 1998). Through the acquisition metaphor learning is described in terms bringing to mind the accumulation of material goods, while through the participation metaphor learning is seen as the process of becoming a member of a certain community. The framework

used in our earlier study is very much based on the acquisition metaphor. The present work, however, takes a participationist view on learning.

In recent years, Sfard has developed a participationist theory of thinking (Sfard, 2008), drawing on ideas from Vygotsky and Wittgenstein. The foundational tenet of participationism is “that *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual.*” (Sfard 2008, p. 78, *emph. in original*) Based on this idea, Sfard defines thinking as “an individualized version of (interpersonal) communicating” (ibid, p. 81), and coins the neologism *commognition* in order to encapsulate both inter- and intrapersonal communication. Different types of communication are called discourses, and these discourses are in constant development, growing and increasing in complexity. Within the commognitive framework, then, learning may be defined as individualizing discourse, becoming ever more capable at communicating within the discourse, with others as well as with oneself (Sfard 2006, p. 162). This is achieved through a process of adjusting one’s discursive activities to fit the leading discourse (or, more rarely, the other way around). The unit of commognitive analysis is the discursive activity, the “patterned, collective doings” (ibid, p. 157). Hence, what I will be looking at in this study is the discourse of function, as it is manifested in the communicative practices of the teachers (and students). But what characterizes specific discourses? Sfard presents four characteristics which can be used to describe and distinguish different discourses (Sfard, 2008, p. 133ff):

*word use* - words specific to the discourse or common words used in discourse-specific ways

*visual mediators* - visual objects operated upon as a part of the discursive process. Examples from mathematical discourse could be diagrams and special symbols.

*narratives* - Sequences of utterances speaking of objects, relations between and/or processes upon objects, subject to endorsement or rejection within the discourse. Mathematical examples could be theorems, definitions and equations.

*routines* – Repetitive patterns characteristic of the discourse. Typical mathematical routines are for instance methods of proof, of performing calculations, and so on.

A more thorough presentation of the commognitive theory of mathematical discourse is beyond the scope of this paper, but a few words about routines and rules of discourse are needed. The discursive patterns are the result of processes governed by rules. Sfard distinguishes between object-level and meta-discursive rules of discourse. The former regard the properties of the objects of the discourse, while the latter govern the actions of the discursants. A routine, then, is a set of meta-rules describing a repetitive discursive action (ibid, p. 208). This set can be divided into the *how* and the *when* of the routine, determining in the first case the course of action and in the second case the situations in which action would be deemed appropriate.

So, given this, the question this study aims to answer is: What characterizes the discourses about functions presented by the teachers, primarily regarding narratives and routines?

## PREVIOUS RESEARCH

The teaching of mathematics in higher education, while not as well-researched as the function concept, has seen an increase in research activity over the last decade. An early example of research focusing on mathematics teaching at university outside of a teacher education context is the work of Burton (2004), studying professional mathematicians as learners, and possible implications for university mathematics teaching. More recently, the actual practice of mathematics teaching at the university level has earned growing research interest, in particular with the work of Jaworski and Nardi (e.g. Nardi, 2008; Nardi, Jaworski & Hegedus 2005), investigating university mathematicians' views about the teaching of mathematics. There are also a number of other studies, including some (e.g. Weber, 2004; Wood, Joyce, Petocz & Rodd 2007) focusing on so-called traditional mathematics instruction. In Sweden, research on the teaching of mathematics at university is rare, but one example is Bergsten (2007), discussing ways of investigating the quality of mathematics lectures, building on a case study of one calculus lecture on limits of functions.

Since Sfard's commognitive theory is relatively recent, and still under development, not that many studies have been reported using this framework, and those that do exist tend to focus on the mathematical learning of younger children (e.g. Sfard 2001, 2007; Sfard & Lavie 2005) or on elementary mathematics, like arithmetic (Ben-Yehuda, Lavy, Linchevski & Sfard, 2005). However, very little has been published on university mathematics learning from a commognitive standpoint. The one example that I'm aware of is the work of Ryve (2006), which makes use of, but also critiques, an early version of Sfard's theory, as presented in for instance Sfard (2001), in order to investigate student interaction in problem-solving. As far as I know, there is yet no published research using commognitive theory to investigate university mathematics *teaching*.

## METHOD

The empirical data in my study consists mainly of videotaped lectures and lessons given by teachers in freshman year mathematics courses at three Swedish universities, chosen for diversity – one old, large university, one more recently established, and one smaller, regional university. The teachers were then selected among those giving freshman courses on relevant topics during the time available for data collection. In two cases, this, together with the obvious fact that the teachers had to agree to participate in the study, effectively made the choice for me. At the large university, where the number of possible participants was greater, I again aimed for diversity, both in topics covered and in teaching experience. One thing all teachers in



the study have in common, however, is an active interest in teaching. In the present paper, I have chosen to focus on three 45-minute excerpts from lectures given at the same university, one of the larger in Sweden. The collection and transcription of data is ongoing, and the excerpts chosen were simply the most extensive transcriptions available at the time of writing. The first excerpt is from an introductory course, mainly preparatory for calculus. The teacher (referred to as teacher A below) is a woman in her fifties, who got her doctoral degree in the 1980's, and has taught at the university for about 20 years. The second is from a course in algebra, and the teacher (B) is a male graduate student in his twenties, giving his first course as a lecturer, having earlier only served as a teaching assistant. Finally, the third is from a course in linear algebra, given by a male teacher (C) in his thirties, having recently gotten his first position following some years of post-doctoral work. The students in all three courses were first semester engineering and computer science students. The excerpts were transcribed verbatim, speech as well as the writing on the board. The transcribed lectures were then analysed, using the four characteristics of discourses described above to try to distinguish the discursive patterns characterizing the teachers' respective discourses of functions. I first analysed each lecture separately, and then looked at all three together, searching for differences and similarities. Since the unit of commognitive analysis is the discursive activity, I have intentionally chosen an outsider perspective, trying to view the enfolding discourse in as unbiased a way as possible. At the same time, I am of course aware of, and also making use of, the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is *not* present in the discourse, except in contrasting different teachers' discursive activities.

## RESULTS

My focus in this paper will be on the narratives and routines characterizing the discourses of the teachers. Observations regarding words and visual mediators used will be referred to whenever they are relevant to the analysis.

A central type of narrative in all three discourses is the *definition*, which can be formal or informal in character, with informal definitions often relying heavily on metaphor. In this respect certain differences between the discourses can be seen. Teacher A consistently introduces new concepts in informal terms, and then presents a formal definition, as in the following example (all excerpts have been translated from Swedish by the author):

Teacher A    maybe it's the whole of B, but maybe it's just a small part of B that actually comes out, that the machine spits out. Then we speak of the domain of  $f$ .

Teacher A    (...) and that you could write like this if you want:  $f$  of  $a$ , the set of all  $f$  of  $a$  when you let  $a$  vary over all [writes:  $\{f(a) : a \in A\}$ ]

Teacher B, however, generally starts by stating a formal definition, and then giving an informal interpretation.



## Working Group 14

Teacher B but if  $C$  is a whole subset of  $A$ , then we define  $f$  of  $C$  as simply the set of all values of the function starting in  $C$ , the set of all  $f$  of  $x$ , where  $x$  lies in  $C$ . And this leads us to our last definition [writes:  $f(A)$  is called the domain of  $f$ ]  $f$  of  $A$  –  $f$  of the whole set  $A$  – is called the domain of  $f$

Teacher B so the domain is all the elements in  $B$  that the function gets to

In fact, in the case of the definition of 'function', he gives no informal interpretation at all. But there is also one case (definition of injectivity) where he introduces the concept informally before giving the formal definition. Teacher C, finally, uses a third approach. He presents the definitions formally in writing, while simultaneously explaining them verbally in an informal fashion, like this:

Teacher C [writes: Definition:  $R^m \xrightarrow{F} R^n$ ] a function from  $R^m$  to  $R^n$  is called a linear map if it is linear, that is if it satisfies two conditions

Teacher C [writes: if 1)  $F(v+w)$ ] the first one is that it respects addition in the following way: if you have two vectors  $v$  and  $w$ , and you add them, and then you apply a function  $F$ , and get vectors in  $R^n$  then that is the same thing as if you take each of the vectors, toss them into  $R^n$  and add them there.

Teacher C [writes:  $=F(v) + F(w)$ ]

Hence, for teachers A and C, the informal definitions provide endorsement for the formal ones. As for teacher B, he often endorses his definitions through reference to a lack or need. Before giving his formal definition of function, he says that “what we haven't learned is how we connect two sets in the sense that to each element in one set we associate an element in the other set”. Interestingly enough, his definition doesn't mention the “one-valuedness” property central to the modern concept of function (to each element in the domain there is exactly one element in the range). Instead he introduces this later, endorsing it partly through another reference to need – without it we wouldn't be able to work with functions at all – and partly by referring to the metaphorical description of the function as a machine.

As for other central narratives of scholarly mathematical discourse, *theorems* and *proofs*, these are much more rare in the discourses of the three teachers. Only teacher C actually states and proves a theorem. Instead the narratives through which the discourses are developed are mainly examples illustrating specific properties. For teacher A, these examples are almost invariably given primarily by formulas. The one exception, a fairly convoluted function, used to illustrate the fact that the rule in the definition of function doesn't require an actual method of computation, is explicitly referred to as a “silly example”. The examples given by teachers B and C are more varied, including geometrically constructed functions (rotations and projections), and functions given by tables of values.

Regarding the routines, the differences in discursive practices are more pronounced. One common category of routines is what we could call *substantiation* routines,

## Working Group 14

aiming at verification or rejection of statements. These often concern the use of definitions. For instance, all three teachers present routines for checking whether a specific example satisfies a given definition. An example, from Teacher B:

Teacher B This function is injective

Teacher B Such a statement, bold as it is, you show by assuming that the values of the function are equal, and from that showing that this implies that  $x_1$  and  $x_2$  are equal

He then goes on using formal algebraic manipulation to show that the last statement follows from the first. On the other hand, in a passage, which unfortunately is too long to be quoted here, teacher C uses graphical representations of plane vectors to show that rotation in the plane by an angle  $\pi/2$  satisfies the two conditions in the definition of a linear map. Having done one vector addition, he then appeals to the geometric intuition of the students:

Teacher C yes, I have just taken that picture and moved it here through rigid rotation, so that parallelogram addition up here has to function just like that one

Teacher C So that rotation satisfies that condition

Both are examples of substantiation routines, but the means of substantiation are different.

Other substantiation routines are those using definitions to exclude non-examples:

Teacher A if we return to this crazy example which wasn't a function, the circle

Teacher A It wasn't a function because even if we insert something between -1 and 1, we would perhaps want that it was a function with  $A$  being -1 to 1, the interval, but then when we insert something which isn't 1 and not -1 [she clearly marks the origin in an already drawn picture of the unit circle in the  $x$ - $y$ -plane] then we get two  $y$ -values that fit, it is different

Teacher A So that the function doesn't give us exactly one, it gives us more than one

Teacher A And then it isn't a function

Here, the teacher uses a graphic argument, directly showing why the circle cannot be the graph of a function. Examples of this kind of routine can be found elsewhere in her lecture, as well as in that of teacher B. They are however not found in the discourse of teacher C.

A second category of routines commonly found in the discourses of the three teachers, but taking very different forms, are *construction* routines, concerned with the construction of discursive objects or narratives. In the discourse of teacher A we can find repeated use of routines for constructing the graph of a function given an algebraic formula, and for finding the value of a function at a point given the graph. These constructions are all very sketchily outlined, however, and it is clearly stated ("we have already talked about this when we did quadratic curves") that this is

## Working Group 14

something which they have been doing for some time. Also there are several instances of routines for determining the range or the largest possible domain of a function given a formula. A typical example:

Teacher A it doesn't say on the board what  $V_f$  is. You have to look, what function is it, and then you have to try to figure out what values can come out here then

Teacher A and in this case, there's a square here, plus 2, squares can be 0 or larger

Teacher A and it can be any number larger than 0 this square, so it can be 2 plus something positive, so it can be any number larger than or equal to 2. So  $V_f$  in this example is the interval starting at 2 and continuing upwards

As we can see, this is something done by looking at the formula, not by for instance looking at the graph, which had been drawn just 20 minutes earlier (although it had now been erased). Routines for determining domain and range are also present in the discourse of teacher B, and handled in pretty much the same way.

The construction routines in the discourse of teacher C are mainly concerned with constructing maps from matrices, and vice versa. This is a central topic of the lecture, being the subject of the theorem which gets stated and proven. In fact, the proof is basically an instance of such a routine. There are variants using vector algebra and geometric reasoning, but both follow this basic pattern:

Teacher C it is easy to solve that type of problem

Teacher C if you know the images of the basis vectors under a certain linear map then you put them as columns, and that is the standard matrix of the map

A general trait of the discourse of teacher C is his frequent use of geometric reasoning. The other two teachers are much more reliant on algebraic methods.

A third, less common category of routines, which however is very much present in the discourse of teacher B, could be called *motivation* routines, dealing with presenting motives for developing new mathematics. The discursive pattern can be described in the following way: OK, so now we know this, but here is something else that we don't know. What would we need in order to find this out, or be able to do this? We would need this. OK, let's define it. We have already seen one example, in the motivation given for introducing the concept of function, and here is another:

Teacher B now that we know what a function is, then maybe one wants to illustrate it in a better way than just writing like this, like in this way you don't see, ok you know what the function does, but you don't really get any idea of what it looks like

This is followed by a description of the “function machine”, and by the definition of the graph of a function. This type of motivational pattern is not found in the discourses of the other two teachers.

## DISCUSSION

First I want to comment on the invisibility of the students in what has been presented here. This is partly due to the fact that the excerpts are from lectures, where students tend to be less active. However, there is a certain amount of teacher-student interaction in the excerpts, some of it very interesting, but an analysis of this is beyond the scope of this paper. Here I will instead give some reflections on what I see as the character of the teachers' discourses.

All three teachers give traditional lectures, speaking and writing at the board. However, the traditional “definition-theorem-proof” (DTP) format (Weber, 2004) is not so apparent, theorems and proofs being largely absent. There are at least two possible reasons for this. Firstly, in all three lectures, the function concept is introduced for the first time in the course, leading to a greater prevalence of definitions and examples, with theorems possibly following later. Secondly, all three courses are first-semester courses, and as such situated somewhere between elementary and advanced courses, perhaps making the DTP format less applicable.

While sharing the overall lecture format, the teachers differ in the discourses they present. These differences are obviously to a great extent predicated by differences in course content. In the discourse of teacher A, teaching to prepare for calculus, the formula is very much the preferred realization of function. This is seen in the way she introduces examples and also in her choice of words. She almost consistently speaks of curves rather than graphs, indicating that what she has in mind is continuous functions. When she does give an example of a function not given by a formula she describes it as “silly”. Teacher B, teaching algebra, uses the language and methods of formal logic in a way neither of the other teachers do. He also presents a much more general idea of what a function might be. I have already mentioned the prevalence of geometric reasoning in the discourse of teacher C (linear algebra). But there are some surprises as well. In Nardi (2008, p. 167), the mathematician (actually a composite portrait of a number of mathematicians interviewed by Nardi) claims that the graph as a realization of function is essential in analysis but meaningless in algebra. Still, teacher B devotes quite some time to the concept of graph, discussing it at a high level of detail. A possible explanation for this could be laying groundwork for the use of functions in other courses. If so, this aim is not made explicit, something which is typical of all three lectures. Connections between different branches of mathematics are seldom made, despite the fact that functions are central to so much of mathematics. This lack is also noted in Nardi (2008, p. 167). I would also like to comment on the process/object duality (Sfard, 1991) of the function concept in the context of the discourses of the three teachers. All three primarily talk of functions in process terms. However, at the same time, they obviously view them as objects:

Teacher A    it is a function; it is the function  $x^2$  which I move one step to the right and two steps up

Teacher C    we speak of the domain of the function, where the function starts (...) and of the target set, where the function is going

It would be interesting to know what effect, if any, this might have on students' learning.

Sfard (2008) speaks of the “how” and “when” of routines. The “when” is seen as more problematic, due to the fact that “Presenting the *when* of routines, that is, constructing exhaustive lists of conditions under which given patterns tend to appear in a discourse of a given group or person, is more complicated, if not altogether unworkable.” (ibid, p. 209) For analytical purposes the questions of “how” and “when” are natural, since they are what can be observed. But in the actual discursive practices of the teachers, the question of “when” to use a certain discursive pattern is mostly addressed by discussing “why” it should be used. Perhaps a reason for this is that explaining why something is done can give more comprehensive criteria of when it should be done, avoiding the problem of presenting conditions of use.

## REFERENCES

- Ben-Yehuda, M., Lavy, I., Linchevski, L & Sfard, A. (2005). Doing wrong with words: What bars students' access to arithmetical discourses. *Journal for Research in Mathematics Education*, 36(3), 176-247.
- Bergsten, C. (2007). Investigating Quality of Undergraduate Mathematics Lectures. *Mathematics Education Research Journal*, 19(3), 48-72.
- Burton, L. (2004). *Mathematicians as Enquirers: Learning about Learning Mathematics*. Dordrecht: Kluwer.
- Harel, G. & Dubinsky, E. (Eds.). (1992). *The Concept of Function: Aspects of Epistemology and Pedagogy*. Washington, DC: Mathematical Association of America.
- Nardi, E. (2008). *Amongst Mathematicians. Teaching and Learning Mathematics at University Level*. New York: Springer.
- Nardi, E., Jaworski, B. & Hegedus, S (2005). A spectrum of pedagogical awareness for undergraduate mathematics: From ‘tricks’ to ‘techniques’. *Journal for Research in Mathematics Education*, 36, 284-316.
- Ryve, A. (2006). *Approaching mathematical discourse: two analytical frameworks and their relation to problem solving interactions*. Department of Mathematics and Physics. Mälardalen University.
- Schwarz, B. & Hershkowitz, R. (1999). Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools. *Journal for Research in Mathematics Education*, 30, 362-389.

- Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics*, 22, 1-36.
- Sfard, A. (1998). On Two Metaphors for Learning and the Dangers of Choosing Just One. *Educational Researcher*, 27(2), 4-13.
- Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, 46, 13-57.
- Sfard, A. (2006). Participationist Discourse on Mathematics Learning. In J. Maasz & W. Schloeglmann (Eds.), *New Mathematics Education Research and Practice* (153-170). Rotterdam: Sense.
- Sfard, A. (2007). When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning From a Commognitive Standpoint. *The Journal of the Learning Sciences*, 16, 565-613.
- Sfard, A. (2008). *Thinking as Communicating. Human Development, the Growth of Discourses, and Mathematizing*. Cambridge, U.K.: Cambridge University Press.
- Sfard, A. & Lavie, I. (2005). Why cannot children see as the same what grown-ups cannot see as different? Early numerical thinking revisited. *Cognition and Instruction*, 23, 237-309.
- Tall, D. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with particular reference to Limits and Continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Viirman, O., Attorps, I. & Tossavainen, T. (2011). Different Views – Some Swedish Mathematics Students' Concept Images of the Function Concept. To appear in *Nordic Studies in Mathematics Education*, 16(1).
- Vinner, S. & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education*, 20, 356-366.
- Weber, K. (2004). Traditional instruction in advanced mathematics courses: a case study of one professor's lectures and proofs in an introductory real analysis course. *Journal of Mathematical Behavior*, 23, 115-133.
- Wood, L. N., Joyce, S., Petocz, P. & Rodd, M. (2007). Learning in lectures: multiple representations. *International Journal of Mathematical Education in Science and Technology*, 38, 907-915.

# SECONDARY-TERTIARY TRANSITION AND EVOLUTIONS OF DIDACTIC CONTRACT:

## THE EXAMPLE OF DUALITY IN LINEAR ALGEBRA

Martine De Vleeschouwer, Unité de didactique des math., Université de Namur

Ghislaine Gueudet, CREAD, IUFM de Bretagne UBO

*This contribution concerns the teaching and learning of duality in linear algebra. Combining an institutional, and a didactic contract perspective we argue that some of the novice students' difficulties can result from specific features of the university contract, at different levels. Analyzing university textbooks, we identify such features, in the case of duality. Drawing on these observations, we design an experimental teaching, aiming to support the students' entrance in the new contract, at different levels. We investigate the impact of this experimental teaching. Analyzing students' productions, we observe that they developed abilities specific to the university contract, concerning duality or more generally mathematics.*

**Keywords:** Secondary-tertiary transition, Linear Algebra, Didactic contract, Institutions

Duality in linear algebra is recognized as an arduous topic for novice students. The general aim of our work is to understand the difficulties they meet in duality, and to propose a teaching of duality likely to overcome these difficulties. Duality can be considered as a content specific of university mathematics, far away from secondary school. It led us to situate our study within the wider issue of secondary-tertiary transition. In previous work (De Vleeschouwer 2010a), we studied the students' difficulties, and proposed categories of difficulties, using an institutional focus. The work we present here corresponds to two new directions of research. On a theoretical level, we propose to consider the change of didactic contract between secondary school and university, and to combine it with the institutional perspective. In our empirical work, we have designed and implemented an experimental teaching intervention, aiming to support the students' entrance in the new didactic contract. We investigate here its impact on the students' outputs.

In part 1, we outline our theoretical propositions, articulating the didactical contract and the institutional perspective. In part 2, we specify our analysis to the context of linear algebra, and especially to duality in linear algebra. We present in part 3 the experimental teaching; we analyse its impact, drawing on students answers to a questionnaire, in part 4.

## DIDACTIC CONTRACT, INSTITUTIONS AND THE SECONDARY-TERTIARY TRANSITION

The notion of didactic contract was introduced by Brousseau (1997), to describe 'a system of rules, mostly implicit, associating the students and the teacher, for a given

piece of knowledge” (Brousseau 1997). Another interpretation of the contract, which is especially relevant in our study, is formulated in terms of sharing responsibility towards knowledge, between the students and the teacher. It seems thereof straightforward to claim, like Artigue (2007), that when a student enters university “the didactic contract is no longer the same”. Several authors retain this perspective to study novice students’ difficulties (Bloch 2005, Grønæk, Misfeldt & Winsløw 2009). Nevertheless, the contract features identified are often very general: the students must show more autonomy, they must be able to develop reasonings involving several frames (Douady 1987) etc. These features seem to characterize general institutional expectations and not a particular mathematical content.

Considering the work of Chevallard (2005) can enlighten this last issue. According to him, a subject encounters a given mathematical knowledge in an institution. The institution frames this knowledge as a mathematical organisation, or praxeology, entailing four components: a type of tasks, a technique to accomplish this type of tasks; a technology, which is a discourse justifying the technique, and a theory. Mathematical organisations exist at several levels, from specific to general.

Considering the didactic contract with this perspective leads to distinguish several levels of contract, in a given institution:

- a general contract, independent of the knowledge at stake (Sarrazy, 2005, terms it the pedagogic contract). For example, at university in some countries attending the courses is not compulsory; taking notes is under the students’ responsibility etc.;
- a didactic contract for mathematics, concerning generally mathematics in the institution: for example, the requirement of rigorous proofs;
- a didactic contract for a given content, concerning particular mathematical notions.

With these distinctions, the main question studied in this article can be formulated as: is it possible to support the students’ entrance in a new contract at different levels, and how? We address this issue in the context of duality in linear algebra. Firstly, we identify features of the didactic contract at university, corresponding to different levels, for the teaching of duality.

## **INSTITUTIONAL DIDACTIC CONTRACT AND DUALITY IN LINEAR ALGEBRA**

We do not consider here the general contract; we start with the level of the didactic contract for mathematics. Considering several research works about transition (Praslon 2000, Bloch 2005, Bosch *et al.* 2004, Winsløw 2008) we retain that the following difficulties of the students correspond to changes of the didactic contract for mathematics, between secondary school and university:

- difficulties with building examples;



- difficulties with working in different frames, with moving between different representations ;
- difficulties with working at the technology-theory level, which means in particular producing a discourse justifying a technique.

According to the authors mentioned above, at university the student is (at least sometimes) responsible for these issues, which were only under the responsibility of the teacher at secondary school.

On a more precise level, about linear algebra and duality, we infer rules of contract by analysing textbooks (De Vleeschouwer 2010*b*).

A central change is that several concepts, in linear algebra, can change status, according to the context. For example, a matrix can be considered as representing a linear function in given bases; it can also be considered as an element of a vector space. A function can be seen as process acting on given objects; it can also be an element of a vector space. This last example is crucial in duality, where the students will have to determine the dual of a given vector space: a set of linear forms. In Belgium where our study takes place, students also encounter matrices and functions at secondary school. But these matrices and functions are not considered as elements of sets. At university, the student must be able to switch between both statuses, which are moreover not explicitly presented.

In 2008-2009 we elaborated and tested a teaching of duality taking into account these features of the contract, both at the discipline level for mathematics and at the content level for duality.

### **SUPPORTING THE ENTRANCE IN A NEW CONTRACT: AN EXPERIMENT AT NAMUR UNIVERSITY**

We present below the main choices made with regard to the experimental teaching. Its focus is on duality, but also develops some prerequisites (as a minimum repertoire of vector spaces). We first want to situate its context, both in terms of the students involved and of teaching organisation.

The University of Namur has set up a device called “springboard operation”, aiming to support novice students, entering university (De Vleeschouwer 2008). It consists in remedial sessions proposed to the students, lasting between 2 and 4 hours each week. The first author of this paper participated as a teacher in the springboard operation, for first year students seeking a Master's degree in mathematics at the university of Namur (26 students in this first year in 2008-2009). She implemented the experimental teaching mainly in the context of this springboard operation (the variety of vector spaces was developed in a group work). This choice is the result of institutional constraints: setting up an experimental teaching in the “normal” course would have been refused by the mathematicians responsible for this teaching. Usually, only some of the students attend the springboard sessions. For the

experimental teaching, all the students were invited to participate; 20 of them finally followed the sessions. Our analysis concerns these 20 students.

Before the teaching of the duality, students had already seen, in the theoretical course and in the exercises concerning algebra, the vector spaces (algebraic structures, linear dependence and dimension, sub-vector spaces); the linear functions and the associated matrices.

The experimental teaching within the springboard operation starts before the teaching of duality with a mandatory group work, aiming to provide the students with a minimum repertoire of vector spaces. Duality itself was tackled one month after the start of the academic year, in October 2008, and the corresponding teaching lasted five weeks:

- during week 1, students received a theoretical course (1.5h) concerning linear forms and dual space. Then they participated in an activity, which purpose is to make students aware of the various statuses that a matrix may have in linear algebra: element of a group, a ring, a vector space or representing a linear function;
- during week 2, we proposed to the students an activity, “linear forms and dual”, described below (1h). Moreover, a theoretical course was given (1.5h), concerning the bidual space, the reflexivity theorem and the transpose transformation;
- during week 3, illustrations of dual and bidual spaces are presented (1h);
- finally during weeks 4 and 5 students had sessions of exercises (2x2h) on duality.

We focus in this paper on the “linear forms and dual” activity. We now present this activity and the corresponding choices, described in detail in (De Vleeschouwer 2010b).

As mentioned above (§2) a function, and thus a linear form, can change status, according to the context. In the context of duality in linear algebra, different statuses of linear forms can appear in the same task.

A linear form  $\varphi_i$  belonging to a dual basis  $X'$  of a basis  $X$  of a vector space  $E$  combines indeed two statuses:

- the status of process, operating on the elements of a vector space  $E$ . This status appears in the relationship linking basis  $X = \{x_1, \dots, x_n\}$  of vector space  $E$  to its dual basis  $X' = \{\varphi_1, \dots, \varphi_n\} : \forall i, j = 1, \dots, n : \varphi_i(x_j) = \delta_{ij}$ , where  $\delta_{ij}$  is Kronecker's delta;
- the status of element of a vector space: the dual space of  $E$  (denoted  $E'$ ), as an element of a basis of  $E'$ .

This combination of statuses can be considered as an aspect of the institutional didactic contract, at the level of a specific content. “A linear form is a process, and an element of a vector space, and students should be able to switch between these two statuses” is a rule of this contract. It is certainly linked with the process/object dialectics (Dubinsky 1991), but we do not retain here a cognitive focus: we consider

how the institution shapes the content. This rule remains implicit; and this change of status yields difficulties of the students. In the experimental teaching we organized, we have chosen to make this rule explicit to students.

We introduced for this purpose a specific vocabulary, presented to the students during the teaching in the springboard sessions. This vocabulary is thus not an analysis tool for our research; it can be seen as a meta-language proposed to students. From the researcher's point of view, it is directly connected with the levels introduced by Chevallard (2005); we can not develop the point here, details can be found in De Vleeschouwer (2010b).

We say that a linear form  $\varphi$  is considered at a *micro level* when it is seen as a process operating on the elements of a vector space  $E$  (on a field  $K$ ). We explain to the students that this choice of vocabulary is a metaphor, indicating that  $\varphi$  is considered in detail, which permits to observe the transformation it operates on the vectors of  $E$ . At this *micro level*, we can consider its kernel, range, rank amongst others.

When a linear form is considered as an element of a vector space, we call it the *macro level*. In this case this linear form can be considered amongst other linear forms on the same space  $E$ , constituting thus a set. In this set, one can define addition and product laws; with these laws one obtains a vector space, the dual of  $E$ .

On both levels, the same object  $\varphi$  is considered, but under different statuses. We **explicitly** presented to the students these levels using the vocabulary "micro" and "macro" during teaching, and connected them by saying that the macro level is obtained by a zoom out, the micro level by a zoom in (see De Vleeschouwer 2010b for the figures associated with this metaphor).

In order to evaluate the precise impact of this experimental teaching, we proposed, four months after the experimental teaching, a questionnaire to the students who attended it, and analysed their answers. We present this work in the next section. More than their success or failure, we try to identify in the students' answers indices of their entrance, or non-entrance, in the new contract.

## IMPACT OF THE EXPERIMENTAL TEACHING

We present here an extract of the questionnaire, and analyse the corresponding students' answers in terms of didactic contract.

34 Let  $\mathcal{P}^3$  be the vector space of polynomials of degree less than or equal to 3, with real coefficients. Let, for  $i=1, \dots, 4$ ,  $p_i: \mathbb{R} \rightarrow \mathbb{R}$  such as  $\forall x \in \mathbb{R}$ :  $p_1(x) = x^3 + 2x^2 + 4$ ,  $p_2(x) = 2x^3 - x + 2$ ,  $p_3(x) = x^3 - 1$ ,  $p_4(x) = 2x^3 + 3$ .

Prove that  $A = p_1, p_2, p_3, p_4$  is a basis of  $\mathcal{P}^3$ , and determine its dual basis.

35 Let  $f_1, f_2, f_3$  such as:

$$f_1: \mathbb{R}^5 \rightarrow \mathbb{R} \\ v = (a, b, c, d, e) \quad f_1(v) = 3a - 2e$$

$$f_2: \mathbb{R}^5 \rightarrow \mathbb{R} \\ v = (a, b, c, d, e) \quad f_2(v) = a - b + 2c$$

$$f_3: \mathbb{R}^5 \rightarrow \mathbb{R} \\ v = (a, b, c, d, e) \quad f_3(v) = 3b + 6c - 2e$$

- 1 Give an example of vector space comprising  $f_1, f_2, f_3$ .
- 2 Does  $f_1, f_2, f_3$  form a linearly independent set of vectors?
- 3 Give an example of linear form which does not belong to  $\text{Span}\{f_1, f_2, f_3\}$ .

36 Choose a vector space different from polynomials,  $\mathbb{R}^n$  or  $\mathbb{C}^n$  ( $\forall n \in \mathbb{N}$ ), and give an example of linear form over this space.

**Table 1 : Extract of the questionnaire proposed to the students**

The methodology we employ here is based on the *a priori* analysis (Hejny *et al.* 1999) of the questionnaire. We identify, in the questionnaire, specific aspects of the university contract, and observe in the students' answers if these issues raised difficulties, or if they evidence on the opposite an entrance in this contract.

The first question is related with polynomials and refers explicitly to duality, since the students have to determine the dual basis of a given basis. The students must consider jointly the micro and the macro level of linear forms, which is typical of the new contract at the level of the 'linear forms' content. The second question concerns functions. The proposed functions  $f_1, f_2, f_3$  are defined at a micro level. They are linear forms; nevertheless, this term is not used in the text in order to avoid that the students answer 'the dual space' to question 2.a). For their answers to question 2. b), the students must consider these functions as vectors, which means changing levels, to work at the macro level. They have to place their reasoning at a level different from the level of the text.

Question 3 requires that the students work in a frame different from the polynomials (question 1) or the algebraic frame (common in the courses). The change of frames is also typical of the university didactic contract at the discipline level. In the second part of question 3, a linear form must be proposed at a micro level, as a process acting on the elements of the chosen set.

Moreover, questions 2.a), 2.c) and 3 require to build an example. Such a task is typical of the university contract, at the discipline level; the same statement holds for the variety of frames, another feature of the questionnaire.

We now consider the students' answers, focusing on the issues identified in the *a priori* analysis.

Two main techniques have emerged from students' responses to determine if the polynomials in question 1 constitute a basis: working with polynomials or with  $n$ -tuples. Eight students (40%) prefer to work with 4-tuples instead of polynomials, but only two of them justify their reasoning (for instance a student cites the theorem asserting the existence of an isomorphism between  $\mathcal{P}^3$  and  $\mathcal{R}^4$ ). We can consider that in doing so, these students comply with the didactical contract at the discipline's level which considers that the different steps of a mathematical reasoning should be justified. All students who responded to the questionnaire were able to determine that the given polynomials were linearly independent. It does not seem to be a problem for them to work in a non-usual frame, as often required at University.

70% of the students present the linear forms of the dual basis in a complete, detailed form (departure space, arrival space and image of any vector, see Figure 1). We interpret these kinds of answers as typical from the university contract, at two levels. At the discipline level, a mathematical answer has to be as complete as possible. At the content level, a function must be characterized by these three elements, whereas at secondary school generally only the expression " $f(x) = \dots$ " or the graph is given.

Twelve students (60%) describe analytically the four linear forms  $p_i'$  and, at the same time, consider them as elements of a set (the dual base): they write explicitly " $A' = p_1', p_2', p_3', p_4'$ " (see Figure 1). In doing so, students consider the linear forms both at the macro level and at the micro level.

$A' \text{ base dual arrive } A = \{p_1', p_2', p_3', p_4'\}$   
 $p \in P^3 : \mathcal{R} \rightarrow \mathcal{R}$   
 $p(p) = a_0 + a_1 + a_2 + a_3$   
 $p_1' : \mathcal{P}^3 \rightarrow \mathcal{R}$   
 $p_1'(p) = \frac{a_2}{2}$   
 $p_2' : \mathcal{P}^3 \rightarrow \mathcal{R}$   
 $p_2'(p) = -a_1$   
 $p_3' : \mathcal{P}^3 \rightarrow \mathcal{R}$   
 $p_3'(p) = -\frac{2}{5}a_0 + \frac{2}{5}a_1 + \frac{1}{5}a_2 + \frac{3}{5}a_3$   
 $p_4' : \mathcal{P}^3 \rightarrow \mathcal{R}$   
 $p_4'(p) = \frac{0.1}{5}a_0 - \frac{0.2}{5}a_1 + \frac{0.3}{5}a_2$

**Figure 14: Conclusion of a student's answer to the second part of question 1**

Concerning question 2.a) fifteen students (75%) succeed in giving a vector space comprising the given linear forms, and nine of them (45%) cite the dual space. It requires to consider at the macro level functions which have been described in the text as processes (micro level). This change of status does not seem to constitute a difficulty for a majority of students. The same statement holds for the sub-question 2.b): while the linear forms were given at the micro level, seventeen (85%) students

## Working Group 14

succeed to consider them at the macro level, and answer correctly that they are linearly independent. Moreover six students convert the linear forms into 4-tuples before starting calculations (see Figure 2). We can interpret this fact as the conversion of a function from the micro level to the macro level. In Figure 2, we see that the student concludes question 2.b) by writing "the *vectors* are linearly indep." instead of "the *linear forms* are linearly indep."

Exercice 2:

a)  $f_1, f_2, f_3$  appartenant à  $\mathbb{R}^5$  par exemple car  
ils sont des formes linéaires.

b)

$$\begin{aligned} f_1(v) &= 3a - 2e \\ f_2(v) &= a - b + 2c \\ f_3(v) &= 3b + 6c - 2e. \end{aligned}$$

$$\alpha(3, 0, 0, 0, -2) + \beta(1, -1, 2, 0, 0) + \gamma(0, 3, 6, 0, -2) = (0, 0, 0, 0, 0)$$

$$\Rightarrow \begin{cases} 3\alpha + \beta = 0 \\ -\beta + 3\gamma = 0 \\ 2\beta + 6\gamma = 0 \\ 0\alpha + 0\beta + 0\gamma = 0 \\ -2\alpha - 2\gamma = 0 \end{cases} \Rightarrow \begin{cases} 3\alpha + \beta = 0 \\ \beta = 3\gamma \\ 6\gamma + 6\gamma = 0 \\ -2\alpha - 2\gamma = 0 \\ \alpha + 0\beta + 0\gamma = 0 \end{cases} \Rightarrow \begin{cases} \beta = 0 \\ \beta = 0 \\ \gamma = 0 \\ \alpha = 0 \end{cases}$$

$\Rightarrow$  les vecteurs sont linéairement indep.

c)  $f_4(v)$  ?

$$f_4: \mathbb{R}^5 \rightarrow \mathbb{R}$$

$$v(a, b, c, d, e) \rightsquigarrow a + b + c + d + e = f_4(v).$$

**Figure 2 : Example of a student's answer to question 2**

Analyzing students' answers to question 3 shows that they seem to have built a variety of vector spaces: amongst the vector spaces cited by the students, eleven are vector spaces of matrices (square matrices of size 2 or 3); two are vector spaces of functions (transformations of  $\mathbb{R}$  or of  $\mathbb{R}^2$ ), one can be considered as algebraic ( $\mathbb{R}^3$  built over  $\mathbb{R}$ ). The unsuccessful attempts concern  $\mathbb{R}^2$  or  $\mathbb{R}^3$  (cited by three students) or  $\mathbb{R}$  (cited by two students); one student makes a non-relevant answer. Note that the module structure, which generalizes the vector space structure, was not presented to students. This perhaps explains the presence of proposals involving  $\mathbb{R}^2$  or  $\mathbb{R}^3$  in the responses of three of them. The variety of frames for linear algebra is typical of the new institutional contract at the discipline level. Moreover eight students *justify* the label 'linear form' given to their example although seven of them give only a partial explanation: "*arrival space is the field*". It seems that the part of justification in the didactic contract at the discipline level is not obvious.

## CONCLUSION

In this work we attempted to determine rules of the didactic contract at university, in the case of duality. Using previous research works, and a textbooks analysis, we identified contract rules at different levels: some of these rules correspond to precise notions, like linear forms, while others concern more generally mathematics.

We designed a teaching intervention aiming to support the entrance of students in this new contract. The objective of such a teaching is not to change the contract, by reducing the students' responsibility. In the case we presented, we chose to make explicit a usually implicit rule, at the level of a specific content (linear forms), introducing a meta-language (*micro/macro*) for students' use. We have also proposed to the students exercises where they were required to change frames, to build examples, and more generally to comply with new university expectations, at the level of the discipline. We do not claim that all the rules should be made explicit. Some of the contract rules have to remain implicit, this well-known paradox is an essential condition for learning (Brousseau 1997). Introducing the micro-macro meta-language, we did not only unveil a rule about linear forms; we contributed to raise the awareness of the students about the different statuses of mathematical objects at university, and the possible need for change of status, according to the context. The meta-language makes this new responsibility explicit.

The analysis of the students' answers to our questionnaire evidence that they at least started to enter in the new contract. We did not carry out a comparison with other students (the conditions of our study did not offer such a possibility); but the first author of the paper, as a teacher, noticed that the students do not seem to meet the usual difficulties, and we consider that the experimental teaching significantly contributed to this progress. We now intend to extend our study to other topics: the precise rules of the didactic contract at university remain largely unknown.

## REFERENCES

- Artigue, M. (2007). *Teaching and learning mathematics at university level*, presentation at the conference 'The future of mathematics education in Europe', Lisbonne.
- Bloch, I. (2005). *Quelques apports de la théorie des situations à la didactique des mathématiques dans l'enseignement secondaire et supérieur*. Habilitation à diriger des recherches, Université Paris VII.
- Bosch, M., Fonseca, C. & Gascon, J. (2004). Incompletud de las organizaciones matematicas locales en las instituciones escolares, *Recherches en Didactique des Mathématiques*, 24(2-3), 205-250.
- Brousseau G. (1997). *Theory of didactical situations in mathematics 1970-1990*, Kluwer Academic Publishers, Dordrecht.

- Chevallard, Y. (2005). Steps towards a new epistemology in mathematics education, In Bosch, M. (ed.) *Proceedings of 4<sup>th</sup> Conference of European Research in Mathematics Education ( CERME)*, (pp.21-30), Sant Feliu de Guíxols, Espagne.
- De Vleeschouwer, M. (2008). *Aider les étudiants à entrer dans un nouveau contrat didactique institutionnel en mathématiques? L'exemple de l'opération tremplin à l'université de Namur*. Communication au colloque Questions de pédagogie dans l'enseignement supérieur, Brest.
- De Vleeschouwer, M. (2010a). Secondary-tertiary transition and students' difficulties: the example of duality. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (eds.) *Proceedings of 6<sup>th</sup> Conference of European Research in Mathematics Education ( CERME)*, (pp.1359-1368), Lyon, France.
- De Vleeschouwer, M. (2010b). *Enseignement à l'Université, perspective institutionnelle et contrat didactique : le cas de la dualité en algèbre linéaire*. Thèse de doctorat de l'université de Namur, Belgique.
- Douady, R. (1987). Jeux de cadres et dialectique outil/objet, *Recherches en didactique des mathématiques*, 7/2, 5-32.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In Tall D. (ed.), *Advanced mathematical thinking*, 95-126. Kluwer: Dordrecht.
- Grønbaek, N., Misfeldt, M., & Winsløw, C. (2009) Assessment and contract-like relationships in undergraduate mathematics education. In O. Skovsmose et al. (eds), *University science and Mathematics Education. Challenges and possibilities*, (pp. 85-108). New York: Springer Science.
- Gueudet, G. (2008). Investigating the secondary-tertiary transition, *Educational Studies in Mathematics*, 67/3, 237-254.
- Hejny, M., Shiu, C., Godino, J. D., & Maier, H. (1999). Research paradigms and methodologies and their applications in mathematics education. In Schwank, I. (ed.) *Proceedings of 1<sup>st</sup> Conference of European Research in Mathematics Education ( CERME)*, (pp. 211-219) Osnabrück, Germany.
- Praslon, F. (2000). *Continuités et ruptures dans la transition Terminale S / DEUG Sciences en analyse. Le cas de la notion de dérivée et son environnement*, Thèse de doctorat de l'Université Paris 7.
- Sarrazy B. (1995). Le contrat didactique. *Revue Française de pédagogie*, Note de synthèse, 112, 85-118.
- Winsløw, C. (2008). Transformer la théorie en tâches : la transition du concret à l'abstrait en analyse réelle. In Rouchier A. et Bloch, I. (eds.) *Perspectives en didactique des mathématiques. Cours de la XIII<sup>ème</sup> école d'été de didactique des mathématiques*, La Pensée Sauvage, Grenoble.



# A DIDACTIC SURVEY OF THE MAIN CHARACTERISTICS OF LAGRANGE'S THEOREM IN MATHEMATICS AND IN ECONOMICS

Sebastian Xhonneux, Valérie Henry

University of Namur, Belgium

*Abstract: Because of its many uses, the constrained optimization problem is presented in most undergraduate mathematics courses dealing with calculus for both mathematicians and economists. Our research focuses on the teaching of Lagrange's Theorem in both branches of study, mathematics and economics. This paper addresses two objectives. First, we describe the methodology of our research project concerning the didactic transposition of Lagrange's Theorem in university courses. Secondly, we compare two mathematics courses dealing with calculus given at the universities of Namur and Louvain by means of the Anthropological Theory of Didactics.*

*Keywords:* Lagrange's Theorem, optimization, didactic transposition, Anthropological Theory of Didactics

## INTRODUCTION

Constrained optimization plays a central role in optimization theory but also in economics. In fact, constrained optimization can be seen as one of the fundamental techniques that economists use to solve economic problems. Lagrange's Theorem and the derived method of Lagrange multipliers [named after Joseph-Louis Lagrange (1736-1813)] provide an appealing strategy for finding the maxima and minima of a function subject to equality constraints; so we are interested in studying the teaching of this theorem in both branches of study, mathematics and economics.

Based on the author's own teaching experiences at the University of Namur (Belgium), it is apparent that a considerable number of first year students struggle with calculus courses and, in particular, with Lagrange's Theorem. Furthermore, the mathematical exercises in these classes involve students using a large number of standardized procedures for obtaining answers to clearly delimited types of exercise questions. Dreyfus mentions in this context that

they end up with a considerable amount of mathematical knowledge but without the working methodology of the mathematician, that is they lack the know-how that

allows them to use their knowledge in a flexible manner to solve problems of a type

unknown to them. (Dreyfus, 1991, p.28)

Hence, we question in our research project whether the choice of the didactic transposition of Lagrange's Theorem may influence the students' perception and understanding. Therefore, we question whether our findings help to enlighten mathematics professors concerned with increasing students' comprehension of this theorem. In fact, we also would like to show how teachers' practices are influenced inside the didactic transposition by a combination of didactic reasons and mathematical reasons.

Next, we describe the theoretical framework used to guide our research. In order to investigate the constraints under which a professor should operate when conceiving and carrying out the teaching of Lagrange's Theorem, we analyzed existing didactic transpositions by means of the “Anthropological Theory of Didactics” (ATD) of Chevallard (1992, 1999). This model describes mathematical activity in terms of *mathematical* (or *didactic*) *organisations* or *praxeologies*. The third section provides a description of our methodology, which used ideas from the ATD, a useful tool for the analysis of mathematical and teaching activities. In the fourth section, we briefly describe the *epistemological reference model* (ERM), which constitutes our basic theoretical model used to describe the didactic transposition. Related, mathematical praxeologies are then used to describe and compare the knowledge to be taught around the Lagrange's multiplier rule as it is proposed at the universities of Namur and of Louvain. Finally, we provide conclusions and a brief survey of perspectives of our research work.

## THE ANTHROPOLOGICAL APPROACH

As we utilize an institutional perspective<sup>76</sup> in our research, the choice of the “Anthropological Theory of Didactics” (ATD) proposed by Chevallard (1992, 1999) appears pertinent to investigate characteristics of teachers' practices. This model of mathematical knowledge envisions mathematics as a human activity involving the study of types of problems. Below, we provide a brief summary of the principal content of this theory based on a paper written by Barbé, Bosch, Espinoza and Gascón (2005).

In the anthropological approach, an object exists from the moment one person or an institution individually recognizes this object as existing, and more precisely, if someone relates to it. These relationships can be established through activities making use of the object. We identify two inseparable aspects of mathematical activity. First, the *pratico-technical block* (or *know-how*) is formed by types of problems or problematic tasks,  $T$ , and by the techniques,  $\square$ , used to resolve them. Studying problems of a given type (with an aim of solving them) is considered to be “doing mathematics”. Furthermore, in the anthropological approach,

---

<sup>76</sup> Institutional perspective means that we observe practices relative to a (mathematical) object, these practices are expected to differ from one institution to another.

procedural methods, resolutions of problems or accomplishments of tasks suppose the existence of a technique. This holds true even if the given technique can scarcely be explained or shown to others or even to ourselves.

Secondly, we assume in the anthropological approach that one can rarely find human practices without a copious environment of discourse. The objective of this “spoken surrounding” is to describe, explain and justify what is done. Therefore, there is the knowledge block of mathematical activity that offers the mathematical discourse necessary to justify and interpret the practical block. The knowledge block is divided into two parts: the technology, □, refers to the technique used and the theory, □, establishes profound justifications of the technology. In ATD, this second block is called the technological-theoretical block.

Types of problems, techniques, technologies and theories can be seen as the fundamental elements of the anthropological model of mathematical activity. We also employ them to describe mathematical knowledge, which can be considered both a means and a product of this activity. When we examine various types of problems, techniques, technologies and theories together, we entitle them mathematical praxeological organisations or, in short, mathematical organisations or mathematical praxeologies. An examination of the etymology of the word “praxeology” shows how practice (praxis) and the discourse about practice (logos) are closely connected.

ATD posits that we can analyse more than only mathematical activities. Any form of human activity can be interpreted in terms of praxeological organisation. Therefore, we also introduce the concept of didactic praxeologies when speaking about the process of study of mathematical constructions.

Given the increasing interest and demand to investigate teachers' practices and their role in the didactic relationship, an investigation on these didactic praxeologies appears warranted.

In order to achieve the aforementioned objectives, our research was guided by the following research questions amongst others:

- 37 What are the essential characteristics of the teaching of Lagrange's Theorem in mathematics and economics?
- 38 Are there similarities and differences between the teaching of Lagrange's Theorem in mathematics and in economics with respect to the mathematical praxeologies?
- 39 May it be possible to improve students' understanding/interpretation of Lagrange's Theorem by exchanging ideas between the two disciplines?

## FOUR KINDS OF KNOWLEDGE

Teaching and learning are not isolated, but take place in the complex process of *didactic transposition* (Chevallard, 1991). With regard to this transposition, we need to distinguish among four kinds of knowledge: “scholarly” mathematical knowledge, mathematical knowledge “to be taught” and mathematical knowledge “as it is actually taught” by professors to students. When including students' comprehension and learning into the process, we have to add the fourth kind of knowledge – mathematical knowledge “learnt”, which is generally difficult to access. A basic theoretical model, the *epistemological reference model* (ERM) (Bosch & Gascón, 2005), is used to analyse at the same time the scholarly knowledge, the knowledge to be taught, and the knowledge actually taught and learned.

We used the following procedural methodology in order to investigate our research questions and related questions:

### **“Scholarly” mathematical knowledge**

To understand “scholarly” mathematical knowledge, our first step consisted of an epistemological analysis of Lagrange's Theorem and of associated mathematical literature in mathematics and economics.

### **The mathematical knowledge “to be taught”**

To gain deeper insight in the mathematical knowledge to be taught, we analyzed textbooks and course notes about Lagrange's Theorem from different mathematics courses, both for mathematics and economics students, using the ATD. In doing so, we exercised caution because only the “knowledge to be taught” can be reproduced from these textbook elements. The “knowledge actually taught” unfortunately appears only in the students' notes and in the specific teaching practices carried out in the day-to-day teaching praxeologies in classrooms.

### **Mathematical knowledge “taught” and “learnt”**

Next, we explored teachers' and students' conceptions about Lagrange's Theorem. We contacted professors who are (or were) responsible for teaching Lagrange's Theorem in either an economics or a mathematics course (or both) dealing with calculus, or more particularly with optimization. This was done at Belgian universities in the French-speaking part of the country. By means of a questionnaire composed of 27 multiple-choice and open-ended questions, we attempted to identify, as precisely as possible, the environment and conditions of the teaching of Lagrange's Theorem. A second questionnaire built upon the first one was then designed in order to obtain information about students' conceptions and ideas. This questionnaire was composed of 14 multiple-choice and open-ended questions. Course observations and students' presentations and evaluations at exams (involving three different tasks to solve) completed our data collection process.

Note that our research is still in progress. “Scholarly” mathematical knowledge and mathematical knowledge “to be taught” are already analysed, mathematical knowledges “taught” and “learnt” are the topics we are currently concerned with.

## **EPISTEMOLOGICAL REFERENCE MODEL OF LAGRANGE'S THEOREM**

One hypothesis to be developed, explored and possibly amended in our doctoral research study is a reference mathematical model for Lagrange's Theorem that includes five mathematical organisations  $MO_1$ ,  $MO_2$ ,  $MO_3$ ,  $MO_4$  and  $MO_5$ . These mathematical organisations respectively address the following types of tasks:

- $T_1$ : Find candidates for optimal solutions for a constrained optimization problem subject to equality constraints.
- $T_2$ : Solve a constrained optimization problem subject to equality constraints.
- $T_3$ : Develop the theory concerning Lagrange's Theorem.
- $T_4$ : Use an interpretation of Lagrange's multiplier.
- $T_5$ : Develop the theory concerning Lagrange's multipliers.

Many relationships among these mathematical organisations can be described but we limit ourselves in this paper to the following brief remarks:  $MO_1$  originates from the original works of Lagrange, whereas  $MO_2$  is concerned with the solving of particular constrained optimization problems. In fact, Lagrange's Theorem may or may not intervene in the technique of  $MO_2$ . Accomplishing  $T_1$  can therefore be one step in the process of accomplishing  $T_2$ .  $MO_3$  can be regarded as part of the theory of  $MO_1$  (and also  $MO_2$ ), but it is also the self-contained praxeology that deals with the proving of Lagrange's Theorem, among other tasks.  $MO_4$  uses Lagrange's multipliers as a mathematical tool (Douady, 1986), whereas  $MO_5$  constitutes an additional mathematical organisation concerned with Lagrange's multiplier being seen as a mathematical object in the sense of Douady (1986).

## **COMPARISON BETWEEN TWO TEXTBOOKS**

To describe emerging mathematical and didactic praxeologies we are going to analyse existing textbooks with regard to our ERM. However, due to space limitations of this paper, the presented analysis only considers two mathematics textbooks, one from each discipline – economics and mathematics. Nevertheless, we will be able to show how the anthropological approach renders a comparison possible.

The components of Lagrange's Theorem we are considering are integrated in the course “Mathematics for Economic Analysis I” (Thiry, 2006) for first-year students in economics and business management at the University of Namur and in the course

“Mathematical Analysis II” (Ponce & Van Schaftingen, 2010) for first-year students in mathematics at the University of Louvain (Belgium).

### Mathematics for Economic Analysis I (Thiry, 2006)

Twelve pages in the textbook deal with Lagrange's Theorem (in Chapter 4, “Multivariable Optimization”). We analysed these pages in terms of praxeologies. Looking at this knowledge to be taught, we can name it “Analytic solving of equality constrained optimization problems”. In fact, the textbook starts with one well-known economics-based optimization problem: the utility maximization problem<sup>77</sup> the consumer faces, and tries to solve it:

- first, by the substitution method,
- then, by noticing that substitution is not always possible. Therefore, the use of the method of Lagrange multipliers is announced.

After this short introduction, the textbook mathematically defines the problem it is going to solve. The textbook poses the following type of problems:

$T_I$ : Find all the candidates for optimal solutions to the following constrained optimization problems

$$(P_1) \begin{cases} \max & f(x, y) \\ \text{SC} & g(x, y) = k, \end{cases} \quad \text{and} \quad (P_2) \begin{cases} \min & f(x, y) \\ \text{SC} & g(x, y) = k. \end{cases}$$

$T_{II}$ : Solve  $(P_1)$  (or  $(P_2)$  respectively).

$T_{III}$ : Approximately determine the maximum of  $(P_1)$  (or the minimum of  $(P_2)$ ) when  $k$  is increased (or decreased) by  $\square$ .

The section about Lagrange's Theorem is followed by exercises where we can find one additional type of problem: exercises that are mainly like  $T_{II}$  but require *mathematical modelling*. We do not consider these problems in the paper due to space limitations.

Let us start with  $T_I$ . Before obtaining an appropriated technique to solve this type of problem, we switch to the technological-theoretical block and read the technology  $\square_1$  used to justify the appropriated resolution process. In terms of a geometric interpretation and by the use of the Implicit Function Theorem, we get a characterization of the solution: a solution  $(x^*, y^*)$  of the equality constrained optimization problem has to verify

- $g(x^*, y^*) = k$  and
- there exists  $\square \square \mathbb{R}$  such that  $\square f(x^*, y^*) = \square \square g(x^*, y^*)$ .

This necessary optimality condition is then formally stated in the third section, “Lagrange's Theorem”.

<sup>77</sup>

This problem can be resumed by "How should I spend my money in order to maximize my utility?"

[...] the observation of the contour lines shows us that a point  $(x^*, y^*)$ , extremum of  $f$  under the constraint  $g(x, y) = k$  necessarily verifies the aforementioned equations. We now formulate these conclusions in the form of a theorem (that will not be proved here). (Thiry, 2006, p. 164)

The author does not give a rigorous proof, which is why we say that the theory closest to the given statement and which justifies the technology presented before is nearly absent from the notes. The section continues by defining Lagrange's function, and reformulates the theorem by means of this function. The whole section can be considered as technology justifying the technique for “finding candidates for optimal solutions”. Section 4 finally establishes the algorithmic technique  $\square_I$  to follow to find candidates as stated in  $T_I$ . The type of tasks  $T_I$ , the technique  $\square_I$  and the technology  $\square_I$  constitute a first praxeology  $[T_I, \square_I, \square_I, /]$  where “/” symbolizes the absent or implicit theory.

Finding all the candidates does not solve the problems of type  $T_{II}$ , even if a solution to  $T_I$  is necessary to solve  $T_{II}$ . We therefore need a second step in the resolution process, which is presented in the textbook in Section 5. No technological-theoretical block is presented for this second step in the resolution process, but only a proposition is provided that details the technique  $\square_{II}$  used to identify whether a candidate is effectively an optimum for the equality constrained optimization problem.

The following proposition (that will not be proved) furnishes a test based on second-order derivatives to decide whether a stationary point is effectively a maximum or a minimum. (Thiry, 2006, p. 169)

This technique is illustrated by means of an example. We say that we obtain the following praxeology  $[T_{II}, \square_I \square \square_{II}, \square_I \square /, /]$ . Again, the “/” denotes that these praxeological elements are not rendered explicitly.

Then, as it is the case for  $T_I$ , the technology  $\square_{III}$  concerned with the solving of type of tasks  $T_{III}$  is presented before the technique  $\square_{III}$  and it is summarised in the proposition 4.12 (Thiry, 2006, p. 171). The associated technique then is exposed in the proposition 4.13 (Thiry, 2006, p. 172) and is completed with one exercise. We get a third praxeology  $[T_{III}, \square_{III}, \square_{III}, /]$  where the slash indicates again that the theory is practically absent in the textbook.

Combining the three aforementioned praxeologies, we find that the considered knowledge to be taught is principally composed of the traces left by  $MO_1$ ,  $MO_2$  and  $MO_4$ . In fact, the first type of tasks is a particular case of tasks of  $MO_1$ . Furthermore, with regard to our ERM, the first praxeology constitutes a particular reconstruction of  $MO_1$ . The theories  $\square_I$  and  $\square_{II}$  not having appeared explicitly, we say that the scope of tasks emerging from  $MO_3$  is not at all covered by the textbook. The second praxeology arises from  $MO_2$  and covers a task that can't be solved only by the technique of Lagrange's multiplier rule. Hence, students have to seek techniques

from “external arguments” that are out of range of Lagrange's Theorem. We make the assumption that students, who give up on the resolution of a constrained optimization problem subject to equality constraints after having applied the multiplier rule, do not distinguish between the solving of tasks  $T_I$  and  $T_{II}$ . This may be caused by the presentation of Lagrange's Theorem as the main technique to solve equality constrained optimization problems without insisting that this theorem only describes necessary, not sufficient, optimality conditions. The third praxeology then clearly comes from  $MO_4$  as regarded in an economical context. Finally, only a few remarks and definitions in the text can be considered as traces left by  $MO_3$  and  $MO_5$ .

### Mathematical Analysis II (Ponce & Van Schaftingen, 2010)

Chapter 5 in this textbook covers optimization problems. The first section deals with unconstrained optimization problems, whereas the second deals with equality constraints and the third deals with inequality constraints. A last section offers multiple exercises. We are interested in the six pages treating optimization problems with equality constraints (Section 5.2). We call this knowledge to be taught “Analytic solving of equality constrained optimization problems by penalty method”.

Before looking at that particular section, let us mention that each chapter of this textbook starts with a list of questions students are going to be confronted with at the final exams. The final section of each chapter provides more exercises. For the constrained optimization problem and Lagrange's Theorem in particular, we find two relevant types of problems:

$T_{IV}$ : Give a geometric interpretation of Lagrange's multipliers.

$T_V$ : Given function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}: (x, y) \mapsto f(x, y)$  determine the minima and maxima of  $f$  constrained to  $g(x, y) = 0$ .

Most of problems of type  $T_V$  in the textbook treat geometric problems and use therefore technology from geometry (e.g., distances, planes, surfaces). Furthermore, the type of problems  $T_V$  resembles  $T_{II}$ , affirming that mathematics and economics students are confronted with the same type of problems.

The section about equality constrained optimization problems opens directly with a theoretical discourse and gives the mathematical formulation of Lagrange's Theorem (Proposition 5.4, Ponce & Van Schaftingen (2010), p. 162-164). Its proof needs lemma 5.5 and both propositions are rigorously proved by a so-called *penalty method*<sup>78</sup>. One remark is then given which concerns the values the multipliers can take<sup>79</sup>. No further explanations are provided and the section ends with two example

---

<sup>78</sup> This method consists of neglecting the constraints while adding a penalty term to the function to be optimized if the constraints are violated. A consequence of this method is the avoidance of the Implicit Function Theorem.

<sup>79</sup> See p. 164, Ponce & Van Schaftingen (2010).



tasks and their resolution  $\square_{VI}$  and  $\square_{VII}$ . We add these problems to the list of types of problems and therefore define:

$T_{VI}$ : Minimize the function  $f: \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto f(x)$  under the equality constraint  $g(x) = 0$ , where  $g: \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto g(x)$ .

$T_{VII}$ : Prove the “inequality of arithmetic and geometric means”.

In summary, we conclude that, for the types of problems  $T_{IV}$  and  $T_V$ , only the most relevant theory  $\square$  is stated and proven even if, from a mathematical point of view, technique  $\square_{VI}$  could be used to solve  $T_V$ . The formulation of Lagrange's Theorem is then used as technology (without giving further argumentation) to solve  $T_{VI}$  and  $T_{VII}$ . However, neither the complete pratico-technical block nor the complete technological-theoretical block is explicitly presented in the textbook to solve the introductory tasks. We obtain the following praxeology  $[T_{IV} \square T_V, /, /, \square]$ . As far as the two problems solved as examples are concerned, the associated technique is furnished, so that we get  $[T_{VI}, \square_{VI}, \square, \square]$  and  $[T_{VII}, \square_{VII}, \square, \square]$  where  $\square$  is the formulation of Lagrange's Theorem. The technology  $\square$  is a minimal discourse in the sense of furnishing only the theorem that justifies the technique. We presume that the understanding of Lagrange's Theorem may be hindered if no supplementary information is given (for example, during lectures), and that students will encounter problems in solving tasks of types  $T_{IV}$  and  $T_V$ . However, this analysis only concerns the textbook and the mathematical knowledge “to be taught” and does not represent the mathematical knowledge “actually taught”.

With regard to our ERM, the textbook does a nearly complete presentation of task of  $MO_3$  “Proof Lagrange’s Theorem” (with one possible proof amongst others). As this trace of the mathematical organisation is presented before solving particular tasks arising from  $MO_2$ , we see that the technological discourse concerned with these tasks of  $MO_2$  is replaced by adding  $MO_3$  to the theoretical discourse. Furthermore, the students have to reason the associated technique to solve tasks of type  $T_V$  by themselves. We do not find traces of  $MO_1$  regarded as self-contained praxeology in the textbook. Finally, as far as  $MO_4$  is concerned, the type of tasks  $T_{IV}$  arises from this mathematical organisation. However,  $MO_5$  is completely absent.

## Comparison

In order to compare the knowledge to be taught as it is presented in textbooks provided to students, we have to take into account that they target different audiences. We therefore obtain a first discrepancy between the teaching of Lagrange's Theorem in mathematics and economics. Students in economics are confronted with a detailed pratico-technical bloc of  $MO_1$  and often obtain profound technological arguments to justify the technique of Lagrange's multiplier before completely solving the equality constrained optimization problem. Traces of  $MO_1$  and  $MO_2$  can be found. Conversely, mathematics students are directly confronted with the more “general” task of finding solutions and tasks of  $MO_1$ , which are

incorporated in the solving process of tasks of  $MO_2$ . Students in mathematics directly have access to a praxeology of type  $MO_3$ . This affirms that proving is one of the dominant activities in mathematical studies. The technological discourse of  $MO_2$  is then reduced to the formulation of the theorem in question, and students are left to find the associated technique by themselves (or by assisting at the theoretical course or in exercise sessions).

## CONCLUDING REMARKS

Data collected from our experiences still need to be analysed in more detail in order to answer our research questions. The methodology presented will be pursued to obtain more insight into the didactic transposition of Lagrange's Theorem. This paper highlighted some discrepancies between the mathematical knowledge to be taught in mathematics and in economics due to the fact that, first and foremost, the role of mathematics in each discipline is different. Even if this is not a surprising result, it demonstrates the descriptive power of ATD as a tool for our ongoing analyses. In fact, the second objective of the paper was to show how ATD can render an analysis of the complex process of didactic transposition possible. It provides a classification of the didactic material presented in the textbooks and, with regard to our epistemological reference model, makes a comparison between these textbooks possible. As already mentioned, we need to be cognizant that textbooks do not represent the mathematical knowledge “as it is actually taught”. We therefore have to refine our epistemological reference model and to carry out further analyses concentrating on the latter to get deeper access in teachers' practices and students' perceptions. Furthermore, in order to investigate the mathematical knowledges “taught” and “learnt” we aim at deploying a second theoretical framework with a cognitive perspective that ATD alone has not.

The intended outcome of this research project does not include a didactic engineering in the sense of Artigue (1989), but we expect to understand better the essential qualities of Lagrange's Theorem and how teachers can improve their practices.

## REFERENCES

- Artigue, M. (1989). Ingénierie didactique. *Recherches en Didactique des Mathématiques*, 9(3), 281-308.
- Barbé, J., Bosch, M., Espinoza, L. & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics*, 59, 235-268.
- Bosch, M. & Gascón, J. (2005). La praxéologie comme unité d'analyse des processus didactiques. In A. Mercier & C. Margolinas (Eds.), *Balises pour la didactique des mathématiques* (pp. 107-122). Grenoble, France: La Pensée Sauvage.

- Chevallard, Y. (1991). *La transposition didactique - du savoir savant au savoir enseigné*. Grenoble, France: La Pensée Sauvage.
- Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques*, 12(1), 73-112.
- Chevallard, Y. (1999). Analyse des pratiques enseignantes et didactique des mathématiques en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, 19(2), 221-265.
- Douady, R. (1986). Jeux de cadre et dialectique outil-objet. *Recherches en Didactique des Mathématiques*, 7(2), 5-31.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer.
- Ponce, A. & Van Schaftingen, J. (2010). *Analyse mathématique 2 – Fonctions de variables vectorielles*. Université catholique de Louvain. Louvain-la-Neuve, Belgium. Retrieved, June 3, 2010, from <http://www.icampus.ucl.ac.be/index.php>.
- Thiry, S. (2006). *Mathématiques pour l'économie et la gestion I*. Facultés universitaires Notre-Dame de la Paix. Namur, Belgium: Librairie des Sciences.

# A QUESTIONNAIRE FOR SURVEYING MATHEMATICS SELF-EFFICACY EXPECTATIONS OF PROSPECTIVE<sup>1</sup> TEACHERS

Marc Zimmermann<sup>1</sup>, Christine Bescherer<sup>1</sup>, and Christian Spannagel<sup>2</sup>

<sup>1</sup>University of Education Ludwigsburg, <sup>2</sup>University of Education Heidelberg

*Knowing of the traditionally high drop out rates in mathematics courses students often expect to fail. This results in a low motivation for learning mathematics and a low mathematics self-efficacy expectation. However, self-efficacy beliefs of a person have been identified as an important factor for performing tasks successfully therefore high self-efficacy beliefs are especially important for students who are training to be mathematics teachers. In order to determine the mathematics self-efficacy expectations of students of math education and to measure the influence of pedagogical interventions on self-efficacy, an adequate instrument is needed. This paper describes the development and validation of a scale for measuring the mathematics self-efficacy expectations of prospective teachers (MaSE-T).*

Keywords: self-efficacy, secondary trainee teachers, pedagogical intervention.

## INTRODUCTION

Negative attitudes and expectations towards mathematics are mostly founded in low performances at school or bad experiences in math classrooms. These negative beliefs lead to an expectation of not being able to handle mathematics in general. As a consequence, for many ‘non-mathematicians’, e.g. students of medical, biological or economic study courses, mathematics is incomprehensible or inaccessible. Some are even scared when they see mathematical formulas or they have to solve a mathematical problem, even if they use mathematics subconsciously in everyday life. These low self-efficacy expectations in mathematics can lead to bad results in exams, which will again decrease the student’s self-efficacy in mathematics. As a result the students will drop out of the study course.

The belief in one's own mathematical competence (skills and knowledge) is an important factor to perform a given task or problem successfully (Bandura, 1977). Increasing mathematics self-efficacy will get ‘non-mathematicians’ to dare more mathematics in everyday life as well as in the mathematics courses of their program of study. In general, it can influence their behavior and attitudes when they are faced with mathematical tasks. For these reasons a learning scenario which increases the mathematical self-efficacy besides the content knowledge should be more valuable than a course which ‘only’ increasing the knowledge. This means that the success of mathematical learning scenarios, e.g. at school or at university, should not only be measured by testing knowledge. Pre- and post-testing the mathematics self-efficacy expectations during a course in mathematics will provide information whether the learners' mathematics self-efficacy expectations have increased. Besides, the

development of self-efficacy beliefs of students in a learning scenario can provide an opportunity for assessing the success of learning scenarios.

Especially for students of mathematics education it is important to possess adequate self-efficacy beliefs in mathematics. On the one hand, even as a teacher, they have to be able to understand mathematical publications on low levels. On the other hand, teachers should set an example for students concerning their mathematical beliefs and serve as models. Teachers with low self-efficacy expectations will hardly motivate students to do mathematics and to believe in their own competence. This is one of the reasons why we focus on the higher mathematical education of prospective teachers in our research.

Measuring the mathematics self-efficacy expectations of students who want to become teachers, demands an adequate instrument for measuring the right level. Existing instruments in German language are either on middle school level or focus on engineering students and do not meet the requirements of German mathematics teacher education. Thus, a new instrument has been developed and validated. In this article we first give a review of the theoretical basics of (mathematics) self-efficacy. After that we describe the development of an instrument for measuring the mathematics self-efficacy of prospective teachers. Finally we validate our 15-item mathematics self-efficacy questionnaire (MaSE-T) in an adult student population. Additionally the relationships between MaSE-T and gender, preparation for primary or secondary schools, as well as the level of specialization in mathematics (major, 'middle', or minor) are investigated. The paper ends with concluding remarks and an outlook on future research.

## **SELF-EFFICACY AND MATHEMATICS SELF-EFFICACY**

Self-efficacy can be defined as the judgement of one's capabilities to successfully perform a particular given task (Bandura, 1977; Bandura 1997; Zimmerman, 2000). These expectations and beliefs influence whether somebody starts working on a task and the intensity of the performance (Pajares & Kranzler, 1995). As a consequence people with low mathematical self-efficacy will avoid mathematical tasks or situations or will give up solving very fast.

Self-efficacy beliefs are a main factor in someone's decision making process, e.g. the choice of academic courses or career decisions (Hackett & Betz, 1981; May & Glynn, 2008). Especially low self-efficacy beliefs lead to 'negative' decisions in the related domain. Consequently, successful learning scenarios – at school or at university – should increase learners' self-efficacy expectations as well as their skills and knowledge. A main source of self-efficacy expectations is one's own successful performance. If a student completes a task autonomously with more or less feedback or gets high marks in exams, s/he develops positive expectations to handle new and unknown situations or problems. However, the effect could be weaker due to the

non-existing own performance if learners only ‘consume’ information about how to solve the task (Bandura, 1977).

In general, self-efficacy expectations ‘are task and domain specific’ (Pajares & Miller, 1995, p.190). For that, measurements of self-efficacy expectations should be always fitted to the related domain or task. For example, questionnaires have been proposed to measure self-efficacy expectations in the field of computer usage (Compeau & Higgins, 1995; Cassidy & Eachus, 2002; Barbeite & Weis, 2004) or mathematics in general (Betz & Hackett, 1983; Pajares & Miller, 1995; May & Glynn, 2008). Mathematics self-efficacy expectations indicate the belief of a person in his/her own competence to solve mathematical problems and tasks successfully. Mathematics self-efficacy is positively related to math performance (Pajares & Miller, 1994; Kabiri & Kiamanesh, 2004; Liu & Koirala, 2009). This means that the higher a person rates on mathematics self-efficacy scales, the better this person can perform on solving mathematical problems and vice versa. There are also gender differences in mathematics self-efficacy expectations. Males are usually scoring higher in mathematics self-efficacy questionnaires than females (Betz & Hackett, 1983; Randhawa & Gupta, 2000). It can be assumed that gender effects are based on social and cultural roles and the masculine image of mathematics.

The survey proposed by Betz and Hackett (1983) and the revised version by Kranzler and Pajares (1997) have widely been used in research. These surveys mainly consist of three kinds of items: math problems, math tasks used in everyday life, and performance in college courses. However, these surveys have several drawbacks: (1) some items are formulated on a level too low for students studying mathematics at German universities, for example: ‘Fred’s bill for some household supplies was \$13.64. If he paid for the items with a \$20 bill, how much change should he receive?’ It is important that questionnaires for measuring mathematics self-efficacy expectations require an adequate level of mathematical content knowledge. (2) Math problems and real-world math tasks are not clearly separated. For example, the item mentioned before is included in the ‘math problem’ scale, but could also belong to the scale with real-world math tasks because it is a everyday situation. (3) It would be interesting to see whether there is a difference between the expectation to solve a mathematical problem without context and a corresponding task which is contextualized in a real-world setting. In the existing surveys, there is no link between mathematical problems and real-world tasks. Therefore we decided to construct an own questionnaire described in the next section.

### **THE MATHEMATICS SELF-EFFICACY SCALE FOR SURVEYING PROSPECTIVE TEACHERS (MASE-T)**

The first version of the questionnaire consisted of 25 items: mathematical problems without contexts (10 items), real-world mathematical problems (10), and reasoning problems (5). In contrast to the scales by Betz and Hackett (1983) and Pajares and

Miller (1997), each mathematical problem was related to a real-world mathematical problem. In table 1 there are some example items for math-problems (Xa) of the questionnaire with their corresponding real-world math problem (Xb). In addition, a third scale has been included with 5 reasoning problems, for example ‘I am confident in proving that the square root of 2 cannot be represented as fraction’. All items were arranged randomly and had to be rated on a 5-point Likert scale ranging from 1 (‘I am not at all confident.’) to 5 (‘I am totally confident.’).

item	
1a	‘I am confident in determining the approximation formula of a somewhat cuboidal geometric body's diagonal.’
1b	‘I am confident in estimating whether a 2.5 meter long board can be transported in a van.’
2a	‘I am confident to solve the systems of equations with $x + y = 7$ and $x \cdot y = 30$ .’
2b	‘I am confident in calculating the length and width of a rectangle, if the perimeter counts 72 cm and the area is 288 cm <sup>2</sup> large.’
3a	‘I am confident in calculating the side length of a perpendicular triangle by using a trigonometric function.’
3b	‘I am confident in calculating the covered difference of height by knowing the incline of the street.’

**Table 1. Example items of the questionnaire.**

## EMPIRICAL STUDIES

In a first study, the 25-item questionnaire has been tested and analyzed aiming to reduce the number of items. In a second study, a reduced questionnaire (15 items) has been tested for reliability and validity.

### Study 1: 25-item mathematic self-efficacy scale

#### *Sample*

831 first year university students of mathematics education at the German universities of education in Ludwigsburg (526 students), Schwäbisch Gmünd (126), and Weingarten (179) participated in the study. The sample consisted of 215 males and 615 females (one missing) with a mean semester count of 1.33 ( $SD=0.86$ ; range 1-8). The overrepresentation of female students is normal in study courses of teacher education at universities in Germany.

#### *Procedure*

All students were asked to complete the questionnaire anonymously during a mathematics lecture at the beginning of the winter semester in 2008/2009. The participation was voluntarily and without any reward.

### *Results*

Internal reliability over all items was very good (Cronbach's  $\alpha = 0.89$ ). This value indicates a high degree of internal consistency of the items. Despite having a three-dimension scale (mathematical problems, real-world mathematical problems, and reasoning problems), each item contributes to the measurement of a single construct (mathematics self-efficacy expectations).

Factor and item analyses conducted on the collected data lead to a four factor model (PCA method, 50.7% of total variance). All items loaded on the first factor, which suggests that all items are related to mathematics self-efficacy. Hence, the remaining three factors can be considered to be related to the three mathematical dimensions. As a result, based on reliability coefficients and factor loadings, the 25-item scale was reduced to a 15-items scale<sup>2</sup>. Therefore now each of the three dimensions consists of 5 items. As a consequence, some of the former corresponding mathematical problem items and real-world problem items have been removed. Thus, for a few items the corresponding items are missing.

### **Study 2: 15-item mathematics self-efficacy scale**

The second study aimed at testing the psychometric properties of the reduced 15-item scale. In addition, the validity of the questionnaire had been tested by correlating the mathematics self-efficacy score with the grades of the 'Abitur', the final high school exam.

### *Sample*

The total sample ( $N$ ) of the main study consisted of 1318 participants (320 males, 995 females). Participants were again first year students (*mean* semesters 1.43,  $SD=0.97$ ; range 1 to 9) of students of mathematics educations at the universities of Education in Ludwigsburg (493 students), Schwäbisch Gmünd (166), Weingarten (174), Heidelberg (165), and Karlsruhe (320).

Additionally, the sample can be divided into different groups with regard to two dimensions. On one hand it can be separated into four groups of students by their study course specialization. Therefore the first group were students who will be future teachers for primary school ( $n=640$ ; 90 males, 550 females). Further groups are future teachers for secondary general school ( $n=189$ ; 58, 181), for intermediate secondary school ( $n=362$ ; 145, 217), and future teachers of schools for special needs children ( $n=120$ , 26, 94). It can be assumed that students for secondary schools perform best on the MaSE-T scale. On the other hand the sample can be divided by the students' level of specialization in mathematics for study. In Baden-Württemberg, Germany, students of education can choose whether they take



mathematics or not ( $n=237$ ; 37, 200). If mathematics is chosen as a subject it can be studied as major ( $n=401$ ; 88, 313), ‘middle’ ( $n=396$ ; 116, 280) or minor ( $n=275$ ; 78, 197) subject.

### *Procedure*

Students from the participating universities were again asked to complete the questionnaire anonymously and voluntarily during 20 minutes of their mathematics lecture at the beginning of the winter semester in 2009/2010.

### *Results*

After the reduction of items the internal reliability, measured by Cronbach’s Alpha, was nearly the same as in the first study ( $\alpha=0.84$ ,  $n=1273$ ) and indicates a high internal consistency. The alpha values of the three dimensions of the scale are still acceptable (see Table 2).

The validity of the questionnaire was tested by relating the MaSE-T scores to different variables such as grades in the final high school exam, gender, chosen school type, and the level of specialization of mathematics.

First, mathematics self-efficacy scores were correlated with the grades in the final high school exam. At a first glance marks in exams don’t give information about the belief in one’s own mathematics self-efficacy expectations. At a closer look mathematics self-efficacy can be seen as a predictor for mathematical academic outcomes (Multon et al., 1991; Betz & Hackett 1983). It can be hypothesized that the final exam grade in math can also be a predictor for mathematics self-efficacy expectations. A correlation between the two scales was significant ( $r=0.47$ ,  $p<0.01$ ,  $n=345$ ). This means that students with higher grades in their final school exam rated higher in the mathematics self-efficacy scale.

Second, gender differences have been investigated in order to test whether male subjects have higher mathematics self-efficacy scores than female subjects. Gender differences have been reported in earlier studies (Betz & Hackett, 1983; Randhawa & Gupta, 2000). Table 3 shows the total score of mathematics self-efficacy and the mean scores of the subscales separated into males and females. Males scored significantly higher in total MaSE-T and the subscales ‘real-world mathematical problems’ and ‘reasoning problems’ than females.

dimension	$n$	Cronbach’s Alpha	items
total MaSE-T	1273	0.84	15
mathematical problems	1309	0.77	5
real-world mathematical problems	1302	0.70	5
reasoning problems	1294	0.74	5

**Table 2. Reliability of the MaSE-T scale.**

### Working Group 14

	males ( <i>n</i> =319)		females ( <i>n</i> =992)		test of significance
scale	<i>Mean</i> <sup>2</sup>	<i>SD</i>	<i>Mean</i>	<i>SD</i>	p
total MaSE-T score <sup>1</sup>	52.1	9.7	49.5	9.1	<0.001
mathematical problems	18.5	4.3	18.4	4.1	0.812
real-world mathematical problems	19.3	3.6	17.9	3.5	<0.001
reasoning problems	14.3	4.1	13.1	4.0	<0.001

<sup>1</sup>Higher scores in mathematical self-efficacy indicate a greater confidence in the ability to accomplish the mathematical task.

<sup>2</sup>Minimum score 15 / maximum score 75.

**Table 3. Gender differences in MaSE-T scores.**

Comparing the MaSE-T scores of the groups aiming for different school types (table 4), also significant main effects for groups ( $F=7.80$ ,  $F_{(3,14)}<5.56$ ,  $p<0.01$ ) were identified. A post hoc analysis reveals that prospective teachers for intermediate secondary schools have a significantly higher MaSE-T score ( $p<0.01$ ) than the other groups. Lowest MaSE-T scores had students who will become teachers for schools for children with special needs, which were significant lower than the scores of the other groups<sup>3</sup> ( $p<0.05$ ).

Comparing groups according to students' with different levels of specializations in mathematics with the MaSE-T score (table 5), an ANOVA revealed a significant main effect of the factor *group* ( $F=33.24$ ,  $F_{(4,14)}<9.73$ ,  $p<0.001$ ). Post hoc tests showed that students who didn't choose mathematics as a subject had significantly ( $p<0.001$ ) lower MaSE-T scores than students who have chosen mathematics, as predicted. In addition, students who have chosen mathematics as minor have significant lower MaSE-T scores ( $p<0.001$ ) than the major and 'middle' students (see Table 5).

These two results between groups, study course specialization as well as the subject choice of mathematics, are covered by the findings of Hackett and Betz (1989). They found that mathematics self-efficacy beliefs are predictive for the choice of major. On the one hand, students with low self-efficacy expectations in mathematics avoid studying mathematics or don't choose mathematics as their major (Table 4). On the other hand, in the group of students who have chosen mathematics as a subject, the students with the highest MaSE-T score are those who will become teachers for intermediate secondary school. In contrast, students with low mathematics self-efficacy beliefs often choose to become primary school teachers because they often think that only basic arithmetic skills are needed. Secondary school teacher students

## Working Group 14

know they'll have to do some 'real' math, consequently the group of prospective teachers for intermediate secondary school score the highest at MaSE-T.

Group	<i>n</i>	<i>Mean</i>	<i>SD</i>
Primary school	640	49.3	9.1
Secondary general school	189	49.2	9.7
Intermediate secondary school	362	53.3	8.5
School for special needs children	120	46.4	9.9

**Table 4. Mean MaSE-T scores of different study course specializations**

Group	<i>n</i>	<i>Mean</i>	<i>SD</i>
major subject	401	53.0	8.3
'middle' subject	396	52.6	8.5
minor subject	275	49.0	8.3
math not as a subject	237	42.7	9.2

**Table 5. Mean MaSE-T scores for different levels of specializations in mathematics.**

## DISCUSSION

In this paper, a questionnaire for measuring mathematics self-efficacy expectations of prospective teachers (MaSE-T) has been introduced. The refined 15-item MASE-T scale achieved a suitable level of internal reliability ( $\alpha=0.84$ ). Validity of the scale was indicated on one hand by producing significant gender differences. On the other hand, a positive correlation between MaSE-T and the final school exam grade has been found. The differences between samples grouped according to their chosen school type and to the different levels of specializations in mathematics also indicate the validity of the MaSE-T scale. While students with higher MaSE-T score chose mathematics as their major subject, students with lower scores avoided mathematics as field of study.

In future studies, the change of students' mathematics self-efficacy in different learning scenarios for students in mathematics education can be measured. By this, lectures and tutorials can be evaluated regarding to the change of the mathematics self-efficacy.

## NOTES

<sup>1</sup> In this paper 'prospective teachers' means students at universities in the first phase of the German teacher education.

<sup>2</sup>The reduced 15-item questionnaire can be downloaded on [www.sail-m.de](http://www.sail-m.de) in a German and English version. This article is based on the German version, the English one hasn't been validated yet.

<sup>3</sup>This group contained a relative high number of students which didn't choose mathematics as subject. A closer look at the data showed that the mean of the major subject students is similar to the mean of intermediate secondary school students.

## ACKNOWLEDGMENTS

Work described in this article is funded by the German Federal Ministry of Education and Research.

## REFERENCES

- Bandura, A. (1977). Self-efficacy: toward a unifying theory of behavioral change. *Psychological Review*, 84(2), 191-215.
- Bandura, A. (1997). *Self-efficacy. The exercise of control*. New York: Freeman.
- Barbeite, F. G. & Weiss, E. M. (2004). Computer self-efficacy and anxiety scales for an internet sample: testing measurement equivalence of existing measures and development of new scales. *Computers in Human Behavior*, 20, 1–15.
- Betz, N. E., & Hackett, G. (1983). The relationship of Mathematics Self-Efficacy Expectations to the selection of Science-Based College Majors. *Journal of Vocational Behavior*, 23, 329-345.
- Cassidy, S. & Eachus, P. (2002). Developing the computer user self-efficacy (CUSE) scale: investigating the relationship between computer self-efficacy, gender and experience with computers. *Journal of Educational Computing Research*, 26(2), 133–153.
- Compeau, D. R. & Higgins, C. A. (1995). Computer self-efficacy: development of a measure and initial test. *MIS Quarterly*, 19(2), 189–211.
- Hackett, G., & Betz, N.E. (1981). A self-efficacy approach to the career development of women. *Journal of Vocational Behavior*, 18, 326-339.
- Hackett, G., & Betz, N.E. (1989). A exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research Mathematics Education*, 20(3), 261-273.
- Kabiri, M. & Kiamanesh, A. R. (2004). The role of self-efficacy, anxiety, attitudes and previous math achievement in students' math performance. In *Proceedings of the Third International Biennial SELF Research Conference*, Available at [http://www.self.ox.ac.uk/Conferences/2004\\_Kabiri\\_Kiamenesh.pdf](http://www.self.ox.ac.uk/Conferences/2004_Kabiri_Kiamenesh.pdf) (last accessed on September 15th 2010)

- Kranzler, J. H. & Pajares, F. (1997). An exploratory factor analysis of the mathematics self-efficacy scale – revised (MSES-R). *Measurement and Evaluation in Counseling and Development*, 29, 215–228.
- Liu, X. & Koirala, H. (2009). The effect of mathematics self-efficacy on mathematics achievement of high school students. In: Proceedings of the NERA Conference 2009. Available at [http://digitalcommons.uconn.edu/cgi/viewcontent.cgi?article=1029&context=nera\\_2009](http://digitalcommons.uconn.edu/cgi/viewcontent.cgi?article=1029&context=nera_2009) (last accessed on September 15th 2010)
- May, D. & Glynn, S. M. (2008). *A mathematics self-efficacy questionnaire for college students*. Paper presented at the Conference on Research in Undergraduate Mathematics Education, Mathematics Association of America, San Diego. Available at <http://mathed.asu.edu/crume2008/Proceedings/May%20SHORT.pdf> (last accessed on September 15th 2010).
- Multon, K., Brown, S., & Lent, R. (1991). Relation of self-efficacy beliefs to academic outcomes: a meta-analytic investigation. *Journal of Counseling Psychology*, 38(1), 30-38.
- Pajares, F. & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: a path analysis. *Journal of Educational Psychology*, 86(2), 193–203.
- Pajares, F., & Kranzler, J. (1995). Self-efficacy beliefs and general mental ability in mathematical problem-solving. *Contemporary Educational Psychology*, 20(4), 426-443.
- Pajares, F., & Miller, M. D. (1995). Mathematics Self-Efficacy and Mathematics Performances: The Need for Specificity of Assessment. *Journal of Counseling Psychology*, 42(2), 190-198.
- Randhawa, B. S. & Gupta, A. (2000). Cross-national gender differences in mathematics achievement, attitude, and self-efficacy within a common intrinsic structure. *Canadian Journal of School Psychology*, 15(2), 51–66.
- Zimmerman, B. J. (2000). Self-efficacy: An essential motive to learn. *Contemporary Educational Psychology*, 25, 82–91.

# **INTRODUCTION TO THE PAPERS OF WG15 TECHNOLOGIES AND RESOURCES IN MATHEMATICS EDUCATION**

Jana Trgalová (INRP-ENS, Lyon, France)

Anne Berit Fuglestad (University of Agder, Kristiansand, Norway)

Mirko Maracci (University of Pavia, Italy)

Hans-Georg Weigand (Wuerzburg University, Germany)

## **INTRODUCTION**

Technologies in mathematics education have been a topic of the working group since the CERME 1999. Since then, technologies as well as research in this field have significantly evolved. In particular, recent years have brought about an important shift in considering technologies within a wide range of resources for students, teachers and teacher educators. There have been new developments concerning software, hand-held technology, online classroom activities, but also more traditional geometry tools, curricular materials, textbooks etc. The introduction of the term “resource” in the title of the technology working group since the CERME 6 congress in 2009 reflects this shift (Gueudet, Bottino, Chiappini, Hegedus & Weigand, 2010).

Technologies and resources have become more abundant and accessible, bringing about some changes in math teaching and learning (Hoyles & Lagrange 2010). They offer new possibilities for representation, modelling, simulation and access to information and knowledge (Tooke & Henderson 2001), and suggest new ways of generating student engagement, motivation and creativity (Passey, Rogers, Machell & McHugh, 2004). Teaching methods with technologies are increasingly focusing on problem-solving approaches (Fuglestad, 2009). Technologies also offer means of collaborating and sharing resources between students and teachers (Trouche, Hivon, Noss, & Wilensky, 2010). But presently, despite the use of digital technologies in the public and business world, and the tremendous number of research and practical classroom papers, the use of technologies in mathematics education and the impact on the change of curricula are still limited (Hoyles & Lagrange, 2010).

The group organized the contributions around four main themes: (1) Design and use of technologies and resources, (2) Technologies, resources and teachers’ professional development, (3) Students’ learning with technologies and resources, and (4) Technology-based assessment in math teaching. Surprisingly, the latter does not appear as a matter of concern since all papers addressed the first three themes.

The group work was organized in two sub-groups discussing respectively the issues of technologies and resources in relation with teachers’ professional development (theme 2), and the issues related to the design of technologies and resources and their impact on students’ learning (themes 1 and 3). In the following, we give a brief overview of the group work according to the three above-mentioned themes.

## THEMES IN WG15

### 1) Design and use of technologies and resources

Papers related to this theme concern on the one hand the design of innovative technological tools or resources. Ladel & Kortenkamp design and develop a multi-touch technology for learning numbers and operations, Sabra & Trouche study the design by a community of teachers of online secondary school math textbook. On the other hand, the use of existing technological tools and resources are addressed, such as the use by teachers of specific curricular material accompanying TI-Nspire (Persson) or of math textbooks (Özgeldi & Çakiroğlu), students' use of TI-Nspire technology (Aldon), of a spreadsheet (Tabach) or of graphic calculators (Consciência & Oliveira; Storfossen). The question of the design of specific tools for students with special needs, such as visually impaired students (Kohanová) or students with dyscalculia, was raised. In the group discussions, *added value* of technology appeared as the most important element determining what kind of tool to use, how and what for. Various roles have been assigned to technology in math education by the participants: fostering motivation, enhancing calculation, visualization, (guided) exploration, hands-on experience on abstract models, allowing validation by providing feedback, but also helping teachers teach with systems providing individualized learning paths based on students' competence diagnosis.

### 2) Technologies, resources and teachers' professional development

Papers related to this theme investigate teachers' use of technologies and resources (both inside and outside the classroom), factors affecting their integration in classrooms, and issues related to teachers' professional development. These issues are addressed focusing on different aspects, from different perspectives and adopting different methodologies. The analysis of the teachers' practice in classroom is investigated at different levels of granularity, each one requiring the design and use of different methodological tools, from fine-grained analyses (Billington), through survey studies (Bretscher), to studies combining both qualitative and quantitative methods (Drijvers). As it clearly emerged, the integration of technologies and resources in the classroom poses a number of challenges to teachers, several of which can be described and discussed in terms of the Technological Pedagogical Content Knowledge that they need to develop (Fuglestad). From the instrumental perspective, some of these challenges can be related to the need for the teacher to transform an ICT tool both to a mathematical and a didactical instrument (Haspekian). All these considerations raise the issue of the teachers' professional development and how it impacts the teachers' use of technology. Studies concerning this issue regarded, for instance, the actual (Amado) or intended use (Pittalis) of technology by pre-service or in-service teachers, who attended specific training courses on technologies in math education. The issue of teachers' professional development is strictly related to the issue of the dissemination of research results

among teachers. Developing communities involving both researchers and teachers is proposed as a means to trigger collaboration and communication among them and to foster the dissemination of research results (Lagrange).

### 3) Students' learning with technologies and resources

Papers addressing issues related to this theme concerned both widespread technological tools, such as dynamic geometry software (Attorps *et al.*; Camacho & Santos), symbolic or graphic calculator (Consciência & Oliveira; Storfossen; Weigand), and more innovative tools like TI-Nspire handheld device (Aldon; Persson) or online games (Kolovou & van den Heuvel-Panhuizen). Some of these papers focus on *student-tool interactions* in order to study processes of students' appropriation of the tool (Consciência & Oliveira; Camacho & Santos) or students' transition between various tools (Tsitsos & Stathopoulou). These studies draw mostly on the *instrumental approach* (Rabardel 2002) as a theoretical framework specifically designed for studying teaching and learning phenomena involving technology. In other papers technologies are considered rather as a *tool mediating mathematics learning*. The choice of a tool and the design of appropriate tasks are the main concerns in these papers (Attorps *et al.*; Kolovou & van den Heuvel-Panhuizen). The task design is supported by two main theoretical approaches: *variation theory* (Marton & Booth, 1997) and *theory of didactical situations* (Brousseau, 1997). Studies involving innovative pieces of software investigate their potentialities for mathematics learning (Ladel & Kortenkamp) as well as for organizing and exploiting the internal resources (Aldon). Another important issue discussed in the group was the question of *assessment* of students' competencies (Weigand): what kind of knowledge and skills developed by the student working with technologies do we want to assess and how can we assess it?

## CONCLUSION

What lessons can be learnt from the work of the WG15? Although the three themes proposed to the working group appeared as strongly articulated, some important issues emerged for each of them. The design of technologies and resources seems to be driven by the consideration of their *added-value* for the teaching and learning mathematics and relies on the *users' feedback*, which becomes part of the design process. The group discussions allowed getting a deeper insight into the *complexity of ICT integration* in teachers' practices, which requires a *double instrumental genesis* in teachers: a personal one, yielding an instrument for teachers' math activity, and a professional one, yielding an instrument for math teaching. This raises the issue of teacher training focusing on the development of teachers' technological pedagogical content knowledge. The idea of teachers' communities sharing resources and practices in using ICT emerges as a powerful means to favour teachers' professional development. In studying students' learning with technologies, the *instrumental approach* becomes widely used to analyse student-tool interaction. It



highlights a strong interconnectedness of mathematical and technological knowledge. However, the impact of technology on students' achievements appeared as very difficult to measure, mostly due to the lack of appropriate methodology.

What perspectives can be outlined for the future CERME conference? There is still a need to develop a comprehensive theoretical framework to support teachers in their integration of ICT to the benefit of students' learning and methodological tools to evaluate the impact of using ICT on students' learning and teachers' practices. Emerging topics, such as communities of practice, quality of resources or best practices, require further theoretical and methodological development. Finally, some topics, which are under-represented, would deserve researchers' interest: use of "new" new technologies, such as interactive white board, mobile devices, Web 2.0, as well as designing ICT for students with special needs.

## REFERENCES

- Brousseau, G. (1997), *Theory of didactical situations in mathematics*. Kluwer Academic Publishers.
- Fuglestad, A. B. (2009). ICT for Inquiry in Mathematics: A Developmental Research Approach. *Journal of Computers in Mathematics and Science Teaching*, 28(2), 191-202.
- Gueudet, G., Bottino, R. M., Chiappini, G., Hegedus, S., & Weigand, H.-G. (2010). Technologies and resources in mathematical education. Introduction to the Working group 7. In V. Durand-Guerrier *et al.* (Eds.), *Proceedings of CERME 6* (pp. 1046-1049), January 28<sup>th</sup>-February 1<sup>st</sup> 2009, Lyon, France.
- Hoyles, C., & Lagrange, J.-B. (Eds.) (2010). *Mathematics Education and Technology - Rethinking the Terrain*. The 17th ICMI Study, Springer: New York.
- Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah, N.J.: Law Erlbaum.
- Passey, D., Rogers, C., Machell, J., & McHugh, G. (2004). *The Motivational Effect of ICT on Pupils*. Research Report RR523, Lancaster University. Retrieved from <http://portaldoprofessor.mec.gov.br/storage/materiais/0000012854.pdf>
- Rabardel P. (2002), *People and technology — a cognitive approach to contemporary instruments*. Retrieved from <http://ergoserv.psy.univ-paris8.fr/>
- Tooke, J., & Henderson, N. (2001). *Using Information Technology in Mathematics Education*. The Haworth Press, Inc.
- Trouche, L., Hivon, L., Noss, R., Wilensky, U. (2010), Connectivity and Virtual Networks for Learning. In J.-B. Lagrange, & C. Hoyles (Eds.), *Mathematics Education and Digital Technologies: Rethinking the terrain* (pp. 439-462), New York: Springer .

# **MATHEMATICS STUDENT TEACHERS' PEDAGOGICAL USE OF TECHNOLOGIES – DIFFERENT TAXONOMIES IN THE CLASSROOM**

Nélia Amado,

FCT, University of Algarve e UIDEF, University of Lisbon, namado@ualg.pt

*This paper refers to the use of technologies by student teachers in the classroom. I present a taxonomy that help to distinguish the various forms of teacher and students' uses of technology. In a pedagogical perspective the use of technologies can be seen in three distinct ways: as an accessory, as a teacher-centred activity and as a student-centred activity. I present three tasks illustrating this taxonomy. The role of the teacher and role of the student, the nature of activities, the environment and the class management are three aspects whose combination is essential to understand what a pedagogical perspective on the use of technologies.*

Key-words: student teacher, technologies, pedagogical perspective, tasks.

## **INTRODUCTION**

Hoyles and Noss (2003) pointed out that technologies emerged to relieve us of the more routine and monotonous, repetitive and even somewhat less interesting work. But while alleviating this work, technology impels us to invest in higher order skills, such as knowing how to interpret a graph, making conjectures, being able to relate concepts and use them, learning to critically analyze the results, and flexibly using different mathematical representations. In turn, an appropriate use of technology may lead students to learn more mathematics and more profoundly.

On the other hand, technology if used in inappropriate manner does not produce major changes in learning. Still, the computer allows some types of activities, such as discovery, and facilitates the development of mathematical intuition in such ways that would be very difficult or even impossible to achieve without the technology.

Thus we arrive at the imperative of a substantial change on the teacher's role in the classroom, which remains a sensible current recommendation (Laborde et al., 2005). We therefore realize how important it remains to investigate how teachers introduce technologies in the classroom.

## **THE USE OF TAXONOMIES TO DESCRIBE THE USE OF TECHNOLOGIES**

Pierce and Stacey (2001) have considered that the introduction of technologies in mathematics teaching and learning can take place at two levels: functional and pedagogical. The distinction between these levels appears to be in the student's role and access to the technology. In the functional perspective the use of technology seems to be confined to the teacher and the students' role is mainly of mere

spectators. On the other hand, a pedagogical perspective would be one that takes place in an educational context, namely a school subject or other, where there is direct interaction of teacher and students with technological tools.

Indeed, the concept of technology as a pedagogical tool entails essential aspects, such as rethinking the methods and purposes of learning mathematics, the roles of the teacher and students, the nature of the activities proposed and the class management (Kokol-Voljc, 2003). The concept of technology as a pedagogical tool is inseparable from the use that is made of it. Technologies can be regarded as pedagogical tools when three crucial conditions are simultaneously assured: the topic, the purpose and the opportunity.

In brief, the notion of pedagogical perspective is connected with the tool use and it depends on who is using it and on the classroom situation where it is used.

Galbraith (2002) and Goos (2005) developed a taxonomy on the use of technology by mathematics teachers. This taxonomy helps to distinguish the various forms of teacher and students' uses of technology. These authors presented a set of metaphors to describe how technologies can pave the way for incorporating new roles in teaching. Thus, when the knowledge and expertise of teachers in the use of technology is small or limited, and the computer use is not their personal initiative and will, but the result of an imposition of the educational system itself, we have a dominating action (technology as master) of technology on teaching practices. Today many secondary schools have interactive whiteboards, forcing the teacher even without genuine desire to use it solely for the purpose of satisfying an imposition of the educational system. The current pressure on the teacher to engage with the use of technologies can lead to concentrating on the basic contents, without any concern about the impact that technology may have beyond the immediate. The use of technology becomes the result of complying with an obligation, on a specific time.

On a second level of use the imposition or recommendation for use of technologies does not dominate. The teacher has some confidence and interest in technological advances, and enjoys knowing and using them. For example, the computer is used for word processing, for the development of materials such as worksheets and tests and technology is used to support lessons. However, there is still no change in the classroom activities. Galbraith (2002) and Goos (2005) consider that this is a use of technology as a servant (technology as servant). In addition to the worksheets or tests designed on the computer, using a PowerPoint in the classroom can be an example of a use of technology as a servant.

These two perspectives of using the technologies seem to fit the functional level, in that it does not introduce major changes in classroom activities or allow students direct interaction with technology. Students are generally spectators, although they may believe that the lesson may have been more colourful, more modern, with a few adornments out of the traditional. Sometimes students find it very nice to have a

lesson with a PowerPoint presentation or with the teacher writing on the IW instead of the blackboard but there is nothing new to the learning process, and the roles of teachers and students do not change. Similarly, the teacher can use the calculator linked to an IW to show to the whole class the results of certain operations and procedures or to display the graph of a function, managing to do it with more quickly, accurately, and using a great diversity of examples instead of just one.

A third perspective of using technology is a partnership (technology as partner). This situation occurs when technologies are used occasionally in the classroom by the teacher and students, enabling them to achieve some knowledge that would otherwise be very difficult or even impossible. In this case, teachers develop a partnership with the technologies as tools to help solving the problems and activities and as a means of promoting learning. This way of using technology gives the students more power over their own learning, but for that to happen it is necessary that the tasks should be designed and adjusted to the purposes of learning. This is where the big issue is - the nature of the tasks and how they are presented to students. These should enable the student to experiment, investigate and draw conclusions. Students need to have access to the computer in the class, and the activities proposed should be rich enough and appropriate to promote learning. Sometimes there is the simple transposition of a paper and pencil assignment to a task with technologies. A task planned for paper and pencil cannot simply be proposed to be done with technology with unchanged learning aims. It is well known that some paper and pencil problematic tasks may become trivial when solved with technology. When tasks are technology based, the situation is more complex, as adding technology deeply affects the task itself (Laborde, 2008).

Finally, the last way to use the technologies to which these authors designate by extension of self (technology as an extension of self) is the highest level of technology use. Such use must occur in a mathematics laboratory equipped with computers, adequate software, graphing calculators, view screen, sensors, and possibly an IW. At this level, effective and creative use of technology is an integral part of the repertoire of the teacher, along with their teachers' competence and knowledge of mathematics. In this case, it is very important to know how to put the technologies for students learning and promote their ability to use them in a timely, intelligent and critical way. The use of technology should not be reduced to a means of confirming answers or illustration of mathematical objects but have the function of raising questions, creating situations that lead students to think and encourage their participation in class work.

## **AIMS OF THE STUDY**

Thomas and Cooper (2000) argue that there is great inconsistency between what prospective teachers learn about technology and their work later on in the classroom. The transposition of the experience offered by the pre-service training into the

classroom practices is a complex process. Hence, the purpose of this study is to understand how prospective teachers, with a solid and consistent preparation for the pedagogical use of computer, integrate technologies when they start teaching at the beginning of their professional practice. Several questions were formulated including:

- What is the importance and value of the knowledge and training they have acquired in the pedagogical use of technologies?
- How is their use of technology integrated in their teaching?

## **THE EMPIRICAL DATA**

This study involved two teams of two student teachers each, in two different schools. The data collected included classroom observation, and interviews with the student teachers at different times of the school year. It has also integrated the collection of teaching materials produced to support lessons and other documents relevant to the study. The data here referred represent a small portion of all the empirical material obtained during the research that lasted for one academic year.

These student teachers were on their final year of training to become professional secondary mathematics teachers, after having taken four years of university studies in mathematics and education courses. During their academic studies they had a specific training on technologies in mathematics education as part of a course in mathematics education. They all had the same preparation on the use of several tools and software for mathematics teaching (dynamic geometry, spreadsheets, graphical calculators, applets...)

I present and analyze three lessons involving 10<sup>th</sup> grade classes (students aged 15-16) with the use of computers, each one based on a different assignment. The lessons reflect different characteristics that illustrate diverse ways of realizing the pedagogical perspective of technology use.

The taxonomies developed by the authors mentioned above have proved insufficient to categorise the data collected. As seen in other studies a combination of theoretical ideas and data supported variability justified the reason to develop a new taxonomy. This situation resembles, for example, the work developed by Drijvers, Doorman, Boon, Reed and Gravemeijer (2010), whose empirical data led to six orchestration types, placing the focus on how the teacher intentionally and deliberately organises the classroom and the use of technological artefacts in a given mathematical task. The authors describe the six types in terms of their specific features but they also divide them into two broad classes: three of the types can be seen as teacher-centred while the other three can be described as teacher-centred orchestrations.

In the case of the present study, substantial differences identified are located in the proposed tasks and in the form of presenting them to the class by the student teachers. Therefore it became necessary to distinguish and rethink about the

classroom practices of the student teachers with the use of technologies in connection to the tasks presented and the particular ways in which they were formulated. The analysis sustained the perception that technology use under a pedagogical perspective can be seen in three distinct ways: as *an accessory*, as *a teacher-centred activity* and as *a student-centred activity*. In certain aspects, this categorisation is in tune with the orchestration types (Drijvers et al, 2010) but it places an increased focus on the nature of the task as a crucial condition to promote a certain use of the computer.

### **Pedagogical perspective I – The equation of the circle “appears”**

In this class, the student teachers (team A) wanted to lead their students to the discovery of the Cartesian equation of the circle. The activity asked students to construct circles and with the help of Cabri Geometry find the equation of the circle and obtain the centre coordinates. This is the translation of a paper and pencil activity to the computer that does not seem to have added anything new to students' learning. In the next lesson they did the same activity with paper and pencil.

#### **Circles**

Using **CABRI GEOMETRY II** follow the steps:

Select the option “Show axes” and then mark a point.

With the option “Equation or coordinates”, identify the coordinates of the point.

With the option “Circle”, construct a circle by centre in the point and any radius.

Mark a point on the circle, with the option “Point on object”.

With the option “Distance or length” measure the distance from the point to the centre of the circle.

Get the equation of the circle, with the option “Equation or coordinates”. Write the equation of the circle.

With the option “Pointer”, change the position of the centre, by drawing and check the changes in the equation. What do you conclude?

With the option “Pointer”, select the point on the circle and change the radius. What do you conclude?

### **Figure 1: The task about the equation of the circle**

The main intention for using the computer in this class was to lead students to discover by themselves an algebraic representation of a geometric object, avoiding the algebraic manipulation. The student teachers expressed this idea as follows:

ST: We wanted that the students discovered the equation of the circle. We used the computer because they could do a series of experiments and arrive at the equation without having us writing on the board and saying: this is the equation of a circle. They could try and reach that conclusion by themselves

The equation of the circle emerges not by the hand of the teacher, but by Cabri. I argue that an essential part of mathematical knowledge was omitted; students did not have the chance to develop a mathematical concept that is associated with another one already known the distance between two points in a Cartesian plane. In an elusive way the computer allowed these students to "reach" the Cartesian equation of

the circle. The computer use can be seen as a misleading servant in that it exhibited the equation of the circle but did not allow students to perceive or understand its basis. Thus we can consider this use of the computer as *an accessory*.

### **Pedagogical perspective II – The area of the square in 29 steps**

The area of the square besides being a dynamic problem, allows multiple connections between mathematical contents studied in 10th grade. However, there seems to be an excess of instructions from the student teachers (team A) that totally affects the students thinking, not giving them space to reflect and to establish their own strategy for addressing the problem. It is worth noting that students were used to working with this software since the beginning of the school year and that this situation was presented at the end of the year. One possible explanation for this self-directed activity can be found in the way the student teachers conceive the roles of the teacher and students in the use of technology in the classroom. These student teachers seem to feel the need to provide the path to the students and leave them no room to find unforeseen situations.

In the twenty first steps nothing new emerged comparing to other constructions made with Cabri in previous classes. The last nine steps were those who brought something new to the students.

With **CABRI GÉOMÈTRE II**, follow the steps:

We want to make the construction shown. So we start by constructing the square [ABCD] and, afterwards we construct the square [EFGP].

- 1) With the option Segment, construct the line segment [AB].
- 2) Now we have to construct the line segment [BC] with the same length of [AB]. Use the Circle and construct a circle by center B and radius [AB].
- 3) Use the option Perpendicular Line and construct the line perpendicular to [AB], and passing through B.
- 4) Construct the point of intersection of the circle and the perpendicular line (option Intersection Points), naming it C (option Label).
- 5) With the option Parallel Line, construct the line passing through C and parallel to [AB].
- 6) Construct a perpendicular line to [AB], passing through A.
- 7) Construct the intersection of the last line and of the parallel line to [AB], and name it D.
- 8) Construct the segment lines [BC], [CD] e [DA].
- 9) Hide the circle and all the lines.

How?

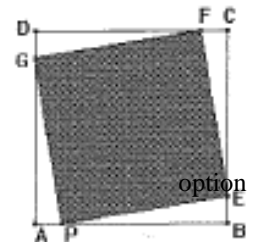
- Select the option Hide/Show.
- Select the objects to be hidden.

- 10) Mark a point P on the segment line [AB], using the option Point on Object.
- 11) Measure the length of [AP], with the option Distance or Length.

How?

... ..

- 22) Tabulate the values of the area while point P moves on the segment line [AB].



### How?

- Select the table, with the option Pointer.
  - Activate the option Animation and click on point P, holding down this button and drag.
  - To stop the animation, click the left mouse button or the Esc key on your keyboard.
  - The values should appear in the table. If you want to view a larger number of registers increase the number of rows.
- 23) Construct a horizontal line passing through point O (the origin of axes).
- 24) Construct a perpendicular line to previous line passing through point O.
- 25) Use the option Measurement Transfer to transfer the length of the line segment [AP], to the x-axis and the area of the square [PEFG] to the y-axis.
- 26) Name the previous points by Q and R.
- 27) By each of these points construct perpendicular lines to the axes and construct the point S as the intersection the two perpendiculars.
- 28) Activate the option Trace On/Off and click on point S.
- 29) Animate the point P using the option Animation.

**Observe the model and answer to the following questions:**

- 1. Between what values can vary the displacement of point P?**
- 2. What reflects the graph obtained?**
- 3. When is the maximum area reached? And when is the minimum reached? Interpret it in the context of the problem.**
- 4. Indicate the co-domain of this function.**
- 5. Define algebraically the function.**

### **Figure 2: The task about the area of the square**

The initial orientation seems to have been both excessive and unnecessary, given that students had done similar constructions before. This activity could have been an opportunity for students to practice and investigate ways to make the construction, using their own imagination and their understanding of the diagram, rather than following a course set by the hand of the teachers. These student teachers, like other more experienced teachers, apparently feel difficult to give their students the opportunity to create, to think independently, to imagine, and to find a way other than perhaps the one they have thought of. For these reasons, I believe that we are facing a pedagogical perspective that is *teacher-centred*.

### **Pedagogical Perspective III – Slicing the cube**

In the study of cross sections, the students usually work with paper and pencil or manipulative materials. Students have great difficulty in visualizing the cross sections even when they are given opportunities to view and manipulate the physical object. A dynamic geometry environment is an excellent choice to promote the manipulation and visualization of the cross sections. In this case (team B), the software GEOMETRY was suggested to students, and they were given a reference guide to use the program.

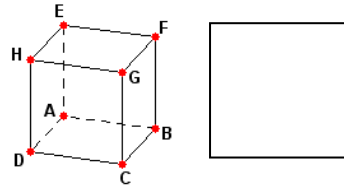


### Cross Sections

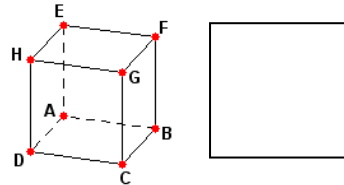
In this task we use the software GEOMETRIA. See the reference guide and follow the steps:

Draw a cube. In each case draw the cross section obtained by slicing the cube; identify and draw the geometrical shape produced.

a) The slicing plane is DBE.



b) The slicing plane is HFB.

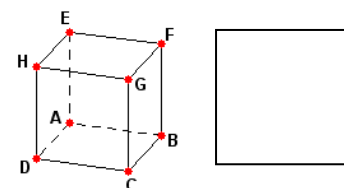


c) The slicing plane is IJK, where:

I is the midpoint of [HG];

J is the midpoint of [GF];

K is the midpoint of [FB].



**Figure 3: The task about cross sections**

In solving the various tasks proposed the students had the opportunity to choose their own strategy, issues have arisen naturally through the class and discussed with the teacher. The student teachers and their students were partners in a process of knowledge development, and together shared questions and difficulties. The students felt that technology helped them understanding the mathematics, gained confidence, and believed the computer was a useful tool in solving the problems. The student teachers themselves were pleasantly surprised with the involvement of students in the pursuit of answers and in the discussion even for students who usually seemed less motivated in mathematics classes. Thus with satisfaction they said:

ST: I am surprised. Even some of my more indifferent students were working as well as those more interested! I never saw them like this!

This was a clear case where the computer was used from a pedagogical perspective that is *student-centred*.

### FINAL REMARKS

The students teachers involved in this study had the same initial training, the same preparation regarding the use of technology, which included a discussion of their potentialities, possible approaches and educational purposes, but as shown in the data, their practices diverged in many respects.

Research has shown that preparation acquired in one or more academic courses are not enough for future teachers to make an immediate transfer of this knowledge to the classroom. Adler (1996), while accepting that formal academic courses should be

part of the education of future teachers argues that knowledge about teaching – in this case, how to use technology in the classroom – is not solely acquired in formal courses but rather it is developed through a continued participation in a community of teachers. Appropriate use of technology in the classroom actually requires learning through participation in a practice that is neither linear nor immediate. In each of the teams of student teachers, a pedagogical perspective of technology use in the classroom was marked by some differences that can be justified or explained by their own ways of thinking about mathematics teaching and learning and, particularly, by the way the student teachers saw the roles of teacher and students in the classroom.

These student teachers shared a willingness to have their students enjoying mathematics and engage them in learning; however they seemed to have different ideas of how to do it. One of them (from team B), a great supporter and user of technology, argued that schools should integrate all technologies that are now part of our daily life. All of them seemed to hold the idea that technology can be an important way to engage students and make them enjoy mathematics more. In team B a greater concern with the purposes and goals of the computer use in classes was prominent. The practices of these two student teachers revealed a view on using the computer that was according with the recommendations offered in the curriculum. It was clear their intention to inquire about and search, to find the best suited software to work on a particular content and the care taken in the preparation of the tasks. They were skilful in the ways in which they shared their power in the classroom, and gave their students a greater and better opportunity to engage in learning.

The other team put technologies at the same level of other materials, for example, manipulatives. Moreover, they showed a great concern with student disruptive behaviours and the fear of indiscipline was a cause for some initial tension. The second downside is related to how these student teachers perceived their role. The teacher was conceived as someone who holds all the knowledge and must pass it on to the students, which entailed to define the path that students should follow, and avoiding all difficulties in students' minds.

As mentioned above, the role of the teacher and role of the student, the nature of activities, the environment and the class management (Kokol-Voljc, 2003) are three aspects whose combination is essential to understand what a pedagogical perspective on the use of technologies means and what it may consist of.

## REFERENCES:

Adler, J. (1996). Lave and Wenger's social practice theory and teaching and learning school mathematics. In L. Puig, & A. Gutiérrez (Eds.), *Proceedings of the 20<sup>th</sup> Conference of the International Group of the Psychology of Mathematics Education*, (Vol. 2, pp. 3-10). University of Valencia, Spain.

- Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, 75, 213-234.
- Galbraith, P. (2002). 'Life wasn't meant to be easy': Separating wheat from chaff in technology aided learning? In *Proceedings of the 2nd International Conference on the teaching of mathematics (at the undergraduate level)*. University of Crete, Greece. Retrieved from <http://www.math.uoc.gr/~ictm2/>
- Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teacher's pedagogical identities as users of technology. *Journal of Mathematics Teacher Education*, 8, 35-59.
- Hoyles, C., & Noss, R. (2003). What can digital Technologies take from and bring to research in mathematics education? In A. Bishop, M. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Second International Handbook of Mathematics Education*, (pp. 323-249). Dordrecht: Kluwer Academic Publishers.
- Kokol-Voljc, V. (2003). What makes a tool a pedagogical tool? In T. Triandafillidis and K. Hatzikiriakou (Eds.), *Proceedings of the 6th International Conference Technology in Mathematics Teaching*, (pp. 92-96). University of Thessaly, Volos, Greece.
- Laborde, C. (2008). Multiple dimensions involved in the design of tasks taking full advantage of dynamic interactive geometry. A. Canavaro, D. Moreira, & M. I. Rocha (Eds). *Tecnologias e Educação Matemática*, (pp. 36-50). Lisboa: SPCE.
- Laborde, C. & Perrin-Glorian, M. (2005). Introduction: Teaching Situations as Object of Research: Empirical Studies within Theoretical Perspectives. In C. Laborde, M. Perrin-Glorian, & A. Sierpiska (Eds.), *Beyond the apparent banality of the mathematics classroom*, (pp. 1-12). New York, NY: Springer.
- Pierce, R. & Stacey, K. (2001). Reflection on the changing pedagogical use of computer algebra systems: Assistance for doing or learning mathematics? *Journal of Computers in Mathematics and Science Teaching*, 20(2), 143-161.
- Smith, D. (2002). How People Learn... Mathematics. *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the Undergraduate level)*. University of Crete, Greece. Retrieved from <http://www.math.uoc.gr/~ictm2/>
- Thomas, J. & Cooper, S. (2000). Teaching Technology: A New Opportunity for Pioneers in Teacher Education. *Journal of Computing in Teacher Education*, 17(1), 13-19.

# SYMBOLIC GENERALIZATION IN A COMPUTER INTENSIVE ENVIRONMENT: THE CASE OF AMY

Michal Tabach

Tel-Aviv University

*Learning beginning algebra is difficult for many students as they must master the use and interpretation of symbolic expressions. This study investigates the use of spreadsheets in a beginning algebra course and tracks the development of one student, Amy, as she moves towards the use of explicit symbolic expressions. Results indicated that the spreadsheet served as a scaffold, helping to bridge between arithmetic and algebra.*

Keywords: beginning algebra, spreadsheet environment, symbolic expressions

Introducing algebra to middle grade students is a challenge. Beginning algebra students must learn to incorporate symbolic expressions in their language, manipulate them, and make sense of them (Arcavi, 1995). Several research projects have employed the use of technological tools in order to overcome some of these hurdles by bypassing the need to communicate symbolically. One such example is Kaput's *SimCalc* project (Kaput, Carraher, & Blanton, 2007) which connected graphical, numerical, and visual representations in order to make sense of changing phenomena. A different approach was adopted by Tabach, Hershkowitz, Arcavi, and Dreyfus (2008), who implemented a functional approach within a spreadsheet environment. Regardless of the use of technological tools, in many countries, students are still expected to be able to generalize or model a given situation using symbolic notations, as well as manipulate symbolic expressions in a paper and pencil environment. Under these conditions, what is the place of computerized tools in a beginning algebra course? Can a computerized environment be used as a scaffolding tool, a tool that will help bridge the gap between arithmetic and algebra but which may be released from use as knowledge evolves?

This paper follows one student, Amy, during a year long beginning algebra course, which took place in a *Computer Intensive Environment* (CIE). This environment had the following characteristics: (1) full and unconstrained access to spreadsheets (Excel) in class and at home, (2) freedom of choice about if, when, and how to use the computerized tools, as well as the strategy employed to solve the given problem situations, and (3) an ad-hoc learning textbook consisting of a sequence of problem situations and simple tasks. An explicit invitation to use spreadsheets as a working tool (including the technical instructions thereof) was only given in approximately one-fifth of the problems, mostly in order to get acquainted with the tool. In all other cases, the teacher supported and legitimized any choices made by students regarding the ways, means, and strategies used to solve the problems. Thus, the intensiveness of the environment relates to the availability of the computerized tools at all times,

but not necessarily the intensiveness of their use. By focusing on one student, this paper aims to gain insight into the learning experiences and processes afforded and supported by such an environment.

## THEORETICAL BACKGROUND

Algebraic thinking involves describing phenomena and generalizations. There are several representations with which one may represent a given problem situation – story, table, graph, or symbolic expression. In a computerized environment, these representations are all dynamic notation systems. However, in a pencil and paper environment, symbolic expressions are the only notation system which stays dynamic (Kaput, 1992); hence its power and the importance of mastering it. The following sections briefly discuss student's development of symbolic generalization, including the use of symbolic expressions within a spreadsheet environment.

**Symbolic generalization.** In a pencil and paper environment, three stages of generalization processes by pre-algebra students may be identified (Arcavi, 1995; Friedlander, Hershkowitz & Arcavi, 1989). At the first stage, when students are aware that they should use symbols, they tend to represent quantities involved in a given situation by using different letters, disregarding existing relationships between the quantities. To demonstrate, let us consider the following situation:

Moshon had \$30 in his saving box at the beginning of the year. Moshon added \$5 to his saving box each week.

Algebra beginners may denote the amount in Moshon's saving box at the end of the first week by  $a$ , and the amount at the end of the second week by  $b$ . Such use conceals the existing connections between the amounts. At the second stage, students express relationships, but only partially. To follow our example, the amount in Moshon's saving box by the end of the first week is still denoted by  $a$ , but the amount at the end of the second week is denoted by  $a+5$ . Thus, the connection between the week and the amount of money is still hidden, while the connection between the amounts in consecutive weeks is explicit. Only at the third stage are students able to express full relationships among changing quantities in a symbolic way (Friedlander & Hershkowitz, 1997). That is, in our example,  $a$  will denote the week and  $30+5a$  denote the amounts in the saving box at the end of week  $a$ . The first two stages explore local connections, whereas only in the third stage general connections among variables appear. Unfortunately, pencil and paper manipulations of algebraic expressions which express only partial relationships may not lead to insightful results.

**Symbolic generalization in a spreadsheet environment.** Within a spreadsheet environment, students at all three stages are able to operate, experiment with, reflect on, and learn about relationships between the quantities involved (e.g., Sutherland & Rojano, 1993). This is possible because a large amount of numbers representing a certain variable can be produced and arrayed in such a way that the relationship is

“visible” even without the explicit and general symbolic expressions. This can be produced by one of the following options [1]:

(a) *Multi-variable approach* — “dragging” and considering a whole array of numbers (i.e. a column in the spreadsheet) as a “variable” dependent on another one (Dreyfus, Hershkowitz, & Schwarz 2001). Consider once again Moshon from the previous example. Students may use column A by entering the constant 5 for each cell. In column B, B1 will be 30. B2 will then be the joined amounts  $A_2 + B_1$ . (b) *Recursive generalization* — the use of recursive expressions, which emphasize local relationships between consecutive elements such as contiguous cells of the same column (Stacey & MacGregor, 2001). In this case, students may describe Moshon's savings by writing 30 in cell  $A_1$ , and in the next cell  $A_1 + 5$ . (c) *Explicit expression* — explicitly expressing the full general relationship among the variables. In our example, it means expressing the savings in B1 as  $30 + 5A_1$  (where column A represents the number of weeks passed).

It seems that students prefer recursive formula over explicit formula (Stacey & MacGregor, 2001). Such preference may indicate initial difficulties encountered when handling explicit formula even when, in many situations, it proves more efficient than recursive expressions. In fact, from a practical point of view, there is no actual difference between the output of working recursively or explicitly in a spreadsheet environment. That is, after entering any of the three types of symbolic expressions to a spreadsheet cell, and “dragging” the expression down, relevant numbers will be displayed whereas the inputted algebraic expressions remain hidden.

A main concern regarding students who use spreadsheets in their beginning algebra course is that they may come to feel too comfortable with the tool, becoming used to multi-variable and recursive expressions which express only partial relationships. In their study of students learning in such an environment, Tabach, Arcavi and Hershkowitz (2008) found that towards the end of the course, “... most students had shifted or started to shift to a pencil and paper environment, especially when they were confronted with situations requiring them to state and solve linear equations” (p. 69). This shift takes place gradually. What changes can we detect along the way? While the previous study reported on the class as a whole, this study focuses on one student in order to gain insight regarding the use of symbolic generalization in an environment which allows for both spreadsheet and paper and pencil tools.

## METHODS

**Amy** Amy was one of 27 students in her seventh grade beginning algebra course. She was chosen for this study because of the changes she underwent during the course. In the beginning of the year, students were given a test in order to assess their problem solving skills and to investigate their initial use of algebra in this process. Amy scored below the class average on this test. By the end of the year, she had progressed and scored above average on the final exam. Throughout the course, Amy

exhibited a positive attitude towards the use of computer software while also incorporating other sources during her algebraic work.

Amy, along with the other students in her class, had been initiated into the world of symbolic generalization during the very first algebra lesson. During this lesson, they worked in a paper and pencil environment on a problem situation involving changing phenomenon. During the second lesson, the use of Excel was introduced. Students were taught how to enter an expression (formula) into a cell and drag it down in order to produce a column of values. It should be noted that, as far as the students were concerned, Excel was a tool which could be used by entering an expression for the purpose of producing a string of values.

**Data Collection** Several tools were used to document classroom learning processes. Classroom observations were conducted and a detailed research diary was written immediately after each lesson. All working files that students saved in their computers were collected. Written works and assessment tasks of all students were collected.

## RESULTS

To follow Amy's advancement during the year we bring examples from her work on different activities which took place at different times during school year. Each example is related to one of three connected issues: (1) the use of symbolic expressions and Excel, (2) generalizing from a sequence of numbers, and (3) generalizing from a story.

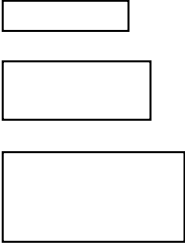
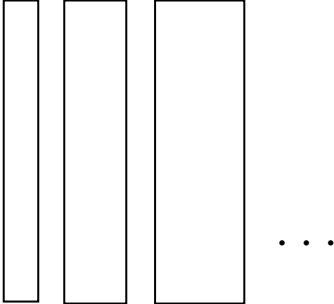
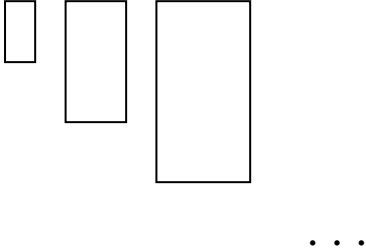
**Example 1: September 27, Growing Rectangles activity.** The activity was concerned with the process of growth in 3 rectangles during a period of ten years and entailed comparing the areas of the rectangles during this time period (see Figure 1). Students were requested to organize their data regarding the growing rectangles in a spreadsheet table and record the formulas used to construct their table. No instructions were given as for how to organize the data.

Amy organized her data in an *Extended Table* (Tabach & Friedlander, 2004), which contained the following ten columns (i.e. variables): the year, six columns for the linear dimensions of each rectangle, and three columns for the area measures. The columns were ordered by variable (Figure 2). Amy used symbolic expressions to describe each changing phenomena. Other than the length of rectangle C (column G) all linear changes were described by recursive expressions, which express a relationship between two consecutive numbers in a sequence (columns). Each of the first elements in columns B-G was a number. Thus, for the most part, Amy missed the opportunity to express the connections between the widths (columns B, D, and F) and the year (column A) as well the relationships between the lengths (columns C, and E) and their respective widths. The areas of the rectangles were described by multivariate expressions, which used the letters corresponding to the length and



## Working Group 15

width columns (e.g.,  $= B_2 * C_2$ ). The use of multivariate formulas conceals the relationships between the length and area of the rectangles. That is, at this time in learning, Amy was able to generate multivariable expressions or recursive expressions.

<u>Rectangle A</u>	<u>Rectangle B</u>	<u>Rectangle C</u>
		
<p>At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year.</p> <p>The length of this rectangle is always longer than its width by three units.</p>	<p>At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year.</p> <p>The length of this rectangle is constantly 10 units.</p>	<p>At the end of the first year, the rectangle's width is one unit, and it grows by an additional unit each year.</p> <p>The length of this rectangle is always twice the length of its width.</p>
<p>At what stages of the first ten years does the area of one rectangle overtake another's area?</p>		

**Figure 1. Problem situation of the *Growing Rectangles*.**

	A	B	C	D	E	F	G	H	I	J
1	Year	Wid. A	Len. A	Wid. B	Len. B	Wid. C	Len. C	Area A	Area B	Area C
2	1			1	10	1	2	$=B_2 * C_2$	$=D_2 * E_2$	$=F_2 * G_2$
3	$=A_2+1$	$=B_2+1$	$=C_2+1$	$=D_2+1$	10	$=F_2+1$	$=F_3 * 2$	$=B_3 * C_3$	$=D_3 * E_3$	$=F_3 * G_3$

**Figure 2. Amy's Algebraic generalizations in Excel for the *Growing Rectangles*.**

**Example 2: October 14, Buying a Walkie-Talkie activity.** The problem situation presented here concerned the weekly savings of eight children. Students were given the linear expressions which described these weekly savings (e.g., Moshon  $30+5x$ , Dina  $7x$ ) and were asked to add pairs of children's savings in order to determine which pair will be the first to buy a Walkie-Talkie which cost \$400, as well as when this will occur. Amy organized her work in a spreadsheet table (Figure 3). She used three columns to describe each joined savings: two columns to describe each child's savings, and a third column to describe the joined amounts. Somewhat redundantly, she copied the savings of the same child in different columns, as can be seen in columns B and E in Figure 3. This organization of data is similar to the one observed



in the *Growing Rectangles* activity, in the sense that she used many more columns than the minimum needed. Yet, her symbolic generalization is somewhat different here. In this example, Amy used explicit expressions (columns B, C, E, and F) which reveal the full connections between the variable and the phenomena, to describe the individual savings (while in the previous activity mainly recursive expressions were used). On the one hand, these expressions were given to her and she did not have to create them on her own. On the other hand, other students in the class created recursive expressions in this activity, despite the fact that explicit expressions were given (Tabach, Hershkowitz, & Arcavi, 2008). Finally, it should be noted that the symbolic expressions created on her own in this activity used multivariate formulas to describe the joined amounts (Columns D & G).

	A	B	C	D	E	F	G
1	Week #	Dina	Karin	Dina & Karin	Dina	Moshon	Dina & Moshon
2	1	$=7*A_2$	$=10*A_2$	$=B_2+C_2$	$=7*A_2$	$=30+5*A_2$	$=E_2+F_2$
3	2	$=7*A_3$	$=10*A_3$	$=B_3+C_3$	$=7*A_3$	$=30+5*A_3$	$=E_3+F_3$
4	3	$=7*A_4$	$=10*A_4$	$=B_4+C_4$	$=7*A_4$	$=30+5*A_4$	$=E_4+F_4$

**Figure 3: Amy's Algebraic generalizations in the *Buying a Walkie-Talkie* activity.**

**Example 3: October 18, Number Sequence task.** This example concerns generalizing from a sequence of numbers. Figure 4 present a given sequence. The task was to find an expression that can be written in cell B<sub>6</sub>, using A<sub>6</sub> as a variable.

	A	B
1	0	30
2	1	27
3	2	24
4	3	21
5	4	18
6	5	

**Figure 4: The Number Sequence task**

For Amy, finding an explicit expression for a given sequence at that time of the course was difficult. In trying to find the requested expression, Amy invented the following "story": "At the beginning of the year there were \$30 in a savings box. Each week, the amount decreased by \$3". Then she wrote the following expression in cell B<sub>6</sub>:  $30-3*A_6$ . It seems that creating a story context served as mediation between the numbers and the expression, which then enabled Amy to create an appropriate symbolic expression. The idea of reverting to a story problem might have stemmed from previous mathematical tasks which were situated within a story context. In general, switching between different representations of a mathematical

problem, between verbal, numerical, symbolic, and graphic representations, was common classroom practice. Still, Amy's solution method for this task was unique.

**Example 4: Feb [2]. 19, Identical Columns activity.** This example concerns making generalizations based on numbers and expressions. At this point in the year students had learned to present changing phenomena with symbolic expressions but had not yet learned to simplify these expressions. Children were given two sequences of numbers (in columns A and B) as well as instructions as to how the other columns should be filled in (see Figure 5 which displays the beginning of the Identical Columns activity).

Copy the following pairs of numbers to your spreadsheet:

	A	B	C	D	E	F	G
1	4	16					
2	2	13					
3	0	10					
4	-2	7					
5	-4	4					
6	-6	1					
7	-8	-2					
8	-10	-5					

1. a) Write in Column C the sum of the numbers from Columns A and B ( $\boxed{=A+B}$ ), and Write in Column D the sum  $\boxed{=2*A + 2*B}$ 
  - b) Use Columns A, B, or C in different ways, to create two additional columns that are identical to Column D.
2. a) Write in Column G the product  $\boxed{=10*(A + B)}$ .
  - b) Use only Columns A and B in different ways to create other columns that are identical to Column G.

**Figure 5: Identical Columns activity.**

Amy's first answer to question 1b was  $2*C$ . Reviewing classroom transcripts revealed that this expression was based on numerical considerations, as the first number in column C is 20 and the first number in column D is 40. Amy's second answer was  $A+A+B+B$ . This expression was based on a more symbolic consideration - decomposing the expression  $2A+2B$ . Regarding question 2b, Amy used the same logic that proved to work in the first task,  $A+B+A+B...$  that is decomposing the symbolic expression. Her second suggestion was  $A*10+B*10$ , which may have resulted from the original expression  $10*(A+B)$  or from her first answer. Either way, this example demonstrates that Amy is working with symbolic expressions to create new explicit expressions. Amy used the Excel spreadsheet to verify the correctness of her expressions, checking if the created columns were indeed identical.

**Example 5: March 16, The Secret Number activity.** This example demonstrates generalizing from verbal instructions. During a whole class discussion, the teacher requested students to, "Choose a number, add 3 to the chosen number, multiply the sum by 2, and subtract 6". To the children's surprise, the teacher was able to guess each child's original number from the final result. Asked to solve this "mystery", Amy said that the obtained number is twice the original number. The teacher agreed, and asked for an explanation. Amy began, "Let's choose 2 as our number". She then followed the steps with her chosen number and upon reaching the step "multiple the sum by 2", explained that no matter which number is given, the result of this step is an even number. Although Amy was not able to fully explain why the resulting number is twice the original number, we see in her attempt to do so a form of "generalizing using a generic example" (Mason and Pimm, 1984). She used the number 2 as an example of "any number".

Students were then asked to invent a similar "game". Amy's game was: "choose a number, add 1,000,000 and multiply by 2, subtract 2,000,000 and then subtract the number you choose. You will get back your chosen number." This invented "game" is close to the original game which was presented by the teacher. However, her use of the number 1,000,000 shows exaggeration, as if once again, the specific example she chose was really a generic example. The use of a generic example is considered to be a step towards symbolic generalization (Mason and Pimm, 1984).

**Example 6: April 15, Train story.** This example was taken from a test and demonstrates generalizing from a story. Students were presented verbally with a story about a train. The train consisted of one engine and several cars. The engine had 6 wheels and each car had 4 wheels. Students were first asked to calculate the number of wheels on a train with 5 cars, and show their solution method. Students were then asked to find the amount of cars contained in a train with 34 wheels. In order to answer this question, Amy generated an explicit expression connecting the number of cars in the train to the total number of wheels,  $6+4x$ . She then wrote a relevant equation,  $6+4x=34$ , solved it, and correctly interpreted the solution in the given situation. Although computers were available for use during the test period, Amy solved this problem with pencil and paper without referring to a spreadsheet.

## DISCUSSION

The discussion begins by summarizing the progress Amy made in her use of symbolic expressions. It then examines how Amy chose to use the available spreadsheet tool in this environment as she progressed in her use of symbolic expressions. In the first example, Amy used mainly recursive and multivariable expressions. In the second example, Amy was able to use given explicit expressions in a meaningful manner although on her own, she created multivariable expressions. In the third example, Amy was able to create explicit expressions, by making up a verbal story as a mediator between the sequence of numbers and the explicit

expression. The fourth example showed initial considerations which were numeric, followed by transitions among explicit expressions. In the fifth example, Amy attempted to reveal hidden connections with the use of a generic example. Finally, in the last example, Amy is able to write an explicit symbolic expression related to a given problem, write an appropriate equation, and solve it.

Amy's use of symbolic generalization and the transition between the ones she used (multivariate recursive, and explicit) is dialectic – during the same activity she implemented more than one kind of generalization. This dialectic process indicates the complexity of working with symbols in general and with explicit expressions in particular. Symbolic expressions are condensed, especially explicit ones, and hence more sophisticated. Although Amy weaved back and forth between the types of expressions used, as her familiarity with the symbolic notations increased, we can detect a shift towards working more with explicit expressions.

Amy acted as a resourceful student. From the beginning of the school year she used Excel's symbolic notations in a flexible manner according to her needs. In the first two examples, she used the spreadsheet as a number generator to act upon while in the fourth example she used it as a verification tool to check her symbolic generalization. This represents a shift in the role of the tool – from a tool used in the process of production to a tool used in the process of control. Amy also had the freedom to choose when not to use the tool. In the third example, Amy turned to a different form of expression, verbal, in order to assist her in forming a symbolic expression. By the end of the school year, she was able to form an explicit symbolic expression without any mediating tools.

A tool is not automatically an instrument. "... it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it with his activity" (Verillon & Rabardel, 1995, p. 84). For Amy, the spreadsheet tool became an instrument as she began to use it in a selective manner with control. By the end of the course, Amy was able to act without the tool and in this sense, the spreadsheet acted as a scaffold which was allowed to recede into the background. Amy's progress in symbolic generalization demonstrates that Excel did not limit her knowledge growth. In fact, it seems as though it facilitated her development.

## NOTES

1. One may enter two numbers in two consecutive cells in the same column, and "drag" both of them together. The resulting numbers in the column will follow a sequence with fixed difference. We do not consider such an approach symbolic. For more information, see Tabach, Arcavi, & Hershkowitz, 2008).
2. During the months November, December, and January, students learned about negative numbers and the four operations, as well as Descriptive Statistics. Hence no data from this period is provided.

## REFERENCES

Arcavi, A. (1995). Teaching and learning algebra: Past, present and future. *Journal of Mathematical Behavior*, 14(1), 145–162.

- Dreyfus, T., Hershkowitz, R., & Schwarz, B. B. (2001). Abstraction in context: The case of peer interaction. *Cognitive Science Quarterly. An International Journal of Basic and Applied Research*, 1(3&4), 307–358.
- Friedlander, A., & Hershkowitz, R. (1997). Reasoning with Algebra. *Mathematics Teacher*, 90, 442–448.
- Friedlander, A., Hershkowitz, R., & Arcavi, A. (1989). Incipient “algebraic” thinking in pre-algebra students. In R. G. Underhill (Ed.), *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1 (pp. 283–290). Paris, France;
- Kaput, J. (1992). Technology and mathematics education. In D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 515-556). Reston, VA: National Council of Teachers of Mathematics.
- Kaput, J., Carraher, D. W., & Blanton, M. (Eds.). (2007). *Algebra in the early grades*. Hillsdale/Reston: Erlbaum/The National Council of Teachers of Mathematics.
- Mason, J., & Pimm, D. (1984). Generic examples: Seen the general in the particular. *Educational Studies in Mathematics*, 15(3), 277-289.
- Stacey, K., & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on school algebra* (pp. 141–153). Dordrecht, The Netherlands: Kluwer.
- Sutherland, R., & Rojano, T. (1993). A spreadsheet approach to solving algebra problems. *Journal of Mathematical Behavior*, 12, 353-383.
- Tabach, M., & Friedlander, A. (2004). Levels of student responses in a spreadsheet-based environment. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the international Group for the Psychology of Mathematics Education*, Vol. 2 (pp. 423–430). Bergen University College: PME.
- Tabach, M. Arcavi, A. & Hershkowitz, R. (2008). Transitions Among Different Symbolic Generalizations by Algebra Beginners in a Computer Intensive Environment. *Educational Studies in Mathematics*, 69(1), 53-71.
- Tabach, M. Hershkowitz, R. & Arcavi, A. (2008). Learning beginning algebra with spreadsheet in a computer intensive environment. *Journal of Mathematics Behavior*, 27(1), 48-63.
- Tabach M., Hershkowitz, R., Arcavi, A. & Dreyfus, T. (2008). Computerized environments in mathematics classrooms: A research - design view. In L. D. English, M. B. Bussi, G. A. Jones, R. A. Lesh, B. Sriraman, & D. Tirosh (Eds.), *Handbook for International Research in Mathematics Education*, 2nd edition (pp. 784-805). NY, USA: Routledge.
- Vérillon, P., & Rabardel, P. (1995). Cognition and artifact: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education*, 9(3), 1-33.

# TEACHERS' AND STUDENTS' FIRST EXPERIENCE OF A CURRICULUM MATERIAL WITH TI-NSPIRE TECHNOLOGY

Per-Eskil Persson

Malmö University

*In a pilot research project, a curriculum material intended for two mathematics courses at Swedish upper secondary school has been constructed. The material is written for the use of TI-Nspire technology, with which it forms a dynamic system. Three teachers and three classes replaced their textbooks with this material during a half semester, and their experiences, as well as the general learning outcomes for the students, were investigated using various methods. This paper describes some of the main findings of the study, along with the more important conclusions that could be drawn.*

Keywords: CAS, curriculum, resource, practice, technology

## CURRICULUM MATERIAL AND TECHNOLOGY

In mathematics instruction, textbooks can play a central role. This is especially true in Sweden, where it often defines the curriculum for both teachers and students. There exist strong beliefs among them that if you do not follow the textbook, you might not fulfil the curriculum and then you fail in the National Tests. These have to some degree rewarded the use of technology in mathematics education, but this has not yet provoked any more extensive changes of the textbooks. It is therefore of great interest to evaluate new types of material for classroom use that integrate technology in a more distinct way.

Within the project 'Nspirerande matematik', especially developed curriculum material was used for parts of the courses Matematik A and Matematik B at Swedish upper secondary school (student age 16-17). This material consists of both traditional texts and tasks, as in a common textbook, and of interactive material for the TI-Nspire technology (Fig. 1). These interactive files give the students opportunities to discover mathematical principles and rules, to make conjectures and justifications, to exercise their

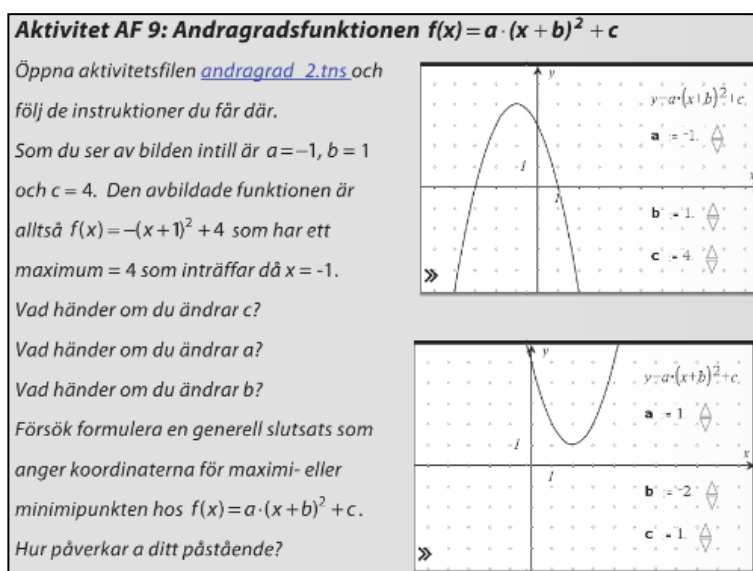


Figure 1: Example of a task for TI-Nspire

skills and to make self-tests of what they have learned. Some tasks are especially designed as activities for inquiry and collaborative learning. The material has been used in three classes from the theoretical programmes at different schools in the middle and southern parts of Sweden during spring 2010.

Both teachers and students had full access to both handheld units and computer software within the TI-Nspire technology, including links between the two. The technology used was in the form of a computer algebra system (CAS). It is important to note that the students were familiar with the handheld units prior to being introduced to the curriculum material.

## **PILOT TEACHERS AND CLASSES**

The three teachers involved in the project were selected as well experienced and have taught mathematics for many years at upper secondary level. Technology in mathematics teaching was in no sense new to any of them at the initiation of the project. Graphing calculators have been standard equipment in all courses at upper secondary level in Sweden for many years, and skilful use of technology in different forms is especially promoted in mathematics curriculum. In the project, CAS was used in the curriculum material and also presupposed for solving some of the tasks and working with the special activities in the material. This, of course, could present different challenges to the teachers, depending on their prior experience of this technology (Weigand, 2007). A brief presentation of the teachers and their classes:

*Anna* is teaching at a school in the centre of a rather large city. The students at the school are mixed in several respects, and the motivation and the ability of the students also vary to a great extent. Some students have considerable problems with their mathematics studies, at the same time as some show great ambitions. In the class, which studied their first year at the Social Science programme, the students represented the whole scale, and the curriculum material was therefore chosen to be presented only in a smaller group (7-8 students). The material used was 'Nspirerande matematik – Ma A' (2010), more specifically the section with functions.

*Carl* and *Eric* are working at secondary schools in two middle-size towns. The students come from these towns and the surrounding rural areas, and both classes ( $\approx 25$  students each) consist of combinations of those who study in the Natural Science and in the Technological programmes. The students are of mixed ability but generally rather motivated for studies in mathematics. The material used was 'Nspirerande matematik – Ma B' (2010), the section with algebra and functions, which represents the larger part of the course, and with probability.

## **AIMS FOR THE STUDY**

The intention was to make a first evaluation of the use of curriculum material for the two mathematics courses, which is specially designed for the interactive use of TI-Nspire technology, based mainly on the experience of teachers and students. Of

special interest are the ways it was used in the classroom work and in the teachers' instructional practice, as well as how it affected the students' achievement. Both students and teachers had the opportunity to express their opinions of how well this material and this technology have functioned in a real educational situation. They have also been able to pinpoint possible problems and obstacles that they have encountered when using the material and/or the technology, as well as how it has affected the students' own motivation, interest and self-confidence when working with mathematical activities.

As being a pilot study, its intention was furthermore to form a basis for a larger evaluation study, which can involve more teachers and classes, and can stretch over a longer time period. In such a study, it can be possible, in other ways, to do research of more subtle outcomes of education, such as deeper understanding of mathematical concepts and methods and how robust knowledge is over time.

## THEORETICAL FRAMEWORK

The theoretical background for this study rests on the classical *didactic triangle* with its three main elements student-teacher-mathematics, discussed for example by Steinbring (2005). This model has, however, been presented in various ways, depending on the overarching theory of learning and on the special context. The focus here lies on processes of mathematical interaction between individuals in the classroom (Cobb & Bauersfeld, 1998), a mainly social constructivist view. Learning takes place through experiences that are mediated by tools (Vygotsky, 1978), that can be mental (like spoken language), symbolic (like mathematical signs) or physical (like compasses), and with assistance drawn from other, competent individuals. Calculators and computer software hold a special position here, as they can be seen as tools within all three aspects.

A tool can develop into a useful instrument in a learning process called *instrumental genesis* (Verillon & Rabardel, 1995; Guin & Trouche, 1999). This process requires time and effort from the user. The user must develop instrumented action schemes that consist of a technical part and a mental part (Guin & Trouche, 1999; Drijvers & Gravemeijer, 2005). The teacher must actively guide the students in a controlled evolution of knowledge, achieved by means of social construction in a class community (Mariotti, 2002). Of special interest is the *instrumental orchestration*, which is defined as the intentional and systematic organisation and use of the artefacts available in a learning environment by the teacher, in order to guide students' instrumental genesis (Drijvers *et al.*, 2010). In the present research project, TI-Nspire CAS calculators together with emulating computer software are the physical parts of the instrumentation process. But the setting for this is within the curriculum material, which is intended as the basic mediating tool for the learning process, replacing the ordinary textbook.



*Affective* factors have been found to play a profound role in the outcomes of mathematical education. Debellis and Goldin (1997) suggested four facets of affective states: emotional states, attitudes, beliefs and values/morals/ethics. Especially the intentions and goals for the mathematical education that students and the teacher have are vital (Hannula, 2002). They are not always coinciding, and this is particularly the case when technological tools and mathematical texts are used in instruction. There are also other elements of attitudes and beliefs that teachers hold that can present obstacles and cause problems, such as the perceived change in their classroom practice or how they believe such teaching will impact on students' learning (Brown *et al.*, 2007; Pierce & Ball, 2009). This is especially true for CAS, which also has the problem of becoming legitimized within the school culture (Kendal & Stacey, 2002).

### RESEARCH QUESTIONS

The research questions of the evaluation study are structured into three groups in accordance with the didactic triangle, and are generally based on the theoretical background and the aims for the study:

*Effects on teaching practice and learners.* How is the integrated system of technology and written content used in the classroom by teachers and students? What effects on classroom dialogue, student-student and teacher-student, can be detected when working with it?

*Teacher experience of the system.* How has the 'Nspirerande matematik' resources supported new approaches to teaching for the teachers involved in the project? Which difficulties or obstacles with using the material and/or the technology can be found?

*Learning outcomes.* How do the teachers in the project estimate the effects of the curriculum material on students' development of conceptual understanding? How does the use of the material together with the technology affect students' motivation, interest and self-confidence when working with mathematical activities?

### RESEARCH METHODS AND DATA COLLECTION

This pilot study has the intention of giving a first and rather general view of the outcomes of the use of the curriculum material and the TI-Nspire technology, both in terms of students' and teachers' views of these and of possible learning results that can be connected with them. Thus a pragmatic use of mixed research methods has been appropriate, mainly focussing on qualitative approaches, but also with some quantitative elements concerning the ways the material and the technology are used in the classrooms.

The classes and the teachers that participated in the project were each visited twice during the project, and the methods used were the following:

*Teacher lesson evaluations and observations.* The teachers should fill in one form or log for each lesson in which they have used the material and the technology.

*Teacher interviews.* A deep, semi-structured interview with the teachers was made in connection with each visit at the schools. All interviews were transcribed.

*Classroom observations.* With each visit a lesson was to be observed by the researcher, using a special observation protocol.

*Student interviews.* Two students were chosen from each class to be interviewed in connection with the observed lesson at each visit, and these were also transcribed.

*Teacher questionnaire.* After finishing this pilot project each teacher was presented with questions concerning their overall experience of using the material and the technology in school instruction.

The teachers and the students used the material full time from an introductory meeting in March until the end of the semester. Visits were made as planned at the three schools by two occasions, one near the end of April and one near the end of May, at which time interviews were made and recorded. The three teachers as well as the students presented in most cases extensive answers to the questions asked, and also gave a clear impression of honesty in them. It was possible to discuss both their progress and their shortcomings with the material and the technology, and they reflected on what they had done in class, both in the middle of the project and at the end of it.

## MAIN FINDINGS AND CONCLUSIONS

The acquired data from this study will not be presented in detail here. Instead some of its main findings are briefly stated and partly compared with other research.

### Effects on teaching practice and the learners

*The teachers made no essential alterations of their approaches or their working styles due to the system.* Their lesson logs showed that they used essentially the same lesson plans and organisation of instruction during the project as before (mainly based on whole-class teacher's presentation, individual or cooperative students' work with teacher's support and finally a summary of the lesson). Maybe this was to be expected in such a short period, even if the material opens for change in many ways. For one of the teachers, this material was in line with what she was used to, but for the others it presented a challenge. However, some signs of progress and change could be detected at the end of the project for one teacher, in the direction of more exploring activities for the students. This could imply a partial change of attitude to the ways mathematics education can be performed (Brown *et al.*, 2007).

*Two of the teachers did not use the calculators as tools for students own learning and exploring to any larger extent.* Balling (2003) distinguishes between the use of calculators as calculating tools, teaching tools and learning tools. In the project, these two teachers mainly used them as extensions of the calculators they had used before,

which means as calculating and teaching tools. The CAS technology was also a problem for especially one teacher, who even doubted the positive impact on students' learning. This is also in accordance with what Pierce and Ball (2009) write about teachers' attitudes. Using CAS can be considered by the teacher as an extra burden, especially for weak students, and distract them from core mathematical learning.

*The curriculum material replaced the usual textbooks and was printed out as compendia.* In their evaluations of the material both teachers and students compared it with the textbooks in all details. In those cases where differences could be seen, many of the students, and also their parents, questioned if the material could provide the 'right' knowledge for the course, and there were worries about the results in the National Tests. Also the attitudes from colleagues and/or the principal for the school could have been of importance for the pre-attitudes (see Pierce & Ball, 2009).

*The combination of the calculator and the material promoted students' discussions and presented opportunities to cooperate.* In all three pilot classes, the students were used to work in pairs or in spontaneous groups. Much of the material was challenging for some students, in ways that the usual textbook did not match sufficiently. The technically advanced calculators also called for cooperation in acquiring proper action schemes (Drijvers & Gravemeijer, 2005). These two factors together created good opportunities for the students to develop a dialog on a rather advanced mathematical level, both with the teacher and with other students. Students from Erik's class:

Female student: Sometimes you want to work by yourself to get into it and understand, but it is very good to be able to help each other.

Male student: Yes, most of the time more will be done when you work in pairs. It is easier when you don't know.

### **Teacher experience of the system**

*During the project, no major obstacles or difficulties with the calculators appeared.* Neither the teachers nor the students mentioned any larger problem with the technique or in the way the calculator software appears. Interesting was the description some students gave of the instrumental genesis with its different parts (Guin & Trouche, 1999) they had experienced, and what this also had taught them in terms of mathematics.

Anna's (female) student: I have no problems with new technology. I understand the calculator very well! The only thing is that you must remember all the steps, but the calculator in itself is quite simple to understand.

*One teacher criticised the way CAS technology was used in the material.* His impression was that this was written for the use of CAS and not for the mathematical knowledge.

## Working Group 15

Erik: For the understanding of the concepts, I abandoned the technology. It is typical that CAS do it in this way, and therefore the material is adjusted for that. CAS in the centre. It rules over the material.

Belfort and Guimares (cited in Dick & Burrill, 2009) give a list of four possible shortcomings in constructing technology-based material, of which the first is: ‘The author’s interest is on mastering the use of the technology where the mathematics is secondary’ (p. 11). However, none of the other two teachers gave the same critique, and this might be misdirected in this case. The problem could instead lie in the fact that the use of CAS may change the classroom practice, which can be perceived by the teacher as a threat (Pierce & Ball, 2009).

*The content of the material in combination with the technology caused problems for some students.* The long texts and explanations with alternating sections of tasks were perceived as difficult by some students. This was especially true for students with lesser mathematical ability, for whom the longer activities or tasks became obstacles.

Erik’s student: It is frustrating if you are in a class and others have difficulties but not I. Then it takes very much time to get it to work for all students in the group.

Research points in diverse directions here. Ruthven and Hennessy (2002) report that access to technology enables less-able students to participate in exploration, while Tynan (2003) concludes that the technical overhead when learning new technological features could present an extra burden for these students. This is also mentioned as an obstacle by two teachers in the project, while the third one instead has observed this category of students succeeded quite well with such activities.

*The opinions of how easy a teacher who is rather new to technology could start using it together with the material varied.* One of the teachers perceived it as being fairly easy, calling for only a shorter introduction, another that a more thorough course was needed. The third teacher said that it was easier than he thought. One explanation to this could be that the material was used in different ways by them, and that the demands of certain knowledge therefore differed. This is in line with what other researchers have found (e.g. Ball, 2004), especially in the connection with the implementation of CAS into the classrooms. All three teachers also mentioned the need for an extensive teacher’s guide accompanying the material and that different kinds of support, mainly when they start using the material, would facilitate this implementation. Implementing in a larger scale has been researched for example by Ball (2004), and Pierce and Ball (2009). One conclusion these authors have drawn is: ‘The responses to this survey confirm that professional development for teachers needs to address attitudes and perceptions as well as technological skill development’ (p. 315). This is also the experience as expressed by the teachers in the present study.

## Learning Outcomes

*Two of the pilot teachers would not draw any certain conclusions about their students' development of deep conceptual understanding of mathematics.* One obvious reason for this is the fairly short time-span of the project. To the teachers' meaning, such effects can only be detected in the longer perspective. However, their estimation was that the more mathematically able students have taken advantage of the material and the calculators in a way that in fact it has made it possible for them to develop this deeper understanding of concepts. At the same time, the lesser able students might be at the risk of having learned less than with the usual textbook.

*One of the pilot teachers declared that her students had shown clear signs of a deeper conceptual understanding.* This was also confirmed in the interviews with the students, who gave detailed descriptions of the laborious process to obtain this understanding. The students' results at the National Test gave more evidence that they really had understood mathematics at a more advanced and deeper level, and so did also the mathematical texts that the students handed in to the teacher. It is important to note that these students had not appeared to be among the most mathematically able ones when the project started.

Anna: The material is more stimulating than 'ordinary' books, and the interaction with the calculator develops a more investigating and inquiring attitude.

Anna: Before, I thought that you had to use the pencil to understand, but I see many today who 'think' with the keys.

*The calculators have been stimulating for the interest, motivation and curiosity towards mathematics.* This is common for all three pilot classes, even if it was not true for all individual students. It is also in accordance with what most research of the use of technology in mathematics education shows (see e.g. Persson, 2009). For most students, it was also their first contact with a more advanced calculator. In one of the groups, the students' attitudes and feelings towards mathematics had changed dramatically. Their comparison was mainly with what they had experienced of mathematics instruction at compulsory school. The details in the interviews give evidence that it was, at least to some degree, the text, the tasks and the activities in the material that caused this change. The students found the mathematics there interesting, challenging and useful, and solving the problems gave them better self-confidence and self-esteem (Hannula, 2002).

Anna's (female) student: This material has been extremely good and incredibly useful. Before, I had a hard time with maths, but now I have Anna and the material and new ways of thinking. So now I got a 'Pass with special distinction' at the National Test, which I had never dreamed of getting before. So it has improved my and the whole class' attitude towards math. And you have to think in new ways all the time, otherwise you don't find the right solutions.

Presently, a continuation of this pilot study has been initiated, in the form of a larger study involving more classes and teachers. It will take place during a whole semester, and the learning outcomes of the students will be in special focus (Artigue & Bardini, 2009). What are the conceptual and affective effects of using this kind of technology and curriculum material (now available at Nspirerande matematik, 2010), especially compared to common textbooks and the material linked to these?

## REFERENCES

- Artigue, M. & Bardini, C. (2009). New didactical phenomena prompted by TI-Nspire specificities – The mathematical component of the instrumentation process. In V. Durrand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.). *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1071-1180). Lyon, France: INRP.
- Ball, L. (2004). Researchers and teachers working together to deal with the issues, opportunities and challenges of implementing CAS into the senior secondary mathematics classroom. *Zentralblatt für Didaktik der Mathematik*, 36(1), 27-31.
- Balling, D. (2003). *Grafregneren i gymnasiets matematikundervisning – Lærernes holdninger og erfaringer* (in Danish). Doctoral thesis. Aarhus: Syddansk universitet.
- Brown, E. T., Karp, K., Petrosko, J., Jones, J., Beswick, G., Howe, C., & Zwagnig, K. (2007). Crutch or catalyst: Teachers' beliefs and practices regarding calculator use in mathematics instruction. *School Science and Mathematics*, 107 (3), 102-116.
- Cobb, P. & Bauersfeld, H. (Eds.) (1998). *The emergence of mathematical meaning – Interaction in classroom cultures*, Vol. 2. Hillsdale, N.J.: Lawrence Erlbaum.
- DeBellis, V.A. & Goldin, G.A. (1997). The affective domain in mathematical problem solving. In E. Pehkonen (Ed.), *Proceedings of the 21st conference of the international group for the psychology of mathematics education*, Vol. 2 (pp. 209-216). Lahti: Helsingfors Universitet.
- Dick, T. & Burrill, G. (2009). *Tough to teach/tough to learn: Research basis, framework and principles for a practical application of TI-Nspire technology in the Math Nspired series*. Dallas, TX: Texas Instruments Inc.
- Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, 75(2), 213-234.
- Drijvers, P. & Gravemeijer, K. (2005). Computer algebra as an instrument: Examples of algebraic schemes. In D. Guin, K. Ruthven & L. Trouche (Eds.). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp.163-196). Dordrecht: Kluwer Academic Publishers.

- Guin, D. & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, 3(3), 195–227.
- Hannula, M. (2002). Attitude towards mathematics: Emotions, expectations and values. *Educational Studies in Mathematics*, 49, 25-46.
- Kendal, M. & Stacey, K. (2002). Teachers in transition: Moving towards CAS-supported classrooms. *Zentralblatt für Didaktik der Mathematik*, 34(5), 196-203.
- Mariotti, M. A. (2002). The influence of technological advances on students mathematics learning. In L. English (Eds.), *Handbook of international research in mathematics learning* (pp. 695- 723). Mahwah, NJ: Lawrence Erlbaum Associates Publishers.
- Nspirerande matematik* (2010). (In Swedish) Retrieved March 14, 2011, from <http://education.ti.com/ebook/>
- Persson, P-E. (2009). Handheld calculators as tools for students' learning of algebra. *NOMAD, Nordic Studies in Mathematics Education*, 14(2), 101-129.
- Pierce, R. & Ball, L. (2009). Perceptions that may affect teachers' intention to use technology in secondary mathematics classes. *Educational Studies in Mathematics*, 71(3), 299-317.
- Ruthven, K. & Hennessy, S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning. *Educational Studies in Mathematics*, 49(1), 47-88.
- Steinbring, H. (2005). Analyzing mathematical teaching-learning situations – The interplay of communicational and epistemological constraints. *Educational Studies in Mathematics*, 59(1/3), 313-324.
- Tynan, D. (2003). Student caricatures in a CAS classroom. In B. Barke, A. Bishop, R. Cameron, H. Forgaz, & W. T. Seah (Eds.), *Making mathematicians. Proceedings of the 40th Annual Conference of the MAV* (pp. 311-319). Melbourne: Mathematical Association of Victoria.
- Verillon, P. & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education*, 10(1), 77-101.
- Vygotsky, L. (1978). *Mind in Society*. Cambridge, MA: Harvard University Press.
- Weigand, H-G. (2007). Teaching with a symbolic calculator – The first years of a long-term project. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education*, (pp. 1540-1549). Larnaca: University of Cyprus.

# TEACHERS TRANSFORMING RESOURCES INTO ORCHESTRATIONS

Paul Drijvers

Freudenthal Institute, Utrecht University

*Mathematics teachers may perceive difficulties in orchestrating education which makes use of technological tools. With instrumental orchestration as a theoretical lens, we investigate into which types of orchestrations teachers transform the technological resources. In a pilot teaching sequence in grade 12 on using applets for practicing algebraic skills, this question is investigated through a case study, questionnaires, and interviews. The results show that teachers privilege orchestrations in which students work individually or in pairs, at the cost of whole-class orchestration types. Compared to their regular teaching practices and their expectations before the pilot, the involvement in the pilot causes teachers to adapt their orchestrations during the pilot.*

Keywords: mathematics education, technology, teachers, orchestration

## INTRODUCTION

Nowadays, teachers are confronted with a myriad of both material and electronic knowledge resources available for mathematics teaching. More and more, such resources can be accessed through technological means and are available on the internet. However, resources do not transform teaching practices in a straightforward way. Several studies show that teachers may perceive difficulties in orchestrating mathematical situations which make use of technological tools and resources, and in adapting their teaching techniques to situations in which technology plays a role (e.g., Gueudet and Trouche, 2009). Also, different teachers may adapt the same set of resources into quite different teaching arrangements.

As Robert and Rogalski (2005) point out, teachers' practices are both complex and stable. Building on this, it is argued that the availability of technological resources amplifies the complexity of teaching practices and, as a consequence, challenges their stability. It is not self-evident that techniques and orchestrations which are used in 'traditional' settings can be applied successfully in a technological-rich learning environment. A new repertoire of orchestrations, instrumented by the available tools, has to emerge. This involves professional development of the teacher, in which both professional activity and professional knowledge may change. This process of transforming sets of technological and other resources into orchestrations is the topic of this paper [1].



## THEORETICAL FRAMEWORK

The main theoretical perspective that frames our investigation of teachers transforming resources into orchestrations is the notion of instrumental orchestration. It is widely acknowledged that student learning needs to be guided by the teacher through the orchestration of mathematical situations. For example, Kendal and Stacey (2002) showed that teachers privilege certain techniques for using technological tools over others and, in this way, guide the students' acquisition of tool mastery and their learning processes. To describe the teacher's role, Trouche (2004) introduced the metaphor of instrumental orchestration.

An instrumental orchestration is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a – in this case computerised – learning environment in a given mathematical task situation, in order to guide students' instrumental genesis (Trouche, 2004). We distinguish three elements within an instrumental orchestration: a didactic configuration, an exploitation mode and a didactical performance (Drijvers *et al.*, 2010). Didactical configurations and exploitation modes were introduced by Trouche (2004). As an instrumental orchestration is partially prepared beforehand and partially created 'on the spot' while teaching, we felt the need to add the actual didactical performance as a third component. Establishing the didactical configuration has a strong preparatory aspect: often, didactical configurations need to be thought of before the lesson and cannot easily be changed during it. Exploitation modes may be more flexible, while didactical performances have a strong ad hoc aspect.

Even if the metaphor of instrumental orchestration is appealing, it has its limitations like every metaphor. If we think of a teacher as a conductor of a symphony orchestra consisting of highly skilled musicians, who enters the concert hall with a clear idea on how to make the musicians play Beethoven the way he himself reads the century-old partition, we may feel uneasy with the metaphor. However, if we think of the class as a jazz band (Trouche and Drijvers, 2010) consisting of both novice and more advanced musicians, and the teacher being the band leader who prepared a global partition but is open for improvisation and interpretation by the students, and for doing justice to input at different levels, the metaphor becomes more appealing. It is in the latter way that we suggest to understand it.

Earlier research focused on the identification of orchestrations within whole-class technology-rich teaching. Drijvers *et al.* (2010) identified six types of such orchestrations, termed Technical-demo, Explain-the-screen, Link-screen-board, Discuss-the-screen, Spot-and-show, and Sherpa-at-work, with the following global descriptions. This categorization, with three more teacher-centred and three more student-centred orchestrations, resulted from a study on the use of applets for the exploration of the function concept in grade 8, and emerged from observation of three teachers in a relatively guided situation (Drijvers *et al.*, 2010). Of course, from

these limited data from a specific context, we cannot claim completeness. Rather, we wonder how specific this categorization is with respect to the type of technology, the mathematical topic, the whole class teaching format, the level and age of the students, and the amount of guidance teachers were provided with. Therefore, the goal of the study presented here is to investigate in another teaching context in which types of orchestrations teachers transform the available technological resources and how these results relate to the above categorization.

## RESEARCH SETTING

The research was carried out in the context of a pilot initiated by the publisher of the main Dutch textbook series for secondary mathematics education. The publisher, seeking for ways to improve their product and to integrate technology, decided to offer to their customers' schools an online, interactive version of a chapter on algebraic skills such as rewriting expressions and solving equations for grade 12, the final year of pre-university secondary education. For this online module, the Freudenthal Institute's Digital Mathematics Environment (DME) was used. DME is a web-based environment which integrates a content management system, an authoring tool and a student registration system, and which already contains content in the form of an impressive amount of applets and modules (Bokhove and Drijvers, 2010). The module for this pilot was designed by the authors of the textbook series, supported by the Freudenthal Institute DME experts. The module includes tasks as well as video clips with elaborated examples. The levels of feedback to students' answers decreases as the module advances. A pdf file of the original textbook chapter was also made available online, with embedded links to the new online activities [2]. Figure 1 shows a part of the book file on the left, and one student's work in the digital environment on the right. The book text includes a reference to the online module and the task to solve two equations. In the right screen, the student makes a mistake in the last line, and gets feedback saying "This step contains both correct and incorrect parts. Remove or replace the incorrect parts".

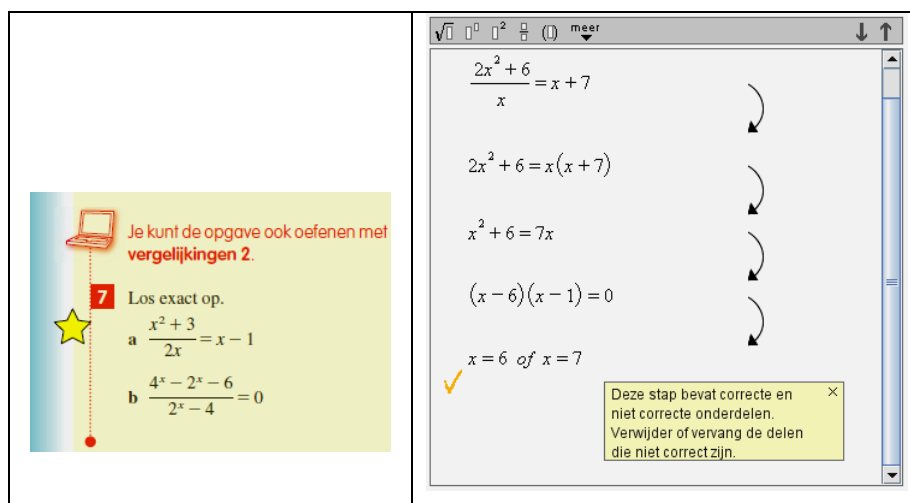


Figure 1: Screen shots from book (left) and digital environment (right)

After a message by the publisher, 69 teachers volunteered to join this pilot teaching sequence. They were provided with online guidelines for the use of the module. For the students, the resources in this pilot include the regular textbook, the online book chapter, digital modules including feedback and video clips, and the traditional resources such as paper and pencil and calculator. As the work is stored on a central server, students can access, revise and continue their work at any time and from any place with internet access. For the teacher, the resources are similar, but with the additional option of access to student work. Overviews of whole class results as well as individual student work can be monitored by the teacher through the internet.

### **METHODS**

The research methods include a case study focusing on one teacher and a survey among all 69 participating teachers. The case study was carried out in two classes of one of the pilot schools, a school in a small, prosperous town in the Netherlands with mainly 'white' student intake. Both classes, with 30 and 14 students, respectively, were taught by the same, experienced teacher. He initially volunteered for the pilot, but later intended to step back, because computer facilities in school were insufficient and his students objected to the idea of practicing algebraic skills with the computer, whereas they would need to master them with paper and pencil in the national central examination. We were able to offer a loan set of 30 netbook computers for the period of the teaching sequence and we convinced the teacher that practicing skills with computer tools was expected to directly transfer into better by-hand skills. Then, both the teacher and his students accepted to participate. During the period of the pilot, this teacher had a heavy teaching load, with 26 50-minutes lessons a week to teach. A technical assistant was available in school to set up the classes with the netbook computers, and to make other practical arrangements such as charging the batteries, et cetera.

Most of the lessons (23 out of 36 during an 8-week period) were observed and videotaped. The video registration was done by a mobile camera person, who followed the teacher very closely during individual teacher-student interactions, so as to capture all speech and screens. Data analysis took place with software for qualitative data analysis and focused on orchestrational aspects of the teaching.

The very specific data from the case study were complemented with data from a survey among all participating teachers. It consisted of two online questionnaires, one before and one after the teaching sequence. The response was 49 out of 69 for the pre-questionnaire, and 41 out of those 49 for the post-questionnaire. Non-response was caused by the fact that not all teachers who originally volunteered for the pilot really started their participation, and that some of the teachers who filled in the pre-questionnaire did not start either, or stopped the pilot before bringing it to an end. Some of them sent messages by email, indicating reasons such as time constraints, lesson cancellation because of illness, or other unforeseen circumstances.

## RESULTS FROM THE CASE STUDY

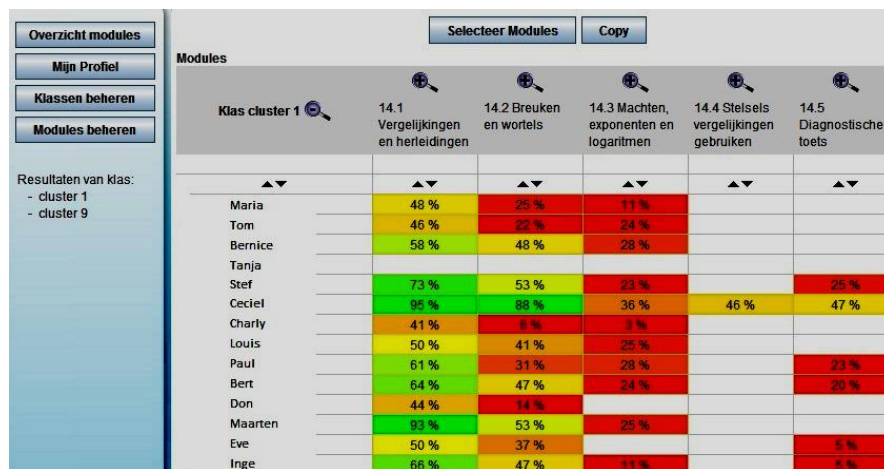
In the case study one particular orchestration type was highly dominant. We call it *Work-and-walk-by*. The didactical configuration and the corresponding resources basically consisted of the students sitting in front of their netbook computers, with wireless access to the online module and their previous work as well as to the textbook chapter in pdf format. In addition to this, a blackboard or whiteboard allowed the teacher to write down additional explanations. A data projector showing the online environment was available in most lessons, but was hardly used. As exploitation mode, the students individually worked through the online module on their netbook computers, and the teacher walked by and sat down with students to answer questions and eventually monitor the students' proceedings (see figure 2). As a reaction to student questions, the teacher in some cases went to the blackboard to write down an algebraic explanation or technique, but still speaking to the individual student who had raised the issue. Concerning the didactical performance, the initiative for teacher-student interaction was taken by the student in almost all cases. If an interaction with a student led to a new insight for the teacher, such as an understanding of a technical issue, he sometimes went back to students whom he had previously spoken to on a similar issue, as to disseminate the news.



**Figure 2: The teacher (left) helping a student (right) individually**

One aspect of this *Work-and-walk-by* orchestration concerns the determination of students' difficulties. If a student has a question while the teacher walks by, the latter is faced with the issue of where the heart of the problem lies: is it a lack of the student's algebraic understanding or skill? Is it a technical problem caused by the student, for example a mistake in entering an expression? Or is it a limitation of the online module, which in some cases gave inappropriate feedback, and was very strict in expecting a specific answer? Determination was difficult and mismatches between student problems and teacher reactions could be observed. Determination in some cases was hindered by the technical issues. As the teacher himself was not familiar with the module, he often was unable to solve students' problems, which led to uncertainty about whether it was a mathematical mistake or a technical problem that caused the technology to report an error. There was little technical guidance or

attention to students' instrumental genesis, and Technical-demo orchestration were not observed.



**Figure 3: Overview of student results generated by the DME**

This Work-and-walk-by orchestration took at least 90% of the lesson time in the lessons we observed, and remained dominant throughout the pilot teaching sequence as a whole without much variation; still, some changes over time in its didactical performance, and in the type of teacher-student interactions in particular, could be noticed. First, later in the teaching sequence, when he had time to find out how it worked, the teacher used the data projector to show the overall advancements of the students, so that each individual student could monitor if he or she was more or less on schedule (see Figure 3). Second, as both teacher and students during the teaching sequence got more familiar with the online module, its technical demands and its feedback, the student questions and the student-teacher interactions gradually focused more on algebra and less on technical issues. As a consequence, the character of these interactions changed from technical discussions into ‘Explain-the-screen’ or ‘Discuss-the-screen’ interactions. Also, the teacher went to the board less frequently, but instead used the online module more often as an environment to check algebraic claims or techniques. He encouraged students to type something in to see if it is correct, and used this as a way to explain the algebra.

If we relate the findings presented in this section to the six whole-class teaching orchestrations types identified above, we already noticed some Explain-the-screen and Discuss-the-screen elements within the didactical performance of the Work-and-walk-by orchestration. The same holds, to a lesser extent, for the Technical-demo orchestration: technical issues regularly emerged in the individual student-teacher interactions, even if the teacher was in many cases not able to solve them. Elements of the Link-screen-board orchestration could also be observed, as the teacher regularly walked to the whiteboard to explain the algebra, or used paper and pencil to do so. The Spot-and-Show opportunities that the didactical configuration offers were

not exploited. The same holds for the Sherpa-at-work, even if the teacher by the end of the teaching sequence invited students to carry out a specific technique in the digital environment, which can be seen as an individual ‘Sherpa-at-work light’.

## RESULTS FROM THE SURVEY

Even if the word ‘orchestration’ was not mentioned in the questionnaires, they provide insight in the orchestrational choices made by the participating teachers. For example, one question on both the pre- and the post-questionnaire was: which ICT-means were used? In the pre-questionnaire this concerned the use of technology in the teacher’s lessons preceding the pilot; in the post-pilot questionnaire, this concerned tool use during the pilot. Participants could click on more than one answer. Table 1 summarizes the findings. Data shows that the technological devices which are most frequently used during the pilot are the computer lab and students’ computers at home, which contrasts to the more teacher-driven ‘regular’ use of ICT before the pilot. Teachers seem to have changed the didactical configurations for the case of the pilot.

ICT-means used (more answers possible)	Pre-pilot (N=47) Frequency (%)	Post-pilot (N=41) Frequency (%)
Data projector	57	46
Teacher’s computer	57	32
Interactive whiteboard	55	37
Computer lab	0	83
Student computers in classroom	0	29
Students’ home computers	0	83

**Table 1: ICT means used during the pilot**

One question on the pre-pilot questionnaire concerned the working formats the teachers were expecting to use during the pilot, and a similar one on the post-pilot questionnaire asking which working formats they used indeed. Table 2 summarizes the findings. It shows that individual work, work in pairs and homework are the most frequently used working formats, whereas whole-class explanations and whole-class homework discussion occurred less than expected beforehand, in spite of the opportunities the didactical configuration offers for it.

Working formats	Expected (Pre-pilot, % of N=47)			Effectuated (Post-pilot, % of N=40)		
	Not	Sometimes	Often	Not	Sometimes	Often
Whole-class explanation	0	36	64	32	48	20
Whole-class demonstration	19	62	19	38	47	15
Whole-class homework discussion	4	47	49	40	47	13
Whole-class presentation	57	38	2	100	0	0
Individual work	6	26	66	2	2	96
Work in pairs	9	30	60	28	25	47
Group work	53	38	4	93	5	2
Homework	23	28	47	7	53	40

**Table 2: Expected and effectuated working formats used during the pilot**

A follow-up question in the post-pilot questionnaire was whether technology was used in the mentioned working formats. The results shown in table 3 confirm the previous impression, namely that technology during the pilot was mainly used for

individual work, work in pairs and homework, and not so much in whole-class orchestrations.

In this light it is somewhat surprising that the option to show students' home work by means of a data projector or interactive whiteboard, and to use it as a catalyst for whole-class discussion, was hardly used, whereas the teachers usually used such technology in whole-class teaching settings according to the pre-pilot questionnaire results. Even if the teachers beforehand expected some more individual work or work in pairs, this seems to have happened to a larger extent, and opportunities for using ICT in the way they were most familiar with, remained unexploited.

ICT in working formats	Post-pilot (% of N=41)		
	Not	Sometimes	Often
Whole-class explanation	58	32	10
Whole-class demonstration	41	24	20
Whole-class homework discussion	61	34	5
Whole-class presentation	98	2	0
Individual work	15	5	80
Work in pairs	51	17	32
Group work	93	5	2
Homework	24	32	44

**Table 3: ICT used in working formats**

To summarize the findings from the questionnaires, we conclude that before the pilot, teachers indicated that they used technology mainly in whole-class teaching settings, probably with the teacher operating the technology. In spite of this preference and experience, during the pilot they privileged individual work and work in pairs, which turn out to be the dominant orchestrations, and thereby neglected options for whole-class teaching offered by the technology. This is in line with the findings by Lagrange and Caliskan (2009). Even if the variety among all teachers seems to be greater than was observed in the case study, the results point into the same direction. They suggest that many teachers in the pilot changed their orchestrations from whole-class teaching using tools such as a data projector or an interactive whiteboard to student-centred orchestrations, for example in computer lab and home settings. Compared to the teachers' experiences with technology in their teaching, this is a shift from the teachers using technologies such as interactive whiteboards and data projectors, towards students using mainly computer labs and home computers.

It is not clear if the six identified whole-class orchestration types also appear in the context of this pilot. The questionnaires do not offer enough information. The focus on individual work and work in pairs is clear, but we do not know what happened besides that. Spot-and-show orchestrations and Sherpa-at-work orchestrations, however, do seem to be very rare, even if some teachers in the interviews reported incidentally using these orchestration types.

## CONCLUSION

A first aim of this study was to investigate in which types of orchestrations teachers transform the available technological resources. The findings from both the case study and the questionnaires – albeit the first to a greater extent than the second – suggest that individual, student-centred orchestrations are dominant when teachers use the resources that were developed in the frame of this pilot. Teachers tended to privilege students working individually or in pairs on the online module tasks, and devoted little time to whole-class explanation or homework discussion, whereas their expectation before the pilot were different. The case study resulted in the identification of a Work-and-walk-by orchestration, which in itself is not very surprising one. However, we were surprised by its dominance and by the fact that other orchestrational opportunities of the available technology were not exploited, whereas more variation could be observed in this teacher's regular lessons.

Several factors may explain this phenomenon. First, the subject, practicing algebraic skills, probably is more suitable for individual work or work in pairs than for whole-class teaching. Second, the computer labs, in which many lessons apparently took place, may be less suitable for whole-class teaching. Third, individual orchestration types are probably the easiest thing to do for a teacher, who is not feeling confident about his or her own technical skills. Fourth and final, it may be the technology itself that invites student work rather than whole-class teaching. Our impression from interviews with teachers is that all these factors play a role. Data is insufficient to decide on the impact of each of them.

A second point of interest is how these results relate to the orchestration types mentioned in the theoretical section. This typology emerged from whole-class teaching episodes, whereas in this pilot mainly individual orchestrations were found. Still, the case study observations suggest that the six whole-class teaching orchestration types identified earlier have their counterparts, or at least similar aspects, in the context of the present study. The overall conclusion, therefore, is that the six whole-class orchestration types of course are not exhaustive, but do contain elements that can be observed in other orchestrations as well. We expect the list of possible orchestrations to be extended in future, not as to strive for a complete list, but as to provide teachers with a diverse repertoire of possible orchestrations as source of inspiration to their professional activity.

A third and final point of interest concerns the change processes that occur when teachers engage in an experimental setting. The conclusion here is twofold. First, the case study provides insight in the change process during the pilot teaching sequence. The findings suggest a stable and not so dynamic orchestration, in which there is not much change, at least not at the superficial level. Meanwhile, at the level of didactical performance some professional development was observed, showing for example an increased focus on the algebra and on what we might call 'Explain-the-screen', at the cost of attention to technological issues. Second, the findings of the



questionnaires shed light on the change that takes place when teachers engage in such a pilot, compared to their regular teaching practices before the pilot. The data suggest that many teachers, who were used to integrating technology in a teacher-centred way – the teacher using a computer connected to a projector, or using an interactive whiteboard – in the frame of this pilot switched to student-centred orchestrations. It seems that most of them did not extend their teaching technique repertoire during the pilot sequence with, for example, a Spot-and-show orchestration type, even if the technology supports the monitoring of student work by the teacher anytime and anyplace.

### NOTES

1. This paper is further elaborated in a chapter with the same title which accepted for publication in Gueudet, G., Pepin, B., & Trouche, L. (Eds.), *Mathematics Curriculum Material and Teacher Development: from text to lived resources?* New York/Berlin: Springer.
2. The module (in Dutch) is available through <http://www.fi.uu.nl/dwo/gr-pilot/>

### REFERENCES

- Bokhove, C. & Drijvers, P. (2010). Assessing assessment tools for algebra: Design and application of an instrument for evaluating tools for digital assessment of algebraic skills. *International Journal of Computers for Mathematical Learning*, 15(1), 45-62.
- Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*. 75(2), 213-234.
- Gueudet, G. & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics*, 71(3), 199-218.
- Kendal, M. & Stacey, K. (2002). Teachers in transition: Moving towards CAS-supported classrooms. *ZDM, The International Journal on Mathematics Education*, 34(5), 196-203.
- Lagrange, J. B., & Caliskan-Dedeoglu, N. (2009). Usages de la technologie dans des conditions ordinaires: le cas de la géométrie dynamique au collège. *Recherches en Didactique des Mathématiques*, 29(2), 189–226.
- Robert, A. & Rogalski, J. (2005). A cross-analysis of the mathematics teacher's activity. An example in a French 10th-grade class. *Educational Studies in Mathematics*, 59, 269-298.
- Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281-307.
- Trouche, L. & Drijvers, P. (2010). Handheld technology: Flashback into the future. *ZDM, The International Journal on Mathematics Education*, 42(7), 667-681.

# THE USE OF MATHEMATICS SOFTWARE IN UNIVERSITY MATHEMATICS TEACHING

Iiris Attorps, Kjell Björk and Mirko Radic

Department of Electronics, Mathematics and Natural Sciences, University of Gävle,  
Sweden

*In this paper we report on a teaching experiment regarding the definite integral concept in university mathematics teaching. The experiment was carried out at a Swedish university by using the free dynamic mathematics software GeoGebra. In our theoretical framework we apply Variation Theory, originating in the phenomenographic research tradition. The data of this study consist of the lecture plan and the engineering students' answers to pre and post tests. In the analysis of the data we applied statistical methods. The experiment revealed that by using GeoGebra it is possible to create learning opportunities of the definite integral concept that support the students' learning.*

Keywords: definite integral, geogebra, learning, variation theory

## introduction

There is a constantly increasing number of software packages that can be used as powerful tools in mathematics teaching. Recent research shows that computer programs such as Maple, Mathematica, Derive, Geometer's Sketchpad, GeoGebra, when used in a classroom, support creative discoveries and mathematical generalizations (Lavicza, 2006).

It has been shown that students who use technology in their learning had positive gains in learning outcomes over students who learned without technology (Camacho-Machín *et al.*, 2010; Camacho-Machín & Depool Rivero, 2003; Touval, 1997). Regardless of evidences of several benefits of using technology, the process of applying technology for mathematics education is slow and complex (Cuban *et al.*, 2001).

Several studies have highlighted difficulties that students encounter with the integral concept. In an early study carried out by Orton (1980), it was observed that students

had difficulty with the integral  $\int_a^b f(x)dx$  when  $f(x)$  is negative or  $b$  is less than  $a$ . In

another study by Orton (1983), it was noticed that some students found it difficult to solve problems related to the understanding of integration as a limit of sums.

Our earlier research (Attorps *et al.*, 2010), in line with other studies (Blum, 2000; González-Martín & Camacho, 2003), has even pointed out that students have an intention to identify the definite integral as an area. Further, Rasslan and Tall (2002) verified that a majority of the students cannot write meaningfully about the definition of the definite integral. In a similar way, studies concerning learning of calculus

concepts (Attorps, 2006; Rösken & Rolka, 2007; Viirman *et al.*, 2011) have shown that definitions play a marginal role in students' learning whereby intuition inherent in concept images dominates the concept learning.

Transformation from procedural to conceptual understanding of the concept of integral requires gradual reconstructions of students' perceptions. Research has however documented the limitations of standard teaching methods, showing that students become reasonably successful on standard tasks and procedures but have difficulties in developing a solid and conceptual understanding of the topics itself (Artigue, 2001).

### The Variation Theory

The variation theory is a theory of learning, which is based on the phenomenographic research tradition and described by Marton and Booth (1997). The main idea in the phenomenography is to identify and describe qualitatively different ways in which people experience certain phenomena in the world, especially in an educational context. There are two main principles in variation theory. The first one is that learning always has an object, in our case the definite integral concept. The second one is that the object of learning is experienced and conceptualized by learners in different ways.

The object of learning can be seen from teacher's, student's and researcher's perspectives. The *intended* object of learning is the object of learning as seen from the teacher's perspective. It includes what the teacher says and wants the students to learn during the lecture. The students experience this intended object of learning in their own way and what they really learned - the outcomes of learning - is called the *lived* object of learning. Hereby, it is easy to understand that students' learning does not always correspond to what the teacher's intention was with the lecture. The *enacted* object of learning as seen from the researcher's perspective defines what is possible to learn during the lecture, to what extent, and in what forms the necessary conditions for specific object learning appear in a classroom setting. The enacted object of learning describes the space of learning, which students and teacher have created together. In this space it is possible for students from their previous learning experiences to discern critical aspects of the object of learning (Marton *et al.*, 2004).

In the variation theory, the experience of *discernment*, *simultaneity* and *variation* is a necessary condition for learning. Variation is the main concern in this theory and a primary factor in supporting student's learning. In order to understand what variations to use in the classroom to promote student's learning, it is first necessary to understand the varying ways students experience something. We can use this information to identify ways to encourage students to discern other aspects of the learning object, the aspects they have previously not discerned.

Every concept, situation or phenomenon has particular aspects and if an aspect is varied and another remained invariant, the varied aspect will be discerned. The

understanding of the object of learning in a certain way requires the simultaneous discernment of critical aspects of the object of learning (Marton & Morris, 2002; Marton *et al.*, 2004). The theoretical elements - *discernment*, *simultaneity* and *variation* related to learning and supposed to be critical for learning to happen - can be also used as an analytical tool for analyzing teaching (ibid). As a result, learning and teaching is brought closer together.

### The purpose of the study

The aim of our study is to design teaching sequences for definite integrals, using GeoGebra, which can support the students' learning in university mathematics. To that end, we seek an answer to the following question: Is it possible to use GeoGebra as a pedagogical tool within the variation theory, in order to vary critical aspects of the concept of definite integral during a lecture in an introductory calculus course?

### Method and design of the study

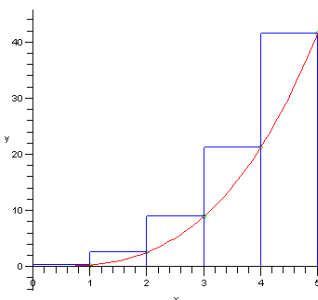
The study took place during a lecture in mathematics at a Swedish university. A total of 17 Chinese engineering students were involved in our study. The data were gathered by doing pre and post tests concerning the definite integral concept. In the analysis of the pre and post test results, statistical and qualitative analysis methods were applied.

### The questionnaire

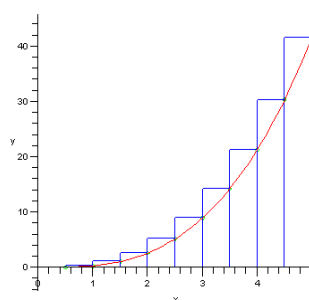
The questionnaire contained 6 questions, including not only 'typical' questions (Questions 2, 3, 5 and 6), but also intuitive questions (Questions 1 and 4). Maximal point in each question was three. Students had 25 minutes to do this test. In both pre and post tests, the same questionnaire was used. It was not allowed to use technical facilities.

*Question 1.* If you want to calculate the area between the curve and x-axis when  $x=0$  and  $x=5$  (see the graphs below) you can get an approximate value of this area by calculating the areas of the columns and by adding them.

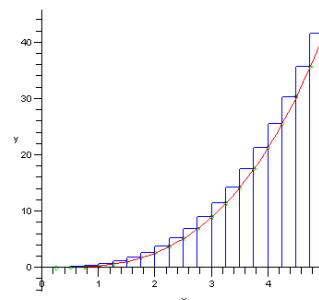
- a) Which of the following graphs should you choose in order to make the error as small as possible?



Graph 1



Graph 2



Graph 3

- b) Can you answer why?

## Working Group 15

The aim of the first question was to test the students' intuitive conception about the definite integral concept as a limiting process. In this case it is about the upper Riemann sum.

*Question 2.* What is this  $\int_a^b f(x)dx$  (the definite integral of the function  $f(x)$  in the interval  $[a, b]$ ) according to your opinion?

The second question was to test how the students grasp the conception of the definite integral.

*Question 3.* There are some approximate values of  $x$  and  $F(x)$  below:

$x$	1	2	3	4	5
$F(x)$	-1	-0.61	0.30	1.55	3.05

You know that  $F'(x) = \ln x$ . Approximate the value of  $\int_3^5 \ln x dx$ .

The purpose of the third question was to test how the students can apply the Fundamental Theorem of Calculus [1].

*Question 4.* The following is given:  $\int_{-1}^5 f(x)dx = 2$  and  $\int_{-1}^7 f(x)dx = -1$ . Evaluate  $\int_5^7 f(x)dx$ .

The intent of the forth question was to test the students' knowledge of the properties of the definite integral concept.

*Question 5.* Can you find any error in the following reasoning?

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^1 x^{-2} dx = \left[ \frac{x^{-1}}{-1} \right]_{-1}^1 = \frac{-1}{1} - \frac{-1}{-1} = -2$$

The aim of the fifth question was to test if the students grasped when it is possible to apply the Fundamental Theorem of Calculus.

*Question 6.* Find the area of the region, which is limited by the functions  $f(x) = 0,5x^2$  and  $g(x) = x^3$ . Give an exact answer.

The idea of the last question was to test how the students comprehended an ordinary high school example of the definite integral. We also wanted to find out how the students could master calculation with fractions.

## Results

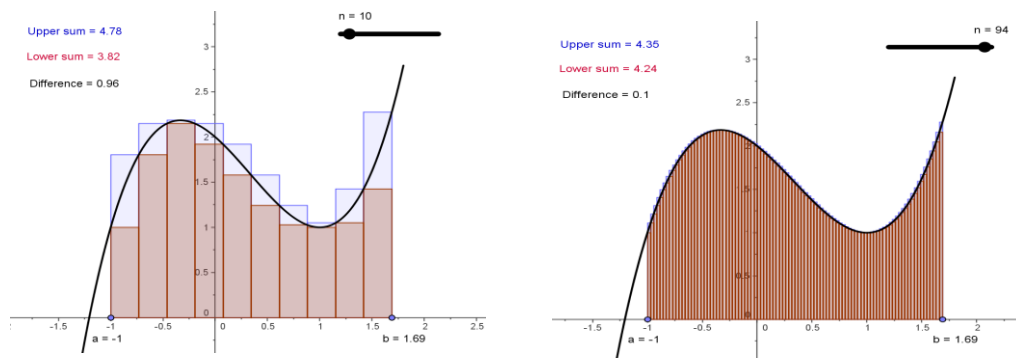
Before designing our lecture, we analysed carefully the pre test results in our study. The aim of the pre test was to identify if the students have some initial conceptions concerning the definite integral concept. Although the concept in China is normally introduced first at the university level (Wang, 2008), we could find that most of the

students showed quite a good intuitive conception of the concept as a limiting process (Question 1) and of the properties of the definite integral concept (Question 4). By reading earlier research, we could find that many students often have only an area conception of the definite integral (see e.g. González-Martín & Camacho, 2003). Furthermore they cannot meaningfully define the definite integral (Rasslan & Tall, 2002). Having this information about the students' conceptions, we designed our lecture. In order to create different teaching sequences that could encourage students to discern varying aspects of the object of learning, we used GeoGebra.

### Teaching sequences

Teaching sequences were implemented in an ordinary lecture with teacher manipulating the computer and students observing the screen. In the first application of GeoGebra (Fig. 1), we introduced numerical approximation of the area (Lower and Upper Riemann sums), as well as the definition of the definite integral with inherent infinite processes.

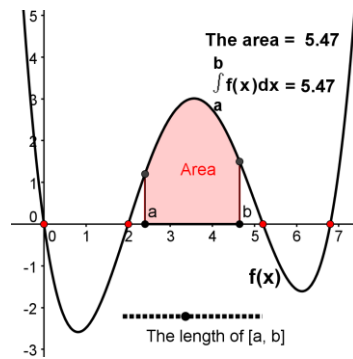
Figure 1 visualizes the concept of the Riemann integral using lower and upper sums. Two points  $a$  and  $b$  are shown that can be moved along the  $x$ -axis in order to modify the investigated interval. The upper and lower values together with their differences are displayed as a dynamic text which automatically adapts to modifications. In this case we keep  $f$  and the interval invariant and vary the number of subintervals. By increasing the number of subintervals, we shorten their length. Our intention was to show that increasing the number of subintervals decreases the difference between the Lower and Upper Riemann sums, which shows that the Lower and Upper Riemann sums eventually coincide with the value of the integral.



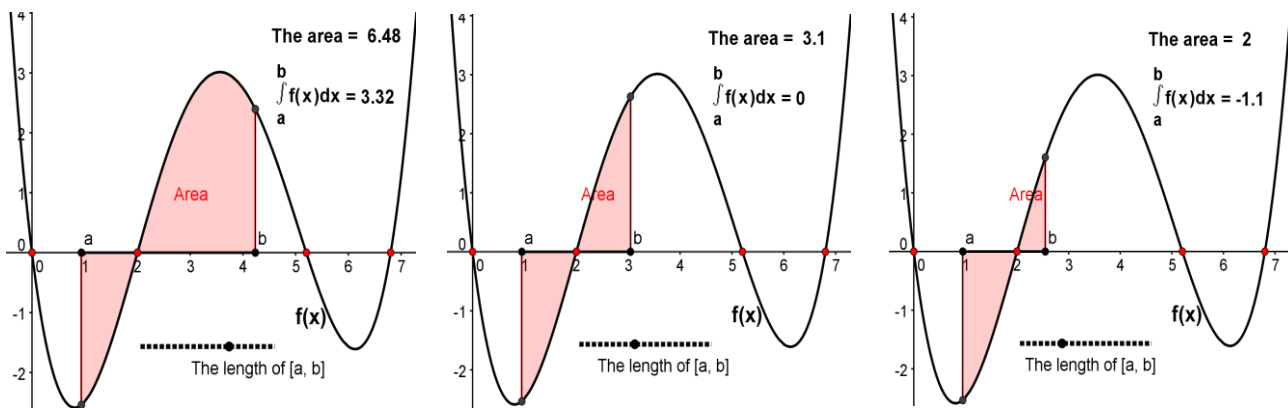
**Figure 1: Lower and Upper Riemann sums and inherent infinite processes**

The second example (Fig. 2 and 3) should help the students to get a wider conception of the definite integral concept. Figures 2 and 3 visualize the Riemann integral related to the area between the function  $f$  and the  $x$ -axis. Two points  $a$  and  $b$  are shown that can be moved along the  $x$ -axis in order to modify the investigated interval. The area and the integral values are displayed as a dynamic text which automatically adapts to modifications. This time we keep only  $f$  invariant and vary both the length of the interval and the upper and lower limit points. Our aim with this teaching sequence was to show that the value of the area between the function and

the x-axes and the integral-value not always coincide. While the area is always a non-negative (but not necessarily constant) real number, the integral-value can be any real number.

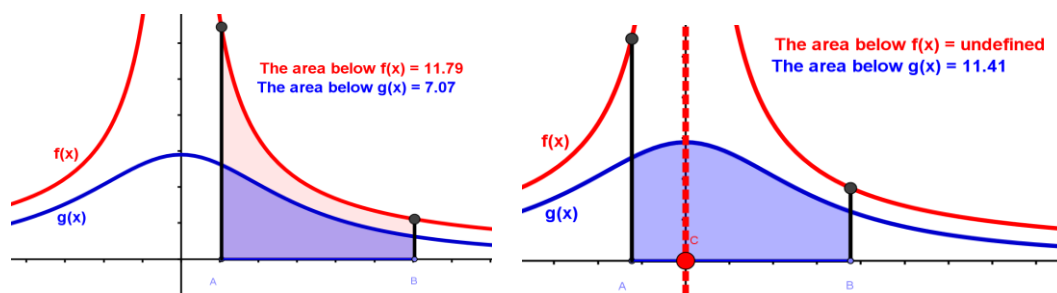


**Figure 2: The value of the definite integral is identical to the area between function and x-axis in the interval  $[a, b]$**



**Figure 3: The definite integral as a real number, which can be positive, zero or negative**

Our ambition with the third presentation (Fig. 4) was to help the students discern situations when it is possible to apply the Fundamental Theorem of Calculus and when it is not.



**Figure 4: Visualization of the application of Fundamental Theorem of Calculus**

In Fig. 4 on the left, it is possible to apply the Fundamental Theorem of Calculus, because both of the functions  $f$  and  $g$  are defined and continuous in the closed and bounded interval from  $A$  to  $B$ . In Fig. 4 on the right, it is not possible because  $f$  is not defined and by that not continuous in the interval. By moving the point  $A$  along the

x-axis we can vary the position of the investigated interval. In this teaching sequence we keep the length of the interval and the functions  $f$  and  $g$  invariant.

### Pre and post test results

The analysis of the data from the pre and post tests was done with a statistic program, the Minitab package. Using the significance level of 95% and a paired-samples one-tailed  $t$ -test, we compared the means of the test results for each problem. The group consisted of 17 participating Chinese students.

The pre and post test results indicate that there are statistically significant improvements in all the questions after the GeoGebra designed teaching practice.

Question number	Pre test mean	Post test mean	p	Maximum scores
1	1.588	2.059	0.014*	3
2	0.059	0.647	0.000*	3
3	0.176	2.471	0.000*	3
4	0.529	2.647	0.000*	3
5	0.000	0.941	0.007*	3
6	0.118	2.765	0.000*	3

\*  $p < 0.05$

**Table 1: Pre and post test results for the group of Chinese students**

The students' scores in question 1 show that their intuitive understanding of the definite integral concept was quite good already at the beginning. One of the students explained in the post test the question 1 in the following way:

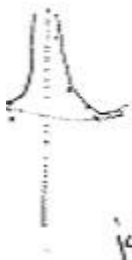
“The difference between the area of columns and the curve is smaller as the columns become smaller and smaller, more and more”.

The scores in questions 2 and 5 remained low in both pre and post tests. The most typical explanation to question 2 was:

“Points a and b are the intersection points of the two functions and we can calculate the area”.

In our qualitative analysis of the results we could notice that most of the students still grasped the integral concept as an area. Pre and post test results in question 5 showed that many of the students failed to give an adequate response in question 5; most of them could not find any errors at all. One of the students who succeeded to motivate his answer in post test explained:





The Integral is undefined on that interval, because the function is not define when  $x=0$ , so, the calculation is wrong, we can not calculate the integral on that interval.

We also noticed that nearly all the Chinese students in the post test were successful in question 6, showing good ability to calculate with fractions.

### Discussion

The integrating of mathematical software in teaching and learning at the university level is important due to its ability to give quick feedback and help students visualize and discern simultaneously varying aspects of the object of learning (Marton & Morris, 2002). One of the aspects is the understanding of the definite integral in a wider context as a real number and not only as an area. We could observe that the use of the GeoGebra software during the lecture increased the students' possibilities to experience the intended object of learning, namely the concept of the definite integral as a real number. In our post test results we could notice that most of the students still grasped the integral concept as an area (cf. González-Martín & Camacho, 2003). Hereby, the students' learning didn't correspond to what our intention was with the lecture. In our opinion, the definite integral concept is too tightly connected with the area conception in textbooks used at the upper secondary and even at the university level. Since in traditional classroom settings typical examples from the textbooks are primarily used to introduce a new concept, it is not so surprising that the students often have narrow conceptions of the mathematical concepts. The acquired experiences from the concept learning seem to be too solid and perhaps prevent adequate learning of the intended mathematical theory.

The understanding of the definite integral concept and the Fundamental Theorem of Calculus unavoidably requires that a student must at the same time focus on quite many separate elements of knowledge. Many of them are given in a symbolic or implicit form, which presupposes that learners can distinguish several aspects of the concept simultaneously. Pre and post test results showed that many of the students failed to give an adequate response in question 5. We think that the students should be trained to use definitions as an ultimate criterion in teaching and learning of mathematics. We see a clear potential in using GeoGebra within the variation theory, but we mean that it covers only one of several representations, which could be used in mathematics teaching.

## Working Group 15

Further studies need to be undertaken to identify which other factors than the integration of technology in teaching and learning of mathematics can be of benefit to both educators and students. Therefore, it would be interesting to design a study by choosing two groups of students; in one group lectures on Calculus will be conducted without any software package and in the other group the teaching block of Calculus will be created by using GeoGebra.

### NOTES

1. Suppose that

a) a function  $f(x)$  is defined and continuous on the interval  $[a, b]$

b)  $F(x)$  is an antiderivative of  $f(x)$  on the interval, i.e.  $F'(x) = f(x)$  for all  $x$  in  $[a, b]$ . Then

$$\int_a^b f(x)dx = F(b) - F(a) \quad (\text{Adams, 2006})$$

### REFERENCES

- Adams, R. A. (2006). *Calculus: A Complete Course*. Toronto: Addison Wesley.
- Artigue, M. (2001). What can we learn from educational research at the university level? In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study*, 207-220. Dordrecht: Kluwer Academic Publishers.
- Attorps, I. (2006). *Mathematics teachers' conceptions about equations*. Doctoral dissertation, Research Report 266. Department of Applied Sciences of Education, University of Helsinki. Finland.
- Attorps, I., Björk, K., Radic, M., & Tossavainen, T. (2010). The Learning Study Model and the Teaching of the Definite Integral Concept. In M. Asikainen, P. E. Hirvonen & K. Sormunen (Eds.). *Reports and Studies in Education, Humanities, and Theology* (pp. 77-86). University of Eastern Finland, Joensuu, Finland.
- Blum, W. (2000). Perspektiven für den Analysisunterricht. *Der Mathematikunterricht*, 46 (4/5), 5-17.
- Camacho-Machín, M., & Depool Rivero, R. (2003). Using DERIVE to understand the concept of definite integral. *Int. Journal for Mathematics Teaching and Learning*, 4, 1-16.
- Camacho-Machín, M., Depool Rivero, R., & Santos-Trigo, M. (2010). Students use of Derive software in comprehending and making sense of definite integral and area concepts. *CBMS Issues in Mathematics Education*, 16, 29-62.
- Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining the apparent paradox. *American Educational Research Journal*, 38(4), 813-834.

- González-Martín, A.S., & Camacho, M. (2003). What is first year Mathematics students' actual knowledge about improper integrals? *International Journal of Mathematical Education in Science and Technology*, 35(1), 73-89.
- Lavicza, Z. (2006). The examination of Computer Algebra Systems (CAS) integration into university-level mathematics teaching. In L. H. Son, N. Sinclair, J. B. Lagrange and C. Hoyles (Eds.), *Proceedings of the ICMI 17 study conference: Background papers for the ICMI 17 study* (pp. 37-44). Hanoi, Vietnam: Hanoi University of Technology.
- Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah, N.J.: Law Earlbaum.
- Marton, F., & Morris, P. (Eds.) (2002). What Matters? Discovering critical conditions of classroom learning. *Göteborg Studies in Education Sciences*, 181. Göteborg: Acta Universitatis Gothoburgensis.
- Marton, F., Runesson, U., & Tsui, A. (2004). The space of learning. In F. Marton and A. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-40). Mahwah, N.J.: Lawrence Erlbaum.
- Orton, A. (1980). An investigation into the understanding of elementary calculus in adolescents and young adults. In W. F. Archenhold *et al.* (Eds.), *Cognitive Development Research in Science and Mathematics. International Seminar Proceedings* (pp. 201-215), Leeds: University of Leeds.
- Orton, A. (1983). Student's understanding of integration. *Educational Studies in Mathematics*, 14(1), 1-18.
- Rasslan, S., & Tall, D. (2002). Definitions and images for the Definite Integral Concept. In A. Cockburn A. and E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 89-96). Norwich: UK.
- Rösken, B., & Rolka, K. (2007). Integrating intuition: The role of concept image and concept definition for students' learning of integral calculus. *The Montana Mathematics Enthusiast*, 3, 181-204.
- Touval, A. (1997). Investigating a definite integral – from graphing calculator to rigorous proof. *Mathematics Teacher*, 90 (3), 230-232.
- Wang, L. (2008). *What challenges are we confronted with in high school mathematics?* Paper presented at ICME 11, Mexico. Retrieved September 09, 2010, from <http://dg.icme11.org/tsg/show/23>
- Viirman, O., Attorps, I., & Tossavainen, T. (2011). Different Views – Some Swedish Mathematics Students' Concept Images of the Function Concept. To appear in *Nordic Studies in Mathematics Education* 16(1).

# CALCULATORS AS DIGITAL RESOURCES

Gilles Aldon

INRP (Institut National de Recherche Pédagogique), S2HEP Université Lyon 1

*New handheld calculators can be seen both as artefact allowing calculation and representation of mathematical objects, and resources for students and teachers allowing to store and to share data. Both teachers and students get hold of these potentialities and develop their own uses in distinct dynamics. Studying the interactions between teachers and students gives clues to understand the trajectories of these dynamics and the role played by the calculators in the construction of knowledge. Taking the opportunity of a broad introduction of handheld calculators in classrooms, we observe the conditions allowing transforming the artefacts into documents, part of the set of resources of teachers and students. The methodology takes into account long time observations, in a qualitative case study. The results show how different functionalities can be shared among teachers and students but also why and how other functionalities remain private and hidden.*

Keywords: Handheld calculators, instrumental genesis, digital resources, interactions, documents

## INTRODUCTION

There is a call for renewal of teaching methods in secondary science education as Rocard *et al.* (2007) state. The question of the place and the role of technology in the scientific classes in order to acquire “*good scientific knowledge and an understanding of technology*” (ibid., p. 6) should be addressed. There is a general agreement that technology brings an increase in collaborative forms of work (Peschek & Schneider, 2002; Kieran & Drijvers, 2006; Zbiek *et al.*, 2007). At the same time, new handheld calculators appear in the classrooms with calculation possibilities, functionalities for representation of data, interoperability between applications and data storing and sharing properties. This very new nature of calculators can be considered as an early variety prefiguring capabilities that will soon become available to students and teachers in portable digital work environments. It gives them a particular role in teaching and learning mathematics. The evolution of the role of a calculator in the system of resources of teachers and students, inside and outside the classroom, has to be studied. Even if the activities of storing, retrieving and sharing documentation are subsidiary to the production of meaning and understanding, they play an important role for teachers in the construction of lessons and for students in their learning process.

“The first challenge concerns extending the notions of mathematical situations and their orchestrations to out-of-school learning environments. The second challenge concerns renewing, from a practical and theoretical point of view, the notion of artefacts for learning and teaching. To enter these fields, we need to be aware that HHT [1] is no

longer an isolated artefact, but integrated in and articulated with a network of resources, particularly online resources.” (Trouche & Drijvers, 2010)

In this paper, we take into account the calculator as an element of the network of resources of both students and teachers and study the handheld calculators not only for their calculation properties but also for their documentary properties.

## THE RESEARCH SETTING

We took the opportunity of a broad introduction of handheld technology in different French classes to study the impact of such a technology on teaching and learning of mathematics. In the study reported in this paper we focus on two schools. In the first school during the school year 2008-2009, all the students of the scientific classes (16-18 years old students) have been equipped with a TI-Nspire handheld calculator. In the second school during the school year 2009-2010, two classes (16-17 years old students) were equipped with the same calculator.

In the first school (called S1 in this paper), most of the teachers are experienced teachers with a low degree of technological integration (Aldon & Sabra, 2009). In the second school (called S2 in this paper), the two teachers we observed are experienced with a high degree of technological integration. These experiments appear to be *natural* experiments in the sense that the classes contexts were not built by researchers but by the teachers' teams. Both in S1 and S2 a particular teacher plays a role of a leader: J1 in S1 and J2 in S2 are involved in the French research team e-CoLab (Aldon *et al.*, 2008).

Our choice to study the introduction of this calculator is linked to its particularly new nature: apart from the fact that it includes a computer algebra system, a spreadsheet, a graphical and geometrical environment, this calculator has specific properties which allow storing, sharing and organizing data. Files can be organized into directories, each file being constituted of one or more activities of one or more pages. The different pages, using each a particular software are connected together; for example, starting from different measurements in the geometrical environment, variables can be stored and computed in other applications, such as spreadsheet or CAS, using a different framework. Finally, it is possible to link calculators and computers and to work equally with the calculator or the computer. In the two schools, computer laboratories were equipped with the software.

A team of the INRP (Institut National de Recherche Pédagogique) has monitored these experiments to study the instrumental genesis (Guin & Trouche, 2002) of both teachers and students and to explore:

- how this calculator modifies the teachers' and the students' systems of resources;
- how it impacts the interactions between teachers and students.

## THEORETICAL FRAMEWORKS

The study focuses on the modifications brought by the introduction of such a technology in teaching and learning. We assume that the calculator is both a tool allowing calculation and representation of mathematical objects but also an element of students' and teachers' sets of resources (Gueudet & Trouche, 2008a, b; 2009). As a digital resource, the handheld calculators possess the main functions required for a documentary production:

“The two cognitive functions, memorization and organization of ideas, seem to be the fundamental basis for the documentary production. [...]

The function of creativity comprising enrichment due to the domain of interest related to the document surpasses that kind of organization just mentioned. [...]

The third and last constitution function of the documentary production is the transmission function.” (Pedaque, 2006, p.3)

These properties can be used in different domains of mediation: the private or individual domain, where the resource is part of the user's own documentation, of her personal library and is designed for her own use; the collective domain, where the resource is designed to be shared in a particular community and the public domain where the resource is addressed to the *public sphere*. Crossing the properties and the domains of mediations offers a grid of analysis of the position of the calculator in the students' and teachers' sets of resources. We speak here of sets of resources in a wide sense. In line with Gueudet & Trouche (2009, p. 200), we consider resources as “everything likely to intervene in teacher's documentation work: discussions between teachers, orally or on line; students worksheets, etc.” We extend this citation to students' documentation work taking into account the fact that the handheld calculator plays a specific role, both as an element of the set of resources and an artefact allowing to mediate teachers' and students' mathematical activities. Several studies (e.g., Artigue, 2002; Guin & Trouche, 2002; Laborde *et al.*, 2005) have shown that the integration of technology into the classrooms is a slow process in which the artefact becomes an instrument through a double movement of instrumentation and instrumentalization. The instrumentation is the process where the artefact modifies the user's activity and the instrumentalization is the process where the user modifies the artefact for her own use. This slow process, called instrumental genesis (Rabardel & Pastré, 2005) transforms the artefact into an instrument through the equation: artefact + scheme = instrument. A scheme, following Vergnaud (1996) is an invariant organization of an activity. We introduce a distinction between artefact and resource to stress the different properties of the handheld calculator. The functions of memorization, organizations of ideas and transmission give to the calculator a specificity: a resource may be transformed into a document through the process of *documentational genesis* (Gueudet & Trouche, 2009). This documentational genesis has a dual nature:

“The instrumentalization dimension conceptualizes the appropriation and reshaping processes [...]. The instrumentation dimension conceptualizes the influence on the teacher’s activity of the resources she draws on.” (Ibidem, p. 205)

From this double movement of instrumentation and instrumentalization, resources become documents, that is to say resources with scheme of utilization at a given moment. We assume that documentary production properties are an important element of this transformation of resources into documents through the documentational genesis for both teachers and students. Looking at the calculator with its different potentialities, we consider it as an artefact with possibilities of calculation and representation (properties of creation) and as a digital resource with possibilities of data processing and data sharing (properties of memorization, organization of ideas, and communication).

### METHODOLOGY

The purpose of the methodology is to capture and to monitor the dynamics of the different genesis. In this particular context, we develop our methodology to obtain information on the processes instead of the results of the processes. We have chosen to passively observe the experiments, without intervention of researchers. In S1, the teachers and students that we observed are in the last class of the high school (Lycée, Terminale S, scientific class, 18-year-old students). The choice of this class level comes from the practical examination that students have to take at the end of the year. This practical examination was an experiment carried out by the ministry of education during which students had to solve a mathematics problem with the help of software or calculators. We thought that this examination would be a good opportunity for teachers to develop the use of calculators in their classrooms and for students a sufficiently precise goal which could lead them to use their calculators in and outside the math classroom. In S2, we observed students of *première S* (scientific class, 17-year old students). We wanted to start our observations at the very beginning of both the introduction of the calculator and the beginning of scientific studies. Both teachers have a long teaching experience; this choice was done because we wanted to focus on the uses of technology in the class without being distracted by mathematics teaching difficulties. In our methodology, we cross different observations and data:

- First, the TI-Nspire calculator, as previously said, has a functionality which allows structuring its contents into directories, files, problems... Hence, we decided to observe the students’ calculators contents; more precisely, we chose *representative* students and asked them to send us the content of their calculator half-monthly from December to June. We mean by representative students with different mathematical and technological skills.

- Second, we organized observations in the classroom: in the ordinary classroom as well as in the computer laboratory. During these observations, we mainly focused on interactions between students and teacher.
- Third, interviews with teachers before and after the observations give us information about the role teachers give to the calculators in the observed lessons.
- Fourth, we took advantage of the final examination to observe and to interview students about their personal use of the calculator.
- Finally, we asked students to fill in a questionnaire at the beginning and at the end of the year, focusing on the one hand on their opinions about the calculator and more generally about technology and, on the other hand, about their attitudes towards mathematics.

Although the body of observations and data analysis enlightens the following results, in this paper, we particularly focus on the study of the content of students' calculators in order to draw different types of utilization in relation with the documentary production. These data allow us to follow a part of their documentational genesis and give us information about the use of calculators outside the classroom. The contents of a small number of calculators (eight the first year, six the second year) allow us to formulate hypotheses that have to be confirmed in a future study. They give us information about the instrumental genesis as well as the evolution of the use of calculators as a part of the students' individual set of resources. In the next paragraph, we give some results about these geneses seen through the analysis of the contents of the calculators.

### **SOME RESULTS**

We lean on the analysis of the contents of the students' calculators to draw a parallel between the main functions of a documentary production as described below and the actual students' organization of their calculators. Through the content of the calculators, we assume that we can approach the private domain of mediation and we cross our information with observations in classroom and interviews to reach the collective and public domains.

#### **Memorization of knowledge and organization of ideas**

From the viewpoint of students, the handheld calculator is a means of data storage, which they perceive as helpful:

It's reassuring in the perspective of the exam to have proofs stored, because we have to know very many proofs... and it's also possible to verify our calculations (Interview, May 19, 2009)

The folder structure of this student's calculator is a good example showing his organization of knowledge, and it is interesting to follow the contents of the folders and their evolution during the year. Figure 1 shows the general organization of the



## Working Group 15

calculator which does not change during the year and Figure 2 shows the evolution of a particular folder (*Maths oblig cours* which is an abbreviation for: math lessons): Figure 2a shows its content in December and Figure 2b its content in June. We clearly see in this calculator organization a complete structure of the math course. Looking more deeply into the files, we see that this student uses his calculator as a digital notebook, giving a summary of the lesson (Figure 3) and allowing mathematics experiment: the slider shown on Figure 3 (on the right) changes the value of  $a$  and the curve of the function  $x \rightarrow a^x$ .

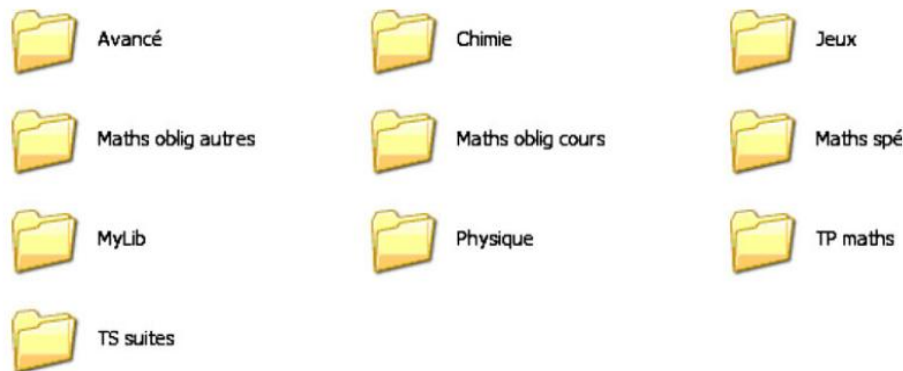


Figure 1: Organization of data in a student's calculator

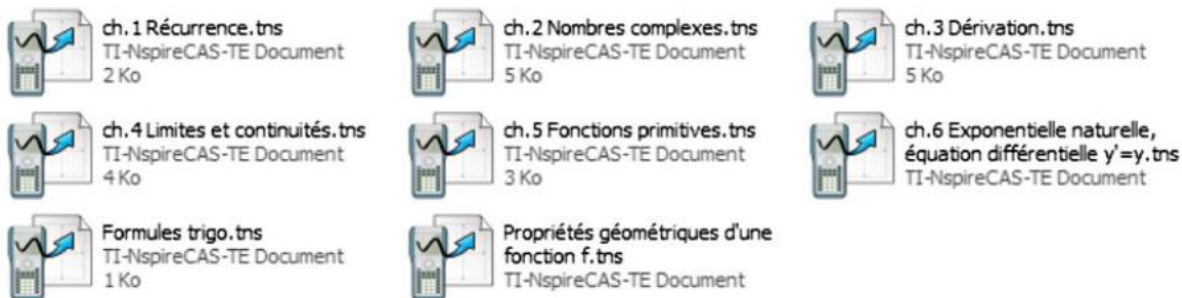


Figure 2a: Evolution of the content of "Maths oblig cours" folder (December 2008)

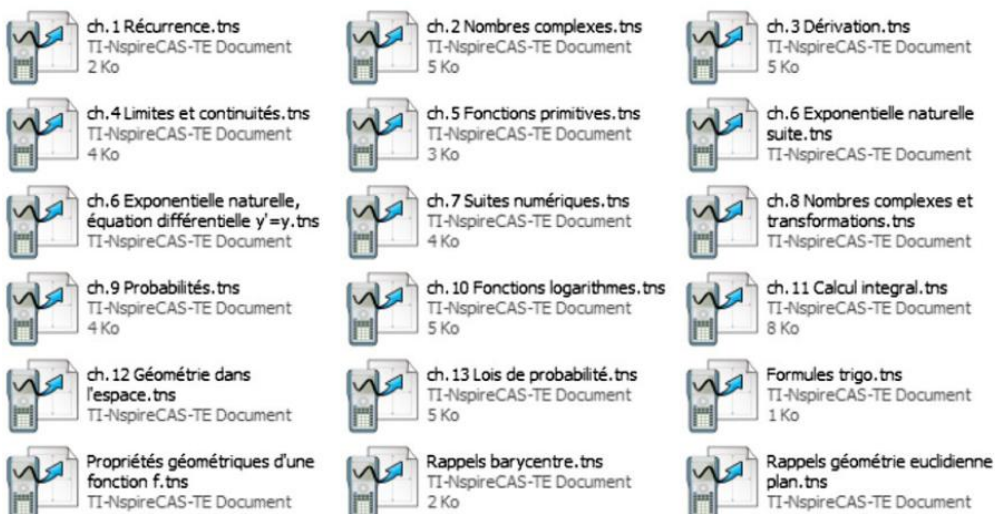
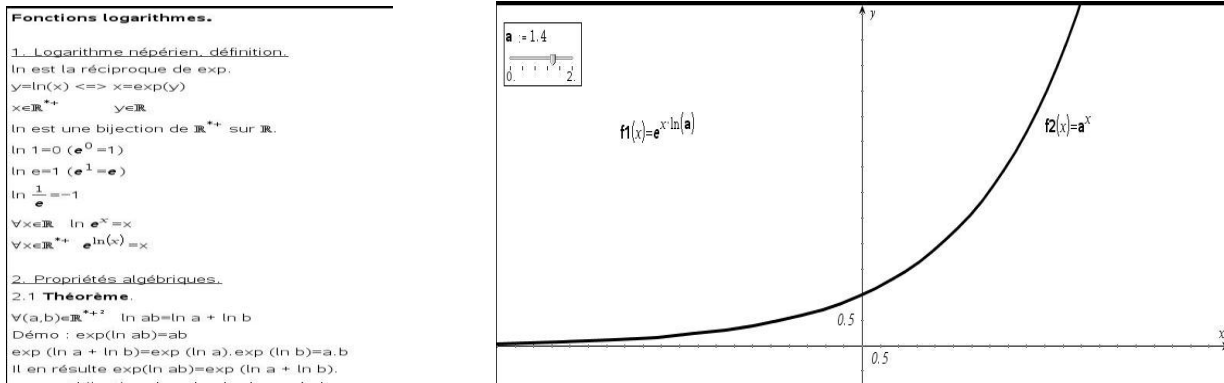


Figure 2b: Evolution of the content of "Maths oblig cours" folder (June 2009)



**Figure 3: Two pages of the file ch.10 “Fonctions logarithmes” (logarithm functions)**

Depending on the teacher’s conceptions, these properties of memorization and organization of ideas are rejected, ignored or promoted and the resource may stay hidden and private or, on the contrary, become collective and shared in the class.

### Property of creation

Looking at the contents of the students’ calculators, it appears that they use them very often as a draft, using a file or a set of files to do the current calculation. For example, one of the calculator structures is made of different folders called *br1*, *br2*..., *br* being an abbreviation for the French *brouillon* (draft in English). The calculator is seen in this case as a direct creation tool bringing immediate feedback to a given question in a personal domain of mediation. The resource at a particular moment cannot become a document because the dimension of organization of ideas is not present. However, in all calculators, the dimension of creation exists through specific pages: calculation pages, representations of data and draft files linked to mathematical problems. The interviews with students confirm this fundamental aspect of the calculator and its use in the classrooms:

Yes, in math lessons, for derivative functions, integrals and so on [...] we really use it (the calculator) during math lessons. (Interview, May 19)

One very important point during the observation in the classroom was to observe the interactions between the teacher and the students related to the organization of the calculators. In S1, the first teacher did not want to interfere; it appears clearly that this teacher wants to give a private status to this property for her own calculator as well as for the students’ calculators. As a consequence, her students’ calculators are organized around the creation property, and all the data memorization remains private, and somewhere hidden. The second teacher, on the contrary, paid attention to the data organization of the calculators. It is in his class that we find the most organized calculators (see Figure 1). In S2, teachers despite their high level of technological skills do not emphasize the organization of data, and the students’ calculators are organized around particular situations that teachers institutionalize as important in the classroom. For example, a particular lesson about statistics has been

stressed, and in all this class students' calculators a folder and files are present and stay present during the year even if there is no visible organization of data.

### **Communication**

The calculator allows for the transmission of information between students and between students and teacher. The orchestration (Trouche, 2004) plays a very important role in the transfer from a private to a collective use of the different properties of the calculator. For example, in S1, the teacher used a particular class organization: students worked with their own calculator in a face to face configuration and one student (called Sherpa in reference to Trouche's work), working under the control of the teacher, calculated and showed through an overhead projector the screen of his calculator. Such a class organization facilitates a collective communication from an individual question as shown in the next excerpt where students have to search the intersection point of two curves:

Sherpa: Madam, here...

Teacher: Yes, we can see nothing much.

Sherpa: Here, the curve is here and after there is no more curve!

Teacher: There is no more curve?

Another student: They are all at the same place, you can't see it.

Sherpa: Yes, but you must see the intersection!

A third student: Yes, but, where is your other curve?

The dialogue begins with an individual question and is continued by a dialogue in the class between students. On the other hand, the Sherpa's calculator appears to be a vector of communication between the teacher and the students, when, taking profit of a question of the Sherpa, the teacher transmits information to the whole class. In this case, the collective calculator appears to be a generic calculator which allows the teacher to regulate the class work. In other words, the teacher, through the orchestration, transforms the creation property from the private domain to the collective domain.

### **CONCLUSION**

The teacher's demeanour has a decisive influence, and the transformation of the calculator as an artefact with calculation and representation potentialities into a resource towards a document giving a help to the students' construction of knowledge depends not only on the teachers' technological skills but also on their awareness of these possibilities and on the pedagogical exploitation of these functionalities, mainly organization of knowledge and memorization. The position of the tool as an element of a system of resources of both teachers and students makes the negotiation of the didactical contract very complex. The availability of the calculator evokes different kinds of tensions:

- tensions between the memorization properties and the teachers' conceptions of students' resources;
- tensions between the property of creativity and the teacher's intentions;
- tensions between the communication property and the teachers' pedagogical organization.

Trying to explain and describe these tensions and exploring the links between the documental genesis related to the calculators and the construction of knowledge offer new perspectives of research that can extend and complete this study.

## NOTES

1. HHT: Handheld technology

## REFERENCES

- 40 Aldon, G., Artigue, M., Bardini, C., Baroux-Raymond, D., Bonnafet, J., Combes, M., Guichard, Y., *et al.* (2008). Nouvel environnement technologique, nouvelles ressources, nouveaux modes de travail : le projet e-CoLab (expérimentation Collaborative de Laboratoires mathématiques). *Repères IREM*, 72 et INRP. English version: [http://educmath.inrp.fr/Educmath/ressources/lecture/dossier\\_mutualisation/ecolab-english.pdf](http://educmath.inrp.fr/Educmath/ressources/lecture/dossier_mutualisation/ecolab-english.pdf)
- 41 Aldon, G., & Sabra, H. (2009). *Intégration des calculatrices dans l'enseignement des mathématiques : nouvelle étape technologique, nouvelles formes d'intégration, nouveau type d'expérimentation*. Lyon: INRP. Retrieved from <http://educmath.inrp.fr/Educmath/recherche/equipes-associees/e-colab/rapport.pdf>
- 42 Artigue, M., (2002). *Learning mathematics in a CAS environment: the genesis of a reflexion about instrumentation and the dialectics between technical and conceptual work*. International Journal of Computers for Mathematical Learning, 7(3), 245-274
- Gueudet, G., & Trouche, L. (2008a). Vers de nouveaux systèmes documentaires des professeurs de mathématiques. Dans F. C. Bloch (Éd.), *Actes de la 14ème Ecole d'été de didactique des mathématiques*, Grenoble : La Pensée Sauvage.
- Gueudet, G., & Trouche, L. (2008b). Du travail documentaire des enseignants : genèses, collectifs, communautés. Le cas des mathématiques. *Education et didactique*, 2-3, 7-33.
- 43 Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *In Education Studies in Mathematics*, 71, 199-218.

- 44 Guin, D., & Trouche, L. (Éd.). (2002). *Calculatrices symboliques : transformer un outil en instrument du travail mathématique, un problème didactique*. Grenoble : La Pensée Sauvage.
  - 45 Kieran, C., & Drijvers, P. (2006). The co-emergence of machine techniques, paper-and-pencil techniques, and theoretical reflection. A study of CAS use in secondary school algebra. *International Journal of Computers for Mathematical Learning*, 11, 205–263.
  - 46 Laborde, C., Perrin-Glorian, M.-J., & Sierpiska, A. (Éd.). (2005). *Beyond the apparent banality of the mathematics classroom*. Dordrecht: Springer.
  - 47 Peschek, W., & Schneider, E. (2002). CAS in general mathematics education. *ZDM - The International Journal on Mathematics Education*, 34(5), 189–195.
  - 48 Pédaque, R. (2006). *Le document à la lumière du numérique*. Caen : C & F éditions.
  - 49 Rabardel, P., & Pastré, P. (Dir.) (2005). *Modèles du sujet pour la conception*. Toulouse : Octarès.
  - 50 Rocard, M., Csermely, P., Jorde, D., Lenzen, D., Walberg-Henriksson, H., Hemmo, V. (2007). *Science Education Now: A Renewed Pedagogy for the Future of Europe*. Luxembourg: Office for Official Publications of the European Communities. Retrieved from [http://ec.europa.eu/research/science-society/document\\_library/pdf\\_06/report-rocard-on-science-education\\_en.pdf](http://ec.europa.eu/research/science-society/document_library/pdf_06/report-rocard-on-science-education_en.pdf)
  - 51 Trouche, L. & Drijvers, P. (2010). Handheld technology for mathematics education: flashback into the future. In P. Drijvers, H.-G. Weigand (Eds.), *The role of handheld technology in the mathematics classroom, Special issue of ZDM – The International Journal on Mathematics Education*, 42, 667-681.
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281-307.
- 52 Vergnaud, G. (1996). Au fond de l'apprentissage, la conceptualisation. In M.-J. Perrin-Glorian, R. Noirfalise (Eds.), *Actes de l'école d'été de didactique des mathématiques* (pp. 174-185). Clermont-Ferrand : IREM.
  - 53 Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester (Ed.), *Second handbook of research* (pp. 1169–1206). Charlotte, NC: Information Age Publishing.

# TRANSITIONS BETWEEN MICRO-CONTEXTS OF MATHEMATICAL PRACTICES [1]

Vasilis Tsitsos, Charoula Stathopoulou

University of Thessaly

*The study presented here is concerned with transitions between micro-contexts of mathematical practices. These micro-contexts are determined by the use of different software. Here we focus on a task in which students had to depict the blueprint plan of their schoolyard. The task completion demonstrates a productive interaction of tool use which combines instrumental approaches to achieve the given purpose. At the same time it provides a framework for the observation of knowledge and skills transfer during transitions from one micro-context to another.*

Keywords: instrumental jumps, transitions, microcontext, mathematical practices, instrumental approach

## INTRODUCTION

The way digital technologies impact or could possibly impact processes of mathematics learning / teaching has been one of the main interests of the educational research over the last two decades (Artigue & Bardini, 2010). The use of digital material is not just an added tool, but creates new conditions which deviate from the established teaching and learning environments. Some of the changes caused involve the cognitive processes, the mode of production and reproduction of knowledge and the communication behaviour of teachers and students. The teachers redefine their roles and the learning process by creating new communication situations. The role of the teacher as knowledge carrier changes and what comes at the heart of the whole process is the interaction between teacher and students as well as between students themselves through the use of digital tools (Milionis & Balta, 2001).

The role and use of tools in the educational process was discussed in the previous CERME WG7 as well. It was considered important to address several issues such as design, articulation between design and use, interaction between resources and teachers' professional practice, technologies, tools and students' mathematical activity. The role of the tools and their transformation when used for the carrying out of specific activities comprise the major issues currently of concern to researchers in the area of ICT. The tools used in the present research were: a) *Geometer's Sketchpad*, a dynamic geometry environment and b) *Microworld pro*, a Logo-based environment / Turtle Geometry software. The teaching contexts in which the materials were converted from artefacts to instruments by the students were considered to be micro-contexts. We investigated how each context relates to the way students manage mathematical operations as well as whether and how mathematical knowledge is transferred from one micro-context of mathematical

practice to another. We also looked into the process of interaction between students as well as between students and teacher as mediator.

The survey was carried out outside school hours and classroom context as part of a school environmental program called "*Environmental journeys through mathematics and technology*" and attended by 24 students, 13 of which were 7<sup>th</sup> Grade and 11 were 8<sup>th</sup> Grade. Meetings were held in the school computer room. The Math/ICT teacher was actively involved along with a Science teacher. The meetings took place for two hours each week for an entire school year.

Utilizing ethnographic and teaching experiment techniques (Chronaki, 2008) we studied the community of practice (Wenger, 1998), formed by the participant students and educators, while we introduced activities. The present paper summarizes the overall research effort and discusses research results focusing on one of the research *tasks* in which students had to depict the blueprint plan of their schoolyard.

## **THEORETICAL FRAMEWORK**

As mentioned in the literature (Artigue & Bardini, 2010; Guin, Ruthven, & Trouche, 2004), researchers have become more and more sensitive to *instrumental approaches* (Verillon & Rabardel, 1995) to mathematical education, that is the processes of *instrumentalisation* and *instrumentation* that drive the transformation of a given artefact into an instrument of mathematical work. This perspective combines both the Piagetian and the Vygotskian theoretical frameworks. The instrumental approach is based on the distinction between an artefact and an instrument. The term artefact describes a human-made object, either material or symbolic. The term instrument describes a mixed entity with artefact-type components as well as *utilization schemes* (Trouche, 2005), which indicate the functional value of the instrument for the individual. These schemes concern the strategies developed by the individual in order to carry out a task. Utilization schemes are formed gradually through the use of the artefact. As a result the instrument is a mental construction of the individual and has psychological qualities. The process of the transformation of an artefact into an instrument is called *instrumental genesis*. This approach, is made of two interrelated processes: an instrumentation process (the artefact shaping a user's activity) and an instrumentalisation process (the artefact shaped by the users' activity).

During the instrumental genesis process the role of the teacher proves to be of paramount importance. Trouche (2004) introduced the term "*instrumental orchestration*" in order to describe the teacher's management of the individual instruments in the collective learning process. He defines instrumental orchestration as a deliberate and systematic process of organizing the different artefacts which are available for a specific project. There are two ways of organizing, namely didactic configuration and exploitation mode. The first relates to the selection and arrangement of the artefacts for the project while the second refers to the decisions



made by the teacher about how the artefacts can be used to achieve the learning objectives. Drijvers, Doorman, Boon and Van Gisbergen (2010) introduce a third way of organizing, which they name "*didactical performance*", in order to stress the ad hoc decisions taken while teaching on how to actually perform the enacted teaching in the chosen didactic configuration and exploitation mode. According to this perspective, instrumental orchestration aims at enhancing the students' instrumental genesis but also causes the instrumental genesis of the teacher/researcher himself/herself both in the process of preparing a deliberate intervention and while it is actually taking place. In this paper we focus on the on-going didactical performance mode of instrumental orchestration and discuss the results with an emphasis on what we would like to describe as *instrumental jumps*, which in our view involve situations where the use of an instrument within a context may act as a step leading to instrumental genesis in a different context. The contexts in the specific research are defined by the two types of software mentioned earlier. We chose these types of software because they mark two different environments both in terms of user access and epistemological foundation. In Microworlds pro, in order to create events students have to describe them in a symbolic way through scripting. In contrast, in a Geometry Sketchpad environment they can operate directly using basic geometric concepts and explore objects and their relationships through dynamic manipulation. Our intention to focus on instrumental jumps stems from the question whether and how knowledge is transferred from one mathematical context to another.

## METHODOLOGY - THE SETTING OF THE RESEARCH

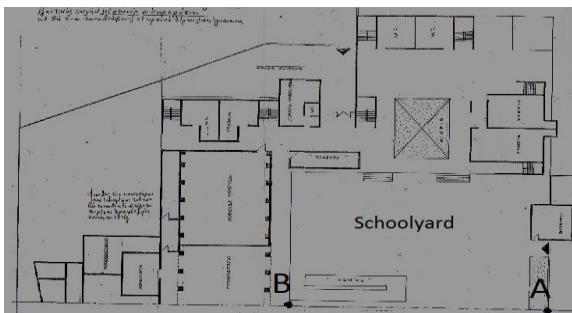
As our interest lies, on the one hand, on studying learning in specific contexts and, on the other, on exploring how students involved in learning activities think, we believe that anthropological research orientation is a suitable kind of approach. The ethnographic survey was supplemented by the implementation of teaching experiment techniques and the researcher intervened at various stages using pre-designed activities which intended to trigger events to be studied. The role of the researcher was principally that of a participant observer, who focused either on observation or on participation accordingly through the assignment of relevant activities or the learning of some software. The ethnographic equipment used included video and tape recordings, field notes, and the material handed in by the students both in final form and during the tackling of an activity.

As mentioned earlier the research was incorporated in an environmental program and so it developed along two mutually assisted directions. The environmental direction sustained the learning community for a long time while the research direction seemed to be augmenting the design of teacher intervention as well as the support of pupils. The students had the opportunity to engage in meaningful instructional situations, and teachers were supported in the design of their interventions. This mode of intervention substantially reduced the researcher's participation in defining

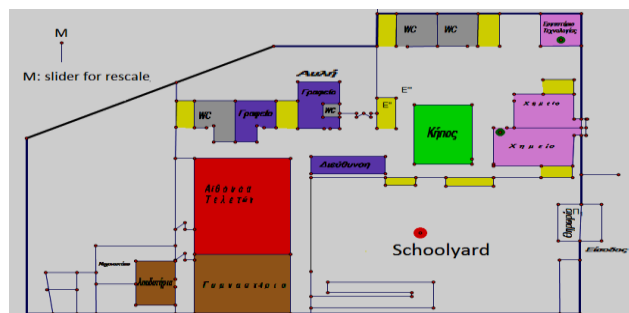


and developing the activities, giving him more leeway for participatory observation. What is more, the participation of students from various grades prevented teachers from applying purely formal teaching. So the community formed differed from a typical class and enabled us to systematically organize communicative relationships between pupils of the 7<sup>th</sup> and 8<sup>th</sup> grades and to observe students in a less formal framework.

The research consisted of three phases based on constructionism theory concerning learning through construction, 'learning-by-making' (Harel & Papert, 1991). The first phase involved designing digital games using symbolic expression software, Microworld pro. The second phase focused on the construction of paper digital cameras pinhole camera [5] and the use of dynamic geometry software for their digital modelling. In the third phase, certain tasks derived from the interweaving of research and the environmental program was given to the students so that they could use the instruments developed in the previous phases. This paper discusses the results of one of the tasks of this third phase, namely the dynamic depiction of the schoolyard. The examples (episodes) used to illustrate the discussion are taken from the work of two groups of 8th grade students who chose to use the environment of dynamic geometry software. This project arose from the need of the environmental program to design a dynamic depiction of the school with the aim of dynamic information encoding. Such information could contribute to immediate awareness of current failures and problems such as burnt bulbs, consumables shortage, and full recycling bins. The potential for dynamic scale manipulation of the floor plan derived from the need to view the site both centrally and in certain parts. To this end, students were given a real architectural sketch of the floor plan of the school as shown in figure 1. The mathematical concepts that were negotiated were those of reduction, growth and measurement.



**Figure 1: Floor plan of the school**



**Figure 2: Final plan of student M5**

## **Episodes / Discussion of the Episodes**

Although the students' dialogues are of great interest as they reflect the students' effort to decode the blueprint given to them, in this paper due to space constraints we focus our attention on some of the episodes associated with the drawing of the plan. Specifically, students in the two groups, three boys (M1, M2, M3) and two girls (M4, M5) are working at adjacent positions and are able to communicate with each other.

Before reviewing a few episodes, we present a typical episode and an initial analysis of it. It focuses on a snippet dialogue developed among three students trying to choose the starting point and design technique. The dialogue is illustrative of the different instrumental approaches that students develop through discussion.

### Episode 1 and its analysis

- 404 M1: Where to start?  
 405 M2: From this angle.  
 406 M2: Let's put a point here.  
 407 M3: We must measure the distance, what to do with the point?  
 408 M2: So I've put the point here, how do I tell the point to move to the left, how long is this? I want the ruler.  
 409 M3: Have you confused the programs? The point is not a turtle.  
 410 M2: There may be a command for the point to move.  
 411 M3: Hey! The ruler, what do I measure?  
 412 M2: This is the yard. Measure, here. Come on, measure.  
 413 M3: 11.2 cm.  
 414 M2: Do we have to tell the point to move 11.2 to the left? I don't know.  
 415 M3: Here we go again, after I told you the point is not a turtle.  
 416 M3: Draw a line (he means segment), measure it.

The students have never before used the specific software to depict blueprint plans that require precise measurements. The dialogue focuses on how the two students differently understand how to start the drawing of the plan. The student M2 wants to start his plan from point A, as shown in figure1, giving this point a command to move [2], while the student M3, as revealed at the end of the dialogue [3], wants to begin his construction with a random segment. The attitude of the student M2, who remains anchored in the culture he had acquired about the movement of the turtle according to specific commands in *Microworld pro*, reveals four different aspects of the instrumental jump process. The first is the symbolic representation of a point as a turtle in the logo-based environment. For the student, this concerns not only a one-to-one point correspondence but also a transfer of the qualities of the turtle resulting from its use. The second concerns the intention of the student to bring that culture to the new situation. The third concerns the communication of his idea to the other students in an effort to elicit ways to adjust the use of the turtle. The fourth aspect relates to the fact that his idea acquires communication features through interaction, a first example of which is the discouraging response of student M3. So starting with a problematic situation in the present work context we observe an attempt of knowledge transfer from a different previous work context. The initial stage of this knowledge transfer is the instrumental jump process described just above. From the probing questions that we asked during the project we made the following realization. Unlike student M3, who rejects the idea due to his knowledge that Geometer's Sketchpad does not have Logo-based characteristics, student M2, who is

not fully aware of this distinction, dares this mental jump and tries to find ways of adjusting his idea. If adjustment of the tool use is achieved successfully, then knowledge transfer has been completed. However, at this point the turtle idea is abandoned for the time being and will be picked up and further explored later.

The different way to start the activity indicates the two different frameworks that provide the springboard for discussion. Student M2 suggests using the turtle and expects the software to have similar functionality with the logo-based environment, while student M3 bases his reasoning [4] on the geometric object of the segment the construction of which he seems to know. This latter idea prevails for the time being and the students work on it.

### **Discussion of the rest of episodes**

Then the group of boys work on the idea of student M3 for the construction of a random segment and try to give it the length of 11,3 cm with suitable manipulations. Their successful effort allows them to proceed with the same rationale to the construction of a closed line consisting of four consecutive variable segments, which resembles a rectangle that corresponds to a part of the schoolyard blueprint.

So, student M3 uses the artefact of the variable segment, a basic element of the work environment, and suggests stabilizing its length through trial and error dragging processes. His active involvement in both groups leads to the appropriation of his idea by the other students. In this way his idea becomes meaningful as it is transformed into an instrument through its collective use for the achievement of the common goal. However, two female students, M4 and M5, have not been involved in the debate so far, but carefully followed the discussion of their peers. The teacher moves a point that represents a corner of the schoolyard and the graph loses the shape of the rectangle. The students' surprise is coupled with the reminder that the goal is to design a dynamic plan that retains its geometric properties. *"But how are we going to do it, Sir? So much work to be wasted!"* student M1 reacts. The reminder of the dynamic floor plan is the cause for a new round of talks, which last for an hour. *"I found it, sir!"* student M3 enthusiastically cuts in. Going on to a different screen he displays the vertical axes of the software and thinking aloud or addressing the others he explains: *"I'll create the first dimension on the axis using the scale and I'll base all measurements on this length, here, when I change this point that other one changes too"*.

Student's M3 idea incorporates both the preservation of the geometric properties of the shape and the potential for dynamic manipulation of the shape. The problem is that he does not know how exactly to put these specific points on the axis and demands that the dynamic geometry program provides a command corresponding to the logo-based environment. *"How do I say 'Set position [11.3, 0]' Sir?"*, he asks. He had already used this Logo-based command in the first phase of the research when constructing a digital puzzle. At this point we intervened and provided information

for the resolution of the specific problem within the dynamic geometry environment. Here we can observe an instrumental jump made by student M3 during the process of instrumentation concerning the depiction of a point on a system of axis in order to achieve the goal of the dynamic blueprint. After that, student M3 makes point B of the plan, as shown in figure 1, correspond to point (0, 0) and works on his own with the tool "*plot points*" thus completing part of the schoolyard. On seeing that student's M3 request for an equivalent tool to depict points in the Sketchpad environment was satisfied student M2 repeats his earlier question: "*How can we move point A, 7.3 cm to the left, is there a command?*" [2]. At this point the idea of the turtle, which was earlier abandoned, is followed up by the group. This time we better evaluate the student's question and realize that, on the one hand, the simulation of the turtle movement command is possible while, on the other, such a simulation requires concepts relevant to geometric transformations. It is a pivotal point in the on-going process of instrumental orchestration as we experience a case of instrumental genesis. The Logo-based turtle movement command turns into a transformation instrument in the dynamic geometry environment. Then the discussion revolves around tools of parallel transfer transformation, which are embedded in the dynamic geometry environment.

The new tools appear to be also accepted by the two girls (M4, M5) slowly getting involved in the drawing of the plan. While the drawing is taking place for the third time, student M3, who has so far managed to draw a dynamic plan, addresses those who have worked with the tool of parallel transfer with the words: "*But yours doesn't zoom in and out! ha! ha!*" Indeed, the students have designed the school playground with accurate measurements and without moving the components affecting the plan. However, they have not given their plan a dynamic character. This idea opens new rounds of talks completed in two consecutive sessions. The focus is on minimizing the measurements as well as on trying to find proportional relations between the sides. "*Let's find all the other distances from segment 11,3*", student M2 characteristically says. Using paper, pencil and a calculator the rest of the students support student M2 and through discussions and individual actions they conclude to proportional relationships. Due to space constraints of the present paper these discussions are not presented here despite their interesting indications of how students perceive the proportional change in shape when they work with combined modeling tools. This area has been adequately illuminated by several other researchers (Noss & Hoyles, 1996; Psycharis & Kynigos, 2004).

Students' discussions and actions highlighted two new ways of working for the drawing of the dynamic plan. The first way adopted by student M2 required the proportional relationship between the sides. To construct the plan he used only one variable (variable-length segment) relating all the lengths to it. It should be noted that the specific student had worked on Microworlds pro in the first phase of the research in a group activity to design and present something that interested them.

When his turn came he said: *"I will not show you something but I'll tell you one way I found to make one and the same shape bigger or smaller without having to write a new program"*. His presentation was, in fact, the structure of a parametric process, which he passionately believed he had invented himself. The reference to this fact is to indicate that the student had learned how to minimize the variables of a parametric shape in a logo-based environment and functionally transferred this knowledge to the environment of dynamic geometry software. The second mode supported by the students M4, M5 involved the multiplicative relationship between each measurement and a number corresponding to the scale factor. The probing questions showed that this method of the female students stemmed from their engagement with the concept of scale in the geography lesson. At this point they could choose one of following three ways to work: based on a scale or on a proportion or on co-ordinations. Students M4 and M5 cooperated to a sufficient extent in the design of the floor plan of the school, but student M5 was the one who completed the plan after a week. Figure 2 shows the final draft of student M5 in which point M on the top left represents the slider of the rescaling. Students M2 and M3 worked at different workstations, each one developing his plan for a while but soon abandoned the task finding it rather tedious to repeat something that they already knew. Careful study of the students' M4 and M5 blueprint produced by the use of the tool "object properties" that is provided by the software as well as the probing questions that followed showed us that the draft floor plan was drawn with a mixed approach. They based their design on the transformations tool but they also used Euclidean geometric constructions whenever they thought they would proceed faster.

There is particular interest in student's M5 effort to name places on the blueprint in a dynamic way. *"Sir, I would like the names of places on the plan to change size along with the plan. Look I wrote Schoolyard and when the size of the plan grows the word remains the same"*, she explains. We were impressed by the fact that although she had repeated the same scale structure to construct the plan components dozens of times she seemed not in need of it in this case. With more profound questions we realized that she firmly believed that if the word was in a closed-dynamic shape it should also zoom in and zoom out along with the plan. She mentioned that the word was on *"a bordered area"*, implying the restricted area of the schoolyard. This restrictive view of the student derived from the instrumental jump she attempted based on her knowledge of the use of *select* and *zoom* buttons available in widely used software.

## CONCLUSION

In the two groups which we could monitor very closely during the set meetings while intervening when we saw fit, we observed the interweaving of different mental schemes, technical knowledge and specific mathematical knowledge, which determined the progress of task completion. These mental schemes became an object of negotiation. The different types of tools we chose gave the students an opportunity

to mentally jump from one instrumental environment to another thus shaping both individual and collective actions. These mental passages led to the creation of mixed instrumental approaches. The knowledge and the utilization schemes transferred from one micro-context to the other were informed both by the inherent features of the former environment and the characteristics of the later environment. This combination resulted in the creation of hybrid tools. Knowledge transfer was achieved in some cases while in others it was hindered, as shown by the brief presentation of the episodes. The conditions under which it is possible to transfer mathematical knowledge from one micro-context of practice to another as well as the role of social interaction are considered attractive research topics that deserve further investigation.

### NOTES

1. The idea comes from the title of the book *Transitions between contexts of mathematical practices* by DeAbreu, Bishop, and Presmeg (2002).
2. Line 408, 410, 414.
3. Line 416.
4. Line 409, 415.
5. Pinhole camera is a simple camera without a lens. Instead of a lens it has a single small aperture.

### REFERENCES

- Artigue, M, Bardini, C. (2010). New didactical phenomena prompted by TI\_Nspire specificities – The mathematical component of the instrumentation process. In V. Durand-Guerrier *et. al.* (Eds.). *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education CERME 6* (pp. 1171-1180), January 28-February 1 2009, Lyon, France.
- Chronaki, A. (2008). *The teaching experiment. Studying learning and teaching process*. In B. Svolopoulos (Ed.), *Connection of educational research and practice*, Athens: Atrapos.
- De Abreu, G., Bishop, A., & Presmeg, N. S. (2002). Mathematics learners in transition. In G. De Abreu, A. J. Bishop, & N. S. Presmeg (Eds.), *Transitions between contexts of Mathematical Practices* (pp. 7-22). Dordrecht: Kluwer Academic Publishers.
- Drijvers, P., Doorman, M., Boon. P, Van Gisbergen, S. (2010). Instrumental orchestration: Theory and Practice. In V. Durand-Guerrier *et. al.* (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education CERME 6* (pp. 1349-1359), January 28-February 1 2009, Lyon, France.

- Guin, D., Ruthven, K., & Trouche, L. (Eds.) (2004). *The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument*. New York: Springer.
- Harel, I. & Papert, S. (1991). *Costructionism: Research reports and essays*. New Jersey: Ablex Publishing Corporation, Norwood.
- Milionis, Ch., & Balta, V. (2001). The communicative dimension of educational procedure and the role of new technologies. In *Proceedings of Hellenic conference New Technologies in Education and Distance Learning* (pp. 346-358), July 8-10, Rethymnon.
- Noss R., & Hoyles, C. (1996). *Windows on mathematical meanings: learning cultures and computers*, Kluwer Academic Publishers.
- Psycharis, G., & Kynigos C. (2004). Normalising geometrical constructions: a context for the generation of meanings for ratio and proportion. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28<sup>th</sup> PME Conference* (vol. 4, pp. 65-72), Bergen: Bergen University College.
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281-307.
- Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculators environments. In D. Guin, K. Ruthven, & L. Trouche (Eds), *The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument* (pp. 137-162). New York, NY: Springer.
- Verillon, P., & Rabardel, P. (1995). Cognition and artefacts: A contribution to the study of thought in relation to instrument activity. *European Journal of Psychology in Education*, 9(3), 77-101.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*, Cambridge University Press.

# FUNCTION CONCEPT AND TRANSFORMATIONS OF FUNCTIONS: THE ROLE OF THE GRAPHIC CALCULATOR [1]

Madalena Consciência

Hélia Oliveira

Escola Sec. Dr. Ginestal Machado

Instituto de Educação da Univ. de Lisboa

*This communication focuses on the activity of two secondary school students in a task involving the concept of function and transformations of functions of the type  $p(x)+k$  and  $p(x+h)$ , and analyses the role of the graphic calculator. The results suggest that students are beginning to have an object-oriented view of function. Concerning the transformation  $p(x)+k$  they have already established the operational invariants, and it seems that the graphic calculator played an important role in that.*

Keywords: Function concept, Transformations of functions, Graphic calculator, Instrumental genesis, Operational invariants

## INTRODUCTION

This communication originates from a study in which the main objective is to understand how secondary school students integrate the graphic calculator in their mathematical activity, and the role it plays in learning of functions, along the 10<sup>th</sup> and 11<sup>th</sup> grades. Here we focus our analysis on the following research question: How do students understand the concept of function and of transformations of functions in a particular task, and what role does the graphic calculator play in this?

## THEORETICAL FRAMEWORK

### The process of instrumental genesis

The availability of a tool does not automatically turn it into an instrument in the student activity. Rabardel (2002) stresses the difference between artefact and instrument: he considers that an artefact only becomes an instrument when the user is able to appropriate it and integrate it into his activity. An instrument is thus a mixed entity, constituted by the artefact and utilization schemes.

The process through which the artefact becomes an instrument is called Instrumental Genesis. It concerns the two poles of the mixed entity (the artefact and the utilization schemes), and integrates two dimensions and orientations: Instrumentalization, directed towards the artefact, is related with the emergence and evolution of the artefact components of the instrument – selection, regrouping, production and attribution of functions, deviations, catachreses, and transformation; and Instrumentation, directed towards the user, is related with the emergence and evolution of utilization schemes (Rabardel, 2002). According to Vergnaud (1998), a scheme always involves: (i) one or more goals, sub-goals and its anticipation; (ii) rules of action, information seeking, and control; (iii) operational invariants –



theorems-in-action (propositions taken as true), which are the means to infer the objectives and the rules to adapt the activity to situations, and concepts-in-action which are central in the schemes organization; and (iv) possibilities of inference. To Rabardel (2002), the analysis of the operational invariants is important from the instrumental point of view, since it allows identifying the situations' characteristics which users take into account.

Several studies have shown that the process of instrumental genesis concerning symbolic calculators or graphing calculators is not trivial (Guin & Trouche, 1999; Drijvers, 2000), and that its integration with educational purposes requires some attention, since the technical and conceptual aspects are intertwined (Drijvers & Gravemeijer, 2005).

### **The learning of functions, representations and multiple representations**

The concept of function is one of the most important concepts in mathematics, but is very difficult for many students (Sajka, 2003). One of the difficulties related with this concept comes from its dual nature. To Sfard (1991), the concept of function can be understood essentially in two ways: operationally, as a process, and structurally, as an object. These two visions should complement each other and form a coherent whole of the concept. The author claims that a mathematical concept is first acquired operationally and that the transition to its structural form is a process accomplished in three phases: internalization, condensation and reification. Gray and Tall (1994) also refer to the duality between processes and mathematical concepts. These authors claim that the mathematical notation involves a certain ambiguity, for example,  $f(x) = x + 2$  may represent a process, i.e., a way to compute the value of the function for a particular value of  $x$ , or it may represent the object that covers the whole concept of function for a general  $x$ . In their opinion, the key to success in mathematics is to interpret the mathematical symbolism in a flexible way.

Slavit (1997) also stresses the importance of students moving from an action view of function to an object-oriented view. He considers that it is very difficult for a student to truly understand an action performed on a function, like the transformations of functions, if an object-oriented view has not yet been achieved. He proposes an approach to the reification of the concept of function based on functional properties, noting that students develop a property-oriented view when the study of functions includes, preferably, graphics and graphics technologies.

Mathematical concepts are closely linked with their representations. The concept of function can hardly be understood and acquired without the use of multiple representations, because each representation only provides information on some of its particular aspects (Gagatsis & Elia, 2005). The graphic calculator (GC) allows to combine multiple representations of functions (numeric, graphical, symbolic and physical), but as underlined by Dick and Edwards (2008), the fact that students have access to multiple representations of a mathematical object does not lead,

automatically, to a deeper conceptual understanding, nor does it lead to a flexibility to determine the most appropriate representation to a particular problem.

## METHODOLOGY

In this paper we are analyzing the responses provided by two 10<sup>th</sup> grade students, in an individual interview with the first author, to Questions 1, 2 and 4 from the following task:

Consider the polynomial  $p(x) = x^3 - 16x^2 - 2037x - 5940$ .

1. Represent it graphically.
2. Indicate two possible values for  $k$ , with opposite signs, so that  $p(x) + k$  has only one zero.
3. How many solutions does the equation  $p(x) = -5940$  have? Determine them with an error less than one thousandth.
4. Determine the solution of the inequality  $p(x+7) \cdot 100 - x^2 \geq 0$ .

The interview was carried out one month after the topic Functions had been covered in the classroom. The task did not aim to enhance functions learning, but to create an opportunity to understand how students mobilize their knowledge about functions and how the GC is integrated in their activity, as they could freely use it during the interview. In the classroom, transformations of functions were explored in same tasks for the case of module and quadratic functions ( $y = a|x+h|+k$ ;  $y = a x+h^2+k$ ), by assigning values to parameters, and using the GC to observe the corresponding effects on the graphical representation. Students used a personal CG, under guidance of the teacher, essentially to solve equations and inequalities graphically. The two selected students, Helena and Diogo, are two of the four case studies, which take part in a broader study. Data collection is still ongoing and involves classroom observation in two consecutive academic years (10<sup>th</sup> and 11<sup>th</sup> grades); clinical interviews with students; and the collection of written documents produced by them in the classroom. The two students have different performances in mathematics and have followed different approaches to the task. Helena is an average student, with ratings around 12 values, in a scale of zero to 20, and Diogo is a student with ratings above average, around 16 values.

## RESULTS

To represent the function graphically, both students used the GC, although Helena did not do it immediately. She represented the axis on her sheet of paper and said:

Helena: Well! I have no idea. [...] I don't know how to represent this graphically!

Researcher: And would the GC not help you in this issue?

Helena: But can we use the calculator? (And immediately picks up her calculator).

The use of the GC was not an assumption made initially by the student, but it was suggested by the researcher.

Both students tried to find a proper display through trial and error, by changing the values directly in the window, without resorting to Zooms, and taking considerable time to achieve a reasonable viewing rectangle.

The students used the algebraic representation to decide whether they had an acceptable graphical representation of the function, however initially both of them had doubts about the degree of the polynomial function. They considered that the degree of the polynomial is given by the sum of the degrees of its terms. When this point was clarified, the degree of the polynomial was a fundamental criterion for them. The criterion used by Helena was based on the following theorem-in-action: *A third degree polynomial function has at most three zeros*. However, she denoted some difficulty in interpreting what she visualized on the screen, putting into question this very theorem-in-action:

Researcher: Are you already seeing the whole graphical representation?

Helena: (Pause) It doesn't have more than three zeros. [...] It has two, so it could be. [...] It has two, if it had.... No, actually, it has four (looks closer at the screen). No, no, it has three. It has three zeros.

The criterion of Diogo draws on a theorem-in-action that is false: *A polynomial function of third degree changes its variation three times*.

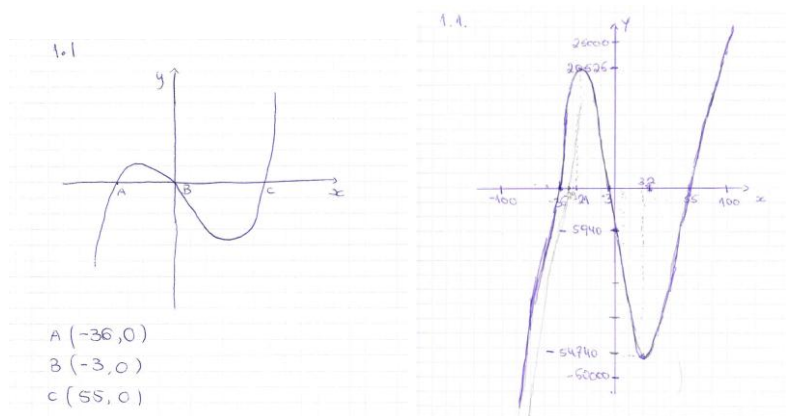
Diogo: Now I know it should be this one [graphical representation], because if it is of third degree, changes three times, is increasing, then decreases, .... Ok, it is of third degree (laughs).

Researcher: Any polynomial function of third degree increases, decreases, and ...? [...] Always?

Diogo: Yes.

The concept image developed by Diogo, related with third degree polynomial functions, does not include monotone functions. This may be explained taking into account that cubic functions which students explored in the classroom involved, almost exclusively, functions with three zeros, with the exception of one example that the teacher showed which had a double zero. In any case, the covered functions always involved change in variation.

The transition from the representation displayed on the GC screen to its register on paper was done differently by the two students (fig. 1). Helena only considered the points of intersection with the  $x$ -axis, and drew the graphical representation as "viewed" on screen, without considering any scale. Diogo, on the other hand, tried to consider a scale, although not properly. In addition, he considered the intersections with the  $x$ -axis, and determined the points corresponding to the relative extremes of the function.



**Figure 1: Graphic representations made by Helena (left) and Diogo (right).**

For determining the zeros and, in the case of Diogo, also the relative maximum and minimum, they used the *Calc* menu of the GC. Students showed no difficulties, suggesting that the utilization schemes have been developed.

For the second question, both students correctly interpreted the influence of parameter  $k$  in the graph of the function  $p$ . It is clear that both students have already established the operational invariants, allowing us to infer that they used the following theorem-in-action: *The graph of function  $p$  undergoes a vertical translation associated with the vector* , as shown, for example, by this extract of the interview with Helena:

Helena: So, I have to find a value for  $k$  (pause). [...] So that the graph goes up or down (pause), and has only one zero.

Researcher: Hum. And why did you say go up or down?

Helena: Because it is ..., it is the influence of  $k$ .

However, she did not seem to have consolidated this knowledge. In fact, in an attempt to understand the influence of the independent term in the graph of the function  $p$ , when viewing the graphical representation of a new function which is the sum of  $p$  with 5940, Helena said that part of the graph went up and another part went down. This shows her difficulty in interpreting what she visualized on the CG display:

Helena: Here went up (pointing to left side) and here went down (pointing to the right).

Researcher: Do you think it went down, in some part?

Helena: (Observes the screen) No, it went up. (Goes to the function editor) This ...

To confirm the type of displacement suffered by the graphical representation, Helena returned again to the GC, taking advantage of the possibility of observing one graphical representation followed by the other: "No, it goes up. It goes up". Even so she showed some difficulties: firstly, to understand that it would be necessary to

## Working Group 15

make the graphical representation of  $p$  go up (or down) in order that the minimum (or maximum) of the function stood above (below) of the  $x$ -axis, and, secondly, in taking advantage of the potential of the GC to determine a possible value for  $k$ :

Helena: So, to have only one zero, going up, haa, this part will have to touch at the  $x$ -axis (points with her pen at the relative minimum and shift it up to the  $x$ -axis).

Researcher: To touch ...?

Helena: At the  $x$ -axis. (Pause) Yes!

Researcher: If this part touches at the  $x$ -axis, will it have just one zero?

Helena: Yes. And going down, haa, it has to be this part (points to the relative maximum and makes a gesture down to the  $x$ -axis), touching the  $x$ -axis.

Researcher: Hum, so? (Long pause) How are you going to do this?

Helena: Well, I don't know! (Long pause).

During the interview, Helena finally understood that it was necessary to shift up the graphical representation of the function  $p$ , in order that the minimum remains above the  $x$ -axis, however, she was not able to find a more effective strategy than to assign values to  $k$  and observe the effects on graphical representation. Here she failed in constructing an efficient instrument that would allow her to answer the question immediately. Diogo, by contrast, was able to use an effective instrument, by linking together the graphical representation of  $p$ , the theorem-in-action and the concept of maximum and minimum of a function. For that purpose, he just looked at the graphical representation that he outlined in the previous question. He responds with confidence, not confirming the answer with the use of the calculator.

Diogo: It must goes up, haa, the function, the graph of the function. And the only way I see for having only one zero, is to shift up the graph, so that the point less 54740 [the minimum of the function  $p$ ] stands above the  $x$ -axis. [...] Then, the value has to be at least ..., haaa, 55000! (Pause) Because if it goes up 55000, there will be only one point that [...] intersects the  $x$ -axis.

Researcher: There will be only one point of intersection with the  $x$ -axis? Why?

Diogo: Because, I shifted the graph up, and only this one stood here (pointing to the left side), that goes down, and which continues.

Similarly, he obtained the value less 22000 as another possible value for  $k$  (with opposite sign to the first). The exploration, done in the classroom, about this transformation of functions, allowed students to establish the operational invariants that have been implemented in this question.

The last question was the one that raised more difficulties for both students. Diogo began by trying to interpret the meaning of  $p(x+7)$ :

## Working Group 15

Diogo: So, first I'll determine  $p$  of  $x$  plus seven, making a function equivalent [to write an equivalent expression] to, to become clearer. I'll replace  $p$  of  $x$  here; I will add seven in this equation (points at the expression of  $p$ ).

Researcher: Are you going to add seven to the  $p$  of  $x$ ?

Diogo: To  $p$  of  $x$ . [...] Seven, no! I can't add seven. [...]

Researcher: Why is that?

Diogo: Because this is influencing the  $x$  and not the  $y$ .

Although Diogo initially misinterpreted the meaning of  $p(x+7)$ , immediately restates his interpretation as he understands that this transformation corresponds to a horizontal displacement. Even so, he chose the displacement in the opposite direction:

Diogo: I have to move, but (pause), it is in the axis of the ... (pause, make a gesture with his hand), of  $x$ . (Pause). The function, which I drew here (points at the calculator display), has to move together, seven "places", to the left.

Researcher: Has to move seven units to the left? [...] Why to the left?

Diogo: Because, it is plus, and as we know that .... (Pause) No, it should be minus! Okay, I don't know. [...] I am thinking how could it be (looks at the statement). I am reconsidering ..., actually is to the right. [...] Because any value of  $x$  plus seven (pause), goes to the right (laughs).

His first answer was correct, however, when questioned, he revealed the fragility of that knowledge, which suggests that the theorem-in-action had been memorized, and when challenged, it was eventually replaced by another one that is false. The GC could have been an important tool for deciding the direction of displacement, however, in this situation, he was unable to take advantage of the GC. Despite having decided the direction of the displacement, he tried to write an expression for that factor, however, he fails in developing the cube properly:

Diogo:  $x$  raised to three plus 21, because is  $x$  cubed, so, I have to repeat three times seven, that is 21! We add three times seven to the three  $x$  here.

Researcher: We add three times seven? (Pause) To the three  $x$ ?

Diogo: Because, usually, if there was only one  $x$ , we add seven units, but as there are three, so we have to add 21. (Pause) (Laughs) It is wrong, ok! (Pause) So, but I ... (he picks up the calculator).

His initial goal was to solve the inequality graphically and to do so, he needed to edit the function in the GC but, during the interview, he changed his strategy and decided to do a table of signs. Even then, he maintained the idea that he had to find an analytic expression for the factor  $p(x+7)$  to edit in the GC. Despite the fact that he had started to draw what he considers to be the graphical representation of  $p(x+7)$ ,

## Working Group 15

which gives all the necessary information to fill the table of signs, he still tried to write an expression to obtain, with the GC, the information that he assumed:

Diogo: [...] (Makes a sketch of the graphical representation) I don't even need to draw, I already know the zeros, and they will be ..., minus 29, [...], four, and the other, 62. Now ... (very long pause). [...] Now I am going to put this in a polynomial. [...] I am writing an expression, with these zeros.

Researcher: Yes. And why do you need that expression? What for?

Diogo: To the table of signs. (Pause) To edit it back in the calculator, so that I can see the resultant graph, calculate the zeros, and make a table of signs.

The intention to write an expression to introduce on the calculator superimposes all his thoughts, leading him to a trajectory that makes no sense.

Helena also chose to solve the inequality using a table of signs, however failed to interpret the meaning of  $p(x+7)$ :

Helena: Hum, I do not understand what that  $p$  is doing there.

Researcher: What is that  $p$ ? (Long pause) OK, if  $p$  was not there, how would you do it?

Helena: If  $p$  was not there (pause). [...] I would just have to multiply this (points at  $x$  plus seven) by this (points at the other factor).

Researcher: Hum. Ok, so how would you do it?

Helena: (Long pause) [...] Maybe I would make a table of signs.

Researcher: Hum, hum. (Pause) Ok but, there is a  $p$  in there. [...] Ok, how do we read it? (Long pause) And if we had, instead,  $p$  of  $x$ ?

Helena: (Pause) Oh! So, it is this ( $p$  of  $x$ ) plus seven, and multiplied by this one (points at the other factor).

The fact that the parentheses include the expression  $x+7$  does not seem to bother her since she immediately decides to add seven to the algebraic expression of  $p$ . Then she uses the GC to determine the zeros and the sign of the two factors, but loses some time in finding an appropriate view window to determine the zeros.

## CONCLUSIONS AND IMPLICATIONS

The two students are beginning to have an object-oriented view of function, which seems obvious when, for example, they refer to the effects that the transformation  $p(x)+k$  has on the graph of the function  $p$ , showing that they can think about the concept as a whole. The exploration they made in the classroom of this kind of transformation on quadratic and module functions, using the GC, seems fundamental in the establishment of operational invariants that emerged in Question 2. Diogo, unlike Helena, was able to establish connections between the effects on the graph of the function  $p$  and various notions associated with the function concept, such as, relative extremes and zeros. Helena considered only zeros as relevant points, and did

not feel the need to determine the relative extremes of the function, which could have been useful for her to solve Question 2.

In the case of the transformation  $p(x+h)$  it is clear that Helena has not developed the operational invariants. In fact, initially, she failed to assign meaning to the symbolic notation  $p(x+7)$ , and only connected it to the function  $p$  after the researcher's suggestion. After that, she addressed the issue without going into any kind of cognitive conflict, and integrated the GC correctly in her activity, i.e., she got all of the necessary information to complete the table of signs. Diogo was able to grasp the transformation produced on function  $p$ , however, he seems to have memorized the respective theorem-in-action, getting into cognitive conflict when questioned about it. Although he has understood that the analytical expression of the new function could be obtained by replacing  $x$  by  $x+7$ , algebraic difficulties did not allow him to obtain the simplified expression. He also failed in using his GC effectively, losing himself in a sort of vicious cycle that could have been avoided if he had been aware that it is not necessary to introduce a simplified expression in the function editor or if he knew that it is possible to edit an implicit function, although this hypothesis has not ever been discussed in the classroom.

The students' difficulties associated with the transformation  $p(x+h)$  compared to  $p(x)+k$  suggest that the approach developed in the classroom did not contribute to the development of operational invariants. Apart from the overall approach, made possible by the use of the GC, it might have been useful to do an exploration which focused more on the local behavior of functions, using numerical representation, so that students would be able to understand the translation direction. The transformations in analytical expression and in editing in the GC could have been simultaneously clarified through that approach. As Sfard (1991) stresses: "the terms "operational" and "structural" refer to inseparable, though dramatically different, facets of the same thing" (p. 9), and although the GC, through the graphical functionality, encourages a structural approach, it is important that, at least at the beginning, students also get the opportunity to work in a more operational way, relating the numerical representation with other representations of functions. The difficulties associated with the horizontal translation have been documented in other studies (e.g., Zazkis, Liljedahl, & Gadowsky, 2003). The GC could be used, in this topic, exploring the possibility of combining multiple representations of functions.

Concerning this particular task, the instruments developed from the GC by Helena are supported by its immediate power for visualization, while the instruments developed by Diogo are essentially supported by the establishment of connections with his mathematical knowledge, although, as noted before, not always successfully. The construction of instruments to perform a mathematical task depends on several interconnected factors, among which we highlight the utilization schemes socially developed in the classroom, the level of students' instrumentalization and instrumentation and their mathematical knowledge.



## NOTES

1. This paper is supported by the Project Improving Mathematics Learning in Numbers and Algebra (Fundação para a Ciência e a Tecnologia - MCTES (PTDC/CED/65448/2006)).

## REFERENCES

- Dick, T. P., & Edwards, B. S. (2008). Multiple representations and local linearity. In M. K. Heid, & G. W. Blume (Eds.), *Research on Technology and the Teaching and Learning of Mathematics: Cases and Perspectives* (Vol. 2, pp. 255-275). New York: NCTM & Information Age Publishing, Inc.
- Drijvers, P. (2000). Students encountering obstacles using a CAS. *International Journal of Computers for Mathematical Learning*, 5(3), 189-209.
- Drijvers, P., & Gravemeijer, K. (2005). Computer algebra as instrument: Examples of algebraic schemes. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculator: Turning a Computational Device into a Mathematical Instrument* (pp. 163-196). New York: Springer.
- Gagatsis, A., & Elia, I. (2005). A review of some recent studies on the role of representations in mathematics education in Cyprus and Greece. In M. Bosh (Ed.), *Proceedings of CERME 4* (pp. 102-111). Sant Feliu de Guixols, Spain.
- Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, 26(2), 115-141.
- Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, 3(3), 195-227.
- Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. Retrieved from <http://ergoserv.psy.univ-paris8.fr>
- Sajka, M. (2003). A secondary school students' understanding of the concept of function - A case study. *Educational Studies in Mathematics*, 53(3), 229-254.
- Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1-36.
- Slavit, D. (1997). An alternate route to the reification of function. *Educational Studies in Mathematics*, 33(3), 259-281.
- Vergnaud, G. (1998). A comprehensive theory of representation for mathematics education. *Journal of Mathematical Behavior*, 17(2), 167-181.
- Zazkis, R., Liljedahl, P., & Gadowsky, k. (2003). Conceptions of function translation: obstacles, intuitions, and rerouting. *Journal of Mathematical Behavior*, 22, 437-450.

# **A SURVEY OF TECHNOLOGY USE: THE RISE OF INTERACTIVE WHITEBOARDS AND THE MYMATHS WEBSITE**

Nicola Bretscher

King's College London

*This study reports the results of a pilot survey of UK mathematics teachers' technology use ( $n = 89$ ) in secondary schools. Previous surveys are confused by a lack of differentiation between hardware and software use. This survey aims to provide insight into the types of software teachers choose to use in conjunction with particular types of hardware. Teachers were asked about their access to hardware and software; their perception of the impact of hardware on students' learning; the frequency of their use of ICT resources and the factors affecting their use of ICT.*

Keywords: technology, teachers, interactive whiteboards

## **INTRODUCTION**

The survey reported in this paper aims to explore the types of software teachers choose to use with different types of hardware and the frequency of their use. The survey was conducted as a pilot study to test the feasibility of a large-scale survey of UK mathematics teachers' technology use and to inform the future collection of qualitative data to contextualise and validate survey findings. The large-scale survey will form part of a wider study into how Information and Communication Technologies (ICT) are used in mathematics teaching and how teachers' perspectives and practices have changed as a result of the introduction of the UK National Curriculum 2007.

Governments around the world have made huge investments in ICT for education (Selwyn, 2000). Despite these investments, the TIMSS 2007 study (Mullis et al, 2008) reports that using computers for any activity as often as in half the mathematics lessons was rare, even in countries with relatively high availability. In the UK, Ofsted (2008) report that opportunities for pupils to use ICT to solve or explore mathematical problems had markedly decreased over the previous seven years of unprecedented investment in technological infrastructure. The gap between investment in ICT and the reality of its use in classrooms seems clear. Investigating the choices teachers make about the technology they use in their classrooms is important in order to understand the apparent failure of ICT to make an impression on school mathematics.

## **TEACHERS' CHOICES: HARDWARE AND SOFTWARE**

The type of hardware and its deployment appears to be an important factor in structuring teachers' choices about technology use in their classroom practice (Ruthven, 2007). In particular, the hardware available affects the types of classroom organisation possible and the nature of pupil interactions with any software used in

conjunction with the hardware. It seems reasonable then that the available hardware might also affect teachers' choice of software and how they choose to integrate the use of such software into their classroom practice.

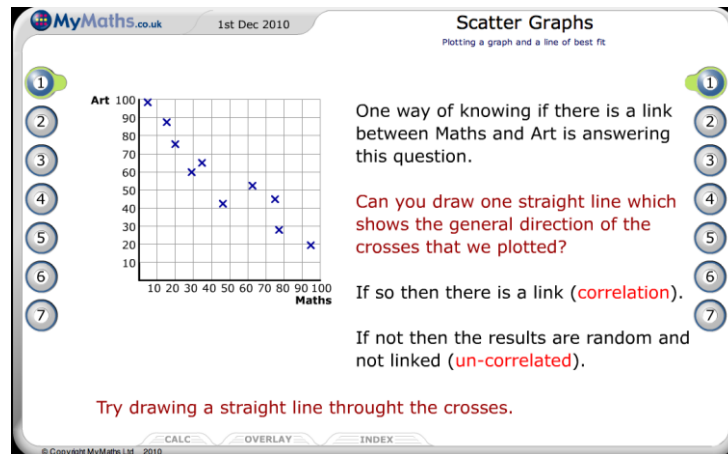
Currently, little is known about what types of software teachers choose to use with particular types of hardware. In terms of hardware, the UK represents a special case since it became the first school-level market to invest heavily in interactive whiteboards (IWBs) (Moss *et al.*, 2007). However, large-scale surveys of technology use within the UK have tended not to report in detail on technology use within subject areas, such as mathematics, nor to differentiate sufficiently between hardware and software use. Thus whilst such surveys provide a broad picture of technology use, they have not provided much insight into the nature of the specific uses by teachers in general or by mathematics teachers in particular. For example, the annual Becta schools survey *Harnessing Technology* reports that 53% of mathematics teachers use subject-specific software in half or more lessons (Kitchen *et al.*, 2007). However, no further detail is given on what types of subject-specific software are used, nor an indication of the hardware involved. Surveys focusing on mathematics teachers' use of technology, such as the survey conducted by Hyde (2004), give a more detailed picture of the types of software used by mathematics teachers; however, this picture is again confused by the lack of differentiation between hardware and software use. Building on such surveys, this study aims to provide insight into the types of software teachers choose to use in conjunction with particular types of hardware. Further, this study aims to investigate the practices of ordinary teachers in ordinary classrooms, continuing the line of research suggested in Bretscher (2009).

## THE SURVEY

The questionnaire design was informed by previous surveys of mathematics teachers' use of ICT, primarily Hyde's (2004) survey of mathematics teachers in Southampton and Forgasz's (2002) survey of mathematics teachers in Victoria, Australia. The questionnaire used both closed and open-ended response formats and contained sections on (a) *About you* - personal details; (b) *ICT in your school* - access to hardware/software and integration of ICT within the department; (c) *ICT use in your own mathematics teaching* - perceived impact and frequency of use of hardware and software; and (d) *Your beliefs about teaching and learning mathematics with ICT* - factors influencing the use of ICT. In line with the aims of this survey and in contrast to Hyde's (2004), teachers were asked separately how often they used software in a whole-class context (e.g. with an IWB or data projector) and how often they gave students direct access to the software (e.g. in a computer suite or with laptops). The list of software was derived from Hyde's list with the notable inclusion of the MyMaths.co.uk website since this site was known anecdotally to be widely used in UK schools. The *MyMaths* website is a subscription site offering teachers pre-planned lessons, on-line homework and many other

## Working Group 15

resources. The lessons and homework are linked to an “Assessment Management system”, allowing teachers to track individual students’ progress.



**Figure 1. Screen snapshot of a *MyMaths* sample lesson on scatter graphs.**

Ten questionnaires were sent to 27 schools working in partnership with King’s College London to offer initial teacher education in secondary mathematics, with 18 schools agreeing to participate in the survey. A total of 89 completed questionnaires were returned, an average of five per school: the lowest number returned by a school was 2 and the highest 9. Since the survey was a pilot study, it was not necessary to select a representative sample of schools. Nevertheless, the participating schools cover a range of characteristics including a wide range of attainment in national tests; some have speciality status and some do not; some are single sex and some are selective. The participating teachers (37 F; 50 M; 2 unspecified) had an average age of 37 years and an average length of service of 10 years. The low percentage of women (42%) is surprising since women tend to outnumber men in teaching. In common with Forgasz’ (2002) findings, no obvious differences in ICT use between the genders was found. The majority of respondents (41) described their main responsibility as classroom teacher. The sample also included 10 heads of department, 9 deputy heads of department and 13 Key Stage coordinators. Comparing themselves to their colleagues, 37.1% of teachers thought they used ICT much more or more frequently; only 11.2% thought they use ICT less or much less frequently. This might suggest that the respondents are relatively well-disposed towards ICT or that they wish to be seen as frequent users of ICT.

Data that could be analysed statistically were entered into PASW Statistics 18.0. This package was used to generate descriptive statistics (i.e. frequency distributions and means). Open-ended responses were analysed manually. In tables 1-3 in the results section below, the findings of this survey are compared with Hyde’s to give a sense of changes in ICT use over time. Hyde sent one questionnaire to each of the 38 schools working with the University of Southampton to deliver initial teacher education (33 returns). Her results give an overview of departmental ICT use whereas this survey reports individual teacher’s responses, thus comparisons should

be treated with some caution. Data was unavailable from Hyde's survey for comparison in tables 4 and 5.

## RESULTS

### Access to hardware and software

Only two schools did not have any IWBs in the mathematics department but each teacher in these schools had access to a data projector. Thus every teacher participating in the survey had access to either an IWB or a data projector. The apparent decline in access to data projectors from Hyde's (2004) survey is likely to be due to the rapid expansion of IWBs over the same period (see Table 1). Only 66% of teachers reported having access to a computer suite shared with other departments. This seems surprisingly low, especially when compared with the coverage of IWBs. In fact, in every school at least one teacher claimed to have access to a shared computer suite. The lack of consistency between teachers in the same school suggest that while some teachers are responding on the basis of the existence of hardware, others are responding according to their perception of availability of the hardware for use. Difficulties in booking computer rooms mean that, although shared computer suites exist, their availability is often severely restricted. The quote below is representative of many teachers' comments on hardware access and neatly summarises the contrast in accessibility between IWBs and computer rooms.

“Computer room access very limited due to lack of resource in school (and monopoly on it by ICT dept lessons). IWBs readily available in all maths teaching rooms.”

Access to hardware	Bretscher	Hyde
IWB	81	64
Data projector	63	76
Computer suite (shared)	66	-
Computer suite (maths only)	16	-
Laptops	26	-
Graphic calculators	26	94

**Table 1: Access to hardware,  $n = 89$ . Hyde's (2004) figures are shown for comparison. All figures are given in percentages.**

None of the teachers from schools with a computer suite dedicated to the mathematics department complained about lack of access to hardware. Although this seems a successful solution to the problem of computer room access, a mathematics only computer suite is still a rare resource (16% have access). Due to their portability, a class set of laptops might be seen as an alternative solution to the access problem. However, access to laptops is also fairly rare (26%) and comments by teachers suggest they may bring additional technical difficulties:

## Working Group 15

“The laptops are of poor quality and not enough for 1 between 2 if you have a full class: there are 14.”

The collapse in access to graphic calculators since Hyde's (2004) survey is impressive – this may be the result of their exclusion from A-level module examinations, whereas previously their use had been encouraged. Again there is a lack of consistency over access to graphic calculators between teachers in the same school. In thirteen of the schools at least one teacher said they had access to graphic calculators. The following comment suggests that the low reported access to graphic calculators may reflect a lack of awareness of their existence, rather than difficulties in booking the resource as in the case of computer rooms:

“We do have a department set of graphic calculators (ie. not explicitly for my classes) but they are rarely (if ever!) used.”

Access to software was not generally seen as a problem: access to generic software such as word processing and presentational software is almost universal (around 90%) and graphical software (81%) also appears to be readily available. Geometry software appears to have declined slightly (-13%) since Hyde's survey, although the majority of teachers (60%) say they have access. Logo has suffered a sharp decline (-49%). In Hyde's survey, 100% of teachers said websites were used in their school, however no further detail was given. The results from this survey suggest that access to the *MyMaths* website (91%) has risen to near ubiquity - it is possibly the dominant resource designed for mathematics teaching in the UK. It is unlikely that any textbook has such a wide coverage of schools, for example. Some teachers did complain about restrictions on downloading software, such as *GeoGebra*, and access to some websites being unnecessarily blocked. Although software was available, teachers expressed uncertainty over whether it had been installed on all computers, thereby adding complexity to conducting lessons in a computer suite.

“Some ICT suites do not have all the mathematical software which can mean plans and resources need to be adapted. Must check prior to booking.”

For many teachers, the software was readily accessible however they lacked training in its use.

“Access not a problem – time to train and develop is a problem. Desperately needs a directory/classification system.”

The time taken to develop and prepare lessons was seen as a considerable hurdle initially; however, once surmounted, teachers found that the resources could be re-used, thus reducing planning time eventually.

“Time required to prepare using ICT is a bar to entry however a number of resources I have spent time developing can then be reused very efficiently in other contexts.”

Access to software	Bretscher	Hyde
Spreadsheet (eg Microsoft Excel)	92	97
MyMaths.co.uk	91	-
Word processor (eg Microsoft Word)	90	79
PowerPoint	90	79
Email	82	-
Graphing software (eg Omnigraph, Autograph)	81	73
CD-ROMs	67	85
Other websites	67	-
Geometry software (eg Cabri, Geometer's Sketchpad)	60	73
Database (eg Microsoft Access)	37	-
Logo	24	73
SMILE mathematics	17	-

**Table 2: Access to software  $n = 89$ . Hyde's (2004) figures are shown for comparison. All figures are given in percentages.**

Software access may present a new issue to consider when applying for a teaching position in the UK. On moving to a new school, one teacher found that his pre-planned lessons had been rendered useless since the IWB software he had used previously was not available. His time investment in planning these lessons had therefore been lost.

“Tech support have not installed Smart Notebook on the maths dept computers, and I have a lot of Smart Notebook files that I made at my previous school that I can't use now.”

### **Perceptions of the impact of ICT on learning**

Teachers were asked whether they agreed with the statement ‘ICT resources can help students to understand mathematics’. In response, 97.8% agreed or strongly agreed with the statement. Teachers were also asked to rate hardware on the impact it has on student learning, using the scale in Hyde's survey from 1 (very little) to 4 (substantial). The results shown in Table 3 suggest that teachers' perception of the impact of ICT on learning varies considerably depending on the hardware being used. IWBs had the highest mean impact score (3.21), followed by data projectors (2.84). These items also came top in Hyde's survey although IWBs scored lower (2.95) and data projectors slightly higher (3.04). The reversal in score is likely to be due to the increased availability of IWBs, since they were a relatively new phenomenon in 2004 and comparatively few schools were equipped with them. Graphic calculators suffered a decline in score of 0.33. Perhaps of most interest is

that a shared computer suite had the lowest impact score (2.34). A dedicated mathematics computer suite scored more highly (2.57), probably in part due to the greater ease in accessing the hardware. However the low impact score of computer rooms is also reflected in some teachers' negative comments about giving students direct access to the hardware:

"ICT maths lessons always seem tedious as the students' development is less than in normal lessons. But with *MyMaths*, as a revision/recap lesson, there are benefits now."

"ICT is generally extremely inefficient."

Not all teachers felt this way, some were more positive although many cited difficulties such as those detailed in the quote below:

"In our school it is not possible to find a venue where there is one computer per child. Therefore this is a very strong de-motivating factor when planning such lessons as I know the group-work element adds a layer of complexity. If it were guaranteed pair-work I might be more motivated but, for example, I recently tried a lesson like this and ended up with 10 computers between 31 students (the IT suite was supposed to have 16 computers!)."

Undoubtedly problems of access reduce the perceived impact of computer rooms, however the results from this survey suggest that teachers remain sceptical of the educational value of giving students direct access to ICT resources.

Impact	Bretscher	Hyde
IWB, $n = 78$	3.21	2.95
Data Projector, $n = 74$	2.84	3.04
Computer suite shared, $n = 73$	2.34	-
Computer suite maths, $n = 51$	2.57	-
Laptops, $n = 57$	2.40	-
Graphic calculators, $n = 59$	2.46	2.79

**Table 3. Mean impact scores for hardware based on a scale where 1 (very little), 2 (some), 3 (significant) and 4 (substantial).**

### Frequency of hardware and software use

The majority of teachers use IWBs and data projectors in most lessons. The ready availability of IWBs and data projectors in normal classrooms makes it unsurprising that they are the most frequently used hardware. It is also likely that their high frequency of use contributes to the high impact scores noted in the previous section. Computer rooms shared with other departments have a much lower frequency of use, with 58% of teachers using them once a term or less and only 17% of teachers using them every week or more. As with IWBs, the frequency of use reflects both the accessibility and impact score of shared computer rooms. Computer suites dedicated



to the mathematics department appear to have much higher frequency of use than shared computer rooms, with 42% of teachers using them every week or more. This is likely to be the case (despite only  $n = 14$ ) since not only is access easier than with a shared computer room, classes are often purposefully timetabled into mathematics only computer rooms to ensure the use of a relatively rare resource.

Frequency of hardware use	Never	Specific topics	Once a term	Once a month	Every week	Most lessons
IWB, $n = 70$	3	1	1	1	10	<b>83</b>
Data Projector, $n = 55$	7	0	4	0	24	<b>65</b>
Computer suite shared, $n = 55$	9	20	<b>29</b>	25	15	2
Computer suite maths, $n = 14$	0	21	7	29	<b>36</b>	7
Laptops, $n = 23$	26	13	13	9	<b>39</b>	0
Graphic calculators, $n = 22$	27	<b>41</b>	18	0	5	9

**Table 4. Frequency of hardware use, with the modal frequency for each item highlighted in bold. All figures are in percentages.**

The number of teachers with access to laptops and graphic calculators is quite low ( $n = 23$  and  $n = 22$  respectively) so it is difficult to draw any firm conclusions from the figures presented in Table 4. However it is worth noting that despite graphic calculators having a higher impact score than either laptops or shared computer rooms, they appear to have the lowest frequency of use with 68% using them for specific topics only or not at all.

Table 5 compares the mean frequency of software use in lessons with an IWB or data projector to lessons where students are given direct access to the software, i.e. those that take place in a computer room or with laptops. A score of above 2 indicates the software is used more than once a term. Email, databases, SMILE and Logo scored very low in both contexts so no satisfactory comparison can be made for these software packages. PowerPoint was the most frequently used piece of software (3.21) in conjunction with an IWB, closely followed by *MyMaths* (3.01). ‘Other websites’ and graphing software also scored above 2 for frequency of use with an IWB.

The frequency of use in lessons where students were given direct access to the software was low in comparison to lessons with an IWB: only *MyMaths* had a frequency score above 2. This is unsurprising given the frequency of hardware use in mathematics lessons reported above: computer rooms are used much less frequently than IWBs. However the decrease in use is not uniform across all types of software. In lessons where there is direct student access, most software packages have a frequency score between 0.9 and 1.1 lower than in lessons with an IWB. *MyMaths* had the smallest drop in frequency use between contexts (-0.5) and geometry software fell by 0.74. The frequency score of PowerPoint dropped the most (-2.51).

Since the main purpose of PowerPoint is for presentation, it appears well suited to teacher exposition in lessons with an IWB but not so relevant in lessons where students have direct access to the software.

Frequency of software use	IWB/Data projector	Direct student access
PowerPoint	3.21	0.70
MyMaths.co.uk	3.01	2.51
Other websites	2.83	1.82
Graphing software (eg Omnigraph, Autograph)	2.01	1.09
Spreadsheet (eg Microsoft Excel)	1.91	0.97
CD-ROMs	1.84	0.74
Word processor (eg Microsoft Word)	1.76	0.84
Geometry software (eg Cabri, Geometer's Sketchpad)	1.44	0.70
Email	0.97	0.45
Database (eg Microsoft Access)	0.90	0.44
SMILE mathematics	0.54	0.37
Logo	0.22	0.25

**Table 5. Mean score for frequency of software use with an IWB or data projector compared to use in a computer room where students have direct access to the software. Based on a scale where 0 (never) to 5 (most lessons),  $n = 89$ .**

Thus not only are computer rooms and laptops used less frequently than IWBs: teachers appear to use a smaller range of software in lessons where students are given direct access to the software. *MyMaths* appears to dominate in both contexts, with the exception of PowerPoint being used more frequently with IWBs.

## CONCLUSIONS

IWBs are the most accessible hardware and teachers rate them highest for impact on students' learning. Arguably, the introduction of IWBs has coincided with, if not encouraged, the apparent rise of *MyMaths* to near ubiquity in UK classrooms. Whilst positive about ICT resources in general, some teachers appear sceptical about the benefits of giving students direct access to software. Shared computer rooms scored lowest for impact and are used infrequently and although computer rooms dedicated to the mathematics department improve matters, they are still a rare resource. When students are given direct access to ICT, *MyMaths* is the most frequently used resource. The reasons for *MyMaths* apparent dominance requires further research. Research suggests that the use of IWBs coupled with PowerPoint and pre-prepared lessons of the sort available from the MyMaths website can lead to a reduction in the

quality of classroom interactions (Zevernbergen & Lerman, 2008). The Second Information Technology in Education Survey concluded that, given the right conditions, ICT might contribute as a lever for change (Law, 2009). Although the findings presented in this paper should be treated with some caution, they suggest that, in the UK, the conditions may be right for ICT to act as a lever for change in a direction that should be of some concern to both researchers and policymakers.

## REFERENCES

- Bretscher, N. (2009). Dynamic Geometry Software: the teacher's role in facilitating instrumental genesis. In V. Durrand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education CERME 6* (pp. 1340-1348). France: Lyon.
- Hyde, R. (2004). A snapshot of practice: views of teachers on the use and impact of technology in secondary mathematics classrooms, *International Congress on Mathematics Education*, Copenhagen, Denmark
- Forgasz, H. (2002). Teachers and Computers for Secondary Mathematics, *Education and Information Technologies*, 7(2), 111-125
- Kitchen, S., Finch, S. and Sinclair, R. (2007). *Harnessing Technology: Schools survey 2007*, Coventry, National Centre for Social Research
- Law, N. (2009). Mathematics and science teachers' pedagogical orientations and their use of ICT in teaching. *Education and Information Technologies*, 14(4), 309-323.
- Moss, G., Jewitt, C., Levaic, R., Armstrong, V., Cardini, A., and Castle, F. (2007) *The Interactive Whiteboards, Pedagogy and Pupil Performance Evaluation: An Evaluation of the Schools Whiteboard Expansion (SWE) Project: London Challenge*. Institute of Education, London
- Mullis, I., Martin, M., and Foy, P. (2008). *TIMSS 2007 International Mathematics Report*. (TIMSS & PIRLS International Study Center, Boston College).
- Ofsted (2008). *Mathematics - understanding the score*. London: Ofsted.
- Ruthven, K. (2007). Teachers, technologies and the structures of schooling. In D. Pitta-Pantazi & G. Philipou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education CERME 5* (pp. 52-67). Cyprus: Larnaca.
- Selwyn, N. (2000). Researching computers and education - glimpses of the wider picture. *Computers and Education*, 34, 93-101.
- Zevernbergen, R., & Lerman, S. (2008). Learning Environments Using Interactive Whiteboards: New Learning Spaces or Reproduction of Old Technologies? *Mathematics Education Research Journal*, 20(1), 108-126.

# **GRAPHIC CALCULATOR USE IN PRIMARY SCHOOLS: AN EXAMPLE OF AN INSTRUMENTAL APPROACH**

Per Storfossen

Hedmark University College, Norway

*This paper presents an empirically based case study design within a sociocultural theoretical framework. The research aimed to describe the implementation of a graphic calculator in a fifth grade primary school class when four students are engaged in mathematical activities performing tasks and challenges given by the researcher. The students' activities associated with the process of appropriation of a technological artifact played a prominent role in the data analysis. The distinction between artifact and instrument through the instrumental approach is in focus. Of special interest is the concept of protocol related to (the size of) the graphic calculator screen display associated with the types of tasks and challenges the students were given.*

Keywords: graphic calculator, protocol, instrument genesis

## **INTRODUCTION**

The study of the type of use and implementation of an advanced graphic calculator in a fifth grade primary school class was the focus of research described herein. Specifically, the responses of four students to mathematical tasks and challenges given by researcher are documented. The researchers' intervention was limited to the introduction of the graphic calculator in class, where a task portfolio adapted for handheld calculator was designed in collaboration with the teacher. The choice of research strategy can be described as a case study design (see the next section). Data were collected in video-taped interviews with students undertaken by the researcher. The study is presented in a sociocultural theoretical framework. The artifact properties, such as the size of the screen were variables considered in this study. The instrument-mediated activity approach was in focus.

Our research question is the following: In what ways the graphic calculator is incorporated, integrated and appropriated in the student's activities in the process of solving mathematical tasks and challenges?

## **A CASE STUDY ON TWO LEVELS**

The framework of the study design applied to the research strategy and concerned the conduct of social research with the selection of methodological approaches to the gathering of empirical data (Bryman, 2008). The choice of research strategy in this study can be described as a case study design on two levels. On the first level, there were 23 students in a fifth grade primary school. The first level formed a backdrop for the next level, the selection of four students. There were two boys and two girls placed into two small groups one of each gender. The choice of the two boys and two

girls was partly based on pre-tests, achievement tests from a pilot study involving a representative sample of the class, but also on the students' motivation as two groups of friends wished to participate in this study that lasted over two years. We meant that the latter was a good argument for the participants not losing interest during the period of data collection. The second level of the case study consisted of the two respective groups, and was a continuation and an elaboration of the study where the entire class constituted the background. In this project, each student had two handheld calculators at their disposal. The four-function pocket calculator with one-line display which the students have been using for about three years and then there was the sophisticated icon-driven graphic calculator Casio fx9750G PLUS introduced by the researcher including a 21-character x 8-line display. The former calculator has a solar panel as an energy source, and the latter is powered by battery. The data collected and analyzed in this research study was from video-taped interviews when each of the four students were working individually, and when they were collaborating in small groups engaged in mathematical activities, challenges and tasks provided by the researcher. The mediated artifacts, among them calculators, were available and were included in their mathematical activities.

## **THEORETICAL FRAMEWORKS**

### **A sociocultural approach**

The Vygotsky-inspired sociocultural tradition constituted the theoretical framework of this study. The research question drew the attention to the analysis of students in a technical environment addressing the way mathematical activity was supported by material artifacts like computers, calculators, slide rules or abacus. The term artifact refers to an abstract (linguistic) or a material object. An artifact comprises objects, things or products made or modified by human beings specified for a certain purpose in reference to a historical and cultural setting (Säljö, 2006). The aim of utilizing an artifact is to sustain human activity in order to perform various types of tasks. From a sociocultural perspective, the artifact calculator is regarded as an external cognitive aid to mental processes. According to (Wertsch, 1998), we should focus on the process of turning an artifact into a useful mediated tool, the process of appropriation. In a sociocultural theoretical framework, the study has a two-sided perspective on focus. On the one hand, the focus is on individuals organized in pairs to exchange experiences and learn from each other both by applying the graphic calculators (tools) and doing mathematics. On the other hand, there are aspects concerning the individual process of appropriation of the artifact graphic calculator to become a mediated tool for the competent user. In the literature both the terms artifact and tool are used. In accordance with (Trouche, 2004, p. 282),

"when speaking of a tool before considering its users and its uses, I will speak of an artifact".

The terms artifact and tool in this study are viewed inseparable and are understood as ‘two sides of the same coin’. When the calculator is taking part in students’ mathematical activities, the calculator becomes a tool for the user. Artifacts are mediation means or cultural tools, devices that convey, shape, and transform physical processes with the purpose of mastering mental processes (Vygotsky, 1978). In a sociocultural perspective, the term mediation presupposes or requires the use of mediation means, and it is the human beings’ utilization of cultural tools that makes mediation possible. It is through or via mediation means the human agent interacts with the outside world, and the tools exert their impact when an agent uses them. Mediation refers to human interaction with one other in symbiosis with external tools as mediators (Daniels, 2001). “Mediation through artifacts applies equally to object and people” (Cole, 1996, p. 118), and artifacts “have been modified by human beings as a means of regulating their interactions with the world and others” (Cole, 1999, p. 90). According to Cole, mediation through artifacts refers to people acting with mediating artifacts which are emerging from historical and cultural settings. Throughout history human functions and competencies have been accumulated and transferred into external aids named artifacts. The mediated artifact graphic calculator was, in this study prominent, present and situated.

### **The unit of analysis**

From the research perspective it is explicitly important to communicate the unit of analysis of the study. Data collection and data analysis are intertwined in social science research. It is therefore crucial for the researcher to be aware of this relationship and sort out the ‘what’ and ‘whom’ that is being studied (Lee & Fielding, 2004). According to (Wertsch, Rio, & Alvarez, 1995, p. 56),

“The goal of sociocultural research is to understand the relationship between human mental functioning, on the one side, and the cultural, historical, and institutional setting.”

The authors refer to a relationship between a sociocultural situation and mental processes, and

“it is essential for the research to formulate its position vis-à-vis the individual-society antinomy in some way” (ibid., p. 59).

Mental functioning and the sociocultural setting are understood as aspects of dialectically interacting moments, and the human action is the connection and is considered as the unit of analysis. (Vygotsky, 1978, 1986) focused on thinking, speech, i.e. mediated action, while (Bakhtin, 1986) maintained focus on utterances as a form of action. (Wertsch *et al.*, 1995) highlighted the research of human actions as a “dynamic human actions existing in real spatiotemporal and social context” (p. 62). Wertsch (1998) proposed mental functioning and sociocultural setting to be understood as dialectically interacting moments, aspects of a unit of analysis. In sociocultural research, the human actions serve as the cornerstone to be described and interpreted. The kinds of actions that concerned Vygotsky have been termed

“mediated actions” as the unit of analysis (Wertsch *et al.*, 1995). To provide an adequate foundation for an account of the action carried out, neither the mediation means nor the individual operates in isolation. The individual is not acting alone as an agent of actions. An agent refers to who is carrying out the actions, e.g. who is speaking. An appropriate description may be characterized by agents of actions like “individual-operating-with-mediation-means” as the unit of analysis, a framework grounded on actions. Mediated action can be transformed by mediation means like the case with the introduction of an advanced calculator. However, it is not so say that the mediation means, in a way, acts alone. The individual using mediated means has to change or adapt as well in response to new techniques and skills. In this study the research question draws the attention on an analysis of students in a technically supported environment, working in small groups engaged in mathematic activities, with natural verbal communication in the interaction. According to theorists like Vygotsky and Wertsch, the unit of analysis in this study can be modified and transformed to individuals-appropriate-technological-artifacts-in-cultural-practice.

### **The instrumental genesis approach**

In this study, the sociocultural perspective was the theoretical framework. Mediated activity, transformed to individuals-appropriate-technological-artifacts-in-cultural-practice constitutes the unit of analysis. (Vérillion & Rabardel, 1995) and (Rabardel, 2002) drew on Wertsch’s key construct of mediated activity (Wertsch, 1998). The process of appropriation of an artifact to become an instrument, a tool for the competent user is further elaborated by (Trouche, 2004). The instrumental approach with the human/computer interaction and human use of tools constitutes two main processes of instrumented mediated activity. The process directed from the artifact towards the human agent is named instrumentation. The tools convey, shape and transform the human agency and the subject adapts to constraints and affordances the tool possesses. The other process, instrumentalization, is directed from the human agent towards the artifact (graphic calculator) which includes stages such as familiarization with the instrument, mastering the instrument and adoption of the instrument to one’s own personal specific needs. In this study the graphic calculator in becoming an instrument for the student, individual and collective activities have drawn on the instrumental genesis approach influenced by Rabardel (2002).

### **Protocols of actions**

Dörfler in his work (2000) emphasized reflection and attention on actions. His constructed protocol of action consisted of records of actions, which can contain manipulations and interactions with objects-like models. Oral expressions and all kinds of counting activities are examples of protocol of actions.

“I use the term protocol to designate the related cognitive process of focusing attention on those stages, phases, results, and products of one’s action and constructions, and of describing and notating them by some means”. (ibid., p.111)

The protocol of action (its schematic carrier) will include a kind of notation-suitable system that will make it possible for the individual or agent to reconstruct his or her activities for which it is a protocol. The protocol, the record of action, makes possible replication and reconstruction of the essential phases, stages of the actions, and transforming the result into a suitable notation system. A counting system is an example of notation system, like the decimal system. For example, a student has counted up the number of pens lying on a table; there are 38 pens, which are grouped into 3 groups of 10 pens and 8 individual pens. Another example of a protocol is given by a transformation; the rotation of a point in a plane is uniquely determined by an origin, the coordinates of a point and an angle. Records of physical objects or figures rotated using coordinates of the vertices is further a carrier (protocol of action) to abstract mathematical rotations. When an agent has developed a *specific protocol*, like the protocol for rotation of a point in a plane, the agent is able to replicate and reconstruct the mathematical actions from the record of actions which constitute the protocol. The specific protocol for an agent constitutes a characteristic, a quality the agent possesses. On the other hand, and on another level, Dörfler's *concept of protocol* of is a theoretical construct and is an analytic tool for the researcher to describe and to study an agent's performance in carrying out the mathematical activities like rotation of a point in a plane.

## THE TASK PORTFOLIO

A task portfolio was designed in collaboration with the teacher of the class. The portfolio was adapted from the textbook and the curriculum for the 10-year compulsory mathematics education. An objective in designing the portfolio was to make the tasks concrete and in a context familiar to the students. The tasks were based on their everyday life experience, with the intention of students conceiving the tasks as realistic and a part of their daily life. A type of task in the portfolio was named exploration tasks (exploratory work) organized by mathematical topic. One of these topics was linear two-variable Diophantine equation that takes the form  $ax + by = c$ ,  $(a,b)=1$ . The parameters  $a$ ,  $b$  and  $c$  are integers, where  $a$  and  $b$  are relatively prime numbers. Two examples of this type of equation with which the four students were challenged are outlined as follows:

Task 1) A train is 78 meters long. There are two kinds of train carriages, carriages of length 7 meters and carriages of length 11 meters. How many carriages of 7 meters and how many carriages of 11 meters will give the length of 78 meters?

Task 2) An ice cream costs 3 euros and a hot dog costs 5 euros. How many ice creams and how many hot dogs can you purchase when you have 70 euros at your disposal?

The students were unfamiliar with a general solution for this type of equation. A method to find all the solutions for this kind of problems required theoretical knowledge and insight in mathematics at a level they did not have, but could later learn in an advanced course in mathematics. However, videotaped interviews show



that some students were able to find one or more solutions of this type of equation on their own, when they utilized the graphic calculator as an external cognitive aid to sustain mental arithmetic operations (See the example with the game 'Dart' below).

### **CALCULATOR DISPLAY AS A PROTOCOL**

When a student is typing in numbers, signs or characters into the calculator, the results of these actions will appear on the screen display. The latter is a record of the actions performed by the student, a mathematical text or a syntax consisting of keystrokes. However, the calculator can interact with the student by giving a spontaneous response in the form of an answer to arithmetic operations or eventually give a syntax error (convention error) message to incorrect typing. The screen is displaying a semi-permanent record of actions. An interesting aspect associated with the type of use of the calculator is when the calculator is displaying more than one line of arithmetic operations with its corresponding answers. The protocol constitutes the activities the students were doing, signs and symbols they typed in the calculator. The calculator displays a mathematical text, a notation system which gives meaning through conventions. They may be read off the display as a semi-permanent inscription. Through the example of a transcript from a video clip excerpted below, we will give an account on how the screen display of a graphic calculator, for a student, become a protocol to develop strategies to find solution(s) to a mathematical task or challenge (see in particular the line 106 below). Here, the concept of protocol is functioning as an analytical tool for the instrumental approach when the focus is on part of the artifact, namely the size of the screen display and the activities that constitute the process of instrumental genesis.

The following excerpt is a transcript of a video clip of an interview of the two students Kate and Signe. The video clip is part of a comprehensive video material.

- 101 Interviewer: Can you say something about what you appreciate when using the graphic calculator?
- 102 Kate: It is pretty nice actually. It has a large screen and you can have several calculations simultaneously.
- 103 Interviewer: You said that with the graphic calculator, the screen made it possible to display several calculations simultaneously. But, is this possible with the one-line calculator?
- 104 Kate: No, you can only have one calculation at a time
- 105 Interviewer: What does it mean for you that it can display several calculations at a time?
- 106 Kate: It means a lot! Because, if you are going to figure out and calculate something important, you need to make miscellaneous calculations...and to move forward and get closer and closer. And then it is important to see the calculations that have been done before, so you don't need to calculate them once more
- 107 Interviewer: How about the one-line calculator?

## Working Group 15

108 Kate: With the one-line calculator, you can only come with... have one calculation at a time. And then it is hard to know...

Later on in the interview the researcher asked the students about the type of task they preferred when using the graphic calculator.

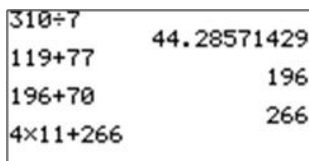
190 Researcher: What type of task might suit this calculator?

191 Kate: Maybe the one with hot dogs and ice cream

192 Signe: Maybe the one with trains...

In the lines 190-192, the students referred to, and considered two prototypical situations of the ‘class of situations’ and tasks contained in the portfolio, two examples of linear two-variable Diophantine equations (see task 1 and task 2 outlined above). The students associated the type of calculator according to the type of task, and they preferred the multi-line display calculator rather than the calculator with one-line display with the Diophantine equation challenges. However, when the students were challenged with simple tasks such as calculating  $38 \times 17$  which required only one arithmetic operation (calculation) at a time, or when there was no need for investigation, they prefer the simpler calculator. This may be the reason why Kate, in the line 108, says: “with the one-line calculator, you can only have one calculation at a time”. She is aware that the one-line display will be definitely sufficient when working with simple tasks or when there is no need to develop a strategy to solve an arithmetic problem. Otherwise, when handling complex tasks like the Diophantine equation problems (task 1 and 2), she will choose the calculator with the multi-line display. This process, instrumental genesis, of awareness of the type of task that can or cannot be solved with a particular type of calculator is a process of an artifact becoming part of an instrument in the hands of a student.

### Develop a strategy by use of a protocol



310÷7	
119+77	44.28571429
196+70	196
4×11+266	266

Figure 1: Display as a protocol for Signe

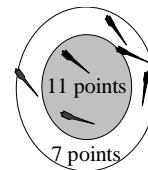


Figure 2: Game ‘Dart’

One of the tasks in the portfolio concerned the game ‘Dart’ (see fig. 1 and fig. 2). A hit with an arrow within the inner circle gives 11 points, while a hit with an arrow within the outer circle gives 7 points. The task was: How many arrows of 7 points and how many arrows of 11 points will give a total of 310 points? This is another example of a Diophantine equation. The description and analyses of the screen display above are based on the same interview with the other student Signe. The screen display shows part of Signe’s solution strategy which addresses how she managed to solve the problem using the calculator. Signe came up with one of the four solutions, here the solution  $38 \times 7 + 4 \times 11 = 310$  or  $\{38, 4\}$ . The screen functioned as a protocol for her. Signe divided 310 by 7 to investigate the number of

7-arrows there would be. The result of the previous calculations 119 ( $17 \times 7$ ) she added to 77, yielding 196. The number 119 was previously calculated and is from a previous screen display. She then added 70 and calculated 266 ( $38 \times 7$ ) points. Finally, she went back to the first line in the display. Typed 44 ( $4 \times 11$ ) and added 266. The calculator gave her the answer 310. The calculator functioned as an external support for her mental arithmetic.

The result of Signe's activities consisting of keystrokes entering numbers, arithmetical operations, signs or other characters into the calculator and displayed on the screen are examples of an inscription. The semiotic display on the calculator screen is regarded as an inscription composed of a mathematical notation system. A student using the calculator dynamically and interactively is able to construct a system out of screen display lines showing arithmetic calculations within a single arithmetic problem, applying to the screen display a protocol of actions (Dörfler, 2000). A calculator screen displaying one or more lines of an arithmetic calculation is not a protocol for each and everyone. It depends on the person's internal processes, on how an individual reasons something out and the cognitive processes. For example, a student who performs isolated single calculations does not necessarily use the calculator screen as a protocol. Others may focus on the entire process to which they relate by using the calculator. For them it is not just a simple calculation they are dealing with, but they focus on the entire process of calculations when utilizing a calculator. Students who are working with the aim of developing a solution process without "going blind," may have found an overall strategy. It is therefore important to achieve a good interplay between the internal cognitive processes and the external artifacts. By having access to a larger screen with the ability to display multiple-lines of arithmetic calculations and corresponding answers simultaneously, Signe could read off the display, control and compare the results of several mathematical expressions she has typed in, keystroke by keystroke. A larger screen and use of protocol makes it possible for her to jump forwards or backwards between the calculations. For each calculation, Signe gets a quick response and she can immediately check the results. She can also return to the previous calculations and then continue where she was. In this way the mathematical process continues in a search for solution(s). As Kate describes in text line 106, and as Signe demonstrated in the game 'Dart', mathematical activities using the artifact can be described as an interactive process like the text in web pages on the internet. Searching in such text or amongst lines of arithmetic calculations is not a linear process because the student can move one or more steps backwards or forwards from where she was in the mathematical text. The availability of a larger screen display, listing up to 8 lines, opens up new opportunities because the user gets an overview of more information at once compared with one-line calculators. The graphic calculator gives her an advantage; the size of the screen display allows her to see several lines of arithmetic operations and opens up for articulation, reasoning and problem solving processes. "So you don't need to do the calculations once more" as she expressed in

the line 106. The interpretation is that the part of the artifact, the screen display can take part in her mathematical activities via the lenses of the protocol. In the ‘Dart’ example, the screen is a protocol for Signe and made it possible for her to reconstruct and replicate the arithmetic calculations on the screen display. The calculator becomes an external cognitive aid for the competent user to find solution(s) to a mathematical problem. The dynamic and creative aspects by utilizing the screen display in this manner can be described as instrument-mediated, appropriating the artifact as an instrument, turning the artifact into an instrument in the hands of a student. The interpretation is that the screen display has become a protocol for Signe. The concrete protocol is a property, a stage in a development in and is a quality she possesses. The protocol says something about how Signe relates to the part of the calculator that is the screen display, and how she is able to utilize the constraints and affordances, the potential of the screen. The concept of protocol is an analytical tool for the researcher to describe and to study how far the student has come in the process of instrumental genesis by studying their ‘protocol scheme’ through the student's use of the screen according to type of task.

## CONCLUSION

The development of mathematics has always been dependent on the material and symbolic tools available for mathematical calculations (Artigue, 2002). A sociocultural tradition can shed light on how students can master and appropriate the graphic calculator in their mathematical activity and to make the artifact an instrument for the competent user. This is an individual process. The graphic calculator can be effective in helping students sustain their mathematical activities in the process of instrumental genesis. The process of instrumental genesis is suitable to analyze the processes of appropriation, the transformation of an artifact to become an instrument in the hands of a student. This study illustrates the influence the type of tasks and challenges given by researcher can have on student’s instrument-mediated activity. The concept of protocol is used as an analytical tool to study student appropriation of the part of the artifact calculator, the screen display. The construct of protocol can thus give substance to instrument genesis by studying the constraints and affordances, through the student's use of the artifact screen display. There is a further need for empirical resources to develop theories that elucidate complex human-machine relationships and to better understand how they may function in different educational settings.

## REFERENCES

- Artigue, M. (2002). Learning Mathematics in a CAS Environment: The Genesis of a Reflection about Instrumentation and the Dialectics between Technical and Conceptual Work. *International Journal of Computers for Mathematical Learning*, 7(3), 245-274.

- Bakhtin, M.M. (1986). *Speech genres and other late essays*. Austin, TX: University of Texas Press.
- Bryman, A. (2008). *Social research methods*. Oxford: Oxford University Press.
- Cole, M. (1996). *Cultural psychology: a once and future discipline*. Cambridge: the Belknap Press of Harvard University Press.
- Cole, M. (1999). Cultural psychology: Some general principles and a concrete example. In Y. Engeström, R.-L. Punamäki-Gitai & R. Miettinen (Eds.), *Perspectives on Activity Theory* (pp. 87-106). Cambridge: Cambridge University Press.
- Daniels, H. (2001). *Vygotsky and pedagogy*. London: Routledge Falmer.
- Dörfler, W. (2000). Means for Meaning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and Communicating in Mathematical Classrooms: Perspective on Discourse, Tools and Instructional Design* (pp. 99-131). Mahwah, NJ: Lawrence Erlbaum Associates.
- Lee, R. M., & Fielding, N. G. (2004). Tools for qualitative data analysis. In M. A. Hardy & A. Bryman (Eds.), *Handbook of data analysis* (pp. 529-546). London: Sage Publications.
- Rabardel, P. (2002). *People and Technology – A Cognitive Approach to Contemporary Instruments*. Retrieved from <http://ergoserv.psy.univ-paris8.fr>
- Säljö, R. (2006). *Læring og kulturelle redskaper: om læreprosesser og den kollektive hukommelsen*. Oslo: Cappelen akademisk forlag.
- Trouche, L. (2004). Managing the Complexity of Human/Machine Interactions in Computerized Learning Environments: Guiding Students' Command Process through Instrumental Orchestrations. *International Journal of Computers for Mathematical Learning*, 9(3), 281-307.
- Vérillon, P., & Rabardel, P. (1995). Cognification and artifacts: A contribution to the study of thought in instrumented activity. *European Journal of Psychology of Education*, 10(1), 25.
- Vygotsky, L. (1978). *Mind in society: the development of higher psychological processes*. Cambridge, Mass.: Harvard University Press.
- Vygotsky, L. (1986). *Thought and language*. Cambridge, Mass.: MIT Press.
- Wertsch, J. V. (1998). *Mind as action*. New York: Oxford University Press.
- Wertsch, J. V., Rio, P.D., & Alvarez, A. (1995). *Sociocultural studies of mind*. Cambridge, Mass.: Harvard University Press.

# AN ONLINE GAME AS A LEARNING ENVIRONMENT FOR EARLY ALGEBRAIC PROBLEM SOLVING BY UPPER PRIMARY SCHOOL STUDENTS

Angeliki Kolovou, Marja van den Heuvel-Panhuizen

Freudenthal Institute for Science and Mathematics Education, Utrecht University, the  
Netherlands

*The paper reports on a study of students working with an online archery game. We invited 318 students from grades 4, 5 and 6 to use the game at home to solve a series of contextual problems addressing covarying quantities. In particular, the study investigated how the students utilized the computer environment and the strategies they applied. The analysis revealed that the online working was positively related to success in a written posttest on early algebra problems. The learning environment stimulated the application of various strategies and the exploration of relations and structure.*

Keywords: Early algebra, online game, problem solving strategies

## INTRODUCTION

In primary school mathematics the focus is mainly on developing numeracy and calculation skills. However, several researchers (Goldenberg, Shteingold, & Feurzeig 2003; Harel, 2008) have pointed out that mathematics also involves ways of thinking or mathematical habits of mind, including seeking and exploring patterns, making conjectures and performing experiments, and applying heuristics to solve nonstandard problems. These ways of thinking should also be a vital part of mathematical instruction at primary school level. Along these lines, primary school students can be engaged in algebraic activities that provide them with the opportunity to practice both basic and more sophisticated thinking skills such as generalization (Kaput, Carraher, & Blanton, 2007). The integration of algebra in the primary grades is essential for adding coherence, depth, and power to school mathematics and can prepare students for the learning of algebra in the later grades (Kaput, 2007).

However, the inclusion of algebra in primary school does not imply adding traditional algebra to the primary school curriculum; rather, it means providing entry points for algebra through treating existing topics in a deeper and more connected way (Kaput *et al.*, 2007). For example, patterning activities can help students move towards understanding functional relations (NCTM, 2000). Rich problem contexts can play an indispensable role herein, as experience and reasoning in particular situations may support students in generating abstract knowledge (Carraher & Schliemann, 2007).

Although tables (Schliemann, Carraher, & Brizuela, 2001) and function machines (Warren, Cooper, & Lamb, 2006) have been employed for studying functional relations at primary school level, new technologies might bring further improvements herein. Computers can perform calculations and diminish the routine workload so that

students can focus more on exploring relationships. Furthermore, computer tools can surpass the constraints of paper-and-pencil methods by providing dynamically linked notations and instant feedback (Roschelle, Pea, Hoadley, Gordin, & Means, 2000).

In this study we offered primary school students an online environment including a dynamic game with which they could solve problems with covarying values (which we call early algebra problems). In particular, we sought to answer the research questions:

How do fourth to sixth graders utilize an interactive online environment including a dynamic game to solve early algebra problems?

Is students' working in this environment related to their performance in a written test on early algebra problem solving?

## **THEORETICAL UNDERPINNINGS**

### **Algebra in primary school**

Kieran (2004) argued that algebraic thinking in primary school entails mathematical activities such as problem solving, generalizing, analyzing relationships, and studying change; these are general activities for which algebra is used as a tool, but which are not exclusive to algebra. Van Amerom (2003) argued that students can acquire algebraic concepts before studying formal algebra through solving nonstandard problems. Their informal approaches can be seen as ways in which they try to deal with algebraic situations without having to rely on formal algebra (Johanning, 2004).

### **Variation theory**

According to variation theory, for learning to occur, the learner should be able to experience and discern critical aspects of variation in the phenomenon under study (Runesson, 2006) and become simultaneously aware of the possible values that these aspects can take (Marton & Tsui, 2004). Although experiencing patterns of variation is significant for learning mathematics in general, it might be especially relevant to the teaching and learning of algebra, since the ability to generalize from particular instances implies that one can distinguish between what varies and what remains invariant. Variation theory can be used for designing learning situations by creating patterns of variation and invariance in relation to critical aspects of that learning (Runesson, 2006). According to Watson and Mason (2006), in comparison to unstructured sets of tasks, tasks that display constrained variation are likely to result in progress.

### **The role of computers in the teaching and learning of mathematics**

Technology influences not only how mathematics is taught and learned but also what is taught and when it is taught; technological tools afford access to powerful visual models and enable the quick and accurate execution of routine procedures, which allows more time for conceptualizing and modelling (NCTM, 2000). Particularly, in simulations students are engaged in active exploration and discovery learning (Lou, Abrami, & d'Apollonia, 2001) and they are challenged to formulate strategies to deal with complex mathematical systems (Crown, 2003). Dynamic computer

environments offering animation-based feedback (Nathan, 1998) might be especially suitable for comprehending how quantities relate to each other, thus supporting algebra problem solving. Likewise, the use of spreadsheets can assist students in understanding the concept of variable as a varying quantity and in generalizing problem situations (Lannin, 2005). However, more research into the use of technology in the early grades is needed (Kilpatrick, Swafford, & Findell, 2001).

## SET UP OF THE STUDY

### Procedure

A pretest-posttest experiment was set up to investigate the influence of a computer-based intervention, including an online game, on students' early algebra performance. For working in the online environment each student received a login account and eight problems split in three sets. The students were requested to solve the problems in the online environment at home and write down their answers on a worksheet. This online activity was not compulsory. The students had approximately a week at their disposal to work on each set of problems. At the end of the week the students presented their answers in short follow-up discussions in class. The purpose of these discussions was not to teach students how to solve the problems, but to provide them with feedback on their answers and sustain their participation. Before and after the intervention a written test on algebraic problem solving was administered at school.

### Participants

In total 318 students from grade 4, 5 and 6 from five schools in a major Dutch city were invited to participate in the study. These schools are situated in five city districts that cover a wide socio-economic range of student population.

### Pretest and posttest on algebraic problem solving

The written test on algebraic problem solving that served as pretest and posttest included seven contextual problems, such as the following:

*Quiz:* In a quiz you get 2 points for each correct answer. If a question is not answered or the answer is wrong, 1 point is subtracted from your score. The quiz contains 10 questions.

Tina received 8 points in total. How many questions did Tina answer correctly?

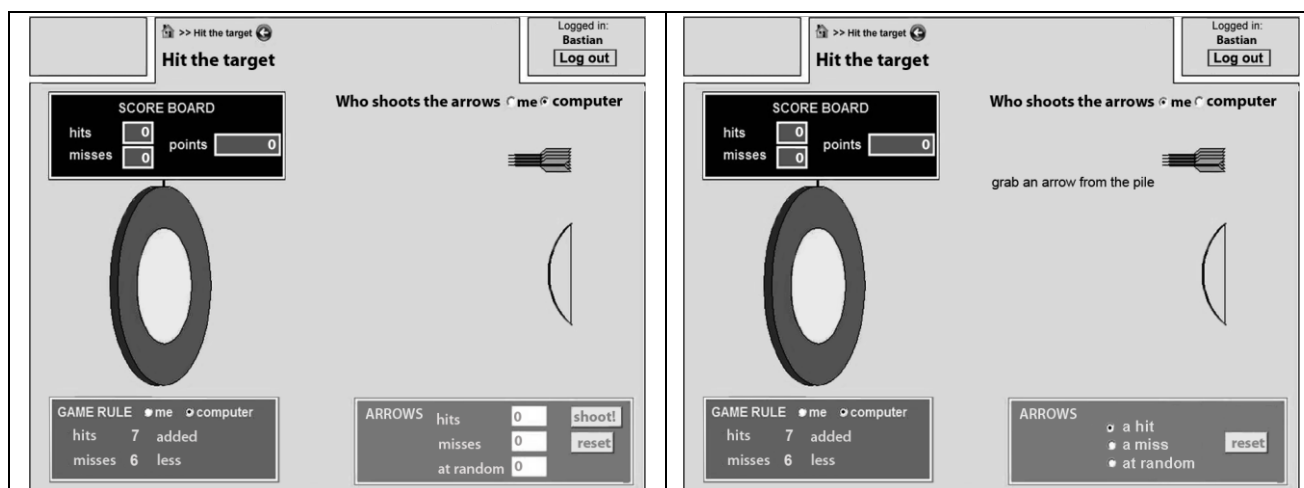
The problems of our study can be solved by a formal algebraic approach that is setting up and solving a system of two linear equations with two unknowns, or by an arithmetic method in which students have to reason informally about the relationships between the quantities. Since formal algebra is introduced in the Netherlands in the first year of the secondary school, we expected that the primary school students in our study had only arithmetic approaches at their disposal.

### The online environment

The environment developed to offer students experience in dealing with interrelated variables includes a dynamic game called *Hit the target* [1] (see Figure 1a/b), which



is a simulation of an archery game. In this game the students can set the shooting mode (user or computer shooting) and the game rule mode (user or computer defined). The features of the game are dynamically linked. In the course of the game the values on the scoreboard update rapidly to provide information about the total score. In this way, students may become aware of the fact that the arrows, the score, and the game rule are related to each other so that a modification in the value of one of these variables has a direct effect on the other variables. Moreover, the game offers instant visual feedback by displaying the consequences of students' actions.



**Figure 1a: Screen view of game in the computer shooting-mode**

**Figure 1b: Screen view of game in the user shooting-mode**

The problems that the students had to solve in this environment varied from finding the number of hits and the number of misses that produce a particular score, to generating a general solution by systematizing all possible answers. In general, two types of problems were included: problems that can be translated into a linear equation with two unknowns and problems that can be translated into a system of two linear equations with two unknowns. This series of problems aimed to support students in detecting patterns and generalizing, and confront them with situations in which multiple conditions should be taken into account simultaneously. Variation in the numbers among the problems was intended to help students grasp the invariant structure of a problem, despite the fact that the particular numbers can vary. Furthermore, the variables that were allowed to vary and the variables that were kept invariant differed between the problems. A selection of the problems is the following:

*Problem 3:* What is the game rule to get 15 points in total with 15 hits and 15 misses? Are there other game rules to get 15 hits, 15 misses, and 15 points?

*Problem 4:* What is the game rule to get 16 points in total with 16 hits and 16 misses? Are there other game rules to get 16 hits, 16 misses, and 16 points?

*Problem 5:* What is the game rule to get 100 points in total with 100 hits and 100 misses? Are there other game rules to get 100 hits, 100 misses, and 100 points? Can you explain your answer?

*Problem 8:* For every hit you gain 2 points and for every miss 1 point is taken away from your score. You have 10 arrows in total. How many hits and misses do you have to shoot to get 5 points in total? Are there any other solutions possible?

The online environment offered students the opportunity to solve these problems by testing their ideas and receiving instant feedback. Moreover, the students' online activity was captured by special software, the so-called *Digital Mathematics Environment (DME)* [2]. The log data used in this study consisted of a list of the actions carried out by the students in the online environment classified into events (i.e., shooting actions) and sessions (i.e., series of events that students perform each time they are logged in) (Figure 3). A coding scheme for analyzing students' log files was developed in several rounds. Its final version consisted of 12 strategies. The interrater reliability was determined by calculating Cohen's kappa, which was considered to be sufficient (.84).

## RESULTS

In total, 253 students logged in at least once. The frequencies of the logged-in students from grades 4, 5, and 6 were 74%, 88%, and 78% respectively. The students who did not log in had the same pretest performance as the students who logged in ( $t(309) = -1.72, p > .05$ ). Table 1 gives an overview of the means and standard deviations of the pre and posttest performance, the login time, the number of events, the number of focused events (i.e., events related to answering the online problems), the percentage of focused events (ratio focused events to total events) and the number of problems students worked on per grade.

	Grade 4 (N=75)		Grade 5 (N=73)		Grade 6 (N=84)	
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Pretest (proportion correct answers)	.06	.14	.23	.26	.38	.30
Posttest (proportion correct answers)	.11	.20	.30	.33	.48	.36
Login time (minutes)	35.70	41.63	25.75	18.69	32.22	54.46
N of events	17.04	14.77	17.81	13.34	13.34	9.90
N of focused events	5.68	7.61	9.60	8.15	7.48	8.07
% of focused events	26.7	26.9	50.6	31.1	46.2	33.7
N of worked problems	3.10	3.82	4.92	3.71	4.46	3.91

**Table 1: Pre and posttest performance and characteristics of students' online working**

The gain score calculated by the difference of the score in the items in the post and pretest was significant across all grades with moderate effect sizes (grade 4:  $t(74) = 2.46, p < .05, d = .16$ , grade 5:  $t(72) = 2.93, p < .01, d = .24$ , grade 6:  $t(83) = 5.19, p < .001, d = .43$ ). The standard deviations show considerable variation in the way students utilized

the online environment. The fifth graders exhibited the highest number of events, focused events and worked problems and the highest percentage of focused events.

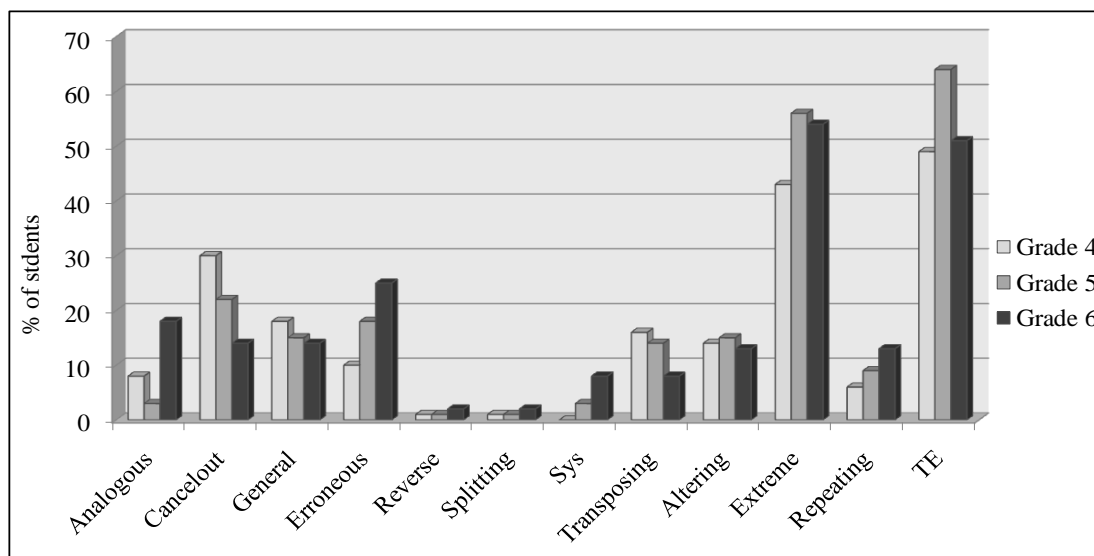
The partial correlations (Table 2) between the characteristics of the online working and the posttest performance controlling for the pretest performance reveal significant relationships between the posttest performance and the percentage of focused events and the number of worked problems for the students of grade 4 and 6.

	Posttest		
	Grade 4	Grade 5	Grade 6
Login time (minutes)	-.03	.03	-.03
N of events	.03	.07	.03
N of focused events	.17	.06	.07
% of focused events	.30**	.11	.21*
N of worked problems	.36**	.16	.22*

\*\*  $p < .01$ , \*  $p < .05$  (one-tailed)

**Table 2: Partial correlations between students' online working and posttest performance**

Students applied various strategies to solve the problems (Figure 2). The mostly used strategy was trial-and-error (TE) (49%–64% of the students). Systematic trials (Sys) were, however, performed by only a few students. A high percentage of students (43%–58%) applied the Extreme strategy, which is a way of reducing the complexity of a problem by putting one variable to zero so that another variable in the linear relationship gets the maximum possible value.



**Figure 2: Relative frequencies of strategy use in grades 4, 5, and 6.**

The Analogous strategy (e.g., in *Problem 5*: shooting 10 hits and 10 misses), the Splitting strategy (e.g., in *Problem 5*: first shooting 100 hits and 0 misses, then 0 hits

and 100 misses, and adding the partial scores to calculate the total score) and the Transposing strategy (i.e., the values of the arrows and the values of the game rule are exchanged) were less frequently applied. These strategies were particularly evoked when students tried to solve Problem 5. Since the maximum number of arrows to be shot at once in the game is restricted to 150, the students cannot shoot 100 hits and 100 misses at once, but should come up with other approaches to solve this problem.

Strategies like Altering (adapting part of the problem information), Extreme, Repeating (replicating a correct solution), and trial-and-error imply a focus on providing the (correct) answer. In contrast, the use of more sophisticated strategies is an indication that students explored relations and structure. Such strategies are Analogous, Cancelout (i.e., the total of hit-points cancels out the total of miss-points), General (looking for a general rule), Erroneous (i.e., applying an erroneously derived rule to produce additional solutions to a problem), Reverse (i.e., reversing a rule to produce additional solutions), Splitting, systematic trials, and Transposing.

Moreover, we could identify three levels of online working: free playing, (just) looking for answers, and exploring relations and structure. Level 1 includes the students who put little or no effort in answering the given problems, in particular, the students who performed less than three focused events or tried to answer less than three problems. Level 2 includes the students who mainly (just) tried to answer the problems. These students exhibited an activity which was equal to or beyond the threshold of minimum effort, but their main concern was to provide correct solutions as indicated by the use of less sophisticated strategies. Level 3 includes the students who also exceeded that minimum amount of effort and used sophisticated strategies, which indicates that the students explored relations and structure.

	Grade 4	Grade 5	Grade 6	Total
Free playing (Level 1)	60.8	29.5	39.6	43.1
(Just) looking for answers (Level 2)	12.7	41.0	28.1	27.3
Exploring relations and structure (Level 3)	26.6	29.5	32.3	29.6

**Table 3: Percentages of students who applied a particular level of online working per grade**

The percentage of students who performed free playing was the highest in grade 4, while the majority of the students in grade 5 were categorized at the level of just looking for answers. In grade 6 free playing was again the most prevailing activity. Nevertheless, the percentage of students who explored relations and structure increased slightly from grade 4 to grades 5 and 6.

Figure 3 shows the log file of a sixth-grade student's problem solving process that eventually led to the discovery of the general rule. In *Problem 3* the student found one solution by trial-and-error. In *Problem 4* she came to a general rule (i.e., the points per hit and the points per miss should add up to 1), which she also applied in *Problem 5*.

<b>session: 1   date: 2008/11/17 04:13:58   duration: 00:05:59   total events: 2</b>						} Problem 3
<i>event: 1</i>						
who shoots: computer   hits: 15   misses: 15   at-random: 0   removed: 2 (hits: 2 misses: 1)						
game rule: student   hit: 4 added   miss: 2 less						
<i>event: 2</i>						
who shoots: computer   hits: 15   misses: 15   at-random: 0   removed: 0						
game rule: student   hit: 1 added   miss: 1 added						
<b>session: 2   date: 2008/11/17 04:20:43   duration: 00:02:52   total events: 1</b>						
<i>event: 1</i>						
who shoots: computer   hits: 15   misses: 15   at-random: 0   removed: 0						
game rule: student   hit: 5 added   miss: 4 less						
<b>session: 3   data: 2008/11/25 04:02:24   duration: 00:36:03   total events: 6</b>						} Problem 4
<i>event: 1</i>						
who shoots: computer   hits: 16   misses: 16   at-random: 0   removed: 0						
game rule: student   hit: 2 added   miss: 1 less						
<i>event: 2</i>						
who shoots: computer   hits: 16   misses: 16   at-random: 0   removed: 0						
game rule: student   hit: 3 added   miss: 2 less						
<i>event: 3</i>						
who shoots: computer   hits: 16   misses: 16   at-random: 0   removed: 32 (hits: 16 misses: 16)						
game rule: student   hit: 100 added   miss: 99 less						
<i>event: 4</i>						} Problem 5
who shoots: computer   hits: 10   misses: 10   at-random: 0   removed: 0						
game rule: student   hit: 2 added   miss: 1 less						
<i>event: 5</i>						
who shoots: computer   hits: 10   misses: 10   at-random: 0   removed: 0						
game rule: student   hit: 1000 added   miss: 999 less						
<i>event: 6</i>						
who shoots: computer   hits: 10   misses: 10   at-random: 0   removed: 0						
game rule: student   hit: 50 added   miss: 49 less						

**Figure 3: Log file of online student working.**

## DISCUSSION

This study looked at how an online learning environment with a game and a series of problems can contribute to students' ability to solve problems with interrelated values. The analysis revealed that the ability to solve the problems increased over the grades with the highest gain in the grade 6. Despite the short duration of the intervention, the online working was positively related with success in the posttest for grades 4 and 6. Furthermore, the environment brought about various types of strategies. The dynamic character of the game offered students the opportunity to apply sophisticated strategies to explore relations and structure. In addition, the problems sequence turned out to be important as well. The log files revealed that by working on the series of problems, the students could experience how the values covariate, which prompted the discovery of the general relation between the values.

However, some limitations should be kept in mind. To begin with, our conclusions are based on the log files of the students' online activity, which might not entirely capture their cognitive processes such as the mental calculations. Nevertheless, the

collected data provided a step-by-step account of students' interaction with the computer environment. Furthermore, because the changes in the levels of online working across the grades followed an irregular pattern, the question is raised whether the students who participated in the study are typical. Due to cluster sampling, there is a higher probability of getting a non-representative sample.

Moreover, we could not control whether the students worked on the online problems on their own or with the help of a peer or family member. However, because the activity was voluntary and low-stakes, it is likely that the students did not feel the need for assistance. In addition, the diversity in the students' engagement in the online activity is an indication that the students worked on their own; had parents been involved, it would have been more likely for the students to try to solve all the presented problems.

However, in general, our results suggest that algebraic reasoning in the primary grades could be stimulated through computer environments. Furthermore, home computing may create an effective learning environment supporting and extending school learning. Yet further investigations are needed, in particular to examine the influence of the game when introduced as a compulsory activity in the classroom.

## NOTES

1. The game 'Hit the target' is developed by the first author and programmed by our colleague Huub Nilwik.

2. The Digital Mathematics Environment (DME) is developed by our colleague Peter Boon.

## REFERENCES

- Crown, W. D. (2003). Using technology to enhance a problem-based approach to teaching: What will and what will not work. In F. K. Lester & R. I. Charles (Eds.), *Teaching Mathematics through Problem Solving, Prekindergarten-Grade 6* (pp. 217–228). Reston, VA: NCTM.
- Carraher, D. W., & Schliemann, A. D. (2007). Early Algebra and Algebraic Reasoning. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 669–705). Charlotte, NC: Information Age Publishing.
- Goldenberg, E. P., Shteingold, N., & Feurzeig, N. (2003). Mathematical habits of mind for young children. In F. K. Lester & R. I. Charles (Eds.), *Teaching mathematics through problem solving: Prekindergarten-Grade 6* (pp. 15–29). Reston, VA: NCTM.
- Harel, G. (2008). What is mathematics? A pedagogical answer to a philosophical question. In B. Gold & R. A. Simons (Eds.), *Proof and Other Dilemmas: Mathematics and Philosophy* (pp. 265–290). Washington, DC: MAA.
- Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *Journal of Mathematical Behavior*, 23, 371–388.

- Kaput, J. (2007). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, & M. Blanton (Eds.) *Algebra in the Early Grades* (pp. 5–18). New York: Lawrence Erlbaum.
- Kaput, J., Carraher, D. W., & Blanton, M. (Eds.). (2007). *Algebra in the early grades*. New York: Lawrence Erlbaum.
- Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, 8, 139–151.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics*. Washington, D.C.: National Academy Press.
- Lannin, J. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, 7(3), 231–258.
- Lou, Y., Abrami, P.C., & d'Apollonia, S. (2001). Small group and individual learning with technology: A meta-analysis. *Review of Educational Research*, 71(3), 449–521.
- Marton, F., & Tsui, A. B. M. (2004). *Classroom Discourse and the Space of Learning*. Mahwah, N.J.: Lawrence Erlbaum.
- Nathan, M. J. (1998). Knowledge and situational feedback in a learning environment for algebra story problem solving. *Interactive Learning Environments*, 5, 135–159.
- National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Roschelle, J. M., Pea, R. D., Hoadley, C. M., Gordin, D. N., & Means, B. M. (2000). Changing how and what children learn in school with computer-based technologies. *The Future of children*, 10(2), 76–101.
- Runesson, U. (2006). What is possible to learn? On variation as necessary conditions for learning. *Scandinavian Journal of Educational Research*, 50(4), 397–410.
- Schliemann, A. D., Carraher, D. W., & Brizuela, B. M. (2001). When tables become function tables. In M. Van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the international group for the psychology of mathematics education* (Vol. 4, pp. 145–152). Utrecht: Freudenthal Institute.
- Van Amerom, B. (2003). Focusing on informal strategies when linking arithmetic to early algebra. *Educational Studies of Mathematics*, 54, 63–75.
- Warren, E. A., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom: Foundations of early algebraic reasoning. *Journal of Mathematical Behavior*, 25, 208–223.
- Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, 8(2), 91–111.

# FRAMING A PROBLEM SOLVING APPROACH BASED ON THE USE OF COMPUTATIONAL TOOLS TO DEVELOP MATHEMATICAL THINKING

Manuel Santos-Trigo  
Cinvestav. Mexico

Matías Camacho-Machín  
University of La Laguna. Spain

*We outline problem-solving episodes dealing with mathematical tasks in which the use of the tools becomes relevant. The episodes involve the use of an inquiring process to approach problem statements or comprehend mathematical concepts. Questioning becomes a form or way of making sense of situations and of representing them in terms of using the tools' affordances. The episodes characterize a framework that teachers can use to structure and reflect on ways to use computational tools to foster their students' development of mathematical thinking.*

## INTRODUCTION

It is widely recognized that the use of computational technology could offer students and teachers various ways to represent and explore mathematical problems or concepts. There is also evidence that different tools might offer learners different opportunities to think of problems in order to represent, explore, and solve them. Recently, different research programs have analyzed and documented the role played by the use of diverse digital tools in the students' development of mathematical knowledge (Hoyles & Lagrange, 2010). For example, the use of dynamic software offers advantages to construct models of situations or problems in which the model components can be displaced within the representation to identify and explore mathematical relations, while the use of calculators offers advantages to represent and deal with problems algebraically. In general, research results indicate that it is important for teachers to incorporate in their teaching scenarios a systematic use of several computational tools to help students develop mathematical comprehension and problem solving proficiency. To this end, several recent curriculum proposals recommend that students use computational tools in their learning activities.

What type of tools should students use and how should they use them to enhance their problem solving approaches? What types of tasks should students work and discuss in order to transform technological artefacts into effective problem solving tools? What types of mathematical reasoning do students develop as a result of using a particular tool in problem solving activities? The discussion of these and similar questions is relevant to orient or guide teachers during the design and implementation of activities and tasks fostering the use of computational tools. The aim of this article is to identify dimensions and processes that characterize students' problem solving approaches fostering the use of several computational tools. In this context, we identify and discuss common math features that distinguish students' use of computational tools to solve problems. In order to do this, we focus mainly on problems where students use the tools to construct a dynamic model of the problems.



The model is then used as a departure point to look for mathematical relationships that become important to solve and extend the problem. It is important to investigate the extent to which the use of particular tools helps students develop knowledge and ways of thinking that are consistent with mathematical practice. In particular, we argue that teachers need to reflect on key issues and concepts concerning the process involved in using the tools in their problem solving approaches.

## ON THE STUDENTS' PROCESS OF APPROPRIATION OF THE TOOLS

How should teachers decide what computational tools to use in their practices? What type of tasks and instructional strategies should they consider to use the tools in problem solving approaches? And what types of reasoning do students construct as a result of using certain tools in their learning experiences? Kaput (1992) stated that “[m]ajor limitations of computer use in the coming decades are likely to be less a result of technological limitations than a result of limited human imagination and the constraints of old habits and social structures” (p. 515). Nowadays, after almost two decades since Kaput’s statement, the math education community faces a challenge not only to clearly incorporate a systematic use of computational tools in curriculum proposals, but also to characterize learning scenarios in which students can efficiently use tools to learn and develop math knowledge. In other words, the use of the tools requires not only the transformation of classroom settings, but also the validation of emerging explorations including visual, empirical and formal reasoning. It is also important to recognize that the tool itself does not provide the media or ways needed for students to efficiently use it in problem solving activities; it involves an appropriation process in which the students transform an artefact into an instrument. This appropriation depends on cognitive schemata that students develop while using the tool to represent and explore the problem. Trouche (2004) stated that “an instrument can be considered an extension of the body, a functional organ made up of an artifact component (an artifact, or the part of an artifact mobilized in the activity) and a psychological component” (p. 285). The artefact characteristics (ergonomics and constraints) and the schemata developed by the students during the activities are important for them to transform the artefact into a problem-solving instrument. In this respect, Trouche (2004) related the students’ psychological component to the construction of a scheme with three functions: “a *pragmatic* function (it allows the agent to do something), a *heuristic* function (it allows the agent to anticipate and plan actions), and an *epistemic* function (it allows the agent to understand what he is doing)” (p. 286). Indeed, these three functions become essential to construct dynamic models of problems. The use of dynamic software plays an important role in constructing models of situations and tasks where the movement of particular elements can be examined and explained in terms of math relationships (Santos-Trigo, 2008). Models might involve configurations made of simple math objects (points, segments, lines, triangles, squares etc.) in which some elements of the models can be moved within the configuration in order to identify

and explore math relationships. These relationships and conjectures become a source that engages students in math inquiry and reflection. As a consequence, the process of model construction and exploration also incorporates a variety of ways to represent, formulate, and examine math relationships. For instance, with the use of the tool, the situations or problems are now analyzed in terms of the facilities and affordances offered by the tool such as dragging particular components, finding loci of points or lines, quantifying certain relations, using a Cartesian system to model the problem algebraically (Santos-Trigo, 2008). Indeed, the use of the dynamic software seems to offer problem solvers the opportunity to explore diverse routes to develop or reconstruct and examine basic mathematical results. In particular, the visual approach becomes relevant to identify mathematical relationships that later can be analyzed in terms of numerical and graphical approaches (Santos-Trigo, 2010).

## **BACKGROUND AND GENERAL CONTEXT**

This study is part of an ongoing project whose general goal is to foster service and pre-service high school teacher use of computational technology as part of their regular practices. These teachers participate in regular problem solving sessions where they have the opportunity to use diverse computational tools to work on math tasks. The tasks include dealing with situations where the participants construct dynamic models or geometric configurations (formed by simple objects such as segments, points, lines, circles, etc.) that lead them to identify math relationships by moving elements within the model (Santos-Trigo & Espinosa-Pérez, 2010). In this type of task, there is no initial well-defined problem statement, and the problem solver formulates and pursues questions as a result of observing the behaviour of the involved objects. Other types of tasks include those that appear in regular textbooks, where the idea is to use tools to represent, explore and look for different ways to approach the tasks.

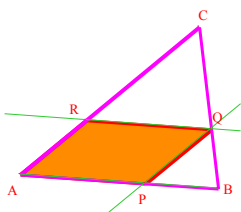
In this paper, we discuss math thinking features we have identified during the implementation of problem solving approaches in teaching scenarios where teachers systematically use diverse computational tools. Our synthesis is based on discussing with teachers and colleagues problem solving behaviour that our research group initially identified as being consistent while the project participants used the tools. This involves a retrospective account of what we observed in problem solving sessions that promoted the use of technology. Thus, by observing and analyzing teachers' and students' use of the tools in problem solving activities, we aim to identify common math behaviour that characterizes their approaches to the tasks (Santos-Trigo & Camacho-Machín, 2009). Our method of inquiry relies on observing teacher's behaviour when using computational tools in problem solving sessions. Thus, in order to introduce and discuss elements of the framework, we chose a task to illustrate different approaches to think of the task in terms of using a dynamic geometry software (DGS). The task is representative of a series of problems in which students deal with phenomena that involve parameter variations to determine

minimum or maximum values of a certain parameter (area, perimeter, length, etc). The task discussion leads us to identify components of a framework that teachers can use to structure and guide their students' use of several tools in problem solving approaches (Schoenfeld, 2010). The framework is presented in terms of episodes that extend the phases used by Polya (1945) to explain problem solving behaviour.

## A PROBLEM-SOLVING FRAMEWORK TO DEAL WITH PHENOMENA OF VARIATION

An example is used to illustrate elements of a framework to organize and guide students' use of computational tools to represent and explore the area variation of an inscribed parallelogram. Another example of a task analysed in terms of the framework appear in the appendix. In particular, we distinguish episodes that show the relevance that it has for students to comprehend and think of the problem statement in terms of math resources. This initial comprehension of the task becomes crucial to construct a dynamic representation of the problem that can help them visualize parameter behaviour. The NCTM (2009) identifies both sense making and reasoning as crucial processes that students need to develop in their problem solving approaches. The exploration of dynamic models of the task provides useful information for students to think of the problem in terms of analytical and geometric knowledge and to reflect on the concepts and processes that appear throughout all the episodes.

The task: In a given triangle ABC, inscribe a parallelogram by selecting a point P on one of the sides of the given triangle. Then from point P draw a parallel line to one of the sides of the triangle. This line intersects one side of the given triangle at point Q. From Q draw a parallel line to side AB of the triangle. This line intersects side AC at R. Draw the parallelogram PQRA (Figure 1). What happens to the area of the inscribed parallelogram APQR when point P is moved along side AB? Is there a position for point P where the area of APQR reaches a maximum value? (Justify).

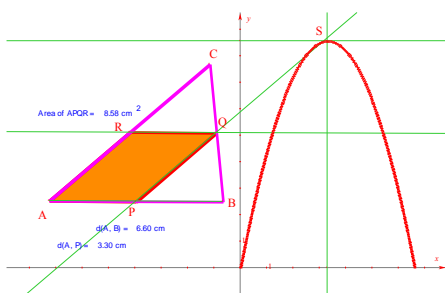


**Figure 1: Drawing a parallelogram inscribed in a given triangle**

**Comprehension Episode.** Polya (1945) identifies the process of understanding the statement of a problem as a crucial step for thinking of possible ways of solving it. Understanding means being able to make sense of the given information, identify relevant concepts, and think of possible representations to explore the problem mathematically. If students are to comprehend and make sense of the problem, they need to problematize the problem statement. They need to think of the problem in terms of questions to be explored and discussed with other students and the teacher. For example, in this problem, the comprehension stage involves discussing questions

such as: What does it mean for any given triangle? What information does one need to draw any triangle? Are there different ways to inscribe a parallelogram into a given triangle? For example, in Figure 1, one can draw from P a parallel line to CB (instead of AC) and this line intersects side AC and from that point of intersection, one can draw a parallel line to AB that intersects BC, thus, the two intersection points and point P and B form an inscribed parallelogram, the problem solver can ask: how is the former parallelogram related to the one that appears in Figure 1? Do they have the same area? How can I recognize that for different positions of point P the area of the parallelogram changes? This problem comprehension phase is important not only to think of the task in terms of using the software commands, but also to identify and later examine possible variations of the task.

**A Problem Exploration Episode.** The comprehension phase provides useful information to identify ways of representing and exploring the problem. The use of a DGS, becomes a powerful means to represent and construct a dynamic model of the problem. To begin with, students can draw a triangle by selecting three non-collinear points. Thus, they can discuss the conditions needed to draw a triangle. In addition, the use of DGS allows them to move any vertex to generate a family of triangles. In this case, they can select a point P on side AB to draw the corresponding parallel lines to inscribe the parallelogram. With the help of the software it is possible to calculate the area of the parallelogram and observe area value changes when point P is moved along side AB. Thus, it makes sense to ask whether there is a position of P in which the area of the inscribed parallelogram reaches either its maximum or minimum values. By setting a Cartesian system with the software, it is possible to construct a function that associates the length of segment AP with the area value of the corresponding parallelogram. Figure 2 shows the graphical representation of that function. That is, the domain of the function is the set of values that represents the lengths of AP when point P is moved along side AB. The range of that function is the corresponding area values of the parallelogram associated with the length AP. With the software, this graphical representation can be obtained by asking for the locus of point S (the coordinates of point S are length of AP and area of APQR) when point P moves along the segment. Here, it is important to observe that the graphical representation can be obtained without explicitly defining the algebraic model of the area change of the parallelogram.

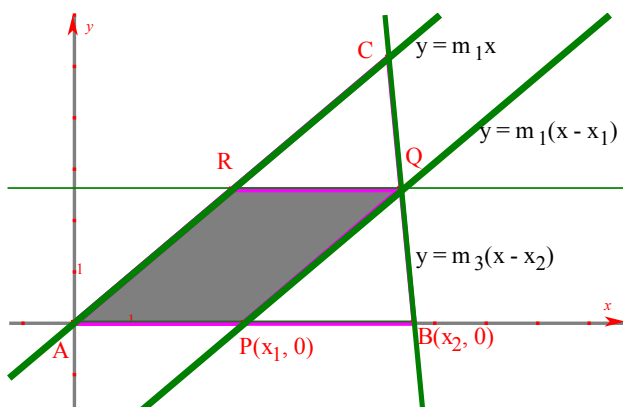


**Figure 2: Representation and visual exploration of the problem**

This graphical approach to solve the problem provides an empirical solution since it is visually and numerically possible to observe that in the given triangle the maximum area of the inscribed parallelogram is obtained when P is situated at 2.30 cm from point A. Here the area of the parallelogram is  $8.56 \text{ cm}^2$ . Indeed, a conjecture emerges based on this information: *When P is the midpoint of segment AB, then the corresponding inscribed parallelogram will reach the maximum area value.*

**The Searching for Multiple Approaches Episode.** We argue that for students to develop a conceptual understanding of mathematical ideas and problem solving proficiency (Kilpatrick, Swafford, & Findell, 2001), they need to think of different ways to solve a problem or examine a mathematical concept. To develop this understanding, students should have the opportunity of using different concepts and resources to represent, explore and solve problems. In this context, the visual and empirical approach used previously to explore the problem provides the basis to introduce other approaches. For example, in the next two approaches the goal is to construct an algebraic model to represent the variation phenomenon. The first one relies on introducing the Cartesian system to represent and operate the main objects associated with the problem, while the second approach is based on the use of geometric properties (similar triangles) to represent relationships among objects. Thus, these two ways of thinking about the problem represent an opportunity for the problem solver to reflect on strengths and limitations associated with the use of different concepts and resources to solve the task.

**1. Analytical approach.** In this approach, the students' initial goal is to represent and examine the problem in terms of algebraic means. The use of the Cartesian system becomes important to represent the objects algebraically. The use of the software also directly provides the equations associated with the lines that are needed to determine the expression of the parallelogram area. The problem can be thought of in general terms as is shown below.

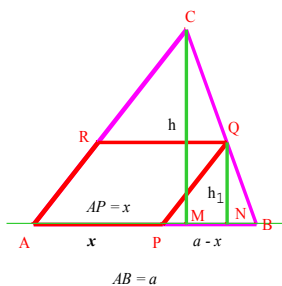


**Figure 3: Using a Cartesian system to construct an algebraic model of the problem**

Without losing any generality, we can always situate the Cartesian System in such a way that one side of the given triangle can be on the x-axis and the other side on line  $y = m_1x$  (Figure 4). Point P will be located on side AB and its coordinates will be  $P(x_1, 0)$ . Point  $B(x_2, 0)$  is vertex B of the given triangle (Figure 3). Based on this

information, the equation of line that goes through P and Q is  $y = m_1(x - x_2)$  and the equation of line BC is  $y = m_3(x - x_2)$ . Solving the system of equations leads to  $x = \frac{m_1x_1 - m_3x_2}{m_1 - m_3}$ , then, substituting this value in  $y = m_1(x - x_1)$  we obtain the y-coordinate of point Q, That is,  $y = \frac{m_1m_3(x_1 - x_2)}{m_1 - m_3}$  (this value corresponds to the height of parallelogram APQR). Then, the function area will be  $A(x_1) = \frac{m_1m_3(x_1^2 - x_2x_1)}{m_1 - m_3}$  (quadratic function whose roots are 0 and  $x_2$ ). This function also has a maximum value if and only if  $\frac{m_1m_3}{m_1 - m_3} < 0$ . We are assuming that  $m_1 > 0$ . The assumption on the triangle location guarantees that  $m_3$  and  $m_1 - m_3$  have opposite signs. Using a symmetric argument,  $A(x_1)$  reaches its maximum at the midpoint of the interval  $[0, x_2]$ , that is, at  $x_1 = \frac{x_2}{2}$ . To determine the maximum value of this expression by using calculus concepts, we have that:  $A'(x_1) = \frac{m_1m_3(2x_1 - x_2)}{m_1 - m_3}$ , now, the critical points are obtained when  $A'(x_1) = 0$ ,  $x_1 = \frac{x_2}{2}$  which is the solution of the equation, then the function  $A(x_1)$  will reach its maximum value at  $x_1 = \frac{x_2}{2}$  ( $A''(x_1) = \frac{m_1m_3}{m_1 - m_3} < 0$ ). Thus, this result supports the conjecture formulated previously in the graphical approach.

**2. A Geometric approach.** The focus here is to use geometric properties embedded in the problem representation to construct an algebraic model of the problem.



**Figure 4: Relying on geometric properties to construct an algebraic model**

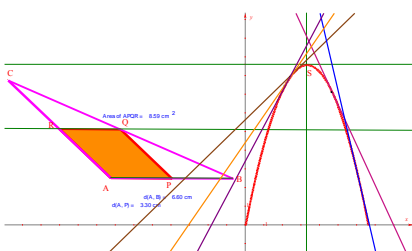
For example, in Figure 4, it can be seen that triangle  $\triangle ABC$  is similar to triangle  $\triangle PBQ$ , this is because angle  $PQB$  is congruent to angle  $ACB$  (they are corresponding angles) and angle  $ABC$  is the same as angle  $PBQ$ . Therefore, we have it that  $\frac{PB}{AB} = \frac{QN}{CM}$ , that is, if  $\overline{AP} = x$  and  $\overline{AB} = a$ , then  $\frac{a - x}{a} = \frac{h_1}{h}$ . Based on the former relationship,  $h_1 = \frac{h(a - x)}{a}$ , area of APQR can then be expressed as  $A = xh_1$ , this latter

expression can be written as  $A(x) = xh - \frac{hx^2}{a}$ . This expression represents a parabola.

$A'(x) = h - \frac{2hx}{a}$ , now if  $A' = h - \frac{2hx}{a} = 0$ , then  $x = a/2$ . Now, we observe that  $A'' < 0$  for any point on the domain defined for  $A(x)$ , therefore, there is a local maximum for that function.

**An Integration Episode.** It is important and convenient to reflect on the process involved in the distinct phases that characterize an approach to solving mathematical problems that fosters the use of computational technology. Initially, the comprehension of the problem statements or concepts involves the use of an inquiring approach to make sense of relevant information embedded in those concepts or statements. This enquiry process provides the basis to relate the use of the tools with ways of dynamically representing the problem or situation. Thus, a dynamic model becomes a source by which the behaviour of parameters is explored visually and numerically, as a result of displacing certain elements within the problem representation. In particular, it might be possible to construct a functional relationship between a variable, for example the variation of the side AP of the parallelogram and its corresponding area.

An interesting feature of this functional approach is that the model can be represented geometrically without having expressed it algebraically. The graphical representation of the task provides an opportunity for the problem solver to understand and discuss the domain of the function and the behaviour of the parameters from a visual approach. For example, by moving point P along side AB, one can observe that there will be two different positions for point P in which the corresponding areas of the inscribed parallelogram will be the same except when the point is located at the midpoint of the side AB. Graphically, this means that a parallel line to the x-axis will cut the graph in two points except when the line passes across the maximum point. At that point, the value of the slope of the tangent line to the curve will be zero (Figure 5). In addition, it is noted that for any triangle with side AB and P situated on AB, then the maximum area for the inscribed parallelogram will be reached when point P is the midpoint of side AB. In general, the visual and numerical approaches to the problem become important to generate a series of conjectures or relationships that needs to be supported by formal arguments.



**Figure 5: Examining properties of the area variation graphically.**

Thus, the idea at this stage is to look for different arguments to formally support not only the visual solution to the problem, but also to justify the conjectures or



relationships that emerged during both the visual and numerical approaches. It is clear that even when the goal here is to represent the parallelogram area symbolically, the ways used to construct the model rely on using different concepts and resources. As a result, the problem solver has the opportunity to relate contents that are often present in different subjects studied separately. In this case, both the analytic and geometric approaches converge in the search for the algebraic model. The algebraic model represents the general case and it can be “validated” by considering the information of the triangle used to generate the visual model. In addition, it can be used to explore some of the relationships that were detected during the visual approach.

In conclusion, the systematic use of computational tools in problem solving approaches led us to identify a pragmatic framework to structure and guide learning activities in such a way that can help students develop mathematical thinking. A distinguishing feature of this framework is that constructing a dynamic model of the problems provides interesting ways to deal with the problem from visual and empirical approaches. Later, analytical and formal methods are used to support conjectures and particular cases that appear in those initial approaches. Thus, the use of computational tools provides a basis not only to introduce and relate empirical and formal approaches, but also to use powerful heuristic tools such as dragging objects and finding loci of particular objects within the dynamic problem representation. Prospective and practicing teachers can use the framework to focus their attention on the activities involved in each episode. In particular, they need to conceive of a task or problem as an opportunity for their students to represent, explore and examine the task from diverse perspectives in order to formulate conjectures and to look for ways to support them.

## ACKNOWLEDGEMENT

The authors acknowledge the support received by Conacyt, reference 80359 and by Spanish Ministry of Science and Innovation in the National Research Plan, reference EDU2008-05254.

## REFERENCES

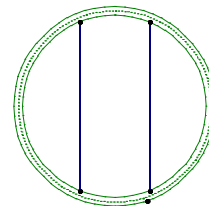
- Hoyles, C., & Lagrange, J-B. (2010) (Eds). *Mathematics education and technology- Rethinking the terrain. The 17<sup>th</sup> ICMI Study*. International Commission on Mathematical Instruction. NY: Springer.
- Kaput, J. (1992). Technology and mathematics education. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515-556). NY: Macmillan.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001), *Adding it up: Helping children learn mathematics*. Washington, DC: National Academia Press.



- NCTM (2009). *Focus on high school mathematics: Reasoning and sense making*. Reston VA: The Council.
- Polya, G. (1945). *How to solve it*. Princeton: Princeton University Press.
- Santos-Trigo, M. (2008). An inquiry approach to construct instructional trajectories based on the use of digital technology. *Eurasia Journal of Mathematics, Science & Technology Education*, 4(4), 347-357.
- Santos-Trigo, M. (2010). A mathematical problem-solving approach to identify and explore instructional routes based on the use of computational tools. In J. Yamamoto, J. Kush, R. Lombard, & J. Hertzog, (Eds.), *Technology implementation and teacher education: Reflective models* (pp. 208-313). IGI Global: Hershey PA.
- Santos-Trigo, M., & Camacho-Machín M. (2009). Towards the construction of a framework to deal with routine problems to foster mathematical inquiry. *PRIMUS Journal*, 19(3), 260-279.
- Santos-Trigo, M. & Espinosa-Pérez, H. (2010). High School Teachers' use of dynamic software to generate serendipitous mathematical relations. *The Montana Mathematics Enthusiast*, 7(1), 31-46.
- Schoenfeld, A. H. (2010). Reflections of an accidental theorist. *Journal for Research in Mathematics Education*, 41(2), 104-116.
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of computers for Mathematical Learning* 9(3), 281-307.

**APPENDIX:** Another example of tasks used in the project.

Three students got together to prepare their calculus final exam. After much hard work, when they were feeling hungry, they ordered a pizza and decided to divide it into three equal parts by making two vertical cuts (Figure 6). The pizza they ordered had a 40 cm radius. Where should they make the pizza-cuts so that each student gets the same amount of pizza? (Santos-Trigo & Camacho-Machín, 2009)



**Figure 6: Pizza cuts viewed from the top**

# ANALYSING TEACHERS' CLASSROOM PRACTICE WHEN NEW TECHNOLOGIES ARE IN USE

Mary Genevieve Billington

University of Agder

*This paper presents an analysis of teachers' practice in classrooms where new technologies are in use. An example of an analysis of data collected in mathematics classroom lessons in an upper secondary school is given. The analysis revealed how teachers structured the lessons and how the different tools were employed to achieve didactical and mathematical goals. The research contributed to research within the project "Teaching Better Mathematics" at the University of Agder.*

Keywords: classroom, analysis technique, didactic process

## INTRODUCTION

Newer technologies are increasingly used in schools. In Norway these technologies are termed "digital tools" and their use is now compulsory in all subjects and at all levels of schooling. It is argued therefore that it is important to learn more about how teachers integrate these tools into their classroom practice.

The classrooms studied were advanced mathematics classrooms in the second year of upper secondary school. In 2002 the school visited had initiated an internal ICT project. All students in advanced mathematics classes were issued with a laptop PC and classrooms were equipped with a screen and a projector. By the spring term of 2007 when the observations took place the classrooms had evolved to paperless classrooms; paper textbooks and notebooks were no longer in use. In the Norwegian context such classrooms were special but the project school was a normal school in that it operated in accordance with all formal regulations and guidelines. Practices described in this paper had evolved in a normal school environment with inherent functioning forces and tensions, not influenced by any external developmental project in mathematics education. I claim that this was an opportunity to observe and learn about ordinary teachers integrating new technology into their practice from the teachers' perspective. Monaghan (2004) and others have seen a need for such research. This feature of the data is seen to be especially interesting given the generally limited success in the integration of digital tools into mathematics classrooms (Lagrange, Artigue, Laborde, & Trouche, 2001).

This paper addresses the following research question: "How do teachers structure mathematics lessons where newer technologies are in use and which technologies and tools are employed to achieve which didactical and mathematical goals". An analysis technique was developed to analyse the classroom data. A detailed description of the technique is given before proceeding to the results of the analyses.

## THEORETICAL FRAMEWORK

In the main study, the Instrumental Approach in Didactics provides the theoretical perspective. This approach combines elements from the Anthropological Theory of Didactic in mathematics education (Chevallard, 2005) and the Instrumental Approach from the field of cognitive ergonomy (Verillon & Rabardel, 1995). The approach is a middle range theory, domain specific to studying the use of newer technologies in mathematics education. Three main notions from the approach served to structure the analysis; *praxeology* and *didactic process* and *instrumental genesis*. In this brief paper *praxeology* and *didactic process* are discussed in depth.

A *praxeology* is defined as the basic unit of human activity. This therefore includes mathematical activity and didactical activity. A praxeology is constituted of two inseparable parts, the praxis part and the logos part and each of these parts consists of two components (Chevallard, 2007, p. 133). The praxis block is formed by types of problems/tasks and by the techniques used to solve these tasks. The knowledge block, the so called discursive environment (logos) is structured in two levels: the technology (the discourse about the techniques used) and the theory that constitutes a deeper level of justification of practice (Barbé, Bosch, Espinoza, & Gascón, 2005, pp. 235-238). In studying teacher action in the classroom I use the term didactical praxeology to refer to teacher activity directed towards promoting the students mathematical activity.

The term *didactic process* is used to refer to the process directed towards the setting up of the mathematical praxeologies for the students. The process is claimed to be made up of six moments; the moment of first encounter, the moment of exploration, the technological theoretical moment, the technical moment, the institutionalisation moment and the evaluation moment (Barbé *et al.*, 2005, p. 239). The analysis attempts to identify these moments in the classroom activity.

Two other notions used to structure the data analysis are *didactical configuration* and *didactical exploitation mode* (Drijvers, Doorman, Boon, van Gisbergen & Reed, 2009; Trouche, 2004, 2005). Drijvers *et al.* (2009) define these terms in the following way:

*A didactical configuration* is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. These artefacts can be technological tools, but the tasks students work on are important artefacts as well. Task design is seen as part of setting up a didactical configuration.

*An exploitation mode* of a didactical configuration is the way the teacher decides to exploit it for the benefit of his didactical intentions. This includes decisions on the way a task is introduced and worked, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students (*ibid.*, p. 2).

## METHODOLOGY

### Data collection

Data was collected over a two week period, alternating between two parallel classes. Two researchers were present at each observation allowing for one researcher to video-record the lesson and the other to take comprehensive field notes.

### The Analysis Technique

The complexity of analysing classroom activity has been widely recognised (Abboud-Blanchard, 2008; Barbé *et al.*, 2005; Monaghan, 2004; Robert & Rogalski, 2005). In line with Robert and Rogalski (2005) my analysis focuses on teacher action rather than modelling the global system in the classroom.

The analysis technique involves four elements; organisation of space and time; the mathematical content; didactic episodes; and instances of instrumented activity. These elements are explained here.

1. *Organisation of space and time.* In this part of the analysis a detailed description of the physical environment and how this is organised is given. In addition, an overview of the time disposition in the lesson sequences is presented in table form.

2. *The mathematical activity.* In this part of the analysis the mathematical content of the lesson sequence is described by firstly presenting the relevant goals in the official syllabus and secondly identifying the four elements of the mathematical praxeologies; the tasks, techniques, discourse and theory. The mathematical problem and the techniques suggested by the teacher to solve the problem are identified and discussed. The main notions, concepts and mathematical notation employed in the lesson are presented. (Barbé *et al.*, 2005, pp. 236-238)

3. *Detailed analysis of didactic episodes with focus on tool use.* In reviewing the data material it was noted that each lesson sequence was clearly divided into time periods signalled by the teacher comments, such as: “ But now we are going to do something fun and this is something you have not done before” (My translation)

I decided to use these teacher made divisions in the analysis. I therefore define a *didactic episode* as a part of the lesson where the class engages in one primary activity as signalled by the teacher. For example: the first didactic episode in a lesson maybe an episode with recap of theory from the previous lesson; the second, correction of homework; the third, introducing new theory and so on. For each lesson sequence observed in the reported study a table, as shown below, was completed.

Episode Time min.	Teacher's goal – Moment of didactic process	Mathematical organisation	Didactical configuration		Didactical exploitation mode	
			Work mode	Tools used by teacher	Tool use - Didactical goal	Tool use - Mathematical goal

**Table 1: Didactic episodes**

The *episodes* are numbered and the time used on each episode is given. The *teacher's goal* is the goal perceived by the observer to be the goal of the episode, for example; a recap of theory. This goal is interpreted in relation to the moments of the didactic process. In the third column the parts of the *mathematical praxeology* presented in the didactic episode are identified as: task, technique, discourse, theory. Under the heading *Didactical configuration*, two sub-headings are included: *Working mode* and *Tools in use*. Working mode describes how and when the teacher devolves the mathematical task to the students. Four physical/material tool sets were observed to be in use by the teacher: the blackboard, a digital textbook, PC+ program Derive, body gestures. Each of these was used in conjunction with the voice and assumingly schemas (cognitive apparatus). The column *Didactical exploitation mode* has two sub-headings: *Tool use - didactical goal* and *Tool use- mathematical goal* included to describe how the teacher exploits the tools and with which didactical and mathematical intentions. The tools were sometimes used together. Instances when the teacher appeared to favour one tool over another or moved between the available didactical tools are identified. This practice of moving between tools within a didactic episode is termed weaving (Billington, 2009).

#### *4. Analysis of instrumented activity in relation to mathematical tasks*

In this part of the analysis focus is on instances of instrumented activity in the lesson sequences (Guin & Trouche, 1999, p. 201; Verillon & Rabardel, 1995). In each lesson sequence a few exemplary episodes are chosen for presentation and the manner in which the teacher uses the digital tools to solve the didactical task and the associated mathematical task/s is described. The instrumented technique/s used or advocated is/are identified and discussed.

### **APPLICATION OF ANALYSIS TECHNIQUE**

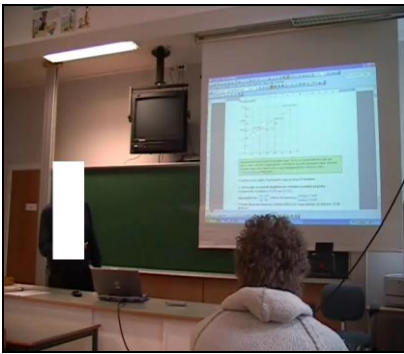
This section contains an illustration of the application of the analysis technique. This is done by presenting **two** elements of a particular lesson sequence with Teacher1: organisation of space and time and didactic episodes.

#### *Illustration starts*

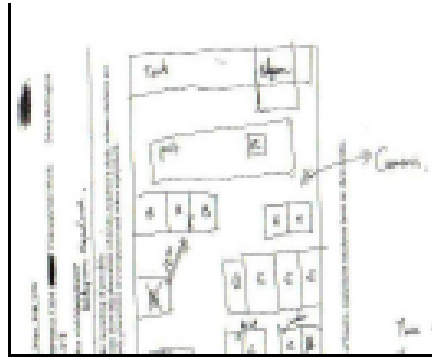
#### **Organisation of space and time**

A large blackboard covered almost the entire front wall of the classroom. A screen was positioned on the right hand side in front of the blackboard covering part of the blackboard. A laser pointer was available. The teacher's portable PC was placed on the large raised desk directly in front of the blackboard and screen. Students had only their portable PC on the desk and perhaps a backpack on the floor. The class came quickly to order. Students were not observed to use pencil and paper. A picture of the classroom and a diagram of the student seating arrangement are shown here.

## Working Group 15



**Figure 1: Picture of classroom**



**Figure 2: Student seating**

Time disposition was analysed as Table 2. In the fourth column the activity is interpreted in terms of the didactic process (Barbé *et al.*, 2005).

Time (m.)	Time %	Activity	Moment of didactic process
14	18	Recap of theory and homework	Institutionalisation moment
14	18	Introducing new task	Moment of first encounter + exploratory moment
11	14	Students draw graph (Derive)	Technical moment
6	8	Trying example together	Technological-theoretical+technical moment
34	43	Students work with problems: teacher goes around	Institutionalisation moment
Tot. 79			

**Table 2: Time/activity analysis**

### Didactic episodes

A table showing the analysis into didactic episodes was completed. A section of the table is given in appendix 1. The completed table showed that there were fourteen shifts in activity ranging from one to seventeen minutes in the duration of the lesson sequence. The lesson sequence could be divided into three main parts. In the first part: introduction with homework correction and recap of previous work (episodes 1-4: 15 min.) the digital textbook was the main or preferred tool used by the teacher. These episodes are interpreted as representing the institutionalisation moment of the mathematical organisation from the previous lesson. In part two, the new mathematical problem and the new techniques to solve the problem (episodes 5-9: 26 min.) were introduced using both the program Derive and the blackboard. The phenomenon of weaving was observed in episode 9. Episode 5 of 5 minutes duration encapsulated both the moment of first encounter and the exploratory moment. The technical-theoretical moment and the technical moment were contained in episodes 6 to 9. In final part of the lesson sequence, students worked individually or in small groups with the teacher assisting individual students in applying the new techniques

(episodes 10-14: 29 min.). These episodes are interpreted as the institutionalisation moment of the didactic process.

In seven of the fourteen episodes the teacher used the digital textbook. It was used as a pre-written blackboard to display the formal mathematical theory and completed and uncompleted exercises. The data indicates that the tool occupies a strategic place in the classroom praxeologies of this teacher. The digital textbook provided support with exercises and formal presentation of the mathematical theory.

The work mode descriptions indicate that the teacher clearly steered the lesson, presenting and explaining the mathematical concepts, tasks and techniques carefully without posing many questions to students. Questions posed required short answers. Student work was not displayed to the whole class. Students engaged actively in the discussion when the mathematical problem was posed in contextual terms. The mathematical tasks were only devolved to the students in the final part of the lesson which involved working on problems.

*Conclusion of illustration*

## **DISCUSSION OF RESULTS**

All observed lesson sequences were analysed in the manner as described above and then compared. In this section I will discuss some of the results of the full analysis with respect to the research question presented earlier: “How do teachers structure mathematics lessons where newer technologies are in use and which technologies and tools are employed to achieve which didactical and mathematical goals”. The full analysis revealed that the teachers had actively taken in use the new technologies and had developed similar patterns of practice and usage in the classroom.

### **The physical environment and time allocation**

The physical classrooms in use were traditional in design except for the rather haphazard installation of a projector, a screen and a large number of electrical contact points. Individual teachers may not have much say in such issues but the seating arrangement could be altered easily by the teacher. The seating allocation observed and described above lends itself to an expository mode of teaching as all desks were directed towards the teacher. The arrangement also made discussion between students difficult. The screen was visible to all students and this allowed the teachers to demonstrate techniques instrumented on the PC. This was observed constantly. It was also possible for students to present their work to the rest of the class by coming forward connecting their PC to the screen. This was not observed.

A comparison of the time allocation tables revealed an apparent lesson script of: teacher review; teacher leads to formula/technique; practice. This script is documented in research as a standard lesson script (Jacobs & Morita, 2002). The introduction of the new technologies had seemingly not altered the traditional lesson script. The further division of the lessons into didactic episodes showed that there

were a large number of teacher initiated shifts, varying from eight to fourteen shifts, in activity in the observed lessons. The effect of so many changes in activity on student learning may warrant further investigation. Further analysis is also required to ascertain if these shifts were caused or stimulated by the availability of the digital tools.

### **The didactic process**

The moments of the didactic process were identified within each didactic episode. The analysis revealed that generally only a few minutes in each lesson sequence were given to the *moment of first encounter*, that is introducing the mathematical tasks and that the blackboard was the preferred tool when introducing the mathematical problem. In some of the documented lessons the students indicated that they did not really understand the problem with such comments as: “What is it actually that we calculate, because I don’t understand?”

The *exploratory moment* was allocated more time but the teacher always led the explorations with demonstrations. These demonstrations were enriched by the facility of the digital tools to reify the mathematical objects and relations. The technological-theoretical moment, that is the moment where the techniques are justified in reference to theory, tended to merge with the technical moment where techniques are practiced. These two moments together were given the largest time allocation in the public part of the lesson sequences. All techniques were instrumented through the digital tools. The largest relative percentage of time in lessons was given to the students solving exercises. Both *the institutionalisation moment* and the *evaluation moments* were realised through short public summaries by the teachers and through the longer periods of students working alone or individually on exercises. The analysis showed that teachers favoured an expository teaching mode; giving explanations, demonstrating, with elicitation of answers from students followed by students working alone.

### **Teacher tool use**

The blackboard appeared to be the preferred tool when the teachers presented the mathematical task, gave an overview, illustrated notation; presented and/or discussed contextual examples and gave responses to spontaneous questions. A subjective observation was that the teachers were livelier when using the blackboard: moving around, using arm movements, tracing over important features of a curve with the chalk and so on. In contrast to its use in classrooms without PC and screen, the blackboard was used in a rough way for sketching out problems and solutions. The blackboard seemed to acquire the status of conceptual sketch pad.

The formal mathematical knowledge; both theory and the argumentation in worked exercises was presented pre-prepared through the digital textbook. This practice required extensive planning and preparation. The students thus did observe the teacher mathematician actually conducting, carrying through a mathematical



argumentation. The teachers clearly appreciated the word processing facilities offered by the digital tools as expressed in the following quotation.

Teacher1: Now everyone has the possibility to work down, now it is the mathematical, it is the logic that causes a stop; it is not all the writing that causes problems. (My translation from SM\_ 070212)

The CAS program was employed in exploring the properties of mathematical objects in the exploratory moment and to demonstrate and justify techniques to solve the mathematical problems in the technological-theoretical and technical moments. Teachers planned for the technological- theoretical moment with the production and preparation of the interactive illustrations and demonstrations. All problems were solved with techniques instrumented through the CAS program although some of these techniques represented simulations of paper and pencil techniques.

A phenomenon, which I term *weaving*, was often observed whereby the teacher moved between the available tools of the blackboard and the digital toolkit when holding public discourse.

## CONCLUSIONS AND FURTHER RESEARCH

It could appear that the new technology was used to strengthen rather than alter existing practice as has been found in other research (Cuban, Kirkpatrick, & Peck, 2001). The lesson observations were discussed with teachers informally and in a more formal meeting and these discussions provided insight into the logos behind some of seemingly unchanged practices.

Regarding the analysis technique, one strength of the technique, as I see it, is that it divides the lessons according to the teacher determined shifts in activity. The imposition of the theoretical notions on the data may perhaps restrict the interpretation. I would like to further develop and adapt the analysis technique to analyse lessons other than mathematics lessons with a goal to identify commonalities and differences in practice.

## REFERENCES

- Abboud-Blanchard, M. (2008). How mathematics teachers handle lessons in technology environments. In C.Winsløw (Ed.), *Nordic research in mathematics education: Proceedings from NORMA08* (pp. 237-244). Rotterdam: Sense Publishers.
- Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. *Educational Studies in Mathematics* 59(1), 235-268.
- Billington, M. (2009). Establishing didactical praxeologies: Teachers using digital tools in upper secondary mathematics classrooms In V. Durand-Guerrier, S.

- Soury-Lavergne & F. Arzarello (Eds.), *Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1330-1339). Lyon
- Chevallard, Y. (2005). Steps towards a new epistemology in mathematics education In M. Bosch (Ed.), *European research in mathematics education IV. Proceedings of CERME4* (pp. 21-30): Fundemi -IQS, Universitat Ramon Llull.
- Chevallard, Y. (2007). Readjusting didactics to a changing epistemology *European Educational Research Journal*, 6, 131-134,
- Cuban, L., Kirkpatrick, H., & Peck, C. (2001). High access and low use of technologies in high school classrooms: Explaining an apparent paradox. *American Educational Research Journal*, 38(4), 813-834.
- Drijvers, P., Doorman, M., Boon, P., van Gisbergen, S., & Reed, H. (2009). Instrumental orchestration theory and practice. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1349-1358). Lyon
- Guin, D. & Trouche, L. (1999) The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning*, 3(3), 199-227
- Jacobs, J. & Morita, E. (2002) Japanese and American teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education*, 33(3), 154-175
- Lagrange, J. B., Artigue, M., Laborde, C., & Trouche, L. (2001). *A meta study of IC technologies in education: towards a multidimensional framework to tackle their integration*. Paper presented at the 25th Conference of the Inter-national Group for the Psychology of Mathematics Education.
- Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. *International Journal of Computers for Mathematical Learning*, 9(3), 327-357.
- Robert, A., & Rogalski, J. (2005). A cross-analysis of the mathematics teachers activity. An example in a French 10th-grade class. *Educational Studies in Mathematics*, 59(1-3), 269-298.
- Trouche, L. (2004) Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9(3), 281-307
- Trouche, L. (2005) Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven & L. Trouche (Ed.) *The didactical challenge of symbolic calculators: turning a computing device into a mathematical instrument* (pp. 197-224). New York: Springer
- Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity *European Journal of Psychology of Education* X(1), 77-101.

Working Group 15

E p. (m in)	Teacher's goal – Moment of did. process	Mathematical Organisation  MO	Didactical configuration		Didactical exploitation mode	
			Work mode	Tools in use by teacher	Tool use - Didactical goal	Tool use – Mathematical goal
1 (5 )	Introd. researchers, prepares for lesson		Whole class lecture	Digital textbook: theory page + examp.	Pre-written blackboard/public textbook	Presents formal math. knowledge
2 (3 )	Recap of concepts from previous lesson  Instit. moment M <sub>5</sub>	<i>MO</i> Finding rate of growth linear function. $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ Use of f(x) instead of y, focus on “rate of growth”& new notation	Whole class lecture: no questions to/or response from students	Digital textbook: theory page  Laser pointer + hands: gesticulate “how fast”	Pre-written blackboard/public textbook	Presents formal mathematical knowledge.  Mathematical notation.
3 (5 )	Correct homework: testing application of knowledge  Tech. moment M <sub>4</sub>	<i>Tech:</i> using formula to calculate rate of change of linear function. <i>Tech:</i> fractions and sign change with arith. operations involving negative numbers T <sub>2</sub>	Whole class lecture: short questions and response to individual students	Digital textbook: worked examples	Pre-written blackboard/public textbook	Presents formal mathematical argumentation + mathematical notation.
4 (2 )	Recap/ application prior know.  Instit. moment M <sub>5</sub>	<i>Tech.</i> Rate of growth linear func. $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$ , Practical ex.,  <i>Discourse:</i> math. terminology, rate of change T <sub>2</sub> + T <sub>3</sub>	Whole class lecture: question from one student	Digital textbook:  Uses hand and laser pointer to graph screen.	Pre-written blackboard/public textbook.	Presents formal mathematical argumentation + mathematical notation.

Table 3. Didactic episode extract

# IMPLEMENTATION OF A MULTI-TOUCH ENVIRONMENT SUPPORTING FINGER SYMBOL SETS

Silke Ladel, Ulrich Kortenkamp

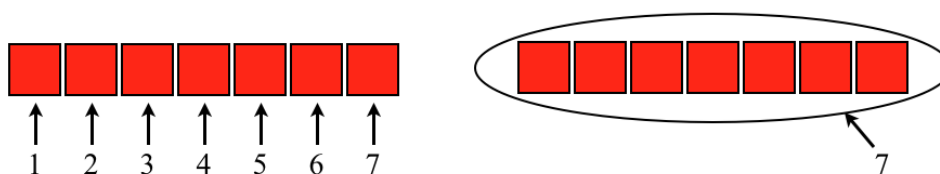
CERMAT, University of Education Karlsruhe, Germany

*Basic concepts of numbers and operations are fundamental for mathematical learning. Suitable materials for developing such basic concepts are hands and fingers. Among other things, this is because of their natural structure of 5 and 10. To support the development of concepts and the process of internalization a linking between different forms of representations by the computer can be helpful. To benefit of both, the advantages of the hands and fingers and the automatically linking, we suggest using multi-touch technology, i.e. computer input devices that are able to recognize several touch gestures at the same time. Here, children can present numbers with their fingers that produce virtual objects. These objects can be automatically linked with the symbolic form of representation.*

Keywords: number concepts, finger symbol sets, multi-touch, early math, Cinderella

## THE ORDINAL AND CARDINAL CONCEPT OF NUMBERS AND OPERATIONS

“How many things are there?” – For parents as well as for mathematicians, this is a common question to pose, if a child already has knowledge about numbers. For the child, this question is almost always the initiation to start counting verbally by saying the number words in a row (Fuson, 1988). The fundamental principles needed for answering the question are a) the one-one principle that relates every single object to exactly one numeral (Gelman & Gallistel, 1978), b) the stable-order principle prescribing the correct order of numbers (Fig. 1, left), and c) the last-word rule that assigns the last said numeral not to the last counted object, but to the quantity as a whole (Fig. 1, right).



**Figure 1: Ordinal (left) and cardinal (right) concept of numbers**

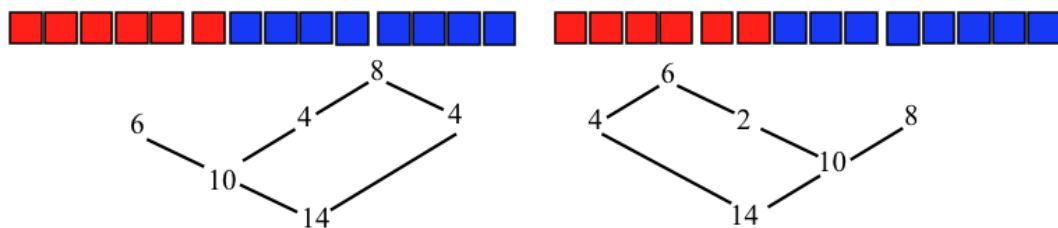
Here, the change from the ordinal concept of numbers, where the numeral is part of the numeral row, to the cardinal concept of numbers, where the numeral identifies a quantity, is necessary. It is not necessary to count a quantity in order to know it, that is, the ordinal concept is not a necessity for the cardinal concept. Resnick, Bill, Lesgold and Leer (1991) distinguish the development of mathematical knowledge by two components that are developed independently: *protoquantitative schemata* and the *mental number line*. To build up a well-developed concept of numbers, these two

threads of development have to be linked. For many children this is a critical problem (Fuson, 1992, p. 63).

Children who do not have a proper linking between the two concepts can misinterpret addition and subtraction as a demand to count forwards or backwards. As long as the children calculate with numbers smaller than 20 they can apply this strategy successfully. But, for instance, when they want to add 55 to 27 and begin to count „28, 29, 30, 31...“, there is no chance to come easily and quickly to the correct result.

„The protoquantitative part-whole schema is the foundation for later understanding of binary addition and subtraction and for several fundamental mathematical principles, such as the commutativity and associativity of addition and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.“ (Resnick *et al.*, 1991, p. 32).

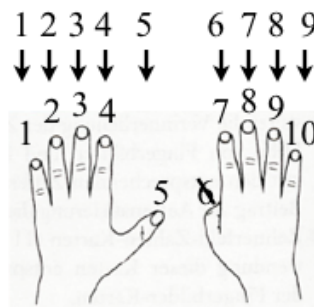
For example when you want to add 6 and 8 with the use of the part-whole schema you can split and add in lots of ways (e.g. Fig. 2).



**Figure 2: Different ways to add with the part-whole schema**

## FINGER SYMBOL SETS

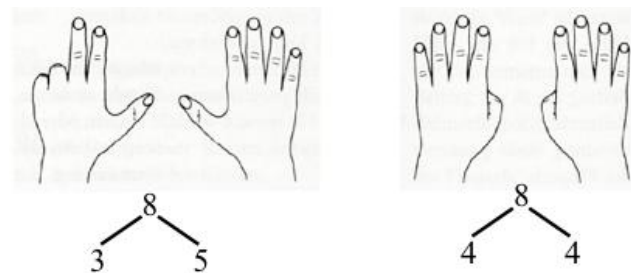
Calculating with fingers has a very bad reputation in mathematics lessons, as it is usually seen as an indicator for counting. Most children do as they have learned from young days on and count objects by „Counting-Word Tagging to Number“ (Brissiaud, 1992). According to the ordinal concept of numbers each finger is related to exactly one numeral. To illustrate this we ask what happens if the sixth finger is buckled? The „name“ of the last finger, that indicated the quantity, was „10“ before, but now the finger has to be renamed into „9“ (Fig. 3).



**Figure 3: Order-irrelevance principle**

The child has to know that it is irrelevant which fingers it uses to present a quantity. To present „3“, the thumb, the index finger and the middle finger can be used as well as the little finger, the middle finger and the thumb, or any other combination of three fingers. As we point out below, the cognitive process behind this fact can be experienced and thus supported by the use of multi-touch technology.

Amongst others, the advantages of fingers and hands are their permanent availability and their natural structure in 10 fingers per child with 5 fingers per hand. The 10 fingers qualify the hands to work out questions about the decimal number system, e.g. „*How many children do we need to see 30 fingers all at once?*“ The „power of five“ (Krauthausen, 1995) is due to the ability to instantaneously recognize quantities (subitizing) up to 4. Applying this to the hands, the shown quantity of the fingers of one hand can be conceived simultaneously and hence the fingers of both hands can be conceived quasi-simultaneously. Furthermore, one hand gets a special status because children tend to present numbers greater than five sequentially (Brissiaud, 1992, p. 61). For example, to present „7“, they tend to use one full hand and then add two fingers of the other hand. In this way the decomposition of the numbers from 1 to 10 with the power of five can be worked out. But not only these, also all other decompositions are possible (Fig. 4) and can be conceived quasi-simultaneously.



**Figure 4: Decomposition of numbers with finger symbol sets**

If the fingers are used like this, in sense of the part-whole schema, they are a qualified working material for a well-developed concept of numbers and operations (Steinweg, 2009). Brissiaud (1992, p. 56) coined the notion „From Finger Symbol Sets to Number“:

„Certain children who were not exposed early to the use of finger symbol sets may become counters, whereas children who were encouraged to use finger symbol sets may preferentially choose finger strategies“.

If children have a part-whole schema of numbers the transition to addition and subtraction is easy. It is just another way of nonverbal symbolic representation of the fact that „two parts make a whole“.

Further strategies like variation in the opposite or in the same direction can than be worked out easily: If one finger is buckled, than another finger must be stretched to keep the same quantity. To get the difference of two quantities, e.g. of 9 and 7, you can vary the numbers in the same direction. For example, a whole hand can be

omitted, which corresponds to subtracting five from each quantity. It is evident that the difference of 9 and 7 is the same as the difference between 4 and 2. Based on such strategies the decadic analogy can be build up.

It is important to pay attention to the fact that the children stretch their fingers *simultaneously* to represent quantities with them. If they show them one-by-one the positive effects of these strategies are lost and the children will still use counting for addition and subtraction.

This introduction can only serve as a small insight into the possible representations of numbers and operations by hands and fingers and their usage in early arithmetic. It is the process of internalization that is of essential importance: How can the children benefit from the mathematical content of these representations and actions and use them in their mental processes?

## **THE PROCESS OF INTERNALIZATION SUPPORTED BY THE USE OF MULTI-TOUCH-TECHNOLOGY**

The process of early mathematical learning follows four stages, independent of the arithmetical subject (Aebli, 1987, Grisse mann & Weber, 2000). Coming from concrete manipulations with different objects (stage 1), the children have to abstract these manipulations and operations to pictorial representations (stage 2). Subsequently they pass over to symbols (stage 3) with the aim to automate their actions (stage 4). For us, stage 2 is of special importance, because there the process of internalization takes place. The child has to comprehend the manipulation of concrete objects as a representation of a quantitative structure and it has to capture the structure and the relations of the concrete manipulation in an intellectual activity (Gerster & Schultz, 2004, p. 47). Lorenz calls this process „focus of attention“. To facilitate this process of focus and abstraction and to develop it, a dialog is essential (Lorenz, 1997, p.93):

„In talking about the working material and the relations between numbers and operations that it represents, the concepts in development of the learner are going to be clarified by verbalisation.“

In this sense, Aebli (1987) suggests that the children should review their concrete manipulations and make forecasts about further actions. Doing this, they comment their own manipulations by iconic illustrations till they are able to reproduce the structures and relations of the manipulations in conceptions. To support this process Aebli (1987, p. 238) established the following rule:

„Every new, more symbolic representation of the operation must be linked as closely as possible with the precedent one.“

The enactive form of representation with finger symbol sets should be related to the nonverbal symbolical form of representation (MER [1]) (Ainsworth, 1995; Mayer, 2005). But as studies show some of the children even don't link the different forms

of representations when they are designed in form of MERs (Clements, 2002). For them, an automatic linking designed with the computer (MELRs [2]) can help them to experience the relations (Thompson, 1992; Clements, 2002; Ladel, 2009). This experience should be as natural and direct as possible. In this article we suggest to use multi-touch technology for this experience, where the children can manipulate with their hands and fingers and an automatic linking with all other forms of representation can take place. In the remainder of this article we assume the availability of a multi-touch-enabled table. Such a table consists of a display surface connected to a computer and some tracking hardware that can recognize several touches on the display simultaneously and report them to the computer software. Similar technology with a different form factor is available in desktop monitors, tablet computers and devices like the Apple iPad, or mobile phones. With the availability of hardware as already imagined by Kay (1972) we now have to answer the question of the educational implications more than ever.

The basic underlying idea for all the activities sketched only briefly in the following is that the computer can track the children's actions on the table and give nonverbal symbolic representations of either the current situation or the action that lead to it in form of a written protocol.

In a first scenario, the children represent numbers with their hands and fingers as described before. This enactive form of representation shall produce an iconic one on the display. The computer creates quadratic pads on the surface of the multi-touch table. Through the contact of the fingers with the multi-touch interface there is not only a link between the enactive form of representation with other forms of representation but also between the tactile and the visual sense. While representing numbers enactively and thus iconically, there is an automatic link to a nonverbal symbolic form of representation. This representation can be imagined like a paper tape or sales slip and serves as a kind of protocol for the manipulations the children do. Such a protocol can support the focus of attention and the numerical aspects of a task (Dörfler, 1986).

In this activity it is possible for children to experience that it is of no particular importance which fingers they use to present quantities. At a table, it is also possible that the children work in teams: Two children can "share the work" to present two fingers if each touches the table with one finger. While this sounds funny for the number two, it is of great importance for partitions of larger numbers. Two partners can try to find all ways to partition numbers up to 20 into two numbers up to 10.

Working in teams or groups the children are also able to present numbers greater than 10, emphasizing the social aspects of learning. Because the protocol immediately reflects the actions of the children their focus of attention is on the mathematical content of their actions automatically, guiding them to abstraction.



It is also possible to support the four basic arithmetic operations and their basic concepts in such an environment. Regarding addition, students can develop the basic concept of a union by manipulating the virtual objects (pads) and arrange them close to each other. For example, if the child merges a group of 3 pads and a group of 5 pads the protocol will show the symbolic representation of this action as „ $3 + 5 = 8$ “. Here the focus of attention lies on the fact that this action constitutes a basic concept of addition, together with its nonverbal symbolic form of representation. In multi-touch technology there is also the possibility to draw a circle around some pads with the effect that these pads are bundled (a so-called lasso-gesture). This again is a manipulation based on the basic concept of union. Another task in the realm of addition and subtraction may be that 3 pads are shown and the child should create so many pads that in the end there are 7 ( $3 + \_ = 7$ ).

The basic concept of balance can be represented as well. Children can create quantities, remove from them, manipulate them with their fingers, and see the consequences of the manipulation at the same time in the nonverbal symbolic protocol. Likewise it is possible to give instructions in the nonverbal symbolic form and to see the output in the iconic forms with the pads.

It is rather easy to imagine that addition and subtraction can be done in such an environment, and we have shown some ways how the action or the state can be linked to a nonverbal symbolic representation. For multiplication and division it is advisable to take advantage of the time as another dimension. The temporal-successive idea of multiplication that can be traced back to a repeated addition is mapped to a repeated touch action of the same quantity of fingers several times. The protocol may then show, for four touches with five fingers, „ $5 + 5 + 5 + 5 = 20$ “ as well as „ $4 \cdot 5 = 20$ “. Thus the children can see that there are different ways to protocol their manipulation. If several children are working together they can take advantage of the spatial-simultaneous idea of multiplication, creating the same quantity by several children at the same time. For division, one example activity would be to move pads and build piles of the same amount to divide a given number of pads.

## TECHNOLOGICAL IMPLEMENTATION

In order to implement prototypical environments and for recording experimental data of children's interaction with a multi-touch-enabled screen we used the interactive geometry software (IGS) Cinderella (Richter-Gebert & Kortenkamp, 2006), which acts as a standard tool for rapid prototyping of learning environments. The customization of the learning environments is done via the integrated scripting language CindyScript (Richter-Gebert & Kortenkamp, 2010). CindyScript is a functional programming language that was designed to match standard mathematical expressions as closely as possible, while still providing all the structural elements of imperative programming.

As CindyScript can be triggered by user actions (like pressing the mouse or a key, moving the mouse, or starting a simulation) it is possible to change the standard behaviour of an IGS into the required interaction for an experiment.

A striking example for such a change in user interface behaviour is the method of adding points in the DOPPELMOPPEL learning environment (Ladel & Kortenkamp, 2009). Here, instead of having a dedicated mode to create points, points can be created in drag mode by pulling them from a never-ending stock onto a virtual table, and they can be deleted by just moving them off the table. For finger symbol sets we adapt this technique: Pads can be created without referring to a stock pile, but just by placing fingers in an area next to the table. This allows for multiple pads to be created simultaneously, as is necessary for quasi-simultaneous representations of numbers.

This modeless operation of the learning environment (see Raskin (2000) for a discussion of modal operations in software) is necessary for any multi-touch environment: As one of the goals is the collaboration of several children, and the actions of the children cannot be differentiated, i.e., the computer cannot know which child is associated to which touch event, any mode would have to globally valid for all children at the same time. Switching to another mode (for example, switching between dragging pads and creating pads) would have to be announced and negotiated. Such negotiation would introduce too many obstacles in the user interaction and counters the collaborative advantages of multi-touch.

The latest version of Cinderella offers multi-touch support by adding the TUIO protocol for input events (Kortenkamp & Dohrmann, 2009). Currently, this support is restricted to allowing several elements to be dragged at the same time. Other modes, like the add-point mode or the add-line mode, are not multi-touch enabled. For the modeless operation, as pointed out above, we are using a helpful extension of the scripting facilities of Cinderella: Touch events (finger detected, finger moved, finger released) are translated into mouse events (mouse down, mouse move, mouse up). Using CindyScript, custom actions can be added to these touch events as it is possible with mouse events.

It is not straightforward to adapt a scripted interface to the fact that several press-drag-release sequences can happen simultaneously. It is customary to program user interfaces under the assumption that mouse events are exclusively delivered in the prescribed order of press, drag (possibly repeated), and release. This is relevant for example if a program assumes a “currently moved element”, like a currently moving point in an IGS. Designing software without that general assumption is much more difficult as it involves keeping track of all the current objects and states and their association to the touching fingers. CindyScript facilitates this design process by offering *touch-local variables*: Declaring a variable to be touch-local using the `mtlocal()`-function assures the availability of a different instance of that variable for each press-drag-release sequence of a finger. This is very similar to the concept

of switching contexts in recursive programs.

As an example, consider a program that will record the current mouse position in the mouse down event and then connect that position with the current mouse position in the drag event. A simple CindyScript implementation would be to place `start=mouse().xy;` in the mouse down event and `draw(start,mouse().xy);` in the drag event. Without declaring the `start` variable touch-local this would fail with multi-touch events, with the declaration `mtlocal(start);` it will work flawlessly with any number of simultaneous touches by recording the start position of each finger separately.

The exchange of global information (like the number of total touches) is easily possible by *not* declaring variables touch-local. Placing the commands `count=count+1;` and `count=count-1;` into the mouse down resp. mouse up events will keep track of the current number of fingers touching the surface.

We found the prototyping facilities of CindyScript with the touch extension to be very appropriate for our needs. The final behaviour of the learning environment is not yet determined and should be easily adaptable to empirical findings during the process of interface design. Also, any professional software programming services would need a full specification and, besides being too expensive in this early research stage, could not reflect the didactic considerations as described above.

## FORECAST

We are currently working on implementing the above scenarios using a multi-touch table built at CERMAT. A first study that examines the critical point in translating numbers and operations from and in different forms of representation has taken place in October 2010. At the same time we conducted a pre-study about the way children touch with their fingers and present quantities on a table. The data analysis is in progress.

Finally, we aim to answer the research question about the impact of the availability of such multi-touch learning environments regarding the diagnosis and the support of acquiring basic concepts of numbers and operations.

## NOTES

1. MER: multiple external representations (Ainsworth, 1999)
2. MELRs: multiple equivalent linked representations (Harrop, 1999)

## REFERENCES

- Aebli, H. (1987). *Zwölf Grundformen des Lehrens*. Stuttgart: Klett-Cotta.
- Ainsworth, S. (1995). *The role of multiple representations in promoting understanding of estimation: An empirical investigation*. Technical report number 30, ESRC Centre for Research in Development, Instruction and Training.

University of Nottingham.

- Ainsworth, S. (1999). *Designing effective multi-representational learning environments*. Technical report number 58, ESRC Centre for Research in Development, Instruction and Training. University of Nottingham.
- Brissiaud, R. (1992). A Toll for Number Construction: Finger Symbol Sets. In J. Bideaud, C. Meljac, & J.-P. Fischer (Eds.) (1992). *Pathways to number. Children's Developing Numerical Abilities* (pp. 41-65). New Jersey: Lawrence Erlbaum Associates.
- Clements, D. H. (2002). Computers in Early Childhood Mathematics. *Contemporary Issues in Early Childhood*, 3(2), 160-181.
- Dörfler, W. (1986). Zur Entwicklung mathematischer Operationen und gegenständlicher Handlungen. In *Beiträge zum Mathematikunterricht 1986* (pp. 88-91). Bad Salzdetfurth: Franzbecker.
- Fuson, K. C. (1988). *Children's Counting and Concepts of Number*. New York: Springer.
- Fuson, K. C. (1992). Research on Learning and Teaching Addition and Subtraction of Whole Numbers. In G. Leinhardt, R. Putnam, & R. A. Hattup (Eds.). *Analysis of arithmetic for mathematics teaching* (pp. 53-187). Hillsdale: Lawrence Erlbaum Associates.
- Gelman, R. & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.
- Gerster, H.-D., & Schultz, R. (2004). *Schwierigkeiten beim Erwerb mathematischer Konzepte im Anfangsunterricht*. Bericht zum Forschungsprojekt Rechenschwäche – Erkennen, Beheben, Vorbeugen. Freiburg im Breisgau.
- Grissemann, H., & Weber, A. (2000). *Grundlagen und Praxis der Dyskalkulietherapie*. Bern: Hans Huber.
- Harrop, A. (1999). ENCAL: a prototype computer-based learning environment for teaching calculator representations. Extended abstract. *Psychology of Programming (PPIG), 11<sup>th</sup> Annual Workshop*, January 5-7, 1999, Leeds. Retrieved from <http://www.ppig.org/papers/11th-harrop.pdf>
- Krauthausen, G. (1995). Die „Kraft der Fünf“ und das denkende Rechnen. In G. Müller, & E. C. Wittmann (Eds.). *Mit Kindern rechnen. Arbeitskreis Grundschule – Der Grundschulverband – e.V* (pp. 87-108). Frankfurt am Main.
- Kay, A. (1972). A Personal Computer for Children of All Ages. In: *Proceedings of the ACM National Conference*, Boston. Retrieved from <http://www.mprove.de/diplom/gui/Kay72a.pdf>
- Kortenkamp, U., & Dohrmann, C. (2009). User interface design for dynamic

geometry software. *Acta Didactica Napocensia*, 3(2), 59-66.

- Ladel, S. (2009). *Multiple externe Repräsentationen (MERs) und deren Verknüpfung durch Computereinsatz. Zur Bedeutung für das Mathematiklernen im Anfangsunterricht*. Didaktik in Forschung und Praxis, Bd. 48. Verlag Dr. Kovac, Hamburg.
- Ladel, S., & Kortenkamp, U. (2009). Realisations of MERS (multiple external representations) and MELRS (multiple equivalent linked representations) in elementary mathematics software. In V. Durand-Guerrier *et al.* (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1050-1059), Jan. 28 – Feb. 1, 2009, Lyon (France).
- Lorenz, J.-H. (1997). Der gescheiterte Rechenunterricht: Rechenversagen – Ursachen und “Therapie”. In W. Stark, T. Fitzner, & Ch. Schubert (Eds.) *Grundbildung für alle in Schule und Erwachsenenbildung* (pp. 90-97). Stuttgart: Klett.
- Mayer, R. (Ed.) (2005). *The Cambridge Handbook of Multimedia Learning*. Cambridge University Press. New York.
- Raskin, J. (2000). *The human interface: new directions for designing interactive systems*. New York: ACM Press/Addison-Wesley Publishing Co.
- Resnick, L. B., Bill, V., Lesgold, S., & Leer, M. (1991). Thinking in arithmetic class. In B. Means, C. Chelemer, & M.S. Knapp (Eds.). *Teaching advanced skills to at-risk students: Views from research and practice* (pp. 27-53). San Francisco: Jossey-Bass.
- Richter-Gebert, J. & Kortenkamp, U. (2006). *The Interactive Geometry Software Cinderella*, Version 2.1, URL: <http://cinderella.de>.
- Richter-Gebert, J. & Kortenkamp, U. (2010). The power of scripting: DGS meets programming. *Acta Didactica Napocensia*, 3(2), 67-78.
- Steinweg, A.-S. (2009). Rechnest du noch mit Fingern? – Aber sicher! In *MNU Primar*, 4(1), 124-128.
- Thompson, P. W. (1992). Notations, conventions and constraints: contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education*, 23(2), 123-147.

# **RESEARCHING TECHNOLOGICAL, PEDAGOGICAL AND MATHEMATICAL UNDERGRADUATE PRIMARY TEACHERS' KNOWLEDGE (TPACK)**

Spyros Doukakis, Maria Chionidou-Moskofoglou, Dimitrios Zibidis

University of the Aegean, Rhodes, Greece

*Twenty-five final-year undergraduate primary education students, who were attending a six month course on mathematics education, participated in a research project during the 2009 spring semester. The course, based on the Technological, Pedagogical and Content Knowledge framework and design experiment procedure, was organised so as to incorporate educational software and educational mathematical scenarios in teaching approaches created by undergraduate students. This article presents the course design, the research procedure and some research results concerning the integration of educational software and mathematical scenario in students' teaching approaches.*

**Keywords:** Mathematics, TPACK, Undergraduate Primary Education students

## **INTRODUCTION**

Over the past few decades, one of the most important issues related to educational change and educational innovation is the integration of Information and Communication Technologies (ICT) (Hoyles, Noss, & Kent, 2004). ICT constitute an essential tool for teachers, since it can be used as: a) an educational method to support student learning; b) as a personal tool to prepare material for his/her lessons, to manage a variety of projects electronically and to search for information; c) as a tool to collaborate with other teachers or colleagues (Da Ponte, Oliveira, & Varandas, 2002). The 2003 reformed Greek National Curriculum in Mathematics has been implemented in the nine-year compulsory education since 2006, as 'Cross Curricular/Thematic Framework (CCTF)'. One of its general principles is "to prepare pupils to explore new information and communication technologies (ICT)" (Official Government Gazette, 2003, p.1). The Pedagogical Institute (Ministry of Education) has developed a compulsory national mathematics textbook for each school year, which is accompanied by national educational software. This is the case for all teaching subjects in the nine-year compulsory education. Despite significant political will and spending by governments on technical equipment and teacher training, ICT integration in schools is often low (Jimoyiannis & Komis, 2007).

Therefore, from a constructivist viewpoint (von Glasersfeld, 1995; Cobb, Stephan, McClain, & Gravemeijer, 2001), integration of educational software into undergraduate students' teaching practice is a crucial factor for teachers' future 'establishment' and improvement in classroom practices. During the 2008-2009 spring semester, a six-month course on primary maths teaching during practicum (school attachment) was organised by the researchers with the aim of integrating ICT

and especially-designed mathematical scenarios (Kynigos, 2006) in students' teaching approaches.

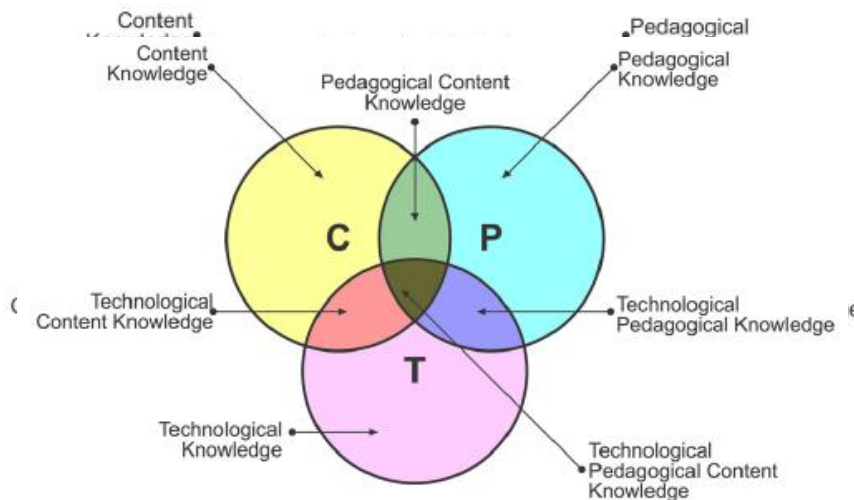
## LITERATURE REVIEW

Cobb *et al.* (2001) explain - according to a constructivism theory - how learners make sense of their environments and experiences to create their own knowledge, while Schoenfeld (1998) argues that whenever the student is actively involved in an activity then s/he is more likely to learn its content. However, this process requires teachers to pose meaningful and worthwhile tasks to facilitate students' learning. Nowadays, research in educational technology suggests the need for Technological Pedagogical Content Knowledge (TPCK or TPACK) so as to incorporate technology in pedagogy (Mishra & Koehler, 2006; Angeli & Valanides, 2009). TPACK is based on Shulman's (1986) idea of 'pedagogical content knowledge', which is related with the Knowledge Quarter (Rowland, Turner, Thwaites, & Huckstep, 2009). This interconnectedness among content, pedagogy and technology has important effects on learning as well as on professional development.

Mishra and Koehler (2006, p. 1020) suggest that

“...a curricular system that would honour the complex, multi-dimensional relationships by treating all three components in an epistemologically and conceptually integrated manner”

and they propose an approach which is called 'learning technology by design'.



**Figure 1: Diagram representing the overlapping components of technological pedagogical content knowledge (Source: Mishra & Koehler, 2006, p. 1025)**

They propose a model suggesting three unitary components of knowledge (content, pedagogy and technology), three dyadic components of knowledge (pedagogical content, technological content, technological pedagogical) and one overarching triad of knowledge (technological pedagogical content). Therefore, Pedagogical Content

Knowledge (PCK) is the knowledge of pedagogy that is applicable to the teaching of a specific content (Mathematics). Technological Content Knowledge (TCK) is the understanding of how technology and content both aid and limit each other. Technological Pedagogical Knowledge (TPK) is the understanding of how teaching and learning changes when particular technologies are used. The authors have represented TPACK through the use of a Venn diagram (Fig. 1), where the individual circles represent the knowledge components of content (C), pedagogy (P), and technology (T) and the overlapping area of all three circles represents TPACK.

During the last two decades many research projects have made significant contributions to the teaching and learning mathematics. For example, researchers claim that when students are working with ICT, they are more able to focus on patterns, connections between multiple representations etc. (Laborde, 2002), but the integration of ICT progresses slowly in everyday school practices (Artigue, 1998; Laborde, 2002). In Greece, where technological tools are used, they are often used by teachers in whole class teaching rather than by the students themselves (Jimoyiannis & Komis, 2007).

Concerning TPACK in mathematics classrooms, research projects have already been done, exploring a) teachers' development model on TPACK (Niess, 2005), b) pre-service teachers' TPACK development (Cavin, 2007) and c) the use of TPACK to teach probability topics and data analysis (Lee & Hollebrands, 2008).

## RESEARCH METHODS

In order to explore the development of TPACK, we have employed design experiments which constitute an effective methodology for studying teacher development in the setting of an education university department (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The researchers have taken a triangulation multiple-method approach (qualitative and quantitative) to ensure greater validity and reliability.

The participants were 25 final-year undergraduate primary teachers (16 females and 9 males) in the Department of Primary Education at the University of the Aegean, who were attending the compulsory course 'Teaching Mathematics - Practicum Phase' during the 2008-2009 spring semester. The first two authors used to have a three-hour meeting with the participants in mathematics lab, twice a week. The lab held twelve PCs, with Windows XP, MS Office 2003, internet access, mathematical software (Educational Software of Pedagogical Institute for Mathematics (ESPIM), Geometer's Sketchpad) and presentation tools. The need for a technologically elaborate working environment that would encourage students to use technology led the research team to use many technological tools (the author's website, the course's electronic mail, Moodle as the course and learning management system, a forum, a blog and mobile SMS).

The research work was developed into five stages:



1. During the first stage and before the beginning of the first lesson, quantitative data regarding undergraduate primary teachers a) background (studies etc.); b) individual learning style according to index of learning styles instrument (Felder & Silverman, 1988); c) attitudes towards ICT, based on Greek computer attitudes scale (GCAS) (Roussos, 2007); d) self-efficacy in ICT according to Greek computer self-efficacy scale (GCSES) (Kassotaki & Roussos, 2006); e) attitudes towards ES - for this purpose, we have designed an educational software attitudes scale (ESAS) based on Roussos' GCAS; f) self-efficacy in mathematics according to the content principles of the CCTF were gathered (GMSES). The same data were gathered from the participants at two more instances (after three months and at the end of the semester), in order to measure possible quantitative differences.

2. Cobb *et al.* (2003) experiment design procedure constituted the second stage. In particular: a) The participants were given a suitable student's worksheet and they worked on geometry tasks about square, rectangle, polygons, cube and parallelepiped (area, perimeter, volume, edge etc). b) After or before their paper and pencil work, they tried to work the same tasks by using the national ESPIM. Each lesson consisted in the teaching of those strategies that incorporate the usage of ICT, so as to involve undergraduate primary teachers in the investigation of geometrical shapes and forms. Teaching was limited to the investigation of geometry problems so that when the teachers come up with their own teaching scenarios (Kynigos, 2006) they will be able to use suitable technological tools that are both efficient and investigatory. The microworlds used were: geo-board, 3D solid manipulation (solid-board), calculator and table tracking from the ESPIM. c) In each lesson, researchers used technological tools while the teachers participated as students taking a lesson in class. d) At the end of each lesson, the teachers were asked to fill out an electronic feedback form, contributing thus further to a discussion of the three-hour lesson that had just finished. The form focused on the development of TPACK in mathematics, with questions on technological tools, teaching strategies and benefits gained from the lesson. This procedure was repeated eight times during the spring semester 2008/2009. For example, one of the worksheets proposed the following task to the students:

“An a-edge cube is transformed to another one with n-times the edge. What happens with the volume of the new cube?”

While the students worked on this worksheet it turned out that some of them had misunderstood the concept of the volume, so according to Cobb *et al.* (2003) procedure, an alternative design worksheet was given to the students to work on it and overcome this misunderstanding. A circled procedure like the latter one was followed by the researchers when it emerged from students needs.

3. The teachers had to write a first assignment that consisted in the search for all geometry problems, activities and exercises involving geometrical shapes and solids in the national math textbooks for the grades 5 and 6, as well as the grades 7, 8 and

9. They also had to work on two activities, two exercises and two problems of their choice (from the above units) using ESPIM. Furthermore, they were asked to create a lesson plan spontaneously for teaching the "area of a parallelogram" chapter or "the volume of a parallelepiped" for grade 6 mathematics.

4. The teachers had to be taught the notion of the 'educational scenario (ES)' so they were asked to participate and act as students in an educational scenario created by the research group for the purposes of the lesson. The title of the scenario was 'Creating Mobile Phone Networks' and it constituted a holistic picture of a learning environment, without limitations but with the ability to focus on those aspects that the educator judged to be of importance (Kynigos, 2006). Then the teachers were asked to create their own ES, to be used with the chapter of the lesson plan they had already created. Therefore, with the theoretical and practical knowledge and the experience gained, the teachers produced their own ES over the following two weeks. Each ES was presented to their peers, who acted as students of a class. The latter provided feedback and assessed the ES on an especially designed form by the researchers. After that, the teacher, creator of the ES, having taken his/her peers' comments into consideration, returned two weeks later and presented his/her improved ES version. Security and originality were safeguarded as all ESs had been posted before the beginning of the presentations. ES presentations were audio recorded on a digital camera so they could be further analysed. Finally, the teachers were self-assessed and gave feedback on their own ES. In their fourth assignment (assessment) students had to create an ES to be used with the chapter of the lesson plan they had already created for grade 8 students.

5. During the above process, semi-structured interviews were conducted very frequently. The initial students' interview took place after the submission of the first assignment and the final interview was conducted after the completion of the second presentation of the ES. The purpose of these interviews was twofold; on the one hand, to investigate procedures followed by the teachers during the writing up of their first assignment and their ES, their perceptions of TPACK in math and the reasons for their inclusion or non-inclusion of ICT in the lesson plan. On the other hand, the purpose of the interview was to determine whether or not this constructivist design experiment procedure was personally suitable for them. Interviews were recorded for further analysis.

6. During the last meeting, the teachers were asked to anonymously complete a questionnaire regarding their satisfaction with the course. Twenty-four completed questionnaires were returned out of the twenty-five that were handed out.

7. Finally, we evaluated the teachers in "paper and pencil" and ESPIM work.

During the next school year (2009-2010), 11 out of 25 participants were hired as primary education teachers. Having completed their first year of teaching, we asked

them to fill out an online questionnaire to investigate if and how they integrate ESPIM and ES in their teaching.

In the next section, the results from a) the analysis of quantitative data on students' computer attitudes, self-efficacy in ICT, attitudes toward educational software, and self-efficacy in math, and b) the analysis of quantitative data on course satisfaction of participants will be presented.

### **SOME RESEARCH FINDINGS**

In order to analyse the quantitative data gathered about course satisfaction of participants we applied a set of powerful ordinal regression methods. The most important results focus on the determination of the course weak and strong points, according to the MUSA methodology (Grigoroudis, & Siskos, 2002). The teachers' global satisfaction with the course was characterized as extremely high. The mean satisfaction value, as measured by the method, reached 98%, while it is of great importance to note that all comments were positive. The teachers also appeared satisfied in the partial (per criterion) satisfaction survey (Table 1), where negative comments were sparse.

Criteria	Satisfaction
Educational Program	90.02
Professor	97.10
PhD Researcher	92.05
Mathematics Lab	97.00
Educational Material	97.09

**Table 1: Satisfaction per criterion**

The research results from the study of the teachers': a) attitudes towards ICT (GCAS), b) self-efficacy towards ICT (GCSES), and c) self-efficacy towards mathematics (MSES) are the following:

a) The 30 items of GCAS (Roussos, 2007) were summed to provide a total score (from 30 to 150) representing the participants' overall attitude toward computers. Descriptive statistics of the first and last GCAS scores are reported in Table 2. The results show an improvement of teachers' attitudes toward ICT, which was not statistically significant [ $F(1.4, 32.28)=2.28, p=.13$ ].

b) The GCSES (Kassotaki & Roussos, 2006) scores represent the participants' self-efficacy toward ICT (scores ranged from 29 to 145). The results (Table 2) again a statistically non-significant improvement [ $F(1.57, 36.26)=1.43, p=.25$ ].

c) Finally, in order to explore the teachers' self-efficacy toward math (GMSES), we used the 7 content principles of the CCTF (problem solving, numbers and operations, measurement and geometry, gathering and processing data, statistics, ratios and

proportions and equations). The GMSES provided a total score representing the participants' self-efficacy toward math (scores ranged from 7 to 35). The results (Table 2) show that the teachers' self-efficacy towards math improved significantly during the semester [ $F(1.58, 36.44)=3.98, p=.036$ ]. Post hoc comparisons using t-tests with Bonferroni correction demonstrated a statistically significant difference between the first and the second measurement stages ( $p=.021$ ).

GCAS	Measurement Stages	Mean	SD	N
	1	103,08	20,61	24
	3	109,25	18,60	24
GCSSES		Mean	SD	N
	1	109,17	24,12	24
	3	113,46	20,74	24
GMSES		Mean	SD	N
	1	22,58	5,70	24
	3	24,21	5,32	24

**Table 2: Means and standard deviations of the four scales for the two measurement stages (beginning and end of semester)**

## DISCUSSION AND CONCLUSIONS

Regarding to our final-year undergraduate primary education students' attitudes and self-efficacy towards ICT, it seems that the participants had already acquired the necessary knowledge of ICT usage before entering university or during their university studies and they were comfortable with its use, as the GCAS and GCSSES means from the research were consistent with Roussos (2007) and Kassotaki & Roussos (2006) research findings. Additionally, these findings were consistent with the Bahr, Shaha, Farnsworth, Lewis, and Benson (2004) results, who reported that pre-service teachers had positive attitudes towards technology and technology integration. Moreover, it seems that the participants of the present study had already reached high level knowledge of technology (TK). These findings are also consistent with the Wentworth, Earle, and Connell (2004) results. The positive attitudes towards ICT and ES had a positive impact on the university faculty who organise educational technology courses (Jimoyiannis & Komis, 2007). Moreover, it seems that the course experiment design and the involvement of undergraduate primary teachers with educational software for math improved their self-efficacy towards math. Also, undergraduate primary teachers improved their mathematical content knowledge. It seems, therefore, that the teachers' attitudes and self-efficacy constitute a force that needs strengthening if ICT is to be incorporated into their teaching practices.

The extremely high results on students' satisfaction lead us to posing of new research questions. The high satisfaction level might be attributed to: a) The small number of undergraduate primary teachers-participants (Krentler & Grudnitski, 2004); b) The support the teachers received during the entire course via blog, forum, website and e-mail services. It is worth mentioning that the professor and the PhD researcher gave responses at the latest within the next day to the 400 e-mails received during the course. Furthermore, the forum received 140 messages (not counting those sent by the professor and the PhD researcher); c) The everyday communication between the teachers and two individuals (the professor and the PhD researcher); d) The possibility of 'self-defensiveness' on the part of the participants might have resulted in inaccurate responses since this was their first time to participate in a satisfaction research study.

It is our belief, therefore, that undergraduate primary teachers' satisfaction in a learning environment that combines teaching in the university classroom and support via an appropriate learning environment plays a crucial role in the sustenance of programmes that incorporate ICT in teaching and learning. Additionally, the correlation between satisfaction and undergraduate primary teachers' characteristics (learning style, attitude towards ICT and self-efficacy in the use of ICT) constitutes a crucial parameter in the improvement of the education provided. On the other hand, teachers' characteristics, their method of making undergraduate primary teacher contact and their teaching style seem to affect the teachers' satisfaction. The above mentioned findings reveal that each new educational establishment needs to adopt an evaluation programme for its provided services, in order to obtain, amongst others, the necessary data on undergraduate primary teachers' satisfaction about the course services (Elliott & Shin, 2002) so that a circled process will take place for the new course improvement.

In addition, it seemed that the crucial factors for the integration of educational software and scenarios into the teaching of mathematics are the students' positive attitudes towards ICT and educational software and the self-efficacy in technological tools and math. Further analysis of qualitative data (interviews, narrative observations and essays) concerning these quantitative research findings and also the students' scenarios structures, is currently under way so that these triangulation research methods will deeper our understanding of primary teacher training on Technological, Pedagogical and Content Knowledge in Mathematics education.

## REFERENCES

- Angeli, C., & Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development, and assessment of ICT-TPCK: Advances in technological pedagogical content knowledge (TPCK). *Computers & Education*, 52(1), 154-168.

- Artigue, M. (1998). Teacher training as a key issue for the integration of computer technologies. In J. D. Tinsley & D. C. Johnston (Eds.), *Proceedings of the IFIP TC3/WG3. 1 Working Conference on Secondary School* (pp. 121-130), London, UK: Chapman & Hall, Ltd.
- Bahr, D. L., Shaha, S. H., Farnsworth, B. J., Lewis, V. K., & Benson, L. F. (2004). Preparing tomorrow's teachers to use technology: attitudinal impacts of technology supported field experience on pre-service teacher candidates, *Journal of Instructional Psychology*, 31(2), 88-97.
- Cavin, R. M. (2007). *Developing technological pedagogical content knowledge in preservice teachers through microteaching lesson study*. Doctoral Dissertation, Florida State University, Tallahassee, FL, USA. Retrieved from <http://www.chipola.edu/instruct/math/cavin>.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9-13.
- Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices, *Journal of the Learning Sciences*, 10(1), 113-164.
- Da Ponte, J. P., Oliveira, H., & Varandas, J. M. (2002). Development of Pre-Service Mathematics Teachers' Professional Knowledge and Identity in Working with Information and Communication Technology, *Journal of Mathematics Teacher Education*, 5(2), 93-115.
- Elliot, K. M., & Shin, D. (2002). Student satisfaction: an alternative approach to assessing this important concept. *Journal of Higher Education Policy and Management*, 24(2), 197-209.
- Felder, R. M., & Silverman, L. K. (1988). Learning and teaching styles in engineering education. *Engineering Education*, 78(7), 674-681.
- Grigoroudis, E., & Siskos, Y. (2002). Preference disaggregation for measuring and analysing customer satisfaction: the MUSA method. *European Journal of Operational Research*, 143(1), 148-170.
- Hoyles, C., Noss, R. & Kent, P. (2004). On the integration of digital technologies into mathematics classrooms, *International Journal of Computers for Mathematical Learning*, 9(3), 309-326.
- Jimoyiannis A., & Komis V. (2007). Examining teachers' beliefs about ICT in education: implications of a teacher preparation program. *Teacher Development*, 11(2), 181-204.
- Kassotaki, S., & Roussos, P. (2006). The Greek Computer Self-Efficacy Scale. In D. Psillos & V. Dagdillelis (Eds.), *Information Communication Technologies in Education, Proceedings of the 5th Panhellenic Conference* (pp. 726-733), 5-8 October, Aristotle University of Macedonia, Thessaloniki, Greece (in Greek).

- Krentler, K. A., & Grudnitski, G. M. (2004). Moving beyond satisfaction: perceived learning as an assessment measure. *Journal of College Teaching & Learning*, 1(10), 7-16.
- Kynigos, C. (2006). *The lesson of exploration*, Ellinika Grammata, Athens (in Greek).
- Laborde, C. (2002). Integration of Technology in the Design of Geometry Tasks with Cabri-Geometry, *International Journal of Computers for Mathematical Learning*, 6(3), 283-317.
- Lee, H. S., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing TPACK. *Contemporary Issues in Technology and Teacher Education* 8(4), 326-341.
- Mishra, P., & Koehler, M. J. (2006). Technological Pedagogical Content Knowledge: A new framework for teacher knowledge, *Teachers College Record*, 108(6), 1017-1054.
- Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: A focus on pedagogical content knowledge. *Teaching and Teacher Education*, 21(5), 509-523.
- Official Government Gazette (2003). *Cross-thematic Curriculum for Compulsory Education*, Pedagogic Institute, Article No 210, 303, Issue B' (in Greek).
- Roussos, P. (2007). The Greek computer attitudes scale: Construction and assessment of psychometric properties. *Computers in Human Behavior*, 23(1), 578-590.
- Rowland, T., Turner, F., Thwaites, A., & Huckstep, P. (2009). *Developing primary mathematics teaching*, London, UK: Sage Publications Ltd.
- Schoenfeld, A. H. (1998). Toward a theory of teaching-in-context, *Issues in Education*, 4(1), 1-94.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- von Glasersfeld E. (1995). A Constructivist Approach to Teaching. In L. P. Steffe (Ed.), *Constructivism in Education* (pp. 3-15). Hillsdale, NJ: Erlbaum.
- Wentworth, N., Earle, R. and., & Connell, M. (2004). *Integrating information technology into the teacher education curriculum*, New York: The Haworth Press Inc.

# THE CO-CONSTRUCTION OF A MATHEMATICAL AND A DIDACTICAL INSTRUMENT

Mariam Haspekian

Laboratoire Didactique André Revuz, Paris, France

*Spreadsheet is not given as didactical tool to serve mathematics education. It may progressively become such an instrument along a professional genesis of use on teachers' side. Using both the notions of distance and double genesis, the case study described in this paper illustrates the beginning of such a genesis and the complexity that comes along with it.*

Keywords: ICT, instrumentation, double (professional / personal) genesis, instrumental distance.

## INTRODUCTION AND DATA PRESENTATION

For several years now, research communities have paid an increasing attention to technology use in math education (see Drijvers *et al.*, 2010 for a historical overview). Theoretical frames as the instrumental approach (Artigue, 2002; Guin, Ruthven & Trouche 2004) have been developed around the concept of *instrumental genesis* focusing on ICT impact on pupils' work. This paper moves from this focus towards the impact of technology on *teachers'* practices. It focuses on the case of the spreadsheet which is not given as a didactical professional tool but that may progressively become such a didactical instrument through a professional genesis on teachers' side. In order to study these geneses, I introduced two theoretical constructs within the frame of Instrumental Approach. The paper illustrates a case study on how these two concepts are useful in analyzing the teacher's geneses with the spreadsheet and in describing the complexity of these geneses. The data on which we are illustrating the concepts are based on observations (on two consecutive years) of an experienced teacher, named Dan in the following, integrating spreadsheet for the first time in her classroom. The observation of Dan's spreadsheet integration shows some evolutions from a year to the next.

### Few words about Dan

Dan is an ordinary teacher, having more than 10 years of experience, also involved in teacher training and having integrated dynamic geometry software, but spreadsheet is a new tool for her. During the first year, Dan was motivated by her participation in a research project focusing on spreadsheet use for *algebra* learning (Haspekian, 2005a). At the end of the research, an interview collected her thoughts and feelings about this experience. The following year, she used the spreadsheet by her own choice, without any research protocol. On that occasion, we recorded her first spreadsheet session and the following session in a paper-pencil environment. Some phenomena during this observation and the way Dan evolved in her practice with



spreadsheet as a didactical tool provide interesting data. Let us first present the evolutions at stake and then describe the theoretical frames to analyse these data.

### **Dan, year 2, or “the second trial of using spreadsheet in mathematics lessons”**

During the second year, Dan introduced spreadsheet not with algebra but with statistics (headcounts, frequencies and cumulative frequencies after having seen these notions in paper-pencil). In this context, some of the observed elements are surprising: the lesson, showing very little statistics, is mostly centred on the tool use and functionalities, and reveals unexpected mathematics (notions of variable, formula, distinction between “numeric/algebraic” function...). The latter reflect the influence of the year 1 experience, centred on algebra, but this does not explain completely the evolution year 2 (variations and regularities) summarized in Table 1:

Use of spreadsheet	Year 1	Year 2
<b>VARIATIONS</b>		
Class level	7 <sup>th</sup> Grade (12 year old)	8 <sup>th</sup> Grade (13 year old)
Old/new content	New	Old
Mathematical Domain	Algebra	Statistics
Spreadsheet location	Limited to computer lab	Computer lab +ordinary classroom
Synthesis	No	Yes
Interactions Teacher-Students	Mostly individual	Individual and collective
Use of the video and collective presentation	Piloted by teacher, limited role	Teacher and student. Important role
Students Configuration	Work by pairs	Work by pairs + collective work: one student at the board
<b>REGULARITES</b>		
Maths objectives, teacher aims	Algebra	
Additional material	Worksheet for pupils and pre-organised spreadsheet file	
Institutionalisation	In an ulterior lesson, in ordinary classroom	

**Table 1: Comparison Year 1- Year 2**

How can we explain these observations, the variations and the emerging regularities?

### **THEORETICAL FRAMES TO UNDERSTAND THE OBSERVATIONS**

To understand teacher’s practices, the “didactic and ergonomic approach” developed by Robert and Rogalski (2002) describes teacher’s activity through 5 components: personal, mediative and cognitive dimensions, as well as institutional and social constraints. The *cognitive* and *mediative* components relate to the choices made by the teacher in the spatial, temporal and mathematical organisation of the lessons. For Robert and Rogalski (2002) teachers are not totally free in these choices; they are more or less constrained by *personal*, *institutional* and *social* dimensions: the personal component relates to the teacher as a singular subject, with his own history, practices, vision of mathematics learning. The institutional and social dimensions relate to curricula, lessons duration, school social habits, math teachers habits etc.

This first frame is used at a global level. In the case of ICT practices, instrumental aspects seem to interfere with each of these components. This leads to the use of the instrumental approach in order to analyse at a more local level some of the phenomena observed during the year 2.

### **What are Dan's evolutions through these 5 components?**

Table 1 shows an evolution of the mediative and cognitive components (math domain chosen, way of introducing spreadsheet, class level, etc.). This indicates (and it is confirmed by the phenomena observed during the lesson) that Dan's personal component evolved too. How can we explain this evolution? Our analyses lead us to consider two main results: first, a phenomenon of *double instrumental genesis* takes place in the evolution of Dan's personal component, and second, all changes operated by Dan in her mediative and cognitive choices go towards a *reduction of the instrumental distance*. Let us precise more these theoretical issues.

The idea of *distance* has been introduced (Haspekian, 2005a) to take into account, beyond the "computer transposition" (Balacheff, 1994), the set of changes (cultural, epistemological, institutional) introduced by the use of a specific tool in math "praxis". For a given tool, a too big distance to the "current school habits" is a constraint on its integration (Haspekian, 2005b). On the other hand, didactical potentialities of technology rely on the distance it introduces as regards to paper-pencil (providing new representations, new problems...). In (Haspekian, 2005a) we have brought out 4 types of elements that can generate some distance. Some are directly linked to the *computer transposition*, as the representations and the associated symbolism. They can also be of an *institutional nature* [1], *didactical nature* (vocabulary, field of problems they allow to solve...), or *epistemological one* (what gives the tool an epistemological legitimacy). This is linked to teacher's personal component (her representations of math, of math teaching, of the role this tool plays in the development of math).

Then, the way teachers orchestrate and support pupils' instrumental geneses evolves year after year. Considering spreadsheet as an instrument for the teacher, allowing her to achieve some teaching goals, we consider a process of instrumental genesis *on teacher's side*. The same artefact, the spreadsheet, becomes an instrument for pupils' mathematical activity and an (other) instrument for teacher's didactical activity.

In Dan's case, this process is even more complicated since it is split in two. There is a *double instrumental genesis* because Dan is not developing one but two instruments from the artefact spreadsheet. The personal instrumental genesis leads to the construction and appropriation of a tool into an instrument for math work, which differs from the *professional instrumental genesis* that leads to the construction and the appropriation of the previous instrument into a didactical instrument for math teaching. The didactical functionalities of this tool are not pre-defined, the teacher must develop and integrate them in her usual teaching practices and habits. My

hypothesis is that these two processes are not independent for Dan as they happened simultaneously in her case. Neither are they independent of pupils' geneses as we will see. Let us first describe more precisely the *professional genesis*:

Applying the instrumental approach to the spreadsheet seen as a *teaching* instrument that the teacher builds along a professional genesis, we can bring out two processes:

- An instrumentalization process: the tool is instrumentalized by the teacher in order to serve her didactic objectives. It is distorted from its initial functions and its didactical potentialities are progressively created (or “discovered” and appropriated in the case of an educational tool);
- An instrumentation process: a teacher will have to incorporate in her teaching schemes that were relatively stable some new ones integrating the tool use. She will progressively specify spreadsheet use to a particular class of situations (as “take advantage of spreadsheet for algebra learning”) and organise her activity in a way that will become progressively invariant for this class of situation (the Dan's case already shows some regularities from year 1 to year 2).

Dan builds up *schemes of instrumented action* [2] aiming at using spreadsheet to teach algebraic concepts (e.g., variables, formulae through the use of the copy, or by taking benefits of the numerical feedback to infer the equivalence of two formulae). This brings into play some usage schemes concerning material aspects, as the tool integration in a larger set of instruments (with the video projector), the organisation of the lessons, that are schemes that will undertake the modes of exploitation and the orchestrations, for instance, using a video projector at the beginning of the session for collective explanations, making pupils communicate and work by pairs, giving a sheet of instructions and a pre-built computer file to gain time, but also regularly “clicking on cell to check whether the pupils have edited a formula or numerical operation, or even directly the numerical result...).

I said this professional genesis was not independent of Dan's personal genesis, but interfered one on the other [3]. I also said that the professional genesis is made more complex by the fact that Dan wants her pupils to manipulate spreadsheet too (one could imagine a spreadsheet usage only under the teacher's control) and learn math through this activity. Thus pupils' instrumental geneses are part of the teacher's instrumental genesis. Here again, the two phenomena are imbricate and interfering. Some of our teacher's activity observations during the year 2 result from these interferences. We will show an example in the next section.

## **TEACHER'S DOUBLE GENESIS INTERFERING WITH PUPILS' INSTRUMENTAL GENESES**

As we mentioned, Dan has inscribed the introduction of spreadsheet in her class within the domain of statistics. Fig.1 is an excerpt of a pupil's exercise that shows the corresponding spreadsheet file with the pre-edited formula built by Dan:

## Working Group 15

### Step 2 : usage of formulae et the «handle of recopy».

distance (km)	$0 < d \leq 5$	$5 < d \leq 10$	$10 < d \leq 15$	$15 < d \leq 20$	total
headcounts	16	14	12	8	50
frequencies (%)	32				100

- 1) a) What is the total number of items? \_\_\_\_\_  
 Where is this number located? What is the formula to calculate it? \_\_\_\_\_  
 b) If one changes the headcounts for  $0 < d \leq 5$ , does the frequency change?

	A	B	C	D	E	F
1						
2						
3	<b>Etape 2 :</b>	<b>calcul de fréquences</b>				
4						
5	distance (km)	$0 < d \leq 5$	$5 < d \leq 10$	$10 < d \leq 15$	$15 < d \leq 20$	total
6	effectif	16	14	12	8	<b>=SOMME(B6:E6)</b>
7	fréquence (%)	<b>=B6/\$F\$6*100</b>				100

**Figure 1: Dan's final version of formulae**

It is interesting to notice that this file has been modified three times by Dan. In its first version, the formula calculating the frequency (in B7) was: **=B6\*100/50**. This formula, if copied along line 7, calculates the correct frequencies for the corresponding data of line 6. But it is not adequate regarding the question b) [4].

The day before the lesson, Dan realised the mistake and changed the formula into: **=B6/F6\*100**. She confided she did not feel yet totally comfortable with spreadsheets. If her own instrumental genesis with spreadsheet as a mathematical instrument probably plays a role here, we also see that the key point of the problem comes from the spreadsheet as a *didactic-oriented* instrument. From the spreadsheet as a “calculus-oriented instrument” point of view, the formula was adequate. It is the didactical aim (showing the mathematical dependency between the number and the frequency) that leads Dan to ask the question b), which turns the formula wrong. Dan did not realise this when she built first her formula. At that moment, the personal instrument stands at the front of the scene, and covers up the professional instrument and its didactical aims (the question b).

Interference between the personal and the professional instrument can be seen again in the continuation of the story. The new formula, “**=B6/F6\*100**”, is now adequate for question b, but it is **still not convenient** if we consider the next question (Fig. 2) for inverted reasons! Dan wants pupils to copy the formula in order to fill line 7 and meet this functionality and the automatic incrementation of cell references (B6 becomes B7...). This time, this is part of her goals for students' instrumental geneses.

- 3) Complete the table using the formula in B7:  
 Recopy the formula on the right. (see instructions below for the “cell recopy”).  
 What is the formula contained in C7? D7? E7?

**Figure 2: The continuation of the task**

But the formula above, if copied on line 7, does not fit anymore, as the cell referring to the total, F6, will be changed into F7, F8... along the copy. A solution to this problem is to fix the cell F6 in the recopy by using the “\$” functionality. But Dan did not want this functionality to appear in the first spreadsheet session. It was over the level of instrumentation she wanted for her pupils at this moment. When she built her new formula for question b, the \$ was not in her mind and she did not put it, forgetting that it will create false results at question 3. The day before the session, after discussing over a phone call she realised the mistake and integrated finally the \$ in a last-minute decision!

This time the formula was wrong regarding an instrumental goal: the \$ symbol was over Dan’s instrumental objectives and she did not keep it in mind. It is neither easy nor trivial to bend to all the constraints, all the more that she had already changed her very first version of formula for a mathematical aim, now she had to change it again for an instrumental aim... This time, it is the professional-oriented instrument that overrode on the personal one, by taking into account pupils’ geneses and the level of instrumentation that she wanted them to reach.

These successive formulae somehow disrupted the session: Dan put the \$ sign in the formula and wanted to avoid speaking about it with pupils, but it has been of course highlighted during the session! Facing pupils’ questions, she was compelled to explain but she just said that it is not important to write it in paper-pencil environment. Then, when a pupil came to the board to write the spreadsheet formula, he forgot the \$, the “division by zero Error” appeared after copy and Dan said “*now you happy?*” but did not explain the message nor the division by zero![5].

### **Interpretation: the complex and “split in two” geneses on teacher’s side**

This example shows how the double genesis on the teacher’s side may interfere with pupils’ geneses: spreadsheet constraints interact with the teacher’s goal and didactical expectations (she wanted to introduce only a basic level of spreadsheet functionalities). She has not yet turned her personal instrument into a math teaching one. This process is made more complex by the different geneses at stake. As we saw in the example, it is constrained by:

- The mathematical learning the teacher aims at (statistics and algebra),
- Pupils’ instrumentation, that is how to make them work math through spreadsheet (as the mathematical headcount-frequency dependence through the change of the frequency cell after changing the value of the headcount cell),
- Pupils’ instrumentalization, that is which functionalities are aimed at, which schemes of use do we want them to build - here: relative references and automated incrementation of cell references with the copy, but not yet the absolute references, the \$ sign and its different behaviour in the copy.

Managing all these constraints at once is not easy: spreadsheet is not given as a

didactical instrument, the case of Dan shows that such an instrument is only progressively built along a complex professional-oriented genesis.

### **Understand the global coherence in the evolution of the teacher's practices**

As a synthesis, we can say that institutional and social components, together with Dan's own reflection on her practices, lead her to evolve in her conception of the spreadsheet use. Here are the observed evolutions:

1. **Higher level of class** : she uses spreadsheet with 8<sup>th</sup> graders instead of 7<sup>th</sup> graders
2. **Lower quantity of « new » concepts**: avoid mixing the introduction of the spreadsheet with the introduction of new math notions
3. **Domain change**: introduce the tool with statistics which seemed to Dan more appropriate than algebra
4. **Contents shifted towards the instrument but conditioned by the level of instrumentation fixed by Dan** (she did not want to stress the \$)
5. **Deeper articulation between social and individual schemes**, the importance of the articulation in instrumental geneses has been mentioned by Trouche (2005). In the interview, Dan says she did not organise moments of mutualisation enough and she explicitly wished to take care of this point the 2<sup>nd</sup> year.

Observing deeper these evolutions, they all appear to converge in the direction of *reducing* the instrumental distance. The next section develops this point.

### **REDUCTION OF THE INSTRUMENTAL DISTANCE ALONG TEACHER'S PROFESSIONAL GENESIS**

At different levels, Dan's modifications year 2 tend to decrease a too big instrumental distance of the spreadsheet.

#### **1. Changing the class level: Higher level of class**

This modification comes with the change of the domain (point 3): in French curricula, spreadsheet is explicitly mentioned with statistics for 8th Grade pupils. In the 7th Grade curriculum, spreadsheet appears in a more general and vague way. It requires from teachers a deeper work and thought to define its potentialities for learning math notions, these latter appear more distant from spreadsheet math than in the 8<sup>th</sup> Grade, where spreadsheet appears clearly in relation with precise notions of the curriculum. Thus, choosing this level allows Dan to reduce the distance and match more easily with the official prescriptions. Besides, year 1, Dan found pupils' instrumentalisation not easy in 7th Grade (difficulty to use the "recopy", select a single cell, edit a formula). Older pupils seem to be more skilful and problems linked to instrumentalisation should be less interfering with the math work. With 7 graders, manipulations of the tool seemed more difficult and the tool appeared less transparent.

## **2. The “old/new” game in the mathematical and in the instrumental contents**

Year 1, Dan introduced a new instrument when she introduced new math contents (algebraic notions). The ratio old/new is different in year 2 and also goes in a direction of reducing the distance by reducing the part of “new”: all the math notions at stake in the spreadsheet session (headcounts, frequency, cumulative frequency) had already been seen previously in paper-pencil. This work (new environment with “already-seen” concepts) will then serve Dan as a basis to work later algebraic notions (new concepts in an “already-seen” instrument).

## **3. Domain changing**

The math domain chosen by Dan in year 2 also reduces the distance as regards to algebra, for at least three reasons. The domain of statistics is usually seen to be more in conformity with the representations of a spreadsheet work than algebra. Furthermore institutional pressure is less important in statistics than algebra, a more classical and traditional domain strongly linked to paper-pencil math. On the contrary, statistics are nowadays seen as more fitted to technologies. At last, in the spreadsheet language, one can find more common terms with statistics whereas the distance to the traditional algebraic vocabulary is important (Haspekian, 2005b).

## **4. A deeper care to instrumental aspects**

Distinguish the instrumental genesis on pupils’ side and the dual genesis on teacher’s side allowed us to understand better the teacher’s activity and the phenomenon observed during the session. Dan’s didactical instrument is improving along the experiences and the examples above (click on the cell to see if pupils edited a formula, use the recopy, introduce the relative references but not the absolute ones, not to introduce the logical functions...) show that spreadsheet sessions require careful consideration of all these instrumental aspects. But this is not evident and the first section illustrates difficulties and interferences between these three geneses. Dan’s evolution shows a deeper care of the instrumental aspects with the pre-determination of a level of instrumentation for pupils, well defined, not too high and not too far from usual (not using the \$, not introducing the “IF” function etc.).

## **5. Moments of mutualisation and articulation with paper-pencil mathematics**

Dan introduced year 2 some moments of mutualisation in spreadsheet sessions. In the interview, she affirmed her will to increase the similarity with the traditional sessions. She said having the feeling that it is necessary to multiply the links with the paper-pencil math (e.g., she started the sequence by a paper-pencil session, then worked the same notions in a spreadsheet session, then she came back on the work done with spreadsheet in a paper-pencil session, etc.). All these actions contribute to reduce the distance with paper-pencil, to mix these two environments in a greater proximity. This is a key point to integrate spreadsheet: in (Haspekian, 2005a), teachers who used to integrate spreadsheet had these characteristics. It is thus

interesting to notice that Dan's professional genesis follows the same line. The "old/new" game mentioned above is another characteristic found in expert practices.

## CONCLUSION AND PERSPECTIVES

Many reports deplore the poor integration of ICT in math teaching and researchers stress a phenomenon of "disappointment" after an enthusiastic period where pioneers claimed ICT benefits to learn math. One of the reasons is the "teacher barrier" (see Ruthven, 2007 or Balanskat, Blamire & Kefala 2006) and particularly the importance of teachers' practices, which is seen as a key issue in ICT integration. This is why it seems crucial to progress in understanding of practices and instrumental geneses. In this understanding, the previous study appears to shed light on two elements.

The notion of distance to the referential environment plays a role in technological integration. As we saw, it explains some of Dan's evolutions in terms of a *reduction* of the distance (either by making this distance more explicit or by multiplying moments that alternate work in the two environments enriching both of them). This constitutes a significant creative task for teachers as the tool is not given with any didactical functionalities, it requires a professional instrumental genesis on teacher's side different from the personal genesis and also different from that on pupils' side.

Thus the dual genesis on teachers' side is another interesting element to consider in order to study both local phenomena that can be observed in a session, and the more global evolution of practices, year after year. Dan's emerging practice with the tool is understood better in this frame of double geneses.

Several questions remain, as delimiting more precisely different criteria that create some distance. To understand practices, it is also necessary to determine which elements may counterbalance the distance and play in favor of the tool integration (such as institutional injunctions, or tool epistemic value, didactical design...). We also have to characterise better geneses on teacher's side compared to pupils' one.

## NOTES

1. Beyond the computer transposition that modifies math objects, the modification, from an institutional point of view, concerns the whole ecology of these objects (tasks, techniques, theories). The idea of "distance" reflects the gap between praxeologies associated to two different environments (paper-pencil being a peculiar environment of the math work).
2. Rabardel (2002) distinguishes two types of schemes: *usage schemes* (related to the *material* dimension of the tool) and the *schemes of instrumented action* (related to the global achievement of the task, with goals and intentions).
3. It may not be the case for all teachers: unlike Dan's case, the first instrument can already be constituted in a more advanced way, long before trying to make it a didactical instrument. .
4. The formula refers to the value 50 for the total. If one changes the value of any headcount, then the total will change and the formula becomes wrong.
5. Increment of references after copying make the formula refer to empty cells, by default, empty cell are treated in formulas as if they contain the value 0, this is an option that can be changed.



## REFERENCES

- Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *Int. J. of Computers for Mathematical Learning*, 7, 245-274.
- Balacheff, N. (1994). La transposition informatique. Note sur un nouveau problème pour la didactique. In M. Artigue (Ed.), *Vingt ans de Didactique des Mathématiques en France*, (pp. 364-370). Grenoble : La pensée sauvage.
- Balanskat, B., Blamire, R., Kefala, S. (2006). *The ICT Impact Report. A review of studies of ICT impact on schools in Europe*. Report written by European Schoolnet in the framework of the European Commission's ICT cluster. [http://ec.europa.eu/education/pdf/doc254\\_en.pdf](http://ec.europa.eu/education/pdf/doc254_en.pdf)
- Drijvers, P., Kieran, C., & Mariotti, M. A. (2010). Integrating technology into mathematics education: theoretical perspectives. In C. Hoyles, & J.-B. Lagrange (Eds.), *Mathematics education and technology - Rethinking the terrain* (pp. 89-132). New York/Berlin: Springer.
- Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2004). *The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument*. New York: Springer.
- Haspekian, M. (2005a). *Intégration d'outils informatiques dans l'enseignement des mathématiques, Etude du cas des tableurs*. Doctoral dissertation, University Paris 7, France. Available from [tel.archives-ouvertes.fr/tel-00011388/en/](http://tel.archives-ouvertes.fr/tel-00011388/en/)
- Haspekian, M. (2005b). An "Instrumental Approach" to study the integration of a computer tool into mathematics teaching: The case of spreadsheets, *International Journal of Computers for Mathematical Learning*, 10, 109–141.
- Robert, A. & Rogalski, J. (2002). Le système complexe et cohérent des pratiques des enseignants de mathématiques : une double approche, *Revue canadienne de l'enseignement des sciences, des mathématiques et des technologies*, 2, 505-528.
- Rabardel, P. (2002). *People and technology -a cognitive approach to contemporary instruments*. <http://ergoserv.psy.univ-paris8.fr>
- Ruthven, K. (2007). Teachers, technologies and the structures of schooling. In D. Pitta-Pantazi, & G. Philippou (Eds.), *Proceedings of the fifth congress of the European Society for Research in Mathematics Education*, (pp. 52-68), Larnaca.
- Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: turning a computational device into a mathematical instrument* (pp. 197-230). New York, NY: Springer.

# **WORKING WITH TEACHERS: COLLABORATION IN A COMMUNITY AROUND INNOVATIVE SOFTWARE**

Jean-Baptiste Lagrange

IUFM, University of Reims, France and LDAR, University Paris-Diderot

*This paper is based on an experience of a small scale community around Casyopée, innovative software produced in the ReMath project. A first assumption was that dissemination of research outcomes is possible through the development of communities involving researchers and teachers. Another assumption was that all teachers are not to be considered at the same level. “First-adopters” are teachers that chose to be associated with the project development from the beginning. “Mid-adopters” are teachers that can be interested by using innovative software in the classroom, but will choose to do it only when it brings a real added value. All other teachers potentially users of Casyopée make a third layer. This paper is a first step in the study of communication and collaboration in a “multi-layer” community.*

**Keywords:** teachers, innovative software, collaboration, community, dissemination

## **INTRODUCTION**

Relatively to the integration of digital environments in mathematics teaching and learning, the gap between researchers' expectations and the reality of classroom uses is often stressed. In my view, this gap is not merely quantitative. In a country like the United Kingdom, where uses by teachers are quantitatively high, researchers have concerns relatively to the quality of uses: Miller and Glover (to appear) regret that

“many teachers make progress using the presentational capability of interactive whiteboards but that failure to use associated digital technologies and to make a significant pedagogic change will (...) lead to wasted opportunities”.

Among the opportunities wasted, I count the difficult dissemination among teachers of innovative software designed by research. Soon after the computer appeared and was regarded as useful in the classroom, researchers in mathematics education thought than designing digital learning environments was part of their work, considering that design principles and decisions are of didactic nature and have a deep effect on classroom use of technology. This ‘tool design’ dimension was important in the European project ReMath (IST4-26751). The development of six ‘Dynamic Digital Artefacts’ (DDAs) that is to say innovative software for mathematics teaching and learning, was central in this project. Bridging the gap with classroom and teachers was also an objective of the project, that was aimed (1) through the elaboration of an integrated theoretical framework that should help to communicate between researchers and also between researchers and teachers (2) the elaboration of scenarios of use of the DDAs with a special version for teachers written after experimenting in the reality of actual classrooms (3) the publication of all this material on a communication platform.

While these outcomes of Remath are significant, it would have been very naïve to consider that, by them, they allow teachers to appropriate DDAs and associated educational strategies. A more reasonable assumption is that this appropriation is possible through the development of communities involving researchers and teachers. The aim of this paper is then to discuss an experience of development of a small scale community around one of the ReMath DDAs: Casyopée. The creation of communities of teachers working to produce varied resources associated to the use of technology has been studied by authors like Guin & Trouche (2008). On the one hand, the problematics of the work presented in this paper is convergent with these studies, in the sense that the goal is also to develop the use of a given technology by way of collective work of teachers. On the other hand, the issue here is dissemination of research production and the real question is then the process of collaboration between actors in different positions towards research, rather than the internal functioning of a teacher community. In this process, the outcomes of Remath could act as “boundary objects”, a notion introduced by Fuglestad, Healy, Kynigos and Monaghan (2009) to make sense of the collaboration between researchers and teachers in the integration of technology.

### **CASYOPÉE: EPISTEMOLOGICAL CHOICES AND CONSTRAINTS**

Casyopée has been developed for ten years in a project involving researchers from the Didirem team and teachers after they experimented with ‘standard’ Computer Algebra Systems (CAS) (Lagrange, 2010). The aim was to take up the challenge of teaching about functions at upper secondary level. The team was concerned that while technology is able to offer multi-representational and symbolic manipulative capabilities very effective for solving problems and learning about functions, no tool really adapted presently exists for students’ use. CAS are designed for more advanced users. Dynamic Geometry (DG) offers means for constructing operational figures and exploring co-variations and dependencies in these figures, but exploration is limited to numerical values. Students are neither encouraged nor helped to use algebraic notation and to work on algebraic models. The focus of ReMath was upon computer representations of mathematical objects. Thus it was an opportunity to extend Casyopée representational capabilities, in order to consider functions as models of non-algebraic dependencies. The choice has been to consider 2D geometry and magnitudes as a field of experience. The result is that Casyopée has now two main windows. In the symbolic window the fundamental objects are functions, defined by their expressions and domain of definition. Other objects are parameters and values of the variable. Casyopée allows students to work with the usual operations on functions like: algebraic manipulations; analytic calculations; graphical representations; support for proof... A new window offers usual DG capabilities and also distinctive features: geometrical objects can depend on algebraic objects and it is possible to export geometrical dependencies into the symbolic window in order to build algebraic models.

On the one hand, these epistemological choices are consistent with the notion of function as considered at secondary level and thus they should facilitate Casyopée integration. On the other hand these choices introduce non trivial constraints and differences with software tools generally proposed for classroom use at this level. First, offering symbolic calculation has been decided in order to help students to identify the different stages in a solution of a problem; this decision also implies that Casyopée relies on computer algebra algorithms which are not always deterministic: for instance, it happens that two expressions are mathematically equivalent but also that none of the algorithms implemented in the computer algebra kernel can recognize the equivalence. More generally, Elbaz-Vincent (2005) points out the “decidability limitation” inherent to computer algebra, and concludes that an ‘intelligent usage’ of CAS in mathematical courses is not obvious. Second, DG in Casyopée is designed to help students in modelling algebraically geometrical dependencies. This implies to carefully distinguish between “variable objects” involved in the dependency and “generic objects” that have to be handled as parameters of the problem. For instance in the following problem: *ABC a triangle, [CH] an altitude of this triangle, find a rectangle MNPQ with M on [HA], N on [AB], P on [BC], Q on [HC] and with the maximum area*, in DG environments, numerical exploration of the dependency between point *M* and the area of the rectangle is possible with *A*, *B* and *C* defined as free points. With Casyopée, this exploration is possible, but, to model the dependency as an algebraic function of one variable, which is Casyopée’s potential, the area has to depend univocally of one point. Then, in situations proposed by the Casyopée team, a “generic object” (here the triangle) is defined by way of parameters and a “variable object” (here the rectangle) is defined by way of a free point.

## COMMUNITIES AROUND CASYOPÉE; GOAL AND HYPOTHESES

Before and during the ReMath project, Casyopée was mainly a research tool. During ReMath, Casyopée like the other DDAs has nevertheless been a ‘boundary object’ for small communities especially through the methodology of cross-experimentation: Casyopée has been “cross experimented” by an Italian team and our French team (Maracci, Cazes, Vandebrouck, & Mariotti 2009). Only by the end of the project, the development of a “real world” community around Casyopée could be considered. It was based on the assumption developed in the introduction that researchers have to consider the dissemination of their production as a research work in itself. The issue of collaboration also raised in the introduction implied that all teachers are not to be considered at the same level in this process of collaboration. Teachers that have been associated with the project development can be considered as “first-adopters”, or “experts”. I consider also “mid-adopters”, that is to say teachers that can be interested by using innovative software in the classroom, but will choose to do it only when they will be convinced that it brings a real added value to their teaching strategies. All other teachers potentially users of Casyopée (that is to say, every

mathematics teacher at upper secondary level) would make a third layer. The idea was then to use classroom scenarios as means for communicating between layers: the elaboration and experimentation of scenarios by mid-adopters would be first a way of collaborating between experts and mid-adopters: mid-adopters would propose uses corresponding to their needs and ask the experts for their advice and support. The scenarios would be built in order to be proposed to all teachers and thus designed as a way to communicate between the second and third layer. The goal of the research was then to investigate what particular approaches those “mid adopters” take when working at these interfaces with the first and the third layer. A first hypothesis was that, like in the collaboration between the French and Italian team, the design of scenarios with the help of experts would help the understanding of Casyopée’s potential and constraints. The second hypothesis was that like in the case of Italian teachers, the mid adopters would adapt the use of the software to their needs and sensibility. I expected that these teachers would be primarily interested in easy-to-achieve and close-to-curriculum applications of Casyopée, and sensible to problems and constraints related with the time required by implementing technologies in their classes, with curriculum requirements, with training needs, etc. Thus their production would provide useful material for an easy integration by teachers in the third layer.

## METHODOLOGY

Expert teachers: the first-adopters or “experts” were two teachers that had been involved in the cyclical process of specifying functionalities for Casyopée, contributing to the software development, and experimenting with their classes. Crucial steps in the project were undertaken as a consequence of dissatisfactions they expressed after classroom experimentations. The decision to develop a software environment around a symbolic kernel for classroom use of symbolic computation derived from the difficulty they felt when using standard CAS. The decision to append a DG window was taken after a long term experiment of Casyopée in their tenth grade classes. Modelling geometrical dependencies was a central problem in the series of lessons on which the experiment was based. Students had to explore a dependency between magnitudes using a DG package. Then the modelling process implied to choose a variable and calculate an algebraic formula representing the dependency before students could study algebraically this dependency with Casyopée. Because of the lack of links between the DG package and Casyopée, this crucial step had to be done in paper pencil. It was quite difficult for students and it had to be managed with a too strong mediation of the teacher.

Mid adopters: in the same region of Brittany where the experts teach, a group of six teachers had been constituted in the IREM (Institute for Research in Mathematics Teaching) to experiment the use of the Interactive White Board (IWB). During two years they used software packages (DG and CAS) on the IWB and they were keen to enter a new project. For me they were good candidates to be “mid adopters”: they

were convinced that technology can support mathematical teaching and learning, they were relatively experienced in the classroom use of technology but were not involved in Casyopée history. I refer to these teachers as “group members”.

The third layer: the region of Brittany was developing its professional platform for teachers, students and families. The group could use this platform for its internal communication, but also for communicating with all mathematics teachers in the Brittany region. It was judged better to address a specific group of teachers via their professional platform rather than to disseminate through an open website. The third layer was then mathematics teachers at upper secondary level in the Brittany region, a group of around 500 teachers.

Data: the group met 12 times along two years in 3 hour sessions gathering researchers, the two experts and the six group members. Six experimentations of scenarios were done in common, one group member offering his class and the rest of the group acting as observers. Communication between the sessions was done via the professional digital platform using a virtual group feature. The data consists in the video recording of the sessions and experimentations, the messages and files exchanged on the platform and the scenarios proposed and experimented by the group as published on the professional digital platform.

### PRELIMINARY FINDINGS

The data exploitation is still in progress. In this paper I report on features that I found relevant or surprising in the meetings and in the productions relatively to the above hypothesis and illustrate by outlining the evolution of two group members, Chris and Rose. The next step in the research will be the systematic exploitation of the data in order to strengthen the findings. The main features in the first year were:

1. The group members engaged in scenario design slower than expected. The reason is that it took time for them to understand the potential of Casyopée. An expectation was that modelling geometrical dependencies for optimisation problems would be an attractive perspective and that they would work from situations they experimented before using standard dynamic geometry software, adapting these situations to take advantage of Casyopées’ symbolic features. Actually, the teachers were keen to exploit Casyopée’s potential to go beyond numerical exploration, but they could not achieve this goal. For instance, in the second meeting Rose protested that she could not implement Casyopée for an optimisation problem she used to propose her students (figure 1). She considered two free points in the plane  $A$  and  $B$  and a free point  $M$  on the segment  $[AB]$ . The exploration was done numerically like in other DG systems. After that, Rose wanted to take advantage of Casyopée for exporting the dependency into the algebraic window in order to solve the problem algebraically. She chose  $AM$  as the independent variable and the calculation :  $\frac{AP^2 + MQ^2}{2}$  as dependent variable. But Casyopée replied that the calculation depended of more than

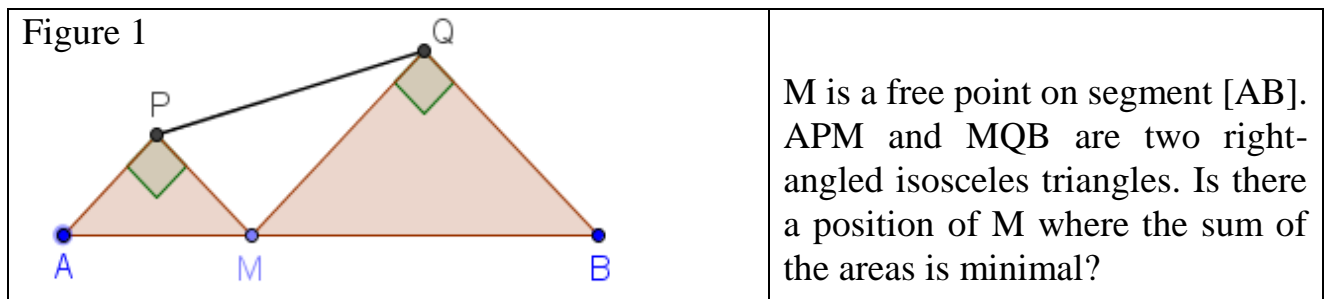
a free point and the exportation was unsuccessful. Rose expected a function like  $x \rightarrow \frac{x^2}{4} + \frac{(AB-x)^2}{4}$  but in Casyopée, functions cannot depend on geometrical objects.

The approach is to define the two points  $A$  and  $B$  as coordinated points, possibly depending on a parameter representing the distance  $AB$ . For instance  $A$  might be at the origin and  $B$  might be  $(0; a)$ . With these settings, Casyopée exports the dependency as  $x \rightarrow \frac{x^2}{4} + \frac{(a-x)^2}{4}$ . In contrast to Rose, the expert teachers understood the constraint of a distinction between “variable objects” involved in a dependency and “generic objects” and conceived adapted scenarios.

2. Group members also thought that a symbolic system like Casyopée would be particularly useful for problems with complicated algebraic calculations. That is why they proposed scenarios with situations resulting in very complex expressions that the symbolic kernel and other modules of Casyopée had difficulties to handle. It took time for these teachers to realise how the limitations of the symbolic kernel stressed above influence Casyopée’s operation. For instance in the third meeting, Chris expressed her concern that, for two functions with equivalent expressions, Casyopée was not directly able to recognize their equality, and that only one of the symbolic calculations provided by Casyopée returned zero for the difference. This is different of expert teachers, who saw the potential of Casyopée in the security and easiness that symbolic calculation provides in ‘ordinary’ calculation.

3. Many scenarios proposed by the group members in the first meetings could not be directly implemented because of limitations of Casyopée. Some of these limitations were inherent to fundamental epistemological choices like those raised above, and some other to the current state of the software. Casyopée is an evolving project and although many efforts had been devoted to software development in ReMath, it could not be considered as an achieved stable environment. This character had not been an obstacle when working with expert teachers who understood well the difference between fundamental choices and minor defects and adapted the scenarios accordingly. Also expert teachers were used to ask for corrections of minor defects before using in the classroom. In contrast, the group members claimed that they could not update the software in the school computer rooms under short notice. The experts tried to propose alternative feasible scenarios and procedures for updating without success. A consequence is that a good part of the discussions in the meetings was devoted to the software itself. These discussions and the pressure that group members put through the scenarios they wanted to implement implied more work on the software than expected. A part of this work was devoted to adapt Casyopée for the intended scenarios and was achieved during the year. Roughly, the scenarios developed and experimented in the first year were of two types: (a) scenarios at tenth grade about linear functions and product of these, taking advantage of the symbolic window only, and of specific features like the display of a table with signs of

functions, in close connection with rules and ways of proving that the teachers want students to discover and remember (b) scenarios at twelve grade involving advanced functions (logarithms) and curves of these. In these scenarios, the teachers seek to take advantage of representing and symbolic computing facilities of Casyopée. Considering curves as geometrical objects and treating symbolically its elements was seen as an attractive capability.



These scenarios are more sophisticated than what was expected. Note also that during the first year, in spite of the expectations, there was no outside communication towards the third layer of teachers. The group members were reluctant to publish material not really achieved. They insisted on publishing reports on successful sessions with a deep didactical analysis, rather than “raw scenarios”.

In the second year a significant part of the meetings was again devoted to comments and discussion by group members about the software in parallel with the preparation and exploitation of classroom experiments. After adaptation by the designers, the features in development were tested in the experts' classes, group members feeling they could not deal with provisional versions. Like in the first year, more efforts than expected had to be devoted to software development. After the experimentation of scenarios these efforts could be directed towards curricular conformity and students' understanding of commands as well as towards the general ergonomics. The appropriation of Casyopée by the group members progressed notably especially with regard to the relationship between the algebraic and the geometric windows. This appropriation was done through the preparation and experimentation of scenarios but also seems inseparable from the discussion on the software itself. Why is it designed like that? Could other options be decided? In the discussion, reference was often made to other CAS or DG.

The communication through the professional platform was organised around eight mathematical themes. Two of these themes are related to the experiments in the first year, two derive from the scenario experimented in ReMath, one was imagined by the experts and myself as a way to introduce 10<sup>th</sup> grade students to modelling. It was consistent with an evolution of the curriculum at this grade, but considered too difficult to handle by group members. Three others illustrate an evolution of the group members and two will be detailed in the next section. Communication itself was done through eight high quality “mini web sites” [1]. Group members insisted on publishing detailed scenarios with precise objectives and account of the



advantages brought by Casyopée. The group members' concern for a strong didactical added value clearly appears in these productions. It implied deep reflection after the experiments and could be achieved only by the end of the year.

## TWO EXAMPLES

### Optimizing pipes (11th grade, non scientific students)

This problem was originally proposed for 12th grade, non scientific students. The context is drain pipes on a rectangular wall, and the optimal point is the Torricelli point in an isosceles triangle. As said above, Chris, the teacher who initiated this theme and experimented the associated scenario expressed often her concern for “how Casyopée does symbolic computation”. Her work in the mini site, as well as her declarations in the meetings, helps to understand that this concern was linked with her ambition to offer non scientific students realistic optimisation problems and her view that Casyopée should be used to scaffold these students' weakness in algebraic computation. In 12<sup>th</sup> grade, studying the Torricelli point generally involves a variable angle. But this has not been implemented in Casyopée and 11th grade non scientific students do not know about oriented angles and trigonometry. Chris realised that on the one hand, Casyopée allows an easy modelling using a variable length with two possible choices, an interesting feature for her students. The resulting algebraic forms can be handled by Casyopée, especially Casyopée calculates different expressions of the derivatives, a task that these students could not tackle alone. On the other hand, she was worried that Casyopée could not calculate the zero of the derivatives because it is the difference of two square roots. Then she adapted her scenario in order that the students realize that the equality of two square roots implies the equality of the expressions and then a polynomial equation that Casyopée could solve. This work on modelling and on symbolic expressions is very unusual in non scientific classes because these students are generally scared of algebra. Through this experience Chris realized the actual potential of symbolic computation in Casyopée: it does not everything by miracle, but it can help when one understands its limits and considers relevant “techniques of uses” (Lagrange, 2000).

### Towards the quadrature of the parabola (12<sup>th</sup> grade)

Rose saw the potential of Casyopée for “advanced topics”. Her previous experience was using separately DG to make students express conjectures and CAS to help for an algebraic proof. Considering geometric properties of curves was for her an appealing field for using Casyopée because it involves geometry *and* calculus. She chose a task based on a result by Archimedes: the area enclosed by a parabola and a cord  $AB$  is  $\frac{4}{3}$  times the area the triangle  $ABC$ ,  $C$  being the point where the tangent is parallel to the cord. She considered the exploration of the figure and the discovery of the linear relationship between the two areas as very important and then she complained that Casyopée did not offer the means for exploration that she was used to. She met also the same issue as in the problem of figure 1 a year before: the figure

depends on the two points  $A$  and  $B$  that she wanted to define as two free points on the parabola, but, as explained above, a function cannot be exported from a calculation depending on two free points! The experts and I explained that the two points could be defined as coordinated points depending on parameters, but Rose wanted that students explore by way of free points and not parameters. We also suggested a simpler definition for  $C$ , because the calculations were heavy with the definition above. There was a long discussion and finally Rose accepted the suggestion. She maintained a free point for  $B$  and chose to define  $A$  as a fixed point. With these changes, the scenario was feasible.

This is how Rose describes the use of Casyopée in the mini site. *“The steps of the proof derive from those highlighted by the exploration. Throughout the exploration, students have to identify these steps and propose calculations. To achieve the algebraic proof they have to establish different expressions. The software provides several aids: it exports geometric calculations into formulas for the areas. The student can either prove them or admit them temporarily and continue, it helps for different forms of expressions (development, factoring) or controls the expressions obtained by the student.”* This helps to understand Rose’s position towards Casyopée. She really wanted to take advantage of aids provided by the software for proving, but she also wished that students base their proof on a thoughtful exploration and she found hard to implement this exploration with Casyopée. While the experts and I tended to consider that adapting to Casyopée constraints could be done without much loss, Rose saw this as a dilemma. Many changes in the recent development of Casyopée result from discussions we had on this point.

## CONCLUSION

This experience of a small scale community around Casyopée was surprising by many aspects. It was expected that the collaboration would be around scenarios as it was the case in the ReMath Franco-Italian experiment, rather than on the software itself. Actually, the “mid adopters” had to reflect deeply on their expectations with regard to the software and often to reconsider what they saw as Casyopée potential. As stressed above, it implied also reconsideration for the experts and I, and more work on the software than expected. Kynigos (2007) conceptualised the work between researchers and teachers on software, using the term of “half baked” microworld. Casyopée was not conceived as a “half baked” microworld, but certainly one has to think of innovative software as a never finished product, evolving through communities. The communication with the third layer consisted in the publication of the mini-web sites. It seems that “mid adopters” privilege a communication with their colleagues based on high didactical quality production. This is also different of what was expected. As said in the introduction, I am interested in the gap between expectations and the reality of classroom use of technology. I have the feeling that this experience can help to identify reasons for this gap, because it shows what obstacles “mid-adopters” positively oriented towards innovative software can meet.

More work has to be done in order to capture its specificity. At a practical level, the exploitation of the data has to be completed and at the theoretical level an “activity theory” framework has to be built in order to conceptualize a “multi-layer” community around innovative software.

### NOTES

1. These web sites are internal to the Brittany professional platform, but can be accessed publicly via <http://code.google.com/p/casyopee/wiki/Activites>

Acknowledgment: this work was done thanks to the help of the European Community (IST4-26751) and of the Institut National de Recherche Pédagogique, Lyon, France.

### REFERENCES

- Elbaz-Vincent P. (2005). A Cas as an Assistant to Reasoned Instrumentation. In D. Guin, K. Ruthven and L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators* (pp. 41-65). Springer.
- Fuglestad, A. B., Healy, L., Kynigos, C., & Monaghan, J. (2009). Working with teachers: context and culture. In C. Hoyles & J.B. Lagrange (Eds.) *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain* (pp. 293-310). Heidelberg: Springer.
- Guin, D., Trouche, L. (2008). Un assistant méthodologique pour étayer le travail documentaire des professeurs: le CDrom SFoDEM 2008. *Repères-IREM* 72, 5-24.
- Kynigos, C. (2007). Using half-baked microworlds to challenge teacher educators' knowing, *Int. Journal of Computers for Mathematical Learning* 12(2), 87-111.
- Lagrange, J.B. (2000). L'intégration d'instruments informatiques dans l'enseignement: une approche par les techniques, *Educational Studies in Mathematics* 43, 1-30.
- Lagrange, J.B. (2010). Teaching and learning about functions at upper secondary level: designing and experimenting the software environment Casyopée. *Int. Journal of Mathematical Education in Science and Technology* 41(2), 243-255.
- Maracci, M., Cazes, C., Vandebrouck, F., & Mariotti M. A. (2009). Casyopée in the classroom: Two different theory-driven pedagogical approaches. In V. Durand-Guerrier et al. (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1399-1408), Lyon (France).
- Miller, D. & Glover, D. (to appear). Teacher development in digital technologies in secondary mathematics in England and research on the use of the interactive whiteboard. To appear in J. B. Lagrange (ed.), *Utiliser les technologies numériques pour les apprentissages des élèves: dispositifs institutionnels et genèses de pratiques enseignantes*.

# **EXTENDING THE TECHNOLOGY ACCEPTANCE MODEL TO ASSESS SECONDARY SCHOOL TEACHERS' INTENTION TO USE CABRI IN GEOMETRY TEACHING**

Marios Pittalis & Constantinos Christou

University of Cyprus

*The purpose of this study was to modify and extend Technology Acceptance Model to assess secondary school teachers' intention to use Cabri in geometry teaching. We developed a measurement-structural model and enriched TAM by integrating in the model a new significant parameter, "perceived pedagogical-learning fit", which refers to evaluating the pedagogical-learning appropriateness of teaching geometry with Cabri based on a cognitive-learning model. One hundred and five pre and in-service secondary school mathematics teachers answered a questionnaire that was developed based on related research studies. The results of the study proved that perceived pedagogical-learning fit and attitude towards the use of Cabri are key determinants of teachers' behavioural intention to use Cabri in geometry teaching.*

Keywords: technology acceptance model, Cabri, task technology fit, teachers' view

## **INTRODUCTION**

The technology acceptance model (TAM) is well known and widely accepted in the study of specific behaviours to understand how users' beliefs and attitudes affect their technology usage behaviour (Teo, Lee, Chai & Wong, 2009; Venkatesh & Davis, 2000). TAM has been used widely in different domains and cultures to test models in technology acceptance (Dishaw & String, 1999; Lee, Yoon & Lee, 2009) and has received extensive empirical support. TAM applications are mostly related in the outcome chain on intention to use or actual use by taking into consideration as fundamental determinants of user acceptance the variables perceived usefulness and perceived ease of use (Davis, 1989). Having in mind the business and commercial origins of TAM, not surprisingly, it has had limited applications in education. Recent research studies of TAM in education have explored students' or teachers' acceptance towards new technologies such as online learning, and technology in education (Lee, Yoon, & Lee, 2009; Stols, 2007).

Dishaw and Strong (1999) support that a weakness of TAM is its lack of task focus and extended the model to integrate task-technology fit (TTF), which refers to matching the capabilities of the technology to the demands of the task. In the present study, we integrate TAM and TTF theoretical considerations to extend and propose a structural and measurement technology acceptance model that assesses the intention of pre and in-service secondary school teachers to use Cabri, a dynamic geometry software, in geometry teaching. To evaluate the intention of use of Cabri based on a theoretical geometry learning model, we modified TAM by adding in the model

teachers' perceived learning fit of Cabri on a task-technology fit theoretical assumption.

## **THEORETICAL CONSIDERATIONS**

### **Technology Acceptance Model**

TAM is based on the Theory of Reasoned Action (Ajzen & Fishbein, 1980) and explains how users' beliefs and attitudes affect their intention to use a specific technological device. TAM explains the interactions among attitudes, beliefs and intention to use technology. The two belief variables refer to perceived usefulness and perceived ease of use (Teo *et al.*, 2009). Perceived usefulness refers to the subjective belief that the use of the new technology will improve job performance and productivity. Perceived ease of use refers to the subjective belief that the use of the new technology does not demand considerable time and effort. Recent studies have shown that the above variables affect users' intention to use and their attitude towards technology use (Cheung & Huang, 2002; Raaij & Schepers, 2008). Attitude has been doubtfully hypothesized to influence the behavioural intention to use the technology and was therefore not considered in later assessments (Venkatesh & Davis, 2000).

Although TAM'S perceived usefulness concept implicitly includes task, the model has been criticized for the lack of task focus and its application revealed mixed results in information technology evaluations (Dishaw & Strong, 1999). In contrast to TAM, the theoretical foundation of the task-technology fit model (TTF) lies in the assumption that technology will be used if, and only if, the functions available to the user fit the activities and needs of the user. Thus, TTF explicitly includes task characteristics and tests for direct effects of task and technology characteristics on utilisation.

### **Mathematics Teachers' views about using DGS**

Dynamic geometry software (DGS) has been considered as an effective tool in the teaching and learning of geometry and proved to have the potential to regenerate geometry in schools (Hollebrands, Laborde, & Straesser, 2008). DGS have become beneficial tools in geometry teaching, because they support students' visualization of the features of geometric shapes and facilitate the interaction of geometrical objects. However, research studies have shown that DGS classroom use has remained limited (Ofsted, 2004). Researchers noted that a significant parameter of the problem is the absence of teachers' contribution (Lagrange, 2008). Jones (2002) asserted that, in the DGS field, there is a need for research on teachers' input and impact. Therefore, the success of a DGS geometry teaching program depends on the extent to which educational decision makers take into consideration teachers' needs and beliefs and educational objectives. Thus, the development of an appropriate DGS teaching program is a complicated task and requires a multidisciplinary approach. Teachers'

evaluation of the DGS and their beliefs and attitude towards the DGS should be a fundamental pillar of the initiative.

Cabri is one of the dominant software in the domain of dynamic geometry for dynamically creating, exploring, elaborating, analysing and synthesizing geometrical concepts (Laborde, 2001; Laborde & Laborde, 2008). It could facilitate the process of discovering geometrical concepts by first visualizing, analyzing and then making conjectures. Several research studies have shown that geometry teaching and learning in Cabri could promote and enhance students' visualization, reasoning and construction processes (Olivero, & Robutti, 2007).

Despite the potential of Cabri as a tool to enhance students' geometric thinking and improve their geometry performance, its value will not be realized if teachers do not accept it as an effective learning tool. TAM has been utilized and extended for research purposes in education to assess pre or in service teachers' acceptance of other information technology innovations, such as e-learning (Lee *et al.*, 2009) and virtual-learning (Raaij & Schepers, 2008). Results showed that perceived usefulness and perceived ease of use proved to be critical parameters of the acceptance and usage of the innovation as an effective and efficient learning technology.

## **THE PRESENT STUDY**

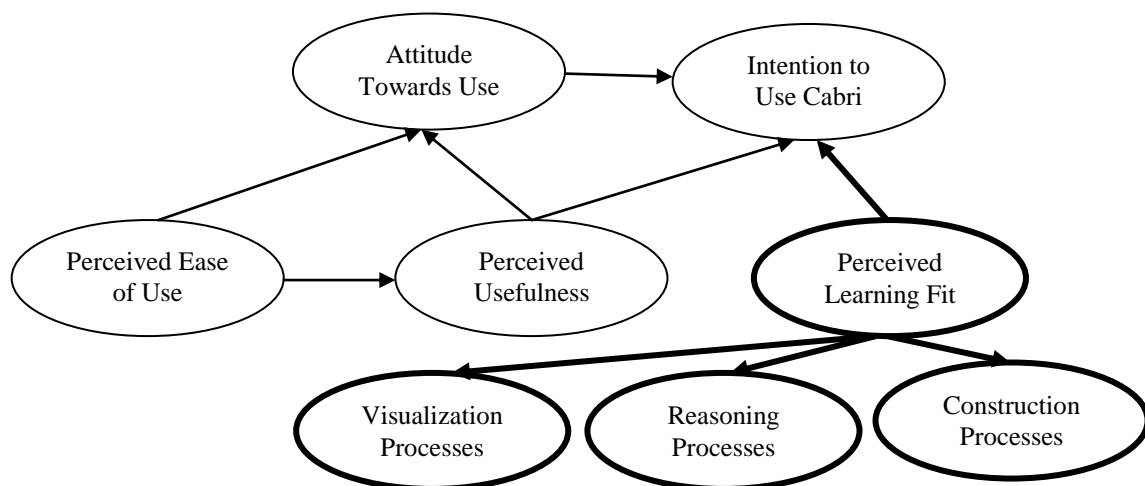
There is a research need to establish an empirical link between TAM and specific mathematics geometry software. Thus, the main purpose of the study is to extend TAM and propose a structural and measurement technology acceptance model that could be used to evaluate the intention of teachers to use Cabri in geometry teaching. The present study adds to the research literature on TAM and DGS in a number of ways. First, it integrates TAM and TTF theoretical considerations by proposing a model that evaluates the task-technology fit of Cabri based on teachers' perceived pedagogical-learning fit of the software. By the term "perceived pedagogical-learning fit" we refer to teachers' perception about the quality of teaching and learning of geometry with Cabri and whether the specific software could meet the learning needs of students in geometry. To do so, the study proposes a model that evaluates Cabri's perceived pedagogical-learning fit based on Duval's cognitive geometry reasoning model (Duval, 1998). Second, the study may provide a worthwhile starting point in mathematics educational technology field in developing appropriate evaluation models that could be used to evaluate the pedagogical value of DGS.

### **Aims of the study and the proposed model**

The purpose of the present study is to propose a model that extends TAM to assess teachers' intention to use Cabri in geometry teaching based on teachers' perceived pedagogical fit of the software. Specifically, the aims of the study were to (a) to validate the measurement model that describes teachers' perceived pedagogical fit of

Cabri based on Duval's geometry reasoning model and (b) to extend and modify TAM so it could potentially be used to assess, on a task-technology fit basis, the intention to use Cabri by integrating in the model, as a task-technology fit parameter, the effect of teachers' perceived pedagogical fit.

In this paper, as it is highlighted in Figure 1, we hypothesized that an additional parameter, "perceived pedagogical-learning fit", influences teachers' intention to use Cabri in geometry teaching. Specifically, based on the literature we assumed that a theoretical construct "perceived pedagogical-learning fit" describes teachers' perceived pedagogical and learning appropriateness of geometry teaching with Cabri to develop students' visualisation, reasoning and construction processes. Based on Duval's model (1998), geometrical reasoning involves three kinds of cognitive processes which fulfil specific cognitive processes; (a) visualization processes that refer to the visual representation of a geometrical concept, (b) construction processes that can be developed in Cabri by appropriate tools and (c) reasoning processes that are necessary for the extension of knowledge, for explanation and proof. Thus, the latent construct "perceived pedagogical-learning fit" consists of three first-order latent factors that refer to teachers' perceived visualization, construction and reasoning processes fit of teaching with Cabri. In addition, based on TAM theory (Dishaw & Strong, 1999), we hypothesized that teachers' intention to use Cabri in geometry teaching is influenced by teachers' (a) perceived usefulness of Cabri and (b) their attitude towards the use of Cabri. Perceived usefulness was also hypothesized to be influenced by perceived ease of use and attitude towards the use of Cabri was assumed to be predicted by the factors perceived ease of use and perceived usefulness.



**Figure 1: The hypothesized model**

## Subjects

The sample of this study consisted of 105 pre and in-service secondary school mathematics teachers; 45 pre-service and 60 in-service teachers. Forty two teachers were males and 63 females. All the subjects attended a compulsory 9-hours module regarding Cabri and its pedagogical applications during their teacher training program in the University of Cyprus during spring 2010. The questionnaire was administered after the completion of the Cabri module. None of the subjects had previous experience with Cabri.

## Instrument construction

A questionnaire instrument was developed for this study. TAM scale items were adopted from previous studies (Dishaw & Strong, 1999; Lee *et al.*, 2009; Teo *et al.*, 2009) and were modified to meet the needs of the present study. Our research TAM model consists of 12 items (see Table 1) that measured “perceived ease of use” (3 items), “perceived usefulness” (3 items), “attitude towards use of Cabri” (3 items) and “use intention” (3 items). In addition, based on the existing literature on geometry reasoning discussed in the previous sections, we developed 10 items that measured teachers’ perceived pedagogical-learning fit. For example (see Table 2), the item “Teaching geometry with Cabri helps in visualizing geometrical concepts” was used to measure “visualization processes” fit, the item “Cabri’s measurement and dragging tools help students making generalisations” was used to examine the “reasoning processes” fit and the item “Cabri’s tools make easy the construction of complex geometrical constructions, such as locus” was developed to examine the “construction processes” fit. We developed multi-item Likert scales which have been widely used in the questionnaire-based perception studies, using the seven-point Likert scale, with 7 being “Totally Agree” and 1 being “Totally Disagree”.

Factor	Items
Perceived usefulness	<p>Q1. Using Cabri in geometry teaching will enable me to accomplish my tasks more quickly.</p> <p>Q2. Using Cabri in geometry teaching will enable me to enhance my effectiveness in teaching.</p> <p>Q3. Using Cabri in geometry teaching will enable me to increase my productivity in teaching.</p>
Perceived ease of use	<p>Q4. My interaction with Cabri tools will be clear and understandable.</p> <p>Q5. I will find the Cabri tools to be flexible to interact with.</p> <p>Q6. I will find the Cabri tools easy to use.</p>
Attitude towards	<p>Q7. I think it would be very good to use Cabri in geometry teaching rather than traditional methods.</p>



Use	Q8. In my opinion it would be very desirable to use Cabri in geometry teaching rather than traditional methods. Q9. Teaching geometry with Cabri makes the lesson more interesting.
Use intention	Q10. I will use Cabri in geometry teaching rather than traditional methods of teaching geometry. Q11. My intention is to use Cabri in geometry teaching rather than traditional teaching methods. Q12. In geometry teaching, I would rather use Cabri than traditional methods.

**Table 1: TAM items****Data Analysis**

The goal of the analysis was to estimate the relative strength of the proposed models. Because we proposed a theoretically driven model about the components of “perceived pedagogical-learning fit”, our first interest was in the assessment of fit of the hypothesized a priori measurement model to the data. Then, we examined the validity of the hypothesized structural model. One of the most widely used structural equation modelling computer programs, MPLUS, was used to test for model fitting (Muthen & Muthen, 2007) and three fit indices were computed: The chi-square to its degrees of freedom ratio ( $\chi^2/\text{df}$ ), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for  $\chi^2/\text{df}$  should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit (Marcoulides & Schumacker, 1996).

Factor	Items
Visualization processes	Q13. Teaching geometry with Cabri helps in visualizing geometrical concepts. Q14. Cabri facilitates the dynamic visualization and understanding of geometric theorems. Q15. Cabri’s functions (i.e. dragging) help students to “see” the properties and characteristics of geometric shapes. Q16. Cabri offers dynamic images that promote dynamic visualisation of geometrical concepts.
Reasoning processes	Q17. Teaching geometry with Cabri helps in developing students’ reasoning and conjecturing thinking. Q18. Manipulating shapes in Cabri contributes in understanding geometric shapes’ relations. Q19. Cabri’s measurement and dragging tools help students making generalisations.
Construction processes	Q20. Cabri’s tools make possible the construction of geometric shapes based on their properties.

- 
- Q21. Cabri's tools make easy the construction of complex geometrical constructions, such as locus.
- Q22. Constructing geometric shapes in Cabri is not a mechanical process, but it develops students' construction abilities.
- 

**Table 2: Perceived Pedagogical-Learning Fit items**

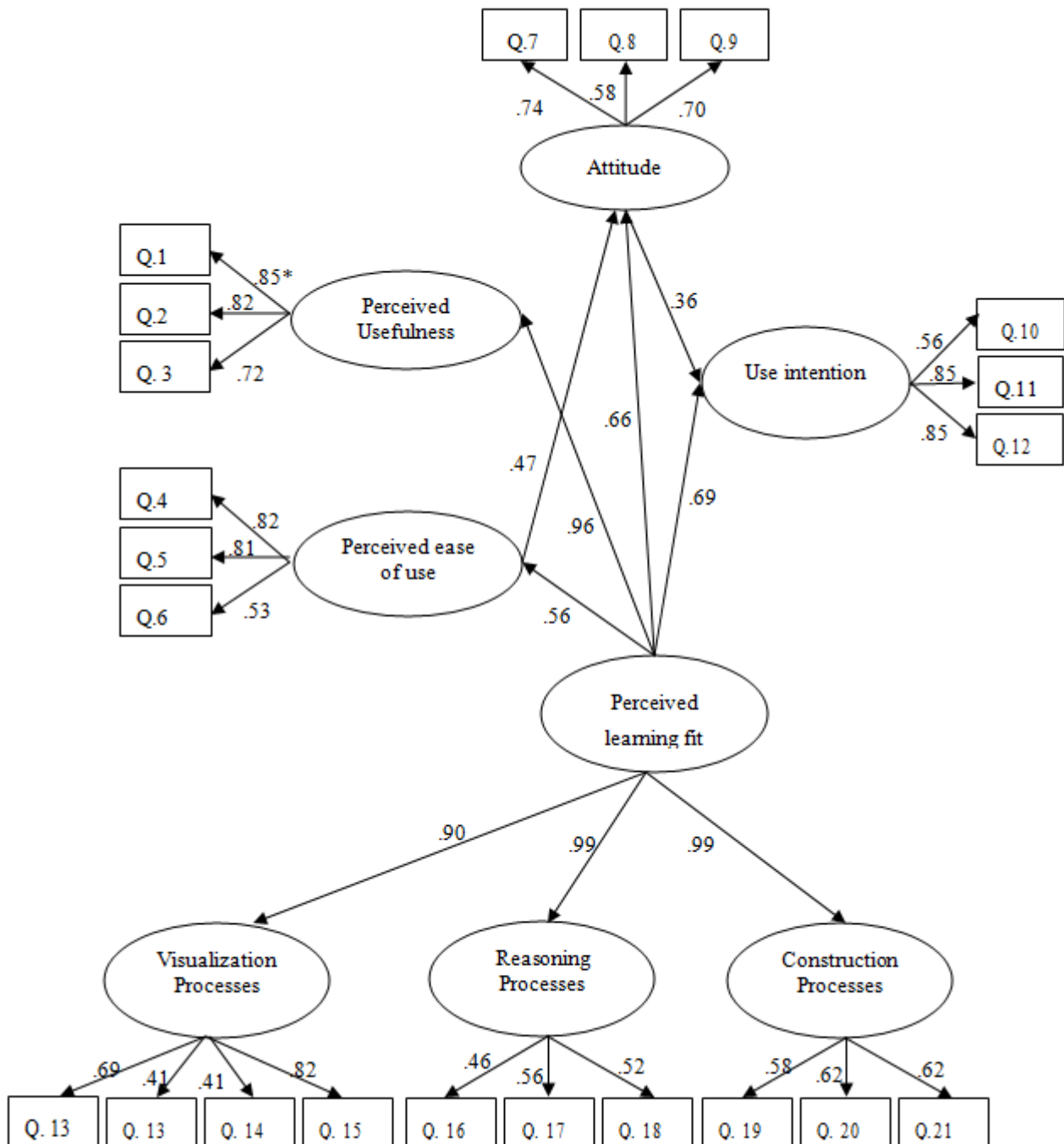
## RESULTS

In this section, we refer to the results of the analysis, establishing the validity of the latent factors and the viability of the structure of the hypothesized latent factors. In this study, we posited an a-priori measurement model and tested the ability of a solution based on this structure to fit the data and then conducted a structural analysis to examine the relation between the factors of the modified TAM.

To examine the first aim of the study, we conducted a confirmatory factor analysis (CFA) to validate a measurement which should have been able to model teachers' perceived pedagogical-learning fit of Cabri. The descriptive-fit measures indicated support for the hypothesized measurement model ( $\text{CFI}=.94$ ,  $\chi^2/df=1.32$ ,  $p>0.05$ ,  $\text{RMSEA}=.06$ ). The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large. The analysis showed that each of the 10 perceived learning fit items employed in the present study loaded adequately only on one of the three geometry processes fit factors, giving support to the assumption that the three first-order factors could represent three distinct dimensions of teachers' perceived learning fit. Moreover, the factor loadings of the first-order factors (visualization, reasoning, and construction processes learning fit) that corresponded to teachers' perceived pedagogical-learning fit were extremely high (.90, .99 and .99 respectively), claiming that a general type of belief that refers to teachers' perceived pedagogical-learning fit could explain very accurately teachers' variances in evaluating Cabri.

To examine the second aim of the study, we tested the validity of the hypothesized structural model, which claimed that the intention to use Cabri is influenced by the factors "perceived usefulness", "attitude towards use of Cabri" and "perceived learning fit". The descriptive-fit measures did not support the hypothesized structural model ( $\text{CFI}=.86$ ,  $\chi^2/df=1.56$ ,  $p<0.05$ ,  $\text{RMSEA}=.09$ ). Thus, we examined the validity of alternative structural models to trace the relations between the factors of the model. Figure 2 presents the modified model that best fitted the empirical data ( $\text{CFI}=.92$ ,  $\chi^2/df=1.34$ ,  $p<0.05$ ,  $\text{RMSEA}=.07$ ). As it is highlighted in Figure 2, the results of the study revealed that the factor "perceived pedagogical-learning fit" is a strong predictive factor of teachers' intention to use Cabri ( $r=.69$ ,  $z=4.02$ ,  $p<0.05$ ). In addition, teachers' perceived pedagogical-learning fit predicts (a) teachers' attitude towards the use of Cabri ( $r=.66$ ,  $z=3.44$ ,  $p<0.05$ ), (b) teachers' perceived ease of use ( $r=.56$ ,  $z=3.09$ ,  $p<0.05$ ) and (c) teachers' perceived usefulness ( $r=.96$ ,  $z=4.02$ ,

$p < 0.05$ ). It could be concluded that teachers' perceived usefulness is strongly affected by their perceived pedagogical-learning fit. The structure of the modified model showed that attitude towards the use of Cabri is also predicted by teachers' perceived ease of use ( $r = .47$ ,  $z = 3.71$ ,  $p < 0.05$ ) and attitude towards the use affects directly teachers' intention to use Cabri ( $r = .36$ ,  $z = 1.97$ ,  $p < 0.05$ ). Further, the solution of the modified model did not validate the direct effect of the factor "perceived usefulness" on other variables. On the contrary, it was deduced that teachers' perceived usefulness does not influence neither their attitude towards the use of Cabri, nor their intention to use it.



**Figure 2: Modified Technology Acceptance Model**

## DISCUSSION

This study attempts to modify and extend TAM by integrating in the model a new significant parameter, “perceived pedagogical-learning fit”, which refers to assessing the pedagogical-learning appropriateness of teaching geometry with Cabri based on a cognitive-learning model. In examining the relations among the constructs in the modified TAM, this study found that perceived pedagogical-learning fit and attitude towards the use of Cabri were key determinants of teachers’ behavioural intention to use Cabri in geometry teaching. Attitude towards the software proved also to be significant predictor in other research studies that examined users’ intention to use computers in education (Teo *et al.*, 2009; Venkatech, Morris, Davis & Davis, 2003).

It is important to note that adopting non-educational derived behavioural intention models to assess teachers’ intention to use software in teaching might give misleading information. Our results yielded that an important parameter that should be examined is teachers’ perceived evaluation of the pedagogical learning fit of the software. The results of this study suggest that teachers’ perceived ease of use, their attitude towards the use and especially their perceived usefulness are significantly influenced by their pedagogical-learning evaluation of the software. For teachers what matters to use the software is the additive, learning value of the software. Thus, although for the past two decades, numerous studies using the TAM as a research framework have been conducted, there is a need for future research in mathematics education domain that modifies TAM according to the learning needs that the innovative technology should meet, based on well established learning and cognitive models.

## REFERENCES

- Ajzen, I., & Fishbein, M. (1980). *Understanding attitudes and predicting social behaviour*. Englewood Cliffs, NJ: Prentice-Hall.
- Cheung, W. M., & Huang, W. (2002). An investigation of commercial usage of the world wide web: A picture from Singapore. *International Journal of Information Management*, 22(5), 377-388.
- Davis, F. D. (1989). Perceived usefulness, perceived ease of use, and user acceptance of information technology. *MIS Quarterly*, 13(3), 319-340.
- Dishaw, M., T., & Strong, D., M. (1999). Extending the technology acceptance model with task-technology fit constructs. *Information & Management*, 36, 9-21.
- Duval, R. (1998). Geometry from a cognitive point of view. In C. Mammana and V. Villani (Eds.), *Perspectives on the Teaching of Geometry for the 21<sup>st</sup> century: An ICMI Study*. Dordrecht: Kluwer.
- Hollebrands, K., Laborde, C., & Strasser, R. (2008). Technology and the learning of geometry at the secondary level. In M. K. Heid & G. Blume (Eds.), *Research on Technology in the Learning and Teaching of Mathematic, volume 1: research syntheses* (pp. 155–205) Greenwich CT: Information Age.

- Jones, K. (2002). Research on the use of Dynamic Geomtery Software: implications for the classroom. *Micromath*, 18(3), 18-20.
- Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, 6(3), 283–317. doi: 10.1023/A:1013309728825
- Laborde, C., & Laborde, J.-M. (2008). The development of a dynamical geometry environment: Cabri-géomètre. In G. Blume & M. K. Heid (Eds.), *Research on Technology in the Learning and Teaching of Mathematic*, volume 2: cases and perspectives (pp. 31–52) Greenwich CT: Information Age.
- Lagrange, J. B. (2008). *Ordinary teachers using technology: concerns, theoretical approaches, case studies*. Lecture presented at the 11th International Congress on Mathematical Education, July 6-13, in Monterrey, Mexico.
- Lee, B. C., Yoon, J. O., & Lee, I. (2009). Learners' acceptance of e-learning in South Korea: Theories and results. *Computers & Education*, 53, 1320-1329. doi:10.1016/j.compedu.2009.06.014
- Marcoulides, G. A., & Schumacker, R. E. (1996). *Advanced structural equation modelling: Issues and techniques*. NJ: Lawrence Erlbaum Associates.
- Muthen, L. K. & Muthen, B. O. (1998-2007). Mplus User's Guide. Fourth Edition. Los Angeles, CA: Muthen & Muthen.
- Ofsted (2004). *2004 report: ICT in schools – the impact of government initiatives, Secondary Mathematics* London, A report from the Office of Her Majesty's Chief Inspector of Schools <http://www.ofsted.gov.uk/assets/3646.pdf>
- Olivero, F. & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*, 12, 135-156. doi: 10.1007/s10758-007-9115-1
- Raaij, E. M. V. & Schepers, J. J. L. (2008). The acceptance and use of a virtual learning environment in China. *Computers & Education*, 50, 838-852. doi:10.1016/j.compedu.2006.09.001
- Stols, G.H. (2007). Designing mathematical-technological activities for teachers using the Technology Acceptance Model. *Pythagoras*, 65, 10 – 17.
- Teo, T., Lee, C., B., Chai, C., S., & Wong, S. L. (2009). Assessing the intention to use technology among pre-service teachers in Singapore and Malaysia: A multigroup invariance analysis of the Technology Acceptance Model (TAM). *Computers & Education*, 53, 1000-1009.
- Venkatesh, V., & Davis, F. D. (2000). A theoretical extension of the Technology Acceptance Model: Four longitudinal field studies. *Management Science*, 46(2), 186-204.
- Venkatech, V., Morris, M., Davis, G., & Davis, F. D. (2003). User acceptance of information technology: Toward a unified view. *MIS Quarterly*, 27(3), 425-478.

# CHALLENGES TEACHERS FACE WITH INTEGRATING ICT WITH AN INQUIRY APPROACH IN MATHEMATICS

Anne Berit Fuglestad

University of Agder

*During more than two decades Norwegian educational authorities have promoted the use of computers, calculators, and more recently, digital tools in teaching in general in various school subjects. This paper reviews efforts to use digital tools in teachers' and pupils' work and problem solving in mathematics within the frame of two research and development projects in six lower secondary schools in Norway. The analysis sets out to characterise and contribute to a deeper understanding of the challenges teachers face with inquiry approach to mathematics.*

Keywords: digital tools, TPACK, teachers' role, teachers' competence, development

## BACKGROUND AND RESEARCH QUESTION

### A brief history of ICT in Norwegian schools

Since 1984 educational authorities in Norway have promoted the use of Information and Communication Technology (ICT) such as computers and calculators in schools. Some teachers and schools introduced computers on a trial basis in the 70's, but the major effort from educational authorities came with the Stortingsmelding 39, 1983-84 (KUF, 1984) which included a four-year plan of action to implement computer technology in some selected trial schools, starting in 1984. The aim of introducing computers was to learn about computer technology in society and to use computers as a teaching aid, particular attention was directed to vocational education and pupils with special needs. A goal was to avoid introduction of new reasons to create differences between pupils. Emphasis was placed on introducing computers with software that represent new opportunities to support learning which other material or teaching aids cannot provide. A number of schools became experiment schools and were provided with equipment. Furthermore an office for coordinating and leading the effort was established on national basis. This happened at a time (1984) when micro computers were fairly new and with limited graphics, memory and processing capacity. To educate teachers for use of ICT a course of 40 hours was offered and half year courses (30 ECTS) were developed for teacher education. The content was dominated by programming, use of databases, and limited education about teaching with ICT. The implementation of ICT as a teaching aid was not clear and purposeful. "Let a thousand flowers bloom..." and find out what works seemed to be the idea.

The first plan of action was later followed by new strategic plans for ICT in schools, and computer technology was built into national curriculum guidelines (UFD, 2004; Krumsvik, 2007). Despite the official policy and initiatives the introduction of ICT in education has been slow in particular related to specific subjects (ITU, 2009).

In the most recent curriculum guidelines, the “Knowledge promotion” from 2006, facility with digital tools [1] was made compulsory as one of five basic competencies and to be utilised in all school subjects. Furthermore, it is implied that use of digital tools will be accessible in examinations with the requirement that some test items have to be solved with computers.

### **Access to and use of digital tools**

Recent surveys show that computers are now widely accessible in Norwegian schools and many pupils have their own portable computer, either private or provided by the local school authorities. On average there are 4.7 pupils per computer in primary and lower secondary schools, and 1.8 pupils per computer upper secondary schools. Internet is accessible from 90% and 98% of the computers respectively. Until recently use of digital tools, except calculators, has been limited in school subjects, and notably less in mathematics than in Norwegian, English and Social studies. This is documented in biannual reviews of technology use in schools (ITU, 2010; Arnseth, Hatlevik, Kløvstad, Kristiansen & Ottestad, 2007). Furthermore, teachers seem to be happy with learning digital competence through trying out themselves and getting guidance from colleagues.

In mathematics a spreadsheet like Excel is the most commonly used software. Dynamic geometry like Cabri, and more recently Geogebra has become popular. Internet is used like in several subjects to search for information and numerical data, e.g from Statistics Norway, (<http://www.ssb.no>). Drill and practice software is used to develop skills with numbers, perhaps most common with younger children. Recently computers have been introduced in the final examinations in mathematics in lower secondary schools, so far this is mainly the use of spreadsheets, and this raises the demands to use digital tools in teaching.

So, the problems related to lack of equipment are mainly solved in Norwegian schools, and suitable software is available. The limited use of digital tools for mathematics teaching and learning indicates there is a challenge related to understanding and appropriate use of digital tools for teaching and learning mathematics. Related to this some questions arise: What is the role of ICT in learning mathematics? What is the nature of digital tools and how can they be utilised? This leads to the following research question related to teaching mathematics:

### **Research question**

What challenges do teachers meet in teaching mathematics supported by digital tools and to provide learning experiences with an inquiry approach for their pupils to develop mathematical understanding and skills? Digital tools here are computers with open software.

The question can be divided into several sub questions related to understanding of the role of digital tools in work and learning of mathematics, challenges related to the digital tools per se or related to organising computer use integrated in the mathematics classroom. In the following I will focus on the subject oriented use of computers with an inquiry approach to mathematics. The research question will be investigated by reanalysing/ review cases from two projects on ICT in mathematics with regard to teachers' understanding of technology, mathematical representations and teaching approach.

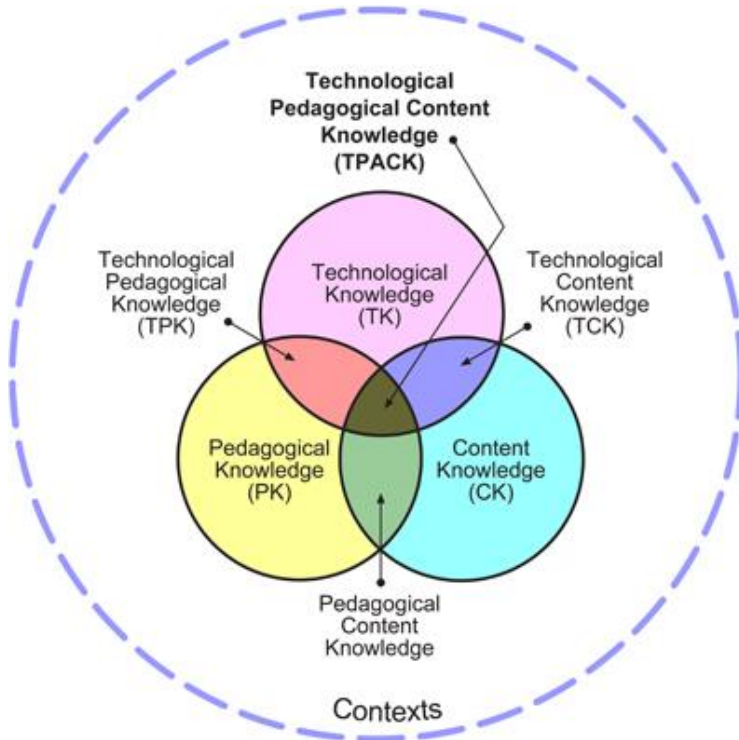
## THEORETICAL BACKGROUND

Teaching in general and with digital tools particularly is a complex activity for the teachers; it involves deep insight in the subject, knowledge of the digital tools, and understanding of pupils' thinking. Furthermore, it involves pedagogical approaches and relating these to the curriculum plans and policy involved and so on, all in a dynamic changing environment. Shulman (1986) introduced the term *pedagogical content knowledge*, PCK, to denote the intersection of pedagogical and content knowledge in order to consider the complex interaction between pedagogy and subject content. The mathematical content and pedagogy, including the teaching approach or considerations related to teaching a specific content or subject, cannot and should not be separated. For mathematics it involves for example how to approach particular mathematical ideas and processes and make the subject accessible for pupils, to understand common errors and various ways pupils think about the subject. Ball, Thames & Phelps (2008) study teaching practice to understand what they call *mathematical knowledge for teaching* and found this knowledge is highly specialised for mathematics. The teacher needs to know more than to find a solution; they need to know various approaches and to be able to follow up pupils thinking. Mishra and Koehler (2006) extended Shulmans' model for teacher knowledge to include technology and introduced the term *technology pedagogical content knowledge*, TPCK, later recast as TPACK (Koehler & Mishra, 2009). Figure 1 (from <http://tpack.org/>) indicates several areas of knowledge. In particular there is a need to understand and develop the knowledge related to technology and content TCK, technology and pedagogy TPK – or how technology affords new approaches to teaching. In the centre is the integration and dynamic interplay of content, i.e. mathematics, with technology and pedagogy (Niess, Ronau, Shafer, Driskell, Harper, Johnston, Browning, Özgun-Koca, & Kersaint, 2009).

Introduction of technology or more specific, digital technology, in teaching implies not just learning to handle the computers with software and other digital tools, but relating the technology to the other knowledge areas, pedagogy and mathematical content knowledge. General knowledge of computer software, i.e, TK, like handling menus and keyboard commands, handling file system and even perform operations specific to the software is not sufficient. In order to develop TCK for mathematics it is necessary to understand how mathematical concepts and relations are, or can be



represented within the software. Furthermore, insight is needed to realise how the software bring in new affordances and constraints related to investigation and solving mathematical problems. An example is spreadsheets where it is possible dynamically to manipulate and experiment with a whole table generated by formulae, and illustrate the distribution of values with a diagram based on the table.



Spreadsheets have common mathematical and statistical functions built in and provide search and sorting facilities. However, as pointed at by Dettori, Garuti, Lemut & Netchitailova (1995), spreadsheets have limitations with lack of symbolic manipulation and are numerical, not algebraic in nature in spite of the use of

### **Figure 1: TPACK - Technological Pedagogical Content Knowledge**

formulae for calculations. Another example of mathematical software is dynamic geometry like Cabri, Geogebra and others. After constructing a figure using built in features in dynamic geometry, part of the figure can be dragged to test if the construction is robust, i.e. not falling apart when dragging one part of the figure. A constructed figure in Cabri is not the same as a drawing which can be represented with various constructed figures in Cabri with different properties (Laborde, 1995). The order of the single constructions for composing a figure make the figure specific and different from the same shape constructed in another way.

The development of deep knowledge TCK can be expressed with terms from the instrumental approach as turning the artefact into an instrument for the user (Trouche, 2005). For a specific artefact this implies to develop utilisation schemes, knowledge on the level of simple usage schemes and more elaborated instrumented

action schemes. Without going in detail I find the terms helpful to indicate the depth and complexity of knowledge involved.

Mishra and Koehler (2006) point to how the TPACK framework can be used both as an analytic tool and as a tool for designing curriculum development. In particular they point to learning by the design based on principles of learning by doing and with less emphasis on overt lecturing and traditional teaching.

## **RESEARCH AND DEVELOPMENT WITH ICT IN MATHEMATICS**

During the last decade ICT in mathematics has been studied in several developmental research projects at the University of Agder (UiA) with close collaboration between teachers and didacticians. During the years 2001 - 2004 the *ICT competence project in mathematics in lower secondary schools* [2] was set out develop pupils' competence with ICT tools so they could judge and choose for themselves which tools to use for a mathematical task, just pencil and paper or computer. Software in use was a spreadsheet, dynamic geometry and a graph plotter, intending to stimulate an investigative approach to teaching and learning.

*ICT and mathematics learning* (ICTML)[3] ran through 2004 – 2007 in parallel with the project *Learning Communities in mathematics* (LCM), both with emphasis on developing a learning community with teachers and didacticians, with inquiry into mathematics and mathematics teaching as fundamental basis for the projects (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild & Grevholm, 2007). The projects employed a socio cultural view of learning. The pedagogical (or didactical) content of the projects emphasised pupils and teachers inquiry, to wonder and ask questions, investigate and explore and develop an inquiry attitude to teaching and learning.

ICT has also been an interest in the most recent project, *Teaching Better Mathematics* (TBM) (2007-2010) which build on the same fundamental principles and extend the work on ICT to upper secondary school. The projects have been concerned with how ICT can provide learning experiences for pupils through an inquiry approach to teaching and learning. Work with teachers in workshops and school team meetings to stimulate the development have been central for the development and research carried out on all areas of the work. The research methodology was developmental research (Gravemeijer, 1994) with strong interaction and integration of development and research. In all the projects mentioned a qualitative research approach was employed utilising field notes, pupils' computer files, video and audio recording of meetings, workshops and interviews. All kinds of events were recorded and additional information in e-mails and pupils work on paper or computer files were collected

## **SMALL CASES FROM PROJECTS**

In this paragraph I will not provide the full background of the projects, but will present some episodes that can illuminate challenges teachers face as they use digital

tools in their classrooms. The teachers all had some background with ICT, mainly on spreadsheets, but had hardly any experience from using ICT for mathematics teaching (Teaching here includes providing learning experiences for pupils).

### Case 1: The teacher solved the task

In a working period over a couple of weeks to summarise and evaluate the effects of the ICT competence project, the pupils were given a booklet of 10 tasks from which they could choose which tasks to work on, and whether they would like to use just pencil and paper or computers. Most pupils worked in pairs or small groups.

Some pupils in one class worked on the stamps task: to combine some stamps of two kinds worth 2.50 NOK [4] and 1.80 NOK to get the value 20.40 NOK for sending a small parcel (Fuglestad, 2007). Several pupils worked on simple tables in a spreadsheet to combine some of each kind of stamps adding up to 20.40 NOK. They experimented with swapping the values and how many stamps they needed. However, I observed one group of three pupils preparing a large table in Excel to make all possible combinations of number of stamps, to be sure to get all the different solutions. When I asked them to tell about their work, the pupils described their table and how they could find all the solutions. Then they commented they really did not understand, but when they asked for help the teacher, Ivar, instructed them how to make this table. Later I learned from Ivar that he made this table for himself the evening before as a preparation, and he interpreted the task to find all possible, not just one solution. The task invites to trial and error more than setting up an algorithm, and asks only for a combination of stamps, not all possible. In fact there is just one solution, but several with slight misreading of the numbers given, and so the teacher moved on to find a more general solution. Ivar seemed to think, he had to be able to present a complete solution, and when pupils asked he showed his solution.

Through his thorough preparation the teacher tells that he feels responsible to have a solution ready. When the pupils asked perhaps he too quickly provided a solution, and did not urge the pupils to investigate with sufficient time for their own solution. There is also the pressure from pupils to give help, and difficulty to know how to help except give a solution. The situation indicates the teacher put a lot of effort to solve the task, but perhaps used less time thinking of how to tackle the pedagogical task.

### Case 2: Exploring Snow man in Cabri

Trude, a teacher at Fjellet School taking part in ICTML, decided to implement Cabri with her class of grade 8 pupils over some weeks (Fuglestad,

- 1) Snow man with a stick, hanging together
- 2) Make a figure by reflection
  - a) Reflection in a line
  - b) Reflection on a point
- 3)  $\triangle ABC$      $AB = 5 \text{ cm}$   
                           $BC = 4.5 \text{ cm}$   
                           $AC = 2.5 \text{ cm}$
- 4) Make the circumscribed circle

**Figure 2: Task on flipover**

2010). She did not use any prewritten material but wrote the task on a flip-chart (translated in Figure 2), planned for pupils inquiry into Cabri. She gave only a short introduction to the tasks and encouraged the pupils to investigate and find out how to solve it. The pupils were asked to experiment and find out about facilities in Cabri, including meaning and effect of some menu items and in this way build usage schemes, the simple one step operations. The pupils worked together in small groups or alone, and had opportunity to discuss their solutions with peers or the teacher. Trude wandered around and supported the pupils with questions or hints and in this way supported their further development of Cabri as an instrument, developing their instrumented actions schemes by combining several operations. In particular she emphasised the feature of dragging objects to test if the figure “hangs” together, in particular relevant for the Snow-man task.

Trude later presented her work in a workshop in the ICTML-project. She did not know Cabri before but was first introduced to it through the project, and worked a lot on her own to prepare before using Cabri with her pupils. She emphasised the need to learn Cabri before she “dared to use Cabri in the class”. Trude became stressed by preparing for the Cabri sessions, but also expressed great enjoyment arising from her work on Cabri with her pupils. Also on other occasions she talked about excitement over how pupils’ became engaged with investigations. Trude’s own development during the work seems related both to the technology and the pedagogy component of TPCK. The technological mathematical knowledge or more precisely TCK is also affected in this as it is necessary to know the way mathematical relations are represented in Cabri.

### **Case 3: Cabri and classical constructions**

At another ICTML school, Austpark, the teacher team talked about introducing Cabri, but had experienced problems to get started due to several practical obstacles. Two factors seemed to encourage their decision to implement Cabri in their classes. A course day for preparation was provided from the ICTML-project at their school and a new teacher, Jacob, started working at the school. Jacob had experience with ICT and dynamic geometry. All teachers at grade eight and two on grade nine started to use Cabri. The teachers choose to use some worksheets that were introduced by Jacob and they regarded this material to be suitable as they found it covers a substantial part of the syllabus and it was close to their own way of working (Erfjord, 2008). The material consisted of seven two-page worksheets and was structured with explanations and step by step instructions of how to carry out constructions in Cabri. The approach characterised as direct instruction with supportive comments was also observed in the teaching of classes at Austpark (Erfjord, 2008). The teachers implemented constructions using Cabri in parallel with use of compass and ruler which is the traditional way in plane geometry in schools. In addition to this convenient approach, they also argued that they expected the classical constructions to be necessary for the coming final examination for pupils. They seem not to

consider other more investigative material that had been presented in a workshop in the project.

In conversations with a didactician the teachers claimed they found it difficult to be more innovative and make their own tasks. It seems difficult to both change style of teaching into an inquiry approach and introduce digital tools at the same time. Erfjord (2008) characterised this as a *double innovation* embedded in the project goal of ICTML. The teachers developed their competence with Cabri for mathematics, the TCK, but preferred to develop their TPK close to their tradition. The reasons are understandable and reflect the pressure for curriculum, coming examination and time constraints, but also their reluctance to engage deeply to change teaching style.

#### **Case 4: Develop material for investigation using Excel**

Three teachers at Dalen school set their goal for their participation in the ICTML-project to develop their own library of Excel-tasks to complement the textbook they used (Fuglestad, 2010). The tasks would support pupils to investigate mathematical connections and properties of percentages, fractions and decimal fractions, area and volume of specific object and the like. The Excel-tasks were to support pupil' inquiry and experiments, and follow on with support for pupils own work to prepare new spreadsheets for other tasks. During the work, tasks were tried out in the class with one or two didacticians observing together with colleagues, and later follow up with discussing experiences and further or new development in the school team which met regularly every second week. Development of the tasks challenged the teachers both on technology and on pedagogical approaches. For example with a task to investigate equal-valued fractions and display several of the same value, it was necessary to develop complicated formulae with nested if-sentences. This was solved by discussing ideas and consulting another colleague who had long experience with Excel. Pedagogical challenges like judging how complicated a comparison would look for pupils, came up after observations in discussions. The interaction between didacticians and teachers was seen useful for both parts in the collaboration and also for stimulating the further development.

Several spreadsheet tasks were made in a similar way, with setting up an environment for the pupils to explore, to insert numbers and observe the results, discuss and experiment to find relations. The tasks may seem fairly directive, setting limits for investigations, directed to certain relations planned for inquiry in a limited knowledge area. However, there were also more open tasks, and tasks to develop their own set up or models on a spreadsheet after using the pre-made tasks for investigations.

## **DISCUSSION AND CONCLUSION**

Although the cases presented are all different they show some similarities. Some key words are: preparation, confidence and control. Furthermore: limitations in understanding of the TPCK as integrated knowledge.

The teachers feel the responsibility to have solutions to the tasks they present for the class. Both Trude and Ivar put a lot of work into their preparation, solved the tasks for themselves and worked on learning the software. This is probably reflecting their wish to feel confident and in control over what will happen or feeling the responsibility for having a solution at the end of the lesson. Trude expressed this as she needed to learn Cabri for herself to dare to use it in class. The pedagogical/didactical challenges related to implementing an inquiry approach with use of ICT seemed to put extra pressure on the need to prepare to feel confident.

Some teachers were reluctant to change teaching approach and therefore tried to implement ICT alongside traditional methods, to avoid large changes. Even if the teachers want to use digital tools with the resulting changes, there seems to be some deep seated traditions and external pressure that make changes difficult. At Austpart this was expressed and commented on when they choose the teaching material that was close to their tradition, and wanted to be sure pupils also could handle ruler and compass for construction.

Also the teachers at Dalen invested a lot of time and effort to learn to use the tool, Excel, and to develop their own teaching material on file, prepare the lessons and discuss experiences and development. Reflecting on issues of control and inquiry, they seemed to find a way between the two sides, and planned for inquiry into a limited mathematical topic. On the other hand, the teachers seemed open and confident to use an inquiry approach in their classes.

As noted in the introduction of this article the process of implementing ICT tools in mathematics teaching has been slow and is still limited. Experiences in general, and from the projects point to hindrances and resistances experienced even when teachers wish to use ICT tools. Perhaps this is due to lack of understanding of what is involved in developing the technological pedagogical content knowledge, the TPCK for mathematics. Affordances and constraints are related to the various areas of knowledge as shown in Figure 1, in particular how mathematics is represented, how the pedagogy if inquiry approach can be implemented with ICT and so on. I think a study of the various knowledge areas and combination of partly and full integration of mathematical content, inquiry approach, affordances and constraints of relevant technology would prove useful to guide further developmental research in the area.

## **NOTES**

<sup>1</sup>. ICT and digital tools will be used interchangeably

<sup>2</sup>.The ICT competence project was supported by a grant from the Department of Education to stimulate ICT in schools

<sup>3</sup>. LCM and ICTML were supported by The Research Council of Norway (RCN), TBM was supported by RCN and The Competence Development Fund of Southern Norway.

<sup>4</sup>. NOK is Norwegian crown, the Norwegian currency

## REFERENCES

- Arnseth, H. K., Hatlevik, O., Kløvstad, V., Kristiansen, T., & Ottestad, G. (2007). *ITU monitor 2007 Skolens digitale tilstand 2007*. Oslo: Universitetsforlaget.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher education*, 59, 389-407.
- Dettori, G., Garuti, R., Lemut, E., & Netchitailova, L. (1995). An analysis of the relationship between spreadsheet and algebra. In L. Burton & B. Jaworski (Eds.), *Technology in Mathematics Teaching* (pp. 261-274). Chartwell-Bratt.
- Erfjord, I. (2008). *Teachers' implementation and orchestration of Cabri-use in mathematics teaching*. Kristiansand, Norway: University of Agder.
- Fuglestad, A. B. (2007). Students' attitudes, choice of tools and solutions of mathematical tasks in an ICT rich environment. In C. Bergsten, B. Grevholm, H. S. Måsøval, & F. Rønning (Eds.), *Relating practice and research in mathematics education. Proceedings of Norma05, Fourth Nordic Conference on Mathematics Education* (pp. 229-242). Trondheim: Tapir Akademisk Forlag.
- Fuglestad, A. B. (2010). Inquiry into mathematics teaching with ICT. In B. Sriraman, C. Bergsten, S. Goodchild, G. Palsdottir, B. Dahl, & L. Haapsalo (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education* (pp. 91-108). Charlotte: Information Age Publishing Inc.
- Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, 25, 443-471.
- ITU (19-8-2010). *ITU monitor 2009 - The digital state of affairs in Norwegian schools 2009* Oslo, ITU, University of Oslo. Retrieved from <http://itu.no>
- Jaworski, B., Fuglestad, A. B., Bjuland, R., Breiteig, T., Goodchild, S., & Grevholm, B. (2007). *Læringsfellesskap i matematikk - Learning Communities in Mathematics*. Bergen: Caspar Forlag AS.
- Koehler, M. J. & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, 9, 60-70.
- Krumsvik, R. (2007). *Skulen og den digitale læringsrevolusjonen*. Oslo: Universitetsforlaget.

- KUF (1984). *St. mld. nr 39 Datateknologi i skolen*. Oslo: Kirke-og undervisningsdepartementet.
- Laborde, C. (1995). Designing Tasks for Learning Geometry in a Computer-based Environment. In L.Burton & B. Jaworski (Eds.), *Technology in Mathematics Teaching - a bridge between teaching and learning* (pp. 35-67). Chartwell Bratt.
- Mishra, P. & Koehler, M. J. (2006). Technology pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, 108, 1017-1054.
- Niess, M. L., Ronau, R. N., Shafer, K. G., Driskell, S. O., Harper, S. R., Johnston, C. *et al.* (2009). Mathematics teachers TPACK standards and developmental model. *Contemporary Issues in Technology and Teacher Education*, 9(1), 4-24.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.
- Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculator environments. In D.Guin, K. Ruthven, & L. Trouche (Eds.), *The Didactical Challenge of Symbolic calculators* (pp. 137-160). Springer.
- UFD (2004). *Program for digital kompetanse 2004-2008*. Oslo: Undervisnings-og forskningsdepartementet.



# TECHNOLOGIES AND TOOLS IN TEACHING MATHEMATICS TO VISUALLY IMPAIRED STUDENTS

Iveta Kohanová

Faculty of Mathematics, Physics and Informatics, Comenius University

Bratislava, Slovakia

*In this paper we describe the effort of the Support Centre for Visually Impaired Students in Bratislava to design an effective and helpful digital tool for learning and teaching mathematics to visually impaired students. In the first phase of our research methodology, which follows the design-based approach, we focus on mapping of the actual situation of teaching math to those students at each level of education in Slovakia. The emphasis is placed on tools and technologies these students use, as well as on problems that occur when using these tools. Later on we focus on LAMBDA editor, which seems appropriate for learning and teaching math to those students. Its adaptation to local conditions, which consisted of several iterative cycles of analysis, development, evaluation and revision, is reported in the second part of the paper.*

Keywords: visually impaired students, digital tools for teaching mathematics, LAMBDA editor

## INTRODUCTION

The changes in society, brought up by an inflow of liberty and humanism, have led to the integration of handicapped people in Slovakia in 1993, and it has become an actual problem. Besides new social relations and situations it was also important to pay more attention to collective education of sighted and non-sighted students, which is particularly difficult when teaching mathematics. It is obvious that reading and writing standard text is completely different than reading and writing mathematics. One can consider mathematics even as a language on its own relying on different types of representations. This raises a question: how can we teach math to a student who is visually impaired? Practice shows that it requires using various tools that can give to the visually impaired students new view on mathematical objects, in order to support their imagination and simplify their manipulations with these objects. General approaches to the teaching math to visually impaired students in the world use tactile representations, audio aids, tonal representations, haptic devices and integrated approaches. The study of the actual Slovak situation, reported in the next section, points out the most serious problem, which is the absence of a uniform linearized notation. We thus focus on looking for a digital tool that could enable visually impaired students to write mathematical texts and perform calculations.

***Overview of the actual situation*****Primary school**

Most of the Slovak visually impaired children (blind and partially sighted) attend special primary schools (in Slovakia, primary school lasts 9 years). During mathematics lessons, students use Braille books with tactile pictures, electronic notebooks to take notes and mechanical typewriters for calculations. The disadvantage of a typewriter is, first, that the way to get a result of a calculation takes too long, the pupils thus try to calculate in their minds and, second, the notation of the calculations is too verbiage, so after a while the pupil is lost. Teaching mathematics at the primary level means first of all helping children to use and organize their experiences, which they gain from actions and interactions with the world around them. Some authors claim that the main goal of mathematics education is to develop an awareness of numbers and coping with different relations and dimensions (Csocsan, Klingenberg, Koskinen and Sjostadt, 2002). The most frequent difficulties blind pupils encounter in mathematics are the following:

- generalizing – finding similarities in different activities in everyday life,
- translating activities and actions into a mathematical language,
- lacking flexibility in problem solving and in calculations,
- translating and transferring three-dimensional objects into two-dimensional iconic forms.

**Secondary school**

There are special secondary schools for visually impaired students in Slovakia, which are mostly oriented on music or some handicrafts (they usually last 3-4 years). If a student wants to come into contact with mathematics then s/he needs to attend a "normal" secondary school, which usually lasts 4-5 years. As we know, mathematics is a subject which is important for studying not only natural sciences such as physics, chemistry, computer science or biology, but it also begins to be popular in human sciences such as psychology, philology, sociology, etc. The direct consequence of this mathematical requirement almost everywhere is that also more and more visually impaired students today start their education in mainstream schools, which is the place where they can study math. Since the teachers in these schools are not specially educated in this field, they often have to use a "trial and error" method to find out the best way of teaching their blind students who are integrated among sighted students. Visually impaired students also face a lack of textbooks and study material, as well as a limited Braille math notation. On the other hand, most of visually impaired students at this level of education do not have difficulties to work with laptops with integrated screen reader and in this way to perform calculations; they already know all basic mathematical operations. However, the complexity of mathematical knowledge increases here very quickly in all fields: algebra, analysis, and geometry.

Hence, the students will have to overcome a lot of other new challenges, especially with the Braille notation of all new symbols. After having studied systems of Braille notation (Kohanová, 2003) in several European countries, we can state that more or less each of the mentioned norms suffers from a lack of rules for the notation of mathematical text. Therefore, the major part of visually impaired students creates their own particular mathematical language that is adapted to their conditions and requirements and the notation of these languages has a linearized form. But this raises new difficulties, because these “languages” are not necessarily comprehensible for people who the visually impaired students communicate with.

### *Communication with visually impaired students*

The integration of visually impaired students among sighted ones in the common schools creates new relations in the classroom. One has to distinguish three types of communication within these special conditions: communication between teacher and sighted students, communication between teacher and non-sighted students, and communication between non-sighted and sighted students. The first case is dominant and it can have various forms. Oral communication is essential in education; however, this fact is not valid in mathematics. In addition, mathematics requires exactness, definiteness, totality and comprehensibility of presentation. Teaching and learning mathematics is very arduous only by oral communication (e.g., when modifying expressions or making geometric constructions) and it is therefore supported by a graphical way - text or picture. It is very common that some students rather prefer notation or picture to talking/argumentation; in the case of visually impaired students it is practically essential. If we talk about graphical communication in the frame of communication between teacher and non-sighted students, we mean communication supported by typhlographic pictures and planes [1] or space models (construction kit, cubes, skewers, paper). Another form is communication supported by laptop; a non-sighted student takes notes or performs computations in the electronic form, mainly linearized. Data from interviews with visually impaired students and their teachers show that it happens quite often that two visually impaired students, who are in the same classroom, use different notations when working with mathematical text (Kohanová, 2003). This phenomenon complicates markedly the communication not only between the student and the teacher, but also between students. Another issue is the question of educational goals. Since in Slovakia there are no standards for teaching mathematics to integrated visually impaired students at the secondary level, the teacher thus has to determine requirements for these students on her/his own, on her/his subjective opinion.

### **University level**

Comparing to the secondary education, there is quite a different situation for blind students at the university in math. The students are supposed to have skills necessary for studying - take notes during lectures, read scientific texts, perform complex calculations, communicate with teachers and other students in written form, etc.

Much more autonomous work is required. If the student graduated at a special school/class and used only the Braille notation and spoken language for the before mentioned purposes, s/he will have to overcome a lot of new challenges. There are very limited sources of scientific literature accessible to a blind student. Therefore s/he should be able to read different mathematical notations. Another way of delivering mathematical expressions in an accessible written form is the electronic text document on a personal/portable computer or a special note taker for the blind users. This sort of document usually contains a linear mathematical notation with expressions built up of ASCII characters. A blind student can access this type of notation in two ways. S/he can either use the refreshable Braille display and read line by line the corresponding Braille cells (groups of 6 or 8 raised dots) by touch, or s/he can listen to a synthetic voice, which reads each written ASCII symbol for him/her. The second method is more difficult for reading complex mathematical expressions, however it could be quicker for longer text with simple mathematical expressions. The ideal is a combination of both methods, when student can choose the appropriate method depending on the current situation (what s/he is reading, writing or calculating). Some solutions, originally dedicated to electronic publishing of scientific text documents (TeX, LaTeX, AmSTeX, HrTeX, MathML), could be read and written by a blind student.

Computer Algebra Systems (CAS), such as Derive, MuPAD, MAPLE, MathCad or Mathematica, are dedicated at the first place to algebraic calculations, e.g., differentiation, integration or solving equations. They are also able to perform numerical calculations and to visualize graphs of functions, curves and 3-dimensional objects. They contain as well a lot of functionalities for analysis, linear algebra, statistics, numerical analysis, number theory, etc. These systems are also useful for visually impaired students, especially the calculation functionalities. It makes no sense to urge visually impaired students to perform calculations that are often just very tedious and mechanical. That is why CAS are helpful. If the commands entered into the command-line are linear, it means that they are fully textual and therefore suitable for visually impaired students. Another advantage of CAS is that the screen-reader does not have any problem to read linear text on the screen; it is thus accessible to the student. Therefore, blind students in Computer Science use CAS for example for calculations during Algebra seminars. It is a useful tool for calculations with matrices, which are time consuming and quite complicated. If they understand the principle, such a tool can save time and a lot of manual work.

## **ITERATIVE DEVELOPMENT OF LAMBDA SYSTEM**

As we mentioned above, the information and communication technologies (ICT) might be very helpful for visually impaired students studying mathematics, since they have largely improved educational opportunities. ICT have begun to be a very important part of the material milieu of didactic situations. The most important requirement for secondary school students is to handle mathematical expressions as

quickly and efficiently as their sighted classmates. Teachers who do not have any knowledge of Braille (usually those in integrated schools) ask for most suitable tools to facilitate the communication with visually impaired students. Later on, at the university, it is important to have a mathematical writing system that is powerful, flexible, and compatible with most common format standards, to enable independent scientific and mathematical work to be distributed digitally. The fact that has to be considered is accessibility. Recently, LAMBDA - Linear Access to Mathematics for Braille Device and Audio-synthesis appeared to supply all needed requirements. The LAMBDA project makes the provision for an integrated system based on a linear code and a software management system (the editor). The editor allows writing and manipulating of mathematical expressions in a linear way. A few facts about Lambda (according to Fogarolo, 2006):

- LAMBDA is intended to be used mostly by young people who learn mathematics, especially visually impaired students.
- LAMBDA is above all, even if not only, a didactical tool. It is the functional component which implements strategies devised in order to make text and mathematical expressions easy to read, write and manipulate by means of vocal output and Braille display, in an educational setting.
- It is important to define didactical requirements needed for a software writing system compared with Braille traditional ones. The change towards mechanic writing systems (such as typewriter and PC) requires skills relative to the management of textual documents using PC. If they are missing, traditional tools are preferable and the passage to new technologies has probably to be put off.

### **Methodology – Setting of the research**

We have first got in contact with LAMBDA system in December 2005 at the international conference in Rome: „*I don't see the problem: new prospects to access Mathematics and Scientific studies for Blind students*“, where the software was presented by its authors and by Italian visually impaired pupils who used and tested it. Later on, on the initiative of the Support Centre for Visually Impaired Students, Comenius University in Bratislava, we decided to investigate whether the LAMBDA system could be used as an educational tool to teach mathematics to visually impaired pupils in Slovakia. In the case that LAMBDA would prove to be a significant tool, the Support Centre planned to find ways of its practical application in Slovak schools. The methodology adopted in this research follows the design-based research approach. According to Plomp (2010), this methodology might be realized in a number of the following phases: preliminary research, prototyping phase and assessment phase. Our preliminary research consisted in mapping the actual situation in Slovakia with focus on tools and technologies used in math education of visually impaired students (see above). The prototyping phase is composed of four iterative

cycles of analysis, development, evaluation and revision of LAMBDA system used in Slovakia. All four iterations had a form of a course for visually impaired students or teachers, who were observed and interviewed in order to get feedback on the actual LAMBDA prototype. The assessment phase should give us an answer on the above-mentioned research question concerning LAMBDA as an educational tool.

### Process of iterations

The first iteration of the prototyping phase was conducted in spring 2006 in one of the two schools for visually impaired pupils in Slovakia. We organized together 5 lessons where the LAMBDA system (English version) was introduced. The study group was formed of 5 pupils of 9<sup>th</sup> grade (4 boys and 1 girl). The pupils were chosen by using a purposeful sampling method (interest in studying mathematics at upper secondary school; basic English and basic computer skills were required). Two of the pupils were short sighted, two virtually non-sighted and one non-sighted. The data for the next iteration were collected from observations during the lessons and from a semi-structured interview after the course.

During the course the participants became familiar with the working environment and the following important and useful features of the software (as advised in Bernareggi, 2006):

- Various possibilities to input characters and mathematical symbols.
- In many situations, it is important to perceive global information, related to the structure or the relation, to define every single time the most suitable paths and methods to face different issues. For example, given the following expression:

Its linear representation  $\frac{x^2 - 4}{x^2 + 4x + 4} + \frac{4}{x + 2}$  in LAMBDA is:

//x^2-4/x^2+4x+4//+//4/x+2//

It is evident that in reading the linear representation it is more difficult to find specific parts and to quickly understand the relations among the structures making the expression, which is an immediate operation for sighted people who use global and bidimensional exploration. Therefore, we presented effective exploration strategies to the students. LAMBDA offers exploration through movement operations, in the sense that one can move to the next numerator, denominator or corresponding separator or tag. The second possibility is a tag structure of an expression that enables to understand the overall structure of the expression and to find its specific parts. The most compressed structure of the above-mentioned expression is:

//+//

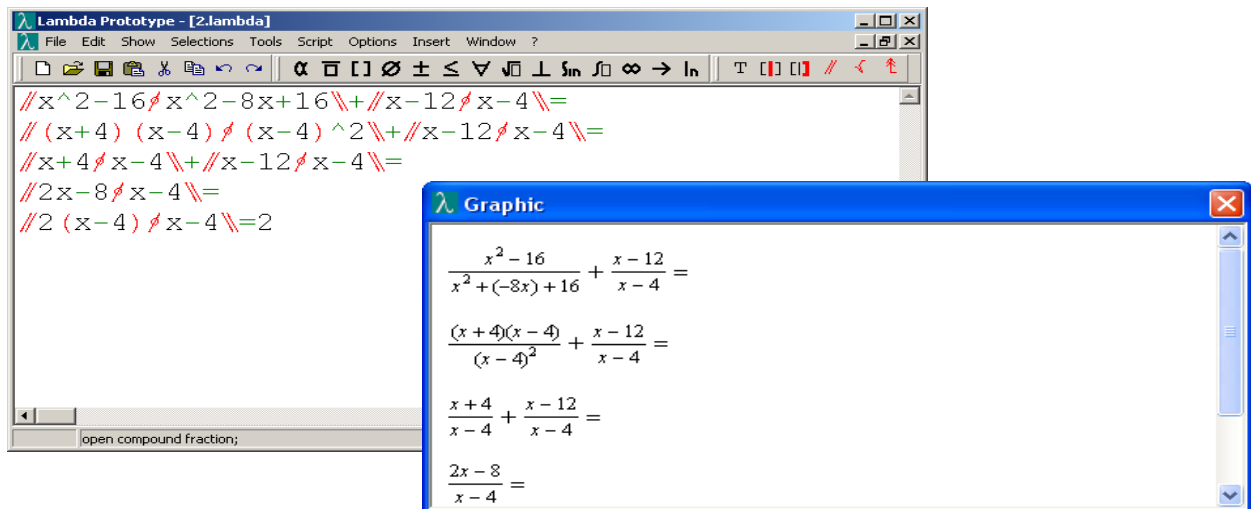
Another

$(( \quad ) \not\! / ( \quad ) \backslash + \not\! / 4 \not\! / x + 2 \backslash \backslash$

depth level looks as follows:

This visualization modality that hides the content of the block by maintaining blank spaces is useful as well to get some information about the size of the hidden blocks.

- Even the linear notation in LAMBDA is more or less intuitive; the graphical visualization (Fig. 1) might be helpful for the teachers, parents or sighted schoolmates in order to be able to help the visually impaired student or correct his/her productions by doing calculations.



**Figure 1: Graphical visualization of the calculation**

- Another beneficial tool that was introduced is a scientific calculator that is able to calculate numerical expressions and paste the result.
- LAMBDA enables automatic double copy of a selected row, we thus get “checking” and “working” lines. Here we have the possibility of checking the steps against a previous, unchanged “checking” line.

Two open-ended and two closed-ended questions were asked to the pupils during the interview after the course:

“Tell me about your experiences with LAMBDA during the lessons.”

“In your opinion, what advantages and what disadvantages LAMBDA has?”

“Do you think you have learnt how to work with LAMBDA?”

“Can you imagine working with LAMBDA when studying mathematics at school or at home?”

The data obtained from observations and interviews confirmed that all 5 pupils had learnt how to work with LAMBDA. Pupils considered LAMBDA as user friendly, more effective and also as a useful tool to do mathematics in comparison with the mechanical typewriter they used before the course. They appreciated the LAMBDA feature of “working” and “checking” lines, as well as the possibility to see what

structure they are working with. As expected, all pupils considered the English language as the biggest disadvantage; they would prefer to work with Slovak version.

The second iteration focused on the translation of the LAMBDA menu into Slovak (September 2006 – August 2007). During the academic year 2007/2008, we continued with courses, whose primary aim was to test the prototype of LAMBDA in the Slovak language and the secondary aim was to get another feedback from pupils on LAMBDA as a tool for studying mathematics. In September 2007, we realized a one-day intensive course for three students who were integrated in a “normal” classroom of an upper secondary school (at that time). All three of them participated in the previous course. This course helped us to discover several errors (terminological and semantic) and imperfections, their corrections were afterwards implemented into LAMBDA system.

The third iteration was realized in December 2007, when 7 students coming from primary or secondary schools in Slovakia attended the third course. This course focused on a revised Slovak version of the software and on the appropriation of some new functionalities of LAMBDA added by the authors. Four participants already knew LAMBDA from the previous courses (2 of them worked with the English version, 2 with the Slovak version) and for the others LAMBDA was new. The latter were observed during the course and asked the same questions as in the first research phase. The data obtained from these interviews showed again that pupils are able to learn how to work with LAMBDA, perform various mathematical calculations and they consider it as an effective tool for doing mathematics. Students who already worked with LAMBDA were interviewed once again. We were interested in what problems and complications they encountered when learning mathematics with LAMBDA at school or at home, that means, in situations when they had to communicate with their sighted schoolmates, teachers or parents, who are not used to a linearized form of mathematical calculations. This was our input data for a forthcoming seminar on how to use LAMBDA organized for mathematics teachers of visually impaired students. After the course, all students were asked to write us feedback on LAMBDA (regarding understanding, complexity, heavy operations, etc.) on the basis of their experiences during the summer term 2008. All highlighted defects were again corrected in the new version of the Slovak editor.

As it was already mentioned, the next iteration focused on teachers at primary and secondary schools who teach mathematics to visually impaired students, because they are also important in the process of studying mathematics. In February 2008, the Support Centre prepared a course for teachers in order to introduce them to LAMBDA as the educational tool. 23 teachers from various parts of Slovakia attended it. They got familiarized with principles of text linearization, LAMBDA basic functionalities, they learnt how to work with it and what are its advantages in comparison with other systems. They could observe one non-sighted student working



with LAMBDA, who demonstrated how quickly and effectively he is able to write dictated mathematical text and, later on, to edit (calculate) it. At the end of the seminar, the teachers expressed their willingness to use LAMBDA in their classrooms, when communicating with their visually impaired students.

In March 2008, we have created a web-forum for students and teachers who work with LAMBDA in order to help them with any problem that could arise. One of the outputs of the realized courses was also a methodical guide of using LAMBDA intended for teachers and parents of visually impaired students, which was distributed to them.

## **DISCUSSION AND CONCLUSION**

In the last years, the solution of the problem of accessibility to mathematics visually impaired students face seems to have an electronic form, which was studied by several authors (Kobolková & Lecký, 2002; Miesenberger, Klaus and Zagler, 2002). There exist systems that are blind friendly so that visually impaired students can do (calculate, read, write) mathematics in the way that is also accessible for their sighted schoolmates and teachers. In Slovakia, we have started to use the LAMBDA system and adapt it to the local conditions in several steps/iterations in accordance with the design-based research methodology. Thus, during the years 2006-2008, we realized several courses for visually impaired students and their teachers in order to find out whether LAMBDA system could be used as an educational tool for visually impaired pupils studying mathematics in Slovakia. Data obtained from observations and interviews first showed the need of LAMBDA localization into the Slovak language. Later on, when LAMBDA was used in math education by a few students, other problems arose. We tried to solve those problems and implement their solutions into the subsequent versions of LAMBDA. Through active web-forum, we got, during the years 2008-2009, a request from students and teachers for further development of the software, which appeared as not usable and quite demanding, since meanwhile LAMBDA became a commercial tool. These facts make it impossible to use LAMBDA as a norm in Slovakia. That is why we decided to develop a new tool for visually impaired users for working with mathematics in the electronic form. It should comply with the following features:

- open-source application, platform independent and providing support for free screen readers;
- usage of standardized formats (XML) to store the results;
- easy language localization and easy extension of new symbols and functionalities;
- standard graphical visualization of the notation for sighted users;
- easy program control not only for visually impaired users, but also for sighted users.

These features should reduce almost all problems that caused the impossibility to adapt LAMBDA software to Slovak conditions. Further research will show whether the new developed tool will prove as effective and helpful tool for learning and teaching mathematics at Slovak primary and secondary schools and universities. In addition, further research should also study teacher's instrumental orchestration (Trouche, 2004) in such specific conditions, where visually impaired students are integrated into a common classroom with sighted students.

### NOTES

1. Typhlographic pictures and planes enable visually impaired people to explore things that are usually not accessible to them. They can touch objects that are distant (sun, moon, clouds), too big (castle, train) or too small (butterfly, ladybird) and they can thus feel their shapes, develop imagination or orient themselves in space.

### REFERENCES

- Bernareggi, C. (2006). *A framework for vocal and tactile access to science in an educational environment*. Università degli studi di Milano, Facoltà di scienze matematiche, fisiche e naturali, Doctoral thesis, Milano
- Csocsan, E., Klingenberg, O., Koskinen, K. L., & Sjostadt, S. (2002). *MATHS "seen" with other eyes: a blind child in the classroom. Teacher's guide to mathematics*. Ekenas Tryckeri Aktiebolag
- Fogarolo, F. (2006). *LAMBDA system and mathematics teaching method*. CSA Vicenza
- Kobolková, M., & Lecký, P. (2002). Experience with access to mathematics for blind students in Slovakia. In K. Miesenberger *et al.* (Eds.), *Computers Helping People with Special Needs, Lecture Notes in Computer Science* (Vol. 2398, pp.159-177), Berlin/Heidelberg: Springer.
- Kohanová, I. (2003). *Možnosti prístupňovania matematických textov v elektronickej forme nevidiacim*. Diploma thesis. FMFI UK, Bratislava
- Miesenberger, K., Klaus, J., & Zagler, W. (Eds.) (2002). *Computers helping people with special needs*, 8th International Conference, ICCPH 2002, Linz, Austria, July 15-20, Proceedings. Springer
- Plomp, T. (2010). *Educational Design Research: an Introduction*. In T. Plomp & N. Nieveen (Eds.), *An Introduction to Educational Design Research* (pp. 9-35), SLO, Enschede
- Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning*, 9, 281-307

# A STUDY ON MATHEMATICS TEACHERS' USE OF TEXTBOOKS IN INSTRUCTIONAL PROCESS

Meriç Özgeldi<sup>1</sup>, Erdinç Çakıroğlu<sup>2</sup>

<sup>1</sup>Mersin University, <sup>2</sup>Middle East Technical University

*This paper provides an analysis of mathematics teachers' use of curriculum materials. 13 elementary mathematics teachers participated in the interviews for how they used the curriculum materials, specifically textbooks. The purpose of this study is to examine what mathematics teachers do with curriculum materials and how they use them for mathematics. The results of the interviews indicated that mathematics teachers used different textbooks to make instructional decisions, and they mostly adapted problems and examples in a constructive way.*

Keywords: mathematics teachers' use of curriculum materials, mathematics textbooks, and teachers' instructional decisions

## INTRODUCTION

One crucial role teachers play in the school context is to transform and implement curricular ideas in classrooms. In their implementation processes, they often benefit from different types of curriculum materials, including textbook and other written resources. Curriculum material is an integral part of teachers' daily work and offer ongoing support for pedagogy and subject matter content throughout an entire school year (Collopy, 2003). They provide ideas and practices which frame classroom activity via text and diagrammatic representations and help teachers in achieving goals that they presumably could not or would not accomplish on their own (Brown, 2009). Certainly, written curriculum materials such as the textbook, worksheet, and teachers' guide are mostly used curriculum materials in the school context. Particularly, textbooks are among the most widely used and trusted curriculum materials that are directly related to teacher's teaching and student's learning (Beaton, Mullis, Martin, Gonzalez, Kelly, & Smith, 1996). Although the term 'curriculum materials' has a general meaning involving a variety of resources, the current study focuses on mathematics textbooks and accompanying student workbooks, teacher guidebooks, and other written resources that are available to teachers.

In general, mathematics curriculum materials such as textbooks, texts, computer software, and geoboards are integrated into mathematical and instructional intentions and possibilities for school mathematics (Adler, 2000). Mathematics curriculum materials have been viewed as critical resources for students' learning of mathematical content and teachers' mathematical instructional decisions; and teachers are accustomed to using them to guide instruction (Stein & Kim, 2009). In this sense, mathematics textbooks are used "as source[s] of problem and exercises, as reference book, and as a teacher in themselves" (Howson, 1995, p.25) because

teachers often rely heavily on textbooks for many decisions such as what to teach, how to teach it, what kinds of tasks and exercises to assign to their students (Robitaille & Travers, 1992). Therefore, it is reasonable to argue that the mathematics textbook is an important part of mathematics learning and teaching context in which students and teachers work.

Studies covering period of 25 years on characterizing and studying how the curriculum is actualized in schools addressed teachers' interactions with curriculum materials and the role of the curriculum materials implementation. The critical point for understanding the curriculum use depends on the process of understanding what teachers do with mathematics curriculum materials and why as well as how their choices influence classroom environment (Remillard, 2009). Their value is likely to depend on the ways they are used (Cohen, Raudenbush, & Ball, 2003). In this sense, understanding the use of textbooks and other relevant written resources by teachers plays an important role in exploring the pedagogical approaches used in the classroom.

Research has shown that when teachers interact with curriculum materials, they do so in dynamic and constructive ways rather than a straightforward process (Brown, 2002; Davis & Krajcik, 2005; Remillard, 2005). Teachers frequently make changes in the curriculum intentions and modify them according to the structure and the purpose of lessons. In doing so, the availability, quality, and flexibility of the curriculum materials play a critical role in teachers' decisions. In general, teachers transform the curriculum ideas, lesson plans, and mathematical tasks into real classroom activities (Remillard, 2005). Therefore, understanding the teachers' interactions with curriculum materials requires an integrated analysis of their uses in the classroom teaching and learning context. For example, Brown (2009) has revealed a kind of interaction between teacher and curriculum materials which involves multiple steps. According to this interaction, teachers first *select* materials; however, the options offered to the teachers are often restricted by higher organs in the educational hierarchy. Second, they *interpret* these materials in planning and during instruction with regard to their perception of materials. Third, they *reconcile* their perceptions of the intended plan with their own goals and with the limitations of the setting. Fourth, they *accommodate* the students' interests, experiences, and limitations. Finally, they *modify* the setting according to their own decisions and to their students' capacities. In fact, these steps proposed by Brown partly reflect the dynamic and constructive relationship between teachers and textbooks.

In sum, understanding teachers' use of textbooks and other relevant curriculum materials provides insight into the contribution of such materials to classroom learning. In this context, the purpose of this study is to reveal what curriculum materials-specifically the textbooks- are crucial to mathematics teachers, and how they utilize them for mathematics. The specific research problems addressed in this study are the following:

- What are the uses of written curriculum materials (mathematics textbooks and accompanying student workbooks, teacher guidebooks, and other written resources) in classroom mathematics for the middle grades mathematics teachers?
- For what reasons and purposes do mathematics teachers select and utilize the written curriculum materials?

## **METHOD**

### **TEXTBOOKS IN TURKISH SCHOOLS**

Any textbook to be used in Turkish elementary schools needs to be officially approved by the Turkish Ministry of Education. The state publishes textbooks for each subject matter included in the curriculum. In addition, private publishers may also publish textbooks for schools. Among the approved textbooks, the ministry of education decides which textbook can be used by which schools, and distributes them free of charge to students and teachers. Typically, a set of mathematics textbooks includes a student course book and student workbook, as well as a teacher guidebook.

### **PARTICIPANTS**

In this study, the data were collected through interviews. For the interviews, all middle grades mathematics teachers working in 11 different schools selected from a district of the western Turkish town İzmir were invited to participate in this study. Among them, 4 male and 9 female mathematics teachers were invited to participate in the study. A purposeful sampling method was used to ensure that a variety of teachers with different teaching experience were questioned. The interview participants had a minimum of 5 years of experience in mathematics teaching. In particular, 7 teachers had taught for over 10 years, and 2 teachers had taught for 25 years or more at the elementary school level. At the time of the data collection, the teachers were working at sixth through eighth grade levels. Furthermore, they were using the same mathematics textbooks.

### **DATA COLLECTION**

During the fall semester of 2009, 13 mathematics teachers were interviewed about how they used mathematics textbooks, and what other resources they used to plan and implement the mathematics lessons. The interviews were conducted for the purpose of obtaining data about how teachers used mathematics textbooks for planning, implementing, evaluating the lesson in the context of the Turkish school culture, as well as their perceptions of the textbook. Interviews were conducted in teachers' schools and took about 40 to 60 minutes with each teacher. Teachers were asked semi-structured questions, and the interviews were tape recorded and transcribed verbatim for the data analysis.

## DATA ANALYSIS

The researchers drew on the interview data to identify the use of textbooks by mathematics teachers. In analyzing the interviews, teachers' utilization of student course book, student workbook, teacher guidebook, and other resource books were analyzed in light of the research questions. Particularly, the researchers analyzed their use in the process of planning, preparing, enactment, and assessment of mathematics instruction.

## RESULTS AND DISCUSSION

Results of the analysis indicated that mathematics teachers were using student course book (CB), workbook (WB), teacher's guidebook (TG), and other resource books (OB). The reports of teachers showed that teachers generally utilized the TG and CB in preparing the mathematics lesson. Moreover, they looked at the TG to make possible a connection between the curriculum intentions and classroom activities. They stated that their first resources for making decisions which topics to teach and how to present them were the TG and CB. As Ms. Aksu stated:

I use the guidebook very frequently because I think that the guidebook gives useful information. For example, it provides information about how I can teach the lesson.

In this case Ms. Aksu uses the teachers' guidebook to make decisions about how to teach the topic. Similarly, Mr. Çelik stated that he uses the TG to determine content boundary in his classes. He stated:

I look at how the subject is covered [*in the textbook*], how it is explained, what the levels of the examples are, what kind exercises there are? [*I use the course book*] to get an idea about where the unit starts and ends, I mean, in order to determine a framework [*for the class*].

Mr. Çelik's use of the TG was limited to determine the level and depth of the topic to be taught. Other participants also made use of the TG or CB in order to review the lesson objectives. Mr. Salih, on the other hand, focused more on the CB. He stated that he usually searches for engaging and sense-making activities from the CB.

I try to start my instruction with activities and real life connections as this makes it easier to reach my students. I use the textbook for trying to reach ideas for this.

Similar to Mr. Salih, other participants preferred to situate the problems in a real-life context from the CB. Particularly, one of the most important uses of the TG and CB were about the selection of in-class questions and exercises to be studied. For instance, Mr. Emre stated:

We first solve the examples, questions in the course book together with students, and then after teaching the topic, I definitely make my students solve the "your turn" part in the book. For the higher-level students, I solve problems from the test books after covering the rules from the course book.

## Working Group 15

In this case, Mr. Emre used the CB as the main source for examples and problems and looked at the OB to find different and upper level problems. Mr. Hüseyin also mentioned that the teaching experience he had was very crucial in adapting the problems and examples for the level of students in his classroom. He stated that:

The level of the students is important for sure. I select more advanced books if the level is high, and easier books if the level is low... I arrange it by myself, actually based on my experiences. For example, sometimes I change a question while I am writing it. I change the questions considering the level (of the students) because I feel, I can see where a student can get it and where s/he cannot.

In this case, Mr. Hüseyin, who had nearly 25 years of experience in mathematics teaching, expressed that he makes use of his teaching experience in selecting the problems and modifying the setting according to his own decisions and students' capacities. Similarly, the other participants reported that they created problems and questions using those textbooks, and adapted to problems and questions in the CB, WB, and OB. As Mr. Sinan expressed, he focuses more on the WB and creates his own problems by the help of questions in the textbook. He stated:

I change the form of the questions in the course book, or I write sub-questions for the questions in the book and pose them to students.

Moreover, the participants expressed strong messages about the importance of addressing students' needs for preparing high stake national exams-namely SBS in Turkey. They adapted and supplemented the questions in the course book to be in line with the national examinations in order to help students achieve better test scores. As Ms. Mine stated:

I always make my students solve the problems at the end of each unit in the course book. I make use of the book, because it is based on the curriculum and because I do not want my students miss any questions in SBS

In addition, Ms. Tülin stated:

In my classroom exams I ask similar questions to the national test.

Similar to those cases, the participants aimed not only the preparation for high stake tests, but also for in-class examinations. Generally, the participants utilized the CB, WB, and OB to select problems and applications for in-class assessment. They generally used these materials to give homework and prepare the exam questions. As Mrs. Akif stated:

I generally use the workbook to assign homework. Most of the questions (in the workbook) are not like the traditional ones such as 'what is x?', but there are figures and tables in them. The questions in the course book and the student workbook are similar, so I assign the workbook to the students.

Similar to Mrs. Akif, the other participants used the WB in their classrooms to ask similar problems in the CB and they gave the problems in the WB as homework.

## CONCLUSIONS

According to Remillard (1999), when teachers read the textbook, they attend to some parts of text and dismiss others. They bring their interpretation to what they read. This is because reading text involves “a series of tacit decisions about what to attend to and how to interpret it” (Remillard, 1999, p. 324). In this current study, the mathematics teachers focused on the mathematics textbooks and accompanying student workbooks, teacher guidebooks, and other written resources in order to make particular decisions about the instruction. For example, they selected the real-life examples from student course book and decided on the problems from student workbook or other source book. Therefore, the use of textbooks by mathematics teachers was based on their interpretations of textbooks in this study.

The results of the study indicate that teachers mostly relied on the official course books and accompanying teacher guidebooks for planning and preparing mathematics instruction. They were generally helpful for teachers about what topics to teach and how to make connections with real life and the other lessons. During the instruction, the teachers preferred to use again the course books for more process oriented activities such explaining topics, focusing on concepts, and assigning projects. For evaluating mathematics instruction, teachers preferred to utilize the course books, workbooks, and the other resource books in order to evaluate students and give homework. The data indicate that the most of the teachers stated that they followed the mathematical content and sequence presented in the CB by a little change. The teachers reported that they adapted the tasks from the CB and OB to give examples, exercises, and problems. For example, according to teachers, they felt that they had to review problems and examples in all resources because the classroom settings (e.g. the level of the students or time) always restricted the teachers. Most of the teachers also stated that they looked at the OB considering that they could not find challenging problems in the CB. Therefore, they stated that they used the OB when they preferred to solve different kinds of problems. Particularly, the teachers mostly adapted problems and examples when they were enacting in classroom and evaluating students. This would eventually indicate that teachers interacted with textbooks in a constructive way rather than a straightforward process.

Furthermore, the teachers used the CB, WB, and OB to select problems and applications for assessment and evaluation. For instance, they stated that they generally utilized them to give homework and prepare the exam questions. Most of the teachers reported that they created problems and questions using those textbooks, and others made few or no adaptations to problems and questions in the textbooks.

A general conclusion in this study is that the teachers mostly rely on the course book and the other resource books for selecting problems to work in class or use in examinations. These resources are mostly the self-study books that aim to prepare students to high stake exams. The scope of such books is usually focused on the preparation for the tests and the problems are usually in multiple choice format.



## REFERENCES

- Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education*, 3(3), 205–224.
- Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. (1996). *Mathematics achievement in middle school years: IEA's Third International Mathematics and Science Study (TIMSS)*. Boston, MA: Centre for the Study of Testing, Evaluation, and Educational Policy, Boston College.
- Brown, M. W. (2002). *Teaching by design: Understanding the interactions between teacher practice and the design of curricular innovation*. Unpublished doctoral dissertation, Northwestern University, Evanston, IL.
- Brown, M. W. (2009). The Teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17-36). New York: Routledge
- Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, 25(2), 119-142.
- Collopy, R. (2003). Curriculum materials as a professional development tool: How a mathematics textbook affected two teachers' learning. *Elementary School Journal*, 103(3), 287–311.
- Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher*, 34(3), 3–14.
- Howson, G (1995). *Mathematics textbooks: A comparative study of grade 8 textbooks*. TIMSS, Vancouver: Pacific Educational Press.
- Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry*, 29(3), 315–342.
- Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211–246.
- Remillard, J. T. (2009). Considering what we know about the relationship between teachers and curriculum material. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 85-92). New York: Routledge.
- Robitaille, D. F., & Travers, K. J. (1992). International studies of achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of Research on mathematics Teaching and Learning* (pp. 687-709). New York: Macmillan; Reston, VA: National Council of Teachers of Mathematics.
- Stein, M. K. & Kim, G. (2009). The role of mathematics curriculum materials in large-scale urban reform. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 37-55). New York: Routledge.

# COLLECTIVE DESIGN OF AN ONLINE MATH TEXTBOOK: WHEN INDIVIDUAL AND COLLECTIVE DOCUMENTATION WORKS MEET

Hussein Sabra, Luc Trouche

INRP & LIRDHIST (Université Lyon 1, France)

*This paper focuses on the documentation work in the mathematics teaching. We show up the individual and collective components of this documentation. We present a new theoretical framework, the documentational approach, which seems adapted for studying this issue. We applied it on a particular project of the French teachers association Sésamath: the collective design of an online mathematics textbook. We present a methodology for observing both this collective project and the case of Anaïs, a teacher involved in this project. We study her professional interest in Sésamath, her didactical interactions with the other project members, and their effects on her professional knowledge.*

Keywords: documentation work, professional knowledge, resources, community of practice.

## INTRODUCTION

*Digitalization* deeply changes conditions of thinking and sharing knowledge at each level of the society (Pédauque, 2006). Its visible manifestations, in mathematics teaching (Hoyles & Lagrange, 2010), are both the *profusion* of resources available via Internet and the *diversification* of technologies that could be used by teachers (USB keys, interactive whiteboard, calculators, software). These evolutions dramatically modify the conditions of professional practice (for *preparing* as well as for *doing* the teaching). They prompt new collective forms of teachers work: on the one hand, digitalization makes *potentially* easier sharing and exchange of resources between teachers, on the other hand, its complexity (dispersion of resources, rapid technological evolutions) makes *necessary* for teachers to help each other. These new forms of collective work among mathematics teachers change the conditions for their professional growth: *teams, networks and communities* appear as new opportunities for teachers to learn (Krainer & Woods, 2008).

The most significant phenomenon of this trend, in France, is, for us, the appearing of Sésamath [1], a French online association of mathematics teachers, aiming to provide mathematics teachers with free online resources. For achieving this goal, Sésamath develops collaborative work of teachers, around common projects (Sabra, 2009). We address, in this article, two questions: why do teachers engage in Sésamath? How do they articulate their work on resources for individual purpose and for Sésamath purpose?

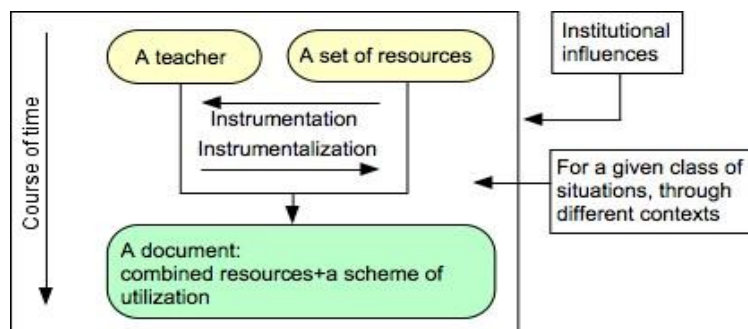
We first introduce a new theoretical approach, which seems necessary for addressing these issues. Then, we present our experimental field, specify our questions and set out our methodology. Finally, we display some preliminary results and propose some new questions that this research raises.

## THEORETICAL FRAMEWORK

Our research relies on a new approach to teachers' work with *resources* and professional development, already presented in CERME 6 (Gueudet & Trouche, 2009) and developed further (Gueudet, Pepin & Trouche, to appear): *the documentational approach of didactics*.

It is built on a distinction between *resources* and *documents*, extending the one introduced by the *instrumental approach* (Rabardel, 1995) between *artefact* and *instrument*. The choice of the word "resource", instead of artefact, aims at catching a great variety of things intervening in teachers' work: textbook, piece of software, student's worksheet, Internet resource, discussion with a colleague, etc. We call *documentation work* what a teacher needs to do for designing her teaching: looking for resources, integrating them in her personal *resource system*, implementing it in practice, sharing it with colleagues, renewing it taking into account various feedback, etc.

A teacher draws on resource sets for her documentation work. A process of genesis (Fig. 1) takes place, producing a *document* [2], made of resources and a scheme (i.e., an invariant organization of the activity to perform a type of task - here a task of preparing and performing a given teaching). Each scheme encapsulates professional knowledge, both shaping teacher's activity and permanently reshaped by this activity. Shulman (1986) proposed a categorization of teacher's professional knowledge. We are particularly interested in Pedagogical Content Knowledge (PCK) that a teacher develops to help her students in their learning (Grossman, 1990).



**Figure 1. Schematic representation of a documentational genesis**

The documentational genesis combines two interrelated processes: the instrumentalization (the teacher acting on resources), and the instrumentation process (resources supporting teacher's activity).

Considering teachers' collective work leads us to articulate the documentational approach to the frame of *communities of practice* (Wenger, 1998). A community of

practice (CoP) is a human group presenting three main features: *mutual engagement* of its members, active *participation* to a joint enterprise, and *reification* of elements of practice (i.e., production of things, results of the common practice, and recognized as a common wealth). This frame appears relevant for studying interplay between a group of teachers and sets of resources they are working on/with. Instead of reification (referring to congealed entities), we coin the expression *community documental genesis* to describe the process of gathering, creating and sharing resources to achieve the community teaching goals. The result of this process, the *community documentation*, is composed of the shared resource repertoire and shared associated knowledge (learnt from conceiving, implementing, discussing resources).

## OUR EXPERIMENTAL FIELD

Sésamath is a math teachers association founded in 2001. Its kernel is constituted of about 100 subscribers (math teachers), sharing a set of principles inscribed in a charter [3]: common philosophy of public service, “math for everybody”... Elected by this kernel, the Sésamath board regularly launches new project groups for designing resources on a given theme corresponding to teachers’ special needs and interests (new curriculum subject, new textbook... Fig. 3). These groups gather about 5000 teachers (a number of teachers belonging to several groups at the same time), working mainly at distance, via a platform and mailing lists. The group members can benefit from the assistance of employees and computer developers hired by the association (Fig. 3). All these groups present the CoP features, at different levels of development: the project groups appear, at the beginning, as *potential* CoP (Wenger *et al.*, 2002), while Sésamath kernel appears as a *maturing* one (*ibid.*). Sésamath, beyond its regular members, thus constitutes a *constellation of CoP* (*ibid.*), each of them sharing a same commitment for the association principles. A questionnaire (Sabra, 2009), proposed in 2008 to 36 members of the Sésamath kernel [4], gave some answers to our first question: why do teachers engage in Sésamath?

What are, for your teaching, your sources of documentation?	Online resources (33/36); Resources that you have developed in previous years (35/36)
What are your professional reasons for engaging in Sésamath?	Training (16/36); Exchange of experience (23/36); Exchange of resources (13/36).

**Table 1. Questionnaire to Sésamath, two questions (among 39) and their answers**

The first item (Table 1) confirms the Sésamath members’ interest for online resources and evidences the place, for them, of resources *reuse* (revealing the importance, for Sésamath work, of exchanging, combining, modifying his/her own existing resources). The question item reveals that the main motivation, for joining Sésamath, is the *exchange of experience*, which evidences Sésamath role for its members’ professional development.

Going further in our research required to monitor the documentation work of one Sésamath project group on the field. We have chosen the group (named DT10), created in 2009, aiming to design a digital textbook for the beginning of the French high school (grade 10). This choice was motivated by two reasons: 1) after having designed textbooks for the French college (grades 7 to 9), Sésamath thus addresses, for the first time, the more complex, grade 10 math; 2) after having designed “classical” textbooks (pdf, OpenOffice files teachers can download and modify) with complements (spreadsheets and interactive applications), Sésamath aimed to create a new type of *digital* textbook, that a teacher can appropriate and adapt to her own needs. We hypothesised that these new challenges (mathematical and technical) could enrich the documentation work of this group, and thus make it more interesting for our research.

Answering our initial questions needed also monitoring the complete documentation work (both for Sésamath and for their own classes) of some DT10 members. For this article, we have chosen to present Anaïs’ case. Anaïs is particularly engaged in DT10. She is 57 years old. After 15 years of various occupations (including a few years as a member of a commune in the countryside), she came back to the university, achieved her math studies and got a position as a math teacher. She has now 18 years of teaching experience (15 years for grade 10). Why does Anaïs engage in the DT10 project? How does her professional knowledge interact with individual and community documentation?

## METHODOLOGY

Observing the individual and collective documentation is a complex task. It requires taking into account several conditions: long time observing to highlight regularities; individual and collective observing; observing *in* and *out* of classroom; following both teachers’ activities and resources. In some methodology of CoP observation, the researcher is engaged in practices (Jaworski, 2009). In our case, we have just observed the practice as an outsider.

### **The reflexive investigation for observing individual documentation**

To observe Anaïs’ documentation, we adopted a methodology designed by Gueudet & Trouche (2010): *reflexive investigation* considering the teacher as an essential actor in data gathering. Among its methodological tools: interview at home, ‘guided tour’ (the teacher being the guide) and schematic representation of her *resource system* (SRRS); questionnaire about her vision of math and of math teaching; follow-up during several weeks including a logbook fulfilled by the teacher, collection of the resources, classroom observations. Anaïs became sick at the beginning of the follow-up, thus a direct classroom observation was impossible. We have adapted the methodology, by observing (and videotaping) Irvin, Anaïs’ substitute in class, who actually used Anaïs’ resources, and discussing afterwards with Anaïs on Irvin uses of her resources. In this paper, we only exploit data from the Anaïs’ interview at home,

from her comments on Irvin's video, from her SRRS and from a set of resources she designed for the teaching of functions (according to the discussion in DT10).

### **Designing a methodology for observing community documentation**

Extending this methodology relying on *individual* reflexive investigation, new methodological tools have been designed for fostering teachers' reflection on their *collective* practice. Among them, an *agenda*, fulfilled by some members of the CoP, chosen for the role that they had in the project (e.g., Adam was selected for its role identified in the mailing list as an effective coordinator of DT10). These agenda aim to identify and analyze, from different points of view of different actors, the effects of the *incidents* (something unexpected needing to reformulate the common goal or to reorganize the community documentation) occurring throughout the common project. We have also asked DT10 members a *schematic representation of collective interactions* (SRCI) in the case of Sésamath, and collect the mathematical resources that they designed.

The methodology takes also advantage of *natural data* which offers the experimental field. We thus exploit the Anaïs' online notebook, the DT10 mailing list offering discussions about project organisation, as well about mathematical, didactical and epistemological aspect of the community documentation work. During our follow-up period (11 months), 627 messages have been exchanged via the mailing list, involving 27 members (including Adam, Anaïs, John, Ben, Pierre and Henry). Anaïs has authored 111 messages out of the 627. We particularly exploit the DT10 *thread of discussions* concerning the "math functions" linked to an incident: a curriculum change occurring in the midst of DT10 work and provoking a more intensive discussion.

## **DATA ANALYSIS AND DISCUSSION**

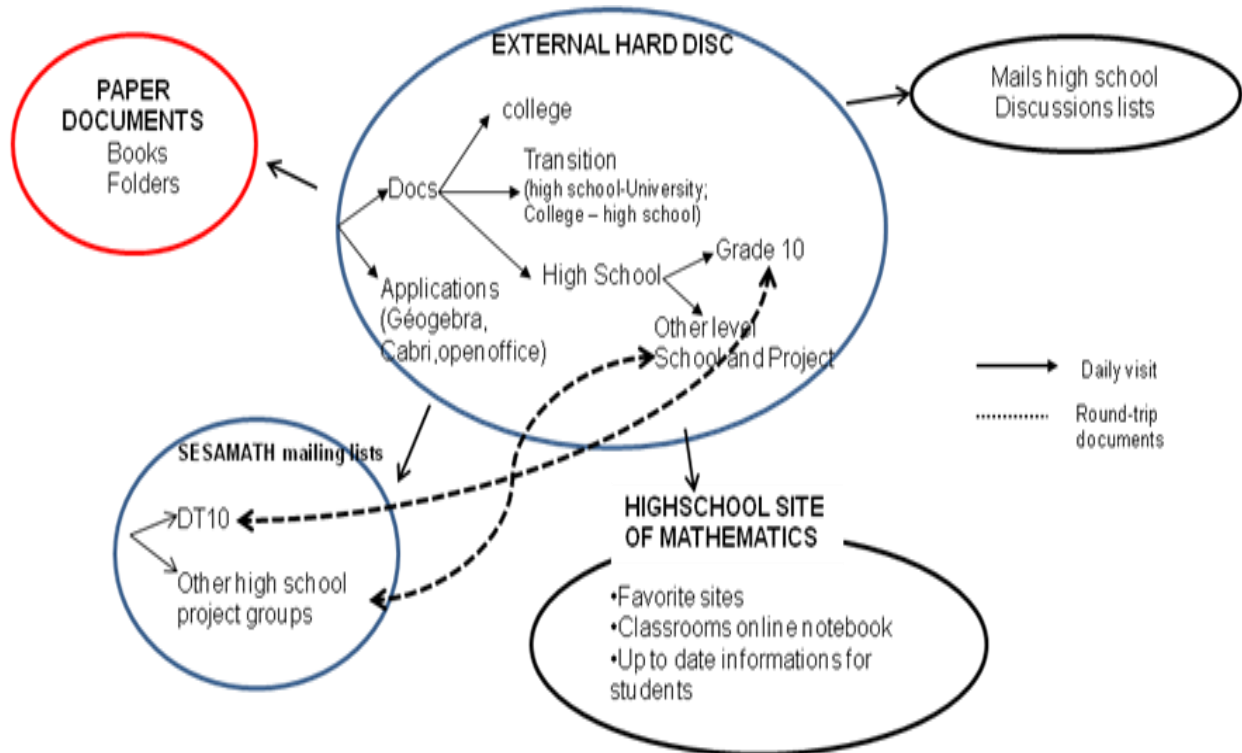
We first analyze Sésamath and Anaïs' documentation, and then present the didactical interactions between Anaïs and DT10.

### **Documentation work: DT10 and Anaïs**

Most of Anaïs' resources (courses, exercises, homework) are digital, stored in an external hard disc. This digital form facilitates the sharing with other teachers via USB key or Internet. The Anaïs' resource system is strongly articulated with Sésamath resources (see Anaïs SRRS, Fig. 2): emails, students' sheets and other resources exchanged with Sésamath seem to have a major role in her documentation. This *osmosis* between Anaïs and Sésamath resources facilitates Anaïs' participation in DT10 work.

Anaïs's documentation is conditioned by three factors: "curriculum changing, institutional recommendations and classroom general level" (Anaïs' interview). For example, her teaching of "function" is strongly related to curriculum change. She said that she lived "*two different spirits*" of teaching functions: "it was, before, more

guided and now the curriculum recommends open questions... so that students have more initiative” (Anaïs' interview). She seems very sensitive to the students' level: this year, I had a low level class. I will modify [the documents of the previous year] by adapting them to the level of my class. But maybe, the next year I will have good students; I will reuse the document in the present form” (Anaïs' interview).



**Figure 2. Anaïs' SRRS, our translation and schema as close as possible of the original**

DT10 documentation is also sensitive to curriculum changes and keeps Sésamath resources as a general background support (particularly when an incident occurs), as it appears in the following excerpt of the DT10 mailing list:

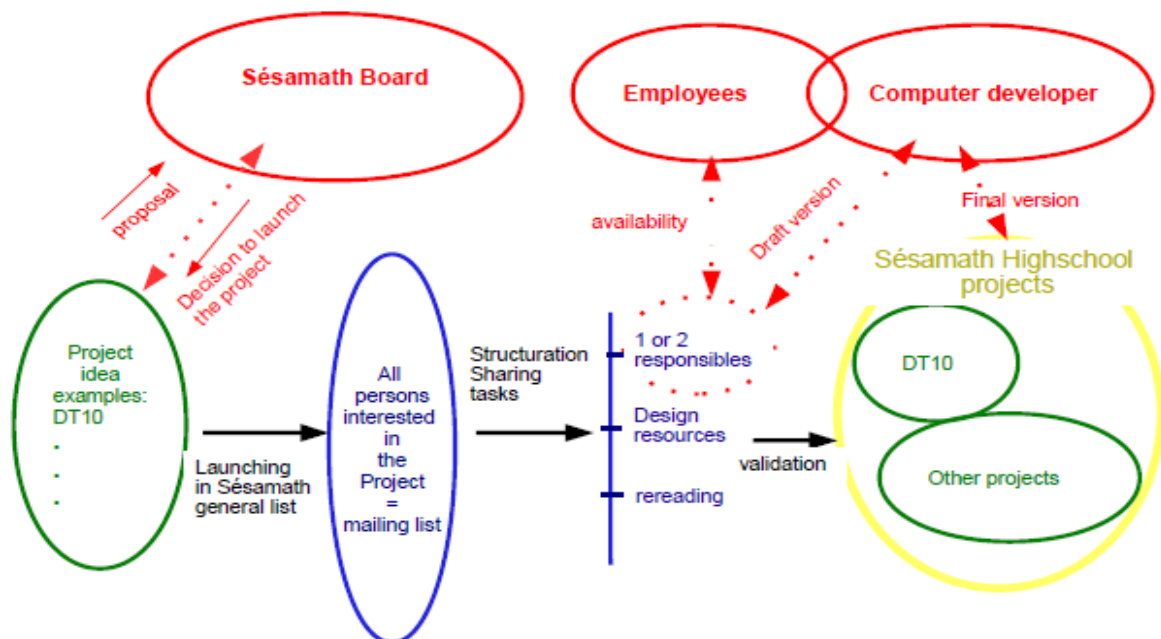
**John:** The curriculum has profoundly changed and we must rebuild the chaptering. This chaptering is very important because it is common to several Sésamath projects that involve the grade 10. Thus, I propose to do a general RESET.

**Henry:** We will create a general mailing list for all Sésamath projects linked to the secondary level as there issues intersect. We will thus share resources and general links that might be useful to all.

**John:** This list must first allow everyone to have a clear idea of what is done in different projects.

DT10 and Anaïs' documentation do not answer to the same constraints: DT10 has to move forward systematically with other high school projects, while Anaïs is attracted by her pupils' level, particularly in the new curriculum spirit. But both documentations are conditioned by the institutional recommendations, which constitute the joint constraint for designing the resources.

Anaïs' interactions with DT10 are linked to her general interest for collective matters, and particularly in this project. Anaïs underlines, in her SRCI for Sésamath (Fig. 3), that each project is constituted by teachers who are interested the project topic. More precisely, her participation resulted from a professional interest: participating in DT10, constitutes, for her, an occasion for searching ideas for her own classroom course. She confirms this idea in the interview, illustrating the productive aspects of instrumentalisation process: “for example, for the chapter “Variations and Extrema of functions” [named 2N2], I don't have constructed the lesson yet. In fact, I will transform it [the lesson designed in DT10] and I will reuse it for my classroom. I will adapt it for my classroom”.



**Figure 3. Anaïs's SRCI, our translation (see comments on project groups §3)**

We analyze in the following section the didactical aspects of the interactions between Anaïs and DT10 members.

### The didactical interaction between Anaïs and DT10 members

The didactical interactions between Anaïs and other DT10 members take place mainly through the project mailing list. Most of the didactical discussion, in the thread “Create textbook chapters”, concerns the theme “mathematics functions”. This discussion came under the PCK category “instructional strategies for teaching the functions”, as it appears in the following extract of DT10 mailing list about the 2N2 chapter (evidencing Adam's role as coordinator):

**Ben:** We need to cut up 2N2 chapter. What are your proposals?

**Pierre:** ... How far will we go with the concept of variations? My proposal: 1) the notion of functions variation (from a curve); 2) Maximum and minimum of a function, 3) graph and table of variation; 4) comparison of numbers.



## Working Group 15

**Anaïs:** My proposals: 1) functions variations and graphical reading... 2) variations and calculations: square function ...inverse and linear functions, 3) Maximum and minimum of a function, 4) table of variation.

**Adam:** I have a slight preference for Pierre's proposal ... because I think that the table of variation is simpler to understand than computations for comparing numbers ... I think that slowly increasing the difficulty is a good strategy.

Anaïs's chapter happens following another structure. She first introduces the concept from what she calls a "start-up activity", with the aim of presenting the semantic values of the "mathematics functions" terminology; then, "the course" where she presents the main definitions (and related examples); at the end, the "series of exercises", and "homework and assessment". This structure appears in her online notebook (table 2).

Monday 21/09/2009	Constructing, from a sheet of paper, a box whose volume is the biggest one
Friday 25/09/2009	Definitions: set of a function definition, variable, image, and antecedent. Table of values and graph using a calculator.
Monday 28/09/2009	Area of a rectangle with a given perimeter. Table of values and graph of <i>reference</i> functions (that are inverse function, square function, square root function...)
Thursday 1/10/2009	Different ways of saying that $f(a) = b$
Friday 2/10/2009	Exercises on image of an interval; first assessment.

**Table 2. An extract of the Anaïs' online notebook**

In the thread of discussion "Extremum 2N2", that concerns designing an exercise for 2N2, Anaïs initiates didactical discussions about the terminology:

**Anaïs:**  $f(x)$  is smaller than (or the image  $f(x)$  is smaller than?). It seems...

**John:** For  $M$  being a maximum, it must be both an upper bound and an image by  $f$ . That is to say, for all  $x$  belonging to a given interval  $I$ ,  $f(x) \leq 4.5$ , then 4.5 is an upper bound. And it is a maximum if it is also an image of some  $x$  belonging to  $I$ .

**Adam:** I propose this formulation: for every real  $x$  belonging to  $I$ ,  $f(x)$  is smaller than, or equal to,  $f(3)$ , which equals 4.5.

**Anaïs:** Either like this: for every  $x$  in the interval  $I$ ,  $f(x) \leq 4.5$  with  $4.5=f(3)$ .

Also, when she discovered Irvin's video, she has particularly commented the language used by the teacher:

**Anaïs:** the expressions ... he says "the straight line  $3x + 1$ ", or the straight line  $Y$ . What does that mean? He should say the line whose equation is  $y = 3x + 1$ .

**Me:** why does it bother you? How this is a problem?

**Anaïs:** I think we should be highly accurate, even when we talk. Being vague or loosely makes fuzzy pupils' mind, especially when introducing new concepts...

We remark a difference between DT10 (more precisely DT10 members taking in charge this question) and Anaïs in the instructional strategies for teaching function. DT10 constructs a “function” concept from very simple tools (reading of curves and table of values), then it moves to more complex tools like calculation and articulation between different types of representation (algebraic, graphical). Anaïs takes care, in her documentation, of the terminology of mathematical concept and its use. This *gap* is an opportunity for Anaïs to discuss her view with other DT10 members.

On another subject, the design of a problem-situation reveals a *convergence* between Anaïs' and other DT10 members' documentation: Anaïs fosters the place of conjecture and experimentation. She presents thus a resource (interview):

**Anaïs:** I like to ask them first to experiment with a calculator... try first with the calculator, guess the number of solutions ... for example  $f(x) = 0$  they speculate with the calculator and ... Once they guess, after they do the proof by calculation. **But I think it is important to first think about from free explorations.**

The thread of discussion “test\_2N2” reveals, within DT10, a shared point of view on problem-situation. The problems designed consist in modelling geometrical situations with a function, graphing this function with a calculator and elaborating conjectures before computing and searching for evidence.

## CONCLUSION

We have presented in this paper a theoretical approach and a methodology aiming to analyze the didactical interactions at stake within a *community documentation work*. We have carried out this approach and this methodology in the case of a documentation project, DT10, of an online teachers association, focusing on Anaïs, a member of this association.

Anaïs and DT10 have a common interest in collaborating for designing resources and collectively facing *incidents* (like changes in the curriculum). Anaïs has also an individual interest linked to her documentation needs: discussing the mathematics terminology and language, and its values in the documentation work. When Anaïs perceives that didactical discussions are part of her interest, she participates strongly in the community documentation.

Three factors seem to stimulate the active participation of a teacher in the community documentation: osmosis between her resource system and the CoP resources; a gap between the teacher and the CoP in the strategies for teaching a subject; a shared interest in the subject of discussion that is the origin of gap. Following an episode of Anaïs's active participation in the community documentation, we identified an aspect of her professional knowledge: the introduction of a concept has to be based on a precise introduction of terminology and language associated, and its uses.

This study revealed a question that deserves more deepening: how identifying the effects of the community documentation on the teacher professional knowledge in the case of convergence between the individual and community documentation?

## NOTES

1. <http://www.Sesamath.net/>
2. The choice of vocabulary intended to match the terminology of document management research. According to Pédaque (2006), “A document is not anything, but anything can become a document, as soon as it supplies information, evidence, in short, as soon as it is authoritative.” (p. 12, our translation).
3. See: [http://www.Sésamath.net/association.php?page=asso\\_charte](http://www.Sésamath.net/association.php?page=asso_charte).
4. See: [http://www.Sésamath.net/association.php?page=asso\\_charte](http://www.Sésamath.net/association.php?page=asso_charte).

## REFERENCES

- Grossman, P. (1990). *The Making of a Teacher*. New York: Teachers College Press.
- Gueudet, G., Pepin, B., & Trouche, L. (to appear). *Mathematics curriculum material and teacher development, from textbooks to 'lived' resources*. New York: Springer.
- Gueudet, G., & Trouche, L. (2009). Teaching resources and teachers' professional development: towards a documentational approach of didactics. In V. Durand-Guerrier *et al.* (Eds.), *Proceedings of the CERME 6* (pp. 1359-1368). Lyon: INRP.
- Gueudet, G., & Trouche, L. (2010). Des ressources aux documents, travail du professeur et genèses documentaires. In G. Gueudet & L. Trouche (Eds.), *Ressources vives. Le travail documentaire des professeurs en mathématiques* (pp. 57-74). Rennes : Presses Universitaires de Rennes and Lyon : INRP.
- Hoyles, C., & Lagrange, J.-B. (Eds.) (2010). *Mathematics Education and Technology - Rethinking the Terrain*. The 17th ICMI Study. New York: Springer.
- Jaworski, B. (2009). The practice of (University) mathematics teaching: mediational inquiry in a community of practice or an activity system. In V. Durand-Guerrier *et al.* (Eds.), *Proceedings of CERME 6* (pp. 1585-1594). Lyon: INRP.
- Krainer, K., & Wood, T. (Eds.). (2008), *Participants in Mathematics Teachers Education: Individuals, Teams, Communities and Networks* (Vol. 3). Rotterdam/Taipei: Sense Publishers.
- Pédaque, R.T. (2006). *Le document à la lumière du numérique*. Caen : C & F éditions.
- Rabardel P. (1995). *Les hommes et les technologies, approche cognitive des instruments contemporains*. Paris: Armand Colin.
- Sabra, H. (2009). Entre monde du professeur et monde du collectif: réflexion sur la dynamique de l'association Sésamath. *Petit x*, 81, 55-78.

### Working Group 15

- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15, 4-14.
- Wenger, E. (1998). *Communities of Practice: Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.
- Wenger, E., McDermott, R.A., & Snyder, W. (2002). *Cultivating communities of practice: a guide to managing knowledge*. Boston MA: Harvard Business School Press.

# DEVELOPING A COMPETENCE MODEL FOR WORKING WITH SYMBOLIC CALCULATORS

Hans-Georg Weigand

University of Wuerzburg (Germany)

*There is a worldwide claim that the actual use of Information and Communication Technology (ICT) is only partially integrated into mathematics learning and teaching. Reasons concerning this issue are fairly diverse. One argument might be the complexity of the integration process in the actual teaching. In the M<sup>3</sup>-project (Model-project with new Media in Mathematics lessons) in Germany, students' knowledge and abilities are evaluated in the frame of a "competence model" while working with a Symbolic Calculator (SC) within the context of the concept of functions. This article demonstrates the way of constructing the (theoretical) competence model while working on different levels of SC-use and while working on different levels of the understanding of functions. It also discusses the results of a first experimental evaluation of this model.*

**Keywords:** Competence model, functions, symbolic calculator, experimental evaluation

There are many suggestions, lesson plans and empirical investigations concerning the use of ICT in mathematical learning and teaching. But despite of the positive results of many of these empirical investigations "the actual use of ICT in real school environments is still having a limited impact" (Bottino & Cerulli, p. 1). This is also noticed quite often in the actual ICMI study 17 (Hoyles & Lagrange, 2010), e. g. "technology still plays a marginal role in mathematics classrooms" (p. 312) or "the impact of this technology (CAS) on most curricula is weak today" (p. 426) and especially in the closing address Michèle Artigue, the president of the ICMI, points out:

"The situation is not so brilliant and no one would claim that the expectations expressed at the time of the first study have been fulfilled" (p. 464).

The complex area of integrating new technology into common school lessons has been underestimated by teachers and researchers (Trousche, 2005).

"Increased technological power, nevertheless, generally goes along with increased complexity and distance from usual teaching and learning environments" (Artigue & Bardini, 2010, p. 1).

We know, especially from the results of the theory of instrumentation and instrumental genesis (Guin *et al.*, 2005), that only local strategies for learning and teaching, for example in the frame of lesson units, will not give successful results. It is necessary to have a global view of the entire learning and teaching process, which includes the teacher, the learner, the content and the learning environment.

In the following, we develop a *competence model*, which shows on the one hand goals of teaching with ICT in the frame of a special content – we have chosen the function concept as one of the most important concepts in the mathematics curriculum. On the other hand, it allows the evaluation, classification or categorization of students' competencies in the use of the digital tool.

## FOUNDATIONS OF A COMPETENCE MODEL

The concepts of *competence* and *competence (level) models* have aroused interest in mathematics education in the past years. Starting with the NCTM Standards (1989) and especially the PISA studies, *competence* and *competencies* are expressions, often used in the context of standards, goals envisaged, knowledge and abilities in mathematics education (OECD, 1999; Niss, 2004).

Concerning our competence model, we decided to concentrate on the concept of functions, as it is a central concept in mathematics and in mathematics education. Moreover, it plays a crucial role in our M<sup>3</sup>-project. These *content competencies* become apparent in understanding functions (UF) and in relation to *tool competencies* (TC) while using the SC.

## DEVELOPMENT OF THE COMPETENCE MODEL

### Understanding of functions (UF)

Concerning the *understanding of functions*, we use a four level model developed by Vollrath (1993), as these levels are already expressed in special competencies (knowledge, abilities) which students are expected to have at each level:

**Level 1:** *Intuitive Understanding*

**Level 2:** *Conceptual Understanding*

**Level 3:** *Relational Understanding*

**Level 4:** *Structural Understanding*

The model includes the dual nature of mathematical concepts: processes and objects (Sfard, 1991) and the instrumental and relational understanding of Skemp (1976) [1]. Compared to the models of Sfard and Skemp, it explicitly describes the competencies a learner should have related to the understanding of functions. For detailed information concerning this model see Weigand & Bichler (2010).

### Tool competencies (TC)

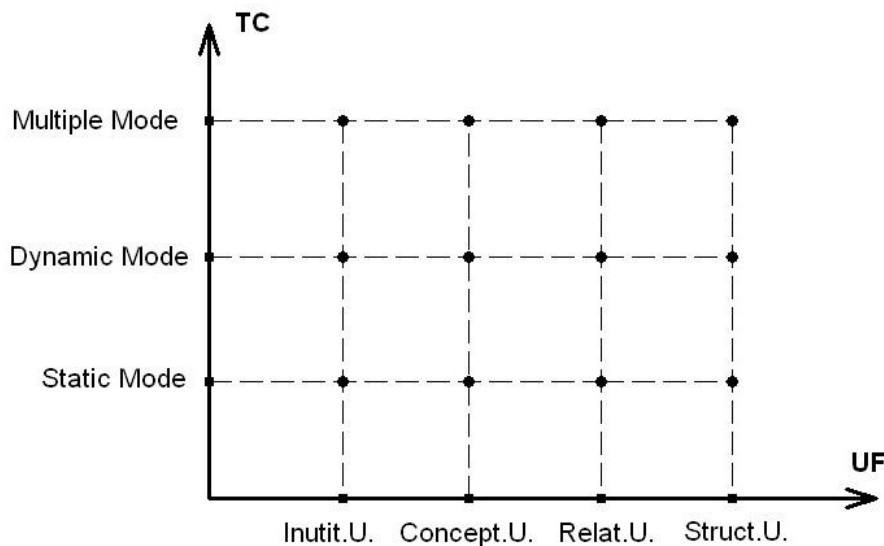
The ability or the competencies *to use SCs* adequately requires technical knowledge about the handling of SCs. Moreover, it requires the knowledge of when to use which features and for which problems it might be helpful. In the following, *the use of SC* is classified concerning the way representations are used. We distinguish three levels, which might also be categorized by using SCs as a (simple) function plotter, as a tool for creating dynamic animations and as a multi-representational tool:

**Level 1:** Using the SC as a tool, which produces static representations. We speak of a *Static Mode* (StaticMode).

**Level 2:** Creating dynamic representations. We speak of a *Dynamic Mode* (DynaMode).

**Level 3:** Using the SC as a multiple representational tool. We speak of a *Multiple Mode* (MultiMode).

### The UF-TC relationship



**Figure 1: UF-TC-relationship**

The relationship UF-TC is established by special problems or tasks, showing on the one hand the level of understanding and on the other hand the way a student uses the symbolic calculator. In Weigand & Bichler (2010), an example for every single cell of the matrix is given.

## THE EMPIRICAL EVALUATION OF THE COMPETENCE MODEL

Concerning the empirical justification of the theoretical model and including the aim of constructing an *empirical* competence model, some questions are to be answered.

- *The validity of the model:* Do the solutions of the problems constructed for each cell of the UF-TC matrix require the competencies the cell stands for? To answer this question, problems dealing with the competencies of each cell will be developed.
- *From a qualitative to a quantitative model:* The PISA studies use a model which is based on the relative frequency with which students are able to solve a problem. The problem that has been solved successfully is taken as a measure of the difficulty of the exercise. The scale is standardized on a mean value 500 with a

standard deviation of 100 (OECD, 1999; 2003). This might also be an aim in the context of this competence model.

## PILOT INVESTIGATION - TEST QUESTIONS

In a first step, we considered the two-dimensional matrix UF-TC (Fig. 1) and created problems concerning the cells of the matrix. The second step – which is still in progress – will be the development of problems on different levels of “cognitive activation” for each cell of the UF-TC-matrix. In July 2010, nearly 700 students of grade 10 had to solve the following problems. We classified these problems concerning the UF-TC-relationship. This relationship is not unique and depends on the solution strategy of the student. We also classified the problems concerning the representations which might be used – in our opinion – in the problem solving process: numerical (N), graphical (G), and symbolic (S) representations.

**Problem A1.** Solve the equation  $\frac{1}{3}x^3 - 3x - 5 = 0$ . Give the solution up to two post decimal positions. Explain!

Intuit.U	Concept.U	Relat.U.	Struc.U		Static M.			Dynamic M.			Multiple M.		
					N	G	S	N	G	S	N	G	S
Functional Understanding					Tool Competencies								

**Problem A2.** Solve the equation  $-\frac{1}{2}x^2 + x - 1 = 0$  graphically! Give the solution up to two post decimal positions. Explain!

Intuit.U	Concept.U	Relat.U.	Struc.U		Static M.			Dynamic M.			Multiple M.		
					N	G	S	N	G	S	N	G	S
Functional Understanding					Tool Competencies								

**Problem A3.** Draw some graphs of the family of functions  $f_a$  with  $f_a(x) = a \cdot x^2 - 4 \cdot a + 1$ ,  $a \in \mathbb{R}$ . Are there any points which are common to all graphs? What are the coordinates of these points? Give reasons!

Intuit.U	Concept.U	Relat.U.	Struc.U		Static M.			Dynamic M.			Multiple M.		
					N	G	S	N	G	S	N	G	S
Functional Understanding					Tool Competencies								

**Problem A4.** Given is the function  $f_c$  with  $f_c(x) = \frac{1}{2}x^2 - 6x + c$ ,  $c \in \mathbb{R}$ . For which values of  $c$  does  $f_c$  have exactly one zero? Give reasons!

Intuit.U	Concept.U	Relat.U.	Struc.U		Static M.			Dynamic M.			Multiple M.		
					N	G	S	N	G	S	N	G	S
Functional Understanding					Tool Competencies								



**Problem A5.** Are there any  $x$ -values, which fulfill the equation  $\sin^2(x) + \sin(x) = 3$ ,  $x \in \mathbb{R}$ ? Give reasons!

Intuit.U	Concept.U	Relat.U.	Struc.U		Static M.			Dynamic M.			Multiple M.		
					N	G	S	N	G	S	N	G	S
Functional Understanding					Tool Competencies								

The participants of this test used the symbolic calculator for one year in the classroom, for their homework and also in tests and examinations. Polynomial and trigonometric functions are regular content in class 10 in German high schools (Gymnasium).

### THE STUDENTS' QUESTIONNAIRE

In order to establish how the calculators were used, we applied a new investigation method: the students completed a questionnaire on the SC-use immediately *after the test*, giving details of whether and how they used the calculator. This test was intended to answer the following questions:

- How do students use the calculator?
- In which phases of a problem-solving process do the students use the calculator?
- Which functionalities (symbolical, graphical or numerical) do the students use?

This research method gives insight into the working style of all students participating in the test. Evidently, it only reproduces the subjective descriptions given by the students. A comparison of the answers of the students and the test results show large conformity and lead us to the conviction that students answered quite seriously. We posed the following general questions:

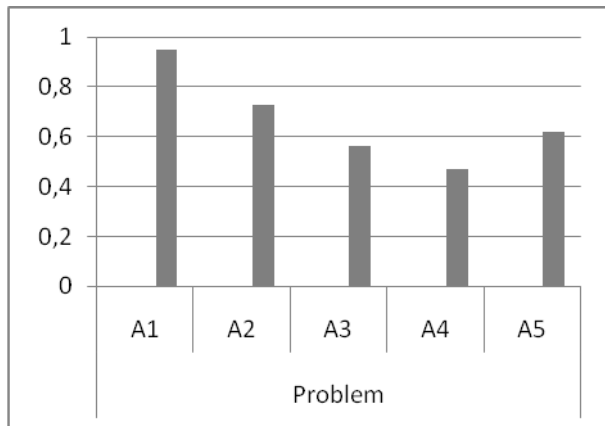
- Did you find the symbolic calculator was helpful when completing the tasks?
- Did you experience any difficulties when recording the use of the symbolic calculator in your solution in written form?
- Did you have any difficulties operating the symbolic calculator?
- Would you agree with the statement that the symbolic calculator gave you a feeling of security when completing the tasks?
- If you reflect your so far experienced lessons with the symbolic calculator, did you find them interesting?

### THE TEACHERS' QUESTIONNAIRE

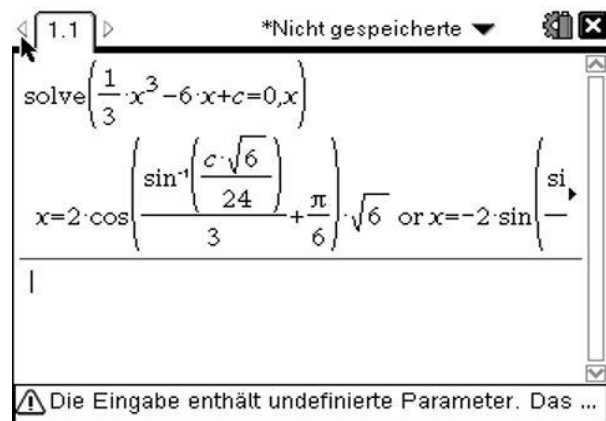
Before each test was carried out, the teachers provided an estimation of the extent to which students would solve the problems and how they would answer the general questions a) – e) (see above).

## RESULTS

Fig. 2 shows the percentage of students' correct answers to the problems A1–A5 (y-axis: 1  $\cong$  100 %). The decreasing percentage of correct answers from problem A1 to A4 confirms our assumption – and so the constructed competence model – concerning the increasing difficulty from static over dynamic to multiple tool competencies. Problem A5 plays a special role (it was labeled as an “additional problem” in the test).



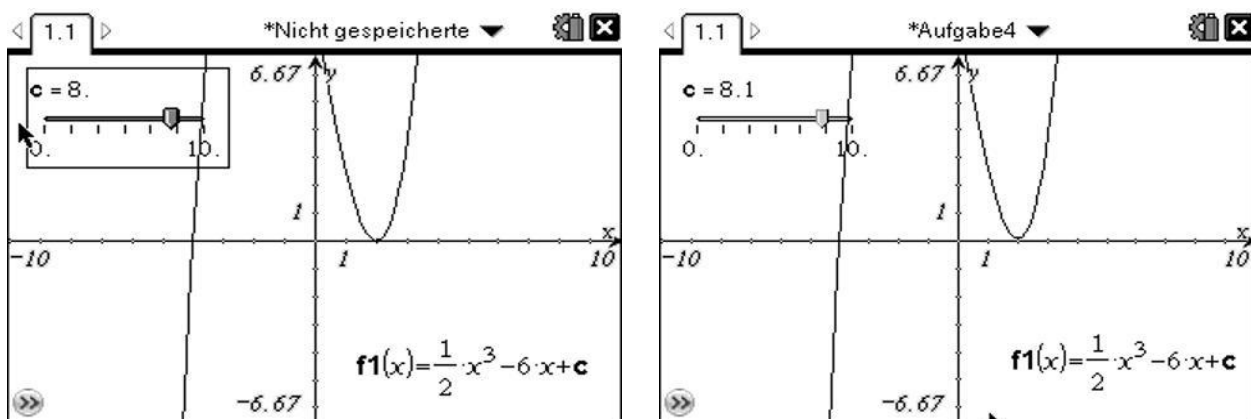
**Figure 2: Percentage of students' correct answers**



**Figure 3: The symbolic SC-solution of  $f_c(x) = 0$  (problem A4)**

Problem A4 reveals some problems while working with the SC. To find the value for  $c$  ( $f_c(x) = \frac{1}{2}x^3 - 6x + c$ ) for which  $f_c$  does have exactly one zero, you can try to find a symbolic solution with the SC (Fig. 3). But this solution cannot be interpreted by the students yet and means nothing to them.

Another possibility in order to get the solution is the dynamic representation, where students change  $c$  with a slider. Fig. 4 shows the graphs of  $f_c$  for  $c = 8$  and  $c = 8.1$ . (For  $c < -8$  and  $c > 8$   $f_c$  does have only one zero.)



**Figure 4: The graphs of  $f_c$  for  $c = 8$  and  $c = 8.1$**

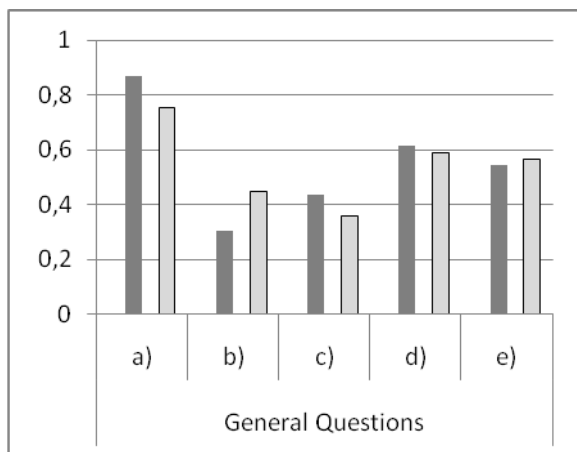
We have got the following incorrect students' answers:

- "For all values from 8 the function has one zero."
- "Slider  $c = 8,1$ . From  $c = 8,1$  (and more) the function  $f_c(x)$  has only one zero."
- " $c > 8,1$  the function has one zero."
- " $c \geq 8,1$ "

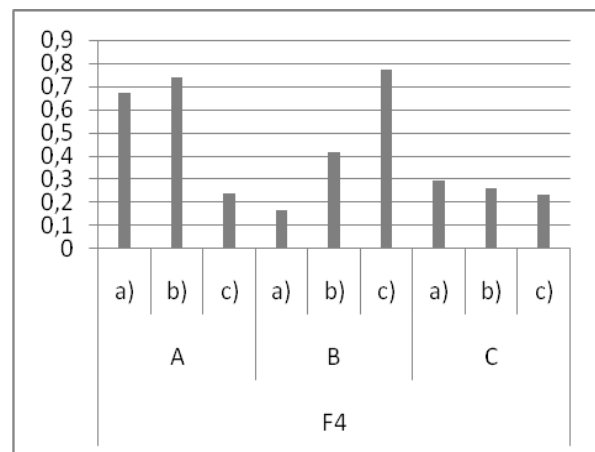
This shows students' missing ability to think in continuous intervals.

Moreover, is it quite difficult – the students did not know calculus – to *give reasons* for the correct answers. You have to know that for  $c = -8$  the graph has  $(-2; 0)$  as a maximum and for  $c = 8$  it has  $(2; 0)$  as a minimum. This gives you the basis for the argumentation.

Fig. 5 shows the percentage of students' answers and the teachers' prediction to the general questions of the questionnaire. It shows especially that the teachers' predictions correlate with the students' answers. This means that the teachers know students' abilities and difficulties quite well. They also notice that solutions and concepts are required to overcome students' obstacles.



**Figure 5: Percentage of students' answers (dark bars) and teachers' prediction (light bars)**



**Figure 6: Students' answers to the questions how they worked during the problem solving process**

Moreover, for each problem, the students had to answer the following multiple-choice questions:

A: When did you use the SC during the problem solving process?

- a) At the beginning      b) During the process      c) At the end

B: If you used the SC. What have you done with it?

- a) I calculated numerically      b) I solved equations      c) I drew graphs

C: Which difficulties did you have with the use of the SC?

- a) I have had no idea how to use it

- b) I have had problems with the numerical commands
- c) I didn't know how to draw the family of graphs

Fig. 6 shows the percentage of students marking the according answer. The percentage of students who still have had problems with the use of the symbolic calculator after one year of practice is surprisingly high.

## FUTURE DEVELOPMENTS

As a diagnostic goal, the competence model has the description of the anticipated aim in the context of the project, but also the detection of the abilities missing whilst students are working with symbolic calculators. But the evaluation or the diagnosis is only the very first step. The second step is to think about consequences based on the evaluation. How can the student be supported to progress to higher levels of understanding and higher competence levels while working with the symbolic calculator?

## NOTES

1. Skemp uses the expression “instrumental” in the sense of “rules without reasons” and in a different sense than it is used e. g. in the theory of instrumentation while working with new technologies. He also uses the expression “relational” in a quite open meaning (“Pupils knowing both what to do and why”), whereas in our model it is limited to relations between objects like functions or relations between properties of functions.

2. [http://www.pisa.oecd.org/document/58/0,2340,en\\_32252351\\_32236159\\_33688954\\_1\\_1\\_1\\_1,00.html](http://www.pisa.oecd.org/document/58/0,2340,en_32252351_32236159_33688954_1_1_1_1,00.html)

3. [http://www.pisa.oecd.org/document/29/0,3343,en\\_32252351\\_32236173\\_33694301\\_1\\_1\\_1\\_1,00.html](http://www.pisa.oecd.org/document/29/0,3343,en_32252351_32236173_33694301_1_1_1_1,00.html)

## REFERENCES

- Artigue, M, Bardini C. (2010). New didactical phenomena prompted by TI\_Nspire specificities – The mathematical component of the instrumentation process. In: V. Durand-Guerrier *et al.* (Eds.). *Proceedings of CERME 6* (pp. 1171-1180), Jan. 28th-Feb. 1st 2009, Lyon France, INRP. Retrieved from <http://www.inrp.fr/publications/edition-electronique/cerme6/wg7-13-artigue-bardini.pdf>
- Bottino, R. M., Cerulli, M. (2010). Methods and tools to face research fragmentation in technology enhanced mathematics education. In V. Durand-Guerrier *et al.* (Eds.). *Proceedings of CERME 6* (pp. 1379-1388), Jan. 28th-Feb. 1st 2009, Lyon France, INRP. Retrieved from <http://www.inrp.fr/publications/edition-electronique/cerme6/wg7-34-bottino-cerulli.pdf>.
- Guin, D., Ruthven, K. and Trouche L. (Eds.) (2005). *The didactical challenge of symbolic calculators*. New York: Springer.
- Hoyles, C. & J.-B. Lagrange (Eds.) (2010). *Mathematics Education and Technology – Rethinking the Terrain*. The 17<sup>th</sup> ICMI Study. New York a. o.: Springer.

- Niss, M. (2004). Mathematical competencies and the learning of mathematics. The Danish KOM project. In A. Gagtsis, S. Papastavridis (Eds.). *3<sup>rd</sup> Mediterranean Conference on mathematical education* (pp. 115–124), 3–5 January 2003. Athens, Greece. Athens: The Hellenic mathematical society. Retrieved from [http://www7.nationalacademies.org/mseb/mathematical\\_competencies\\_and\\_the\\_learning\\_of\\_mathematics.pdf](http://www7.nationalacademies.org/mseb/mathematical_competencies_and_the_learning_of_mathematics.pdf)
- OECD: Organisation for Economic Co-operation and Development (Ed.) (1999). *Measuring student knowledge and skills. A new framework for assessment*. Paris: OECD Publication Service [2]. Retrieved from <http://www.pisa.oecd.org>
- OECD: Organisation for Economic Co-operation and Development (Ed.) (2003). *The PISA 2003 Assessment Framework: Mathematics, reading, science and problem solving knowledge and skills*. Paris: OECD Publication Service [3]. Retrieved from <http://www.pisa.oecd.org>
- Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22(1), 1–36.
- Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 20–26.
- Trouche, L. (2005). Calculators in Mathematics Education: A rapid Evolution of Tools, with differential Effects. In D. Guin, K. Ruthven and L. Trouche (Eds). *The Didactical Challenge of Symbolic Calculators* (pp. 9-39). New York: Springer.
- Vollrath, H.-J. (1993). Reflections on mathematical concepts as starting points for mathematical thinking. In R. Biehler, R. W. Scholz, R. Sträßer, B. Winkelmann (Eds.). *Didactics of mathematics as a scientific discipline* (pp. 61–72). Dordrecht, Boston, London: Kluwer.
- Weigand, H.-G., Bichler, E. (2010). Towards a competence model for the use of symbolic calculators in mathematics lessons – The case of functions. *ZDM - The International Journal on Mathematics Education*, 42(6), 697–713.

# INTRODUCTION TO THE PAPERS OF WG 16: DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN RESEARCH IN MATHEMATICS EDUCATION

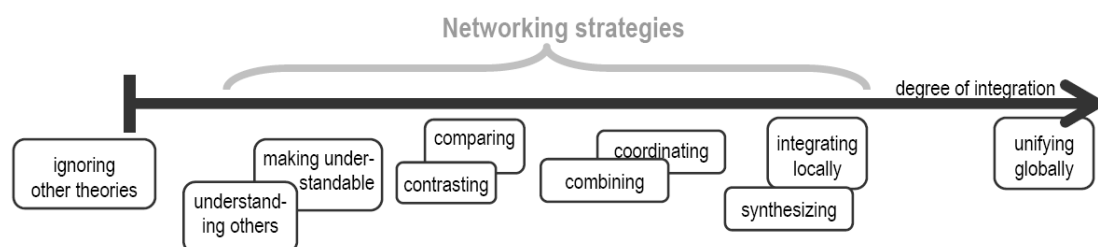
Introduction to the Papers of Working group 16

Ivy Kidron<sup>a</sup>, Angelika Bikner-Ahsbabs<sup>b</sup>, John Monaghan<sup>c</sup>, Luis Radford<sup>d</sup>, Gérard Sensevy<sup>e</sup>

<sup>a</sup>Jerusalem College of Technology (Israel), <sup>b</sup> Universität Bremen (Germany), <sup>c</sup>University of Leeds, (United Kingdom), <sup>d</sup>Université Laurentienne, (Canada), <sup>e</sup> INRP Rennes (France)

The diversity of different theoretical perspectives in mathematics education research can be seen as a problem or as a benefit to the advancement of the field. This Working Group, which was established at CERME 4 (2005), seeks to: explore ways of handling the diversity of theories in order to better grasp the complexity of learning and teaching processes; and understand how theories can be connected or not in a manner that respects their underlying assumptions. The central term that emerged from the CERME 4 working group was *networking*. In order to promote the networking of theories it was suggested: to make explicit the level at which a theory operates; to increase our awareness of the underlying assumptions of each theory. Both of these aspects were revisited in CERME 7 where the question of underlying assumptions also concerned questioning *the nature of mathematical objects*.

After a two stage peer review process, 15 papers were accepted for discussion in the Working Group. We revisit the 15 papers to raise themes that emerged from the conference discussion. The first is strategies for networking which includes the dialectical development/transformation of theory(s) and theorist(s) (see Monaghan, 2011) and the semiosphere (see Radford, 2008). We then discuss the 15 papers with respect to the spectrum of networking strategies developed by Prediger, Bikner-Ahsbabs & Arzarello (2008) represented by the diagram below. Then we devote a subsection on the nature of mathematical objects as discussed in the working group.



**Figure 1: Networking strategies**

## STRATEGIES FOR NETWORKING

### Towards a “theory of networking theories”

A central term that emerged from the working group was *transformation*: after networking researchers are often transformed and are able to see things which they could not see before; negotiating meanings as a sub-issue here concerns the conditions which permit fruitful dialogue; and methods for networking theories can be intentional or implicit. Papers considered under this theme are those of: Artigue, Bosch & Gascón; Mason; Monaghan; and Radford (2008). Although the latter is not a CERME paper, it was repeatedly raised in relation to its idea of *semiosphere*, i.e., a multi-cultural space of meaning-making processes and understandings generated by theories as they come to know and interact with each other. In the *semiosphere*, a theory is considered as a dynamic interrelated triplet (P, M, Q) formed of theoretical principles (P), methodologies (M), and research questions (Q). Strategies for networking depend to an important extent on how “close” or “far” the networked theories are located in the *semiosphere*. Mason’s paper is couched in his discourse of ‘shifts of attention’ (Mason, 2003). He suggests that “theories in mathematics education consist of collections of frame(work)s, which themselves consist of labels for distinctions ... [with] assumptions and values which, when combined with the distinctions, suggest actions that might be taken”. A theory allows one to discern detail through making distinctions. He is critical of the *semiosphere* as a means to compare theories without attending to “the intentions or uses of those theories and to reach mutual clarification of the worlds of experience they occasion, including the phenomena they recognise and at what grain size, and the sorts of conclusions, explanations etc. they afford.” It was argued, however, that the intentional aspect of a theory is already captured in its research question component, Q; in a similar manner, the assumptions and values are embedded in the theoretical principles, P.

Artigue, Bosch & Gascón consider the potential offered by the Anthropological Theory Didactics (ATD) for addressing issues of networking between theories, which they term ‘research praxeologies’. Although ATD developed as a theory within mathematics education, it is sufficiently general to be applied to other spheres of human activity. Praxeologies consist of two blocks each with two elements: (i) practice of research (*types of problems* and *techniques* used in these problems); (ii) the *technological-theoretical* discourse used to describe, justify and interpret both the research practice and the results obtained. An important adjunct is the construct of ‘didactic phenomena’. Some phenomena “enrich the initial theoretical framework to produce new interpretations and techniques or research methodologies, while others remain at the level of ‘results obtained’ and are reinvested to formulate new problems or to propose new diagnostic and practice-development tools”.

Monaghan explores the ‘theoretical genesis’ of an informal meta-theory presented in Ruthven, Laborde, Leach, & Tiberghien (2009), a construct coined in analogy with ‘instrumental genesis’. The analogy has some cogency, both concern an agent

appropriating an artefact for a purpose over time and a dialectic between agent and artefact/theory exists, but Monaghan also explores the limits of the analogy. Factors involved in the theoretical genesis include writing, learning, engagement with research and other voices; making ones theoretical stance explicit to intended journal article readers from a different research culture appeared particularly important. It was recognised, during the working group discussion, that much in Monaghan's paper could be rewritten in terms of research praxeologies.

Some efforts were made to find links between these four approaches to researcher networking; an effort of networking "approaches for networking theories". Initial considerations focused on the possible use of these four approaches in the concrete examples of networking in the papers in the working group, but at this point in time we are not in a position to present an overall schema for how they interrelate (or not) nor to matters which they, collectively, do not address.

## **DISCUSSION OF THE PAPERS WITH RESPECT TO THE SPECTRUM OF NETWORKING STRATEGIES**

### **Understanding/making understandable**

Craig offers a way of exploring patterns of research collaboration within the mathematics education research field. Kaenders, Kvasz& Weiss-Pidstrygach present categories of mathematical awareness (an analytical tool) connected to activity theory.

### **Comparing/contrasting**

LaCroix compares two activity theoretical perspectives with resemblance in their sources: Cultural Historical Activity Theory –Engeström's interpretation and Radford's theory of knowledge objectification. It is argued that the unit of analysis in 3rd generation activity theory is too big to understand micro-genetic development.

Sollervall deals with semiotic representations to negotiate disciplinary and individual perspectives on the notion of meaning in mathematics. He uses Peirce's semiotic triangle and Duval's theory of registers. Pierce's theory is used to develop a unified model of theoretical constructs to account for the role of external representations as mediators of individual meanings in mathematics classrooms.

### **Combining/coordinating**

Douek coordinates socio-cultural theories to answer the question of how to deal with the learning difficulties of poor learners. Goodchild coordinates two socio-cultural perspectives with fundamental differences: the theoretical basis of the Communities of Practice approach is distinct from that of Cultural Historical Activity Theory. The aim is to develop a community of inquiry. Both are (separately) useful for research taking into account two kinds of development: extrapolation and expansion.

Ligozat, Wickman&Hamza and Tabach&Nachlieli use networking theories to analyze mathematical classroom discourses. Ligozat et al. look at classroom activities from



different perspectives: the institutional and the participant and combine two theories with focus on social aspects. Tabach & Nachlieli offer an example of networking two theories with focus on socio-cultural and linguistic approaches.

In Santi's paper two semiotic approaches are coordinated which provide (rather like Goodchild's) complementary views of Duval's and Radford's theories which can only be linked in a diachronic way due to strong differences in theoretical principles.

### **Integrating/synthesizing**

Jay's paper deals with difficulties of networking the cognitivist and socio-culturalist perspectives. The reconciliation of perspectives is permitted by a semiotic approach which appears to offer a neutral arena for negotiation of definitions of crucial terms like concept, understanding, learning.

The paper by Kidron, Bikner-Ahsbahr & Dreyfus deals with a second iteration of networking between two theoretical frames. Their first iteration networking focused on two theoretical concepts: the need for a new construct, and interest. A benefit of this previous networking was the insight that besides interest and the need for a new construct, a more general epistemic need (GEN) can drive students' progress in learning processes according to the challenge they meet within a situation. The second iteration concerns the different roles of the GEN from each perspective.

Font, Malaspina, Giménez & Wilhelmi ask "What is the nature of the mathematical objects?" They explore this question by the use of a synthesis between the onto-semiotic approach (OSA), APOS theory and the cognitive science of mathematics (CSM) as regards their use of the concept 'mathematical object'. APOS theory and CSM highlight partial aspects of the complex process through which, according to OSA, mathematical objects emerge. OSA extends APOS theory by addressing the role of semiotic representations; it improves the genetic decomposition by incorporating ideas of semiotic complexity, networks of semiotic functions and semiotic conflicts; it offers a refined analysis due to the way in which it considers the nature of such objects and their emergence out of mathematical practices.

### **THEMES FOR FURTHER CONSIDERATION**

Further to issues already raised we find, in comparing examples of specific networking strategies, that approaches can be very different concerning the function of empirical content. LaCroix is concerned with the phenomena of learning whereas Sollervall seeks to develop a tool for the analysis of empirical phenomena. A similar situation holds with regard to Jay's and Kidron et al.'s papers. Although both papers aim to connect individual and social learning, Jay aims to offer an integrated tool to analyse empirical phenomena whereas Kidron et al. start with the empirical situation building up to a point of integration. At CERME6, a distinction between top-down networking, bottom-up networking and a kind of mix between both was discussed. Our knowledge is still at a level of craft knowledge, but we have experienced

progress in that examples show the potential of networking for improving the quality of research practices gaining linked results. We hope this work towards a *theory of networking theoretical approaches* helps the community develop in the direction of scientifically based multi-theoretical empirical research.

## REFERENCES

- Mason, J. (2003). On the structure of attention in the learning of mathematics. *AAMT*, 59(4), 17-25.
- Monaghan, J. (2011). People and theories. *Proceedings of Cerme 6*, 16-23. <http://www.inrp.fr/editions/editions-electroniques/cerme6/>
- Prediger, S., Bikner-Ahsbals, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education*, 40(2), 165-178.
- Radford L. (2008). Connecting theories in mathematics education: Challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, 40(2), 317-327.
- Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher*, 38(5), 329-342.

# RESEARCH PRAXEOLOGIES AND NETWORKING THEORIES

M. Artigue,<sup>I</sup> M. Bosch,<sup>II</sup> & J. Gascón<sup>III</sup>

IUniversité Paris 7 (France), IIUniversitat Ramon Llull (Spain),

IIIUniversitat Autònoma de Barcelona (Spain)

***Abstract:** In this contribution, we consider the potential offered by the anthropological theory of the didactic (ATD) for addressing the issue of networking between theories through the extension of the notion of praxeology, which is at the core of ATD to research praxeologies. After introducing such an extension, we discuss its implications in terms of networking, giving a crucial role to the notion of didactic phenomena. We then use this language for reflecting on two networking experiences in which we have been involved.*

## INTRODUCTION

In accordance with the work done by the ‘theories group’ in the last two CERMEs, this contribution joins the efforts made to support a productive cooperation in European mathematics education research, in order to overcome the framework compartmentalization that could hinder the capitalisation of knowledge and its practical exploitation. These efforts have shown that the interaction between researchers working with different approaches has to go further than the ‘communication paradigm’ that dominates exchange activities in most international conferences. It needs real ‘teaching and learning’ activities to explain what one does and to understand what the others do. Experiences in ‘networking theories’, carried out in this sense<sup>80</sup>, have shown that their productive development also requires the consideration of a shared epistemological model, that is, a common way of thinking and talking about what scientific work is and how it evolves. In fact any research activity supposes a particular implicit way of interpreting the nature of problems that are approached, the empirical field to consider, the kind of methodologies that can be used and, more generally, what research is and what it is for. When the exchange between researchers attains the level of the theoretical bases – as is the case in the ‘networking theories’ activities –, then it becomes necessary to question the implicit epistemological model of each approach, looking for a common language to express and discuss the respective epistemologies. In this sense, the Anthropological Theory of the Didactic (ATD) that we use in our research and, more concretely, its central notion of ‘praxeology’ has progressively appeared to us as a useful tool to develop such a common ‘language’ or epistemological model. Here we present the recent work we have undertaken in this direction.

---

<sup>80</sup> We refer more especially to the so-called ‘Networking group’ led by Angelika Bikner-Ahsbals that emerged from CERME4 and to the work carried out by the first author in European projects like TELMA and ReMath.

## **‘THEORIES’ OR RESEARCH PRAXEOLOGIES?**

If, according to ATD, we assume the general anthropological postulate that all human activities can be described in terms of ‘praxeologies’ (Chevallard, 1999, 2006), this must be also the case for research activities. Any kind of research, including ‘networking’ projects, should thus be subsumed under the notion of *research praxeologies*. In this perspective, talking about ‘theories’ (as in the expression of ‘networking theories’) is the result of a metonymy used to point to the whole – *research praxeologies* – by only indicating one part, the theoretical block of praxeologies. As any other praxeology, research praxeologies are indeed composed of an amalgam of pieces that can be described by a set of four elements  $[T/\tau/\theta/\Theta]$ . The pair  $[T/\tau]$  corresponds to the ‘practice’ (or know-how) of research, with the *types of problems*  $T$  that are approached and the *techniques*  $\tau$  used to approach the problems. The block  $[\theta/\Theta]$  forms the *technological-theoretical* discourse used to describe, justify and interpret both the research practice and the results obtained. This theoretical block corresponds to research ‘knowledge’ and is often considered as the representative of the whole praxeology, with the limitations and biases that this reduction can generate in the approach to and treatment of ‘networking’ issues.

We postulate that the notion of praxeology can help overcome these limitations and that it can also be useful to retrospectively reflect on networking efforts. We also find it important to stress that research praxeologies, as any other praxeological form, are ‘alive’ entities that evolve and change, which affects at the same time their four components  $[T/\tau/\theta/\Theta]$  and the interaction of these. The evolution of the practical block  $[T/\tau]$  produces new theoretical needs that make the theoretical block  $[\theta/\Theta]$  progress and, reciprocally, the evolution of concepts, interpretations or ways of thinking and the emergence of new results lead to the construction of new techniques and the formulation of new problems. Research praxeologies can appear as different kinds of amalgams, more or less organised depending on the maturity of the field. It is the historical development of the field that helps structure these praxeological amalgams, making them more coherent and easier to disseminate according to different didactic and institutional transposition processes. Beyond the static description of research praxeologies in terms of their practical and theoretical blocks, processes piloting their dynamics are still to be analysed in depth. Our contribution consists in considering the notion of ‘phenomenon’ and relating it with the ‘technological’ element of praxeologies, which will highlight its crucial role in the dynamics of praxeologies.

## **‘PHENOMENA’ AND THE DYNAMICS OF RESEARCH PRAXEOLOGIES**

**The notion of ‘didactic phenomenon’.** Today the notion of ‘phenomenon’ does not happen to have a central function in many didactic approaches. It did however play a crucial role in the emergence of the theory of didactic situations (TDS) and its vision of didactics as a scientific discipline. In the first developments of TDS in the 1980s,

and through several different formulations, Guy Brousseau (1997) defined didactics of mathematics as the science the essential aim of which is the knowledge of *didactic phenomena*, that is, phenomena appearing in the teaching, learning or, more generally, diffusing of mathematical knowledge in social institutions (including school ones). Hence, didactic phenomena have to be considered both as a construction and as an object of study of didactics, in the same way that physics studies the specific construction ‘physical phenomena’, or sociology studies and also defines social phenomena, etc. –including all the historical controversies about phenomena delimitation in nature and social sciences.

What is the role played by *phenomena* in relation to *research praxeologies* and their evolution? In a first approach, we can characterise didactic phenomena as empirical facts, regularities that arise through the study of research problems. Some of these phenomena enrich the initial theoretical framework to produce new interpretations and techniques or research methodologies, while others remain at the level of ‘results obtained’ and are reinvested to formulate new problems or to propose new diagnostic and practice-development tools. In order to clarify the relation between the notion of phenomenon and the four components of a praxeology, let us start from a very simple example of mathematical praxeology. Let us consider Pythagoras’ theorem or, to be more precise, the phenomenon underlying this theorem, that is, a certain regularity between the measures of the sides of right triangles. At the beginning we can consider a *type of mathematical problem* that could be formulated as the problem of the characterisation of a right triangle or the graphical representation of a right angle. The answer to this problem appears as a *technological element* (the description of a property of a given set of figures) within the mathematical praxeology that emerges around this type of problems. This technological element is not only the description of a regular fact: it also produces new *mathematical techniques*, helps formulate *new problems* and discover new regularities, thus producing more *technological results*. In the long run, if the initial regularity appears to be strong enough, then it comes to integrate the *theory* of the praxeology as a basic principle of certain kind of geometries (those with a Euclidian metric).

It is thus an entire mathematical praxeology, with its types of problems, its techniques and its technological-theoretical discourses that the expression ‘Pythagoras theorem’ refers metonymically to. Behind a technological ingredient such as a theorem – or any other description of a regular fact or *phenomenon* – we can find a whole set of praxeological ingredients (problems, techniques, etc.), which this technological ingredient contributes to structure. Taking all necessary precautions, we will briefly establish a parallelism between this example and research praxeologies in didactics. We will use a concrete example, the phenomenon

of didactic transposition, to illustrate our proposal<sup>81</sup> which is to see the role played by phenomena in the ‘structuring’ of praxeologies, that is to say in their *dynamics*.

**Phenomena and type of problems.** As any scientific discipline, didactics of mathematics aims at identifying and studying a specific kind of phenomena (didactic ones) in order to attain a greater capacity of action and comprehension. Any research question or problem thus has to be related – even if this relation is mostly done a posteriori – to the highlighting of a phenomenon, its delimitation, the conditions needed for its existence and evolution, etc. Even if research takes as a starting point a problem emerging from a very concrete teaching or learning practice (as it often does), an effort is always made to formulate the problem in a more general way, implicitly considering it as a specimen of a given ‘type of problem’. This is a first step in the work that follows: looking for regularities related to the practical issue approached, trying to characterise them and, to some extent, ‘understand’ or ‘act upon’ them. Let us consider, for instance, the phenomenon of didactic transposition as it was characterised by Yves Chevallard (1985). Several new problems have been raised and studied that could not even have been formulated before the identification of the phenomenon (see Bosch & Gascon, 2006 for a recent review).

**Phenomena and technological components** In research processes, the results obtained as an answer to the raised problems generally contribute to enrich the initial research technology by integrating new characteristics of the studied phenomena or even new derived phenomena. There always exists a double-direction effect between the results obtained and the evolution of the technology of research praxeologies, which can be considered at the core of progress of scientific research.<sup>82</sup> For instance, the study of transpositive processes in different mathematical domains has highlighted various phenomena that, in turn, have been used as a starting point to formulate new problems and draw attention to new regularities. A good example is the phenomenon of the ‘algebraisation’ of Calculus at upper secondary school level (Artigue, 1995), a result that has then been used to analyse the teaching of limits of functions (Barbé et al., 2005). Other examples coming from the didactic transposition processes are the derived phenomenon of the ‘stoppage’ of didactic transposition (Assude, 1993) or of ‘detransposition’ (Antibi & Brousseau, 2000).

**Phenomena and technical components** The study of phenomena not only generates the description of regularities, restrictions or ‘paradoxes’ (like those of the ‘didactic

---

<sup>81</sup> We are perfectly conscious of the distance between mathematics and didactics as fields of research. However, commonalities can be established and can be productive in both senses: sometimes the maturity of mathematics hides some evolution phenomena that are more visible in the recent and less developed dynamics of didactics.

<sup>82</sup> This is less true when the theoretical block of the research praxeology comes from a different discipline. We then obtain a single-direction effect which ‘breaks down’ the dynamics of scientific research: for instance, when a given notion of cognitive psychology is used to analyse some facts related to the learning of mathematics, because the ‘external’ character of the results obtained, they will have no effect on the development of the initial psychology notional frame.

contract', for example). It also leads to new ways of doing research, that is, new techniques and new methodologies. In the case of transpositive phenomena, the highlighting of relations and differences, both chronological and diachronic, between the 'scholarly knowledge', the 'knowledge to be taught' and the '(actually) taught knowledge' has now become a technique of didactic analysis by itself. Almost any problem studied within the ATD or the TDS includes, to some extent, a questioning about which is the knowledge at stake, where does it come from, what 'scholarly knowledge' legitimises its teaching, what changes have been operated on it, what 'noospherian' discourses support or hinder its teaching, etc. The notion of didactic transposition has represented an important enlargement of the field of study of didactics because it has pointed out the need to also consider the mathematical activities that exists outside the school (Bosch and Gascón, 2006).

**Phenomena and theoretical components.** In a praxeology, the 'theory' component includes the set of notions and relations that are used to apprehend phenomena (describe them, formulate questions about them, etc.), to develop them and to identify new regularities. The 'theory' appears as the second level of validation of the activity, as an explanation and justification of the 'technology'. It contains the assumptions taken, that is, the technological elements that come up being taken for granted because of their solidity and persistence. At this level we find questions such as: What phenomena are studied? What is a problem in didactics? Why can this or that result be assumed as such? The empirical enlargements mentioned before are also integrated at this level as far as they become basic and implicit assumptions. At the same time, the unit of analysis that is assumed determines the kind of phenomena that can be considered and the kind of data that are being collected to bring evidence to the study. For instance, the existence of transpositive processes between institutions is a theoretical assumption that is not questioned, nor questionable, in ATD. The 'kind' of transpositive processes that are taking place, their main characteristics and the conditions and restrictions they create on teaching institutions are, on the contrary, some of the main problems considered by this approach.

The praxeological dynamics we just described may help understand the processes through which the studied phenomena produce new technological results that partially become new theoretical tools and produce in turn new research techniques allowing the identification or construction of new phenomena. It is this praxeological dynamics that we propose to consider here in order to analyse – and guide – two networking experiences between European research teams.

## **THE EXPERIENCE OF THE GROUP « NETWORKING THEORIES IN MATHEMATICS EDUCATION »**

The working group on 'networking theories' was created in 2005. It includes 12 researchers from six different countries and its work aims at the exchange, comparison, and connection between theoretical frameworks. Results obtained have

been presented in previous CERMEs and more recently in a research forum at the last PME conference (Bikner-Ahsbals et al., 2010). In this section, we analyse an episode of its work, already evoked at CERME6 (Artigue, Bosch, Gascon & Lenfant, 2010), with the aim of showing how research questions and theoretical components of praxeologies influence research methodologies (technical components), the units of analysis considered pertinent, and the didactic phenomena identified.

The initial work of the group was based on a video realized in a grade 10 Italian classroom and additional material considered necessary for its analysis by our Italian colleagues. Each team was asked to analyze the video from its own theoretical perspective but the data provided was judged insufficient by each except the Italian team. The video showed two students working in a pair, with little intervention of the teacher. Additional information provided on the session itself and its context was quite limited, making an analysis supported by TDS very hypothetical and an analysis supported by ATD nearly impossible. A questionnaire was then addressed to the teacher in charge of the classroom, asking for additional information to allow the different teams to complement the partial analyses already carried out. In the teacher's answers, the attention of several members of the group was especially attracted by the following excerpt:

*I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures bring me to become aware of the so called 'semiotic game' that consists in using the same gestures of students but accompanying them with a more specific and precise language in a relation to the language used by students. Semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge.*

This convergence of interests led the group to develop a new strategy for progressing in the collaborative work undertaken: The TDS team should associate a question articulated in the TDS framework to this excerpt, and then each of the other teams should rephrase this question according to its own perspective. We reproduce below the text introducing the TDS question, which in fact also uses some ATD constructs.

*The connection between the mathematics produced by students in what we would label, using the TDS frame, an adidactic situation through interaction with the adidactic milieu of this situation on the one hand, and the institutional knowledge aimed at on the other hand, generally requires at least changes in the ways the mathematics at stake is expressed in order to progressively tune these it conventional forms of expression. The teacher considers that he has a specific mediating role to play for making this connection possible and uses semiotic games as a tool for that. In other terms, semiotic games can be considered as components of the praxeology (or more certainly one of the praxeologies) that he has developed in order to solve this didactic task.*

The expression "semiotic game" thus denotes what can be seen as a technique, a component of a teaching praxeology, resulting from the identification of some particular phenomena of semiotic mediation. Interpreted that way, it shows how a theoretical focus (in this case a semiotic focus) can lead to the identification of



specific phenomena, and from that to theoretical constructs or to didactic techniques, considered as tools for improving the efficiency of learning and teaching processes. A TDS perspective leads to question the efficiency of such a didactic praxeology for two reasons at least. The first one is that very often interactions with the adidactic « milieu » do not guarantee the possibility of establishing a direct connection with the institutional knowledge aimed at. These limitations are the source of different didactic phenomena identified as paradoxes of the didactic contract: Topaze effect, Jourdain effect, meta-cognitive slide. The second reason is that the adidactic situations most often observed in classrooms are situations of action, not situations of formulation. In such situations, some linguistic activity generally takes place but it is not taken in charge in the piloting of the situation through didactic variables.

From this perspective, the video analysis leads to the postulate that, in this particular context, the distance between what the students have autonomously produced and the forms of knowledge aimed at by the teacher, as expressed in his answers to the questionnaire, makes problematic the productive character of such a *semiotic game*. Thus the question proposed to the group:

*Do the episodes at our disposal allow us to identify characteristics of the semiotic game technique that would help us to understand their potential for compensating the possible limits of the interaction with the adidactic milieu for achieving the expected mathematical goals, and linguistic evolution linked to the needs of institutionalization processes?*

In the networking group, each team has rephrased this question from its specific theoretical perspective. This episode shows how the consideration of a new theoretical framework, here TSD, can lead to question a didactic praxeology, legitimately considered as a research result in another didactic culture. For addressing this question, a new research praxeology has been developed, a research praxeology which had no reason to emerge in either of these didactic cultures and only exists because a specific networking activity has been undertaken. In the limited space of this contribution, we cannot present the results produced by this research praxeology, nor their elaboration into didactic phenomena. We nevertheless hope to have shown up to what point the relationships existing between the different components of research praxeologies and the didactic praxeologies emerging from research results, deserve our attention. We also hope to have shown that an approach in terms of praxeologies can be helpful for addressing networking issues.

## **THE EXPERIENCE OF THE EUROPEAN PROJECT REMATH**

The experience of the European project ReMath (<http://remath.cti.gr>) offers complementary insights for putting to the test an approach of networking theories in terms of research praxeologies. An essential goal of this project was to support the capitalization of research on digital technologies in mathematics education, through the development of an integrated theoretical framework, with a focus on the affordances of digital technologies for mathematics learning in terms of

representations, and more globally of semiotic activities. Six European teams worked on this project during four years, relying on the previous experience of the European research team TELMA (Artigue, 2009). A sophisticated methodology was developed for this project. It relied on a meta-language created in TELMA and a system of cross-experiments using the same digital technology in different didactical cultures, whose negotiation, implementation and analysis was taken itself as an object of research. This project also allowed the researchers involved to insert the networking activity into a permanent dialogue between the design of digital artefacts and the design of scenarios for their educational use in different educational contexts.

Looking back at this project through the lens of research praxeologies, it looks clear that the methodology used allowed the ReMath teams to organize their work around the collective study of their respective research praxeologies. These research praxeologies were made explicit enough in their different components for ensuring the productivity of comparative analyses, and particularly that of the cross-case studies of the different experimentations carried out with the same digital artefact. The articulation of common research questions to be addressed by the different cross-experiments and then the addition by each team of questions reflecting its specific concerns, the strict organization of interactions between teams all along the process, from the design of artefacts to the a posteriori analysis of cross-experimentations, the meta-language of concerns, played an essential role. The design of artefacts and the cross-experimentations contributed ipso facto to two different types of research praxeologies: on the one hand, praxeologies inserted in the didactic culture proper to each team and, on the other hand, a “networking praxeology” still in development. The problem addressed by the first ones was the identification of the learning affordances of the systems of representations of mathematical objects implemented in the six digital artefacts, and of the conditions for a possible ecology of these in realistic contexts. The problem addressed by the second one was networking between theories. It situated at a meta-level with respect to the first ones, and the results it produced have a different nature. Some are methodological and a priori regard more the practical block of this networking praxeology, for instance those concerning the technique of cross-experimentation, an essential ingredient of the networking praxis progressively refined. Some are more likely to contribute to its theoretical block. This is the case for the boundary objects identified for facilitating the communication between the theoretical frameworks at stake and for the STF (Shared Theoretical Frame). Some results show the possibility of connections and even offer partial integration of theoretical frameworks while some others identify limits to such ambitions, but it is worth noticing that, at this stage, none of these results has been given a clear status of phenomenon. Due to these characteristics, there is no doubt for us that a systematic a posteriori analysis of the ReMath project using the notion of research praxeology should be helpful

## CONCLUDING REMARKS

The retrospective analysis of the two research projects outlined above seems to confirm our initial postulate that networking between theoretical frameworks must be situated in a wider perspective than that consisting of the search for connections between the objects and relationships structuring these. From this point of view, our approach is fully coherent with that developed by Radford (2008) who, defining a theory as a triplet made of a system of principles, a methodology and a template of research questions, insists on the necessity of considering these three components in networking activity. Space limits do not allow us to enter into a comparison of our approach with that presented by Radford conceiving a network of theories as a semiosphere, but we hope that the discussions in the CERME Working Group will contribute to clarify similarities and differences. In our approach for instance, the notion of phenomenon appears as a crucial notion for understanding the dynamics of research praxeologies and the evolutionary links between their different components, while the word phenomenon is absent from Radford's text. This is an intriguing difference which certainly needs to be collectively analysed and discussed.

Our reflection tends to show that an approach in terms of research praxeologies can be productive for networking between theories, especially because it helps address the essential issue of the functionality of theoretical frameworks, by inserting these in systems of practices. Networking between theoretical frameworks, if considered as a task to be solved, requires nevertheless the development of specific praxeological elements that cannot be separated from research praxeologies. The European projects evoked above attest the existence of such elements, with emerging techniques and embryonic technologies made of classifications, structured landscapes, meta-languages. The model of praxeologies could thus help us compare the different existing efforts of networking and develop more productive ones. This is nevertheless only a hypothesis which has not yet been seriously worked out. Finally, we would like to stress that when adopting such a perspective, one must remain sensitive to the fact that this approach, as any other one, can also introduce some limitations. Considering them is indeed part of the epistemological vigilance required in any research process.

## REFERENCES

- Antibi, A. y Brousseau, G. (2000). La *détransposition* des connaissances scolaires. *Recherches en Didactique des Mathématiques*, 20(1), 7-40
- Artigue M. (1995): La enseñanza de los principios del cálculo: problemas epistemológicos, cognitivos y didácticos. En Artigue, M., Douady, R., Moreno, L., Gómez, P. (Eds.) *Ingeniería didáctica en educación matemática* (pp. 97-140). Grupo Editorial Iberoamérica: México.
- Artigue M. (Ed.) (2009). Connecting Approaches to Technology Enhanced Learning in Mathematics: The TELMA experience. *International Journal of Computers for Mathematical Learning*, 14(3) (special issue).

- Artigue, M., Bosch, M., Gascón, J., & Lenfant, A. (2010). Research problems emerging from a teaching episode: a dialogue between TDS and ATD. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 1535-1544). Lyon: INRP [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6).
- Assude T. (1993). Écologie de l'objet « racine carrée » et analyse du curriculum. *Petit x*, 35, 43-58
- Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: the case of limits of functions in Spanish high schools, *Educational Studies in Mathematics*, 59, 235-268.
- Bikner-Ahsbahr, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In Pinto M. M. F. & Kawasaki T. F. (Eds.), *Proceedings of PME 34*, Vol. 1 (pp. 145-175). Belo Horizonte, Brazil.
- Bosch, M., & Gascón, J. (2006). 25 years of the Didactic Transposition. *Bulletin of the International Commission on Mathematical Instruction* 58, 51-65.
- Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht: Kluwer Academic Publishers.
- Chevallard, Y. (1985). *La transposition didactique. Du savoir savant au savoir enseigné*. Grenoble : La Pensée sauvage (2d edition : 1991),
- Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, 19(2), 221-266.
- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (Ed.), *Proceedings of the 4<sup>th</sup> Conference of the European Society for Research in Mathematics Education (CERME 4)* (pp. 21-30). Barcelona: FUNDEMI-IQS.
- Kidron, I., Lenfant, A., Bikner-Ahsbahr, A., Artigue, M., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *ZDM The International Journal on Mathematics Education*, 40(2), 247-264.
- Radford L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The International Journal on Mathematics Education*, 40 (2), 317-327.

# EXPLORING FRAGMENTATION IN MATHEMATICS EDUCATION RESEARCH

Ayshea Craig

Institute of Education, University of London

*This paper reports on a study which uses techniques from social network analysis to explore patterns of research collaboration within the mathematics education research field, focusing on English language publications. The results suggest that the perceived theoretical or methodological fragmentation in mathematics education does not translate in a straightforward way into a fragmentation between researchers. This result is used to argue for a method for connecting research practices complementary to some of those discussed in previous papers in the CERME working group. Namely, that existing collaborations could be used to promote increased theoretical or methodological coherence across the research field when combined with the strategies already developed by the group.*

## INTRODUCTION

One of the reasons often given for addressing the multiplicity of theoretical perspectives and approaches in mathematics education and for attempting to connect them has been to reduce the perceived fragmentation and diversity in the field (Prediger, Bikner-Ahsbals, & Arzarello, 2008; Prediger, Bosch, Kidron, Monaghan, & Sensevy, 2009). The field has been criticised (from within and without) for lack of coherence, for failing to build on previous results and for susceptibility to fashion and cyclically reinventing old ideas (see for example references in English, 2002; Maasz & Schloeglmann, 2008; Sierpiska, 2003). The fragmentation and diversity is not just a result of lack of agreement about the theoretical framework we should work in, or even about whether we should work in one, or what role it ought to play. It can also be understood as a result of the social organisation of research and the conditions under which it is carried out, in particular, its openness as a field to political and other external influences, the different national contexts (intellectual, social, political) in which it is carried out, and the diverse educational and training backgrounds of those carrying out research.

Any programme for managing the fragmentation and diversity of the field needs to draw on an understanding of the state of the field and the mechanisms which have led to and perpetuate the current fragmented state. Based on such an exploration we can recognise and develop possible mechanisms for change and ask questions about the social conditions necessary for change. Additionally, the social conditions of research place constraints on the goals which are possible in addressing the theoretical and methodological diversity of maths education research. An awareness of this adds an important dimension to discussions of what mathematics education research as a field should aim to be, or how it ought to develop.

Discussions in the papers of the CERME working group on theoretical diversity in the European mathematics education research community and ZDM special issue 40(2) have made a move from the abstract discussion of principles etc. to more concrete attempts to connect theories and this move has been accompanied by questioning of the different ways in which theory can be defined, and (to a lesser degree) the nature of theory and theory/practice relationships. In particular, the idea of theory has been developed and expanded by a number of authors beyond assumptions, principles and concepts, to include methodologies, research questions and even 'ways of seeing'. This is a change of focus from theories themselves to theories-in-use in research and through that to research practice more generally (see, for example, the final remarks in Prediger et al., 2009), where this practice can be seen as a theoretically guided activity. This move from connecting theories to connecting research, or research practices, or even to connecting researchers and the accompanying changing goals, suggests that there may be some value in revisiting the initial assumptions about the theoretical fragmentation and diversity of approaches in mathematics education and to ask what these mean in terms of research practices and researchers. Although we can sketch an a priori argument for the possible value of revisiting these assumptions, and that of exploration of the social conditions of research in mathematics education more generally, any particular implications, for understanding and managing diversity, must be demonstrated empirically.

In the study reported here I use techniques from social network analysis in order to examine evidence of fragmentation or diversity through existing collaborative research links between mathematics education researchers. We might expect that the theoretical and methodological divides would be seen in patterns of collaboration, since we would expect individuals using the same theories or working within the same perspectives to be more likely to collaborate with one another than with those working with different theories. This would then be evident in collaboration patterns with more densely connected groups loosely connected to other groups.

In looking at collaborations using social network analysis we create only one of many possible views of relations in mathematics education research, but an important one. Collaboration which results in a research publication implies a strong connection and investment of time. We might assume that even outside of the connecting theories endeavour, research collaborations are places where ideas meet, are elaborated and negotiated, both in the research process and in order to create a joint production which all parties are happy to put their name to. As such, evidence of connectedness across the field at the level of research collaborations would raise questions about assumptions of fragmentation of theoretical frameworks and research approaches, leading us to question the nature of this fragmentation.

This research is part of a larger project looking at the mathematics education research community in England and so focuses on collaborative links both within the

group of researchers based in England and between these researchers and the broader mathematics education community. My analysis suggests that these mathematics education *researchers* as a group are less fragmented in terms of their research collaborations than perceptions of theoretical and methodological fragmentation in the field, and in particular of mathematics research in England, might suggest.

## METHODOLOGY

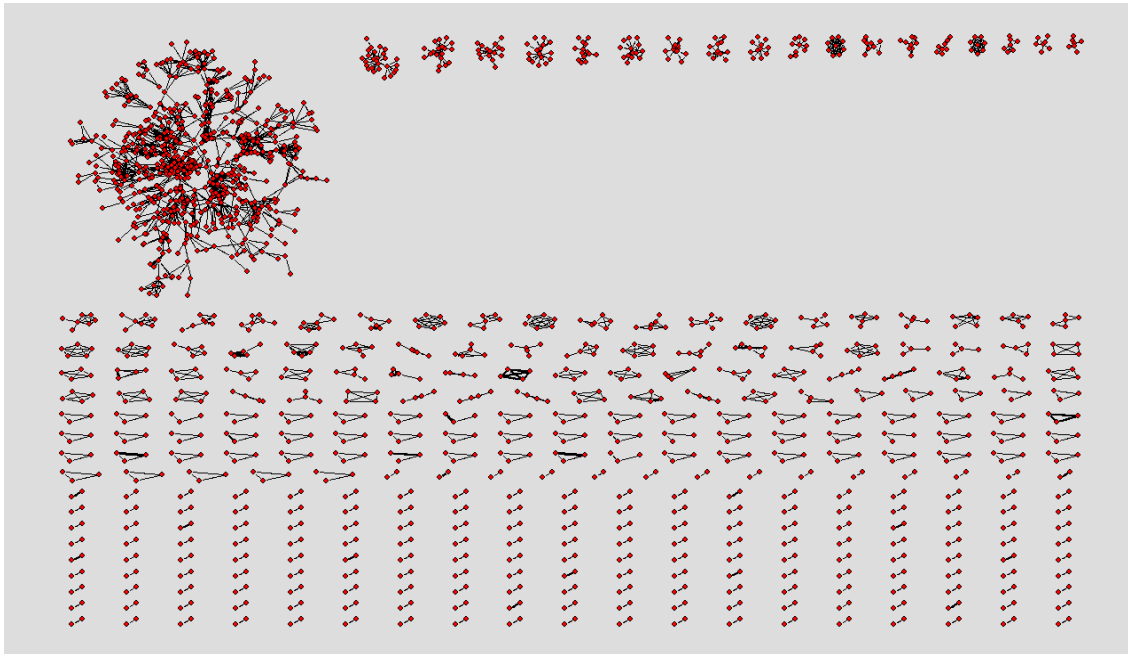
The study reported here using techniques from social network analysis to explore patterns of collaboration within the mathematics education research community using data collected from a large number of co-authored research papers.

The data used consisted of all research papers published in nine, largely English-language, international mathematics education journals between 2000 and 2009. Short editorials, book reviews, and announcements were excluded. Five journals were more general: *For the Learning of Mathematics*, *Educational Studies in Mathematics*, *Research in Mathematics Education*, the *Journal for Research in Mathematics Education* and *ZDM-International Journal on Mathematics Education*, along with four more specialised journals: The *International Journal for Technology in Mathematics Education*, the *International Journal of Computers for Mathematical Learning*, *Teaching Mathematics and its Applications* and the *Journal of Mathematics Teacher Education*. Journals in which English authors more commonly publish were preferentially selected, as the broader study, of which the reported research is a part, focuses on England. The final data set included 2264 papers in total, of which 1098 had more than one author.

In social network analysis (Hanneman & Riddle, 2005; Scott, 2000) individuals and the relations between them are modelled as the nodes and edges of graphs or networks. Tools from graph theory, along with some developed for use in the social sciences, are then employed to analyse and visualise these networks in order to explore or answer particular questions about their structure. In this study, researchers were represented by nodes and two researchers were linked by an edge if they had co-authored at least one paper in the data-set. Exploratory analysis of the resulting network focused on the number and sizes of connected sub-graphs (or components), how centralised the network was, whether there were relatively isolated sections, and how dense the connections were. Some of these measures are based on the distances between nodes within connected components of the network: the *degree of separation* of a node is its average distance from all other (connected) nodes and the *average degree of separation* for the component is calculated by taking the average of this value across all the nodes in the component. This type of measure can be used to make comparisons with other networks. *Cut-points* are nodes or edges which would disconnect sections of the network if they were removed. These can be used to explore how robust the features of the structure are, and as a measure of ‘connectedness’ by asking how hard it is to disconnect groups of nodes.

## RESULTS & DISCUSSION

The list of 2264 research papers was used to generate a list of authors publishing in the journals and to identify who they had written papers with. Overall 2199 unique authors were identified, 1737 of whom had written a multiple authored paper in at least one of these journals within the ten year period specified. A further 462 authors only produced individually authored papers, i.e. they had not collaborated on any papers in the dataset. Most of the authors (66%) published only one paper within these nine journals, with 17% publishing two papers, 12% between three and five papers and only 3% of the authors publishing more than five.



**Figure 1: Visualisation of the collaboration network of links between authors, generated using the freeware package Pajek (<http://pajek.imfm.si/doku.php>)**

Figure 1 shows the visualised network of collaborations produced from this data. In the figure, researchers are represented by nodes and two researchers are joined by an edge if they have published a joint paper (authors who did not collaborate at all are excluded to save space but could be represented as 462 additional isolated points). The most striking feature of the network is that it has a single ‘giant’ connected component, seen here at the top left, containing 28% (or 612) of the authors. The remaining authors are all connected through collaboration to smaller groups of at most 27 authors. This pattern of one giant component and a number of much smaller connected components is characteristic of scientific collaboration networks (Newman, 2001, 2003) and is a pattern which proved to be robust to the addition and removal of particular journals (although the percentage coverage of the giant component varies).

I will use two particular results from my early descriptive analysis to address the question of the fragmentation of the field at the level of researchers. The first result



is that theoretical divides or other fragmenting factors discussed in the literature aren't seen at the level of patterns of collaboration. Of course within small research groups this is the case, with many joint papers turning into a dense cluster in the network. However, looking for larger groups we see little evidence of clustering.

Analysis of the structure of the giant component to identify subgroups of more densely connected researchers reveals a 'small worlds' structure (Newman, 2001). This is as expected; small world structures arise naturally in many diverse systems. The average degree of separation of two nodes within the giant component is 7.6 i.e. researchers are connected to one another by on average a chain of only six or seven intermediate collaborators, with a maximum distance of 18. Overall the giant component is characterised by small distances between researchers and few (significant) cut points meaning that the structure is quite robust to the removal of ties or individuals.

This can be contrasted with patterns of collaboration in the hard sciences which are traditionally considered to be much less fragmented and more theoretically and methodologically coherent than mathematics education research. The measures of distance within the giant component are comparable (although his study used a much larger data set) with measures from Mark Newman's study of collaboration patterns in the hard sciences (Newman, 2001). In fact, Newman's study showed average distances of between 4 and 7 in six databases drawing on different areas of research and of 9.7 in a computer science database. In other words, there is a subset of mathematics education researchers with mutual patterns of collaboration which are not obviously more dispersed than those found among scientists in the hard sciences, particularly when compared to the sciences with lower average numbers of authors per paper. Unfortunately there are few similar studies of other social sciences with which to compare.

The second result from the network analysis relates to the proportion of the authors found within the giant component, in other words the proportion of authors connected to a significant number of other mathematics education researchers by collaborative links. Only 28% of all the authors publishing in the nine journals over the ten year period were connected in this way (or 35% of those who had published collaboratively). This is a low proportion compared with that found in studies of the hard sciences, which was over 50% in all subject areas examined and closer to 80 or 90% in many (Newman, 2001). The low proportion in the data is relatively robust to the removal, interchange or addition of journals (any 8 of the 9 producing a giant component covering 20-30% of the authors). This suggests that the low proportion is not just an artefact of the particular publications selected, nor a result of the relatively small number of publications being considered. Of the 1587 authors outside the giant component it is useful to separate them into two groups: around 70% of these have collaborated, but their collaboration still leaves them relatively isolated. Most are connected through collaboration (meaning that we consider their

collaborators and their collaborators' collaborators and so on until there are no more connections to exploit) to only a small number of other researchers (on average to 4.4 others outside the giant component). The other 30% of authors outside the giant component have not collaborated on any papers within the data set. This significant proportion of *authors* who have not collaborated in the data set (21% of the total authors), along with the large number of *papers* in the original data with only one author (52%), is another point where the results differ sharply from patterns of collaboration familiar from the sciences. Even within subjects like mathematics, where working alone has traditionally been seen as the norm, there has been a trend of increased collaboration (Burton 1999 cited in Burton & Morgan, 2000). It would make an interesting further study to look at whether trends in mathematics education tend towards more or less collaborative research.

Interpreting this result requires more information and further research. We might ask what it means in practice to be inside or outside of the giant component in this maths education collaboration network. A researcher may tend to work alone but be actively involved with the community in other ways, or they may work alone as a result of geographic or institutional isolation; they may largely collaborate outside mathematics education, straddling several fields, or they may have published only one or two papers as part of a doctorate before leaving academia; they may publish occasionally but see the bulk of their work as lying outside research, in teaching, policy-making or administration for example. Clearly there are many different research profiles compatible with a position in the relatively unconnected sections of the network diagram, and so any single account will fail to capture this diversity of experience. Within the UK, education academics work in widely differing institutional contexts, with different patterns of research funding and different balances between their research and teaching functions (Lawn & Furlong, 2007; Oancea, 2005); additionally the career backgrounds of academics differ with many second-career researchers with varied experiences of research training (Mills et al., 2006). It may be that the giant component can be interpreted as representing a core of research-focused academics who focus on mathematics education and that the fragmentation of mathematics education can be partly understood through the relatively small size of this core with respect to the overall number of people publishing in the area.

## **FOCUS ON COLLABORATION IN CONNECTING THEORIES RESEARCH**

Debate about the strengths and weaknesses and mutual compatibility of particular theories has a long history (Bikner-Ahsbahr & Prediger, 2006; Cobb, 2007; Sierpiska & Kilpatrick, 1998; Sriraman & English, 2005) and more recently this tradition of abstract discussion has been joined by moves (within the working group on theory at CERME 4, 5 and 6 and associated special issues of ZDM) to take a more practical approach to the problem by exploring strategies to connect theories in the context of empirical research.

One consequence of the move from debate about theories to practical strategies to connect theories is that it seems to have necessitated a different sort of engagement with the idea of theory. Discussions of the different ways in which theory can be defined for the purposes of connecting theories (Bikner-Ahsbahr & Prediger, 2006; Radford, 2008) or thought about in relation to empirical research (Cerulli, Trgalova, Maracci, Psycharis, & Georget, 2008; Prediger, 2008) can be seen as grappling with the difficulty in separating theory from theory-in-use. The use of ‘static’ definitions of theory has been challenged, introducing the idea that theories are tied up in the work of those who use and write about them and hence that a more fluid or ‘dynamic’ idea of theory is needed for a discussion of connecting theories through research (Prediger et al., 2008). In particular a possible change of emphasis has been suggested from networking theories to ‘the *networking of research practices*’ (Prediger et al., 2009, p. 1534, emphasis in original).

These changes of focus have brought with them the need to reconsider what might be the possible aims or goals for explorations of diversity and attempts to connect theories (ReMath first deliverable quoted in Artigue et al., 2009; Prediger et al., 2009). Across the papers of the CERME working group and the ReMath project there are discussions of important outcomes in terms of the ways researchers see their own work differently as a result of working with others who use different theoretical ideas (e.g. Trgalova, 2008). These experiences are described as a valuable part of the work by many, yet it seems clear from the discussion that it is an individually experienced result of the actual process of engaging with others in theory-focused research collaboration, rather than an insight which can be shared in a straightforward way with others or which might be seen as a tangible result of the research. Given the experiences reported, a valuable additional way of looking at the research of the CERME working group on theoretical and methodological diversity, and of thinking about the ways in which it could contribute to reducing the fragmentation of the field, might be to think of it as an exploration of the potential of collaboration, where that collaboration focuses on connecting theories and approaches to research, to act as a mechanism for change in dealing with fragmentation in mathematics education research. This move is consistent with the change of focus from connecting theories to connecting research practices.

## CONCLUSION

I argue that the significance for mathematics education of the relative ‘closeness’ of researchers working with very different approaches and from different theoretical perspectives within the field of English-language publications is in the potential it suggests for using these collaborations to develop a greater sense of coherence in the field. Strategies developed from the connecting theories literature could play a role in developing the potential of existing collaborations. To explore this potential further suggests the need for work on the nature of existing collaborations and the extent to which researchers bring different theoretical perspectives into focus and debate when

collaborating, and whether this process could come to be seen as part of what is worth reporting about the research process. In doing this it would be important to recognise different ways of collaborating and working. The pattern of working illustrated in the data here suggests that in order to use collaboration as a mechanism to increase the coherence of mathematics education what may be needed is not necessarily more collaborations between researchers working within different theoretical traditions or with different approaches, but the exploitation of existing collaboration and the creation of new collaborative links to researchers without existing links.

The exploration of collaboration patterns in mathematics education research reported above raises some interesting questions about the assumption of fragmentation in the field based on theoretical or methodological divisions. Much fragmentation undoubtedly exists at the level of researchers collaborating, with high proportions of papers single-authored and a large proportion of the researchers publishing in mathematics education relatively isolated within the field in terms of collaborations. However there exists a smaller core of researchers who, despite differences of approach and theoretical perspective, remain quite closely connected through a relatively robust network of collaborative links.

The original study reported here was not designed to explore collaborative research connection across the whole mathematics education research community but instead to explore those of researchers working in England, and consequently it focused on research published in English. The result is that while we can draw conclusions about patterns of collaboration within the English-language literature, researchers who do not publish exclusively in English are systematically excluded or misrepresented, and so we need to take care in drawing conclusions about the whole field of mathematics education research. We might ask what degree of clustering a broader dataset would reveal, in particular around language groups. Education research generally differs from the sciences studied by Newman, in that English is not the only international publishing language although it remains dominant. Despite this limitation I would argue that the lack of evidence of strong disconnection among researchers' collaborations warrants further exploration.

An issue in using network analysis and visualisations of collaborations in this research has been the ease with which one can move between the language of network analysis and the use of overlapping language employed generally within mathematics education research to discuss the state of the field in terms of methodological and theoretical diversity, and more specifically to the language of connecting theories and practices used by some in the CERME working group. The ideas of *fragmentation* and of *connections* or *connectivity* are particularly problematic here and care must be taken to trace differences of meaning between these terms as they are used in different contexts. Also the application of these terms in turn to theory, research, research practices, approaches and researchers can

unintentionally blur arguments and ideas which are specific to one into the others. A question raised here is what it might mean, and whether it is reasonable, to talk about the community of researchers in mathematics education as fragmented, and how this fragmentation might relate to theoretical or methodological fragmentation.

I see the use of network analysis techniques to explore collaboration patterns as just one of many possibilities for exploring the social context of research with a view to inform discussions about possibilities for reducing the fragmentation evident in mathematics education research. This could be an important complement to concrete attempts to connect theories and more abstract debates about the role of theory in research, the range of theories and the different approaches to research found in maths education and their implications for the knowledge production of the field. The social as well as epistemological, ontological or conceptual causes of theoretical fragmentation mean that even if we were to satisfy ourselves about the connectivity or otherwise of theories in mathematics education research, we would still find that fragmentation and diversity remained. The exploration of theories as theories (on an abstract level and through concrete attempts to connect theories and research), and the exploration of theoretical diversity as it has arisen from the social conditions of research, are both necessary in order to address the broader fragmentation of the field.

## REFERENCES

- Artigue, M., Alexopoulou, E., Alshwaikh, J., Cazes, C., Chaachoua, H., Chiappini, G., et al. (2009). ReMath Project Deliverable 18: Integrated Theoretical Framework Version C.
- Bikner-Ahsbahr, A., & Prediger, S. (2006). Diversity of Theories in Mathematics Education - How can we deal with it? *ZDM-International Journal on Mathematics Education*, 38(1), 52-57.
- Burton, L., & Morgan, C. (2000). Mathematicians Writing. *Journal for Research in Mathematics Education*, 31(4), 429-453.
- Cerulli, M., Trgalova, J., Maracci, M., Psycharis, G., & Georget, J.-P. (2008). Comparing Theoretical Frameworks Enacted in Experimental Research: TELMA Experience. *ZDM-International Journal on Mathematics Education*, 40(2), 201-213.
- Cobb, P. (2007). Putting Philosophy to Work: Coping with Multiple Theoretical Perspectives. In F. J. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 1, pp. 3-38). Charlotte, NC: Information Age Publishing.
- English, L. D. (2002). Priority Themes and Issues in International Research In L. D. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 3-16). Oxford: Routledge.
- Hanneman, R. A., & Riddle, M. (2005). *Introduction to Social Network Methods*. Retrieved 24th March 2010, from <http://faculty.ucr.edu/~hanneman/nettext/>

## Working Group 16

- Lawn, M., & Furlong, J. (2007). The Social Organisation of Education Research in England. *European Educational Research Journal*, 6(1), 55-70.
- Maasz, J., & Schloeglmann, W. (2008). *New Mathematics Education Research and Practice*. Rotterdam: Sense Publishers.
- Mills, D., Jepson, A., Coxon, T., Easterby-Smith, M., Hawkins, P., & Spencer, J. (2006). *Demographic Review of the UK Social Sciences*. Swindon: ESRC.
- Newman, M. E. J. (2001). The Structure of Scientific Collaboration Networks. *Proceedings of the National Academy of Sciences of the United States of America*, 98(2), 404-409.
- Newman, M. E. J. (2003). The Structure and Function of Complex Networks. *SIAM Review*, 45, 167-256.
- Oancea, A. (2005). Assessing the Quality of Applied and Practice-based Education Research. ESRC TLRP seminar series 'Quality in educational research'.
- Prediger, S. (2008). How are theoretical approaches expressed in research practices? A report on an experience in comparing theoretical approaches with respect to the construction of research problems. *ZDM-International Journal on Mathematics Education*, 40(2), 277-286.
- Prediger, S., Bikner-Ahsbahr, A., & Arzarello, F. (2008). Networking Strategies and Methods for Connecting Theoretical Approaches: First Steps towards a Conceptual Framework. *ZDM-International Journal on Mathematics Education*, 40(2), 165-178.
- Prediger, S., Bosch, M., Kidron, I., Monaghan, J., & Sensevy, G. (2009). *Introduction: Different Theoretical Perspectives and Approaches in Mathematics Education Research - Strategies and Difficulties when Connecting Theories*. Paper presented at the Conference of European Research in Mathematics Education (CERME 6), Lyon, France.
- Radford, L. (2008). Connecting Theories in Mathematics Education: Challenges and Possibilities. *ZDM-International Journal on Mathematics Education*, 40(2), 317-327.
- Scott, J. (2000). *Social Network Analysis: A Handbook* (2nd ed.). London: Sage.
- Sierpinska, A. (2003). *Research in Mathematics Education: Through a Keyhole*. Paper presented at the CMESG/GCEDM, Acadia, Canada.
- Sierpinska, A., & Kilpatrick, J. (1998). *Mathematics Education as a Research Domain: A Search for Identity*. Dordrecht: Kluwer.
- Sriraman, B., & English, L. D. (2005). Theories of Mathematics Education: A Global Survey of Theoretical Frameworks/Trends in Mathematics Education Research. *ZDM-International Journal on Mathematics Education*, 37(6), 450-456.
- Trgalova, J. (2008). *Integrative Theoretical Framework to address issues of ICT in Mathematics Education: Telma and Remath Projects*. Paper presented at the International Congress on Mathematics Education 11, Monterrey, Mexico.

# COMPLEMENTING AND INTEGRATING THEORETICAL TOOLS: A CASE STUDY CONCERNING POOR LEARNERS

Nadia Douek

IUFM C. Freinet Université de Nice, UMR ADEF Université de Provence

*This paper contributes to the debate about comparing and using different theoretical constructs in Mathematics Education through a case study concerning two poor learners. Complementing and integrating theoretical constructs of different sources resulted in an operational tool to benefit from the resource of a specialised teacher and allow poor learners' to be productive participants to whole class activity.*

Key words: theoretical tools; poor learners; problem solving; mathematical concepts

## INTRODUCTION

In the last two decades of the 20th century the discussion about the nature of Mathematics Education as a scientific domain and its relationships with other disciplines (psychology, epistemology, sociology of education, etc.) developed together with an increase of knowledge and theories in the field. Different positions emerged. According to H. G. Steiner (Steiner, 1985) and others in more recent years (Arzarello & Carolina Bussi, 1998), the development of the field requires theoretical constructs borrowed from different disciplines; and its autonomy as a field of research consists in the appropriate choice and adaptation of those constructs and in the construction of new theoretical tools needed to deal with specific teaching and learning problems. Without denying the importance of borrowed tools, researchers like G. Brousseau and Y. Chevallard engaged in constructing autonomous, specific theories for Mathematics Education. The volume “Mathematics Education as a Research Domain: A Search for Identity” edited by J. Kilpatrick and A. Sierpiska and published in 1998 by Kluwer represents the debate at that moment. Recently, interest shifted towards how to exploit and deal with the numerous theories and tools elaborated within mathematics education or adapted from other disciplines. L. Radford's (2008) construct of semiosphere frames theory connections at the levels of their system of principles, their methodologies and their template of research questions.

I will use the semiosphere framing to underline the connections between theoretical constructs that I used to design specific tools to analyse the learning difficulties of “poor learners” and to organise the teaching sequences in order to help them contributing to the whole class activity, thanks to the mediation of a specialised teacher. So, this paper aims at contributing to the present debate by providing an example (the object of our case study) in which theoretical constructs are adapted and implemented by *complementing* each other or by *integrating* them according to specific needs, and by discussing the level of their connections. By “complementing” we mean using different constructs to deal with different aspects

of a given problem, by “integrating” we mean grafting elements of a construct on another, to elaborate a more powerful and comprehensive analytical tool to deal with a problem.

## **POOR LEARNERS: A CHALLENGE FOR MATHEMATICS EDUCATION**

Following literature, poor learners (Shuell, 1983) are those with no neurological or physical impairment, but considered as having difficulties with learning. They often have difficulties in being inserted in the class community, suffer from unfavourable environmental or family conditions and represent a heavy load for the class teacher. In France poor learners of a classroom are supported by an auxiliary “specialised teacher”, who usually tries to achieve learning goals through small group extra-class work concerning easy fundamental notions and techniques of the curriculum, but seldom allowing students' to productively participate in ordinary classroom work. The result is often an exclusion both from practice of high-level mental processes (like arguing, or producing, contrasting and validating hypotheses), and from culture in a long-term perspective (especially when family cannot provide general enculturation). Also, social insertion in the classroom community may be problematic.

In spite of different orientations (see Dunn, 2004; Perrin-Glorian, 1993), current literature on poor learners in mathematics agrees on some means favouring progress in learning mathematics and inclusion like : engaging them in small group activities supported by the teacher and concerning ordinary classroom tasks; teaching mathematics in context; and promoting awareness of the ways of solving problems. Further work is needed concerning planning, monitoring and adjusting mathematics teaching to their needs when a specialised teacher works with them.

The case study presented in this paper explores the potential of complementing and integrating different theoretical tools, chosen as appropriate to deal with the problem of teaching and learning mathematics to poor learners in primary school. They were used to plan, manage and analyse a sequence of activities performed by a specialised teacher and aimed at developing students' mathematical knowledge and favouring their productive participation in classroom activities.

## **CHOICE AND ADAPTATION OF THEORETICAL TOOLS**

Socio-cultural theories offer suitable general perspectives to tackle the problem of mathematical enculturation of poor learners: in particular, the Vygotskian elaboration about teachers' mediation in student's zone of proximal development (ZPD) engages identifying and exploiting their learning potential. The Vygotskian elaboration about “scientific concepts” (SC) and “everyday concepts” (EDC) dialectic offers a *complementary* tool to model the enculturation problem of a subject as a long term development within tensions between different socio-cultural webs he/she is inserted in. This model favours exploiting what poor learners (like all



students) bring to the classroom from their everyday experience as a resource to be developed in a dialectic relationship with the culture brought (and mediated) by the teacher. Vygotsky's description of SC offers useful criteria to describe some requirements typical for school culture (development of systemic links, conscious and voluntary use, etc...)

These contributions of Vygotsky have been *integrated* by the Genoa research team according to specific mathematics education needs (for all students in general and poor learners in particular) by developing two original interrelated constructs: “experience field” (Boero, 1989; Dapuetto & Parenti, 1999; Douek, 2003) and “experience fields didactics” (Boero & Douek, 2008). “Experience field” construct frames the identification of human culture areas, rich in opportunities for developing mathematical knowledge and skills (like money use in first grades), immediately or potentially accessible to all students. An *experience field* consists of three components (teacher's internal context, student's internal context and the external context - signs, objects, physical or social constraints) mobilised and evolving during classroom activities led by the teacher. Experience fields didactics consists in the long term organisation of teacher's activity exploiting resources of a given experience field to engage in a dialectic relationship between SC and students' EDC. A crucial element is the sequencing of cycles: given a problem contextualised in the experience field, students produce (individually, and/or with teacher's help) written solutions; a few solutions selected by the teacher are discussed under his/her guide and mediation; then a written synthesis is produced (individually and/or with teacher's help). Writing and discussing are also framed in a Vygotskian perspective: see Boero, Douek & Ferrari (2008); Bartolini Bussi (1996) and Boero & Douek (2008).

In the experience field didactics, Vygotsky's EDC-SC dialectics offers a cultural model for conceptualising and guiding the elaboration of the teacher's mediation within the student's ZPD. Further *specific* epistemological and cognitive analytical tools for the process of *mathematical* conceptualisation are provided by *integrating* Vergnaud's construct of concept (Vergnaud, 1990) within Vygotsky's SC-EDC dialectics (see Douek, 2003). Vergnaud defines a concept through its three components: its reference situations; its operational invariants (theorems and concepts in action); and its semiotic representations.

Our integration allows a finer analysis of these components: we consider that they reflect and depend on the subject's activity as inserted in a socio-cultural web and we can point to their possible different cultural roots. We study concepts from a plural point of view and follow the subjects' school activity as in tension within different socio-cultural webs. The resulting analysis tool allows identifying elements of conceptualisation in what students already know and are able to do, and in what the teacher brings through interaction. In more detail, it allows to analyse the evolution of: systemic links between reference situations and semiotic representations of a

concept or of different ones; operational invariants and schemes relying on them; conscious and voluntary mastery of concepts. Such analysis is applied to student's production (verbal or more general activity) and used to organise teaching sequences and particularly teacher/ students interaction.

### **THE ANALYSIS TOOL AT WORK: KEN AND MELVIN'S CASES**

Ken and Melvin were two “poor learners” in a third grade class of 22 students of a school in a “priority education zone” of France (a zone with social problems). Both the classroom teacher and the specialised teacher had taken part in an in-service teacher education activity; they volunteered to adopt the experience field didactics methods and connect the specialised teacher's action to the whole class activity, so that Ken and Melvin may take part in the class activity of solving the (relatively) difficult problems posed to the whole class, as protagonists, overcoming their usual marginal position. Adjustments of the planned teaching were made according to the analysis of students' behaviours, using our tool resulting from integration.

In the reported initial activity (a third of the whole activities planned with the teachers) the mathematical problem was contextualised in gardening. The class regularly practiced it, we can consider it a meaningful experience field. Students had some knowledge about it and about related practical problems, and the two boys were well concerned with it. They were offered interaction with the specialised teacher for a longer period of time than was offered to the whole class. Then they participated in the subsequent collective activities. For all the students, working pace was much slower than usual; contextualised complex problem solving was new, and so was the equilibrium between oral and written verbal activities (they were not accustomed to write down in words their problem solving reasoning).

The class teacher, though interested in the proposed activities, in this phase missed several crucial indications proper to the experience fields didactics. He preferred a traditional exposure of the problem with its stereotyped linguistic form and all the necessary data (whereas we wished that the problem was posed in a familiar way and that students gather data from their everyday practice within the familiar context). He chose light interactions with students. He felt uneasy with ZPD-related practices and feared excessive influence on student's activity. On the contrary the proposition was close enough to the specialised teacher practices and she easily adapted her methods. However the classroom teacher adopted the proposed methodology in a more and more coherent way during the subsequent activities (not reported in this paper).

#### **The teaching experiment**

During the first session the problem was presented to the whole class. Then the specialised teacher took Melvin and Ken to another room to solve it, while the rest of the students worked individually in the main classroom. Two additional sessions with the specialised teacher allowed Melvin and Ken to conclude their work. Then a

second whole class session was dedicated to discuss the solutions and a third to individual synthesis (the boys were attended by the specialised teacher in the main classroom), according to the cyclic structure of the experience fields didactics.

The collected data were: all the students' individual productions; the specialised teacher field notes about her interactions with the students (including utterances, gestures) and remarks about their behaviour (like impatience, enthusiasm, difficulties of expression), and elements of on the spot interpretation that helped us to build the consecutive sessions; videotapes of the collective debates. The methodology used to analyse data and the evolution of students' skills is the interpretation of their oral and written productions and gestures through the cognitive and epistemological components of our integrated theoretical framework concerning conceptualisation: the evolution of Vergnaud's concept components according to a SC-EDC dialectic.

### **First whole class problem solving session**

The problem was presented to the whole class through the following text:

We need to know the dimensions of the parcels of the garden in which we will be planting flowers. We have a 2m60 by 2m80 rectangular parcel in which we want to delimitate 4 identical parcels with two paths crossing between them.

He presented a rectangular schema with the four parcels, and specified that the paths had to be 40 cm wide. He encouraged interactions and noted on the blackboard:

Ingi:  $130+130=260$ . We make the middle to trace the pathway, 130 and 130 to have identical parcels

Antoine: there are two pathways to trace

Emir: yes 130 and 130 and the pathway ?

Victor:  $130+130+40=300$  impossible... we have 260

The problem's difficulty was implicitly visible by the students' remarks in the discussion: the pathways width must be considered in the calculations. After the discussion, the classroom teacher asks the students to solve the problem individually, and to write explanations and justifications. Explanation and justification tasks are essential in experience field didactics and are generally supported by the teacher's mediation, but they are unusual in this class' didactical contract. Insufficient mediation made the task difficult. Meanwhile, Melvin and Ken followed the specialised teacher into another classroom.

### **First interaction session with Ken and Melvin:**

The specialised teacher encouraged the students to recall the task and the data and to try to find a way to solve the problem. She used the schema, accompanied her verbal interaction with gesture and tried to stimulate the students' ideas and verbal expressions. At the beginning they just repeated the statement of the task.

## Working Group 16

Interactions revealed important aspects of the students' conceptualisation, and allowed the teacher to adapt her activity to their ZPD. For instance:

Teacher: the pathway must be right in the middle of the parcel. What to do to place it at the right place? (gestures close to the schema)

M: we could try to find where is the middle of this side (showing the longer side) .....

Teacher: what to do to find the middle?

K: we measure with the ruler

The boys searched middle points by trials, Ken measured to verify and got upset: lengths are not equal. The teacher noticed that they don't use numerical data to find the length of half a side. So, she suggested a numerical procedure reciprocal to their's, based on their gestures and verification efforts.

Analysis: Ken related “middle” and “measurement with a ruler”. The students' systemic links between the concepts of “middle” and of “measure”, with procedures were weak and not used intentionally with a precise goal. A theorem in action (Vergnaud, 1990) seems to guide Ken's action of placing middles and verifying them by measuring. This theorem could be expressed as: *the measures of the segments at each side of the mid-point must be equal*. Ken also used semiotic representations of the middle of a segment. Links between spatial midpoint and numerical half were missing.

Using our integrated analysis tool, we estimated Ken's conceptualisation related to this situation (with these theorem in action and semiotic representation) as his EDC concerning middle point, half and measure and we conceived their potential development in relation to the teacher's references and goals. This guided the design of the next sessions according to the aim that students develop a more scientific conceptualisation: Stabilise semiotic representations (geometric representation, gesture, verbal expression, numerical symbols and schematisation) concerning length, measure, middle, half (they will need them to act upon this situation); direct students to relate explicitly and functionally geometric middle point with measure and numerical half, to favour systemic links typical for a “scientific” conceptualisation of these notions and to approach a numerical procedure; make Ken's theorem in action explicit, linked with useful semiotic representations and with the procedures to be built, and using it for verification; engage in the EDC/SC dialectics. We also had to help students expressing the purposes of the activity, in order to maintain control on meaning. The teacher's explicit use of elements reflecting student's actual conceptualisation engaged them in the activity and reassured them. They could see their activity as constructive and acknowledged.

### **Second interaction session with Ken and Melvin**

A third student, Grace, joined the group. The boys explained their work.

## Working Group 16

K: ... to find the middle of 280

Teacher: what does it mean the middle of 280?

M: for example, we draw a line of 30 and the half is 15 because 15 and 15 makes 30.

Analysis of evolution towards SC: Melvin became able to explain a numerical calculation to find the middle point of a segment. His explanation had a character of *generality*. The link between measuring procedure and numerical calculation was interiorised and the student could call it up *intentionally* for a purpose that was clear to him (generality and intentional mobilisation belong to the scientific level of conceptualisation). And a spatial reference situation for calculation became available.

### Third interaction session with Ken and Melvin

Using the schema and strips of paper representing the pathways, the students concluded the problem solving through interaction, and dictated a synthesis:

*Melvin proposed to find the middle of 280 to put the pathway right in the middle. We prepared a strip of paper.*

*It was difficult, so he proposed to split 28 into halves. Together with Ken he found that it was 14, because  $14+14$  makes 28.*

*Ken said we must add the 0 of 280. So we also added 0 to 14 and the half of 280 is 140, because  $140$  and  $140$  makes 280.*

*Grace showed on the backboard that the pathway had to be moved apart 20 and 20 because it makes 40.*

Analysis of evolution: This text reveals important progress in understanding the problem, the relation between half and mid-point, the problem solving steps, the calculation and the final purpose. The students' improved mastering semiotic tools that helped them dealing with the situation. Using the word "écarter" (move apart) they imagined how to pose the pathway on the ground (the French expression is not appropriate, but is meaningful: it was related to the gesture of joined hands put in the middle and parted to form the pathway).

The problem solving context allowed the students to preserve the goal and a coherent relation between the different working stages. Text production reinforced this. Note that these students had tremendous difficulties to mobilise their mathematical knowledge and often lost the goal of their activity. Written text production has a crucial role in experience field didactics: it dialectically favours and requires reasoning to be organised (see Vygotsky, 1985, Ch. VI), thus scientifically conceptualised. The students were not able to produce the written text on their own, but they had to take the responsibility of producing the text orally and organising it.

## **Second whole class session: mathematical discussion**

The students who stayed in class worked individually in a rather autonomous way. Our analysis of their production revealed that: The majority did not appropriate the schema drawn on the blackboard. Their expression was reduced to calculations; most students who elaborated a correct problem solving procedure were those who offered ideas during the preliminary discussion. Martine is the only student who produced a text explaining a procedure. It was poorly structured, short with a correct schema. Most written productions show limited evolution of ideas when compared with the blackboard's ones. Middle level students did not grasp the problem nor did they engage in the resolution. On the contrary, Melvin and Ken's ideas evolved greatly. We distributed an excerpt of Martine's text and one of the boys:

Martine: I make  $280 : 2 = 140$  to find the middle and I looked for half of the pathway. Half of 30 and  $30=20$   $140-20 = 120$

Melvin and Ken: Melvin proposed to look for the middle of 280 to put the pathway right in the middle. We prepared two strips of paper. Grace showed on the backboard that the pathway had to be moved apart 20 and 20 because it makes 40

We asked the students to compare their own ideas with this document, to compare the two texts, then to interpret Martine's calculation. Her mistake favoured interpretation efforts and allowed to identify the role of the data; it also favoured personal positioning. Melvin and Ken's word "move apart" helped interpreting her text and the reason why 20 was a half. Afterwards we asked Melvin and Ken to explain their text in order to involve them (they did it with some difficulties), then we asked Martine. The aim was to stabilise the boys' expression and understanding, and asserting their role in the collective construction.

Analysis: Discussing procedures and reconstituting reasoning relying on collective interpretations of the texts favoured the enrichment of reference situations and schemes for middle level students concerning parting procedures, and allowed conscious mobilisation of systemic links. Students were encouraged to refer to a well known reality. Acknowledgement of the two texts' contributions and debate were essential ingredients to realise the Vygotskian EDC-SC dialectics and, on the other side, the productive participation of Melvin and Ken.

Collective resolution ended with the first pathway situated and drawn on the schema.

## **Third whole class session: individual production of synthesis**

We asked the students to solve again the problem, individually, drawing the schema on a squared paper and giving explanations. Most students (13) solved the problem and drew the schema correctly. The boys worked individually like their classmates, they drew the pathways and solved the problem only for the first pathway (like it was done collectively). Producing explanations was difficult for all.

## DISCUSSION

Complementing and integrating suitable theoretical tools borrowed from other disciplines (Vygotsky's elaboration about ZPD and SC-EDC dialectics, Vergnaud's definition of concept) together with mathematics education tools (experience field and experience field didactics constructs) resulted in a toolkit that enabled us to plan, manage and analyse activities aimed at developing the two poor learners' mathematical competencies in relation to a complex third grade problem situation and to allow them to take part as protagonists in the subsequent discussion.

Our integrated theoretical tool allowed to frame the specialised teacher practice, and probably to improve her decision making in relation to the students' activity. She was already well aware of the importance of combining different semiotic means, and of referring to student's surrounding reality. She was able to calm their anxiety about failure; the familiar context of gardening experience field was a favourable condition for that. She offered verbal expressions to meet students' utterances and gestures, and legitimate them. Their legitimacy depends on our capacity for analysis: we “saw” in these concept components signs of an EDC related to the aimed conceptualisation. The teacher arranged them and inserted them into a more scientific discourse; she recycled the students' explanations into more complete ones, favouring systemic links between the SC in construction and their own conceptualisation. Local syntheses insisting on semiotic means helped stabilising the new developments on different levels, and keeping control on activity. The specialised teacher engaged in activities and decision making coherent with the theoretical perspective proposed by the researcher; while the classroom teacher only accepted the idea of working on a common task with her. This particular situation allowed comparing the development of problem solving and related conceptualisation processes in different educational settings, showing the potential of intentional, coherent mediation of conceptual elements and levels in students' ZPD on the basis of our theoretical frame of conceptualisation. Getting back to Radford's semiosphere: Experience fields didactics is based on Vygotskian principles; Vergnaud's definition is integrated into the SC/EDC dialectics on the method level, it does not contradict the Vygotskian principles. The question (including poor learners to class complex problem solving) emanates from the observation of a “phenomenon” (see Artigues & Bosch, 2011) through the lenses of experience fields didactics: ordinary class mathematical situations seldom rely on culturally meaningful context. In such cases poor learners are forbidden to do collective meaning making: minimal common meaning due to common cultural background is not available. Whereas experience fields perspective is to engage students in mathematical activity within a culturally meaningful context, and to insure a collectively shared sensitivity to the context and constructive activity. This phenomenon and this question modified the integrated theoretical frame of conceptualisation: it favoured the interpretation and instrumentalisation of the EDC/SC dialectics as a tension between socio-cultural webs of knowledge (see

Douek, 2011) that Vergnaud's component allowed to capture at various levels of evolution.

## REFERENCES

- Artigue, M. Bosch, M. & J. Gascón, J. (2011). Research, praxeologies and networking theories. *Proceedings of CERME7*. Rzeszow.
- Arzarello, F. & Bartolini Bussi, M. (1998). Italian trends in research in mathematics education: A national case study in the international perspective. In J. Kilpatrick & A. Sierpiska (eds.). *Mathematics education as a research domain: A Search for Identity*, vol. 2 (pp. 243-262). Dordrecht: Kluwer A. P.
- Bartolini Bussi, M. (1996). Mathematical discussion and perspective drawing in primary school. *Educational Studies in Mathematics*, 31, 11-41.
- Boero, P. (1989). Mathematics for all: Experiences and problems. In G. Vergnaud, J. Rogalski & M. Artigue (eds.), *Proceedings of the PME-XIII*, vol. 1 (pp. 62-76). Paris: Laboratoire de Psychologie du Développement et de l'Éducation de l'Enfant.
- Boero, P. & Douek, N. (2008). La didactique des domaines d'expérience. *Carrefours de l'éducation*, 26, 99-114.
- Boero, P., Douek, N., & Ferrari, P. L. (2008). Developing mastery of natural language: Approach to theoretical aspects of mathematics. In L. English (ed.), *Handbook of International Research in Mathematics Education* (pp. 262-295). New York: Routledge.
- Dapueto, C. & Parenti, L. (1999). Contributions and obstacles of contexts in the development of mathematical knowledge. *Educational Studies in Math.*, 39, 1-21.
- Douek, N. (2003). *Les rapports entre l'argumentation et la conceptualisation dans les domaines d'expérience*. Thèse. Paris: Université R. Descartes, Paris5.
- Douek, N. (2011). La costruzione argomentativa dei concetti della matematica nei campi di esperienza: inquadramento teorico. *Seminario italiano di ricerca in didattica della didattica*. Rimini.
- Dunn, T. K. (2004). Enhancing mathematics teaching for at-risk Students: Influences of a teaching experience in alternative high school. *Journal of Instructional Psychology*, 31, 316-320.
- Perrin-Glorian, M. J. (1993). Questions de didactique soulevées à partir de l'enseignement des mathématiques dans des classes faibles. *Recherches en didactique des mathématiques*, 13, 95-118.
- Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM Mathematics Education*, 40, 317-327
- Shuell, T. J. (1983). The effect of instructions to organize for good and poor learners. *Intelligence*, 7, 271-286.
- Steiner, H. G. (1985). Theory of Mathematics Education (TME): An Introduction, *For the Learning of Mathematics*, 5, 2, 11-17.
- Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques*, 10, 133-170..



# MATHEMATICAL OBJECTS THROUGH THE LENS OF THREE DIFFERENT THEORETICAL PERSPECTIVES

Vicenç Font\*, Uldarico Malaspina \*\*, Joaquin Giménez\* and Miguel R.  
Wilhelmi\*\*\*

\*Universitat de Barcelona, \*\* Pontificia Universidad Católica del Perú,

\*\*\*Universidad Pública de Navarra

*In this paper we establish a link between the onto-semiotic approach (OSA) to mathematics cognition and instruction, APOS theory and the cognitive science of mathematics (CSM) as regards their use of the concept 'mathematical object'. It is argued that the notion of object used in the OSA does not contradict that employed by APOS theory or the CSM, since what the latter two theories do is highlight partial aspects of the complex process through which, according to the OSA, mathematical objects emerge out from mathematical practices.*

## INTRODUCTION

One characteristic of the research community in mathematics education is its diversity of different theoretical perspectives, and hence there is a need for strategies that connect theories. Each theoretical perspective tends to privilege certain dimensions of reality over others. Thus, it is not always easy to identify links between research questions, descriptions, methodologies and conclusions that are elaborated within different paradigms. In this paper we establish a link between the OSA, APOS theory and the CSM as regards their use of the concept 'mathematical object'.

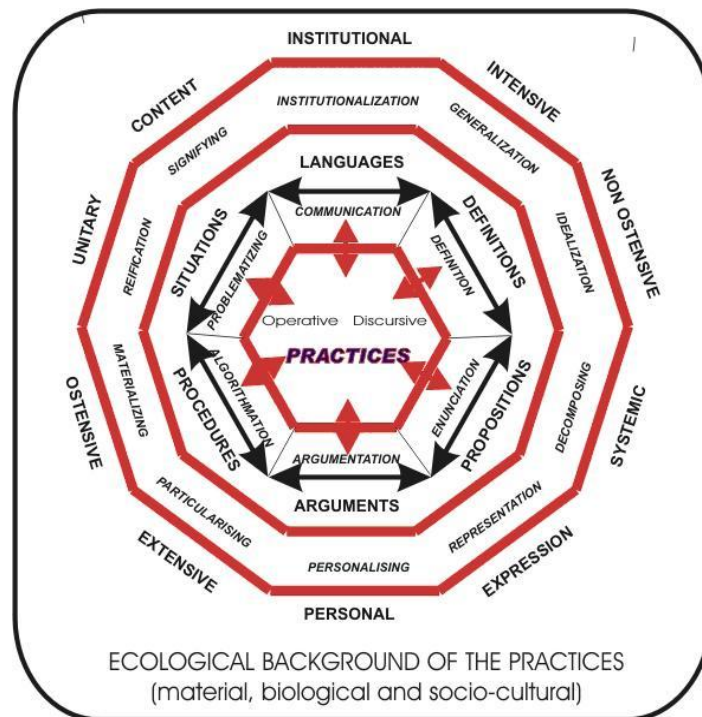
We begin by arguing that the notion of 'object' used in APOS theory is the result of a cognitive process referred to as 'encapsulation', which, in our opinion, is no different from the process of reification. Hence, this is a cognitive view of mathematical objects. We then argue that in the CSM the notion of object emerges through the metaphorical projection of a series of image schemas, and especially that it is the result of the object metaphor. Finally, it is argued that the notion of object in the OSA is broader than that used by the other two theories, although it does not contradict them. This is because what both APOS theory and the CSM do is highlight partial aspects of the complex process through which, according to the OSA, mathematical objects emerge out of mathematical practices.

## THREE DIFFERENT THEORETICAL PERSPECTIVES

### The OSA

Figure 1 (Font, & Contreras, 2008, p. 35) shows some of the theoretical notions contained in the onto-semiotic approach to mathematics cognition and instruction (Godino, Batanero, & Font, 2007). Here mathematical activity plays a central role and is modeled in terms of systems of operative and discursive practices. From these

practices the different types of related primary objects (language, arguments, concepts, propositions, procedures and problems) emerge, building epistemic or cognitive (depending on whether the adopted point of view is institutional or personal) configurations among one another (see hexagon in Figure 1).



**Figure 1: An onto-semiotic representation of mathematical knowledge**

Problem-situations promote and contextualize the activity, languages (symbols, notations, graphs) represent the other entities and serve as tools for action, and arguments justify the procedures and propositions that relate the concepts. Lastly, the mathematical objects that emerge from the mathematical practices depend on the “language game” (Wittgenstein, 1953) in which they participate, and might be considered from the five facets of dual dimensions (decagon in Figure 1): personal/institutional, unitary/systemic, expression/content, ostensive/non-ostensive and extensive/intensive. Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown in Figure 1. Instead of giving a general definition of process the OSA opts to select a list of processes that are considered important in mathematical activity (those of Figure 1), without claiming that this list includes all the processes implicit in mathematical activity; this is because, among other reasons, some of the most important of them (for example, the solving of problems or modeling) are more than just processes and should be considered as hyper- or mega-processes.

### The APOS theory

APOS (Asiala et al., 1996; Dubinsky, & McDonald, 2001) is an acronym that stands for the types of mental structures (Action, Process, Object, and Schema) which students build in their attempts to understand mathematical concepts. A learner, to

whom we usually refer as a student, uses certain mental mechanisms called interiorization, coordination, and encapsulation to construct these structures. According to APOS theory, the formation of a mathematical concept involves applying a transformation to existing objects to obtain new objects. An action is any transformation of objects according to an explicit algorithm in order to obtain other objects, and is seen as being at least somewhat externally driven.

As an action is repeated and the individual reflects upon it, it may be interiorized into a mental process. An important characteristic of a process is that the individual is able to describe, or reflect upon, the steps of the transformation wholly in her/his mind without actually performing those steps. Additionally, once a mental process exists, it is possible for an individual to think of it in reverse and possibly construct a new process (a reversal of the original process).

When an individual becomes aware of the process as a totality and is able to transform it by some action, it is said that the process has been encapsulated as an object. When necessary, an individual may de-encapsulate an object back to its underlying process. In other situations, the individual may think of the transformation in terms of actions.

A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent framework in the individual's mind.

The research method or investigative approach of this framework consists of three-step cycles. The first step is a theoretical analysis of the actions, processes, objects, and schemas that a learner may construct in order to learn a given/specific mathematical concept. The resulting description is called a genetic decomposition of the concept. This is used to design and implement the second step, the instructional treatment. The third step is the collection and analysis of both quantitative and qualitative data.

### **Cognitive Science of Mathematics**

In this paper we are interested particularly in Lakoff and Nunez's account. This is a very particular and, may be, controversial interpretation of "cognitive science of mathematics". Lakoff and Núñez (2000) state that the mathematical structures people build have to be looked for in daily cognitive processes, such as image schemas and metaphorical thinking. These processes allow us to explain how the construction of mathematical objects is supported by the way in which our body interacts with the objects of everyday life. To achieve abstract thinking we need to use basic schemas derived from the immediate experience of our bodies. We use these basic schemas, called image schemas, to make sense of our experiences in abstract domains through metaphorical mappings. Lakoff and Núñez (2000) claim that metaphors create a conceptual relationship between the source domain and the target domain. They distinguish between two types of conceptual metaphors in relation to mathematics: a)

grounding metaphors that relate a source domain outside of mathematics with a target domain inside mathematics; and b) linking metaphors that have their source and target domains within mathematics.

## **THE LINK BETWEEN THESE FRAMEWORKS**

### **The emergence of objects in the OSA**

The OSA considers that the process through which mathematical objects emerge out from mathematical practices is a complex one and that at least two levels need to be distinguished. The first level corresponds to the emergence of representations, definitions, propositions, procedures, problems and arguments (primary objects). As regards the nature of these objects, the OSA, in line with the philosophy of Wittgenstein, considers that the type of existence which can be ascribed to the concepts/definitions, the propositions and the procedures is that of conventional rules. From this point of view, mathematical statements are rules (of a grammatical kind) for using certain types of signs, because in fact they are used as rules. They do not describe properties of mathematical objects that exist independently of the people who wish to know about them or of the language used to know them, even though it may appear that this is the case.

Although the OSA adopts a conventionalist point of view on the nature of mathematical objects, it is acknowledged that a descriptive/realist view of mathematics is implicitly suggested in teaching processes. In order to explain how this vision is generated it is necessary to consider a second level in the emergence of mathematical objects; an example might be the object ‘function’, which is considered as an object that is represented by different representations that may have several equivalent definitions, which have properties, etc.

In order to explain how primary objects emerge, the metaphor of ‘climbing stairs’ proves highly useful. When we climb stairs we have to stand on one foot as we move, but that foot then moves progressively to a higher stair. Mathematical practice can be considered as ‘climbing stairs’. The stair on which we stand in order to carry out the practice is an already-known configuration of primary objects, whereas the higher stair which we then reach, as a consequence of the practice carried out, is a new configuration of objects, one (or more) of which was previously unknown. The new primary objects appear as a result of this mathematical practice and become institutional primary objects due, among other processes considered in Figure 1 (including reification and idealization), to processes of institutionalization that form part of the teaching-learning process being studied.

The second level of emergence is the result of several factors, the main ones being as follows: 1) mathematical discourse, explicitly or otherwise, gives students the message that mathematics is a ‘certain’, ‘true’ or ‘objective’ science; 2) the predictive success of the sciences that make use of mathematics is used, explicitly or

otherwise, to argue in favor of the existence of mathematical objects; 3) the simplicity that derives from postulating the existence of mathematical objects. Their postulation is justified on the basis of the practical benefits, especially as regards simplifying the mathematical theory which is being studied. Indeed, it is highly convenient to consider that there exists a mathematical object that is represented by different representations, which can be defined by various equivalent definitions, or which has properties, etc. 4) The object metaphor is always present in teachers' discourse because here the mathematical entities are presented as "objects with properties". It is common in mathematics discourse to use certain metaphorical expressions which suggest that mathematical objects are not constructed but, rather, are discovered as pre-existing objects; for example, words such as 'describe' or 'find', etc. 5). As discussed in Font, Godino, Planas and Acevedo (2010) it is possible in mathematics discourse (a) to talk about ostensive objects representing non-ostensive objects that do not exist (for example, we can say that  $f'(a)$  does not exist because the graph of  $f(x)$  has a pointed form in  $x = a$ ), and (b) to differentiate the mathematical object from one of its representations. Both aspects lead students to interpret mathematical objects as being different from their ostensive representations.

These five factors generate, implicitly or explicitly, a descriptive/realist view of mathematics which considers (1) that mathematical propositions describe properties of mathematical objects, and (2) that these objects have a certain kind of existence that is independent of the people who encounter them and the language through which they are known. This view is hard to avoid since the reasons why it is adopted are always operating, albeit subtly. More than a consciously assumed philosophical position we are dealing here with an implicit way of understanding mathematical objects.

### **Objects in APOS theory**

In APOS theory (Asiala et al., 1996) encapsulation and de-encapsulation play an important role. APOS theory begins with actions and moves through processes to objects. These are then integrated into schemas which can themselves become objects. The ideas arise from attempts to extend the work of Jean Piaget on reflective abstraction in children's learning to the level of collegiate mathematics.

In the paper titled "Reification as the Birth of Metaphor," Sfard (1994) reports on the interviews she conducted with three renowned mathematicians. In these interviews the three mathematicians talk about the mathematical concepts that they study as if they were concrete in some way. The term that Sfard uses for this cognitive phenomenon is reification, which is similar to Dubinsky's construct of encapsulation.

Reification is a term used in philosophy that means, etymologically, "to treat something like a thing." In the processes of mathematical reification, abstract notions are conceived like objects. To reify (or encapsulate) is to regard or treat an

abstraction as if it had a concrete or material existence. One example of reification is when we assume — or state linguistically — that there is an object with various properties or various representations.

### **The link between the OSA and APOS theory**

In accordance with the onto-semiotic approach (see section 3.1) we consider that reification (encapsulation) is very important in terms of explaining the emergence of mathematical objects, but that it is insufficient to describe adequately this emergence and the nature of mathematical objects. Furthermore, we believe that APOS theory has a number of limitations, two of which we regard as especially important: 1) in APOS theory the construct ‘object’ is considered as the product of the encapsulation (reification) process. However, with this characterization, which basically comes from psychology, it is not clear how to address some issues related to mathematical objects, such as the nature of mathematical objects, their various types, the way in which they are formed and how they participate in mathematical activity.

In order to overcome this limitation it is helpful to consider the proposal of semiotic perspectives, especially the OSA, which regard mathematical objects as emerging out from mathematical practices. 2) The construct of ‘semiotic medium’ is not explicitly addressed by APOS theory, which does not specifically address the role of semiotic representations.

Recent research that has extended APOS theory through the incorporation of semiotic perspectives turns either to Duval’s theory of semiotic registers (Trigueros, & Martínez-Planell, 2010) or to the OSA (Badillo, Azcárate, & Font, 2010; Font, Montiel, Wilhelmi, & Vidakovic, 2010). In line with Badillo, Azcárate and Font (2010), we consider that the OSA complements APOS theory as follows: 1) it extends APOS theory by specifically addressing the role of semiotic representations; 2) it improves the genetic decomposition by incorporating the ideas of semiotic complexity, network of semiotic functions, and semiotic conflicts; and 3) it offers a more detailed notion of mathematical objects due to the way in which it considers the nature of such objects and their emergence out of mathematical practices.

### **Objects in the CSM**

The group of grounding metaphors includes the ontological type, where we find the object metaphor. The object metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and enables the interpretation of events, activities, emotions and ideas, etc. as if they were real entities with properties. This type of metaphor is combined with other ontological, classical metaphors such as that of the “container” and that of the “part-whole”. The combination of these types leads to the interpretation of ideas and concepts, etc., as entities that are part of other entities and which are constituted by them.

Ontological metaphors are considered as a group of metaphors that result from the projection of image schemas (container, whole-part, object, etc.) which, in our view, share a ‘common territory’; therefore, there may be a certain hierarchy among them. One question that remains open (Santibáñez, 2000) concerns the relationship between these image schemas. For example, one could consider that the object image schema is the fundamental schema and that the others are derived from it, or alternatively, that there is some other schema from which all (or some) of those mentioned are derived, for instance, the entity schema (Quinn, 1991) or the notion of thing (Langacker, 1998).

### **The link between the OSA and the CSM**

In line with the onto-semiotic approach we consider that the process through which mathematical objects emerge from mathematical practices is highly complex (see section 3.1). Therefore, we believe that Lakoff and Núñez’s methodology of “mathematical idea analysis” is very important in terms of explaining the emergence of mathematical objects, but that it is insufficient to describe adequately this emergence and the nature of mathematical objects. This limitation was pointed out by various authors in the discussions that followed the publication of Lakoff and Núñez’s book (e.g. Sinclair, & Schiralli, 2003).

The way in which the OSA explains the emergence of mathematical objects not only extends and improves upon the explanation offered by the CSM, but also provides clarification of one of the central processes considered by the latter, namely metaphorical processes (Acevedo, 2008; Malaspina & Font, 2010; Font, Montiel, Wilhelmi, & Vidakovic, 2010). The reasons for this are set out below.

Here we are interested in observing metaphorical processes from the ‘unitary/systemic’ duality proposed in the OSA, since the reification/decomposing processes in the OSA are associated with this unitary/systemic facet or dimension. When a mathematical abstraction is treated as an object, this is equivalent to adopting a unitary point of view on this object. On the other hand, the mathematical object can be treated from a systemic viewpoint, considering the actions that a subject can make on it and on the other objects, parts or processes that compose it. In the work of Lakoff and Núñez (2000), the unitary/systemic duality has a central role. On the one hand, the metaphor is elementary ( $A$  is  $B$ ). However, the metaphor allows us to generate a new system of practices (systemic perspective) as a result of our understanding of the target domain in terms of the source domain. The CSM develops the elementary/systemic duality for different metaphors, a good example of which is the object metaphor (Font, Bolite, & Acevedo, 2010, p. 139): Unitary: “Mathematical objects are physical objects.” Systemic: “Table 1. Metaphor projection.” In fact, most research on metaphors has been mainly targeted at studying such a duality. In other words, given a metaphor, the source and target domains are decomposed to determine what concepts, properties, relationships, etc. from the source domain are transferred to the target domain. The systemic vision of a

metaphor leads us to understand it as a generator of new practices. The OSA considers that in order to carry out a mathematical practice the agent must have the basic knowledge required to do so. If we consider the components of the knowledge that the agent must have in order to develop and evaluate the practice that enables a problem to be solved (e.g., propose and solve a system of two equations with two unknowns), we can see that certain verbal (e.g., solution) and symbolic (e.g.,  $x$ ) language must be used.

Source domain: Object image schema	Target domain: Mathematics
Physical object	Mathematical object
Physical objects are manipulated, found, discovered, etc.	Mathematical objects are manipulated, found, discovered, etc.
Physical objects are different from their material representations (i.e., a clock is different from the drawing of a clock)	Mathematical representations are different from the mathematical objects they represent
Properties of the physical object	Properties of the mathematical object
Physical objects exist	Mathematical objects exist

**Table 1: “Mathematical objects are physical objects”**

This language is the ostensive part of a series of concepts (e.g., equation), propositions (e.g., if the same term is added to the two sides of an equation, an equivalent equation is obtained) and procedures (e.g., solution by substitution) that will be used in making arguments so as to decide if the simple actions that make up the practice (where this is understood to be a compound action) are satisfactory. Hence when an agent carries out and evaluates a mathematical practice, it is necessary that it activates some (or all) of the elements mentioned above: situation-problems, language, concepts, propositions, procedures and arguments. By articulating these types of objects we obtain the configuration (hexagon in Figure 1). If, in addition to the “structure”, it is necessary to analyze the genesis and functioning of the mathematical activity, other tools are necessary, especially some of the processes shown in Figure 1, as well as the metacognitive processes. The epistemic configuration tool allows us to see the structure of those objects that make mathematical practice possible and which regulate it within a specific institutional framework. Because the OSA considers that, among other aspects, an epistemic/cognitive configuration (depending on whether the adopted point of view is institutional or personal) has to be activated in order to perform mathematical practices, and that the systemic vision of the metaphor leads us to understand it as a generator of new practices, it is natural to ask ourselves the following question: How is the metaphor related to the building components of epistemic/cognitive configurations? The conclusion drawn by Acevedo (2008) on linking metaphors, after he had studied in detail the linking metaphor used by Descartes when solving the problem of Pappus within the framework of analytic geometry, is that a linking metaphor projects an epistemic/cognitive configuration onto another one.



The epistemic/cognitive configuration construct allows us to explain and make precise the structure that is projected onto the linking metaphors. There is a source domain that has the structure of an epistemic/cognitive configuration (whether the adopted point of view is institutional or personal) and which projects itself onto a target domain that also has the structure of an epistemic/cognitive configuration. This way of understanding the preservation of the metaphoric projection improves upon the explanation of such a preservation given by Lakoff and Núñez (2000), who simply give a two-column table in which properties and concepts are mixed. The reader can intuit that the properties are projected onto properties and the concepts onto concepts. The question which remains to be resolved is: what structure is projected in the case of a grounding metaphor? We believe that unlike in the case of linking metaphors, only some parts of the epistemic/cognitive configuration are projected. The specific study of each grounding metaphor will allow the identification of these parts.

## FINAL CONSIDERATIONS

A sound theoretical understanding of mathematical objects must be a key part of any research on mathematics learning. The way in which the OSA understands the emergence of mathematical objects enables us to explain: (1) how primary mathematical objects emerge from mathematical practices and they construct among one another cognitive or epistemic configurations; and (2) why a descriptive/realist view of mathematics is usually presented in mathematics classrooms. This account goes beyond the explanations offered by APOS theory and the CSM regarding the emergence of mathematical objects, and shows that what the latter two approaches do is highlight partial aspects of the complex process through which such objects emerge out of mathematical practices.

## Acknowledgements

This research was carried out as part of the project EDU 2009-08120/EDUC.

## REFERENCES

- Acevedo, J. I. (2008). Fenómenos relacionados con el uso de metáforas en el discurso del profesor. El caso de las gráficas de funciones. Doctoral dissertation. Barcelona, Spain: University of Barcelona.
- Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education II* (pp. 1–32). Providence, RI: American Mathematical Society.
- Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In Derek Holton, et al. (Eds.), *The*

*teaching and learning of mathematics at university level: An ICMI Study*, (pp. 273–280). Dordrecht: Kluwer Academic Publishers.

- Badillo, E., Azcárate, C., & Font, V. (2010). Análisis de los niveles de comprensión de los objetos  $f'(a)$  y  $f'(x)$  de profesores de matemáticas. *Enseñanza de las Ciencias* (in press).
- Font, V.; Montiel, M.; Vidakovic, D.; Wilhelmi, M. R. (2011). Analysis of Dimensional Analogy by Means of Different Theoretical Perspectives. In Roberta V. Nata (Ed.), *Progress in Education*, Volume 19. (pp. 39-76) Hauppauge, NY: Nova.
- Font, V., Bolite, J., & Acevedo, J. I. (2010). Metaphors in mathematics classrooms: analyzing the dynamic process of teaching and learning of graph functions. *Educational Studies in Mathematics*, 75 (2), 131-152, doi: 10.1007/s10649-010-9247-4.
- Font, V., & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. *Educational Studies in Mathematics*, 69(1), 33-52.
- Font, V., Godino, J. D., Planas, N., & Acevedo, J. I. (2010). The object metaphor and synecdoche in mathematics classroom discourse. *For the Learning of Mathematics*, 30(1), 15-19.
- Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*, 39(1-2), 127-135.
- Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. New York, NY: Basic Books.
- Langacker, R. W. (1998). Conceptualization, symbolization, and grammar. In M. Tomasello (ed.), *The new psychology of language* (pp. 1-39). Mahwah, NJ: Lawrence Erlbaum.
- Malaspina, U., & Font, V. (2010). The role of intuition in the solving of optimization problems. *Educational Studies in Mathematics* 75, 107-130. doi: 10.1007/s10649-010-9243-8.
- Quinn, N. (1991). The cultural basis of metaphor. In J. W. Fernández (Ed.), *Beyond metaphor. The theory of tropes in anthropology* (pp. 56-93). Stanford, CA: Stanford University Press.
- Santibáñez, F. (2002). The object image-schema and other dependent schemas *ATLANTIS*, XXIV(2), 183-201.
- Sinclair, N., & Schiralli, M. (2003). A constructive response to ‘Where mathematics comes from.’ *Educational Studies in Mathematics*, 52(1), 79-91.
- Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, 14(1), 44-54.
- Trigueros M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational Studies in Mathematics* 73(1), 3-19.
- Wittgenstein, L. (1953) *Philosophical Investigations*. New York: MacMillan.

# USING DIFFERENT SOCIOCULTURAL PERSPECTIVES IN MATHEMATICS TEACHING DEVELOPMENTAL RESEARCH

Simon Goodchild

University of Agder, Norway

*Communities of practice theory and cultural historical activity theory are considered alongside each other and argued to be incompatible on the grounds of their treatment of human agency. It is then demonstrated that it is of value to use both perspectives in the context of mathematics teaching developmental research. The introduction of inquiry into community of practice theory, it is asserted, creates a different type of community, a community of inquiry, and from this perspective developmental research is argued to be consistent with the critical research paradigm. The combination of theories is shown to be useful as theoretically heuristic, and each has a value as teaching is studied at different stages in the developmental process.*

## INTRODUCTION

At CERME5 in 2007 Barbara Jaworski and I presented papers within the “theories” group. In these we discussed theoretical issues relating to a mathematics teaching development project in which we were both engaged. Barbara’s paper (Jaworski, 2007) took a social practice perspective, taking Community of Practice Theory (CPT) as a starting point. My paper (Goodchild, 2007) was from the perspective of Cultural Historical Activity Theory (CHAT). It was our deliberate decision that we should adopt different perspectives in our papers. However, in our collaboration we share both perspectives, and Barbara, at that time and subsequently, has published from both. Barbara concluded her CERME5 paper with the following observation:

We recognize also that employing a multiplicity of theoretical perspectives begs some overall theoretical rationalization – not a unification, but a clarity on how the theoretical perspectives we are using are juxtaposed and what issues of commensurability arise from juxtapositioning. Space here has limited further discussion of such issues. (Jaworski 2007, p. 1696).

The purpose of this present paper is to take up this discussion and consider some of the issues that arise when using these two theories alongside each other.

I use the rather unusual approach here and refer to Barbara Jaworski’s contribution reported in her CERME5 paper (Jaworski, 2007), using her first name only. My purpose is to reflect the fact that within the projects we are close colleagues, developing ideas (with others in our research group) in critical collaboration. However, it should be recognised that Barbara is the principal architect of the ideas she expresses. I will revert to the more conventional form of citation when I refer to her other published work. Barbara has read a draft version of this paper and is largely in agreement with the content but we have not had the opportunity to discuss details.

At the time Barbara and I presented the papers at CERME5 we were coming to the close of the project Learning Communities in Mathematics (LCM, 2004-2007), and it was on this project that our reports were based. Concurrent with LCM ran another mathematics teaching developmental research project ICT and Mathematics Learning (ICTML). Subsequent to LCM and ICTML a follow-on binary project Teaching Better Mathematics / Learning Better Mathematics (TBM/LBM, 2007-2010) continued with a majority of the schools and teachers participating throughout both project periods. LCM, ICTML and TBM were supported by the Research Council of Norway. Further support for TBM and support for LBM came from the Competence Development Fund of Southern Norway. All projects were based on the principle of developing communities of inquiry (CoI) comprising teams of teachers in schools and a team of about 10 didacticians at the university. There were eight schools participating in LCM, four of these also in ICTML; nine schools and four kindergartens, and the leaders of three school authorities participated in LBM/TBM. Details about the methodology and implementation of the projects will emerge in the discussion that follows. This paper draws on experience from all the aforementioned.

The paper continues with a brief outline of CPT and CHAT, and goes on to expose fundamental differences between them. The second (smaller) part of the paper sets out some of the ways the projects are better served by taking more than one perspective; this is illuminated using an illustration from the projects.

## **A VERY BRIEF INTRODUCTION TO CPT AND CHAT**

To fulfil the purpose of this paper it is necessary to provide a brief introduction to the two theoretical perspectives and draw attention to their fundamental differences. This is problematic for several reasons. First the theories are complex and some of their key ideas are subtle and need to be interpreted with care. Moreover, key words are used with different meanings or nuances, for example words such as *practice*, *community*, *activity* and *action*; a word ascribed a special meaning in one theory may be used in an everyday sense in the other, and as Radford observes (in Bikner-Ahsbabs et al., 2010) “the semantic value of a theoretical term ... in a theory results from its position in the main web of dynamic interconnections that characterize the theory as a whole” (p. 169). Second, much has been published on both theories; brief summaries will inevitably be incomplete and therefore liable to mislead. Third, there are differences and developments within the theories themselves. For example, Kaner and Lerman (2008) expose differences within CPT, in particular, between Wenger’s (1998) ‘Communities of Practice’ and Lave and Wenger’s (1991) ‘Situated Learning’. I believe Lave’s (1988) ‘Cognition in Practice’ introduces a third variety or nuance within CPT. Similarly CHAT is traced through three ‘generations’ of development through the works of (amongst many others) Vygotsky, Leont’ev, and Engeström whose names I use here as representatives of successive generations. In the following I combine sources to articulate one version of CPT, and one version of CHAT. This may be unsatisfactory in regards to exploring the theories themselves but I believe there is sufficient internal consistency to enable a discussion of

differences between CPT and CHAT.

First, one must consider whether there is any a priori basis to suppose that there are fundamental differences between CPT and CHAT; they are both considered to be within the family of sociocultural theories, and exponents of both refer to Vygotsky's contribution. I begin, therefore, by attending to what key protagonists of each have to say. In a substantial review of CHAT, Roth and Lee (2007) explicitly exclude works from a CPT perspective because despite 'strong family resemblances' they claim CHAT and CPT to be 'distinct'. Roth and Lee do not, however, go into detail about the distinguishing features. Wenger (1998) claims there are ontological differences between CPT and CHAT. However, in my opinion, his argument is based on an interpretation of 'mediated action' that is not consistent with CHAT, as developed, for example, by Wertsch (1994). Nardi (1996) compares and contrasts CPT (situated action), CHAT, and distributed cognition perspectives, and finds differences between them. Nardi points to the treatment of goals as a distinguishing feature: "Attention to the shaping force of goals in activity theory ... contrasts with the contingent, responsive, improvisatory emphasis of situated action" (p. 79). I argue below that this, in the context of mathematics teaching developmental research, is the kernel of an essential incompatibility between the theories. Further, Kanen and Lerman (2008), in their discussion of differences within CPT note their opinion that both articulations of CPT lack a theory of mediation. This lacking feature emerges in our theorising of the projects as Barbara explains in her CERME5 paper (Jaworski, 2007) as she extends CPT using principles of mediation based on CHAT.

In CHAT, *activity*, Leont'ev (1979) asserts, is "the nonadditive, molar unit of life ... a system with its own structure, its own internal transformations, and its own development" (p. 46), and later he asserts, "there can be no activity without a motive" (p. 59). Activity is thus the unit of analysis. I will illuminate Leont'ev's definition by developing an illustration used by Vygotsky (1986). If the object of study is water, the unit of analysis must be water – one could have more or less water, but it is nonadditive in the sense that more does not make it more waterish. Water is a compound comprising elements of hydrogen and oxygen, but if the object of study is water, little from the perspective of (classical) chemistry or physics will be learned by studying the elements, water is the 'molar unit'. Activity, Leont'ev (1979) explains is comprised of (conscious) goal directed actions, and actions are comprised of (sub-conscious) operations within constraints or conditions. Actions are conscious and mostly observable (maybe mental actions are not observable but then their effect may be), but an action needs to be analysed in the context of the activity-motive in which it arises. Water also has properties (density, transparency, etc.) that distinguish it from other liquids; similarly, one might describe as properties of activity that distinguish one activity system from another, these are: the person or group, and the object of their activity, cultural artefacts or tools that mediate actions, community, and the rules and division of labour within the community. Engeström (1993) refers to these as 'components' or 'elements', which relate dialectically to each other within an 'activity system', and he draws attention to the historical nature

of activity by observing “An activity system ... is composed of a multitude of often disparate elements ... This multiplicity can be understood in terms of historical modes, as well as buds or shoots of its possible future” (1993, p. 68). Thus, whereas actions occur *in time*, activity endures (and changes or develops) *over time*.

In CPT, Wenger (1998) argues that “community of practice constitutes *a level of analysis*” (p. 122, my emphasis). Lave (1988) however, identifies three levels of analysis – “semiotic systems *with* their structural entailments, ... relations among person-acting, setting and activity, ... (and) dialectical relations between an experienced world and its constitutive order” (p. 179). A ‘community of practice’ might also be explained as a characteristically distinct compound of elements, Wenger (1998) describes three ‘dimensions of practice’ which he explains are ‘the property of a community’ these are ‘joint enterprise’, ‘mutual engagement’ and ‘shared repertoire’ (p. 73). *Practice*, Wenger (1998) explains “is doing in a historical and social context that gives structure and meaning to what we do” (p. 47). At this point one might be tempted to question whether there is any substantive difference between *practice* in CPT and *activity* in CHAT. A definition of practice offered by Scribner and Cole (1981) provides a clue to what might fundamentally distinguish the theories. They explain, practice is “a recurrent, goal-directed sequence of activities ... (it) consists of three components: technology, knowledge and skills ... Practice always refers to socially developed patterned ways of using technology and knowledge to accomplish tasks” (p. 236). The point that I draw attention to in Scribner and Cole’s definition is that it is ‘practice’ that is goal directed, rather than the participating person. This is emphasised by Lave (1988) “motivation is neither merely internal to the person nor to be found exclusively in the environment ... goals ... are not *prefabricated* by the person-acting or some other goal-giver as a *precondition* for action” (p. 184, emphasis added). As Nardi (1996) observes, in explaining the entailments of CPT based on Lave (1988), in CPT “goals are our musings out loud about why we did something *after* we have done it” (p.79).

In CHAT actions are goal directed, whereas in CPT actions emerge dialectically between the person and the environment, this is a difference that reveals CPT and CHAT to be incompatible – in other words, they cannot co-exist peacefully. The impossibility of peaceful co-existence is evident in Engeström’s observation “Activity theory contends that ... a notion of context beyond our influence is a fiction, a fetish ... human agency is necessarily realized in the form of (goal directed) actions” (1993, p. 66). I inserted ‘goal directed’ in the above quotation to act as a reminder and emphasise that in CHAT actions are ‘goal directed’. As Roth observes, “Fundamental to CHAT is the human ability to act or agency” (2006, “Power to act, agency” Para. 1). Wenger (1998) is clear that his work does not focus on agency; thus giving the impression that the concept is not so important in his articulation of CPT. I assert that the introduction of ‘inquiry’ to CPT entails a shift into the critical paradigm; because it is in the critical paradigm that the goal is to empower actors to exert their agency and transform their practice. Thus, I conclude that CPT and CHAT are incommensurable. Moreover, CHAT offers a model, in the form of expansive

learning that fits the critical paradigm. The adoption of different theoretical perspectives has been discussed by, for example, Cobb (2007) who proposes that different perspectives can be “sources of ideas that we can appropriate and modify for our purposes as mathematics educators” (p. 29). From outside mathematics education Guba and Lincoln (2005) defend “borrowing” from different theoretical perspectives where it “seems useful, richness enhancing, or theoretically heuristic” (p. 197). In the next section I demonstrate how reference to both CPT and CHAT is useful in the theoretically heuristic sense within our mathematics teaching developmental research.

## **DIFFERENT PERSPECTIVES ARE “THEORETICALLY HEURISTIC”**

A model of development is explicitly articulated within CHAT by Engeström (1987, 2001), and, I believe, a theory of development is implicit in Wenger’s (1998) articulation of CPT. In CHAT, Engeström (2001) explains development in terms of expansive cycles through which activity systems pass as groups (of participants) address the tensions and contradictions (double binds) of their activity system. Double binds might be thought of as ‘lose-lose’ situations, such as faced by teachers with a demanding curriculum and high-stakes examination AND pressure from their professional subject milieu to teach for ‘understanding’. Attending to the demand of one side appears to threaten losing out on the other. Briefly, expansive cycles take the form of successive phases of internalisation and externalisation. First persons engage in activity and appropriate (internalise) the routines and practices of the activity. As they continue in the activity they become increasingly aware of the double binds and embark on a search for solutions to resolve the tensions and contradictions. Initially solutions are found in the form of creative innovations external to the activity system (externalisation), and the activity system then ‘expands’ to accommodate the innovations. Within the projects, inquiry, operationalised as an inquiry cycle, is argued to be a purposeful realisation of expansion. Within CHAT, inquiry can be conceptualised as both action, and a tool mediating action.

To build a theory of development from Wenger’s (1998) CPT, I start with the ‘modes of belonging’ that Wenger describes in the context of an individual’s identity. Identity occupies an important position in CPT because, as Wenger explains, the discussion of identity “narrows the focus onto the person ... from a social perspective” (1998, p. 145). Wenger argues further: “To make sense of ... processes of identity formation and learning, it is useful to consider three distinct modes of belonging: ... engagement ... imagination ... alignment” (pp. 173-174). Imagination enables a person to extrapolate from his/her own experience, or that shared by others to expand oneself and ‘transcend engagement’ (p. 177).

It is through imagination that we see our own practices as continuing histories that reach far into the past, and it is through imagination that we conceive of new developments, explore alternatives, and envision possible futures. ... By taking us into the past and carrying us into the future, it can recast the present and show it as holding unsuspected

possibilities. (Wenger, 1998, p. 178)

However, Wenger also draws attention to possible “trade-offs of imagination” in particular of the “risk of losing touch with the sense of social efficacy by which our experience of the world can be interpreted as competence” (1998, p. 178). This resonates strongly with our observations in the projects when experienced teachers have been encouraged to try out some innovation (which inevitably entails risk because it is a departure from the known) and the outcome has disturbed the teacher’s experience of competence (Jaworski, Goodchild, Daland, & Eriksen, in press).

I find it difficult to perceive of the extrapolation of practice, and possible trade-offs, in the absence of an individual’s intentionality and agency. If it were possible to leave out the issue of agency a synthesis of ‘extrapolation’ and ‘expansion’ might be possible. And such synthesis provides a useful tool for analysing development.

CPT does not explicitly include a consideration of inquiry, thus one is challenged to explore how a theory of inquiry can be integrated with CPT, or indeed how a community of inquiry (CoI) can be expressed in terms of CPT. Here the notion of alignment is valuable in terms of being ‘theoretically heuristic’ as Barbara demonstrated in her CERME paper. Barbara starts from Wenger’s explanation: alignment is about participation and engagement and doing “what it takes to play our part” (1998, p. 179) in the practice. It can be noted that here again, Wenger identifies ‘trade-offs’ in particular that alignment ‘can ... be blind and disempowering’. Barbara continues by explaining how the introduction of inquiry transforms ‘alignment’ into ‘critical alignment’, “inquiry brings with it a critically questioning attitude towards practice and knowledge in practice that allows critical reflection on the practice of teaching and hence can lead to development of teaching” (Jaworski, 2007, p. 1693). The introduction creates a ‘community of inquiry’ which Barbara describes as “extending Wenger’s exposition of community of practice” (p. 1693).

I believe that the combination of inquiry and communities of practice is not just an extension of CPT but that it constitutes a paradigm shift. In Goodchild (2008) I attempt to produce a mapping between critical alignment and Freire’s (1972) articulation of ‘conscientization’. The point is that critical alignment enables teachers and didacticians to become aware of the oppressive force of the double binds that constrain mathematics teaching practice and development, and inquiry is a means of liberation. Furthermore, the inquiry cycle, plan-act-observe-reflect-feedback (Jaworski, 2007, p. 1694) is closely related to the action research cycle, thus I see CoI as a developmental methodology within a critical research paradigm. It was Barbara’s articulation of CPT, inquiry and critical alignment that resonated with Freire’s work.

## **THE VALUE OF USING DIFFERENT THEORETICAL PERSPECTIVES**

The projects in which Barbara and I have collaborated have been built on theoretical constructs of CPT, inquiry and community; it is on the latter of these that I now



focus. It has been a condition of joining in the projects that there are at least three participating teachers in each school and that the project has the support of the school principal. The school ‘project’ community is important as a source of mutual support and encouragement, ideas and critically reflective thinking, opportunities for joint activity, observation, and joint reflection. The school community is also intended to provide a critical mass that will facilitate teaching development beyond the life of the projects. A wider project community is envisioned comprising all participating teachers together with the team of didacticians at the university. The project community comes together for workshops three times each semester. Workshop activity comprises presentations by didacticians and participating teachers, and group discussion activities that focus on mathematical problems and approaches to teaching mathematics. As Barbara illustrates (Jaworski, 2007), a mathematical problem is introduced in a workshop plenary session, and then worked on in small groups. When the teachers return to their school they (as a school team) prepare the mathematical problem to present as a class task, the task is implemented with subsequent reflection in the school team and the cycle returns to the point where teachers report their experience in a subsequent workshop, and possibly initiate a further cycle.

There is a further methodological concept that is central in the implementation of the projects: the relationship between teachers and didacticians is based on a ‘co-learning agreement’ (Wagner, 1997). This emerges from ethical and practical principles; it respects teachers’ expertise in their practice and casts the research enterprise as something in which all contribute and from which all can learn in relation to their own interests and concerns. Further, it recognises that substantive changes in teaching will only be achieved *by teachers in their own classes*. In effect it means that didacticians do not presume to *tell* about or demonstrate better methods of teaching. The intention is to stimulate discussion and inquiry about mathematics and teaching mathematics, mediated by tasks that are designed to stimulate mathematical inquiry, and critical reflection on teaching mathematics. Thus, the projects are not based on notions of novice and master/expert in the sense that Lave and Wenger (1991) use in their description of learning in practice as ‘legitimate peripheral participation’. With this in mind I return to the notion of development as ‘extrapolation’.

It is possible that development is restricted to a modest ‘tinkering’ with practice rather than fundamental changes; this could happen if no-one comes with experience of teaching practice that transcends what ‘normally’ occurs. This can be illustrated by a case study within the projects that focused on teaching developments in three project schools, specifically with reference to ICT use, and the introduction of dynamic geometry software (DGS) (Erfjord, 2009). A little over a year after the projects began there appeared to be only limited signs of development at one of the schools, which prompted the project directors to visit the school to see if further support were necessary, and ‘encourage’ project related activity. One outcome was an arrangement in which an experienced teacher and specialist in using digital

technology with his classes in another project school was asked to lead some school based workshops. In this, one might see an ‘outsider’ providing additional experience from which the teachers could develop their imagination and extrapolate practice. However, it was the appointment of a novice teacher some months later that was decisive. He introduced some teaching resources for using DGS. These resources were closely aligned to the teachers’ existing practice and led the introduction of DGS at the school. The point is, there need to be opportunities to extend participants’ experience in practice. Perhaps a fundamentalist attitude to the co-learning agreement in the context of CPT could hinder this if the principle of ‘co-learning’ were to prevent the purposeful introduction of ‘outsiders’ to offer support and new ideas. However, the project ideal was co-learning within a community of inquiry. Erfjord’s case study of the school, in which the teachers remained aligned to their regular practice, might be interpreted as providing evidence that a CoI had not yet been established. In this instance, CPT provides the analytic categories to make sense of the observations: the development is characteristic of extrapolation.

## DISCUSSION

Evidence from some of the case studies in the projects, such as discussed above, combined with other discussions of teaching development (e.g. Cuban, 2001) suggests that teachers will adapt a new technology (such as ICT, or ‘inquiry tasks’) to their existing practice before they see new technology as an opportunity for creative development within their practice. This supports a conception of development as extrapolation and possibly reveals the strength of alignment to existing practice. In this regard CPT can be seen as providing a useful model, and explanatory tools for the stable situations in which incremental development is observed, as in the example provided. This is consistent with teachers incorporating research into their practice which Jaworski (1998) describes as ‘evolutionary’. However, the goal is to create a CoI and thereby the development of practice that is better characterised as expansion.

‘Inquiry’ transforms alignment into critical alignment. The intention is that inquiry will heighten awareness of the apparent constraints that limit teaching and learning, *and* provide the means of addressing those constraints. One teacher, for example, after six years of participating in the projects claimed that ‘inquiry’ based teaching and learning offers an approach to practice in mixed attainment classes (grouping students by attainment is illegal in Norway). Inquiry empowers, it also destabilises practice by bringing constraints into view, and provides a means to achieve new forms of stability through creative innovation. CPT provided a model for initiating the projects in which inquiry was seen as an important element of practice, CHAT was introduced later as an analytical tool. The combination of CPT and inquiry to theorise the envisioned communities of inquiry is ‘theoretically heuristic’. Consideration of ‘alignment’ from CPT and ‘critical alignment’ developed from the theorisation process resulted in the realisation that the transformation from CPT to CoI is more than a development of CPT, it constitutes a paradigm shift. CPT, and

CoI analysed within CHAT, provide alternative lenses that can explain different forms of development. Thus, the combination of CPT and CHAT, argued here to be both incompatible and incommensurable, is theoretically heuristic because they draw attention to qualitatively different forms of development of teaching practice. CHAT also offers a tool for conceptualising and analysing development, as expansive cycles, which follow creative innovation in response to constraints and tensions experienced in activity.

## REFERENCES

- Bikner-Ahsbahr, A. et al. (2010). Networking of theories of education. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education, Vol 1*. (pp. 145-175). Belo Horizonte, Brazil: PME.
- Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 3-38). Charlotte, NC: Information Age Publishing.
- Cuban, L. (2001). *Oversold & underused: Computers in the classroom*. Cambridge MS: Harvard University Press.
- Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki, Finland: Orienta-Konsultit.
- Engeström, Y. (1993). Developmental studies of work as a testbench of activity theory: The case of primary care medical practice. In S. Chaiklin & J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 64-103). Cambridge UK: Cambridge University Press.
- Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization, *Journal of Education and Work*, 14(1), 133-156.
- Erfjord, I. (2009). *Teachers' implementation and orchestration of Cabri-use: Initial use of a dynamic geometry software package in mathematics teaching*. Saarbrücken, Germany: VDM Verlag Dr. Müller Aktiengesellschaft & Co. KG.
- Freire, P. (1972). *Pedagogy of the oppressed*. Harmondsworth, UK: Penguin.
- Goodchild, S. (2007). An activity theory perspective of didacticians' learning within a mathematics teaching development research project. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the V Congress of the European Society for Research in Mathematics Education* (pp. 1678-1687). Larnaca, Cyprus: Department of Education, University of Cyprus.
- Goodchild, S. (2008). A quest for 'good' research. In B. Jaworski, & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 201-220). Rotterdam, The Netherlands: Sense Publishers.
- Guba, E. G. & Lincoln, Y. S. (2005). Paradigmatic controversies, contradictions, and emerging confluences. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research 3rd. ed.* (pp. 191-215). London: Sage.

- Jaworski, B. (1998). Mathematics teacher research: Process, practice and the development of teaching. *Journal of Mathematics Teacher Education*, 1, 3-31.
- Jaworski, B. (2007). Theory in developmental research in mathematics teaching and learning: Social practice theory and community of inquiry as analytical tools. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the V Congress of the European Society for Research in Mathematics Education* (pp. 1688-1697). Larnaca, Cyprus: Department of Education, University of Cyprus.
- Jaworski, B., Goodchild, S., Daland, E., & Eriksen, S. (in press). Mediating mathematics teaching development and pupils' mathematics learning: the life cycle of a task. O. Zaslavsky & P. Sullivan (Eds.), *Constructing knowledge for teaching secondary mathematics: Tasks to enhance prospective and practicing teacher learning*. New York: Springer.
- Kanes, C. & Lerman, S. (2008). Analysing concepts of community of practice. In A. Watson & P. Winbourne (Eds.), *New directions for situated cognition in mathematics education* (pp. 303-328). New York: Springer.
- Lave, J. (1988). *Cognition in practice*. Cambridge, UK: Cambridge University Press.
- Lave, J. (1996). Teaching as Learning, in Practice. *Mind Culture and Activity*, 3(3), 149-164.
- Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, UK: Cambridge University Press.
- Leont'ev, A. N. (1979). The problem of activity in psychology. In J. V. Wertsch (Ed.), *The concept of activity in Soviet psychology* (pp. 37-71). New York: Sharpe.
- Nardi, B. A. (1996). Studying Context: A comparison of Activity Theory, Situated Action Models, and Distributed Cognition. In B. A. Nardi (Ed.), *Context and consciousness: Activity theory and human-computer interaction* (pp. 69-102). Cambridge, MA: MIT Press.
- Roth, W-M. (2006). Activity theory. In N. J. Salkind, L. Margolis, K. DeRuyck, & K. Rasmussen (Eds.), *Encyclopedia of Human Development* [E-book version]. Retrieved from <http://www.sage-ereference.com/human-development/> on October 5, 2009.
- Roth, W-M., & Lee, Y-J. (2007). "Vygotsky's neglected legacy": Cultural-historical activity theory. *Review of Educational Research*, 77(2), 186-232.
- Scribner, S. and Cole, M. (1981). *The psychology of literacy*. Cambridge, MA.: Harvard University Press.
- Vygotsky, L. (1986). *Thought and language*. Cambridge, MA: The MIT Press.
- Wagner, J. (1997). The unavoidable intervention of educational research: A framework for reconsidering researcher-practitioner cooperation. *Educational Researcher*, 26(7), 13-22.
- Wenger, E. (1998). *Communities of practice*. Cambridge, UK: Cambridge University Press.
- Wertsch, J. (1994). The primacy of mediated action in sociocultural studies. *Mind, Culture, and Activity*, 1(4), 202-208.

# THE POSSIBILITY OF APERSPECTIVAL RESEARCH ON MATHEMATICS LEARNING

Tim Jay

Graduate School of Education, University of Bristol

*The difficulty in reconciling the differing commitments – ontological, epistemological and methodological – of the various perspectives on research on mathematics learning is well established. As an example of this difficulty, there will be some discussion of cognitivist (e.g. Anderson, Reder & Simon, 1996, 1997) versus socioculturalist (e.g. Greeno, 1997) perspectives on mathematics learning. The claim will be made that problems of language, pace Wittgenstein (1953) and Derrida (1997), are at the heart of the apparent irreconcilability of these two and other perspectives, and that this apparent irreconcilability can be at least partly remedied through a post-structuralist, semiotic, approach. Some practical examples of the problem and some advantages of this approach are described and discussed.*

## PERSPECTIVES AND METAPHORS

My introduction, as many others', to some of the knotty theoretical problems faced by mathematics learning researchers, was through the papers of Anderson, Reder and Simon (1996, 1997) and Greeno (1997). Anderson et al. (1996) sets out a set of 4 purported claims of situated learning researchers; that "action is grounded in the concrete situation in which it occurs" (p.6), that "knowledge does not transfer between tasks" (p.6), that training by abstraction is of little use (p.8) and that instruction needs to be done in complex social environments (p.9). Each of these claims is dismissed in turn, the authors referring principally to cognitive psychology literature in their critique. The point of view expressed by Anderson et al. is that learning is fundamentally an individual process. They claim that whilst they unreservedly recognise the "profoundly social nature of the human species" (p.20), this social nature is best researched by analysing the "complex social situation into relations among a number of individuals and study the mind of each individual and how it contributes to the interaction" (p.21).

The response from Greeno (1997) consists in an attack, not on the evidence for the critique, but on the status of the four claims themselves. Greeno argues that the four claims set out by Anderson et al. demonstrate a misunderstanding of "the important differences between cognitive and situative perspectives" (p.5), and constitute something of a straw man for the cognitivists to attack. The substance of Greeno's response is that cognitivist and situative researchers differ primarily in terms of their 'primary focus of analysis'. The primary focus of analysis for the cognitivist is the set of processes and structures that exist within the individual mind, whilst the primary focus of analysis for the situativist is "at the level of interactive systems that

include individuals as participants, interacting with each other and with material and representational systems” (p.7).

Recently, two books have been published, Anderson (2007) and Sfard (2007), that set out updated theoretical frameworks from the two perspectives. There is still a very clear division between the two. Anderson (2007) involves the development of the ACT-R (Adaptive Control of Thought - Rational) architecture for creating models of human thinking, using data from human participants solving problems together with fMRI scans of brain activity in order to provide evidence for the validity of those models. The models created in ACT-R are capable of learning from instruction, but not spontaneous learning, or learning from peers. So whilst Anderson’s perspective has expanded in order to incorporate some neuroscientific thought and method, there has been no incursion into sociocultural thought. Sfard (2007) develops a theoretical framework centred on ‘commognition’, emphasising the fact that cognition is communication. However, the model is firmly sociocultural, and Sfard is explicit about the fact that she is concerned with thought that is exclusively human; that involves language. This approach precludes the researcher from incorporating aspects of cognitivist method into the study of learning.

## **THE LANGUAGE PROBLEM**

This paper focuses on language in the reconciliation of perspectives. In fact, rather than treating this exercise as a reconciliation this paper, as is suggested in the title, takes the position that a more fruitful approach may be to abandon perspectives in favour of an aperspectival approach; binding the researcher to as few ontological and epistemological commitments as possible. This paper has already highlighted language as a key factor maintaining the dichotomy of perspectives. Researchers on both sides claim that their point of view has been misunderstood or misinterpreted by the other side. The problem to solve is that of dealing with a dichotomy that is obstructive and misleading. The solution we want to achieve is a theoretical framework that allows us to talk about individual cognition and learning, about sociocultural objects and processes and learning, and about interactions between these. Two approaches suggest themselves. The first is Derrida's (1997) deconstruction. Deconstruction feeds on dichotomy, subsuming the two sides of a dichotomy within a more comprehensive account. The second is Wittgenstein’s (1953) philosophy of language. Wittgenstein's later period involved a rejection of philosophical problems, and a claim that what appeared to be philosophical problems were in fact linguistic puzzles.

These two approaches, the deconstruction of the cognitivist/socioculturalist dichotomy, and the reconstitution of problems of ontology as problems of language, appear entirely compatible. In fact, it seems that to apply one approach is to apply the other. A demonstration that what appears to be a dichotomy is in fact an incompatibility of two language games appears to be an instantiation of

deconstruction. A deconstruction of a dichotomy appears to be equivalent to the founding of a language game in which that dichotomy dissolves. So we aim to solve the problem of this obstructive dichotomy then, by applying an analysis informed by Derrida and Wittgenstein. The primary resources for the framework are the constitution of the research process as a set of language games and the use of a semiotic account in order to construct the framework. Some previous literature has considered a possible role for this form of analysis in mathematics learning research, although the focus has generally been on the use of post-structuralist, semiotic analysis as a means of describing classroom activity (e.g. Brown, 1997; Evans & Tsatsaroni, 1996); and as such these have taken a sociocultural perspective on mathematics learning. In this paper, we are working at a different level, addressing questions of mathematics learning more generally.

### **THE PROBLEM: A RESEARCH EXAMPLE**

This section presents an example of research in mathematics learning, addresses the issue of what exactly is the problem with the existing dichotomy of perspectives, and introduces questions that are difficult to engage with within this dichotomy.

Jay (2009) makes use of priming protocols, employed within experimental method, to demonstrate a relationship between number knowledge and strategy use in young children. The study focused on children's use of the 'tie' strategy, for solving near-double single-digit addition problems with solutions greater than 10. For example, the problem ' $7+8$ ' could be solved by solving ' $7+7+1$ ', making use of the 'doubling fact' ' $7+7=14$ '. In this study, children between 7 and 9 years of age took part in two activities; a) solving a set of single-digit problems with solutions greater than 10, reporting the strategy used following each problem (children had a free choice regarding strategy), and b) completing a set of priming trials designed to test for the automaticity of activation of doubles in response to single-digit stimuli (testing whether '7' activates '14', for example). The priming trials were based on previous research demonstrating automatic processing of numerals and relations amongst numerical information by, for example, Garcia-Orza, Damas-Lopez, Matas and Rodriguez (2009) and Reynvoet and Brysbaert (2004). The sample of children was divided into two groups, one group consisting of all of the children who used the tie strategy at least once whilst working through the set of addition problems, and a second group consisting of all those who did not. The two groups were then compared with reference to data from the priming trials. This analysis showed that only the children using the tie strategy showed evidence of automatic activation of doubles. This in turn suggests that automatic activation of doubles is a key resource in children's development of the tie strategy.

Up to this point in the description, Jay (2009) will appear to be situated firmly within the cognitivist perspective, utilising experimental psychology methods. However, this study is best seen as situated within the primary school classrooms in which the

research took place. Conversations with the children forming the sample for the study (5 classes in 2 schools took part), and with their teachers and the mathematics coordinators for the schools, made clear a number of points with a bearing on the interpretation of the findings. Firstly, all of the children involved in the study knew about doubles, and were able to give the double of a single-digit number with only a short delay. Secondly, classroom mathematics instruction had involved making children aware of the variety of strategies for solving simple addition problems, and encouraging children to make of use these strategies in order to increase the efficiency of children's problem solving, since at least Year 1 (the children in the study were in Years 3-5). Mathematics lessons took place for one hour every day for all children in both schools.

Within the wider context of the classes and the schools in which the children's mathematics education is taking place, the results of Jay (2009) raise some interesting questions. Firstly, why do some children have automatic activation while others do not? A second question is; what can we do in order to help children develop and use the tie strategy and other efficient strategies for solving arithmetic problems? This is not just a developmental issue; some children in each year group from year 3 to 5 (age 7-9) did not automatically activate doubles in response to single-digit stimuli. It is not just a matter of having had experience of doubling; all children could calculate a double without difficulty and all had significant experience of having been taught about doubles and their relevance for calculation in the classroom. Now, the question I would like to pose next is: within which perspective ought we to proceed? The problem faced at this point is that neither the cognitivist nor the socioculturalist perspective offers an appropriate language for asking the kinds of questions we are going to want to ask. This research situation is a clear example of one in which we will need to ask not about individual thought and learning processes, not about social or group cognition, and not about broader sociocultural process, at least not in isolation from one another. What is needed in order to really address the problem are questions that focus on the *interactions* between these processes. As long as the cognitivist and socioculturalist perspectives are considered separate and mutually exclusive, it is very difficult to ask question that address interactions between biology, cognition, classroom interactions and wider sociocultural objects and processes.

## **A POST-STRUCTURALIST ANALYSIS**

The purpose of this treatment is to reject the dichotomy between accounts of the individual and accounts of the group. This leads us towards a distributed, situated account of thought and learning. It also leads us to deconstruction. A first principle of this approach is to say that all we can talk about are signifiers and relationships amongst signifiers. All thought consists of signifiers and relationships amongst signifiers. This is very much related to Derrida's suggestion that there is 'nothing outside the text' (e.g. Derrida, 1997) – there is nothing we can say about thought that



is outside what we can say about signifiers. If we want to talk about learning, then we consider all aspects of learning part of the one single-order text, whether biological, cognitive, sociocultural or otherwise.

So, what do we have to do in order to demonstrate that a semiotic account functions as a framework for theoretical discussion of learning? What affordances must such a framework have? I want to claim that there are just two criteria for a theoretical framework to meet; we want to be able to demonstrate the fact that we can ask the questions that we want to ask about learning and we want to be able to demonstrate the fact that those questions are answerable. Of course, we will have to be able to say what is meant by conventional uses of terms such as 'knowledge', 'concepts', 'learning', 'mind', or at least construct meaningful and useful definitions of these terms within the framework, but that is just one example of a criterion that is necessary for the two principle criteria to be met.

The purpose for rejecting the dichotomy between individual and social accounts of learning is that it is becoming increasingly clear that any genuine, meaningful, account of learning is going to depend on being able to describe the interplay between individual and social factors. So, the kinds of questions that we want our theoretical framework to help us ask and answer include all those asked by cognitivists (lots of 'what' questions), all those asked by socioculturalists (lots of 'why' questions) and those addressing the interaction between the two.

## **APPLICATION TO EARLY NUMBER**

How is the semiotic framework going to work? What is it going to look like? This section presents an example involving a child's developing understanding of number, taking the number 4 as an example. A child, during the course of their first few years of life, will come to recognise and use several signifiers related to the number 4. Some examples: the Arabic digit '4'; the written English word 'four'; the spoken English word 'four'; the four fingers (without the thumb) of one hand; and the arrangement of 4 dots on one side of a 6-sided die. There are likely to be many other signifiers directly related to the number 4. For example, my son, at the time of writing is three years old. He identifies very strongly with the number 3. If he sees the digit '3', he will often say something like, "I'm that number". I imagine he will continue to say something like that when he is four years old, although I expect it will be interesting to observe how this kind of statement will develop as he comes to adjust his differentiation of his use of number-signifiers for their various purposes.

Now, we have said above that these are examples of signifiers related to the number 4. We might want to ask, what then is the signified? Is it '4'? This is a very important step in the development of our account. We cannot talk about a signified, only about other signifiers and relationships amongst signifiers. If we can't talk about signifieds then how do we deal with questions of meaning? We might normally want to talk about the meaning of a particular signifier associated with a signified. It might still

be useful to think in this way, but we also need to remember that we have no way, within the language system available to us, to talk about signifieds. So we try to describe signifieds – by instantiating new signifiers and new relationships amongst signifiers, in an attempt to, little by little, close the gap between signifier and signified. The gap can never be completely closed, because we are using a language system to describe and define something outside of that language system. In Lacan's (e.g. 2001) framework, we are trying to bridge the gap between the Symbolic order and the order of the Real, the unbridgeable gap referred to as the 'lack'.

So what is the meaning of the Arabic symbol '4'? Its meaning consists in the relationships that it has with other signifiers. '4' has a relationship with the other signifiers given in the list above. It has a relationship with the signifier for 'a sense of four-ness'. It has relationships with signifiers of '3' and '5', because we are habituated to seeing and thinking about '4' in its place in the sequence of natural numbers, or integers. An important point must be made here regarding this account of meaning. The statements above could be read in a relatively simplistic way, if one assumed that relations amongst signifiers are constrained by the bounds of an individual brain or mind. However, once we reject this boundary it is clear that the simplistic reading needs further development. Here again there is correspondence with Wittgenstein's (1953) account of language. 'Meaning is use' and the Private Language argument tell us that no analysis of meaning can take place entirely within the bounds of an individual brain/mind. Meaning of language (consisting of signifiers) consists in the role that language (that set of signifiers) plays in acts of communication amongst brains and minds. So, the analysis of the meaning of '4' can be developed as follows: The meaning of the signifier '4' consists in the role that '4' plays in acts of communication both intra- and inter-personally. This is close to the position taken by Sfard (2008), although Sfard appears to take a position restricting analysis of cognition to exclusively human modes of thought. It is clear from the cognition literature that the deliberative portion of human thought is largely dependent on automatic, uncontrolled processes (see for example, reviews of the Stroop effect [MacLeod, 1991] and of priming studies [e.g. Kinoshita & Lupker, 2003]). Our response to the signifier '4' is largely involuntary, at least in the first few milliseconds after perceiving it. However, involuntary or not, there is no reason for doubting that the meaning of a signifier consists in its role in an act of communication – after all, involuntary responses to a signifier are not limited to intra-personal communication. Involuntary responses are also clearly apparent in inter-personal communication (see again priming studies). This is not to say that these data regarding automatic responses contradict Sfard's theory, but rather to say that Sfard's theory does not *encompass* these aspects of what we know to be a fundamental aspect of thinking and learning.

## Defining key terms

We are in a position now where we need to define some key terms. If we want to be able to talk individual and social learning, subsuming cognitivist and socioculturalist perspectives, then we are going to need to know what we want to mean by terms such as 'concept', 'knowledge', 'learning', 'understanding', 'mind', and so on. These terms themselves have potential to act as barriers to interdisciplinarity, due to substantial differences in their definition and interpretation by researchers operating from different perspectives. Each term is a reification, brought into being by the researcher/observer. A 'concept' is nothing but an observer's definition. So the substantially different definitions of 'concept' by, for example, cognitivist and socioculturalist researchers, constitute a significant barrier to communication. It is important that such terms ('concept', 'knowledge', 'understanding' and so on) are recognised as reifications, in order to provide a means and an arena for the negotiation of their meaning. Some initial sketches of meaning might be as follows:

**Signifier:** A signifier is a unit of meaning. It might be a word, symbol, image or object. It might have a physical instantiation (ink on paper, physical object, photograph, audible sound and so on). The meaning of a signifier consists in its relation to other signifiers.

**Concept:** The network of signifiers and relationships among them, that relate to a given signifier. So the concept 'addition' consists in the set of signifiers that relate to the signifier 'addition', plus the relationships amongst them. Such related signifiers might include words like 'sum', 'add', 'plus', the symbol '+' and addition facts such as '2+2=4'. Also strategies or algorithms for solving particular addition problems (we might, as researchers/observers, refer to these as 'count-on', 'min' and so on – others will not use these names, however the algorithm itself constitutes a signifier). It is clear that any concept is a dynamic entity. Concepts change as the focal signifier is associated with new signifiers, or relationships amongst a set of signifiers alter.

**Mind:** The set of signifiers, and relationships amongst those signifiers, for some definable set of individuals. We can talk about the mind of an individual – that is the set of signifiers for that individual (the external physical objects that have meaning for that individual, plus the neurologically instantiated signifiers for that individual, plus the relationships amongst these signifiers).

**Understanding:** This seems to be a term that refers to the intersubjective aspects of 'concept'. One might be said to have understood a concept when one has developed a signification network that is sufficiently similar to that of a community with which one wants to engage with reference to that concept.

Two important points should be made apparent at this time. One is that it is clearly possible to define these and other terms within the proposed semiotic framework. That is to say that the framework appears sufficient for the discussions that we want to have about learning. The second point is that the framework makes clear the fact

that each term is a reification that, if it is to be used in any meaningful way, must be defined within a shared framework. The proposed semiotic framework appears to offer a sufficiently neutral arena for negotiation of such definitions amongst researchers from traditionally disparate or dichotomous perspectives.

## RETURN TO THE RESEARCH PROBLEM

As a conclusion to this paper, we return to the discussion above regarding Jay (2009). What can be done to address the research problem; why do some children not automatically activate doubles, and what can be done to help them? What kind of research questions arise from the kind of analysis described above? At the heart of the proposed framework is an imperative to focus on the interactions amongst cognitive and sociocultural objects and processes. To address these interactions, we describe the situation in terms of signifiers and relationships amongst signifiers. Let us begin with a description of the research situation within the individual child. Firstly, we are interested in why it is that some children automatically activate doubles in response to single-digit numbers. This is to say that we are interested in why some children exhibit a very strong connection between the signifier '6' and the signifier '12', between '7' and '14' and so on. We are also interested in the role that this set of activations plays in the use of the tie strategy.

On the sociocultural side we are again interested in how number signifiers and relationships amongst them are used, but this time we are interested at the level of interaction amongst actors in the classroom. Stahl's (2005) 'group cognition' might be a useful way to think about some of the processes that might be involved in this activity, as long as we remember that within social processes we are dealing with exactly the same kinds of signifiers and relationships amongst signifiers that we deal with within the individual.

The first thing to realise is that this approach allows now to hold some apparently contradictory things to be true. So within any given individual in the classroom, the meaning of '7' consists in the relationships that '7' has to other signifiers. So one particular child might relate '7' to 'seven', 'holding up 7 fingers', 'my age', '6' and '8' (due to proximity on the number line). Other children might relate it to 'prime', some others to '14' (double 7). So, from the cognitivist perspective, we talk about *an* individual child that doesn't exist; we talk about a generalised, average, child. In fact, just looking at the meaning of '7', it is likely that, firstly, different children will relate different signifiers to '7', and secondly, the strengths of these relationships will be different. So '7' has a different (even if only subtly) meaning for each person in a classroom. Aside from all of the individually constituted meanings, there is also a socially negotiated meaning of '7' for the *classroom*. This is formed through interactions amongst members of the group. The negotiation of meaning of signifiers in the classroom is dynamic; there will be a constant dynamic interaction between socially constituted meaning of signifiers and meanings of those same signifiers held

by individuals. In turn, meaning for the classroom, is tempered by, and interacts with, meanings of wider and external communities and groups. So, for example, each child brings with them aspects of meaning from their own families. In the context of the research situation, this opens us up to dealing with some new questions, focusing on interactions between meanings constituted by individuals, and meanings constituted, or negotiated, by the group. In general, we can ask questions about how children's knowledge, accessible to cognitive and neuroscientific methods, interacts with (influences and is influenced by) classroom activity. Specifically with regard to Jay (2009), it would be informative to ask what differences there are in terms of engagement in classroom interaction between those children who exhibit a priming effect for doubles and those who do not. It might also be possible to investigate how knowledge such as this spreads through a classroom, through group interaction.

### CONCLUSION

This paper has argued for a post-structural, semiotic, treatment of research on mathematics learning in order to reconcile traditional perspectives and allow the possibility of asking questions about interactions between individual/cognitive and group/sociocultural, aspects of learning. These kinds of questions offer an opportunity to engage with research on learning with greater depth than is possible with more restrictive, traditional, perspectives on mathematics learning research. There is much more to do in the development of this approach. Existing research conducted from multiple perspectives can be interpreted within this framework in order to understand points of conflict and potential interactions more clearly. It will be important to thoroughly test the claim made here that this framework offers the potential to fairly represent existing perspectives on mathematics learning.

The poststructuralist approach provides a means of being explicit about what aspects of what questions of learning we are interested in, without excluding other aspects from the arena. We can be free to talk about what meanings are common amongst a particular population of children in a particular domain of mathematics, including trajectories of individual development, relationships between knowledge and understanding, knowing-that and knowing-how, as well as talking about individual differences within that population. We can also talk about classroom activity and the development of socially negotiated meaning through classroom interaction. As a result of using the same language, of signifiers and relations amongst signifiers, in discussions of individual and social thought and learning, some barriers to discussions of interactions between individual and social are removed.

### REFERENCES

- Anderson, J., Reder, L. and Simon, H. (1996). Situated learning and education. *Educational Researcher*, 25 (4), 5-11.
- Anderson, J., Reder, L. and Simon, H. (1997). Situative versus cognitive perspectives: form versus substance. *Educational Researcher*, 26 (1), 18-21.

## Working Group 16

- Anderson, J. (2007). *How can the human mind exist in the physical universe?* New York: Oxford University Press.
- Brown, T. (1997). *Mathematics Education and Language: Interpreting Hermeneutics and Post-Structuralism*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Derrida, J. (1997). *Of Grammatology (corrected edition) trans. Spivak, G.* Baltimore, Maryland: Johns Hopkins University Press.
- Evans, J. & Tsatsaroni, A. (1996). Linking the cognitive and the affective in educational research: cognitivist, psychoanalytic and post-structuralist models. *British Educational Research Journal*, 22 (3), 347-358.
- Garcia-Orza, J., Damas-Lopez, J., Matas, A. & Rodriguez, J.M. (2009). "2 x 3" primes naming "6": Evidence from masked priming. *Attention, Perception & Psychophysics*, 71 (3), 471-480.
- Greeno, J. (1997). On claims that answer the wrong questions. *Educational Researcher*, 26 (1), 5-17.
- Jay, T. (2009). The relationship between number knowledge and strategy choice: what we can learn from the priming paradigm. In Joubert, M. (Ed.) *Proceedings of the British Society for Research into Learning Mathematics*, 29 (3), November 2009, Loughborough University.
- Kinoshita, S. & Lupker, S. (Eds.) (2003). *Masked Priming: The State of the Art*. Hove, East Sussex: Psychology Press.
- Lacan, J. (2001). *Écrits: a selection, trans. Alan Sheridan*. London: Routledge.
- MacLeod, C. (1991). Half a century of research on the Stroop effect: An integrative review. *Psychological bulletin*, 109, pp. 163-203.
- Neely, J. H. (1991). Semantic priming effects in visual word recognition: A selective review of current findings and theories. In Besner, D. & Humphreys, G. (Eds.). *Basic processes in reading: Visual word recognition*. Hillsdale, New Jersey: Lawrence Erlbaum.
- Reynvoet B. & Brysbaert, M. (2004). Cross-notation number priming at different stimulus onset asynchronies in parity and naming tasks. *Experimental Psychology*, 51, pp. 81-90.
- Sfard, A. (2007). *Commognition: Thinking as communicating, the case of mathematics*. New York: Cambridge University Press
- Stahl, G. (2005). Group cognition in computer-assisted collaborative learning. *Journal of Computer Assisted Learning*, 21 (2), pp. 79-90.
- Wittgenstein, L. (1953). *Philosophical Investigations*. Oxford: Blackwell.

# RECOVERING MATHEMATICAL AWARENESS BY LINGUISTIC ANALYSIS OF VARIABLE SUBSTITUTION

Kaenders<sup>a</sup>, R.H., Kvasz<sup>b</sup>, L., Weiss-Pidstrygach<sup>c</sup>, Y.

University of Cologne<sup>a</sup>, Charles University in Prague<sup>b</sup>, University of Mainz<sup>c</sup>

*How are we to understand and distinguish qualities of mathematical acquirement? The authors show with help of a school relevant example how mathematical awareness can be classified. The classification uses linguistic methods to identify factors important for conceptual understanding of mathematics. The analysis relates historical to activity theoretical aspects.*

## INTRODUCTION

Usually mathematical awareness of a student puts itself into effect in his approach to problems, a mathematical topic, in the way the student conceptualizes a mathematical object and uses it as tool. We use this aspect of student's cognition in order to relate two models describing the quality of mathematical acquirement from different perspectives both by linguistic means.

The first model takes an activity-theory approach (Weiss-Pidstrygach, 2011). Symbols, definitions, formulas, skills and approaches are linked to object-oriented activities, externalisation and internalisation. The second model uses different qualities of *mathematical awareness* (Kaenders & Kvasz, 2011, 2010) to describe mathematical aptitude. The classification is given by the three dimensions (content, skill and thinking activity) and by the quality these dimensions are linked together. The classification by different qualities of awareness is based on historical considerations and patterns of change in the development of mathematical language (Kvasz, 2008).

This approach is based on three basic convictions:

- Mathematical awareness is a *holistic* concept that unites such qualities like number sense in arithmetic, symbol sense in algebra and geometrical awareness in geometry.
- It is *topically neutral*, i.e. awareness acquired in one area can be transferred to another.
- It has different degrees that are closely related to *different degrees of rigor*.

In order to motivate a discussion about possible combining frames of these two models, their further differentiation, their relevance for mathematics education we explain our approaches with the help of an example from the school curriculum.

## VARIABLE SUBSTITUTION AT SCHOOL

How do we have to modify the graph of  $f$  in order to find the graph of the function  $g$  which is given by  $g(x) = f(x+3)$ ?

The problem is standard and appears first for a quadratic function  $f$  in Grade 8 and in the above general phrasing in the context of modeling and preparation of the “small chain rule” in Grade 10. (German curriculum)

Understanding of the mathematical question takes for granted some experience with the mathematical objects involved in the formulation of the problem as well as their relations and role in a wider context. Examples of prerequisite mathematical objects in the German curriculum are coordinate system, units, correspondence, linear functions, different representations of functional dependencies with and without calculator (value table, graph, term, text), standard functions, such as rational, trigonometric, exponential functions. The understanding and handling of these objects by students depends a lot on the attitude of the teacher and ranges from recalling definitions or knowing the calculator commands up to their conceptual use as a tool.

In the last decade, the theoretical approach to functional dependencies in textbooks became extremely pictorial, bounded to CAS supported graphs and concrete functions. On the other hand, later on, in the context of modeling, maximum and minimum value problems and integration problems are often formulated in algebraic notations (e.g.  $f(x) = \sin(ax+b)$ ).

The existing teaching and textbook culture defines a certain linguistic frame for the determination of the graph in question.

The Graph of  $g(x) = f(x+3)$  can be found by various thinking activities, using various skills, solving problems formulated in various contexts and answering differently interpreted questions, following instructions or working in a group, trying to solve a problem or to teach the rules to somebody else...

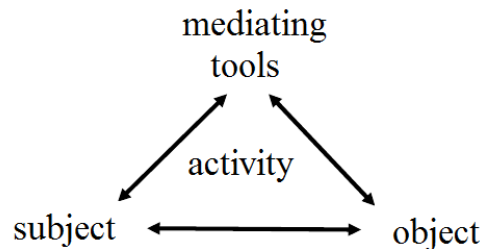
The solution itself tells little about the mathematical acquaintance of the student, it could e.g. be an imitation learned by rote. His mathematical awareness manifests itself in the way the student talks about his solution, his first intuition, the appearance and development of his assumptions, doubts he has, thought experiments he went through. Thereby his possibilities to couch generalizations, analogies, counter examples etc. will depend on the language of the mathematical concepts of which the objects involved make a part.

In different concepts the denotation given to the objects can coincide: the (pointwise defined) *polynomial function* is also the *polynomial function* defined by its coefficients and is also the *polynomial function* as an element of the set of all continuous functions.

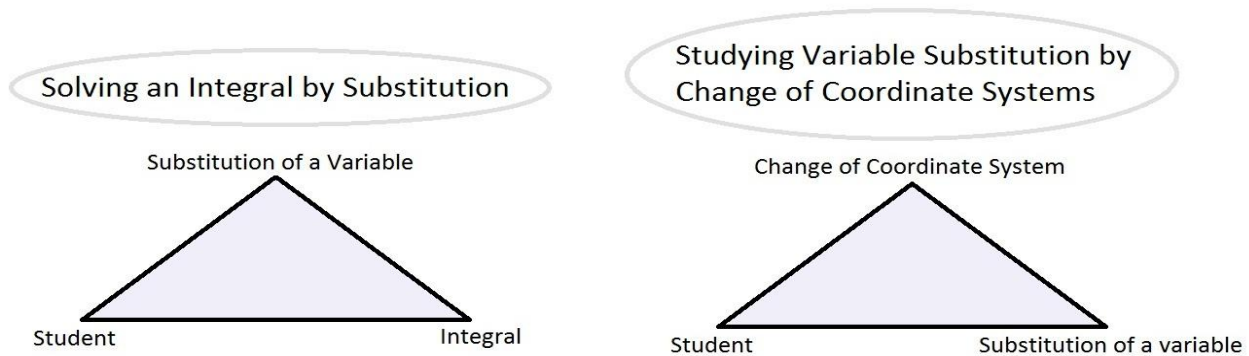


The meaning of the mathematical object as a concept itself is constituted by different contexts the object can be embedded into (part of existing mathematical theories), as well as by the problems which can be solved by using it as a tool. The tool-object-duality was explicitly introduced by R. Douady and used in the context of didactical engineering for concept development (e.g. Douady, 1997). We interpret this principle in an activity frame in order to change the constructivist perspective by a social-historical one.

Speaking in the terminology of activity theory: the mathematical concept “substitution of variables” can in the model of an action appear in two positions: as an object and as a mediating tool. The general model used in activity theory can be represented by the diagram on the right.



This general scheme can be applied to a concrete mathematical problem in two different ways (see fig. 1).



**Figure 1: Substitution of variables as an object and as a mediating tool.**

We can for instance consider variable substitution as a special case of a considerable more general method (concept): structure preserving transformations. Typical problems, which are solved by using structure preserving transformations are: finding a representative with the same structure but easier to handle (e.g. nicer coordinates) or transforming a given object in order to get a whole class with the same properties (possible solutions of an equation). We can also look at variable substitution as special case of the method of introducing coordinates, i.e. coordinatization. Typical problems handled by this method are finding explicit solutions in local coordinates (Taylor expansions, equations on manifolds...)

In other words the given mathematical object can be conceptualized in different ways, creating different methods for problem solving. Also the mathematical object on its own can be defined in different ways using different mathematical languages – and therefore predetermining different conceptual developments.

In the following we sketch different approaches to find the graph of  $g(x) = f(x+3)$  and indicate the concept it belongs to. To this end we represent the respective correspondence given by value table, graph or term. The students are used to situations in arithmetic where '+3' indicates '3 to the right' or '3 upwards'. For the graph of  $g(x) = f(x)+3$  this rule of thumb seems to be consistent. Finding the graph of  $g(x) = f(x+3)$  therefore causes for many students an intuitive conflict.

### POSSIBLE APPROACHES TO FINDING THE GRAPH OF $f(x+3)$

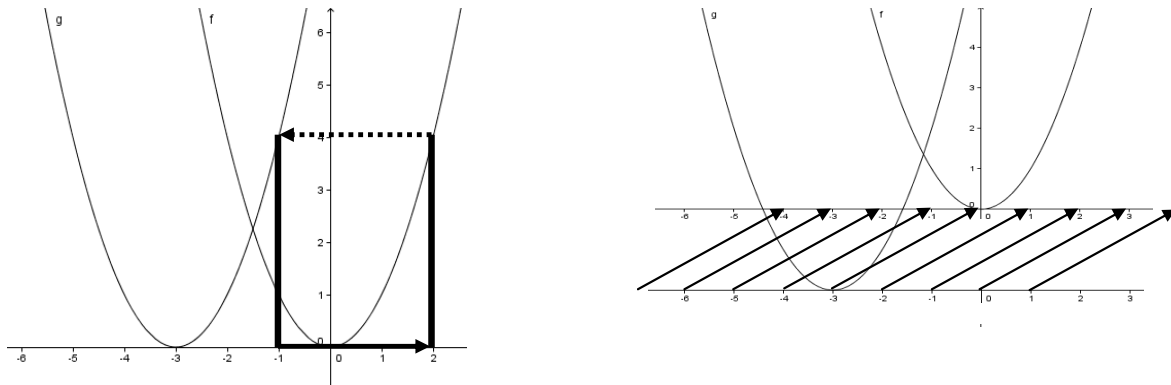
The problem of finding the graph of  $g(x) = f(x+3)$  has a rich mathematical and didactic structure. There are many different approaches which the teacher can adopt. We would like to mention ten *examples*:

- 54 Given a function  $f$  by  $x \mapsto f(x)$ , take the definition of  $g$  and evaluate for any  $x$  the function  $f$  at  $x+3$ . Draw the graph of  $g$  from the table of values.
- 55 Insert the term of a special function  $f$  in a graphical calculator, compute and plot the associated value table of the function  $g(x) = f(x+3)$ .
- 56 We can use path-time diagrams as a metaphor for a general function. For instance think of two motorcyclists driving exactly the same way. The path-time diagram of one of them is given by the graph of  $f$  i.e. at time  $x$  she is at distance  $f(x)$ . Her colleague however has started already 3 minutes earlier and has at time  $x$  already reached  $f(x+3)$ . Of course, the path-time diagram of the early rider has to be drawn three units to the left from the one who departs on time.
- 57 We can also interpret the formula  $g(x) = f(x+3)$  as recipe for *picking up values* of  $g$  (see Fig. 2). If we want to determine the height of the graph of  $g$  at the value  $x$  we go 3 units to the right and pick up the value of  $f$ . When we proceed like this several times it becomes apparent that the graph of  $f$  will be shifted 3 units to the left.
- 58 Another way to look at this particular shift to the left is to use nomograms for the composition with the linear function  $x+3$  and to combine it with the graph of  $f$  (see Fig.2).

Substitution of the variable using the set-theoretic definition of the graph: We can consider the graph of a function  $h$  as consisting of points with coordinates of the form  $(x, h(x))$ . When we move the points  $(x, f(x))$  of the graph of  $f$  with 3 units to the left, then a point  $(x, f(x))$  is moved to  $(x-3, f(x))$ . Now we look for a function  $g$  such that  $(x-3, f(x))$  is a point  $(x', g(x'))$  on the graph of  $g$ . Hence  $x' = x-3$  and  $g(x') = f(x'+3)$ .

These approaches consider the graph of a function as a pointwise defined object.

We can also let  $f$  be the graph of a function, perceived as a curve, drawn or plotted in a coordinate system, representing a function as an entire object.



**Figure 2: ‘Picking up’ values of  $g$  and combination of  $g$  with a nomogram.**

In this situation, examples to find the graph of  $g(x) = f(x+3)$  are the following.

- 59 Using a DGS device for shifting the curve around (e.g. GeoGebra or appropriate applets). Insert the term of a special function  $f$  and plot the corresponding graph. Shifting the graph changes the term and vice versa. It turns out that a movement to the left corresponds to adding a positive constant to the argument  $x$  of the function  $f$ .
- 60 By variation of the graph by e.g. dynamical geometry, using a parameter  $a$  one can see how the graph of a function  $g(x) = f(x+a)$  behaves. This general observation can be confined to  $a = 3$ .
- 61 The shift  $x \mapsto x+3$  can be seen as a shift of the coordinate system where the origin is moved 3 units to the right. The graph of the function  $f$  remains unchanged in the plane and hence moves – seen from the coordinate system – to the left.
- 62 Any function depending on a variable  $x$  can be *developed around*  $x_0 = -3$ . That is to find a function  $g(x')$  with  $g(x') = f(x'+x_0)$ . The approved way to do this is to substitute  $x$  by  $x'+x_0$  or to write  $f(x) = f((x-x_0)+x_0)$ .

The described interrelation between definition, embedding, conceptualization and operationalization of mathematical structures shows that parts of the zone of proximal development for a mathematical activity are defined and can be understood in linguistic terms. The language in which the relevant mathematical objects are named and presented provides a presetting for the local scaffolding, in particular the language for diagnostics of the zone of actual development and variations of the task. The compatibility and transferability as a tool for problem solving depends on the grammar of the concept (relations, hierarchy, dependencies between objects inside the local theory) and possible changes of perspective for the speaker (existence of a

paradigm). It seems to us that these linguistic data structure the zone of proximal development in the way in which speech and motivation are interacting with thinking.

## LINGUISTIC ANALYSIS OF THE EXAMPLE $f(x+3)$

For the conceptual framework of *mathematical awareness* we distinguish three aspects *contents*, *thinking* and *skills* as three main dimensions in which mathematical aptitude can be positioned. The basic idea is: the quality of the respective mathematical awareness is not an additional dimension in this diagram but it qualifies the way in which the contents, skills, and thinking are connected, thus it qualifies *the way in which* for example someone *argues* in *arithmetic* by *visualizing* or someone *proves* in *calculus* by *algebraizing*.

On the level of contents we distinguish arithmetic, synthetic geometry, algebra, analytic geometry, calculus, logic, set theory, probability.

Introduction of other subjects would be possible. The ordering of the topics follows roughly the order of historical development of mathematics, and also the growing complexity of the mathematical language (cf. Kvasz, 2008).

The dimension of skills is separated from the dimension of the thinking activities. We address the following skills: to count, to calculate, to draw, to construct, to symbolize, to algorithmize, to visualize, to recognize patterns, to establish dependencies, to use limit transitions and to employ language. Particular skills are best acquired in corresponding contents, as for instance counting in arithmetic, drawing in synthetic geometry, etc. Nevertheless, one of the reasons for separating skills from contents is to emphasize the possibility (and actually the necessity) of transference of skills from one content to another. Thus we usually apply addition to numbers, but we can add intervals, polynomials, vectors, functions, etc. So mathematical awareness is the awareness of the transferability of skills from one content to another. It is precisely the failure of many forms of mathematical instruction that particular skills are strictly tight to their corresponding content. Competence models, on the other hand, are tight to certain practical requests. Their solution requires different skills but remains on a certain level of thinking. In our model the relations between contents, skills and thinking allow a great variety of combinations.

We distinguish the following possibly not exhaustive list of thinking activities: to observe, to formulate, to argue, to explain, to verbalize, to classify, to define, to prove, to confine, to generalize, to vary, to concretize, to analogize, to structure. As in the previous case, also in the case of thinking, the role of an independent dimension emphasizes the manifold relations of thinking activities with the various topics and with the different skills. The student needs to master sufficiently a particular skill in order to be able to explain, define, or generalize a mathematical

phenomenon. One of the important aspects of the mathematical awareness is the awareness of the adequate degree of rigor and justification that is sufficient for a particular calculation or construction.

Mathematical awareness then is a quality, how our knowledge of some mathematics contents, skills and thinking are linked together. It is not an extensive notion but describes intensity of the amalgam, which links these three aspects with each other. It is rather a color or a taste than a position of some mathematical ability. The list of the different qualities is possibly not exhaustive:

- social awareness
- imitative awareness
- manipulative awareness
- instrumental awareness
- diagrammatic awareness
- intuitive awareness
- experimental awareness
- strategic awareness
- contextual awareness
- argumentative awareness
- logical awareness
- theoretical awareness

Here we want to discuss the possible different linguistic indications to varying qualities of mathematical awareness that we can think of when we consider the different ways to find the graph of  $g(x) = f(x+3)$ .

#### **i. Social awareness**

This is the first possible level of awareness and it might be that many people stay on that level all their life. A solution to the above problem is that ‘+3’ in  $f(x+3)$  has to be interpreted as ‘3 units to the left’ since the teacher did so in class. Moreover, also the fellow students do it that way.

#### **• Imitative awareness**

The above insight can also be explained by the teacher step for step. It is possible to reproduce each step of the argumentation without understanding its complete strand. And, look, it works. Also this very classical type of awareness is of great importance for the understanding of mathematics. To any student this approach gives a possibility to betake oneself in the middle of the subject guided by the authority of the teacher.

#### **• Manipulative awareness**

If we look at the substitution of the variable using the set-theoretic definition of the graph in *example 6* and the development of the function around  $x_0 = -3$  in approach i), then we see that still an insight in the manipulations with variables is necessary. The procedure can be understood just from the manipulations with  $x = x' + x_0$  and  $f(x) = f((x - x_0) + x_0)$ . In particular with polynomials we can use long division. The

solution is obtained by applying these two mechanical rules in the algebraic expressions.

- **Instrumental awareness**

Plotting of a value table or using a DGS device for shifting the curve around yields instrumental awareness. It is true, that the students know the result only thanks to the use of the instrument, but the teacher would compromise his credibility if he would insist that they don't know the solution yet. Of most things in life we neither have a better knowledge nor a higher form of awareness. The instrumental awareness is what the CAS-system or likewise a graphical device tells us: the result is correct – it came out from the computer and I do not know how – not more but also not less.

- **Diagrammatic awareness**

Using  $g(x) = f(x+3)$  as recipe for *picking up values* in **example d)** or the interpretation by a nomogram in **example e)** as well as the shift  $x \mapsto x+3$  of the coordinate system in **example i)** are ways to come to a diagrammatic awareness. In any branch of mathematics pictures, graphs and diagrams play a central role. We can represent for example, arithmetic relations by dot pictures, functions by graphs or nomograms, polynomials by Newton diagrams. It is possible to argue, to define and to perform most thinking activities in a diagrammatic way. Diagrammatic awareness is indispensable for understanding mathematics, although there is always a risk by stopping at the level of diagrammatic awareness to prevent further growth to logical and theoretical awareness.

- **Intuitive awareness**

The expectation, based on experiences in arithmetic, that '+3' indicates '3 to the right' or '3 upwards' is an example of a wrong intuitive awareness. The shift of the coordinate system to the right in **example i)** or the adoption of a nomogram in **example e)** on the other hand are able to foster a better intuitive awareness. In some cases there exists a feeling, which leads to a hypothesis. That is different from experimental awareness since there we only find an assertion to be true in the case of some cases.

- **Experimental awareness**

The calculating and plotting of a value table in **example a)** and **b)** and the shifting of the graph in a DGS device in **example g)** founds an experimental awareness. It is nothing but: I tried it, and it worked out this way. Mathematicians know also the converse experience: Even if one has proven something in a general setting, a concrete calculation with the predicted result is not necessarily superfluous and can very well be an enhancement of one's awareness. More generally, experimental awareness can be the study of particular cases for a more general situation without worrying about the legitimacy of the particular steps supplemented by heuristic arguments.

- **Strategic awareness**

Also to find the graph of the function  $g(x) = f(x+3)$  needs an appropriate strategy depending on the context. In trigonometry other ways seem appropriate than by the idea of moving the coordinate system. Especially for problem solving, at least since Poincaré, Hadamard, Polya and Schoenfeld, mathematicians are conscious of the fact that one needs more than just factual knowledge and skills. What is needed is the knowledge and the experience of steps one can take even if one does not finally know how to proceed successfully. Polya introduced for this the term *heuristics*. On the highest level, also building of a theory requires strategic knowledge: which facts will be put into axioms and which can better be proved. We see that strategic awareness goes beyond problem solving in the narrow sense.

- **Contextual awareness**

The metaphor of the two motorcyclists in *example 3* provides us with a context that allows getting insight into the relation between the two graphs. However, not every graph of a function may be interpreted as path-time diagram of a motorcycle. In general we can speak of contextual awareness when we attribute semantics to a mathematical topic. As well for mathematics teaching as also for professional mathematicians this type of awareness is of crucial importance. Contextual awareness constitutes *mental objects* (Freudenthal, 1991, p.19). In order to build such mental objects Freudenthal formulated the principle of *rich contexts* (p. 73).

- **Argumentative awareness**

Before proving an assertion within a theory we can argue independently of the particular theoretical framework that it must hold. For instance when we argue that it is the coordinate system that is shifted instead of the graph or when we speak of picking up the values of  $f$ , we give an argument which is not a logical one. However it serves for our argumentation. We argue by means of actions, heuristics, algorithms, and estimations etc. that help us to trust our result. We can also argue by thought experiments and or by applying metaphorical arguments.

- **Logical awareness**

Logical awareness is the awareness that ensures us of mathematical proofs and arguments. It enables us to check proofs and to distinguish heuristic arguments from logically necessary ones. But without all the other types of awareness we can hardly think logically. For example the substitution of the variable using the set-theoretic definition of the graph is a logical argument within a theoretical framework.

- **Theoretical awareness**

By theoretical awareness we understand the ability to see mathematical propositions as relative to particular theoretical frameworks, the ability to relate proposition to different frameworks and to relate frameworks to each other. Although theoretical

awareness may in some sense be considered as the highest form of mathematical awareness, it becomes a kind of fata morgana without grounding on other types of mathematical awareness. In our example one could realize that the set-theoretical construction of graph is a very general construction which is used in topology or algebraic geometry to equip the graph of a function with topological or algebraic structure.

### DISCUSSION

The work in our group was most inspiring. Discussions and demonstrations of various ways to link and join theories (cherry picking, organized networking, applied networking theories...) encouraged us to more problem and attention oriented developments of our approach. We are grateful for directing our attention to Regine Douady's model and realizations of the tool-object-duality and to John Mason's complex and multi-purpose work on the development and education of awareness. The latter helped us in particular to place our approach and to structure what is to be done next.

### REFERENCES

- Douady, R. (1997). Didactic engineering. In T. Núñez & P. Bryant (Eds.), *Learning and teaching mathematics: An International Perspective*, 373-400. East Sussex: Psychology Press.
- Freudenthal, H. (1991). *Revisiting Mathematics Education, China Lectures*. Dordrecht: Kluwer Academic Publishers.
- Kaenders, R.H. & Kvasz, L. (2011). Mathematisches Bewusstsein. In: Helmerich, M., Lengnink, K., Nickel, G., Rathgeb, M. (Hrsg.) *Mathematik verstehen – philosophische und didaktische Perspektiven*. Wiesbaden: Vieweg.
- Kaenders, R.H. & Kvasz, L. (2010). *Varying Qualities of Mathematical Awareness*. Preprint – see [www.mathedidaktik.uni-koeln.de/kaenders.html](http://www.mathedidaktik.uni-koeln.de/kaenders.html). Submitted to: *For the learning of mathematics*.
- Kaenders, R.H. & Weiss-Pidstrygach, Y. (2010). Geometrisches Propädeutikum zur Begriffsbildung in der Analysis. In *Tagungsband der Jahrestagung der GDM 2010*. Münster: WTM-Verlag.
- Kvasz, L. (2008). *Patterns of Change, Linguistic Innovations in the Development of Classical Mathematics*. Basel: Birkhäuser Verlag.
- Weiss-Pidstrygach, Y. (2011). Begriffsbildung mit tätigkeitstheoretischen Methoden. *Tagungsband der Jahrestagung der GDM 2011*. Münster: WTM-Verlag.



# HOW A GENERAL EPISTEMIC NEED LEADS TO A NEED FOR A NEW CONSTRUCT

## A CASE OF NETWORKING TWO THEORETICAL APPROACHES

Ivy Kidron<sup>a</sup>, Angelika Bikner-Ahsbahr<sup>b</sup>, Tommy Dreyfus<sup>c</sup>

<sup>a</sup>Jerusalem College of Technology (Israel), <sup>b</sup>Universität Bremen (Germany),

<sup>c</sup>Tel Aviv University (Israel)

*Abstract: Two students were asked to carry out an activity about continued fractions in the presence of an interviewer. We present two separate analyses of a central episode from two theoretical perspectives focussing on what drives the students' progress of coming to know. Making intensive use of the notion of general epistemic need, we show how such needs arise and are expressed, and how they lead to the need for a specific new construct. The paper concludes with reflections on the networking process, highlighting conditions that have supported the process and hence emergence of a more integrated view of the episode.*

### INTRODUCTION

Networking of theories has recently become an important endeavour of mathematics education (Bikner-Ahsbahr et al., 2010). We present a facet of an ongoing research project that investigates effective knowledge construction in interest-dense situations<sup>1</sup>. We found it useful to proceed by networking two theoretical approaches, the perspective of abstraction in context (AiC) and that of interest-dense situations (IDS) with particular focus on two theoretical concepts: the need for a new construct, and interest. In both perspectives epistemic processes play a central role.

AiC focuses on the construction of knowledge by individuals or small groups, using a model based on three epistemic actions: *recognizing* constructs known from previous activity, *building-with* these recognized constructs, for example to solve a problem, and *constructing* new knowledge while building-with recognized previous constructs. While the model serves to analyse the central stage of the emergence of a new (to the learners) construct, the researchers postulated that a necessary condition for such emergence is that the learners experience a need for the new construct (see Schwarz, Dreyfus, & HersHKowitz, 2009, for a detailed description of AiC).

Interest-dense situations (IDS) focus on the epistemic processes within social interactions. These are situations that are constituted by students who are collectively *deeply involved* in a mathematical problem, *constructing mathematical knowledge* in a deep way and *highly valuing* the mathematics they consider. This approach uses an epistemic actions model that comprises three collective epistemic actions: *gathering* and *connecting* mathematical meanings and *seeing* mathematical structures. Any IDS lead to structure seeing, and students who are engaged in an IDS show situational

interest. Situational interest holds when an individual experiences involvement and meaningfulness of a topic (see Bikner-Ahsbahs, 2005, for a detailed description).

In a previous paper (Kidron, Bikner-Ahsbahs, Cramer, Dreyfus, & Gilboa, 2010), we exhibited the methodology of networking the two approaches on the basis of common analyses of interview protocols from both perspectives. One benefits of networking was the insight that besides interest (IDS) and the need for a new construct (AiC), a more general epistemic need (GEN) can be the driving force that makes students to progress in learning processes according to the challenge they meet within a situation, individually and socially. The GEN is the need to develop an initially vague idea further, leading to a more definite one, in accord with Davydov's (1972/1990) view of abstraction. The GEN is experienced according to the demands of the situation and hence becomes specific, for example as a need to be more precise that is shown by student actions and can be observed.

In the present paper, we continue the previous research using a different set of data. We carried out a networked analysis of an epistemic situation from the two perspectives concerning the roles of the GEN for constructing knowledge. The general result of the analysis is a holistic view of the learning process integrating both perspectives. An important specific result is that the GEN develops, at least in this case, into a need for a specific new construct as postulated by AiC.

## SETTING AND TASK

Two grade 10 students work on a task about the continued fraction  $1 + \frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \dots}}}$  in

an interview situation. The main role of the interviewer is to support these students with hints (beginning with very weak hints) when they fail to progress. The students were asked to calculate the first seven fractions, expressing them as simple fractions; it was suggested to denote the  $x^{\text{th}}$  fraction by  $f(x)$ , beginning with  $f(0)=1$ . The students were then asked to represent the first 20 fractions as decimal numbers, to look for patterns, make conjectures, and explain why these conjectures are true.

The students discovered that the decimal numbers in even places ( $x$  even) begin with a 1, the decimal point being followed by a number of nines, and the ones in odd places begin with a 2, the decimal point being followed by a number of zeros. They also noticed that the number of nines, respectively zeros grows as more fractions are computed.

The students later observed the following rule: when the  $x$ -value passes a perfect square, the numbers of nines (zeros) increases by 1 and becomes equal to the root of that perfect square. When formulating this rule, they were talking in terms of a "space of places" (places referring to decimal places). They also noticed that this rule was valid only approximately. The rule may appear as an expression of a polynomial

growth law to the expert, but at least at this stage, the students considered it only locally, for fixed  $x$ .

## THE ROLE OF THE GEN FOR ABSTRACTION IN CONTEXT

### The GEN and the seeds for a constructing action

In the transcript (available from the researchers) we observe some phenomenological identification of patterns (in utterances 719-1353), which we may consider as *seeds for constructing actions* that take place later. By *phenomenological*, we refer to the fact that the elements of the sequence are considered as strings of digits rather than as numbers. The expression “space of places” (which we will abbreviate henceforth as SP) appears in 818 and there is some indication for an epistemic need in the efforts of the students to clarify this notion and to assign it a name. We submit that when somebody expresses the same phenomenon in different ways in an attempt to associate meaning with it, we have an indication for a GEN. This was the case in relation to the expression “space of places”. In addition to the different ways the students expressed the SP, we also discern a double interpretation of the SP: (i) SP as an interval on the  $x$ -axis, which numbers the elements, specifically the interval in which the number of nines/zeros remains the same; and (ii) SP as a part of the decimal expansion of  $f(x)$ , specifically the part containing the nines (or zeros). This double interpretation reinforces our interpretation of the SP as a seed for later constructing, as something that is not precise and needs to be elaborated.

### The Role of the GEN in constructing actions

We first relate to one specific aspect of the GEN, namely the need to understand the present situation in terms of the previous knowledge or previous experience, to engage with the challenges offered by the task. Then, we note how this need and the limitations of the previous knowledge [specifically that the previous knowledge was adequate to empirical computations while the strings of digits were explicitly written and observed] lead to other specific aspects of the GEN, namely the need to be more general and the need to clarify. Finally, we will observe how these specific aspects of the GEN lead to the emergence of the need for a new construct as postulated by AiC.

After the phase described above as phenomenological, we observe a striking change in the students’ attitude and way of thinking: They start giving reasons rather than only phenomenological descriptions. We interpret this as a consequence of the interviewer’s initiative to ask questions concerning the SP for  $f(1'000)$  (in 1354) and for  $f(1'000'000)$  (in 1397). The students express their need to understand the new situation in terms of their previous empirical experience, and express their thinking that they need to do all 1000 computations by recursion (in 1359-1362). The limitation that results (“we cannot do all the 1000 computations”) leads to the *need to be more general* and to apply their “theory” that the length of the SP approximately equals the square root of  $x$  more generally (1382).

The next initiative of the interviewer, the question “How would it work then?” (in 1424) - again causes the students to experience a limitation of their previous experience and it leads them to the need to think in a more general way. This need directed the students towards the beginning of a constructing phase in which infinity plays a role. We note the important role of the interviewer but at the same time also the fact that the notion of infinity was expressed on the students’ initiative. The students express a need to understand the meaning of “infinite[ly many] zeros” (in 1429), and “infinite[ly many] nines” (in 1431) as well as the meaning of a sequence approaching a number: “it keeps on leaning closer to zero-, closer to two, both numbers” (1427). Hence, we observe constructing actions that relate to convergence and accompany the students’ extension of the growth pattern of the SP from the initial 20 elements to 1’000, 1’000’000 and beyond, to what the students call infinity.

A first constructing action that we denote  $C_0$ , *Convergence as coming very close to ...*, appears in 1418 and more clearly in 1427. Then, we observe  $C_1$ , *the Potential Infinite* process view in 1427 (see above) but also, for example, “If one looks at it precisely, it never reaches two, even if there are infinite nines, after it there always comes seven three two whatever, can be anything the following numbers, we have not even looked at them yet, could be that they have a pattern too, but I don’t, personally I don’t see anything there (M laughs)” (in 1473).  $C_1$  is accompanied by  $C_2$ , *Infinite as a façon de parler: very, very large but finite*, as well as  $C_3$ , *the strange infinite object is legitimate only in the mathematical world*, for example in 1356:

1354. T: Because one, one nine ninth is namely one point nine nine nine nine nine nine nine nine, a-and two, because one plus nine ninth is definitely two but nine, one ninth, is zero point one one one one one

1355. M: Yes but then, if you want to make nine ninth, then it would be two

1356. T: Theoretically (M laughs)

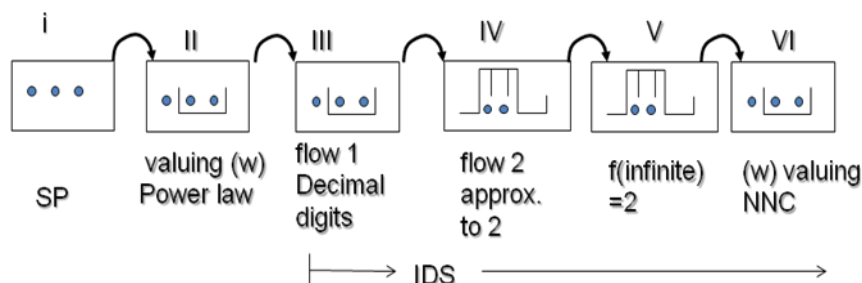
$C_3$  is similarly expressed in “If, if you insert infinity it theoretically equals two” (in 1437), which somewhat later leads them to write “ $f(\infty)=2$ ” on their worksheet. Another aspect of  $C_3$  is the transition from infinite as a *façon de parler* to infinite as a legitimate object in the mathematical world.

$C_1$ ,  $C_2$ , and  $C_3$  develop in parallel: At the same time that the potential infinite process view is developed, the students also begin manipulating the infinite as a legitimate mathematical object as they have done previously for large but finite numbers. The occurrence of so different (and somewhat contradictory) constructing actions in parallel places heavy demands on the students. This leads to a feeling of unease, of confusion, which is expressed in 1473 (cited above) and causes the expression of a need for a new construct in 1478: “The best would be of course if we had a functional equation right? Thus if one could say exactly,  $f$  of  $x$  equals (...)”. We can see this need as a consequence of the limitations of the students’ previous experience: In the present situation, they are not able any more to use what they

know from the finite case. Therefore, a need for a new view is expressed. This need leads to the construction  $C_4$ , *transition from a numerical way of thinking (with empirical results calculated by the students) to a more general way of thinking (which does not depend on specific cases)*. A new construct is needed to permit this transition. The need for this new construct is explicitly expressed in 1478. During this search, the students continue giving reasons rather than only describing phenomena. This new approach in which the students explain their way of thinking provides evidence for the passage from the seeds of constructing to the beginning of the constructing process. The seeds of constructing also influenced later constructions. This appears, for example, when the students express the distance of  $f(x)$  from 2 by means of their idea of SP (in 1534) and point out that the SP is the base 10 logarithm of the distance.

### THE ROLE OF GEN IN INTEREST-DENSE SITUATIONS

The social interactions in IDSs can be regarded from a mathematical point of view as a flow of mathematical ideas that produces mathematical knowledge in an effective way by the epistemic actions of *gathering*, and *connecting mathematical meanings*, which lead to *structure seeing*. Figure 1 shows pictograms of the six phases of the analyzed episode. All are initiated by the interviewer (see the arrows). Phase I mainly consists of gathering, in phases II, III and VI gathering and connecting are merged. In phases IV and V, the students reach structure seeing including validating structures.



**Figure 1: Phase diagram of the analyzed episode (1333-1512), phase I: 1333-1353, II: 1354-1401, III: 1402-1423, IV: 1424-1454, V: 1455-1466, VI: 1468-1512**

Epistemic actions serve to describe the flow of ideas and to investigate what drives the epistemic process in the flow and where it leads. A *flow of ideas* is a horizontal scanning of the mathematical aspects of a problem expressed in the utterances towards oneself and the other in order to describe, concretize, understand, progress, ..., but also to inform the other, to take up her idea and develop it further, negotiate, explain, ... It is an evolution of ideas associated with a given mathematical problem, building on previous experiences. Within a constructing process driven by a GEN the flow of ideas may lead to: *recognizing* an idea as fruitful, which may lead to further developing it; *connecting* the aspects together, which may lead to *building-with* a

comprehensive view; *seeing* a *structure* or *constructing* something new; checking, understanding, making concrete and justifying the structure or the construct.

For example in phase III (1397-1423), the flow of ideas refers to how the digits of  $f(x)$  for  $x=1'000'000$  may look like. The status of the power law for the length of SP had been made clear before as a conjecture that leads to estimations about it. The focus now is on how the decimal numbers look as compared to 2.  $f(1'000'000)$  cannot be computed explicitly. Therefore, the question how  $f(x)$  looks for  $x=1'000'000$ , turns the argumentation into a hypothetical direction, connecting it to the SP and to how the sequence might go on:

1413. /M: But what we do know in any case is that eeh there is a one before the decimal point, well not for one thousand and one, for thou- for one-

1414. /T: No, for one thousand and one there is a one in front of the decimal point, well no wait yes a two

1415. /M: That's an odd number, yes

1416. T: Two point zero zero zero zero zero

1417. /M: Yes because its an odd eeh place

1418. T: Yes, so its very close to two already

1419. M: Yes

1420. T: Those are then about a hundred zero or so (laughs), and then comes some different number

In the end the last idea is confirmed (1422, 1423). This flow of ideas describes horizontally how the decimal number  $f(1'000'000)$  might look without going a step further. However, this horizontal scanning process implicitly produces the aspect of approximation to 2 that is *recognized* as fruitful for answering the interviewer's question "and how would it work then?" (1424). It is further developed in the next flow of ideas as a process leading to *structure seeing*: "it keeps on going" (1425) (seed for infinite as a process), "an infinite number" (1426) (length of the decimal number), "leaning closer to zero, closer to two, both numbers" (1427), (structure seeing because of the leaning-key-idea of approximation, grasping the convergence to 2, and referring to both numbers, meaning converging from both sides), "but no never becomes 2" (sequential process of potential infinity), "there are always infinite zeros" (1429) (here the value of the digits connected to the process directs the view to the actual infinite), "it's infinite that's just it" (1430) (the length of the decimal number), "at the end there are infinite zeros or infinite nines, and there is something" (1432) (the image of the infinite length is rooted in the experience of the finite).

Here again an aspect is further developed that has been prepared by enlarging the size of  $x$ -values from 1'000 to 1'000'000. Before, the flow of ideas was concerned with the infinite length of the decimal numbers, whereas now the students take infinite as an actual value of  $x$ , which they consider substituting in a functional term:

actual infinity is reached (1434), structure seeing took place. This kind of substitution changes the view from function values to the variable  $x$ . Based on that, the students several times use an if-then consideration and a flow of ideas about the conclusion arises: if "we insert infinity" (1435), "will always be the same" (1436), "if you insert infinity, it theoretically equals two" (1437), "then it would be two" (1438). "One point nine period" (1440), "equals two then" (1442), "equals about two" (1443), "equals two" (1444), "so close, ah ok" (1445).

The students agree about the if-then-argumentation and the data but not about the conclusion. Here we have a flow of ideas leading to dispersing views. T is bound to the view of potential infinity (1437, 1443, 1445) whereas M reaches actual infinity ("we can insert infinity" 1434, see also 1442, 1444). The difference between these views is not overcome because of their incompatibility. This is an experience of limitation and brings about a need for certitude referring to and repeating what the teacher has said. M recognizes an argument of their teacher as fruitful and together they reconstruct it: manipulations leading to  $1.9999... = 2$  as learnt in school.

In addition, the flow of ideas can lead to the experience of limitations such as not being able to continue solving the problem. This can happen when the students do not have access to tools that would help them progress, but it can also happen when they have a different understanding of an aspect that cannot be overcome in this moment. However, limitations do not necessarily have to lead students to give up; they can be overcome in different directions. They may lead to changing the conditions, going back to a clearer situation, taking a more general view; they may also lead to the *need for a new construct (NNC)*. We now discuss some of these possibilities.

### 1. Giving up

In this case the students are not driven by a GEN nor do they find an adequate expression of it (no GEN); for example in phase I they are ready to give up since the flow of ideas dries out because they have experienced that the power law is not always valid: "we could probably work on this all day long" (1340) and "yes but I think it's enough" (1341).

### 2. Changing the conditions

Changing the conditions with a potential for progress involves a mixture of a need for clarification and a need to progress: The demand to use the function value of 1000 that seems too big to compute, and the attempt to use the power law that cannot be applied directly, make them change the conditions. Instead of 1000 they take 100 first and then 1024, since these are numbers with well known square roots (1370).

### 3. Going back to a clear situation

This reaction is concerned with a need for certitude: the demand to look at 1'000'000 makes them change the conditions first and then a need for *certitude* leads them to

say what “but what we do know in any case is that eeh there is a one before the decimal point, well not for one thousand and one, for thou- for one-“ (1413), which is confirmed by T in 1414, and again in 1477.

#### **4. Taking a more general view**

This may be connected to a need to be more general: the demand of the interviewer to look at 1000 or 1001 led to a limitation “one can’t say anything else” (in 1359). This caused a need for being more general “wait, we have our theory” (1360) that they applied afterwards.

#### **5. Expressing the need for a new construct**

This need arises when the students describe what they would need in order to continue and why they do not have it yet: In 1473, T experiences a deep personal limitation, “personally, I don’t see anything there” (see above for the full citation) although there is a specific epistemic need to be general. T tries to transfer his images of finite numbers to the infinite ones but this is not observable. Therefore he experiences a lack of tools to continue and that confuses him. The GEN is expressed by a need for a new construct in 1478 (see citation above).

#### **Situational interest empowers the students to act epistemically**

In 1467, the students turn to the task of justifying their conjecture, which makes them laugh. As before, the students do not take the demand to justify seriously. However, they value this task now as being more difficult. In 1469, the interviewer acknowledges “I find your last aspect just now most interesting”. This changes the situation completely. M confirms: “Yes that’s really is interesting how“ (1470), without laughing. T describes what is really interesting: “yes, so theoretically it keeps on leaning closer to two” (1471). Both students take the aspect of approximation as what is most interesting. The number 2 is understood as the leaning point including the experience they have made (1471): theoretically, by hypothetical thinking, approximation to 2 is understood *as leaning to 2* (from potential infinity in the direction of actual infinity). The term *leaning (anlehnen)* is non-conventional in this context also in German, the language of the students. Explaining why causes confusion, a deep personal limitation. The interviewer’s repeated demand to explain why, makes M express the GEN as the need for certitude “let’s look at the beginning again here” (1477) while T expresses a need for a new construct “it would be best if we had a function equation...” (1478), valuing highly the construct he looks for, compared with the less valuable representation they have “we only have a functional equation just dependent on the variable before” (1480). Directly after that, the students refer to the need for a new construct expressing their willingness: ”right, shall we try to discover something like that ‘cause that would be” (1485), “on that depends on x right?“ (1486), “Yes, so f of“ (1487).



The above shows the students' deep involvement accompanied by meaningfulness, ending up with valuing highly what they do not yet have, expressing a *need for a new construct (NNC)* by situational interest. This scene also shows how the students' interest is caught: The interviewer values aspects as being most interesting. This empowers the students to act: they immediately deepen their involvement, express their personal experience of limitation, which caused their need for a new construct, which in turn empowered their willingness to construct it. Thus, interest is held, since the NNC and situational interest are mutually reinforcing the students' progress.

## **REFLECTING ON THE NETWORKING PROCESS**

AiC postulates that without a need for a new construct (Hershkowitz, Schwarz, & Dreyfus, 2001) the process of constructing such a new construct will not be initiated. When trying to identify such a need for a new construct in data sets, we discovered that it was not always clearly identifiable. However, IDS showed that the learning process was driven in such cases by far more general epistemic needs such as described in this paper as GEN. The previous research also showed that the GEN was closely linked to seeds for later constructing actions. The main contribution of the present paper is to show that a GEN cannot only drive the epistemic process but may lead to a need for a specific new construct.

Another important result is an integrated view of how the individual and social construction of knowledge promote and call for each other. The flow of ideas as a social flow describes the process as an evolution of ideas in the community. This can happen by gathering and connecting pieces of knowledge that may bring to the fore an aspect that can be recognized as fruitful to be further developed, driven by a GEN or even a specific need for a new construct. This may lead to limitations that again may reinforce the expression of a GEN meeting the situational challenge. Hence, GEN and experiencing limitation fruitfully interact.

We conclude this paper with a reflection on the networking process and on conditions that influenced it positively and negatively. The networking process concerned the different roles of the GEN as seen from each perspective. For AiC, the role of the GEN is its relationship to the seeds of constructing and to the emergence of the need for a specific new construct that marks the beginning of the construction process. In IDS, on the other hand, the role of the GEN is related to situational interest, empowering the students to progress in the construction of knowledge. The GEN is transformed into actions progressing within the flow of mathematical ideas.

The networking process was basically enabled by the following features, in which there was a synergy between the two research teams, already at the outset:

- a. The research questions asked by the two teams are rather closely related and refer to how knowledge is constructed by means of epistemic processes and what factors influence processes of constructing knowledge.

- b. The theoretical frameworks used by the both teams are based on epistemic actions; while these are not identical, they are complementary.
- c. The methodologies used by both teams include fine-grained microanalyses of interview protocols. This afforded us the opportunity to consider a single data set at a similar level of depth from different points of view: We used a three stage cross-analysis methodology that helped overcome some of the difficulties listed below: (i) separate analyses by each team; (ii) re-analysis by each team, in view of the other team's analysis; (iii) common analysis by both teams in meeting.
- d. The notion of GEN that emerged from a first stage of networking (the previous study) has become an integral part of both perspectives and hence a catalyst for further stages of networking (the current study). Hence, the role of the GEN has turned from that of a research result into that of a base for further research using both theoretical frameworks in unison.

On the other hand, some differences between the approaches of the two teams caused difficulties that turned out to be fruitful for challenging our cross-analyses:

- a. The AiC approach puts the cognitive aspects in the centre, considering social aspects as important but secondary, whereas the IDS approach considers the social aspects to be of primary importance in constructing knowledge.
- b. Therefore, there are differences between the natures of the sets of epistemic actions, which turned out to provide complementary insights.
- c. The two approaches espouse somewhat different views of what constitutes construction of knowledge, in particular what kinds of knowledge can be the object of a constructing process and how it could be fostered.

Since the two research teams have already been working together for more than two years, we have reached a state of profound mutual understanding. Hence, each perspective contributes to deepen the insight of the other one into the development of the students' process of constructing knowledge.

- 1 Project supported by the German-Israeli Foundation for Scientific Research and Development (GIF) under grant number 946-357.4/2006.

## REFERENCES

- Bikner-Ahsbals, A. (2005). *Mathematikinteresse zwischen Subjekt und Situation* [Interest in mathematics between subject and situation]. Hildesheim: Verlag Franzbecker.
- Bikner-Ahsbals, A., Dreyfus, T., Kidron, I., Arzarello, F., Radford, L., Artigue, M., & Sabena, C. (2010). Networking of theories in mathematics education. In Pinto, M. M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 145-175). Belo Horizonte, Brazil: PME.

- Davydov, V. V. (1972/1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., & J. Teller, Trans.). Reston, VA: NCTM.
- Hershkowitz, R., Schwarz, B., & Dreyfus, T. (2001). Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education*, 32 (2), 195-222.
- Kidron, I., Bikner-Ahsbabs, A. Cramer, J., Dreyfus, T., & Gilboa, N. (2010). Construction of knowledge: need and interest. In Pinto, M. M. F. & Kawasaki, T. F. (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 169-176). Belo Horizonte, Brazil: PME.
- Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). *The nested epistemic actions model for abstraction in context*. In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11-41). London, UK: Routledge.

# MATHEMATICS LEARNING THROUGH THE LENSES OF CULTURAL HISTORICAL ACTIVITY THEORY AND THE THEORY OF KNOWLEDGE OBJECTIFICATION [1]

Lionel N. LaCroix

Brock University, Canada

*This paper draws upon analysis of an adult student as he learns to read fractions-of-an-inch on a measuring tape with the researcher as tutor. This provides a basis for comparing two activity-theoretical perspectives: cultural historical activity theory as developed by YrjöEngeström and his collaborators and Luis Radford's recent elaboration of activity theory, the theory of knowledge objectification. The former perspective provides a general framework for analyzing activity as a whole, whereas the later draws attention to particular genetic aspects of the mathematics learning and thinking of individuals. Insights gleaned through the use of each theory in analysis of this activity are summarized in-turn and contrasted for the purpose of showing the utility of each approach for the analysis of mathematics learning.*

## INTRODUCTION AND THEORETICAL FRAMEWORKS

Activity theory is a cross-disciplinary framework for examining how humans purposefully transform natural and social reality, including themselves, as a culturally and historically situated, materially and socially mediated process. Originating in the dialectical socio-cultural psychology of Vygotsky, this work was subsequently developed into a theory of activity by his student and colleague, A. N. Leont'ev (cf., 1978) and others. Today in the west this perspective is often associated with the work of Engeström and his collaborators and is referred to as cultural historical activity theory or the acronym CHAT, emphasizing the essential situated nature of activity. It continues to develop in different ways, highlighting both its complexity and the fact that CHAT remains, in many ways, a work in progress.

The impromptu tutoring session involving a pre-apprentice in the pipe trades and the researcher that formed the basis of this analysis focused on learning to read fractions-of-an-inch to sixteenths-of-an-inch; an essential skill in this training program and the workplace. The present paper uses an analysis of this workplace training course as a whole as well as an in-depth analysis of the targeted tutoring session using two complementary perspectives, Engeström's interpretation of CHAT and Radford's theory of knowledge objectification (TO). This, in turn, provides a basis for comparing and contrasting these two perspectives. This analysis is part of a larger study of mathematics learning within apprenticeship training conducted in a number of pre-apprenticeship and apprenticeship training programs in the construction trades.

### **A brief overview of Engeström's interpretation of activity theory**

Engeström's work develops the implications of Leont'ev's ideas and systematizes them in the form of an activity system. (Because of space restrictions, a figure showing Engeström's well known triangular activity system model will be provided only in the results section.) Included as the co-mediating elements within an activity system are the subject, community, tools (including signs and artifacts), rules or norms, and division of labour; all oriented towards the object and outcome of the activity (cf., Engeström, 1987, 1993, 2001). And, for any activity system, it is the meeting of a human need that motivates the activity.

Engeström also articulates a number of key principles of activity. The following of these are most pertinent to the present assessment of this perspective for the analysis of mathematics learning:

- that a collective, artifact-mediated and object-oriented activity system, seen in its network of relations to other activity systems, is taken as the prime unit of analysis...;
- [that] an activity system is always a community of multiple points of views, ... and interests....;
- [that] activity systems take shape and get transformed over lengthy periods of time. Their problems and potentials can only be understood against their own history...; [and]
- [that] contradictions [play a central role] as sources of change and development [of an activity], ...; (Engeström, 2001, pp. 136-137).

The large scale view of activity taken by this perspective considers learning in terms of fundamental qualitative changes in an activity system as a whole, a process that Engeström calls expansive learning. This occurs as a result of deliberate efforts by participants over time to resolve conflicts and contractions that are an inherent part of any activity system. Engeström's theorization provides little, if any, explicit direction for understanding the place of mathematics within activity nor provides details relating to the learning processes of individuals within activity systems. Hedoes, however, acknowledge the need for context-specific concepts and methods to be created and employed when applying CHAT to particular empirical cases (Engeström, 1999,2008).

### **A brief overview of Radford's theory of knowledge objectification**

Based on his reading of Vygotsky's semiotics, Leont'ev's activity theory, and the more recent work of Felix Mikhailov and Evald Ilyenkov, Radford has developed the TO specifically for unpacking nuances and processes of the mathematics activity and learning of individuals from a cultural-semiotic activity perspective (cf., Radford, 2006, 2007, 2008b). In contrast to Engeström, Radford's work focuses on specific aspects of the consciousness, learning, and being of individuals as well as the

semiotic and social dimensions of mathematics activity from an activity perspective. Foremost in Radford's theorization are emphases on:

- 1) the intimate and dialectical relationship between human thinking—including mathematical thinking—and the material and cultural world,
- 2) the central role of semiotic resources used within culturally and historically situated practices and social interaction in mathematical activity and learning, and
- 3) the twin dialectical processes of *subjectification*—the process of becoming, and *objectification*—the process of making sense of and becoming critically conversant with the cultural-historical logic with which systems of thought, such as mathematics, have been endowed (see also Radford, 2008a, 2009);

In the TO, learning is conceptualized as an interactive and creative acquisition of historically constituted forms of thinking. Such an acquisition is thematized as a process of *objectification*; that is, as a process of making sense of and becoming critically conversant with the cultural-historical logic with which systems of thought, such as mathematics, have been endowed (see also Radford, 2008a, 2009). Radford's concept of objectification is a refinement of Vygotsky's notion of internalization in that it emphasizes the dialectical relationship between the subject and the cultural object being attended to. *Semiotic means of objectification* (SMO) is the empirical reflection of this process. This refers to the use of semiotic means to draw and sustain the attention of others and one's own attention to particular aspects of mathematical objects in an effort to achieve stable forms of awareness, to make apparent one's intentions, and/or to carry out actions to attain the goal of one's activity. Radford identifies the following three processes as forms of SMO from his empirical research of collaborative mathematical problem solving and learning:

- 1) *Iconicity*—the process of noticing and re-enacting or re-voicing significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification (Radford, personal communication, September 29, 2008),
- 2) *Semiotic nodes*—places in mathematical activity where multiple semiotic resources are used together and in a coordinated manner to achieve knowledge objectification. "Since knowledge objectification is a process of becoming aware of certain conceptual states of affairs, [changes in] semiotic nodes are associated with the progressive course of becoming conscious of something. They are associated with layers of objectification" (Radford, 2005), and
- 3) *Semiotic contraction*—the process of coming to recognize and attend to the essential elements within an evolving mathematical experience; and making one's semiotic actions compact, simplified, and routine as a result of this acquaintance with conceptual traits of the objects under objectification and their stabilization in consciousness (Radford, 2008a).

It should be noted that the process of mathematics learning or objectification can be accounted for readily within the theory of knowledge objectification through analysis of social interactions and the use of semiotic means of objectification within mathematics learning activity (Radford, 2008b).

## **METHOD**

### **Data collection**

The empirical data that provides the basis for the present analysis was collected in a pre-apprenticeship training class for the pipe-trades conducted at trade union run school in British Columbia, Canada. This program involved classroom work as well as practical work in the shop. The course content was selected to give the pre-apprentices a head start with important skills that would be addressed subsequently in the early years of their formal apprenticeship training in a number of different pipe-trades.

The researcher visited the class extensively over its eight-week duration and served as a math tutor for any pre-apprentices who asked for his help. The researcher also observed pre-apprentices and engaged them in discussion about their mathematics related coursework as they were working on it. The mathematics related activity of individual and groups of pre-apprentices, working either on their own or with the researcher, was documented using a video camera and field notes, and copies of the course print materials and the pre-apprentices' written work were retained for analysis. The researcher also conducted ongoing formal and informal discussions with the course instructor and administrator of the program, as well as many students and kept field notes of these encounters. The data used in the present analysis was drawn from this collection of data. This particular 33-minute tutoring session was selected for fine-grained analysis from the approximately 35 hours of video data from this pre-apprenticeship class on the basis that it was, by far, the longest episode of a student focused on a particular mathematical object (here the pattern of binary fractions-of-an-inch used in linear measure) that was central to this vocational training. This, in turn, provided a unique opportunity to examine ways in which a pre-apprentice's thinking developed in relation to this mathematical object and was reflected both in his actions and in his interactions with the researcher-as-tutor.

Prior to the data analysis (as part of the larger research study involving multiple workplace training sites) the researcher made extensive visits to an earlier session of this same pre-apprenticeship course as well as a fourth-year plumbing apprenticeship program at a local technical college. Extensive visits were also made to all three levels of an iron-working apprenticeship program also at the local technical college and short visits spanning one to a few days were made to a variety of other apprenticeship programs for a variety of construction trades. In all cases the focus was on observing and documenting the mathematics related parts of these programs.

This experience served to inform the researcher's analysis of the activity within the target pre-apprenticeship program.

### **Data analysis**

The multi-semiotic analysis of this activity takes place on two levels, reflecting the breadth of foci of the two theoretical perspectives used. The various elements of the activity system of the pipe-trade training program, for example, were discerned from the entire corpus of related data identified above. At another level the multi-semiotic analysis of the pre-apprentice's and the researcher-as-tutor's joint activity during the tutoring session began with the construction of a verbatim transcript of the entire tutoring session from the video recording. This included the construction of a detailed account of significant actions including the use of various semiotic resources, gestures, body position, and artifacts. Episodes in the data were then coded, first to identify the various aspects of reading the fraction-of-an-inch pattern being attended to, and then to identify actions related to both the pre-apprentice (who will henceforth be referred to as "C") and the researcher-as-tutor's (who will henceforth be referred to as L) objectification of the pattern of fractions of an inch on the measuring tape and subjectification as participants within this activity. At times this process required slow motion and frame-by-frame analysis of videotape to assess the role and co-ordination of spoken language with the use of artifacts and gestures.

## **RESULTS AND DISCUSSION [2]**

### **A summary analysis using Engeström's interpretation of activity theory**

A detailed analysis of the elements of the tutoring session activity system is provided using a triangular activity system model in Figure 1. The lines between the nodes or elements of the activity are intended to draw attention to the mediating relationships amongst them—an essential feature of such a system. While it is not possible to determine the precise mediating roles of each of these elements throughout the activity, there is evidence of each playing a dynamic role in shaping the course of events. The various semiotic resources employed in the discourse, for example, serve to draw C's attention to particular aspects of the object of the activity and to deepen his understanding. The design of the particular measuring tape used (marked in thirty-seconds-of-an-inch up to twelve inches and in sixteenths thereafter) necessitated that this difference be attended to explicitly and negotiated during the activity. And, the conventional design of the measuring tape with the endpoints of subintervals of the inch indicated by a system of signs necessitated that C attend to the intervals between these divisions rather than the division markings themselves in the process of learning to measure.

A number of contradictions in the form of breakdowns, misunderstandings, and complications exist on different levels within this activity. These levels include complications within individual elements of the activity such as the object of the



activity itself—the system of multiple binary fractions-of-an-inch represented by a single inscription pattern on the measuring tape—and with L given his dual roles as tutor and researcher. Between different elements of the activity there are a number of other misunderstandings and breakdowns.

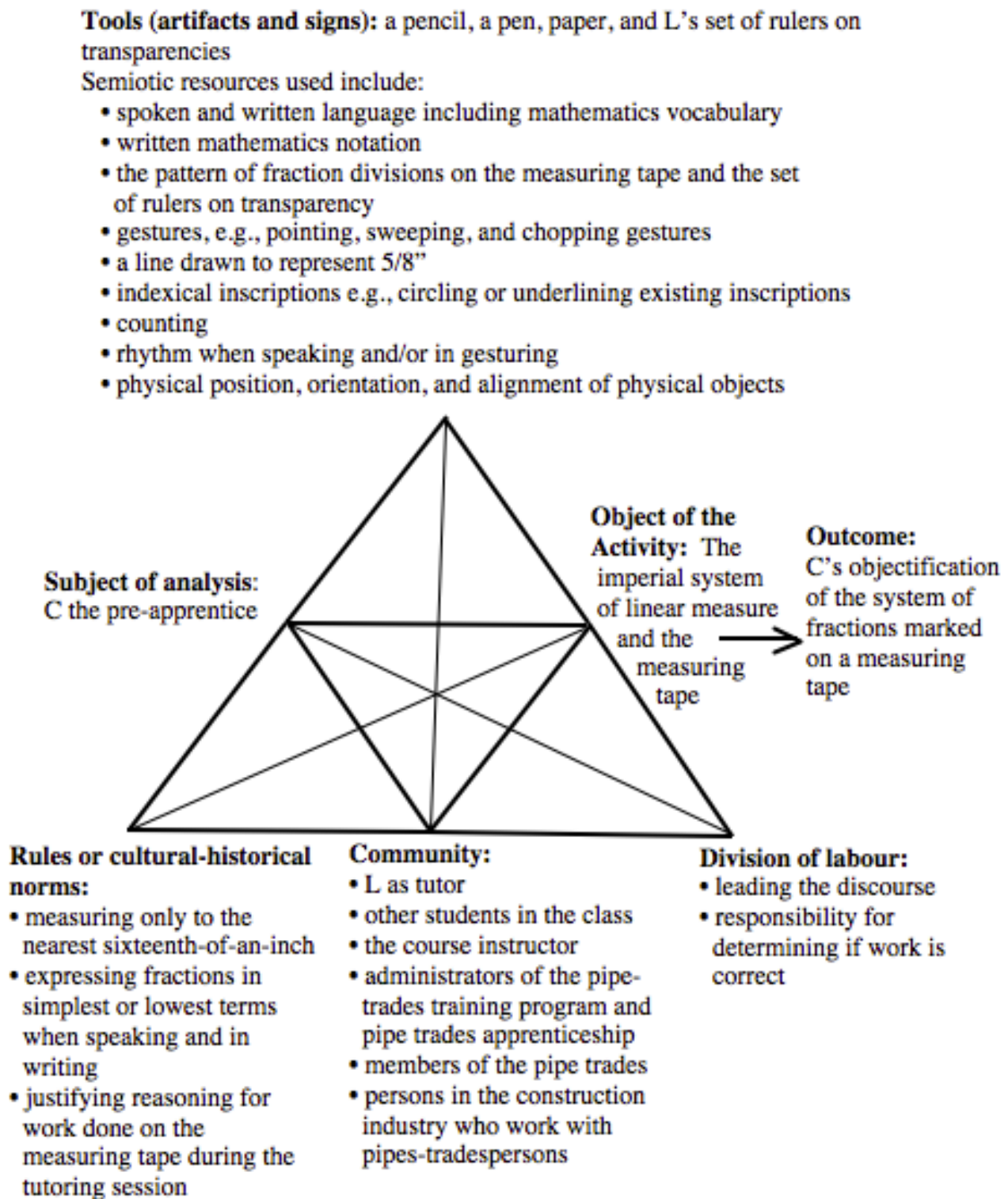


Figure 1. Activity system of the pre-apprentice learning to read the measuring tape

These include C's unfamiliarity with the imperial system of linear measure that he is required to use in his practical work, his limited understanding of the meaning of the denominator of a fraction represented in digit form at the start of the session, communication difficulties between C and L, and L's initial approach to identifying various fraction-of-an-inch intervals on the measuring tape with C. Considerable effort is made by both participants throughout the tutoring session to attend to and resolve these contradictions as they arise within the activity.

### **A summary analysis using the theory of knowledge objectification**

A number of processes that are parts of the mathematics learning process identified by the TO figure prominently within C and L's discourse. For example, C repeats what L says or re-enacts his actions relating to the task at hand on 60 separate occasions during the 33 minute tutoring session. These actions reflect an effort to deepen his sense of—literally, to deepen his sensory experience of—these statements or actions using the same means of semiotic expression used by L or other semiotic means. On one occasion C re-enacts a unique form of interval gesture that he had just used himself and on another occasion L creates a zone of proximal development for C to help bring coherence to his understanding of the division pattern on the measuring tape by inviting C to explain what he (L) had said earlier while providing him with verbal prompts to help him along. And, this is not a one-sided affair. On a few occasions L also re-enacts or repeats what C had done or said earlier. These examples of repeating or re-enacting what another has said correspond to the process of iconicity, identified by Radford as a significant part of the process of attaining a cultural logic of thinking or knowledge objectification. [3]

The ways that L and C use multiple semiotic resources together (semiotic nodes) throughout the tutoring session and, in C's case, the further enactment of semiotic contractions, reflect their understandings of the object of the activity. L's frequent use of various combinations of words, pointing and sweeping gestures, fractions written using digits and words, the fractions-of-an-inch division pattern on the measuring tape and transparency rulers, along with other semiotic resources in a coordinated manner draws and maintains C's attention to/on various aspects of the system of binary fractions-of-an-inch on the measuring tape. And, given L's extensive experience working with the system of binary fractions-of-an-inch extending back to his own elementary school days, it is not surprising that his use of various semiotic resources remains relatively consistent during his explanations to C throughout the tutoring session, reflecting little or no change in his understanding in the process.

In contrast, there is a marked shift over the 33 minutes of the tutoring session in the way that C expresses his understanding using various combinations of semiotic resources as he communicates with L and works to bring clarity to his own thinking. Early on, when C responds to L's request for him to explain what difference he notices in the patterns of divisions below and above twelve inches on the measuring

tape, C's response is predominantly gestural, accompanied by only a single sentence and two sentence fragments. As the tutoring session progresses, C's means of expressing himself shifts completely, at times, to the clear and succinct use of words alone—an ultimate form of semiotic contraction.

The last process from the TO to be summarized here is that of C and L's subjectification. C becomes more active in the way in which he participates in the activity over the duration of the tutoring session. This is evidenced by the collective changes in the patterns of his gaze and attentiveness, his role in the dialogue, his affective responses, and his own expressions of agency and self-reliance regarding his use of the measuring tape. C also nods his head or says "okay" or "yeah" on numerous occasions throughout the session acknowledging to L that he is following what he is saying. This also reflects part of C's process of subjectivity within the activity. L changes during the tutoring session as well, but in a less obvious way. Specifically, L changes in his approach to teaching C how to read the measuring tape from a more general approach (intended for use with a measuring tape or ruler marked to any one of a number of subdivisions of the inch, e.g. sixteenths, thirty-seconds, or sixty-fourths) that is typical of school mathematics teaching, to a much more practical one tailored specifically to workplace demands within the pipe trades where lengths are taken only to the nearest sixteenth-of-an-inch.

### **Comparing these two activity-theoretical perspectives**

Engeström's activity-theoretical perspective focuses on activity systems as a whole, including relationships and contradictions at various levels of activity and the ongoing transformation of activity over long periods of time. His more recent work (e.g. Engeström, 2008) continues at the activity system level focus by addressing interactions between multiple activity systems. These dynamics of activity, while useful for research in many contexts, do not address in a direct way mathematics educators' practical interests in teaching and learning activity, that is, individual students' mathematical enculturation on a day-to-day, if not minute-to-minute basis. Furthermore, Engeström's work does not make clear a means to talk about, nor situate mathematics within activity.

Radford's theory of knowledge objectification, in comparison, focuses on the microgenesis of the mathematics thinking and learning of individuals. It provides a way of defining and positioning mathematics as a cultural practice within particular forms of activity, articulates a clear view of mathematics learning and thinking, and unpacks the dialectical relationship between the subject of activity and the object of activity through the theorization of objectification and subjectification thus revealing an important but often tacit dimension of mathematics learning. (For a detailed discussion refer to Radford, 2008b.) In stark contrast to Engeström who theorizes learning within activity theory as change in an activity itself, Radford focuses on the learning of individuals as they become participants within existing historical-cultural activity.

## CONCLUDING REMARKS

The analysis of a one-on-one tutoring session with a pre-apprentice learning to read a measuring tape summarized here using both Engeström's perspective on CHAT and Radford's theory of knowledge objectification illustrates differences between these two approaches for the analysis of mathematics learning. Clearly both of these complementary activity-theoretical perspectives have contributed and will continue to contribute to mathematics educators' understanding of mathematics activity and learning. But given the objects for which each was developed, on the one hand Engeström's theorization for activity in general and, on the other, Radford's theorization for mathematics learning activities in particular, it should come as no surprise that Radford's theory of knowledge objectification provides a more powerful and directly applicable tool for investigating and understanding mathematics learning at the level of the individual. In pragmatic terms, before embarking on any activity-theoretical analysis in mathematics education, it is advisable for researchers to consider and draw from a range of activity-theoretical perspectives given that this field is evolving in a variety of different ways.

## NOTES

1. This paper is the result of a research program funded by The Social Sciences and Humanities Research Council of Canada / Le Conseil de recherches en sciences humaines du Canada (SSHRC/CRCH).
2. Given the focus of this paper on comparing different activity-theoretical perspectives and space limitations, only a summary can be provided of the analysis done using each perspective.
3. A detailed analysis of the different forms of iconicity evident within a single 21 second clip from this tutoring session was the focus of a paper presented at CERME-6 (see LaCroix, 2010).

## REFERENCES

- Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit.
- Engeström, Y. (1993). Developmental studies of work as a testbench of activity theory: The case of primary care medical practice. In S. Chaiklin, & J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 64–103). Cambridge, England: Cambridge University Press.
- Engeström, Y. (1999). Innovative learning in workteam: Analyzing cycles of knowledge creation in practice. In Y. Engeström, R. Miettinen, & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 377-404). Cambridge: Cambridge University Press.
- Engeström, Y. (2001). Expansive learning at work: toward an activity theoretical reconceptualization. *Journal of Education and Work*, 14(1), 133-156.

- Engeström, Y. (2008). From teams to knots: Activity-theoretical studies of collaboration and learning at work. New York: Cambridge University Press.
- LaCroix, L. (2010). Iconicity, objectification, and the math behind the measuring tape: An example from pipe-trades training. *Proceedings of Sixth Conference of European Research in Mathematical Education*. Available online from <http://www.inrp.fr/editions/editions-electroniques/cerme6/working-group-6>
- Leont'ev, A. N. (1978). *Activity, consciousness and personality* (M. J. Hall, Trans.). Englewood Cliffs, NJ: Prentice Hall.
- Radford, L. (2006). Elements of a cultural theory of objectification [Elements of a cultural theory of objectification]. *Revista Latinoamericana de Investigación en Matemática Educativa*, Special issue on semiotics, culture and mathematical thinking, pp. 103-129.
- Radford, L. (2007). Towards a cultural theory of learning. In Pitta-Pantazi, D., & Philippou, G. (Eds.). *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education (CERME – 5)* (pp. 1782-1797). Larnaca, Cyprus, CD-ROM, ISBN - 978-9963-671-25-0.
- Radford, L. (2005). Why do gestures matter? Gestures as semiotic means of objectification. In H. L. Chick, & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 143-145). University of Melbourne, Australia: PME.
- Radford, L. (2008a). Iconicity and contraction: A semiotic investigation of forms of algebraic generalization of patterns in different contexts. *ZDM—The International Journal on Mathematics Education*, 40(1), 83-96.
- Radford, L. (2008b). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom and culture* (pp. 215-234). Rotterdam: Sense Publishers.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70(2), 111-126.

# USING PRACTICAL EPISTEMOLOGY ANALYSIS TO STUDY THE TEACHER'S AND STUDENTS' JOINT ACTION IN THE MATHEMATICS CLASSROOM

Florence Ligozat<sup>1</sup>, Per-Olof Wickman<sup>2</sup> & Karim Hamza<sup>2</sup>

<sup>1</sup>University of Geneva; <sup>2</sup>Stockholm University

*This paper aims at characterizing the teaching actions that are used in a primary school mathematics lesson, and their consequences for the learning progression. To proceed, we explore the analytical outcomes of combining two analytical frameworks, namely the Practical Epistemology Analysis of classroom discourse (Wickman & Östman, 2002; Wickman, 2004; 2006) and the triple [Meso; Topo; Chrono]-Geneses featuring didactical transactions in the Theory of Joint Action in Didactics (Sensevy, Mercier, Schubauer-Leoni, Ligozat & Perrot 2005; Ligozat & Schubauer-Leoni, 2010; Sensevy 2010). The analytic approach is exhibited through an empirical sample of a mathematical lesson about the learning of surface area with 4th grade students. Analyses are guided by two questions: 1) testing the PEA for identifying the content learnt in transactions in the case of mathematics; 2) examining how PEA may augment the MTC-Geneses description to characterize the learning progression over time within the teacher's and students' joint actions. This later question is crucial to understand the generalization of the students' experience against the teaching process unfolded by the teacher.*

## THEORETICAL BACKGROUND

### Practical Epistemology Analysis (PEA)

In the Swedish pragmatist approach to science learning, the PEA framework was developed as an analytical tool for characterizing the meaning-making process in science-classroom discourse. This approach features learning as the unfolding of purposeful action and change of habits for coping with reality (Rorty 1991). Cultural practices entail epistemologies, as implicit rules for acting adequately in social groups. In designing and carrying out classroom work, the teacher makes explicit and implicit decisions about the situations that the students will experience. Wickman (2004) suggests that as the curriculum unfolds in the teacher's and the students' actions and discourse, a *practical epistemology* is shaped. Hence, from the student point of view, learning content is dependent on the epistemologies developed in the classroom, as a set of epistemic and social norms that guide the selection of relevant actions to achieve a purpose. Of course, such norms are tied to the socio-historical traditions embedded in curricula. We do not aim at describing such rules and power relations *per se* but we seek for the connection between how classroom participants produce meaning and what meaning is produced in a specific practice.

The model of Practical Epistemology Analysis developed by P.-O. Wickman and collaborators relies upon L. Wittgenstein's notion of *language-game* (Wittgenstein

1967) and J. Dewey's theory of *continuity in experience* transformation (Dewey 1938/1997). For the former, meaning is a given in the socially shared rules supported by a language proper to a context. Learning is then mastering a language-game, i.e. the grammar of actions featuring a practice. For the latter, experience is continually transformed by the transactions taking place between an individual and his environment. Subjects build continuity between past and present experiences so that experience earned in a given situation becomes an instrument for understanding and dealing with the situations which follow (Dewey 1938/1997, p44). PEA is grounded on *four categories* for analyzing discourse as a transformation of experience within a language game (Wickman & Östman 2002; Wickman 2004; Wickman 2006).

(i) *Encounter*: an encounter delineates a specific situation in terms of what can be seen to meet or interact in actions and discourse. This involves human beings as the participants of the situation and the "things" that become part of the experience in this situation. These may include physical objects, signs, words, utterances, phenomena like natural facts and events as well as recalled experiences.

(ii) *Stand fast*: in an encounter, certain objects are manipulated without any questions arising about their use. Such objects and words stand fast in the encounter. What stand fast in one situation may later be questioned in another situation. Neither does stand fast necessarily imply a correct use from the observer's perspective. It simply implies that the meaning of certain words in discourse is self-evident for the participants with respect to this specific situation.

(iii) *Gaps and relations*: in an encounter participants notice gaps. They then establish difference and similarity relations to what stand fast. If participants succeed in filling a gap with relations they build continuity between past and present experience. If they fail, the gap lingers and the course of action may change direction toward another purpose.

It is important to notice that the four concepts of PEA enable a first analysis of meaning-making from the interlocutors' perspective. From the observer's perspective (the researcher in this paper), "something" is learnt when the activity moves on, that is when there is evidence that the participants can proceed towards a purpose. Learning proceeds when people notice gaps and fill them with relations to what stand fast in encounters. This inclusive account of learning focuses on what works in the situation in order to overcome it and not solely what is right or wrong with respect to conceptual knowledge. Questioning truth is central in scientific reasoning but it is only one of several ways to proceed in everyday life practices (Habermas, 1984/1990). PEA accounts for the meanings being construed in action without prioritizing what is true / not true and what should be said or done in order to acquire the expected knowledge. This may also be understood as a methodological caution aiming at minimizing the risk of overlooking certain forms of learning just because they were not included in the definition from the outset.

### The triple [Meso; Topo; &Chrono]-Geneses

The Joint Action Theory in Didactics (JATD) envisions the teaching and learning practices as a *didactical game* in which the teacher achieves his/her goal - making the students learn knowledge content- only if the students get involved and act in *a certain way* (Sensevy, 2010). The expected way of acting defines the rules of the learning progression. For the student to learn, the teacher has to design *a set of conditions* made of material and symbolic objects bound to a question, task or inquiry to be attended and featuring the students' ends in view. This set of conditions is viewed as the "primitive" milieu (or a teaching design) from which certain meanings are construed in action. The teacher and the student(s) cannot achieve their respective purposes without paying attention to the action of the other and moreover to the object of the other's action. An "object" is anything that can be the target of attention of an individual and that can be designated by him. The meaning of an object is given to an individual by the way in which the others are prepared to act toward it (Mead 1934/1992). The teacher's action and the student's action are *joint actions* in which each participant adjusts his action to the other's line of action (Blumer 2004). In the framework of the JATD the triple [Meso; Topo; Chrono]-Geneses models the construction of a *common ground* of meanings in the joint actions that are performed by the teacher and the student(s).

(i) *Mesogenesis* - The fitting of lines of actions of the teachers and the students (or within a student group) to achieve their respective purposes generates new meanings through the relations that participants establish to the objects of their environment. In adopting this point of view, the milieu in which actions unfold, is not just the set of conditions defined in the teaching design, regulated by the teacher over time, and against which the student would play a game (i.e. the milieu in TSDM; Brousseau 1997, p55-58). It is *a constant build up of relationships* to objects in discourse and actions, i.e. a mesogenesis in Chevallard's words (1992). Mesogenesis takes up both the students' elaboration of meanings and the teacher's elaboration of meanings on the basis of what the students produce.

(ii) *Topogenesis* – Each category of participants (teacher versus student) lives in distinctive epistemological positions within the classroom collective. They do not share the same perspective on the task at the same time. These positions are movable but they never merge. The moves in the epistemological positions (either towards a reduction of the gap or towards its deepening) feature the topogenesis. Topogenetic moves result from *the division of the activity* between the teacher and the students, but also among students themselves according to their potentialities in a specific situation.

(iii) *Chronogenesis* - The teacher knows the overall direction that learning should take on the knowledge timeline. The learning content expected by the teacher in the first place corresponds to an institutional purpose in terms of contents and values to be conveyed to the students. Such a purpose is described as pieces of knowledge to



be learnt, attitudes to be adopted, competencies to develop...etc. in curriculum texts that reconfigure outer-school socio-cultural knowledge historically construed in the human activities. The overall direction that the learning progression takes in the classroom is described by the chronogenesis. Chronogenetic moves result from the legitimating process of certain meanings made by the students in the collective and /or the introduction of new relations directly made by the teacher.

The primitive milieu designed by the teacher is continuously augmented over time by the meanings arising in the participants' fitting lines of action. Meanings are epistemologically distributed across the classroom collective between the students' position(s) and the teacher's position. Certain meanings construed at a time in the mesogenesis may be judged relevant or not by the participants with respect to 1) their potential to support the ends-in-view structured by the task (epistemic relevance); 2) the expected learning content that is the overarching goal of the teacher (didactical relevance). The outcome of the collective analysis of situated actions is the departure point for the participants to further their activities. We contend that the meanings built in the mesogenesis undergo a selective process to become part of a "supposed-to-be-shared" common ground in the classroom collective (Ligozat & Leutenegger 2008). The ongoing construction of this reference is an institutionalizing process of meanings construed in the situated actions towards a *collective objectivation of knowledge* in discourse.

## EMPIRICAL FINDINGS

In the following, we attempt to use PEA for describing meanings made in the contingencies of mathematical activities with primary school students. In particular, we try the analytical categories of the PEA approach (encounter, stand fast, gaps and relations) for describing the dynamics of purposes in the joint action and the content of the mesogenesis. The chronogenetic and topogenetic moves feature how the teacher directs the students' attention towards certain relevant objects and correlations in the setting in order to achieve a mathematical task. We merely use a short excerpt of classroom discourse to highlight the analytical potentialities of combining both PEA and MTC-Geneses frameworks. The students are working in small groups, with a set of 13 geometrical shapes and with a worksheet bearing the instructions i) "rank the shapes from the smallest to the largest according to their area"; ii) "justify your ranking"). They first make conjectures about the use of the objects provided [Gap 1: what should we do?]. When the teacher comes nearby, the students call upon her for helping. The teacher tells the students to read the instruction and asks them about the meaning of the word "area". A new gap is noticed [Gap 2: what is area?]. The students suggest that "it is the shapes", i.e. a word that stand fast to them in this situation. From the teacher's perspective, the students do not manage to construe any relevant relations to the word "area" (something like "it is the surface of the shapes" or "the space lying inside the borders

## Working Group 16

of the shape" may be expected). At min 3:29, the teacher goes back to the instruction in the following way:

43. TEACHER : [...] it is written / rank from the smallest to the largest  
/// actually from the largest to the smallest-
44. KAM : well / first we know that this one is the smallest (*takes triangle H out of the set*)
45. TEACHER : how do you know that it is the smallest / how can you prove that it is the smallest+
46. MAR : (*puts square C close to triangle H [PICT2]*) because this one is smaller
47. DIA : it is half of it-
48. MAR : (*murmuring*) but we can't prove it-
49. TEACHER : (*looking at Mark*) OK but... / this is your feelings OK+ / but how can you prove it+ // because here (*points at the corner of triangle H – PICT 3*) / one may say / it sticks out a little bit- / so how can you prove that it is really the smallest+
50. DIA : (*getting excited*) I know I know+
51. MAR : oh like this / according to their area / we've got to set in line (*sets the base of triangle H on the same line as square C [PICT4]*)
52. DIA : no look + / I disagree / this is smaller because this (*picks up triangle H*) is half of the square (*puts H onto C [PICT5]*)
53. TEACHER : ah+ / do you think this could be *a proof*+
54. [silence 5 sec]
55. KAM : this / that's two / mmh / the whole square that's twice this one (*points at H*)
56. TEACHER : how did Dina do to prove you this+
57. KAM : she puts it over (*points at H again*)
58. TEACHER : yeah / she puts it over the square / she superimposed the shapes // now / you have some transparent paper / some square grid paper // and by using Dina's technique / you've got to find some tricks / that to prove that / a shape is larger or smaller than another one



PICT 2



PICT 3



PICT 4



PICT 5

Our 1<sup>st</sup> analytic focus looks at encounters, stand fast words and gaps to describe the dynamical structure of the joint action. The encounters delineate the relationships developed on a small portion of time by the participants with respect to certain objects (words, things, signs) that become parts of their environment. A new encounter opens up when we identify some changes in the participants' purposes during the course of action.

## Working Group 16

- Encounter 1 (line 43-45) → Mark, Dina, Kamer and Teacher take into account shape H and shape C (among all the set), smallest (a word from the instruction sentence)
- Encounter 2 (line 46 – 57) → Mark, Dina, Kamer and Teacher take into account shape H and shape C (among all the set), know, prove, proof, stick out, set in line, half, twice, put over.
- Encounter 3 (line 58) → Mark, Dina, Kamer and Teacher take into account shape H and C (among all the set), put over, superimpose, transparent paper, square grid paper, Dina's technique, tricks, prove, larger, smaller.

In this case, changes in purposes and so the openings of new encounters are prompted on the basis of *gaps indicated by the teacher*. As such, they coincide with some chronogenetical moves (CM) and topogenetical moves (TM). Encounter 2 corresponds to an *expansion of teaching time* to attend the need to "prove" (CM); the teacher adopts a feigned "low" position supposed to give some responsibilities to the students in finding "proof" arguments (this TM analysis will be nuanced in our 2<sup>nd</sup> focus of analysis). Encounter 3 corresponds to a *contraction of the teaching time* with the acknowledgement of the superposition technique as a to-be-shared reference (CM); the teacher uses her institutional power to prompt a new purpose in the joint action (TM). Through the dynamics of encounters 1, 2 and 3, we get an insight of the nature of the expectations upon the student's actions: i) sum up what is known from the reading of the instruction, ii) how we get to know it, and iii) extend the use of the "put over" technique. Changes in purposes may also originate in the students' course of action when a gap cannot be filled. For instance, at the very beginning of the group work session, when the students cut off the shapes from the cardboard, one of them tried isolating the smallest shape, another tried to order the shapes according to their alphabetical letters, then they tried ordering 4 shapes on a same line. At some point, the students stopped manipulating the shapes because they could not find a way to determine what to do, or more exactly, what they are expected to do in this situation. Gap 1 [what should we do?] lingers, resulting in a change of purpose from trying to do something with the shapes to getting an explication from the teacher. Hence, encounters, stand fast words and gaps reflect the evolution of the reciprocal expectations in the teacher and students' joint action.

Our 2<sup>nd</sup> analytic focus is put on gaps and relations to describe the epistemic content built up in the mesogenesis. In bold characters, are the teacher participations; in italic characters are the students' participations.

**Table 2: Gaps and epistemic relations developing in the mesogenesis**

line	GAPS	RELATIONS
infra	G1: <i>what should we do?</i> => linger	
infra	G2: <b>What is area?</b> => linger	
44		Rel-a : <i>know- smallest- (triangle H)</i>
45-49	G3 : <b>How do we know/prove [Rel-a]?</b> => filled by Mark and Dina	Rel b : <b>knowing –need to prove</b> => implicit Rel-c : <i>(triangle H) – half – (square C)</i> Rel-d : <i>[Rel-a] – can't prove</i>
49-52	G4 : <b>Could C be the smallest?</b> => filled by Dina	Rel-e : <b>(triangle H) – sticks out – (square C) – (parallel bases)</b> Rel-f : <i>area – (aligned bases)</i> Rel-g : <i>[Rel-a] –because- [Rel-c] – (triangle H over square C)</i>
53-56	G5 : <b>is [Rel-g] a proof?</b> => filled by Teacher	Rel-h : <i>whole square – twice (triangle H)</i> Rel-i : <b>[Rel-g] – a proof</b>
56-58	G6 : <b>how did Dina proved [Rel-f]?</b> => filled by Kamer	Rel-j : <i>prove- put over</i> Rel-k : <b>prove – superimpose the shapes</b> => implicit
58	G7 ( $\rightarrow$ Gap1) : <b>what should we do?</b> => (implicit); partially filled by Teacher	Rel-l : <b>transparent paper – square grid paper- Dina's technique – tricks – prove – smaller /larger shapes</b>

(i) Gap 4 is a challenge to Mark's belief that we can't prove that H is smaller than C (line 48; Rel-d). Mark attempts to fill Gap 4 in establishing some conditions for triangle H to be smaller than C (line 48, Rel-f). He tries an inferential relation between the word "area" for the first time at this point and "set in line". The inference is: if the criteria "according to area" (that does not stand fast) was to mean "set in line" (that stand fast in actions), then the conditions for H to be smaller than C would be warranted. Unfortunately for Mark, this inference cannot be sustained in the common ground of meanings privileged by the teacher because this is not compliant with the mathematical culture. The teacher opposes Mark's relation with another relation (line 46; Rel-e) stemming from a change in the relative positions of the two shapes. The consequence is that in this position, triangle H may not be smaller than square C. Indeed, the order relation between H and C depends on the geometrical objects considered: side length of square C is smaller than base length of triangle H but side length of square C longer than the height of triangle H. These order relations on length are more or less salient depending on the relative position of the shapes in space (parallel bases VS aligned bases of the shape). The teacher uses spatial contingency to increase uncertainty and make the students noticing Gap 4 [Could C be the smallest?]. We have an example of a topogenetical move toward more responsibilities to the students in assessing a relation being made with respect

to the geometrical relationships among the shapes in the primitive milieu managed by the teacher (moving the shapes).

(ii) To fill Gap 4, Dina brings in a new relation that aggregates previous ones into a causal pattern (line 52; Rel-g). This relation ties in discourse the order relation [H is smaller than C] and a new pattern of the relative positions of shapes H and C in space [H over C]. From the researcher's perspective, the statement [H is half of C] is a quantification of the magnitude of the surface area of triangle H with respect to square C as a standard unit. It is an argument to prove that the surface area of H is smaller than the surface area of C. But what is the significance of Rel-g for the students? The mere thing we can do is to track any correlated relations unfolding in the participants' action and check whether these relations stand fast in furthering the activity. In eliciting Gap 5 [is Dina's utterance a proof?], the teacher tries to empower the group with the task to assess Dina's relation with respect to Gap 3 and gap 4. But the word "proof" does not stand fast to the students and Gap 5 cannot be filled by the students themselves. In noticing Gap 6, the teacher introduces a relation (Rel- i; line 56) that implicitly fills Gap 5. The responsibility given to the students is too high about a task (identifying a proof) that is out of reach of the students. The teacher subtly moves back toward high position, to manage the answers and reduce uncertainty. In this topogenetical move, the focus on "what" is proved is drifted towards "how" it is proved.

The mesogenesis is a *series of ephemeral and situated encounters* which are co-elaborated by the participants. Certain words and actions stand fast in these encounters. We contend that the student's experience in which epistemic relations is made is an *individuated* experience stemming from a collective experience made of joint actions and shared meanings. "Smallest", "largest", "set on a line" are recognized with respect to the collective experience shared in the group about ranking objects activities like sticks, blocks, etc. What stand fast describes the reference from the participants perspective. But what stand fast *hic et nunc* in an encounter may also remain contingent for the students. Indeed, observing the students' action in the subsequent encounters of this lesson show that the students go on with making superimpositions of shapes but they hardly draw conclusions from them, in terms of larger/smaller shapes. From this, we understand that "Dina's technique" does not make sense in the collective experience as a generalized content (or knowledge) which in turn, could be a resource for each student in further activity. Furthermore, since the word "area" was not related to this rule, the concept of surface area does not earn significance in action. From the succession of gaps highlighted by the teacher and featuring a fine-grained chronogenesis, we understand that the knowledge of the mathematical concepts (what is surface area?) and practices (knowing in mathematics is proof-based) is prioritized over the relations *effectively made* by the students in the encounter (ordering the shapes by sight estimation, comparing side lengths, finding a numerical ratio between the shapes). Of course, the

teacher takes up these relations (cf- Gap 6), but these relations construed in situated action remain contingent from the student's perspective. Each time that a gap is filled, it is replaced by a new one prompted by the teacher without enabling the students to proceed with the new relations made.

## CONCLUDING REMARKS

We now discuss the implications of our analysis for the (re)conceptualization of PEA tools in the JATD. PEA empowered us with high resolution tools to analyse the content being learnt in classroom joint action from the participants' perspective. From this analysis it can be seen that the basic concepts of the lesson (proof and area) do not make sense to the students as part of the purpose of ordering according to size, without being reformulated as different kinds of doings (putting side by side, superimpose). This demonstrates that the teacher, in joint action with the students, would need to construe relations between those terms which the students are supposed to learn (proof, area) and those that stand fast (putting side by side, superimpose). Here there is no evidence of this in action, and so there is no evidence to the teacher that students have learnt what proof and area means in terms of habitual ways of talking and acting mathematics. [MTC]-geneses augment the analysis in directing analysis on the overall joint action about how the relations sum up (mesogenesis), the role of the student vis-à-vis the teacher (topogenesis), and how the learning progresses over time (chronogenesis). The MTC-analysis offers a means to analyse the social control on the meanings to be learnt from an institutional perspective. Thus, the power of this combined analysis lies in its ability to elucidate the meaning-making process from the participants' perspective (PEA-analysis), and combine this with an analysis of the consequences of the teacher managing the learning progression in certain, specified ways (MTC-genesis analysis). If teaching is organizing "signs" (words, symbols, constellation of artefacts) to make someone learn a content, learning involves making sense of such signs and forms in order to act adequately with respect to the sign-maker/organizer's purposes. But learning cannot be unilaterally controlled by the organization of signs in a teaching design, however genuine it may be. Learning is contingent on the experience of the learner and on the haphazard sequence of events developing in the joint action of the classroom (Hamza, 2010). Meanings arising in encounters are not "controlled" at their source (in the mind of the students) by the teacher but they are shaped in discourse according to a collective process of selection, aggregation and social validation and so needs close empirical examination.

## REFERENCES

- Blumer, H. (2004). *George Herbert Mead and Human Conduct*. Alta Mira Press: Walnut Creek
- Brousseau, G. (1997). *The Theory of Didactic Situations in Mathematics*. Dordrecht: Kluwer.

## Working Group 16

- Chevallard, Y. (1992). Fundamental Concepts in Didactics : perspectives provided by an anthropological approach. *Research in Didactique of Mathematics. Selected papers* (pp131-167). Paris: ADIREM & Grenoble: La Pensée Sauvage édition .
- Dewey, J. (1938 /1997). *Experience and Education*. NY: Touchstone.
- Habermas, J. (1984/2001). *On the pragmatics of social interaction: Preliminary studies in the theories of communicative action*. Cambridge, MA : MIT Press
- Hamza, K.M. (2010) *Contingency in High-School Students' Reasoning about Electrochemical Cells. Opportunities for learning and teaching in school science*. Doctoral thesis in Science Education, Stockholm University. Sweden.
- Ligozat, F. & Leutenegger, F. (2008). La construction de la référence dans l'action conjointe du professeur et des élèves. Le cas de milieux différentiels émergeant dans un problème d'agrandissement de distances. *Recherches en Didactique des Mathématiques*, 28(3), pp 319-378.
- Ligozat, F. & Schubauer-Leoni, M-L. (2010) The Joint action Theory in Didactics: Why do we need it in the case of teaching and learning mathematics? In V. Durand Guerrier, S. Maury & F. Arzarello. *CERME 6 Proceedings* (pp 1615-1624), [<http://www.inrp.fr/publications/edition-electronique/cerme6/wg9-12-sensevy.pdf>] Lyon : INRP
- Mead, G.H. (1934/1992). *Mind, self and society from the standpoint of a social behaviorist*. University of Chicago Press.
- Rorty, R. (1991) Introduction: Antirepresentationalism, ethnocentrism and liberalism. In R. Rorty(ed), *Objectivity, Relativism and Truth* (vol 1, pp1-17). Cambridge University Press.
- Sensevy, G. (2010). Outline of a joint action theory in didactics. In V. Durand Guerrier, S. Maury & F. Arzarello. *CERME 6 Proceedings* (pp1645-1654), [<http://www.inrp.fr/publications/edition-electronique/cerme6/wg9-12-sensevy.pdf>] Lyon : INRP
- Sensevy, G., Mercier, A., Schubauer-Leoni, M.-L., Ligozat, F. & Perrot, G., (2005). An attempt to model the teacher's action in the mathematics class, [Special issue] *Educational Studies in Mathematics*, 59 (1,2,3), 153-181.
- Wickman, P.-O. (2004). The practical epistemologies of the classroom: A study of laboratory work, *Science Education*, 88, 325-344
- Wickman, P.-O. (2006). *Aesthetics Experience in Science Education*. Learning and meaning-making as situated talk and action. NJ: LEA
- Wickman, P.-O. & Östman, L. (2002) Learning as discourse change: A socio-cultural mechanism. *Science Education*, 86, 601-623.
- Wittgenstein, L. (1967) *Philosophical Investigations*. Oxford & Cambridge: Blackwell

# DISCERNING IN AND BETWEEN THEORIES IN MATHEMATICS EDUCATION

John Mason

University of Oxford & Open University

*Abstract: the conjecture is put forward that the most important feature of theories in mathematics education are the frame(work)s, which consist of labels for distinctions. These determine what can be attended to, and what therefore is not attended to. These distinctions incorporate assumptions and values which together suggest actions that might be taken. The discourse of 'shifts of attention' is used to structure an analysis of what theories might and might not be able to contribute to the improvement of teaching and learning, drawing on a background theory called 'Systematics'. My conclusion is that to compare or conjoin theories requires a generally accepted way of agreeing meaning of technical terms so that misunderstanding and mis-appreciation of distinctions is minimised.*

## INTRODUCTION: THEORIZING AS A WHOLE

The plethora of research papers in mathematics education, and the considerable degree of overlap between issues and phenomena studied, using a wide range of theoretical constructs, distinctions and methods is fostering concern about how theories might be integrated, compared and conjoined (Sriraman & English 2010, PME34 research forum, CERME Working Group 2008).

At ICME-2 in Exeter, René Thom put forward a typically Continental position, that "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics". In other words, 'every pedagogical choice rests on a theory about effective teaching and learning', just as 'every research choice depends on a theory about enquiry'. An implied concomitant sentiment is that until the theory is articulated, action ought to be delayed. The action is two-way however, for although theories are how we make sense of experience, experience is possible only through actions involving theory.

Having been bothered for some time by the demands of journal editors to situate studies within acknowledged and named theories, it seems appropriate as a contribution to this working group to reflect on the nature of theories in mathematics education so as to clarify my own actions and choices. While well aware that there have been several similar discussions based on other approaches using semiotics (Radford 2008), and collections of essays (Sriraman *et al. op. cit.*), I choose a structural approach, using constructs elaborated by Bennett (1993) under the heading of *Systematics*, which has its roots in ancient psychology in drawing on qualities of numbers and which focuses on details overlooked in other expositions.



The overall structure of this paper self-referentially follows the discourse of the structure of attention (Mason 1996), since theories direct attention, and the direction, focus and nature of attention can be said to constitute one's theory. Useful distinctions within the discourse surrounding shifts of attention are the states of *holding wholes* (gazing), *discerning details*, *recognizing relationships* amongst what is discerned in the specific situation, *perceiving properties* as instantiated in the particular, and *reasoning on the basis of agreed properties*. Although closely aligned with generalized van Hiele levels (van Hiele 1986), the major difference is that rather than defining 'levels', shifts are likely to take place moment by moment, back and forth in no particular order. The alert reader will discern that 'holding wholes' was the orientation and role of this introduction.

## **THEORISING THEORIES**

What do theories consist of, what are they useful for, and at the core, what really is a theory? Following the structure of attention, and having considered briefly the whole, I consider distinctions, relationships, properties, and potentialities in turn.

### **Theory as Discerning Details through Making Distinctions**

To discern detail is to distinguish some 'thing' from 'something else'. The root meaning of 'theory' lies in the Greek *theorein* meaning 'to see' and hence, through extension, 'to perceive'. Thus at its core a theory enables the discerning of detail, and it does this through providing labels for distinctions considered worthwhile, and embedding these in a larger discourse. For example Bruner (1966) distinguished three modes or worlds of experience, *enactive*, *iconic* and *symbolic* which provide distinctions between how thoughts or awarenesses are (re)presented, while *scaffolding* (Bruner 1966) and *scaffolding&fading*, (Brown, Collins & Duguid 1989) label ways of choosing between teacher actions in relation to learners.

That "observation is theory laden" (Hanson 1958) has been noted many times. Indeed Goodman (1978) suggested that "we want our theories to be as fact laden as our facts are theory laden". In other words, the theoretical constructs proposed need to be observable and present in relevant situations to the same extent that the distinctions made arise from and are based in an articulation of the theoretical stance.

There are severe ontological consequences of the hyparchic act of labelling a distinction. Labelling a situation or action tends to reinforce its existence. For example, distinctions between conceptual and procedural, or between instrumental and relational understanding begin by being helpful, but then mislead people into talking, and then acting, as if the distinctions were natural, rather than regions in a continuum arising from people stressing some features and downplaying others.

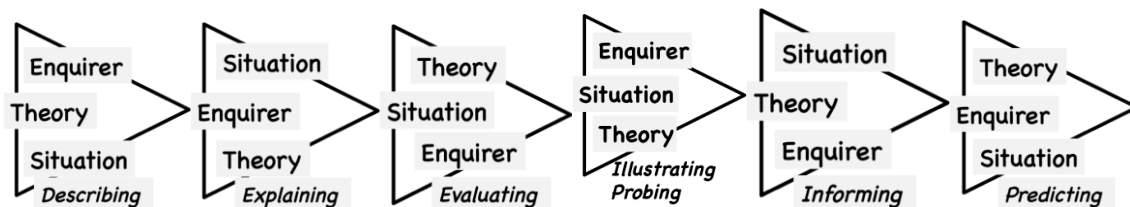
### **Theory as Recognising Relationships and Undertaking Actions**

Developing vocabulary is only part of the sense-making process that drives theorizing. Once discerned, details can be related to each other. A discourse

develops, incorporating (literally) the labels. Collections of related distinctions form *frames* or *frameworks* through which or by means of which one makes sense of and perceives phenomena. But sense-making must involve more than the use of technical terms and metaphors in order to qualify as sense-making. There must be some consequences beyond mere description.

Theories contribute relationships to recognise between discerned elements, including related and relevant actions which may be informed by or be justified by use of the discourse. The more elaborate the discourse, the stronger the impulse to perceive through that discourse. Labels can be used superficially to avoid thinking, to bypass insight, and to masquerade as wisdom, at least as easily as they can support and promote insight.

Theories have been classified by a variety of adjectives, including descriptive, illustrative, explanatory, informative, predictive and evaluative, but these functions are more usefully seen as modes of interaction between an *enquirer* (researcher or teacher) a classroom or pedagogic *situation* and a *theory*. Bennett's Systematics suggests that actions require three impulses (initiator, respondent and mediator), and so can take place in any of six modes derived from the six permutations of the three impulses. Each mode is depicted below as a triangle, with the initiator shown at the top, the respondent at the bottom, and the mediator in the middle. The mediator binds the other two together so that an action can actually take place. The result of the action is an actualisation or amplification of the mediating term. These actions can be read from the following diagrams, though there is not space here to elaborate.

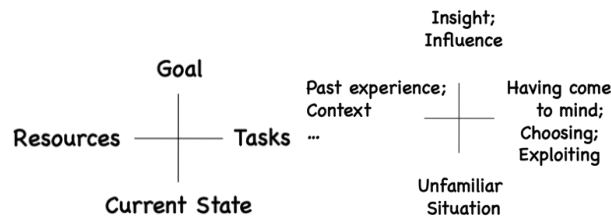


This sixfold structure can be used whenever something is perceived as mediating.

### Theory as Instantiating Perceived Properties in Activity

Activity is recognisable through discerning details and recognising relationships as they emerge dynamically, but not in isolation. There is always a history, both personal and socio-cultural which influences the activity. In Systematics, activity is seen as involving four terms based on two axes: one based on resources and tasks and the other on the motivational gap between present-state (as perceived) and goal-state (as desired). If space permitted, it would be possible to look at the researcher's activity in relation to the teacher's or learners' activity.

## Working Group 16



For example, Gibson's affordances, constraints and attunements (Greeno 1994) relates what is available, what is (necessarily) constrained or excluded, and what is needed in order to make use of the affordances and the constraints.

Sustaining and promoting effective enquiry depends on perceiving properties of methodologically, epistemologically and philosophically well-founded activity as capable of being instantiated in the specific activity. As one use of this way of perceiving activity, the resources a researcher has to call upon include their past experience, their awarenesses, the situation as experienced and interpreted, the mathematical and research processes in which they have engaged, their socio-cultural-historical appreciation of the context and of pedagogic strategies and didactic tactics, and so on. But these have to come to mind, and that happens through metaphoric resonance and metonymic triggering, filtered through a value system. What comes to mind (phrased and framed in the discourse of the theory) informs and dictates what is possible: what specific acts will be instituted, what tasks will be undertaken and so on. Tasks need to be in balance with available resources (too rich or too impoverished a resource base makes choice difficult or limited), and to be appropriate to the desired movement in the motivational axis. Similarly the current and goal states need to be in relation, neither too far apart (too great a challenge) nor too close together (too trivial to be worth bothering about).

Each of the triples in the tetrad can be analysed as an action with its six modes, but there is no room for elaboration here. Theory can be read into a situation by observing the teacher's actions, and by seeking both prospective articulation of choices and retrospective justification of actions.

### **Theory as Providing Potential through Reasoning on the Basis of Agreed Properties**

Theorizing is fed by enquiry, by the wish to 'make sense', by desire to repeat and improve actions or interactions in some situations and not to repeat others, by desire to develop over time, in short by questioning. At the very least, a theory provides distinctions to discern. According to associated epistemic and ontic commitment, a theory articulates what sorts of questions can be asked, what sorts of things are deemed to be data (what details to discern), what form of analysis might be undertaken (what features to attend to), what kinds of narratives are woven around that analysis, and what products (findings, insights, potential actions etc.) might emerge. Concomitantly, theory provides a discourse through which to validate those products. What theories contribute to is a sense of meaning, significance, insight and

## Working Group 16

a format for communicating findings, as well as prospectively informing actions or retrospectively justifying.

At their best, theories provide a complete *weltanschauung*, a way of being as an enquirer, whether teacher, researcher, educator or learner, and they make it possible to communicate effectively both with oneself and with others through negotiated ways of working together so that insight and creativity inform future actions.

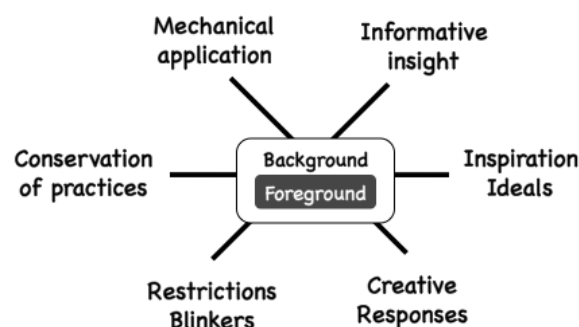
All this amounts to theories as, at their most comprehensive, incorporating (again, literally) worlds of experience. These worlds encompass epistemic and ontic-hyparchic commitment; they generate both a system of values which justify the kinds of products that merge, and the forms of validation that make sense within that world; they entail a view of the psychological and sociological influences and how the environment in its broadest terms influences the actors in the enterprise of mathematics education. For example (Mason 2002),

A world of external facts entails and requires repeatable experiments, as in the natural sciences;	A world of opinion and belief entails and requires surveys, questionnaires, interviews, as in sociology;
A world of others' experience entails and requires participant observation and ethnographic and anthropological approaches;	A world of involvement in action entails and requires change, prediction and evaluation, as in action research;
A world of personal experience entails and requires sensitising oneself to notice what was previously not noticed, as in phenomenological & phenomenographic approaches;	

These 'worlds' encompass assumptions, some of which are implicit while others may be explicit, and values concerning various aspects including what is data, what is done with data and what actions are promoted or critiqued.

## STRENGTHS AND LIMITATIONS

Theories foreground some aspects and so necessarily background others. "To express is to over stress" (Mason 1987), which is a necessary but ultimately limiting feature of human sense-making. All sense made is partial. At any moment attention may be dominated by one or more of the aspects displayed below.



Conservationists of particular theories need to guard against inappropriate expansion, restriction or reinterpretation of distinctions, while eclectic gathering of useful distinctions, actions and relationships into explanatory narratives can all too readily become superficial. Focussing exclusively on the left-hand three 'directions' in the diagram can be stultifying or simply jargon-ridden; focus exclusively on the right-hand three can lead to unsubstantiated claims and the preference for values over facts.

### **Problematic Aspects of Theories in Mathematics Education**

I have already raised the issue of the ontic consequences of hyparchic acts of distinction making and labelling. Two further issues of particular concern are the lack of any generally agreed approach to agree on the meaning and use of technical terms within the mathematics education community, and the paucity of conclusions drawn from theories: can description be a sufficient end in itself?

#### **How is the use of technical terms agreed?**

Developing a discourse is one thing, but getting agreement as to the meaningful use of that discourse is quite another. To 'use a theory' is, at the most basic level, to make distinctions which others who use the same theory would also make. This is a matter of some significance, on a par with the notion of robustness of social-science research instruments. It deserves far more attention than it gets. What mathematics education as a domain of enquiry notably lacks is an explicit and generally accepted method for how meaning is to be agreed or enriched. Curiously, there seem to be widely shared practices involving pre- and in-service teachers working on tasks through which they are likely to gain direct experience of what is being discerned and labelled (Watson & Mason 2007).

A mathematical approach was adopted by the Open University in its many courses for teachers of mathematics over 30 years. We employed a parallel between the mathematical practice of offering examples and discussing explicitly in what way the example actually exemplified the construct, with constructs from mathematics education having origins in different disciplines (Mason & de Geest 2010).

#### **Is theory being illustrated, tested, or used to some other end?**

When data is collected, distinctions made, phenomena identified and labelled, there is a pressing question: are the technical terms from the theory being exemplified and illustrated, are they being validated in some way, are they simply being used as descriptors to classify observed actions, or is the theory being used to draw conclusions? Sometimes it is difficult to work out which if any of these are intended. This applies especially to papers introducing a new or refined distinction, or drawing attention to a previously overlooked relationship that might, if accepted and acknowledged, constitute a phenomenon, a property perceivable as being instantiated. Typically papers do this in the early stages of the growth of a complex

of distinctions, relationships and properties in an effort to persuade others that the theory is pertinent, plausible, or useful. Unfortunately once absorbed into a complex theory distinctions and relationships become less attractive. Can it ever be sufficient to claim simply that distinctions can be made? Does it confirm a theory if the distinctions it comprises are ‘shown’ to be discernible? In what way is a researcher’s perspective transformed simply by using a richly technical language for describing observation?

One weakness of current theories in mathematics education is a lack of definite conclusions, as well as an incomplete set of necessary conditions. But what conclusions might be possible? A plausible conjecture is that there is an often unstated assumption or hypothesis when theories are called upon: if the conditions of the theoretical constructs are met (if the labels really do apply), then learning will take (or have taken) place. However, to make such a conclusion possible it would be necessary to articulate a large number of conditions concerning the psyche of the learner(s) and teacher, their current states, and the many forces acting upon the situation from institutional pressures of all kinds.

Variation theory (Marton 2006) provides a notable exception, because it attempts to articulate certain conditions that are necessary for something to be learned; if those conditions are absent, learning will be at best fragmented and partial.

### **COMING TO APPRECIATE A THEORY OR PARTIAL THEORY**

In order to use and appreciate what a particular theory offers, it is necessary to locate what is discerned and discernible, what relationships are afforded or highlighted, what properties are consequences, and what grain size the theory addresses. This in turn requires perceiving theoretical properties or constructs as instantiated in the particular or specific situation.

#### **Bricolage**

If while reading a paper or attending a session a distinction offered that makes immediate sense of past experience or current enquiries, it may be adopted and adapted for use. This can be called *cherry-picking* (after the expression in English which summons up an image of picking the best fruit but leaving the rest of the tree and its life-cycle alone), but more positively, it contributes to *bricolage* (the use of found objects and available resources to create works of art or to make repairs). Reading around the distinction, finding examples of its use, perhaps becoming informed about its origins and where it arises in the overall thinking of the originator can be thought of as movement along a spectrum of involvement towards *immersion*.

## **Immersion**

To immerse yourself in a theory is to engage with all of its constructs, and, strictly speaking, not to stray outside of those boundaries, at least until it is completely familiar. Acceptance is achieved when papers are published in journals refereed by other experts who are also immersed in the same theory.

## **Along a Spectrum**

Complete immersion is relatively rare. Most researchers depend upon the elaborations of a few scholars who devote considerable time and attention to immersion. A common pattern is for an idea (a distinction, a construct) to be introduced into mathematics education (cherry-picking/bricolage) and when it is picked up and used by a few researchers, people begin to probe its origins, moving along a spectrum of increasing immersion in the setting and the background.

It seems that adopting a distinction is relatively easy to do, especially where it makes sense of past or current experience and where it informs future actions. However it is all too easy to misinterpret or mis-appreciate the core meaning and use of a distinction, and then to use it in ways unintended by the originators. The *ZPD* provides an excellent example: misunderstood through a single example, it became a metaphor for almost any teacher initiated action. Picking up a framework of distinctions is slightly more complex but can be assisted by relating the distinctions to personal experience (Mason 1999, Mason & deGeest 2010).

Absorbing a complex of frameworks and a theoretical perspective that generates or encompasses them is much more difficult, and much less congenial for people who want to get on and act in the world rather than becoming scholars of someone else's ways of thinking, expressing and acting.

## **DISCERNING BETWEEN THEORIES**

In order to distinguish between theories it is necessary to appreciate both, and to work at agreeing the meanings of technical terms (distinctions) to see whether there is overlap or significant difference. What the theory foregrounds, and what therefore its backgrounds, are critical. To do this effectively requires a commonly accepted approach to agreeing meaning through usage.

When searching for a 'suitable theory' on which to base an enquiry, what is really being sought is resonance with underlying assumptions and approach, with distinctions being made as being pertinent to the enquiry and as being likely to be observable, and with the sorts of conclusions, explanations, valuations etc. likely to arise. Too often the search ends when something doable has been located.

## **CONDITIONS FOR CONJOINING THEORIES**

Radford (2008) has coined the notion of a *semiosphere* as a semiotic domain in which different theories might be conjoined, amalgamated or at least compared and

contrasted, through considering the principles, methods and questions associated with them. I propose that in order to compare, conjoin or even confirm theories successfully it is necessary to be clear on the intentions or uses of those theories and to reach mutual clarification of the worlds of experience they occasion, including the phenomena they recognise and at what grain size, and the sorts of conclusions, explanations etc. they afford: in short, what they foreground as worthy of attention. More specifically, this means the details discerned, the relationships recognised, the properties perceived as instantiated, and any consequences.

For me the only underlying problem about conjoining theories (apart from intense investment in and identification with a particular discourse) arises from participants' relation to complexity (Davis, Summara & Simmt 2006). Consistency and compatibility can be valued, but so can contrast/contradiction in a multiplicity of perspectives corresponding to the perceived complexity of the domain. The former stresses orthodoxy, the latter is flexibility.

What blocks the comparison of theories is commitment to and investment in the exclusivity, the universality of different perspectives by different people. Underlying perspectives may differ, and there may be variation in what is considered worth attending to and for what purpose, but no theory can expect to be universal. The ancient teaching-story of the blind men and the elephant (Shah 1970 p119) comes to mind (an example of a situation resonating and-or triggering connections). Multiple theories applied to the same situation are likely to reach multiple conclusions, to characterise situations in different and even apparently incompatible ways.

The real difficulty in conjoining theories is in finding a language which accepts that the situation being analysed, like the elephant, is always greater and more complex than the sum of the component observations. One aim of Systematics is the eventual abandoning or fusing of insights gained through discerning various 'systems', in order to return to a richer in-dwelling of the complexity of the experience. Complexity theory (Davis *et al. op. cit.*) appears to attempt something similar. Whether other theories aspire to their own transcendence is not so clear.

One of the unfortunate features of theories is that the use of assertions such as '... is ...' implies exclusiveness. Thus 'mathematics is social' or 'mathematics is communication' can be taken to imply that other ways of perceiving are invalid, that mathematics is 'nothing but' social or communication. Replacing 'is' by 'can be seen as' is one way to gain flexibility, acknowledge complexity, enrich analysis and be open to insights from other perspectives. To express is, again, to over-stress.

## CONCLUSION

My analysis of theories has used the first five terms of a structural approach based on qualities of numbers known as Systematics (Bennett 1993). My discourse is centred on the structure of attention, suggesting that each individual or group of researchers accumulates a collection of distinctions, relationships, properties, significances and



values, and a discourse of technical terms for these. These then form the discourse in which actions and activity are described, actions planned, observations and experience analysed, salient results reported, and claims validated. Most importantly, developing a common approach to agreeing meaning of technical terms takes place through discerning situations or incidents, recognising relationships, proposing these as instantiations of phenomena (perceived as having properties), and seeking agreement with colleagues about the relevance of the labelled distinctions, relationships and phenomena.

## REFERENCES

- Bennett J. (1993). *Elementary Systematics: a tool for understanding wholes*, Santa Fe: Bennett Books.
- Brown, S. Collins, A. & Duguid, P. (1989). Situated Cognition and the Culture of Learning. *Educational Researcher* 18 (1) p32-41.
- Bruner, J (1966). *Towards a Theory of Instruction*. Cambridge: Harvard University Press.
- Davis, B. Summara, D. and Simmt, E. (2006). *Complexity & Education: inquiries into learning, teaching and research*. Mahwah: Lawrence Erlbaum.
- Goodman, N. (1978). *Ways of World Making*. London: Harvester.
- Greeno, J. (1994). Gibson's Affordances. *Psychological Review*. 101 (2) p336-342.
- Hanson, N. (1958). *Patterns of Discovery*. Cambridge: Cambridge University Press.
- Marton, F. (2006). Sameness and Difference in Transfer. *Journal of the Learning Sciences*. 15(4) p499-535.
- Mason, J. (1987). Refractions of Mathematics Education in the LOGOSPHERE. in J. Hillel (Ed.) *Logo and Mathematics Education: 3rd International Conference*. Montreal : Concordia University p40-55.
- Mason, J. (1998). Researching From the Inside in Mathematics Education. In A. Sierpiska & J. Kilpatrick (Eds.) *Mathematics Education as a Research Domain: a search for identity*, 2 vols, Dordrecht: Kluwer, Vol 2 p357-378.
- Mason, J. (1999). The Role of Labels for Experience in Promoting Learning from Experience Among Teachers and Students. In L. Burton, (Ed.), *Learning Mathematics: from hierarchies to networks*. p187-208. London: Falmer.
- Mason J. (2002). *Researching Your Own Practice: the discipline of noticing*, London: RoutledgeFalmer.
- Mason, J. (2003). On The Structure of Attention in the Learning of Mathematics, *AAMT*, 59 (4) p17-25.

- Mason, J. & DeGeest, E. (2010). How an Experientially Presented Labelled-Framework Approach to Professional Development at a Distance Can Influence Teachers' Practice. In M. Pinto & T. Kawasaki (Eds.) *Proceedings of the 34<sup>th</sup> International Conference on the Psychology of Mathematics Education*. Belo Horizonte: PME Vol 3 p273-280.
- Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM Mathematics Education* 40 p317–327.
- Shah, I. (1970). *The Dermis Probe*. London: Jonathan Cape.
- Sriraman, B. & English, L. (Eds.) (2010). *Theories of mathematics Education: seeking new frontiers*. Advances in Mathematics Education. Heidelberg: Springer.
- van Hiele, P. (1986). *Structure and Insight: a theory of mathematics education*. Developmental Psychology Series. London: Academic Press.
- Watson, A. & Mason, J. (2007). Taken-as-Shared: a review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education* 10 (4-6) p205-215.

# **THEORETICAL GENESIS OF AN INFORMAL META-THEORY TO DEVELOP A WAY OF TALKING ABOUT MATHEMATICS AND SCIENCE EDUCATION AND TO CONNECT EUROPEAN AND NORTH AMERICAN LITERATURE**

John Monaghan

School of Education, University of Leeds

*This paper explores the development of ideas in a recent journal paper that connects theoretical approaches in mathematics and science education. It re-constructs the development of the ideas presented in the journal paper through data obtained from interviews with the authors. This paper considers writing, engagement with research and other voices when people connect theoretical approaches.*

## **INTRODUCTION**

At CERME 6 Michèle Artigue (verbally in the theoretical perspectives working group) and I (Monaghan, 2010) raised a possible analogy between ‘instrumental genesis’ (Guin & Trouche, 1999) and the development of researchers’ theoretical ideas; we referred to this as ‘theoretical genesis’. In this paper I explore the genesis of an informal meta-theory presented in Ruthven, Laborde, Leach, & Tiberghien (2009). With regard to the CERME 7, Working Group 16 ‘call for papers’, the current paper touches upon issues in the 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup> and 8<sup>th</sup> bullet points – strategies, limits, conditions and outside (of mathematics education) influences in connecting theories.

This paper is structured as follows. The first two sections explore the construct ‘theoretical genesis’ and then outlines the Ruthven et al. (2009) paper. The next section outlines my methodology, and the coding, for exploring the theoretical genesis of these four researchers. I then present a ‘story’ of the development of ideas in the course of writing Ruthven et al. (2009). This is followed by a discussion of issues raised by the story. I end with issues for further consideration.

## **THEORETICAL GENESIS**

As the term ‘theoretical genesis’ was coined by analogy to ‘instrumental genesis’ I begin with a summary of the latter term. The root word ‘instrument’ pays homage to Vérillon & Rabardel’s (1995) distinction between a tool, as a material object, and an instrument as a psychological construct – a tool is just an artefact until someone appropriates it and integrates it into activity. Tool and person are interrelated: the tool shapes the actions of the person, *instrumentation*; the person uses the tool in specific ways, *instrumentalisation*. The process of making a tool an instrument is called ‘instrumental genesis’. This basic account, constructed to analyse students, and later teachers, use of technological tools in mathematical activity, is, to me, important, suasive and open to analogies of taking theories as tools which are appropriated by researchers in mathematics education activity. Subsequent

development of instrumentation theory, in which the psychological component, gestures and instrumental genesis are explained by way of the Piagetian notion of a *scheme*, are, I believe, a little problematic in themselves (see Monaghan, 2007) and not particularly useful for the analogy considered in this paper.

Monaghan (2010) was a reaction to Bikner-Ahsbals (2010) which addressed ‘networking theories’. An important part of this reaction was that theories cannot be separated from the people theorising. I stated that researchers come to theories with past experiences and ideas. They interact with theories in the light of these past experiences and ideas and, if the theory, or rather, a part of it, is convincing for the researcher, then the researcher develops and a variant theory, adapted to the past experiences, ideas and needs of the researcher, emerges. This is what I referred to as theoretical genesis. On reading Ruthven et al. (2009) the idea of interviewing these authors about their theoretical genesis arose.

Although the term ‘theoretical genesis’ is new, it relates to the history of ideas. In the history of mathematics Lakatos (1976) challenged then dominant ideas on the development of proofs and the importance of counterexamples. A forerunner in the sociology of scientific knowledge is Kuhn (1970) who characterised the history of science into periods of normal science punctuated by periods of radical reconceptualisation. Further developments, including actor network theory accounts of scientific advance, reconceptualised the nature and importance of agency in the development of ideas/knowledge (see Pickering, 1995). There is not space to do more than acknowledge this field here but I briefly return to this matter in closing.

### **RUTHVEN, LABORDE, LEACH, & TIBERGHIE (2009)**

Ruthven et al. (2009) is an important paper which advances the literature on connecting theories. I summarise it for the sake of completeness but this summary does not do justice to the rich details and arguments. I recommend the reader reads it.

The paper frames its focus in terms of recent interest in design research, e.g. Kelly (2003), and the contribution of European didactical research in the design of teaching sequences that pay high regard to content knowledge. The authors comment that didactical design may be informed by tacit professional knowledge and/or ‘grand theory’. They note a move to develop “specific frameworks, intermediary between grand theory and the process of design” (Ruthven et al., 2009, p.330) and attempt to “develop more overarching ideas about the relations between grand theories, intermediate frameworks, and design tools” (ibid.); in the words of one author, after the paper appeared, they developed “a kind of informal meta-theory of design tools, intermediate theories, grand theories”. Their first example concerns Brousseau’s Theory of Didactical Situations (TDS) as an approach to the design of teaching sequences in mathematics. Grand theories used come from Piaget and Bachelard; TDS is the intermediate theoretical framework; design tools are didactical situations and didactical variables. They then present and contrast two design approaches in science education in which design tools are informed by grand

theories and intermediate theoretical frameworks. Terms and theorists aside there are commonalities in these science education design research approaches though “different cultural emphases and values can act against synthesis of intermediate frameworks and promote a proliferation of design tools” (p.341). Intermediate frameworks and design tools are viewed as ‘instruments’ to guide theoretically informed decisions about the design of teaching sequences. TDS is sufficiently developed and recognised that reference to grand theories behind it are rare; “the final teaching process results from ... what is usually an iterative process of development” (p.332). The science education intermediate frameworks, like TDS, relate epistemology, learning and teaching. They mainly draw on grand theories about epistemology and learning rather than teaching; teaching activity is developed from professional knowledge. The paper notes: not all design tools are strongly theorised; it is important to raise awareness of design tools available; the importance of differences in the grain size of teaching sequences; the value of dialogue between different traditions of design research to develop a more comprehensive perspective.

## METHODOLOGY

I approached the four authors (referred to as AT, CL, JL & KR) about interviewing them regarding their theoretical development in the course of writing this paper. Two group interviews were conducted by *skype*<sup>TM</sup>, the first with AT and CL, the second with AT, JL and KR. Both interviews lasted about 45 minutes. Each interview had three questions: how did this paper begin; what did you learn in the process; how does the paper advance knowledge. I viewed the first question as crucial as it related to theoretical genesis. In both interviews about 2/3 of the time was spent discussing the first question. The interviews were semi-structured in as much as I asked sub-questions to follow up themes the authors introduced. The authors spoke freely and most of the interview transcripts consisted of interviewee text.

The purpose of the interviews was to get an account of the development of ideas, a story of theoretical genesis; there is a sense in which research is story-telling which aspires to objectivity. The story, in my plan for the paper, would be the basis for my discussion of theoretical genesis. A problem (of sorts) was that the interviews provided two (interrelated) stories. I subjected the transcripts to open coding (à la Strauss and Corbin, 1998) but did not follow a grounded theory approach further than this as I had a theoretical framework, instrumentation theory. I marked extracts of transcribed text and gave them names (codes) which resulted in a lot of codes. I then merged codes where possible to produce a manageable number. An example of a merged code is ‘people’ and the following extracts were all ‘people’ codes: I asked several people to help me; AT had been working with several colleagues; JL contacted KR. There was no opportunity for a second person to code the transcripts, so my codes are open to criticism with regard to reliability. The codes were:

Communication, e.g. “to speak with the audience”

Communities of practice (CoP), e.g. “in the European academic community”

Design, e.g. “thinking that would be the design of teaching”

Learn, “it helped me to see what a future agenda for work is”

Meet, e.g. “I popped in to see KR at Cambridge”

People, see above

Purpose, e.g. “trying to put together academic perspectives”

Subject (mathematics/physics), e.g. “I’m on the side of mathematics”

Theory, e.g. “we don’t have the same theoretical framework”

Write, e.g. “we would like to publish this work in English”

There was overlap between the codes, e.g.[our problem] “how to report about research done in very different theoretical frameworks” was coded ‘theory’ and ‘write’. The codes, however, were simply a means towards producing a story. After coding I produced a narrative from each interview, a summary using the interviewees’ own words, paraphrases and my codes. I then merged the two narratives to construct the ‘story’ presented in the next section.

## A STORY

The following is based on the interviews and narratives. It uses the participants’ own words and my codes wherever possible.

AT, CL, JL & KR are senior academics and, as such, write and attend conferences (meet people). AT knows CL through French didactics CoPs, and knows JL through science education CoPs. CL knows KR through mathematics education CoPs. JL knows KR through British education CoPs. About six years ago AT was asked, by a research council, to write a review on naïve knowledge on the design of teaching sequences. She collaborated with others and decided the review should have four chapters. Towards this she worked with CL on one chapter and asked JL to give a keynote speech at a conference in Paris. JL’s work was also focused on the design of teaching sequences. AT and CL worked on their chapter for one year and wanted to write a paper in English. Simultaneous with this AT met JL at several European conferences and they also exchanged ideas via e-mail. Both AT and JL place considerable value on subject knowledge in teaching and learning and felt that they has important things to say on the design of teaching sequences that would advance research but a stumbling block was differences in theoretical perspectives, basically that Francophone researchers embraced theory in a way that Anglophone researchers, in general, did not; communicating nuances of French didactical theory in English was not going to be easy. They needed a person who could help them to disseminate their ideas and KR, as a bi-lingual British researcher with a good knowledge of French didactics, was approached. It was decided that rather than writing a book integrating big strands of the literature, the publication should ‘scale down’ to an aspect that was central to everyone’s interest, the design of teaching sequences.

Thus the group of four was formed: AT and JL as members whose past research formed the basis for an intended journal publication; KR and CL as members who

would find a position from which to reflect on this research, and its theoretical bases, to facilitate effective dissemination across different academic CoPs and their practices in academic discourse, as well as bringing a mathematics education perspective to the intended publication. There were two meetings of this group of four with the single focus of writing a journal paper, one in 2006 and one in 2007, but there were also a number of sub-meetings of two members before, between and after these whole group meetings, as well as e-mail exchanges.

A key event in the 2006 meeting was collective reading of a special issue of *Educational Researcher*<sup>1</sup> on design experiments Cobb et al. (2003) in particular. The group worked intensively on those texts and AT/CL brought perspectives from Francophone didactics literature/theory. They attempted to locate perspectives against and alongside each other (rather than making links between theoretical perspectives) in order to make decisions about which aspects to address in their journal paper<sup>2</sup>. At this time the construct ‘intermediate framework’ was not explicit; both science educators had, with colleagues, published papers on design tools and their relation to grand theories but without mentioning intermediate frameworks. It was on the second day of the 2006 group meeting that the construct ‘intermediate framework’ was formed because of a realisation that “we were mainly speaking about tools without really distinguishing them” (AT). JL likened intermediate frameworks to Cobb’s (2007) phrase “make the theory do the work”.

The writing of the original paper was shared out, by sections, between the four members. This was a learning experience for all in terms of learning about each others research, constructs and theoretical orientations and also in terms of learning more about familiar theories and the new theoretical development. CL’s work illustrates learning through writing. CL was assigned the section on TDS in the original paper. CL knew this theory well. It was not her intention to summarise the entire theory, only aspects relevant to the paper. CL became aware, through writing, that TDS was, in her words, an epistemological obstacle to her as, at the beginning of the collaboration “it was very clear that when we did a design we had to use this theory” but by the end of the collaboration, with the grand theory-theoretical framework-design tools development, “I could see the TDS within a range of different theories, I could structure more a range of theories through the tools we were proposing, through the intermediate theory, through the design tools” (CL). Further to this, the didactical situation tool of the Ruthven et al. (2009) paper was not present in the original draft of the paper (more on this below).

The paper was submitted in June 2008 and revised in December 2008. Writing the original submitted and the revised versions of the paper brought in two other groups of people the audience (*Educational Researcher* readers) and the reviewers.

... we needed to think about ... who is the audience for the work that we want to write, where are they coming from, what are their understanding, what are the ideas that they already know ... [and to show] the design research community that has its base in the US ... what the distinctive contribution might be of the European approaches to that North

American audience, so I think a lot of writing ... [was to bridge ideas] between the European tradition, primarily a didactical tradition, and a North American learning sciences tradition.

The reviewers gave extensive and, to the authors, useful feedback which provided an occasion to develop the ideas further. Commenting on this feedback an author stated:

the notion of design tools was well developed in the science education [work] ... and [in] the version of the article that was originally submitted ... [this idea] was more tied to the science education work ... [in response to reviewer] comment on the fact that this just wasn't just developed in relation to Brousseau's TDS ... [in the process of rewriting] we talk about the a-didactical situation tool. The didactics variables tool was already there and was kind of recognised as a tool even if it hadn't directly been given that name but one of the main aspects of the rewriting was to formulate this notion of the a-didactical situation tool. As the article points out, this is not that way that Brousseau himself ... would talk about it ... so it's a good example of ... a process of writing for an audience and then getting feedback from reviewers, can ...[shift]... the model that you build in writing a paper so that it develops greater coherence and communicates more effectively.

The development of this 'grand theory-theoretical framework-design tools' meta-theory, then, can partially be ascribed to the need to communicate in writing for a specific journal and these other voices. Indeed, all of the authors repeatedly and positively mentioned personal development through the need to be explicit in the writing, as AT noted:

... in France we are very theoretically orientated and it is not easy to disseminate, to make clear our theoretical orientations ... [so collaboration] is very, very fruitful for us because it make explicit our own approach and it is a shared experience.

## DISCUSSION

I raise four interrelated themes from the above story which I consider important to the development of theories before revisiting the construct 'theoretical genesis'. The four themes are writing, learning, engagement with research and other voices.

Writing is a recurring thread in the above story: the paper started when AT was asked to write a review; AT worked on a draft chapter with CL which they wanted to write in English; when the team of four was composed, they decided to write a journal paper and targeted a specific journal; decisions of what to include were partially determined by the constraints of writing journal length papers; the writing was shared out; the submitted paper was rewritten. Whilst this is hardly surprising, as writing is a part of the job of academics, writing as a goal of joint activity of researchers networking theories is not something which I am aware has been raised in the literature on networking mathematical education theories. Writing to these authors was theory directed action and the need to be explicit in their writing acts was a part of their theoretical development, as the AT quote above demonstrates.



The need to be explicit shows that researcher learning in networking and developing theory is linked to writing (for other academics). I am sure that this will resonate with the experiences of most readers as they recall times where they have spent hours trying to get the words in the paragraph they are writing to express the thoughts in their mind. It is evident in the story that each of these four researchers learnt about the research and theoretical constructs of the other three researchers but they also learnt about their own theoretical stances. Networking theories usually involves networking with people and communicating ones own understandings, and sometimes we realise aspects of our own understandings in communicating that were hitherto implicit and we develop theoretically in the communication process. This happened in the case of CL in the section of the story which refers to epistemological obstacles. Brousseau's (1997) writings have popularised this construct amongst mathematics educators but Brousseau adapted the construct from the French philosopher Bachelard who applied it more widely than to just mathematics. An epistemological obstacle is any piece of knowledge that is relevant to a particular stage of a person's knowledge development but becomes an obstacle for further knowledge development. In the act of writing CL realised "when we did a design we had to use this theory" but networking theories enabled her to not only view learning design anew but to see TDS as an epistemological obstacle for her.

Networking theories in mathematics education is a process and a moment in the trajectory of theory development which interrelates with at least two other trajectories, networking researchers' own research and the research of others. In this paragraph I dwell on the research of others but first note that the advancement of and the networking of AT and JL's own research was clearly a catalyst for Ruthven et al. (2009) and the development of the informal meta-theory. The collective reading of the special issue of *Educational Researcher* and the paper of Cobb et al. (2003) was clearly an important event: "this was a kind of source of inspiration for us", AT/CL interview; "one of the key events was actually when we started reading and taking seriously the articles in the special issue of *Educational Researcher*", AT/JL/KR interview. Whilst the special issue and the Cobb et al. paper were valued by the authors, they also felt that related design research had been going on in Europe for some time and that there were differences in what the research communities either side of the Atlantic did, in particular that European researchers, in general, take subject content (mathematics & science) more seriously than North American (NA) researchers. So an emergent issue, reactions to the special issue, was transformed into a goal of Ruthven et al. (2009): to communicate European design research to a NA audience, which leads to the next theme of this discussion, other voices.

In academic writing other voices (Wertsch, 1993) enter our writing: research collaborators, authors of research papers that have influenced us, the audience (for the journal we have selected) and, later, reviewers. These other voices were explicitly mentioned in the interviews, as interview extracts in the story above show. More controversially I hold that the audience and the reviewers' voices actually contributed to the networking and the theory development. *Educational Researcher*

is a respected NA-based journal. In order to communicate with its NA readers it was important, as an interview quote in the story makes clear, to mediate or bridge, i.e. to network, ideas from a European didactical approach and a NA learning sciences approach. In the case of the reviewers, they commented that the notion of design tools was not well developed in relation to Brousseau's TDS which led to the inclusion of the a-didactical situation tool in the revised paper.

I now consider the term 'theoretical genesis', its validity and whether it is appropriate to describe the four authors' development in the writing of Ruthven et al. (2009). I start by noting that the term is just two words and we can describe the process without the use of these two words. Nevertheless, constructs as terms, can be useful to encapsulate a set of ideas, though it is wise to avoid an unnecessarily large number of pseudo-scientific terms. Theoretical genesis, as noted above, was coined by analogy to the term 'instrumental genesis' and I look at it via the main features of instrumental genesis, which I consider to be: it concerns an agent (or agents together) appropriating an artefact for a purpose; it is a process over time (though the duration is not important); there is two way interaction (instrumentation and instrumentalisation) between agent(s) and artefact; the agent(s) recognise and utilise affordances and constraints of the artefact. These features, I argue, were present in the development of Ruthven et al. (2009). A theory is an artefact and just an artefact until a person appropriates it and integrates it into academic activity; the team of four appropriated various (bits of) theory for a purpose (writing the paper). Ruthven et al. (2009) was a process that went through several stages: AT's initial discussions and writing with CL and JL; the recruitment of a 'go between' (KR); meetings and sub-meetings to clarify ideas; reconceiving ideas; various writing and rewriting actions. There was a two way interaction between authors and the theories, the authors shaped (bits of) theory for their purposes and theory advanced their thinking of design issues. The authors learnt many things in the process and this included the affordances and constraints of theories with regard to their goal of understanding the role of theories in the design of teaching sequences, e.g. CL's epistemological obstacle regarding TDS. But there are limits to the analogy, at least with regard to Guin & Trouche's (1999) technological instrument focus, because a theory, unlike a calculator, does not require physical manipulation and personal agency is greater with a theory than with a calculator, i.e. I am free to regard Brousseau as a constructivist or whatever but I am not free to do something on a calculator that it is not wired or programmed to do what I want it to do. Further to this, but neutral with regard to the analogy, instrumentation and instrumentalisation appear more pronounced with a theory-person dyad than with a calculator-person dyad: a theory, even a misunderstood theory, will shape my data analysis and hence my interpretation of data and I will adjust, consciously or not, the theory to my perceived needs. The term 'theoretical genesis' does, to me, appear to be a valid term to apply to appropriation of theory in the course of writing Ruthven et al. (1990). This writing of this paper, however, generated a new theory (artefact), albeit an informal meta-theory, which is not something expected in instrumental genesis.

Regarding the development of the four authors in this theoretical genesis I turn to what they claimed to have learnt. AT, CL and JL learnt about the work (including the theoretical stance) of the others and about their own work, which may be taken as learning about positioning their own work in relation to the work of others (CL “I could see TDS within a range of different theories”). Theoretical genesis appears, for these researchers, to include making connections between theoretical stances. KR’s theoretical genesis also concerned such connections but as an ‘outsider’ “trying to find some position from which to reflect on these theories”. Further learning in this theoretical genesis included using the theory. AT learnt “how to manage, how to construct these intermediate theories”. CL learnt that “didactic variables could be used not only in mathematics”. JL learnt that “you can actually strip away the theory now and use them [design tools] as tools” [in the design of teaching sequences]. The use KR mentioned is encapsulated in the title, a way of talking about mathematics and science education and to connect bodies of literature.

### **ISSUES FOR FURTHER CONSIDERATION**

Space is limited but I briefly raise issues for further consideration concerning this study, the sociology of scientific knowledge and research praxeologies.

The networking and generation of theories/theory explored in this paper is only one example of theoretical genesis and is, I suspect, quite an atypical example. As evidence for this consider Radford (2008), cited by many papers in the theoretical perspectives section of the CERME 6 proceedings. Radford states that “a theory can be seen as a way of producing understandings and ways of actions based on” (p.320) a system of basic principles, a methodology and a set of paradigmatic research questions. I think that this statement holds for much theory-informed empirical research but not for Ruthven et al. (2009) as it does not include a set of research questions. In this respect, then, it is atypical. It would be interesting to research theoretical genesis in research conforming to Radford’s statement but there would be considerable difficulties in conducting such research. More realistic research could monitor theoretical development/genesis in post graduate student research.

At the beginning of this paper I raised links between the focus of this paper and work in the sociology of scientific knowledge. It is interesting that there are few references in the mathematics education networking theories literature, including CERME proceedings, to this field of study. Study of generating and networking theories in mathematics education in the wider setting of the sociology of scientific knowledge would, undoubtedly, be useful. As a small step towards this CL’s epistemological obstacle, considered above, might be seen as a ‘local’ (21<sup>st</sup> C France) instance of Kuhn’s (1970) hold of ‘normal science’, and one researcher’s realisation of this

During CERME 7 WG 16 Artigue, Bosch and Gascón (this volume) was discussed and it was clear that much that I report on here could be recast in terms of research praxeologies but there is, unfortunately, no space left for me to do this.

## NOTE

<sup>1</sup>*Educational Researcher*, **32**(1), 2003.

<sup>2</sup>Not all aspects were included in Ruthven et al. (2009) due to length considerations and the refinement of ideas, e.g. in 2006 the paper had a semiotic dimension that was not in the final paper.

## REFERENCES

- Bikner-Ahsbahr, A. (2010). Networking of theories: why and how? *Proceedings of Cerme 6*, 6-15. <http://www.inrp.fr/editions/editions-electroniques/cerme6/>
- Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Norwell, MA: Kluwer Academic.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R. & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, **32**(1), 9-13.
- Cobb, P. (2007). Putting philosophy to work. In F. K. Lester Jr (Ed.), *Second handbook of research in mathematics teaching and learning* (pp. 3–38). Charlotte: Information Age Publishing.
- Guin D. & Trouche L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning* **3**(3), 195-227.
- Kelly, A.E. (2003). Research as Design. *Educational Researcher*, **32**(1), 3-4.
- Kuhn, T. (1970). *The structure of scientific revolutions*. 2<sup>nd</sup> ed. Chicago: University of Chicago Press. 1<sup>st</sup> ed. published 1962.
- Lakatos, I. (1976). *Proofs and Refutations: The logic of mathematical discovery*. Cambridge: Cambridge University Press.
- Monaghan, J. (2007). Computer algebra, instrumentation and the anthropological approach. *Int. Journal for Technology in Mathematics Education*, **14**(2), 63-72.
- Monaghan, J. (2010). People and theories. *Proceedings of Cerme 6*, 16-23. <http://www.inrp.fr/editions/editions-electroniques/cerme6/>
- Pickering, A. (1995). *The mangle of practice: Time, agency, & science*. Chicago: Chicago University Press.
- Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *ZDM – The Int. Journal on Mathematics Education*, **40**(2), 317-327.
- Ruthven, K., Laborde, C., Leach, J. & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher*, **38**(5), 329-342.
- Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage.
- Vérillon, P. & Rabardel, P. (1995). Cognition and Artefacts: a contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, **10**(1), 77-101.
- Wertsch, J.V. (1993) *Voices of the mind*. Cambridge: Harvard University Press.

# MEANING OF MATHEMATICAL OBJECTS: A COMPARISON BETWEEN SEMIOTIC PERSPECTIVES

Giorgio Santi

NRD, Department of Mathematics, University of Bologna, Italy

*In this paper we present a networking of semiotic perspectives to frame the issue of meaning of mathematical objects. We will connect Duval's structural approach and Radford's cultural semiotic approach to analyse students' difficulties with the meaning of mathematical objects when exposed to semiotic transformations.*

Keywords: semiotics, transformations, networking theories, objectification, meaning

## INTRODUCTION

D'Amore (2006) shows that, at all school levels, students encounter difficulties in coping with the meaning of mathematical objects in relation to their semiotic representations. These results question Duval's claim that *conversion* (transformation that involves 2 semiotic systems) is the key cognitive function that guarantees a differentiation between mathematical objects and their representations, and that conversion is *the* cognitive threshold the student has to overcome in order to reach a correct conceptualization. Furthermore, Duval considers conversion the main source of difficulties when learning mathematics since it clashes with the cognitive paradox that leads to identify the mathematical object with its semiotic representation. D'Amore's results show that at all school levels students find severe difficulties *also* when they are exposed to *treatments* (transformation within the same semiotic system). We have singled out 2 episodes taken from primary school and university.

Primary school students working on probability recognise that  $\frac{3}{6}$ ,  $\frac{1}{2}$ ,  $\frac{50}{100}$ , 50% represent the probability of rolling an even number, when throwing a 6 face die, after performing both treatment and conversion transformations. But after a treatment transformation from  $\frac{1}{2}$  to  $\frac{4}{8}$  students and teacher don't recognise in  $\frac{4}{8}$  the same probability. It is interesting that the teacher declares that « $\frac{4}{8}$  *cannot* represent that event because the die has 6 faces. Dice don't have only 6 faces, there are also dice with 8 faces; in that case yes, the fraction  $\frac{4}{8}$  represents the probability of rolling an even number».

The research problem has been framed within Duval's structural and functional semiotic approach that considers meaning as a relation between the *signifier* and the *signified*, i.e. between the representation and the entity the representation refers to. Since a mathematical object has more semiotic representations there are more signifiers for the same signified, that are equisignificant. D'Amore termed the phenomenon we described above as a *change of meaning due to treatment transformations*, to express the idea that students break the equisignificant relation

that binds the different representations through the common mathematical object they denote.

The intrinsic inaccessibility of mathematical objects makes changes of meaning an expected phenomenon in conversion transformations since the passage from one representation to the other is not sustained by the direct reference to the common object nor by transformation rules that bind the two semiotic systems. It is an unexpected phenomenon in treatment transformations since the passage from one representation to the other is sustained by the specific transformation rules of the semiotic system involved.

This paper addresses the issue of changes of meaning by networking Duval's and Radford's semiotic approaches. Using suitable networking strategies, our aim is to provide a conceptual framework to scrutinize the issue of "changes of meaning". The networking process we propose leads to the broadening both of the notion of sign and the notion of meaning, by integrating different ontological and epistemological standpoints. In the next section we provide basic theoretical tools regarding the connection of theories in Mathematics Education. Then we construct a conceptual framework that we will use to analyse a protocol taken from an experimentation with primary school students. In the last section we propose some concluding remarks.

## NETWORKING THEORIES

Connecting theories is important to reduce the inflation of theoretical perspectives thereby bestowing mathematics education a global coherence, theoretical and methodological unity, effective research design. It also enhances a spin off in education to improve teaching and learning. We introduce some basic conceptual tools that play an important role in the networking of theories.

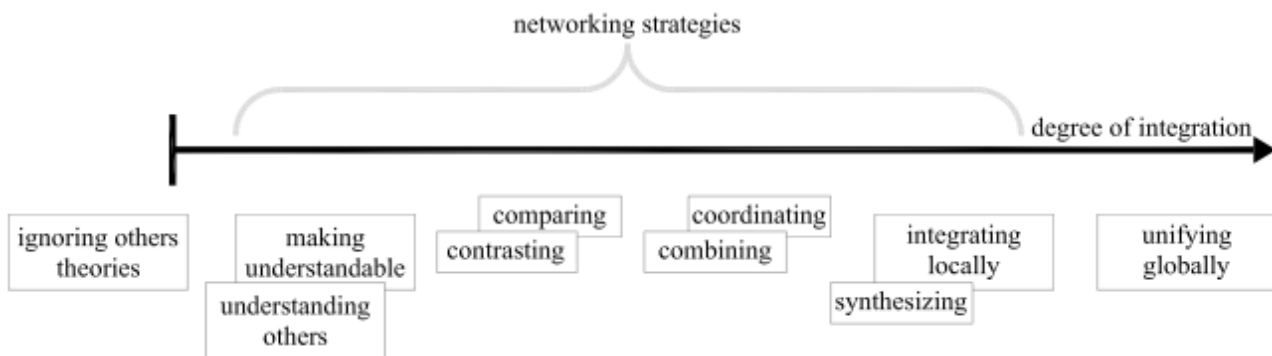
Radford (2008a) develops his analysis of theories within a social-cultural space that he calls the *semiosphere*, a space that fosters interaction and dialogue between different cultural identities. In a networking perspective, the semiosphere blends two important plots, *integration* of its entities in view of a synthesizing objective and *differentiation* that fosters *identity* and self-knowledge.

In a networking perspective it is useful to give an effective characterization of theories in mathematics education. For our purposes we focus on Radford (2008a) who identifies a theory  $T$  in mathematics education as a triplet  $T(P,M,Q)$ , a dynamic structure consisting of a *system of principles*, a *methodology* and a *template of research questions*. The system of principles defines the nature of the theory. It is important to note, especially when connecting theories, that the principles of a theory are not a juxtaposition of claims but they belong to a structured system. We must take into account not only the principles themselves but also their *hierarchical* position in the system. One or more principles can be common to more than one theory but this doesn't imply that such theories are equivalent, if such principles have a different position and relationship in the hierarchy. The system of principles

is very effective to outline the boundaries (Radford, 2008a) of a theory. The *boundary* of a theory is a threshold that cannot be overcome without loss of identity. The boundary sets the limit of discourse of a theory, beyond such limit the theory contradicts its systems of principles.

Networking of theories in the meta-theoretical space of the semiosphere, requires to blend integration and identity/differentiation plots. There are two possible extreme behaviours. On the one hand, if we overweigh the plot of identity, the risk is to have a set of disarticulated theories that ignore themselves on the other hand if we overweigh the plot of integration we end up with “a theory of everything” that is unable to frame the complexity and variety of teaching-learning processes.

Prediger, Bikner-Ahsbahr, & Arzarello (2008) propose a “landscape” of possible connecting strategies that, within the space of the semiosphere, balance identity and integration. The following schema taken from the aforementioned article shows the networking strategies ordered according to their degree of integration:



**Fig. 1: Degrees of integration in networking strategies.**

In this paper we will apply *coordinating* as networking strategy. Coordinating is a strategy that connects two or more theories in view of describing an empirical phenomenon or tackling a particular research issue. The outcome of this connecting strategy is not a coherent complete theory, but rather a conceptual framework that allows to coordinate different theoretical tools for the sake of a specific objective. Coordinating is a viable strategy when it connects theories that share a high level of *complementarity* and coherence.

## A CONCEPTUAL FRAMEWORK FOR CHANGES OF MEANING

Before connecting the two semiotic perspectives we briefly recall Duval’s and Radford’s system of principles, focussing in particular on how the two perspectives frame the meaning of mathematical objects.

### Duval’s Structural and Functional Approach

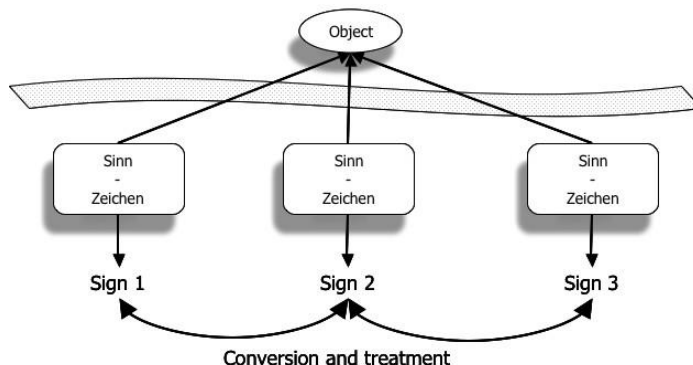
Duval’s (1995) approach stems from a realistic view point that considers mathematical objects *a priori inaccessible* ideal objects. Since mathematical objects are inaccessible entities, the theory pivots around the notion of *semiotic systems* and

the coordination of semiotic systems through *treatment and conversion* transformations. A semiotic system is characterized by a set of *elementary signs*, a set of *rules* for the *production* and the *transformation* of signs and an underlying *meaning structure* deriving from the relationship between the signs within the system.

Mathematical objects, that cannot be referred to directly, are recognised as invariant entities that bind different semiotic representations as treatment and conversion transformations are performed. Duval identifies the specific cognitive functioning to mathematics with the coordination of a variety of semiotic systems. Both the development of mathematics as a field of knowledge and its learning are accomplished through such specific cognitive functioning.

Duval develops Frege's classical semiotic triangle (*sinn-bedeutung-zeichen*) and identifies meaning with the *couple (sign-object)*, i.e. a relationship between a sign and the object it represents. The sign becomes a rich structure that condenses both the semiotic representation (*zeichen*) and the way the semiotic expression offers the object in relation to the underlying meaning of the semiotic structure *sinn*. Meaning therefore has a twofold dimension: *sinn*, the way a semiotic representation offers the object; *bedeutung* the reference to the inaccessible mathematical object. Meaning making processes and learning require to handle different *sinns* networked through semiotic transformations without losing the *bedeutung* to the invariant mathematical object.

The following schema represents the construction of meaning when several semiotic systems are coordinated to conceptualize a mathematical object.



**Fig. 2 Meaning and changes of meaning in Duval's approach**

In this framework, the research question is: how can students recognize the common *bedeutung* as the *sinn* changes through semiotic transformations? What we have above termed a “change of meaning”, is a change of *bedeutung* as the *sinn* changes.

### **Radford's Cultural-semiotic approach.**

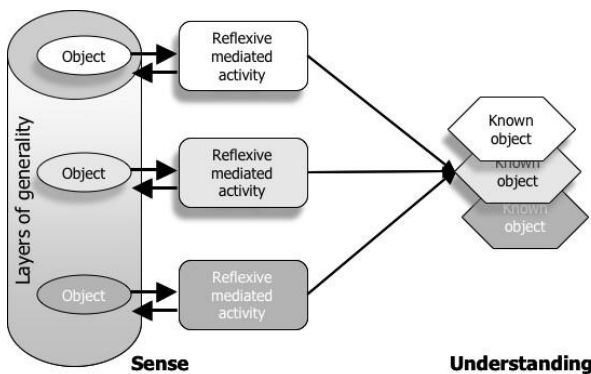
Within a socio-cultural and phenomenological standpoint, Radford's (2008) approach ascribes *reflexive mediated activity*, a central role both in cognition and in



the emergence of mathematical objects. The reflexive activity entangles mathematical objects, semiotic resources, individuals' consciousness and intentional acts, within social practice and a cultural and historical dimension.

Mathematical objects are *fixed patterns* that emerge from the reflexive mediated activity. Mathematical objects lose any ideal and a priori existence but they are ontologically intertwined with the mediated activity from which they emerge. Nevertheless, mathematical objects acquire a form of *ideality* and *existence* in the *culture* that encompasses the reflexive activity.

Learning is considered an *objectification* process accomplished through a reflexive activity, a *meaning* making process that allows to become aware of the mathematical object that exists in the culture, but the student doesn't recognize. The complexity of the objectification process requires to broaden the notion of sign and go beyond its representational role, since signs culturally mediate activity and direct the individual's intention towards the mathematical object. Signs are termed as *semiotic means of objectification* and they include, artefacts, gestures, language, rhythm. Semiotic means of objectification stratify the mathematical object into levels of generality according the reflexive activity they mediate.



**Fig. 3 Meaning and changes of meaning in Radford's perspectives**

Meaning is no longer a mere relation sign-object, but is deeply interwoven with the reflexive activity, with intentional acts culturally mediated by semiotic means of objectification. Meaning is a double sided construct with a personal and a cultural dimension. The personal dimension refers to the individual's intentional acts directed towards a cultural unitary object. The cultural dimension refers to cultural and historical features that are condensed in the general and interpersonal mathematical object brought to the individual by teaching activities. The expected outcome of learning as an objectification process, is the alignment of the *personal meaning* with the *cultural meaning*.

### Boundaries

Although both the theories we analysed are semiotic perspectives - if we look at the relationship with cognition - semiotics has a different hierarchical position in the respective system of principles. Therefore, the two theories have strong boundaries

that separate them. This brings along also differences regarding the nature of mathematical objects and processes. In Duval's approach semiotics plays a *representational* role and it is the very *substance of cognition* that is identified with the coordination of semiotic systems. In Radford's perspective cognition is considered a process of objectification in which *signs mediate* a reflexive activity. Furthermore, the way signs are used is very different. In Duval's perspective, semiotic representations are used *diachronically* through treatment and conversion transformations. Whereas in Radford's perspective, a wide range of semiotic means of objectification are used *synchronously* organized around a particular mediator that changes as the level of generality changes. The different hierarchical position of semiotics allows Radford to broaden the notion of sign to include gestures, artefacts, rhythm, kinaesthetic activity etc. that Duval would never consider semiotic.

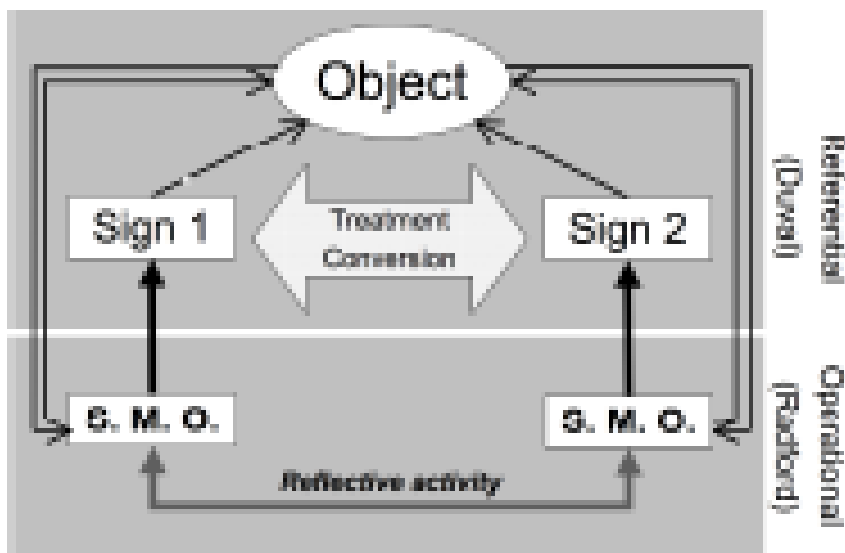
The different hierarchical position of semiotics stems from the different ontologies behind the two theories. The structural and functional approach has a realistic view of mathematical objects that ascribes to semiotics a representational role and to meaning a relation sign-object. The theory of objectification has a pragmatic stand towards mathematical objects that ascribes to semiotics the role of mediating a reflexive activity, the "substance" of ontology, meaning and cognition. Mathematical process are also differently positioned in the system of principles. Duval identifies the mathematical activity with the transformation of signs, subsumed in the robust structure of the semiotic systems that accomplish discursive and meta-discursive functions. Radford considers activity a form of reflection that involves the individual as a whole – his consciousness, feelings, perception, sensorimotor activity etc-immersed in a system of cultural signification that orients his intentional acts.

### **Networking by "Coordinating" strategy**

At a more profound level, any attempt to enlarge one of the theories subsuming elements of the other conflicts with its epistemological foundations. Nevertheless, the boundaries that separate the two theories do not imply an opposition between the two perspectives. Ullmann (1962) highlights two complementary features that characterise the development of mathematical objects: the *operational phase* and the *referential phase*. On the one hand mathematical objects and their meaning emerge from and are objectified by a reflexive activity, on the other hand it is necessary to linguistically refer to the entities that emerge from such practices. The dual nature of mathematical objects – as patterns of activity and as "existing" ideal entities in the culture – implies that also meaning and semiotics have a dual nature. In the connecting theories terminology, the strong differences result in a high level of *complementarity* that accounts for networking by *coordinating* the two perspectives, respecting their *identity*. Coordinating Duval's and Radford's theories allows to encompass the double-sided nature of objects, meaning and representations.

The emergence of a mathematical object and its objectification is described by the cultural semiotic perspective whereas the reference to the object is accounted for by

the structural and functional approach. Meaning as a sense making process of the individual and as the activity culturally condensed in the institutional object are described by Radford's approach; meaning as the interplay between *sinn* and *bedeutung* is framed by Duval's approach. The coordination of the two theories is in turn achieved by the dual nature of semiotics. On the one hand signs mediate reflexive activity on the other hand they represent objects and broaden our cognitive possibilities through semiotic transformations. Our general conjecture is that a successful outcome of mathematical learning processes rests on the dual nature of semiotic resources, i.e. as semiotic registers and semiotic means of objectification; as a semiotic mean of objectification a sign *-synchronically* interwoven with a rich arsenal of mediators – supports the reflexive activity; a sign belonging to a semiotic system can be *diachronically* transformed into another to connote and denote mathematical objects. They are two complementary and interwoven aspects of the same phenomenon.



**Fig. 4 The complementary roles of Duval's and Radford's approaches in framing the meaning of mathematical objects.**

If we disregard signs as semiotic means of objectification, learning is an empty and meaningless manipulation of signs, if we disregard signs as belonging also to semiotic systems, mathematical objects wouldn't have developed into the form of rationality we know today and their conceptual acquisition would be impossible. The changes of meaning can be traced back to semiotic transformations that are not sustained by a mediated reflexive activity that guarantees the relation to the common cultural meaning of the mathematical object. The technology of the semiotic system allows the transformations of signs, but meaning in its cultural and personal sense evaporates, thereby losing also the correct interplay between *sinn* and *bedeutung*.

## ANALYSIS OF A PROTOCOL

### Methodology

The experimentation was carried out with grade 5 students (10 years old) of a primary school in Bologna. The class (20 students) worked according the following schema: presentation of the activity by the teacher and the researcher; students work in small groups with the help of the teacher; general discussion with the whole class. Students worked on sequences represented in a figural register; they were exposed to 3 figural representations of the same sequence. We will analyse the sequence  $a_n = n^2 + 2n$  focussing only on two different figural representations that are reported below.

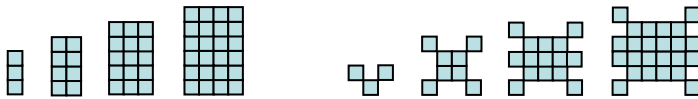


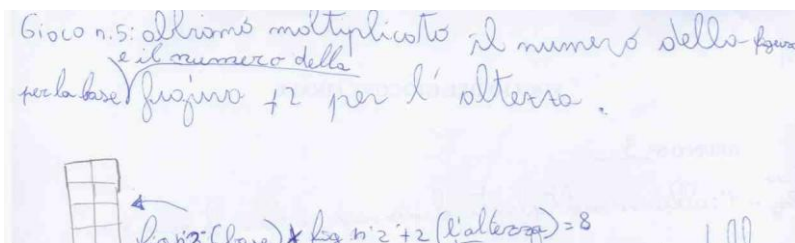
Fig.1 Fig. 2 Fig. 3 Fig. 4

Fig.1 Fig. 2 Fig. 3 Fig. 4

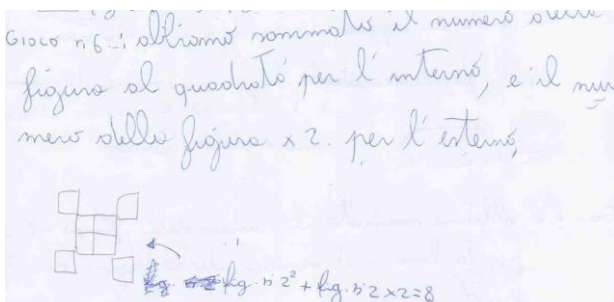
Students were asked to find the general schema to determine the number of elements for any number of the figure. Then they were asked if the different figural representations referred to the same sequence.

### Results

Most of the groups that were able to determine the general schema, also recognized the same sequence as the figural representation changed. Without any explicit request, some students even attempted a first algebraic symbolism to express the general rule. Video tapes testify also students' rich sensory-motor activity, conveyed mainly by gestures, that I cannot relate here but it is clearly condensed in the explanation of the two schemas where the generality of the rule is expressed with spatial-geometrical properties as base, height, inside, outside. Below the protocol of group 5 with the general schema to determine the number of elements of any figure of the sequence.



We multiplied the number of the figure for the base and the number of the figure +2 as the height.



We added the square of the number of the figure for inside and the number of the figure x 2 for outside.

## Interpretation

The pupils were successful in finding the schema for the general term of the sequence<sup>83</sup> and they didn't change meaning when exposed to treatment transformation in the figural register. To understand how students accomplish this result we have to use the conceptual framework we constructed above. Using semiotic resources both as semiotic means of objectification and semiotic registers, pupils grasp the dual nature of mathematical objects and their meaning.

*Cultural-semiotic interpretation:* Students objectify the mathematical sequence within the sociocultural space of the classroom. The use of semiotic means of objectification pivots around the figural representation that allows also the synchronic use of gestures and the sensorimotor activity. The activity was extremely meaningful to the students because it was intimately connected to their embodied experience. As the students are more and more involved in the reflexive activity there is an increasing agreement between the personal meaning and the cultural meaning of the mathematical object, thereby accessing higher levels of generality. This accounts for both the recognition of the same sequence, as the figural representation changes, and the spontaneous attempt to introduce a syncopated algebraic notation for the general term of the sequence.

*Structural and functional interpretation:* Students carry out a complicated network of semiotic transformations that involve both treatment and conversion. The task proposed to students, requires to connect three semiotic systems: the figural register, natural language and the arithmetical register. First of all, a very difficult conversion is necessary to construct the function that associates the number of elements in the figure to the number of the figure. Also to recognize the general schema of the sequence, students perform a conversion that involves the above registers; they first find the number of elements for a small number then they generalize the schema to a big number, thereby arriving to the general term of the sequence. The conversions are carried out passing the following order: figural register-arithmetical register (to calculate the number of elements in the figure)-natural language (to represent the general term). The outcome of the coordination of such semiotic systems is that students recognize the common reference (bedeutung) as the figural representations (sinn) changes.

Our contention is that students are able to handle meaning correctly at the referential level because the semiotic transformation is supported, at the operational level, by a

---

<sup>83</sup> Without an algebraic expression the sequence could not be univocally determined but the figural representation somehow fixes the general schema. Furthermore, the search for a general schema in this type of task is a social practice recognized by students, that allows to objectify the cultural object "numerical sequence" at a lower level of generality.

strong embodied reflexive activity that involves the students consciousness within a sociocultural space of signification.

## CONCLUSIONS

This paper shows the effectiveness of *coordinating two complementary* semiotic perspectives to understand a specific didactical phenomenon. The analysis of the protocol shows that complementarity entails a *diachronic* use of the two perspectives. The systems of principles of the two perspectives fix and determine the phenomena we are looking at and the questions we are addressing. The cultural semiotic approach allows to investigate, from a sociocultural viewpoint, *objectification processes* whereas the structural approach, from a realistic standpoint, allows to analyse the *coordination of semiotic systems*. The boundaries between the systems of principles require a diachronic shift from one analysis to the other and we cannot merge the two theories into a synchronic analysis.

On the one hand networking at the level of coordination fosters the plot of identity without reaching a more encompassing new theory, rather a new conceptual framework. On the other hand the lower degree of integration allows to tackle a specific empirical phenomenon that allows a transformation both of our discipline and researchers. The plot of identity enhances a negotiation between theories and researchers in a process that Radford (2008a) calls *togethering*. Something I personally experienced in my doctoral research from which this paper stems.

## REFERENCES

- D'Amore, B. (2006). Oggetti matematici e senso. Le trasformazioni semiotiche cambiano il senso degli oggetti matematici. *La Matematica e la sua Didattica*, 20(4), 557-583.
- Duval, R. (1995). *apprentissages intellectuels*. Berne: Peter Lang.
- Prediger, S., Bikner-Ahsbahr, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM Mathematics Education*, 40, 165-178.
- Radford, L (2008). A cultural theory of learning. In L.Radford, G. Schubring, & F. Seeger F. (Eds.), *Semiotics in Mathematics Education* (pp. 215-234). Rotterdam: Sense Publishers.
- Radford, L. (2008a). Connecting theories in mathematics education: challenges and possibilities. *ZDM Mathematics Education*, 40, 317-327.
- Ullmann, S. (1962). . Madrid: Aguilar.

# MODELING EXTERNAL REPRESENTATIONS AS MEDIATORS OF MEANING IN THE MATHEMATICS CLASSROOM

Håkan Sollervall

Linnaeus University

*As mental constructs, mathematical objects are of abstract nature and have no references in the physical world. The mathematical objects are experienced and processed through various external representations. We propose a structural model that accounts for external representations as mediators of meanings in the mathematics classroom. During construction of the model, we particularly pay attention to negotiating disciplinary and individual perspectives regarding structure and specific constructs, with the objective to support understanding of the role of external representations that are used and processed in the mathematics classroom.*

## INTRODUCTION

It may be argued that the development of cognitive capabilities is inseparable from the sociocultural practices that constitute the context of their development (Cobb, 2006). On the other hand, we may argue that some structural features of cognitive development are independent of context, for example connecting modes of interaction in the physical world with modes of thought that we acquire through processing of impressions and experiences from this interaction (Bruner, 1966).

The philosopher Charles Sanders Peirce proposed in the late 19<sup>th</sup> century a theory of signs that accounts for human understanding of impressions and experiences based on interpretation of physical entities (Houser & Kloesel, 1992). Peirce's theory of signs is structural and general in character, makes no paradigmatic assumptions about learning, and therefore appears to be consistent with any learning theory. There are several instances of important theoretical constructs in mathematics education that may be interpreted within Peirce's theory of signs, for example *procept* as the amalgam of process and concept (Tall et al., 2001) and *registers* of representations that allow mathematical processing (Duval, 2006). We will propose a unifying structural model, based on Peirce's theory, as a structural foundation for understanding the role of representations in the mathematics classroom.

We will consider mathematics classroom practice from a pragmatic viewpoint, as a practice within which mathematics teachers teach the subject and also teach to their students. The disciplinary learning objectives, as expressed in steering documents and interpreted by the teacher, relate to features of mathematics as a school subject and do not explicitly account for individual variation in knowledge between students. With regard to teaching mathematics, such a *disciplinary* perspective reasonably has to be combined with an *individual* perspective which considers each individual student as a learner in the classroom. The two perspectives relate to the dichotomy between mathematics as a body of knowledge and as a social domain of

enquiry (Ernest, 2004). In the current study, disciplinary and individual perspectives are readily combined as they are considered from a pragmatic viewpoint, simply as two perspectives that have to be accounted for and negotiated with respect to teaching and learning in the mathematics classroom.

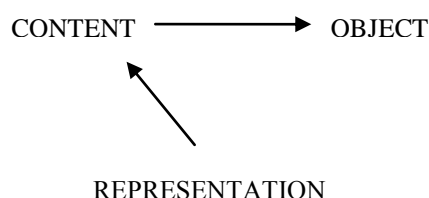
## RESEARCH OBJECTIVE

We aim to develop a unifying model of theoretical constructs and theories to account for the role of external representations as mediators of individual meanings in the mathematics classroom. The model will attempt to account for learning mathematics as a negotiation of individual and disciplinary meanings.

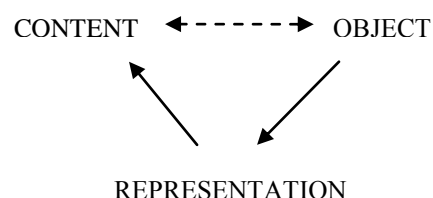
Although such a model puts focus on individual cognition, our ambition is to avoid making paradigmatic choices regarding learning theories, so that the model may be interpreted as a local but universal complement to understanding individual cognitive development in the context of mathematical classroom practice.

## THE ESSENTIAL ROLE OF REPRESENTATIONS IN MATHEMATICS

From an epistemological perspective, mathematical objects have a very specific status compared with objects in many other domains of scientific knowledge as the mathematical objects are never accessible by perception or by instruments (Duval, 2006, p. 107). We gain access to the mathematical objects through their representations. For example, the number “five” cannot be experienced in the physical world, in contrast with the physical object “tree” which we can access directly through our senses. We can understand quite a lot about a tree through our senses and without representing it, for example by taking a picture of the tree, talking about the tree, or by recalling its name and associating the name to known facts about that particular kind of tree. The figures below, following Duval (2006, p. 114), summarize the differences between representations of mathematical and physical objects. In these figures, “content” should be understood as a mental construct or internal representation, while “representation” refers to the external representation which is accessible for inspection. Furthermore, “object” should in both cases be regarded as a mental construct, while the notion “physical object” refers to physical (external) instances of the object.



**Figure 1: Mathematical (mental) objects**

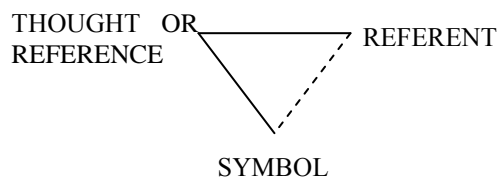


**Figure 2: Physical objects**

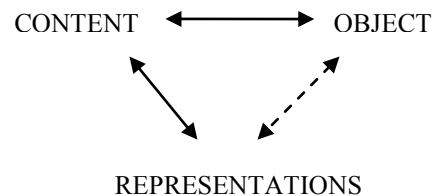


In Fig. 2, there is an obvious causality in that the physical object is a natural reference for the representation. The representation of a physical object may be regarded as a consequence of the object, while the representation of a mental object (Fig. 1) evokes a content which is required to experience the object. A mathematical object such as five cannot be experienced directly through our senses but must be understood in terms of its representations, neither of which should be confused with the mathematical object. To avoid the mistake of confusing the mathematical object with any of its representations, several representations of the same object should be experienced and contrasted by the learner. The mathematical object may be regarded as the invariant of the representations.

When the learner becomes acquainted with a mathematical object through its representations, it is reasonable to claim a direct connection also between object and representation. The two models above may then be merged into one model, where the objects are allowed to be either of mental or physical nature. This unifying model is often referred to as Ogden's semiotic triangle, although there are several similar versions in the literature (Fig. 3, following Ogden & Richards, 1923, p. 11). The symbols stand in direct relation to thoughts (content) and only indirectly to things (objects) in that the symbols direct and organize thoughts, record and communicate thoughts (ibid, p. 9). With regard to mathematical objects, we find it more fruitful to draw on the model in Fig. 1 and the corresponding "mature" model in Fig. 4, which may involve not only one but several representations of the same object.



**Figure 3: Ogden's semiotic triangle**

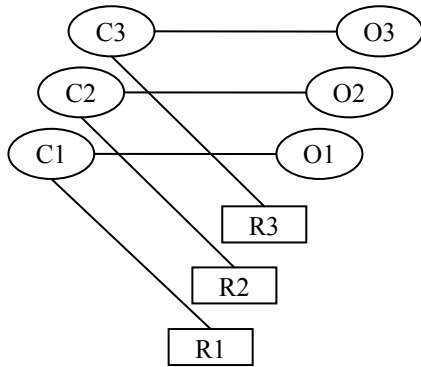


**Figure 4: Physical or mental objects**

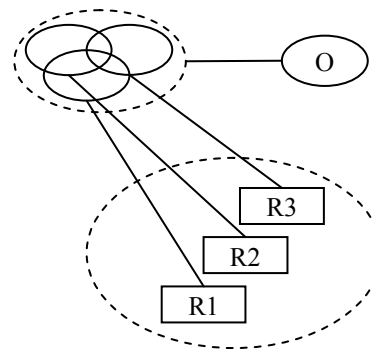
Each representation enriches the content and may both strengthen and extend our understanding of the object, which in turn may influence new representations which add to the content in a cyclic process. Especially with regard to problem solving, it is important to note that a specific aspect of the content may not be supported by an arbitrary representation. For example, it is not possible to make sense of addition of positive whole numbers by representing them *as points* on a number line. However, addition may be understood on the number line if numbers are represented dynamically as repeated one-step jumps or by number arrows (vectors).

When the student processes a new representation, preliminary content and a preliminary object (Fig. 5) are attributed to the representation. The student may spontaneously construct thematized objects of thought (Piaget; in Tall, 1999) by making connections between preliminary objects (Fig. 6). The process of thematizing may for example be supported by baptizing objects (Sfard, 2000) or by supplying

concept definitions (Tall & Vinner, 1981). For example, baptizing five apples as “five” and baptizing further representations (such as five fingers, the symbol 5, five marks on a paper) using the same name, may support students’ associative processes towards perceiving “five” as a thematized object. Another approach to supporting association of representations would be to define the concept “five” as referring to a certain number of objects, for example “as many as the fingers on one hand”.



**Figure 5: Preliminary objects**



**Figure 6: Thematized object**

## NEGOTIATING DISCIPLINARY AND INDIVIDUAL PERSPECTIVES

What is a mental object? According to the model in the previous section, the object emerges as a consequence of thinking about an external representation. Obviously, different individuals may think about the same representation in different ways. Nevertheless, the notion of object is often used from an objectivist perspective, where a pre-defined content is assigned to objects based on (some) disciplinary understanding of mathematics. Such a definition rules out the possibility of relating to students’ preliminary objects as individual mental constructs associated with one or several representations. With regard to our research objective of negotiating the notion of meaning in the mathematics classroom, it is necessary to acknowledge individual differences while still accounting for meanings that may be attributed to mathematics as a discipline. However, we find it neither necessary nor fruitful to define *one* notion of content and object. Rather, we will propose two separate definitions, one from an individual perspective and one from a disciplinary perspective. But first, we would like to discuss two examples which highlight the problems with a one-sided objectivist perspective on the notion of content and object.

Our first example is attributed to Husserl (quoted in Ogden & Richards, p. 271) who claims that equiangular triangles and equilateral triangles “name the same object”.

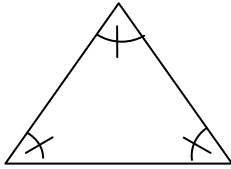


Figure 7: An equiangular triangle

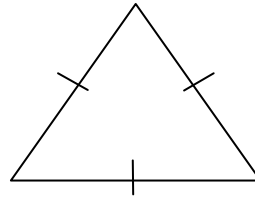


Figure 8: An equilateral triangle

Husserl acknowledges that the two entities have different ontological status in that he states that there are two distinct meanings related to each name. Nevertheless, the claim is that the two expressions *name the same object*. Certainly, it is well known within the discipline of mathematics that equiangular triangles and equilateral triangles name the same class of *mathematical* objects. As *mental* objects (of thought) we argue that the individual student may not conceive these as the same object, since that conclusion requires that the individual acknowledges the identification at a cognitive level. This identification may be considered a mathematical learning objective in itself, and will therefore be accounted for in the constructs that we propose in the current work.

We will elaborate briefly on how to model the process of identification in this case. Initially, the two classes  $\{\text{Triangle} \mid \text{equiangular}\}$  and  $\{\text{Triangle} \mid \text{equilateral}\}$  are defined in the mathematics classroom. The classes are defined according to different criteria and hence have different ontological status. So far we agree with Husserl. But we find it unreasonable to claim that the individual student, who has not yet connected the two classes in his or her mind, may consider these as the same object. However, when the student has made this connection, the (common) class may be interpreted as a thematized object  $\{\text{Triangle} \mid \text{equiangular and equilateral}\}$ , which inherits all established properties of the two previous classes. We consider this to be an important structural feature of mathematics teaching and learning that we would like to account for in our proposed model.

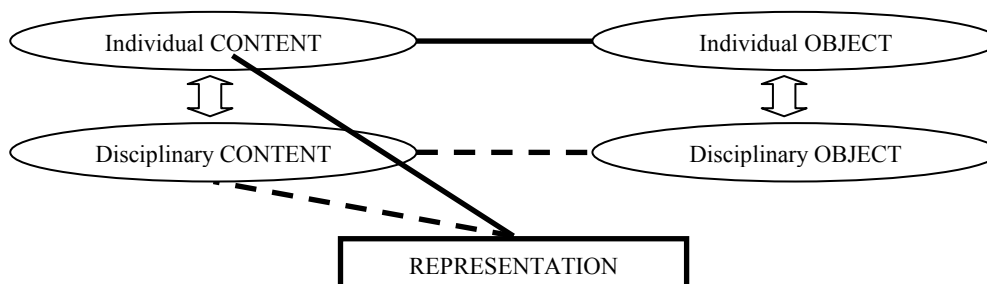
Our second example concerns a different domain of mathematics, namely fractions: “The symbols “ $\frac{2}{3}$ ” and “ $\frac{12}{18}$ ” mean the same because they refer to the same number” (Sfard, 2000, p. 37). As Sfard argues, an indirect procedure is required to prove the equivalence of the two expressions. From a disciplinary perspective, it is well known that the two symbols  $\frac{2}{3}$  and  $\frac{12}{18}$  represent the same number. But fractions are not only numbers. They may, for example, be thought of as ordered pairs of numbers. Proving equivalence of fractions rests upon defining an equivalence relation on the set of fractions. This equivalence relation *may* be defined in terms of comparing fractions *as* numbers, but the equivalence relation may also be defined algebraically based on interpreting the fraction as an ordered pair of numbers. We claim that both understanding the principles of identification (no matter how they are defined) and applying these principles to the specific fractions are learning objectives for each individual student. The two fractions  $\frac{2}{3}$  and  $\frac{12}{18}$  can not “mean the same” to the individual student until the student has noted that

they are (mathematically) equivalent. Furthermore, the example may be viewed as trouble-some not only from an individual but also from a disciplinary perspective, since the conclusion (that the two fractions “mean the same”) rests on the implicit assumption that the fraction as an object is determined based solely on its interpretation as a number. For comparison, Davis (2010) argues in a similar direction regarding justification of  $3 \times (-4) = -12$ , noting that the diversity of implicit assumptions is as important to attend to as the diversity of explicit representations of multiplication.

We note that justifications of  $2/3 = 12/18$  and  $3 \times (-4) = -12$  may rest on different assumptions in different classrooms. Therefore, it is not apparent how a disciplinary perspective on content and object should be defined.

## TWO PERSPECTIVES ON REPRESENTATION – CONTENT – OBJECT

With regard to our research objectives, we would like to connect disciplinary and individual perspectives on representations, content, and object, in such a way that the reader (and the teacher) has freedom of interpretation to balance these perspectives according to her own preferences. Our strategy to achieve this connection is to relate to the semiotic triangle from both perspectives. The representation, as an external construct, is independent of the choice of perspective. On the other hand, the choice of perspective affects the interpretation of content and object as mental constructs. The individual content and object relate to constructs of the individual student, while the disciplinary content and object are not well-defined constructs (as noted in the previous section). As our ambition is to account for learning mathematics as a negotiation of individual and disciplinary meanings, we regard disciplinary content and object as *learning objectives*. As learning objectives, disciplinary content and object may still be defined in several different ways. We invite the reader to interpret learning objectives (for example) either from a student perspective, from a teacher perspective in terms of notions that the teacher aims at establishing with respect to the students, from a community perspective as a joint learning objective for teacher and students, or from an institutional perspective according to steering documents. Independent of choice of perspective, the disciplinary content and object are learning objectives that someone is pursuing, or would like to be pursued, in a general or specific mathematics classroom.



**Figure 9: A model that connects disciplinary and individual perspectives**

The arrows in Fig. 9 indicate a negotiation between individual and disciplinary learning objectives that we do not elaborate on in this short paper. An obvious shortcoming of this model is that it only describes a state and only implicitly implies what is expected to happen in the mathematics classroom. This leads us to consider *how* content and object may emerge in the classroom. While linguistic interpretations depend on the syntactical embedding of each word, mathematical interpretations depend essentially on processing of representations (Duval, 2006). Manipulation and processing of mathematical objects may be interpreted from either an external or internal (cognitive) perspective. The notion of representation is relevant to discuss from either perspective, and it is well known that there is a close correlation between external and internal representations (Baddeley, 2007). In this paper, we consider representations only as external constructs.

## MATHEMATICAL PROCESSES

We will use theories of semiotics as a point of departure for modeling processing of representations. As a point of departure, we briefly relate to the works of the Swiss linguist Ferdinand de Saussure (1857-1913) who defines a *sign* in terms of two components, the signifier (*signifiant*, the form, pattern of the sign) and the signified (*signifié*, the content, thought of the sign). The sign itself is formed by the associative link between the signifier and the signified and gets its meaning in relation to or in contrast with other signs of the same nature (Engström, 2002). The American philosopher, logician, and mathematician Charles S. Peirce (1839-1914) defines the notion of sign in a way that explicitly accounts for relations with other signs, in a way that (not surprisingly) seems to fit mathematical structures particularly well. Our interpretation of Peirce, in comparison with Saussure, is that he includes two dimensions in the thought of the sign. He names these dimensions *object* and *interpretant*, where the latter “fulfils the office of an interpreter” (Houser & Kloesel, 1992, p. 5) and hence involves translation of the sign. The object relates to how the specific sign is understood, as a concept, while the interpretant concerns translations, applications and consequences of the sign, including processes. The dimension of the interpretant is particularly important in mathematics, as argued by Duval (2006, p. 106): “... the leading role of signs is not to stand for mathematical objects, but to provide the capacity of substituting some signs for others.” Peirce’s third notion *representamen* corresponds to Saussure’s signifier, that is, the pattern of the sign. The two models are summarized in the figures below (Engström, 2002).



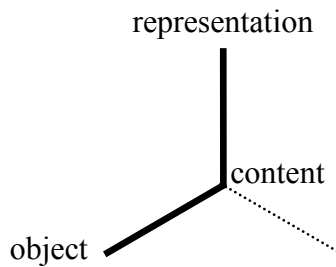
**Figure 10: Saussure's model of the sign****Figure 11: Peirce's model of the sign**

Peirce introduces many other notions of seemingly abstract philosophical nature. One such notion is the distinction between *firstness*, *secondness*, and *thirdness*, which he refers to “as modes or tones of thought” (Houser & Kloesel, 1992, p. 247). Firstness concerns the representamen, which is always present. Secondness requires that the interpreter associates either object or interpretant with the representamen. Thirdness is the desired state of the sign and requires that both object and interpretant are associated with the representamen. Peirce discusses the notion of *degenerated secondness*, where either the object or the interpretant may be weak. An example of degenerated secondness is when a learner is able solve linear equations by following formal rules, but having a weak understanding of what an equation is. In this case, the learner can solve the equations by manipulating symbols but can not interpret the equations, which indicates a strong interpretant but a weak object. Another learner may be able to interpret an equation but not know how to solve it, thus having a weak interpretant but a strong object.

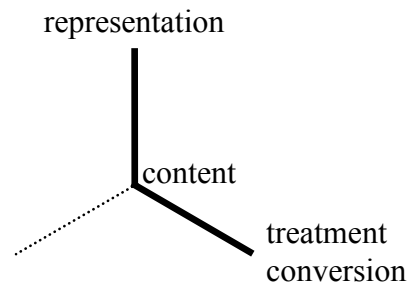
Raymond Duval (2006) highlights several characteristics regarding forms of representations that have important didactical implications. First, he provides a qualitative classification of semiotic representation systems in terms of *registers*, which by definition have (potential) thirdness. Second, he puts focus on two different types of transformations: *treatment* within a specific system (such as  $34+25=50+9$ ) and *conversion* between different systems (for example, pictures to symbols) as two distinct types of processes that describe how learners engage in mathematical activities and specifically when solving mathematical problems. From a disciplinary objectivist perspective, Duval classifies registers according to their degree of pre-defined regulation of treatment. However, from an individual perspective any representation has potential thirdness depending on how an individual interprets and interacts with the representation. For our purposes, we choose to interpret Duval's notion of register as a relational rather than a pre-defined disciplinary construct.

### **CONNECTING PEIRCE, OGDEN & RICHARDS, AND DUVAL**

There are no apparent inconsistencies between Peirce's sign (Fig. 11), the semiotic triangle (Fig. 1, Fig. 3), and Duval's theory of registers. This is not at all surprising, since the latter authors explicitly draw on Peircean semiotics but with focus on different relations, respectively representamen-object (Fig. 12) and representamen-interpretant (Fig. 13). Specifically, Duval (2006) puts focus on the processes of treatment and conversion which both refer to the interpretant dimension of Peirce.

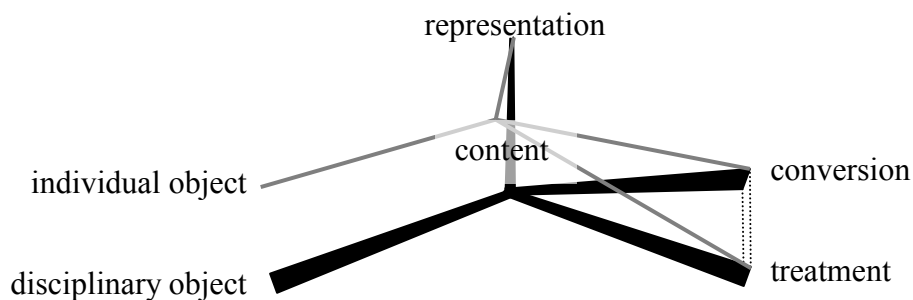


**Figure 12: Fig. 1 in Peirce's sign**



**Figure 13: Processes in Peirce's sign**

While Duval (2006) explicitly discusses the object dimension, he puts focus on the interpretant in terms of treatment and conversion. In Ogden & Richards' semiotic triangle (Fig. 3) we may interpret the referent as including both object and interpretant, but with a clear emphasis on the object dimension. As mathematicians, both Peirce and Duval recognize the importance of processing signs in order to create mathematical meaning, while authors such as Saussure, and Ogden and Richards, whose main interest is natural language, do not explicitly consider the notion of processing. However, the models of the latter authors do not exclude including an explicit dimension related to processing. To account for both a disciplinary and an individual perspective on content and object, we may readily connect the previous models, particularly those represented in Fig. 13, Fig. 11, and Fig. 9, keeping in mind that (the external) representation is common for the perspectives, as are (the resulting external representations of) conversion and treatment, while content and object (as mental constructs) depend on choice of perspective. Furthermore, we choose to highlight conversion and treatment as two separate processes by splitting the interpretant in two separate legs. The resulting model is shown in Fig. 14.

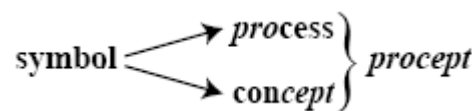


**Figure 14: Integrated model coordinating disciplinary and individual perspectives**

As noted earlier, the semiotic triangle (Fig. 1) with vertices representation – content – object, appears in two versions to the left in Fig. 14. For comparison, Saussure's model may also be interpreted in Fig. 14 with representation as the signifier and object together with treatment and conversion (as processes) as elements within the signified. While the semiotic triangle and Saussure's sign are redundant in Fig. 14, they provide support for our model as a globally unifying construct, in the sense of Prediger, Bikner-Ahsbahs & Arzarello (2008).

## CONCLUDING REMARKS

The investigations that resulted in the current paper originated as a survey of research concerning semiotic representations, particularly stimulated by Duval's notions of registers, and the processes of treatment and conversion. As our work progressed, Peirce's theory of signs emerged as a unifying theory. The final model (Fig. 14) may be interpreted mainly as accounting for underlying theoretical structures rather than innovative networking of theories, in the sense of Prediger et al. (2008). Our contribution to the model is the refinement of the interpretant and coordination of individual and disciplinary perspectives. Peirce's theory of signs seems to have served as a natural foundation for research focusing on structural aspects of semiotic representations. Some authors, such as Ogden & Richards, explicitly refer to Peirce, while others, such as Tall et al., do not. We feel that interpreting specific theoretical constructs within the model provides a context that may guide interpretation of these constructs. For example, the reader may interpret the following visual model of Tall et al. (2001) with respect to Fig. 14 by visually rotating either model. While these visual models do not have any inherent meaning, they may still serve to guide a reader's understanding of underlying theoretical constructs.



## REFERENCES

- Baddeley, A. (2007). *Working memory, thought, and action*. Oxford University Press.
- Bruner, J.S. (1966). *Toward a theory of instruction*. Cambridge, MA: Harvard University Press.
- Cobb, P. (2006). Mathematics learning as a social process. In J. Maas, & W. Schlöglmann (Eds.), *New mathematics education research and practice* (pp. 147-152). Rotterdam, The Netherlands: Sense Publishers.
- Davis, B. (2010). Concept studies: Designing settings for teachers' disciplinary knowledge. In M. M. F. Pinto, & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 63-78). Belo Horizonte, Brazil: PME.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.
- Engström, A. (2002). *Semiotik och matematikdidaktik – En introduktion*. [Semiotics and mathematics education – an introduction]. Arbetsrapporter vid Pedagogiska institutionen, 6. Örebro universitet.



- Ernest, P. (2004). What is the philosophy of mathematics education?' *Philosophy of Mathematics Education Journal*, issue number 18, 17pp.  
<http://www.ex.ac.uk/~PErnest/pome18/contents.htm>.
- Houser, N., & Kloesel, C. (1992). *The essential Peirce. Selected philosophical writings*. Vol. 1 (1867-1893). Bloomington: Indiana University Press.
- Ogden, C.K., & Richards, I.A. (1923). *The meaning of meaning* (1st ed. 1923, 10th ed. 1956). London: Routledge & Kegan.
- Prediger, S., Bikner-Ahsbahr, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches – first steps towards a conceptual framework. *ZDM*, 40(2), 165–178.
- Sfard, A. (2000). Symbolizing mathematical reality into being – Or how mathematical discourse and mathematical objects create each other. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (p. 37-98). Mahwah, NJ: Lawrence Erlbaum.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Tall, D. (1999). Reflections on APOS theory in elementary and advanced mathematical thinking. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 111-118). Haifa, Israel: PME.
- Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., & et al. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics and Technology Education*, 1, 81–104.

# COMBINING THEORIES TO ANALYZE CLASSROOM DISCOURSE: A METHOD TO STUDY LEARNING PROCESS

Michal Tabach

Talli Nachlieli

Tel Aviv University

Levinsky College of Education

*This paper focuses on learning processes of prospective elementary school mathematics teachers who are studying a course on functions and graphs. Specifically, we address the questions – "What were the modifications in students' ways of mathematizing about functions?" and "How did the teacher's and students' actions enable and promote modifications in the students' ways of mathematizing?". For this purpose, we analyze the mathematics classroom discourse that developed in class by combining two theories – systemic functional linguistics and the commognitive framework. In this paper we present the method that we have developed and exemplify its use and advantages by analyzing an episode of a mathematics classroom. By doing so, we provide a lens to think about and capture complexities of instruction.*

## INTRODUCTION

It is widely accepted today to focus research on learning processes in classrooms, rather than just on the outcomes of learning. This raises substantial theoretical and methodological issues. In this paper we suggest a method for analyzing classroom discourse to learn about instructional processes, and specifically, to address questions such as *What were the modifications in students' ways of mathematizing<sup>1</sup> about functions?* and *How did the teacher's and students' actions enable and promote modifications in the students' ways of mathematizing?* Specifically, how did the teacher and students organize the mathematical discourse so that peripheral participants (Lave and Wenger, 1991) could become more active participants of the canonical mathematical discourse? For this purpose we draw on two theories that each seems to have the potential to address different, yet complimentary, aspects of those questions - Systemic Functional Linguistic, (SFL, Halliday, 1978; Halliday & Matthiessen, 2004) and the commognitive framework (Sfard, 2008). This combination enabled us to develop a new perspective with which to investigate classroom learning. The data used to exemplify the suggested tool is taken from a study<sup>2</sup> that aims at identifying instructional processes of prospective teachers, attending a functions and graphs course. In the following sections we outline the basic principles of SFL (Halliday, 1978), followed by basic tenets of commognitive framework (Sfard, 2008). Then we describe and exemplify how we use the suggested method.

## THEORETICAL BACKGROUND

### Systemic Functional Linguistics

According to SFL (Halliday, 1978; Halliday & Matthiessen, 2004), language is a resource for making meaning through choice. This approach is concerned with the analysis of how language is used to achieve certain discursive goals and the analysis of the choices that have been made in any instance of language use (O'Halloran, 2005:61). The sets of possible choices were clustered by Halliday in terms of the functions that they serve (and therefore are called metafunctions): (1) *the ideational* – the content function of language, what is talked about. This metafunction expresses those things in language such as the objects, actions, and relations, of the world and of our own consciousness; (2) *the interpersonal* – the participatory function of language, through which "*the speaker introduces himself into the context of situation, both expressing his own attitudes and judgments and seeking to influence the attitudes and behaviour of others.*" (Halliday, 1978:112); and (3) *the textual* – the organization of the text. This is the metafunction that "*makes language relevant. ... it expresses the relation of the language to its environment, including both the verbal environment – what has been said or written before – and the non verbal, situational environment.*" (Halliday, 1978:112-113).

That is, any language use, serves three functions simultaneously, constructing some aspect of experience, negotiating relationship and organizing the language in a way that it realizes a satisfactory message (Christie, 2002).

### The commognitive approach to study learning

The commognitive framework (Sfard, 2007; 2008) is a socio-cultural approach. Within this framework *thinking* is defined as an individualization of interpersonal communication, although not necessarily verbal<sup>3</sup>. *Discourse* is considered a special type of communication, made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions. To emphasize the unity of cognitive processes and communication, the word *commognition*, a combination of the two, is used to name the framework.

Mathematics, as any academic discipline, may be considered a form of discourse made distinct by four characteristics: words and their uses, visual mediators, routines and endorsed narratives, as detailed below.

*Words and their uses.* Any professional discourse has a unique vocabulary. Some of the words may be used in other discourses, either in the same way or according to a different definition. Words and their uses are central to a discourse as often they determine what one can say about the world. With regard to the area of functions and graphs, we find words such as slope and function with unique uses in the mathematical discourse.

*Visual mediators.* Those are the objects acted upon as a part of the communication. While colloquial discourse is mediated mainly by images of concrete objects that exist independently of the specific discourse, in mathematics, most symbols and other mediators were created mainly for the purpose of communication. Visual mediators of the mathematics discourse include algebraic symbols that mediate ideas such as written numbers and graphs, or other symbols like those that represent variables, coefficients and equality. The mediators used in the communication often influence what one can say about the idea discussed. To illustrate, while solving equations in algebra, students often participate in a different discourse if they use graphs as their visual mediators, or if they refer to the algebraic symbolic equation as their discursive objects.

*Routines.* A routine is a set of meta-rules defining a discursive pattern that is repeated in similar types of situations. Those rules are the observer's construct as they describe past actions that were noticed by the observer. Although they describe past actions, routines are helpful in learning a new discourse as our ability to act in new situations often depends on recalling one's or others' past experiences. An example for a routine often practiced in mathematics regards finding the slope of a given linear function. The specific mediator chosen for a function (e.g. graphs or algebraic symbols) often dictates the routine chosen for that purpose.

*Endorsed narratives.* Endorsed narratives are any text that can be accepted as true by the relevant community. Specifically, in mathematics, the endorsed narratives are those narratives that become "mathematical facts". Narratives such as axioms, definitions and theorems are all endorsed narratives, with each of them being derived differently.

### **Combining the theories**

Our focus is on instructional processes – on the processes by which learning is enabled and enacted. For this we seek a theory that views language as a set of choices. Moreover, the unit of analysis relevant for our suggested studies is the discourse itself (or parts of it). These two requirements are met by both theories. In Gellert's (2009) words, the underlying principles of the two theories are 'near enough'. However, whereas SFL focuses on language, that is, on the verbal aspects of discourse, commognition holds a wider view and considers also non-verbal aspects of the discourse (e.g. routines and visual mediators.) In addition, while SFL explicitly distinguishes between the content function, the participatory function and the organization of the text, studies conducted under the commognitive framework focus on making explicit routines, endorsed narratives, words and visual mediators of the discourse. While commognition is a socio-cultural approach that aims at providing a lens to study learning processes, SFL is a linguistic approach may help researchers focus on specific choices of participants' language use that may be overlooked.

For our purposes, each theory has an added, complementary, value – we wish to distinguish between the three metafunctions, as is called for by SFL, and we wish to identify the various discourse characteristics, as is called for by commognition. We believe that this dual analysis would allow noticing aspects of classroom discourse that were not identified thus far.

## THE METHOD

In the following sections we present the data to be analyzed, the unique method that we developed to analyze classroom discourse and an example of using this method to analyze classroom discourse. We begin by specifying our research questions in light of the theoretical frameworks that we adopt.

### *Research questions*

*What were the modifications in students' ways of mathematizing about functions?* Specifically, what are the modifications and changes in students' use of words, visual mediators, routines, and endorsed narratives, while they participate in a mathematics discourse? The focus of attention here is Halliday's ideational metafunction. We broaden this metafunction to refer not only to what is being said, but also to the actions performed as a part of instruction (e.g. calculations in writing and drawing.)

*How did the teacher's and students' actions enable and promote modifications in the students' ways of mathematizing?* Specifically, what are the actions that the teacher and students perform to organize the mathematics discourse so that peripheral participants could become more active participants, with regard to the ways by which they use words, visual mediators, routines, and endorsed narratives? How do the teacher and students develop social relationships and how do the participants orient themselves to the learning of mathematics and to others? Here the focus is dual – first, the textual metafunction, that makes language relevant (Halliday, 1978: 112-113). We use this category to refer to what it is that one (usually the teacher) assumes others (usually the students) already know and what it is that is assumed new, and therefore – how the teacher organizes the discourse so that students, as novice participants, could participate. The second focus is the interpersonal metafunction, which regards the ways by which the teacher and her students develop social relationships and by which they orient themselves to each other and to the mathematics.

*Data collection and documentation.* The following transcript<sup>4</sup> is an example of the type of data to be analyzed, and of our method of presenting transcriptions. It was taken during a whole class discussion in a 1<sup>st</sup> year course for prospective elementary school teachers in a college of education in Israel. All 14 lessons were video and audio taped and are used in a larger study that focuses on learning about functions. In the following task, the students were asked to compare the steepness of five segments (see Figure1). Noam suggested that AB was the steepest, and was asked to explain why:

## Working Group 16

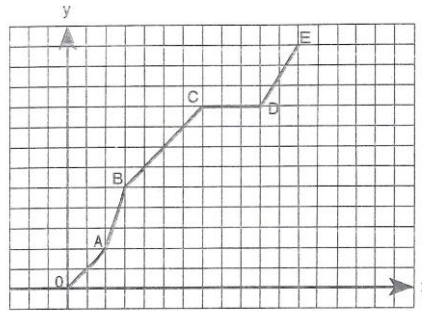


Figure1: The graph presented to the students

49. Noam I, like, used a silly method.
50. Teacher What method?
51. Noam I said that, like, in one x, like, the segment AB, then in one x it went up a lot.
52. Teacher Wait, wait, wait. I want you all to listen to Noam. Noam is trying to explain why, what was the segment that you said is steepest?
53. Noam AB
54. Teacher Come, show it on the graph. Come, come, come. Noam wants to explain something, and, a method she worked according to. Come, stand where I stand, and with the pen you can show here on the, ... you see, the segment AB is here... come, show them.
55. Noam [walks to the overhead. She points with her pen towards segment AB.]
56. Noam Segment AB went up by, like, only, eee... only in this x [moves her hand parallel to the x axis along one unit] it went up a lot [moves her hand parallel to the y axis along three units.] see Figure1
57. Teacher That means, what are you saying, that you moved from A, from point A towards the positive direction of the x axis in one unit [moves her hand parallel to the x axis along one unit].
58. Noam Yes
59. Teacher What did you see, that it went up by how much?
60. Noam Three y [the teacher moves her hand parallel to the y axis along three units.]
61. Teacher That is, y changed by three [the teacher moves her hand along segment AB]
62. Noam Yes
63. Teacher When x changed by one, y changed by three.
64. Teacher As opposed to the other segments... what is it?
65. Noam Let's say, in OA and in BC
66. Teacher OA, look, in OA, what happens in OA?
67. Noam It is equal because there is the same increment, two x, two x. Points at segments OA and BC]
68. Teacher That is, what is there in OA? When x went up by... when x increased by 1, by how much did y change?

69. Student By one  
70. Teacher Thank you.

*Method of analysis and examples of findings.* We use the lens of SFL in general, and specifically, that of the ideational, interpersonal and the textual metafunctions to differentiate between the three discourses that are a part of the mathematical classroom discourse – the mathematical<sup>5</sup>, the social and the organizational. For that purpose, we broaden each of the metafunctions to refer not only to what is being said, but also to the actions performed as a part of instruction (e.g. calculations in writing and drawing). Specifically, we use the textual meta-function to refer to what it is that one (usually the teacher) assumes others (usually the students) already know and what it is that is assumed new, and therefore – how the teacher organizes the discourse so that students, as novice participants, could participate.

For each metafunction, we refer to the words and visual mediators that participants use, the routines that could be identified and the narratives endorsed.

We summarize and exemplify our method in the table below, with regard to the given transcript. We realize that our sayings are limited due to the shortness of the analyzed excerpt.

<b>Discourse characteristics</b> <b>Metafunction</b>	<b>Words and visual mediators</b>	<b>Routines<sup>6</sup></b>	<b>Endorsed narratives<sup>7</sup></b>
<b>The ideational</b>	<p>The words and visual mediators used in class that relate to the mathematics.</p> <p>Examples: Student: "Three y" (60) Teacher: "That is, y changed by three" [the teacher moves her hand along segment AB] (61)</p> <p>Student: "in one x, it went up a lot." (51) Teacher: "when x changed by one, y changed by three" (63)</p>	<p>The routines observed in class that relate to the mathematics.</p> <p>Examples: The student refers to a routine by which she decides which segment is steepest. She calls it "a method" (49). She compares the graphs qualitatively (and not quantitatively.) (turns: 49, 51) The teacher compares the steepness of graphs by referring to their slopes (63).</p>	<p>The narratives that are discussed in this learning community that relate to the mathematics.</p> <p>Examples: at this time there is no evidence that the students and teacher endorse the same narrative. There is evidence regarding the narratives that some of them endorsed:</p> <p>Student: "in one x, it went up a lot." (51) Teacher: "when x changed by one, y changed by ..." (63)</p>
<b>The inter-personal</b>	<p>The words and visual mediators by which people orient to the mathematics, to themselves or to others, and by which they develop social relationships and a learning community.</p>	<p>Discursive routines that relate to the ways by which people orient to the mathematics, to themselves or to others, and by which they develop social relationships and a learning community.</p>	<p>Endorsed narratives that relate to the ways by which people orient to the mathematics, to themselves or to others, and by which they develop social relationships and a learning community.</p> <p>Examples: (our examples are from the rest of the lesson.) "This is great." (487)</p>

## Working Group 16

	<p>Examples:</p> <ul style="list-style-type: none"> <li>- The student evaluates a method that she chose to use to solve a problem: "I use a silly method" (49)</li> <li>- The teacher marks Noam's method as important for the entire class: "Wait, wait. I want you all to listen to Noam." (52)</li> </ul>	<p>Examples:</p> <p>The teacher empowers students by inviting them to stand in front of the class and tell their ideas in their own words. (e.g. 52, 54)</p>	<p>"You are doing a wonderful job"</p> <p>"Do you understand why?" the teacher's disposition towards learning and knowing: – to know includes explaining why. The students' answers repeatedly included explanations or saying of the type: "I know that this is so, I do not know why" even when were not asked to explain explicitly.</p> <p>Another disposition: "you should work in small groups"</p>
<b>The textual</b>	<p>The words and visual mediators that relate to the ways by which the participants organize the mathematics discourse so that peripheral participants could become more active participants.</p> <p>Examples:</p> <ul style="list-style-type: none"> <li>- Choosing the task: comparing steepness of graphs, before actually talking about the slope of a graph.</li> </ul> <p>"that means, what you are saying..." (57)</p>	<p>Discursive routines that relate to the ways by which the participants organize the mathematics discourse so that peripheral participants could become more active participants.</p> <p>Examples:</p> <ul style="list-style-type: none"> <li>- Repeating her way of comparing steepness of graphs.</li> <li>- Revoicing. (e.g. "that means, what you are saying...", (57)</li> <li>- Realizing the need for a publicly seen mediator to communicate ideas (graphs on overhead.)</li> </ul>	<p>The narratives that are endorsed by this specific group that help organize and enable the communication between peripheral and expert participants.</p> <p>Examples:</p> <p>"you should work in small groups"</p> <p>"that means, what you are saying...", (57)</p>

**Table2: A method to analyze classroom discourse**

## FINDINGS

Mathematics classroom discourse interweaves several discourses – the mathematical, social and pedagogical. We identify the mathematical discourse with the ideational metafunction<sup>8</sup>, the social with the interpersonal metafunction and the pedagogical<sup>9</sup> with the textual. That is, for those discourses we focus on each of the table's rows separately. For each discourse, we consider each of its characteristics separately, by considering the table's columns.

The following findings are restricted to the given episode, and are therefore limited in their scope.



*Mathematical discourse.* This focus exposes differences in the teacher's and students' ways of mathematizing – their use of words, visual mediators, routines and narratives. The student's word use is more colloquial ("it went up a lot") and the ideas expressed are imprecise. In the context of comparing slopes, while the teacher compares slopes by referring to it quantitatively, the students compare them visually.

*Social Discourse.* The teacher empowers students by allocating time and place for them to present their ideas to the other students and by evaluating their work. The teacher's disposition towards learning seems to be that small group discussions promote learning and that understanding why and being able to express that are crucial for learning.

*Pedagogical Discourse.* The teacher performs several actions to organize the mathematical discourse so that the students could become less peripheral participants. Those actions reveal her dispositions towards learning mathematics: e.g. using revoice as a pedagogical strategy; this way she shows respect her students' ideas, yet able to use them as a springboard to present mathematically accepted ways of doing and saying.

## FINAL COMMENTS

In this paper we suggested combining two theories, SL and commognition, to learn about instructional processes. The suggested combined method is a practical and coherent way of analyzing classroom discourse to study about learning processes. Thus it provides a conceptual framework by which various aspects of classroom discourse could be observed, identified and thought of. In other words, the combined method helped us to "*direct researchers' attention to particular relationships in providing meaning for the phenomena being studied*" (Silver and Herbst, 2007: 373). As can be seen, it allows one to focus on each of three discourses that develop in class – the mathematical, the social and the pedagogical. Its importance lies at the underlying assumptions that to improve our understanding of mathematics learning and teaching, one should focus on processes rather than on end-results only, and on those processes that take place in classroom rather than in different "laboratory-like" settings.

## NOTE

- <sup>1</sup> Mathematize: participate in a mathematical discourse; "doing" mathematics (Sfard, 2008).
- <sup>2</sup> This study is supported by the Israeli Science Foundation, no. 446/10.
- <sup>3</sup> This is coherent with Vygotsky's theory regarding the higher mental functions that appear first in the social plane and are only then individualized (Vygotsky, 1986).
- <sup>4</sup> The transcript was translated from Hebrew by the authors.

## Working Group 16

- <sup>5</sup> The ideational metafunctions refers to what is talked about. For our purposes, we suggest narrowing the scope addressed by this metafunctions and relate only to the mathematical content discussed.
- <sup>6</sup> To identify a routine we would need longer stretches of text or declarations regarding how one often acts. As we only exemplify a short episode, the routines are generalized from analysis of a larger corpus of data.
- <sup>7</sup> While narratives may be presented in a single sentence, their endorsement by the class community may be identified only in longer episodes. Therefore, our examples are taken from a larger corpus of data.
- <sup>8</sup> We do not claim that the ideation metafunction includes only parts of the discourse that are about mathematics. For our purpose we choose only those parts.
- <sup>9</sup> In the literature, the pedagogical discourse is often used to consider what we refer to as "classroom discourse" (e.g. Christie, 2002). We use it to refer to teacher's choices regarding the organizations of the discourse so that students, as novice participants, could participate.

## REFERENCES

- Christie, F. (2002). *Classroom discourse analysis: a functional perspective*. Continuum Press.
- Gellert, U. (2009). Modalities of local integration of theories in mathematics education. In B. Sariraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 623-646). Hiedelberg, New York:Springer – Advances in mathematics education series.
- Halliday, M.A.K. (1978). *Language as social semiotic: The social interpretation of language and meaning*. Edward Arnold.
- Halliday, M.A.K., & Matthiessen, C.M.I.M. (2004). *An Introduction to Functional Grammar* (Third ed.). London: Arnold
- Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge University Press, Cambridge.
- O'Halloran, K. (2005). *Mathematical discourse. Language, symbolism and visual images*. Continuum Press.
- Sfard, A. (2007). When the rules of discourse change, but nobody tells you: making sense of mathematics learning from a commognitive standpoint. *Journal of learning science: 16*(4), 1-47.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.
- Silver, E. A., & Herbst, P. (2007). Theory in mathematics education scholarship. In F. K. Lester (Ed.), *Second Handbook of Reaserch on Mathematics Teaching and Learning* (pp. 39-67). Charlotte, NC: Information Age Publishing and Reston, VA: NCTM.
- Vygostky, L. S. (1986). *Thought and Language*. MIT Press.

## **INTRODUCTION TO THE PAPERS OF WG 17: FROM A STUDY OF TEACHING PRACTICES TO ISSUES IN TEACHER EDUCATION<sup>84</sup>**

Leonor Santos (Portugal), Claire Berg (Norway), Laurinda Brown (UK),  
Nicolina Malara (Italy), Despina Potari (Greece), Fay Turner (UK)

Overview: There was recognition of the value and complementarities of different approaches to the professional development of teachers. However, it was also recognised that there are constraints and affordances for different approaches, which vary between cultural contexts. Working across cultures on teacher development projects, which employ different strategies, was considered to be a useful way of moving forward our understanding of different approaches. There were some attempts to synergise different frameworks in research and development activities that were reported in the papers. However, there remains considerable work to be done in understanding how different frameworks relate to one another and in supporting researchers in selecting elements of different frameworks that will enable them to answer specific research questions.

Group 17 received 57 proposals (48 for papers and 9 for posters), which involved 129 authors from 28 nationalities. Each paper was reviewed by one of the group leaders and two authors. For most proposals we asked for some revisions. In the sessions of the working group during the conference, 37 papers and 8 posters were presented.

According to what had been proposed in CERME 6, the working group split into two subgroups (WG17 A and WG17 B). The group was all together for only the first part of session one and for the last session. All the papers were grouped into seven topics and distributed to the two subgroups. All the participants were informed in advance of the distribution of the papers in the two subgroups.

All participants of WG17 were expected to have read papers previously to the session in which they were presented. In each session, three or four authors sketched the key ideas of their report (5 minutes each). One of the group leaders or participants then gave a prepared reaction to the set of papers (10 minutes). In most cases, the reactor attempted to make links between the papers and suggested questions arising from the papers that might form the basis for discussion.

The organisation of the sessions was highly rated by the participants, as was the atmosphere. Nevertheless, at the final session, the group coordinator presented several possible scenarios to organize the working group for the future, given that

---

<sup>84</sup> Alena Hošpesová (Czech Republic) contributed as a group leaders, being in charge of part of the review process

participation remains high. Although no decision has been taken, a general opinion was voiced against the possibility of splitting this working group into two new ones.

### **Topics**

We present the issues and ideas that emerged in reference to seven central topics.

#### **Topic I: Mathematical content knowledge for teaching**

The categorisation of knowledge needed for the teaching of mathematics and how such knowledge might be identified in the practice of, or when discussing teaching underpinned many of the papers discussed in WG 17A. Ball et al's 'egg' appeared to be generally accepted as a useful framework for categorising knowledge and Rowland et al's Knowledge Quartet was frequently used as a means for identifying the situations in which such knowledge was revealed in the practice of teaching. This common language and understanding of frameworks represented a clear progression from the discussion in CERME 6.

#### **Topic II: Professional knowledge for teaching**

The papers discussed referred to a number of different approaches to the development of mathematics teaching through both Initial Teacher Education (ITE) and Continuing Professional Development (CPD). Although supporting learning in knowledge about mathematics and mathematics pedagogy was seen as the foundation for developing mathematics teaching, it was recognised that it was developing the application of this knowledge in action (knowing how to) that should be our ultimate developmental and research concern. The balance between focusing on the development of knowledge about and the development of knowing how to was considered in relation to how this varies between ITE and CPD.

#### **Topic III: Reflection in mathematics teachers' professional development**

One approach that seemed to be effective was the use of theoretical concepts and frameworks by the teachers in discussing and analysing mathematics teaching. Although teachers' interpretations of these concepts and frameworks and the way they link them to teaching are often idiosyncratic, the studies indicate that this approach promotes critical reflection of teachers' beliefs, knowledge and practices. Another issue that emerged was that teachers' reflection on a number of mathematical, teaching and learning phenomena can improve both their mathematics and pedagogical content knowledge. Finally, mathematics educator's learning from teachers' reflection is also an important issue that needs further research attention.

#### **Topic IV: Collaboration in mathematics teachers' professional development**

Within teachers' professional development the modes of collaboration between the different actors are of crucial importance for achieving successful development. The three papers related to this topic present different models concerning the

collaboration between teachers and didacticians, teachers with colleagues, and teacher students and teacher educators. These studies share a willingness of considering teachers, colleagues, or teacher students as true partners in the process of change, and this seems to be one of the principles of professional development. However, issues related to sustainability of such collaborative models need to be critically addressed and carefully investigated in future.

### **Topic V: Professional development**

A significant discussion in the working group concerned the long-term effectiveness of professional development programmes, in particular related to mathematics teacher education. It is necessary to understand, in a deep way, the characteristics of professional development for sustainable impact. Is it really possible to know what is the impact after some years? What does sustainability mean in a society where people change profession?

The working group also discussed whether there are different issues concerning pre-service and in-service teachers education programmes.

### **Topic VI: Conceptions and practices**

The papers presented in this session concern mainly the study of particular aspects of teachers' practice, highlighting the relationships between teachers' beliefs, knowledge, didactical and methodological choices and students' learning.

Two studies are devoted to the analysis of teachers' work in their daily practice, with the aim of collecting data useful to find ways to improve teacher training and show the incidence of the teacher's beliefs in management of their class work. Another paper focused on collaborative curriculum management in the context of a school mathematics department focusing on sustainability of the culture. This contrasted with a study involving prospective teachers, evidencing trainees' difficulties in theoretical and methodological analysis, given their lack of teaching experiences and the short length of their course. The personal dimension seems to have a real influence, including negatively, documented by a study involving teachers who, convinced that metacognitive activities can be practised exclusively with more gifted students, were negatively influenced in their teaching by this belief.

### **Topic VII: Interaction in the classroom**

The four papers were in the frame of teachers' educational projects centered on laboratories devoted to classroom practice either in pre-service or in service education. In general terms, these projects are aimed at developing the teachers' ability to enact generative teaching, to refine their communicative practices (posing questions, listening and answering) to control the cognitive implications of their behaviors and to assess the students' mathematical learning. Even if framed in different theoretical studies, all the research projects are realized over the long term (at least one year), through collaborative work between teachers and

mentors/researchers.

## **PAPERS**

### **WG17A**

#### **Topic I: Mathematical content knowledge for teaching**

Bednarz, N. & Proulx, J. *An attempt at defining teachers' mathematics through research on mathematics at work.*

Davis, S. *The Impact of teaching mental calculation strategies to primary PGCE students.*

Fernández, S.; Figueiras, L.; Deulofeu, J. & Martínez, M. *Re-defining HCK to approach transition.*

Ineson, G. *The use of the empty number line to develop a programme of mental mathematics for primary trainee teachers.*

Kaldrimidou, M.; Haralambos, S. & Tzekaki, M. *Readings of the mathematical meaning shaped in the classroom: exploiting socio-cultural and instructional lenses.*

Kleve, B. *Literacy in mathematics – a challenge for teachers in their work with pupils.*

Ribeiro, C. M. & Carrillo, J. *Knowing mathematics as a teacher.*

Tichá, M. & Hošpesová, A. *Teacher competences prerequisite to natural differentiation.*

Tutak, F. *Pre-service elementary teachers' Geometry content knowledge in methods course.*

#### **Topic II: Professional knowledge for teaching**

Kilic, H. *The nature of preservice teachers' pedagogical content knowledge.*

Kuntze *et al.* *Professional knowledge related to Big Ideas in Mathematics – an empirical study with pre-service teachers.*

Rowland, T.; Jared, L. & Thwaites, A. *Secondary mathematics teachers' content knowledge: the case of Heidi.*

Turner, F. *Differences in the Propositional knowledge and the knowledge in practice of beginning primary school teachers.*

#### **Topic III: Approaching reflection in mathematics teachers' professional development**

Liston, M. & Gill, O. *The role of video-based experiences in the teacher education of pre-service mathematics teachers.*

Potari *et al.* *Prospective mathematics teachers' noticing of classroom practice*

*through critical events.*

Sánchez, M. *Concepts from mathematics education research as a trigger for mathematics teachers' reflections.*

**Topic IV: Approaching collaboration in mathematics teachers' professional development**

Berg, C. *Adopting an Inquiry approach to teaching practice: the case of a primary school teacher.*

Gunnarsdóttir, G. & Pálsdóttir, G. *Lesson study in Teacher Education: a tool to establish a learning community.*

Mgombelo, J. & Jamani-Jaipal, K. *Mathematics problem solving professional learning through collaborative action research.*

**WG17B**

**Topic V: Professional development**

Asami-Johansson, Y. *A study of problem solving oriented lesson structure in mathematics in Japan.*

Back, J. & Joubert, M. *Lesson study as a process for professional development: working with teachers to effect significant and sustained changes in practice.*

Canavarro, A. P. & Patrício, M. *Mathematical investigations in the classroom: a context for the development of professional knowledge of mathematics teachers.*

Corcoran, D. *The need to make "boundary objects" meaningful: a learning outcome from lesson study research.*

Koleza, E.; Markopolous, C. & Nika, S. *Helping in-service teachers analyze and construct mathematical tasks according to their cognitive demand.*

Matins, C. & Santos, L. *Planning teaching activity within a continuous training program.*

Regecová, M. & Slavíčková, M. *Curricular changes in preparation of future teachers – financial mathematics course.*

Rubio *et al.* *Preservice teachers learning to assess mathematical competencies.*

Zehetmeier, S. & Kraine, K. *Effective ways of promoting in-service mathematics teachers' professional development.*

**Topic VI: Conceptions and practices**

Arditi, S. *Primary school ordinary teachers using a same manual written by didactician practices' variability.*

Choquet, C. *Why do some french teachers propose to their pupils «problèmes*

*ouverts» in mathematics in primary school?*

Cusi, A. & Malara, N. *Analysis of the teacher's role in an approach to algebra as a tool for thinking: problems pointed out during laboratorial activities with perspective teachers.*

Nowińska, E. *A study concerning the differences between the surface and deep structure of Math lessons.*

Nunes, C. & Ponte, J. *Teachers managing the curriculum in the context of the mathematics' subject group.*

### **Topic VII: Interaction in the classroom**

Guerreiro, A. & Serrazina, L. *Conceptions and practices of mathematical communication.*

Malara, N. & Navarra, G. *Multicommented transcripts methodology as an educational tool for teachers involved in constructive didactical projects in early Algebra.*

Martignone, F. *Laboratory activities in teacher training.*

Tomás Ferreira, R. *Moving beyond an evaluative teaching mode: the case of Diana.*

### **POSTERS**

Bommel, J. *How to teach mathematical knowledge for teaching.*

Ceia, M. *Analysing exams mathematical questions.*

Fransman, J., Mgombelo, J. & Van der Walt, M. *The practices of prospective teachers in South African and Canadian mathematical literacy teacher education programs: What works and what does not?*

Rocha, H. *Teachers' use of graphing calculators in high school mathematics classroom - the influence of teachers' professional knowledge.*

Spencer, P. & Edwards, J. *Deeper mathematical understanding through teacher and teaching assistant collaboration.*

Vanegas, Y., Giménez, J. & Font, V. *Didactical analysis and citizenship with prospective mathematics teachers.*



# **PRIMARY SCHOOL TEACHERS' PRACTICES USING A SAME MANUAL WRITTEN BY DIDACTICIAN**

Sara Arditi

University of Paris VII Denis Diderot

*Our thesis work reviews teachers' practices when using the same manual written by didacticians. The study aims to analyse variability in specific teachers' practices when they use different types of problems from the same manual written by didacticians and also to analyse several teachers' practices when they use the same problem written by didacticians. This paper reports the comparative analysis of ways teachers carry out a problem about fractions proposed in the manual Euromaths with children aged 9 or 10 years. This comparative analysis raises questions about the reproducibility of situations written by didacticians. This study could contribute to a characterisation of didacticians' intervention latitude. It could also contribute to find ways to improve teacher training.*

*Keywords : practices, conceptions, manual, transmission, reproducibility*

## **INTRODUCTION, THEORITICAL FRAMEWORK AND METHODOLOGY**

The question of how problems written by didacticians are transmitted is often raised by the didactician community. According to Artigue (Artigue, 1989) questions about transmissibility of such problems lead to questions about their reproducibility. Manuals are written by didacticians to transmit problems contingent on recent research results. But, the confines of a manual forces the author to transpose the problem and its analysis which may make the transmission more difficult. It is especially difficult for the “*a priori*” analysis to be entirely written in the teachers' book. My thesis work aimed to analyse the way in which problems and teaching sequences written by didacticians are transmitted to practitioners through teaching manuals. So, it reviews the practices of teachers using the same manual written by didacticians. The meaning of “practices” according to Robert & Rogalski (2002) is what the teacher does before, during and after class, encouraging the practices into the class.

In this paper the analysis of teachers' practices using a problem written by didacticians are focused on. The aim of this study is to determine, at least partially, the spectrum of ways teachers carry out the same problem written by didacticians. The spectrum extent will suggest how effectively the chosen problem has been transmitted and if it can be reproducible. A problem will be reproducible if it has the ability to create necessary conditions (expected by the didacticians who wrote it) for the students to learn. But, to talk about reproducibility doesn't really fit the situation of ordinary primary school teachers. So, in order to determine the spectrum we chose to study teachers' appropriations of a *Euromath's* problem. The term appropriation seems to better describe how teachers understand a problem written by didacticians

and how they make it their own (Bolon, 1996). To determine those appropriations, the distance between tasks proposed by the manual to the teachers and tasks proposed and carried out in the classroom has been examined.

I use the theoretical framework “double approche”, developed by Robert (Robert, 2001). It says that “teachers’ practices in class are the result of their work - though partly implicit - which has its own consistency and may not be reduced to a mere analysis of students’ potential learning”. The analysis only of what students learn is not enough to understand teachers’ practices. My teachers’ practices study will then include two points of view: the student’s learning for one part and the teachers’ work for another part. So I analyse expected scenario, mathematics’ tasks proposed to the students and teachers’ speech in order to have an access to students learning possibilities. Then, I adopt the second point of view about teachers’ work to understand teachers’ choices and their consistency. They develop some personal strategies according to goals and pressure of their work. Those depend on several factors such as their beliefs about mathematics, their beliefs about how to learn, their knowledge of the mathematics content of schools curriculum but also their experiences as a teacher or a student (Robert, 2004; Perrin-Glorian, 1995).

The research was carried out with five teachers in CM2 (children aged 9 or 10) or CM1/CM2 (children aged 8, 9 or 10) grades. All the teachers chose to use Euromaths in their class. They had different professional backgrounds, some of them had scientific or didactical background (3 of the 5 teachers). They were different ages, they were both male and female and they had been teaching for more or less time. In order to read the paper more easily, I call the teacher A, B, C, D and E according to the distance between the tasks they carry out and the manual tasks *a posteriori*. In the following table I present some of teachers’ personal data we used to understand the practices. According to Shulman (1986), mathematics knowledge, pedagogical mathematics knowledge and curricular knowledge are three points to look at to understand if a teacher may be able to teach in a certain way. That’s why I focus on the teachers’ scientific background, didactical background and teaching experience with or without *Euromaths*.

Teachers	Grade	Scientific background	Didactical background	Teaching years	Euromaths manual used years
A	CM1/CM2	Yes	Yes	3	2
B	CM2	No	Yes	10	2
C	CM2	Yes	No	10	1
D	CM2	No	No	25	3
E	CM2	No	No	30	1

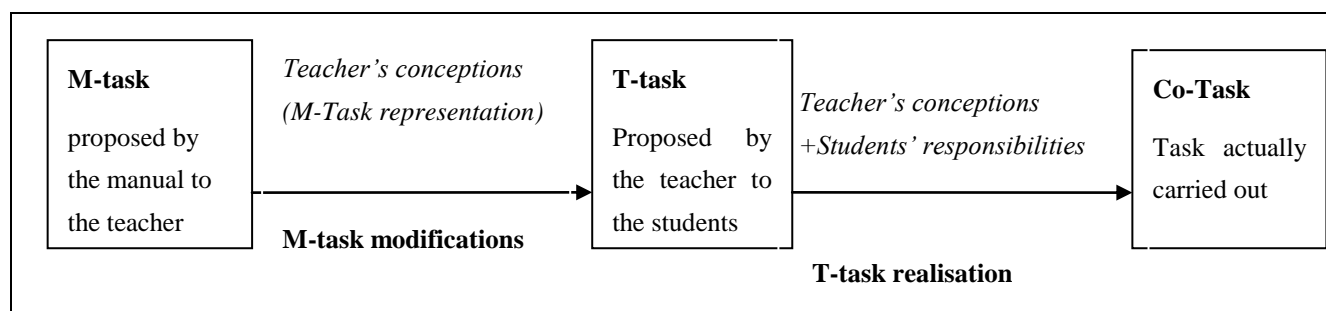
**Table 1: Teachers’ personal data**

I went into each teacher’s class to observe and tape all the work on fractions (5 to 20 session per teacher). I interviewed the teachers before the beginning of this work and

after it ended. The lessons and interviews were transcribed. According to Robert (2004) even if teachers' espoused practice might indicate that their conceptions are close to the authors' conceptions, their actual practices may be really different. For this reason, in order to analyse teachers' practices (and conceptions) I used the lessons transcription rather than interviews. I cut the transcriptions into different episodes like "proposition of the task by the teacher", "students' research", "collective work" etc. and I analysed each episode.

To determine the spectrum of teachers practices when they use a problem from the manual, I identified for each teacher the distance between tasks proposed by the manual and tasks proposed and carried out in the classroom. The M-tasks I consider are the tasks written in the textbook but also the different ways they can be carried out according to the authors' conceptions. Both texts (the problem text in the textbook and the guideline in the teacher book) have been analysed to understand what could happen in the class. According to those analyses we determined the knowledge contained in the tasks, the possible ways to carry the tasks out and I defined the points on which we will especially focus.

The indicators we used to understand teachers' practices are the tasks proposed by the teacher to his student (T-tasks) and the tasks actually carried out in the classroom (Co-Task) which includes the way teachers share responsibilities with their students.



**Table 2: Distance between M-task, T-task and Co-Task**

The T-tasks are the tasks the teacher propose to his students for their individual or group work. To understand what those T-tasks were I analysed the teachers' work episodes happening before the students' work episode. I analysed how the teacher changes the problem text if he does, what kind of help he gave the children before their individual work, of what he reminds his students if he does remind something, etc. Then I did an "*a priori*" analysis of the T-tasks proposed to the students. Indeed, according to Leplat (Leplat, 1997) the Co-tasks actually carried out in the classroom have to be analysed according to the new tasks logic i.e. to the T-tasks logic. The "*a priori*" analysis of the T-tasks also allow us to measure the distance between the T-tasks and the M-tasks. The T-tasks proposed by the teacher to his students for their individual or group work may not match with the M-tasks. Those "new" tasks (T-tasks) will depend on the teachers' representation of the M-tasks which depends on their conceptions of mathematics, of teaching and of their students, etc. So, the

differences between the manual's tasks and the tasks proposed by the teacher to his students gave us some first elements to understand teachers' conceptions and the way they "read" the situation. By "reading" the situation I mean what the teachers think they have to do, what they think the students have to do and the knowledge they think is contained in the problem before carrying it out.

To measure the distance between Co-tasks and M-tasks I therefore analysed the realisation of the T-tasks according to the T-tasks logic but also to the M-tasks logic. To understand how the T-tasks were carried out I especially looked at the students responsibilities for the construction of knowledge. The responsibilities given to children were identified in relation to their individual or group work time, what they are asked for (about tasks they worked on, about questions they had to answer in collective work, about the way the teacher listened to what they say and adapt the work...) and the way they are asked (open or closed questions, often or not and which students are asked). I also looked at how the teacher helps the students and how he or she uses what they do or say.

## THE PROBLEM

The *Euromaths*' problem I chose is about fractions in the measures framework. It is a transposition of a didactical engineering built by Perrin-Glorian and Douady (1986). This problem's transposition is the following.

To measure a segment length, Leïla used the u-segment as unit.

(The segment is drawn in a corner, just below a character who is often used in the manual says : To have  $\frac{1}{3}$  of the unit, I bend in three like an accordion the unit strip.)

She transfers one time the unit u, then the half of u, and finally the third of u.

She wrote :  $1\text{ u} + \frac{1}{2}\text{ u} + \frac{1}{3}\text{ u}$ .

Question 1 : Reproduce the unit strip u. Using this strip find which segment Leïla has measured. (Five segments are drawn which measures are :  $AB = 1\text{ u} + \frac{1}{2}\text{ u}$  ;  $CD = 1\text{ u} + \frac{2}{3}\text{ u}$  ;  $EF = 1\text{ u} + \frac{1}{2}\text{ u} + \frac{1}{3}\text{ u}$  ;  $GH = 1\text{ u} + \frac{1}{2}\text{ u} + \frac{1}{4}\text{ u}$  ;  $IJ = 2\text{ u}$ .)

Question 2 : Find the measures of other segments' length, using the unit u.

Question 3 : Leïla says that the segment [GH] measure  $\frac{7}{4}$ , is she right?

### Table 1: Euromaths' tasks

The aim of first question is for the children to build a technique to measure segments. The way the question is asked might encourage them into the "complex writing" of measurements (a sum of an integer which here will be 1 and of one or several fractions which are less than one). The aim of second question is to practice this technique and to arrive at the complex writing measurement for the [GH] segment. The real aim of the problem is in the third question. The children will have to point out that Leïla's segment measure matches to the measure they obtain in a complex

writing form. They can check that the complex writing measure they obtain is equal to Leïla's measure, i.e. that  $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$  or  $1 + \frac{3}{4} = \frac{7}{4}$ . The aspects of knowledge contained in this problem are the meaning of a half, a third or a quarter according to the split of the unit but also the use of complex writing and the equality seen in the second and third questions (for example :  $1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ ).

The tasks of this problem are open. Students can find several techniques to answer the questions and can find several ways of writing the measures for the segments. The two first questions cannot be validated by the student themselves. The validation episode might be too complex if teachers don't want to give children the answer and if they want to keep the problem open for everyone. Concerning the last question it can be validated by the children themselves.

The different ways to carry out the two first questions depend especially on the way responsibility is shared for the individual work on the questions and on the collective summarizing work. The third question also depends on the sharing of responsibility but also on the framework used to answer the question. It can be the measure's framework, the numerical framework or both.

## RESULTS

In order to understand the analyses results we present what are the T-tasks, Co-tasks, student's responsibilities, teachers' help and responsibilities in the following table.

Teachers	T-tasks	Student's responsibilities	Teachers' help	Teachers' responsibilities	Co-tasks
A	M-tasks (question 1, 2, 3)	All the students have responsibilities	Individual help Don't reduce the tasks	Tutor	M-tasks
B	M-tasks (question 1, 2, 3)	All the students have responsibilities	Individual help Don't reduce the tasks	Conductor	M-tasks
C	M-tasks (question 1, 2, 3)	Students with difficulties have responsibilities	Individual help Don't reduce the tasks	Tutor	Reduced M-tasks
D	Reduced M-tasks : 1. Find Leïla's segments already knowing the technique to measure 2. Measure others segments	None students has responsibilities	Collectives help Reduces the tasks	Knowledge's owner	T-task (question 1) M-tasks (question 2)
E	Radically different from the M-tasks : Measure all the segments	Very good students have responsibilities	Collectives help Reduces the tasks	Knowledge's owner	Reduced T-task

**Table 4: Teachers' practices when using the Euromaths problem results**

The first result according to the distance between the manual's tasks and the tasks carried out in the classroom is that teachers A, B and C share the author's conceptions. On the contrary, teachers D and E do not. They are the knowledge owners and the student's work in their classes seems to be reduced to reproductions of exercises the teacher showed at first. The second result is that the spectrum extent of ways teachers carry out the problem is very large. In each class the students work on different tasks and the knowledge pointed out by the teacher is also different.

### **Teachers proposing the manual's problem to their students**

Three of the teachers (teachers A, B and C) propose tasks which match to the problem in the manual. It means that when they give time to the children to work, it is on the M-tasks. All the questions are proposed like they are written in the manual and the different tasks are not reduced. The meaning of reducing a task is to give students more elements to solve the tasks than are contained in the manual which will make the problem easier. Moreover, for those three teachers the knowledge they want to teach is the same as in the manual.

### **Teacher proposing a reduced problem to her students**

One of the teachers (teacher D) reduced the problem according to the style of her practice. She gave children lots of elements which made the tasks easier, less problematic and no more related to the other tasks. She didn't radically change the problem but she changed the meaning of it. Children simply had to measure the segment and not to think about how to measure it. Moreover the teacher didn't propose the last question which is the real issue of this problem. The problem is reduced to an application exercise which the teacher has shown at first, i.e. measuring segments using a unit strip. Thus, the knowledge contained in the tasks proposed by the teacher is about knowing how to measure segments using a unit strip.

### **Teacher proposing a tasks radically different from the manual to her students**

The last teacher (teacher E) radically changed the manual tasks. Firstly, like teacher D she reduced the problem giving the students the way to measure the segments. Secondly, teacher E changed the problem text. She didn't give students some of the information given in the manual (she didn't tell them to read the text and she didn't read it to them, and most of the students didn't think to read it). So, they missed Leïla's segment's measure and the way she measured it as well as the way to bend the unit in three equal parts. Then, the teacher changed the questions and asked the children to measure all the segments. This task is radically different from the manual's problem. Notably, children had to measure the segment [EF] (Leïla's segment which has not to be measure according to the manual but found in the complex writing form :  $1\text{ u} + \frac{1}{2}\text{ u} + \frac{1}{3}\text{ u}$ ) . It is a new didactical variable. They can find Leïla's measure or the measure  $1\text{ u} + \frac{5}{6}\text{ u}$ . To validate the equality of these two measures might be difficult. Either students could check if the measure is right but

they will have to bend the strip in six parts which is quite difficult at their age, or they have to understand the equality  $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$  in the numeric framework which may be more difficult. Moreover, the knowledge contain in the T-tasks according to the teacher is about a fraction more than one whose numerator is bigger than the denominator. This knowledge is quite different to the knowledge contained in the manual's tasks according to the "*a priori*" analysis.

### **First conclusion about the tasks proposed to the students in the different classes**

The spectrum of tasks' proposed by the teachers to the students seems to be large. Three of the teachers propose the manual's tasks without changing it. They are the three teachers with scientific and/or mathematical background. The two other teachers who do not have this background propose a reduced or radically different problem. Their representations therefore seem to suggest conceptions different from the authors' conception.

### **Teacher carrying out the tasks they wanted to carry out**

Teachers A and B, in different ways (being a tutor or a conductor) give students the responsibility for the knowledge construction. Children have a lot of time for individual or group work and they can validate their results themselves. The tasks carried out by the teachers A and B match to the problem they propose and to the authors' conception. On the contrary teacher D doesn't give any responsibility to her students. The task she proposed to the students is reduced. Children don't have much individual work time and don't speak a lot, contrary to students in teachers A and B's classes. Teacher D therefore controls the realisation in the classroom and carried out the tasks she proposed to the children (a reduction of the manual's tasks).

### **Teacher carrying out different tasks from what they propose at first**

Teacher C gave responsibilities to children who have big difficulties. The rhythm in the class is then very slow and most of the children are bored and off-task. Because of this very slow rhythm, the teacher has to go very fast on important phases. Thus, this teacher who didn't change the manual tasks at first, reduces it when she carried it out because of the way she shares responsibilities with the students. Teacher E appeared to be trying to match with the authors conceptions and gave responsibilities to students. However, she only gave responsibility to the very good students. The rhythm was then really fast and most of the children were lost. Moreover, according to what they do or say she changed her objectives and taught for example how to sum  $\frac{1}{2} + \frac{1}{3}$  which does not need to be known by children aged 9 or 10 years.

### **First conclusion about the tasks carried out in the different classes**

The responsibilities sharing is really different according to the teachers' practices. It has an influence on the tasks they carry out. The teachers who share the responsibilities with all the students carry out the tasks they have proposed, while the teachers who share the responsibilities with one only kind of students do not.

Moreover, when they try to fit with the author's conception (although their own conceptions are far from it) the teachers' practices might be problematic.

## **CONSEQUENCES OF TEACHERS PROBLEM'S APPROPRIATION**

Teachers' A and B carried out the manual's tasks without any problem. But, it seems that teachers C, D and E didn't carry out these tasks which might be problematic for the student's potential learning. We will see that the association of their practices and of this problem can lose the students or even make the differences between good students and students who have difficulties larger. But, I will also see that the use of the problem can change their practices.

### **Practices and manual written by didacticians problematic association**

Teacher C, because of the responsibilities she gives to students with very high difficulties, made the rhythm really slow and lost other students attention. That might be a problem for students' potential learning.

Teacher E proposed the students really different tasks from the manual's tasks which created difficulties, e.g. the question about  $1u + \frac{5}{6}u$  and  $1 + \frac{1}{2}u + \frac{1}{3}u$  equality. This question introduced a new variable and to answer it the teacher introduced knowledge which children aged 9 or 10 would not be expected to have. This is really problematic. Moreover, in this class the students were not all working on the same tasks as they were not working with the same responsibilities. Firstly, some students work on the manual's problem while the others work on the radically different teacher's problem. Unlike the other students, the good students had all the necessary conditions, like Leïla's segment measure, to solve the problem. Secondly, good students had enough time to answer the questions the teacher asked because they worked very quickly while other students didn't have time to think of it. So, good students participate and can debate about their results while the others are just listening. I have also to point out that it's according to their answers that the teacher changes her objectives losing the other students' attention. Thus, in this class, because of the teacher's practices when using this problem, the difference between the students levels increase which is really problematic.

### **Manual problems traces on teachers' practices**

On the contrary, because of the responsibilities given to the children by the manual's tasks, which is not radically changed by the teacher, teacher D's practices seem to change. She gives more responsibilities to her students which might improve their potential learning. Thus, it seems that the manual can change the teachers' practices in the direction of the authors' conceptions. Indeed, we saw that for the teacher D, the knowledge contained in the problem was about measuring segments using fractions (she didn't propose the third question but only the first two which are about measuring segments). I also saw that she didn't share responsibilities with her students. But, when she proposes the second question, children proposed several results for each segment measure. Thus, the teacher has to give more responsibilities



to her students to explain their results and the knowledge about equality of a complex writing form and a factionary form was explored even though the teacher didn't think of it before. When she carried out the reduced tasks proposed to the students the teacher changed her practices and came back to the manual's tasks and to the knowledge contained in them. The problem seems to be constructed in a way which can change the way this teacher carried out the tasks and especially the way she shared responsibilities with students. This new responsibilities sharing may improve student's potential learning and because of the definition of practices which include the potential learning, I may say that the use of this problem might improve teachers' practices.

### **POSSIBLES TEACHERS' PRACTICES / MANUAL'S PROBLEM INCOMPATIBILITY FACTORS**

Even if all the teachers say they share the author's conceptions (that's why they choose this manual). Teachers A, B and C are the only three who seem to really share their conceptions and teachers A and B are the only two who seem to manage carrying out the manual's problem. However, I saw that teacher D changed her practices according to the authors' conceptions. According to the practice's analyses, I point out several things which might be the reasons for an incompatibility of the teachers' practices and the manual tasks.

Firstly, the knowledge contained in the problem might not be understood by the teacher at first sight. Without an "*a priori*" analyses or without reading the teacher's book (most of the teacher don't read it), teachers might not "see" what knowledge is pointed out by the problem. Thus, I saw that two of the five teachers didn't point out this knowledge and made the students work on something else. Unlike the other three, these two teachers didn't have a scientific or didactical background. It could mean that these backgrounds are necessary to understand the problems' objectives.

Secondly, it seems to be necessary to share responsibilities with all the students in order to present the problem in the way intended by the manual. This kind of responsibilities sharing needs the teachers to really pay attention during the realisation of the manual tasks and needs them to be able to adapt themselves really quickly. Moreover, children cannot validate the results themselves for questions one and two. In order to leave the problem open, teachers have to find ways which are not obvious. It means another occasion when they have to really pay attention and to adapt themselves really quickly if they didn't think about it earlier.

The problem seems so to be difficult to carry out, needing teachers to have a mathematical or didactical background, a real attention and capacity to adapt themselves and needs them to give the students some responsibilities. Moreover, the ways teachers A and B carry out the manual's tasks (being a tutor or a conductor) according to the authors' conceptions are not economic which mean that they could be difficult to transpose to other teachers.

## CONCLUSION AND PERSPECTIVES

The spectrum of teachers' practices using the same 'written by didacticians' problem seems to be large. Some of the teachers who share the authors' conceptions make good use of the problem. For such teachers, the manual may be helpful especially to gain time. I also saw that for teachers who didn't change the problem radically and who carried it out with their own conception even if they are far from the authors' conceptions, the use of the problem written by didacticians can improve their practices. It is a result which has to be developed analyzing more than one problem being carried out. But, the use of the manual can also be problematic if the teacher's conceptions are far from the author's conception and if the teacher tries anyway to carry the problem out according to the authors' conceptions without understanding them. Teachers' practices can then lose their consistency which is quite a problem. I therefore have to know if for such teachers the problematic use of the problem can be reduced and how this might be done.

The analysis of only one use of problem is not enough to understand teachers' practices. I will have to analyse several problems with different characteristics to understand what may make the use of the manual by some teachers problematic and what exactly may improve teachers' practices when using such manuals.

## REFERENCES

- Artigue, M. (1989). Ingénierie didactique. *Recherche en didactique des mathématiques*, 9 (3).
- Bolon, J. (1996). Comment les enseignants tirent-ils parti des recherche faites en didactique des mathématiques? Le cas de l'enseignement des décimaux à la charnière école-collège. *Thèse de doctorat*.
- Leplat, J. (1997). *Regards sur l'activité en situation de travail. Contribution à la psychologie ergonomique*. Paris: PUF.
- Perrin-Glorian, M.-J. (1995). actes de la VIIIème école d'été de didactique des mathématiques.
- Perrin-Glorian, M.-J., & Douady, R. (1986). Nombres décimaux liaison école-collège. 62.
- Robert, A. (2001). Les recherches sur les pratiques des enseignants et les contraintes du métier d'enseignant. *Recherches en didactique des mathématiques*, 21 (1/2), pp. 57-80.
- Robert, A. (2004). Que cherchons-nous à comprendre dans les pratiques des enseignants? Quelles analyses menons-nous? In M.-L. Peltier, *Dur pour les élèves, dur pour les enseignants, dur d'enseigner en ZEP* (pp. 15-32). La pensée sauvage.
- Robert, A., & Rogalski, J. (2002). Le système complexe et cohérent des enseignants de mathématiques: une double approche. *La revue canadienne de l'enseignement des sciences des mathématiques et des technologies*.
- Shulman, L. (1986). 'Those who understand, knowledge growth in teaching'. *Educational Researcher*, 15 (2), pp. 4-14.

# A STUDY OF A PROBLEM SOLVING ORIENTED LESSON STRUCTURE IN MATHEMATICS IN JAPAN

Yukiko Asami-Johansson

University of Gävle, Sweden

*This paper presents and analyses “Mondaikaiketsu no jugyou” which translates to “the problem solving oriented – approach”. It is a set of didactic techniques with the aim of motivating the students’ positive attitude toward engaging in mathematical activities and fostering mathematical thinking. As an analytical tool, The Anthropological Theory of Didactics (ATD) will be applied.*

Keywords: Problem solving, Japanese mathematics class, ATD.

## INTRODUCTION

Teaching methods were developed differently in Japan compared to other industrialized countries. Hiebert, Stigler and Manaster (1999) argue that Japanese teachers emphasize mathematical thinking rather than mathematical skills. This goal is reached by having the students discuss with the teacher and peers on the settlement options of problems presented to the whole class. I will call this type of didactic techniques, where students work on whole-class problem solving, for *problem oriented lesson structure* (POLS).

A basic problem in mathematics education, and in the training of teachers, is to find ways to organise the classroom work so as to make the students active learners of mathematics, without losing the focus on the mathematical content. Japanese teaching methods, like the ones described, have attracted attention in Sweden lately (Dagens Nyheter, 2009), and it has been discussed as a possible model to develop in Swedish school system.

Kazuhiko Souma is one of the teacher educators/researchers who has proposed, introduced and elaborated on POLS. He calls his method “*The problem solving oriented*” approach (shortened to PSO; the author’s translation; “*Mondaikaiketsu no jugyou*”, in Japanese). Like POLS in general, it aims to enhance the students’ attitude towards engaging in mathematical activity in the classroom. In Japan, there is a tradition of publishing practical books for mathematics teachers as the target group. This literature aim to present ideas and concrete lesson plans, based on well-constructed mathematical problems, according to the proposed teaching methods (Souma, 1995; Kunimune & Koseki, 1999; Tsubota, 2007). Souma has written and edited a number of such books and his method is actively and widely used by teachers in service. It has received little attention from the academic community, perhaps because of its practical attribute and the lack of clear theoretical base. The lack of theoretical overhead is perhaps part of the appeal, but becomes a problem when one is to describe and assess such approaches.

I conjecture that PSO (and POLS in general) could make a beneficial contribution for pre-service teacher education due to its distinct didactical structure, but the lack of a theoretical base is a hinder. The aim of my paper is therefore to present and analyse PSO in relation to other Japanese POLS approaches. To illustrate, I will analyse an episode of a practical application of PSO in a Japanese classroom.

As an analytical tool, I will use the anthropological theory of didactics (ATD) developed by Y. Chevallard (2006), with the assumption that ATD has the right level of abstraction for the purposes of didactic planning. The focus of this paper is on how the proposed didactic techniques and lesson structure of PSO relates to the description in ATD of the didactic process. In a follow-up paper I intend to shift focus to the mathematical content and in particular to how Souma's ideas regarding the construction of problems can be analysed in ATD.

## **BACKGROUND TO The PSo APPROACH**

### **The Anthropological Theory of Didactics**

The anthropological theory of didactics (ATD) approaches learning and knowledge as institutional issues. Mathematics learning can be modelled as the construction, within a context of social institutions of interlinked *praxeologies* of mathematical activity, which we also refer to as mathematical organisations (MOs) (Chevallard, 1999 in Barbé, et al., 2005). A praxeology is described by its *tasks* and *techniques* (*praxis*), together with its *technology* and *theory* (*logos*). Technology constitutes the tools for discourse on and justification of the techniques and the theory provides further justification of the technology and connections to other MOs. The process, under which a mathematical praxeology is constructed within the educational institution, is called the *didactic process* (ibid.). Chevallard proposes to describe it as being organised in six “moments” that can be thought of as different modes of activity in the study of mathematics. The moments are: (FE) the moment of *first encounter* (or re-encounter) of tasks associated to the praxeology, (EX) the *exploratory* moment of finding and elaboration of techniques suitable to the tasks, (T) the *technical-work* moment of using and improving techniques, (TT) the *technological–theoretical* moment in which possible techniques are assessed and technological discourse is taking place, (I) the *institutionalisation* moment where one is trying to identify and discern the elaborated MO and (EV) the *evaluation* moment which aims to examine the value of the MO. The description of mathematical knowledge (the MO) and learning/teaching of this (the didactic process) is referred to as the *epistemological* component of ATD.

To organise the work of achieving an appropriate MO, the educator faces the task of designing and controlling the didactic process. To this end, one develops a didactical praxeology, or a *didactical organisation* (DO), consisting of didactic techniques together with a technology/theory to describe and justify those techniques.

I think it is fair to say that ATD carries both a *normative* and *descriptive* component. As an analytical framework, it holds that any didactic processes can be *described* as the construction, via the six moments, of a praxeology. Similarly, the didactic organisation can be described in terms of its praxis and logos block, independently of whether the studied DO's have ATD incorporated as an epistemological model of the learning object and the didactic process. But, when used as a design tool, ATD also carries some normative implications, so that the resulting MOs and DOs can be compared with respect to how suitable, structured, and useful (or *legitimate*) they are. A typical implication is that the local MOs should integrate with and reinforce larger superstructures in the form of global praxeologies and that all moments of the didactic process needs to be visited and such local objectives are compatible with the aims of Souma's writings. In this paper, I will attempt to use ATD to describe the DO proposed by Souma and also use the epistemological component of ATD to motivate some of Souma's didactic techniques. Thus, in a way, propose an *extension* of the DO proposed by Souma with a technological-theoretical block from ATD.

### **The PSO lesson template**

PSO has the form of a proposed lesson structure and Souma states that it is instrumental that the PSO approach is applied with the same basic form regularly. The motivation is that familiarity with the situation makes the students feel secure in participating in the discourse and engaging in the didactic process.

Souma, like most Japanese writers of this genre, often gives general didactic advices: to be generous with positive feedback, taking care of shy students, etc., in order to handle the long-term didactic goals, such as “fostering the students to active learners of mathematics”. The explicit motivations are often taken from a technological-theoretical block, which could be referred to as Japanese “didactical/pedagogical common sense”, although, as stated below, Souma explicitly refers to the cognitive theories of Dewey and Polya as a motivation. The epistemological description is usually a concrete mathematical example and listing presupposed knowledge and goals for the lessons. This format is natural for this type of inspirational literature, but has its limitations when one wants to discuss the generalities. A central recurring term is that of “mathematical activity”, which measures the degree of interest, independence and motivation with which students are carrying out the mathematical work.

In order to make a fair description of Souma's approach, it does not seem reasonable to eliminate all such psychological aspects. Therefore, in this description below, I will make qualifications when talking about the didactic process: like the degree of *participation* in the didactic process and “*invigorate* the didactic process” to mean, “increasing the *mathematical activity*” of the didactic process.

According to Souma's example from his book (1995), a typical POLS lesson starts with a teacher giving a problem, for instance, “Show that the difference of the

squares of two integers that follow each other is equal to the sum of the two numbers ( $5^2 - 4^2 = 9 = 5 + 4$ ,  $24^2 - 23^2 = 24 + 23$  and so on).” The students try to solve the task and some students write their solutions on the blackboard and explain their solution orally. Souma wonders (pp. 103-104) if the students in this situation will feel a “necessity” to reflect upon the task. Furthermore, some students might not get any ideas on how to solve the problem and will therefore become alienated from the discourse. As an alternative, he proposes the following variation of the problem formulation: The teacher writes down expressions on the blackboard without any comments;

$$5^2 - 4^2 = 9, \quad 24^2 - 23^2 = 47, \quad (-9)^2 - (-10)^2 = -19$$

and asks the students what they can observe. All students are supposed to be able find such observations, perhaps working in groups. Students may answer, “It becomes odd numbers”, “The differences equal the sum of the integers”, “The differences equals the first integer times two minus one”, “The last integer times two plus one”. After the response of the students, the teacher then controls that all proposals are correct on the blackboard and says; “Now we try to prove each of the statements”. Ideally, the formulated problems have many possible roads to solutions: Several students may use the formula for expanding the square of a sum; and several others, using  $x$  to the first integer and  $y$  for the second integer, the rule of the conjugate.

Souma proposes to use a didactic technique, which I refer to as *guessing*. One should, regularly, let all students guess an answer or formulate hypotheses about the phenomena. It is implied that the “guess” is something that all students can participate in. In the example the students are not asked to simply guess an answer, but they are invited to, discover patterns by themselves, make hypotheses about the phenomena and by implication set their own tasks. By committing to make a guess or a hypothesis, especially in the social context of the class, the student will have a stronger motivation to study the task and follow it up.

We can find explicit theoretical motivation of the DO in Souma’s description (1997) where he declares that he is inspired by John Dewey’s theory of reflective thinking. Dewey (1933) presents five cognitive phases of problem solving. 1. Recognize the problem. 2. Define the problem. 3. Generate hypotheses about the phenomena. 4. Use reasoning if the hypotheses are viable to solve the problem. 5. Test the most credible hypotheses. Dewey’s theory has a general scope and is applicable to any problem context. It is also concerned with the cognitive dimensions, rather than the didactic process as such. Souma states that educators in mathematics may have a tendency to hurry up to address the later phase to “use reasoning”. In this way, the development of reflective thinking and motivation may be impeded. Souma thus feels that it is necessary to pay attention especially to the first three phases. He expresses that (1997), from Dewey’s theory, we may infer that it is important that we should “(a) have an aim for why we solve the task, (b) feel a necessity to solve the task and (c)

have made hypotheses before starting the reasoning process.” (p. 34) Souma also refers to Polya’s (1957) cognitive theories on problem solving and, in particular, Polya’s insistence on the importance of guessing. Polya states that our hypothesis may of course be wrong, but the process of examining the guess should lead to improved hypotheses and a deeper understanding.

The focus on motivation on the first encounter and the exploration, together with the insistence on a well defined and controlled mathematical content, is perhaps the point that, most distinctively, sets PSO apart from other proposed DO’s in the POLS tradition. Souma states that the teacher much take care to plan how the problem is presented and reflect on how students will to act. Souma names (1987) the type of tasks a teacher should aim at, as “*open-closed*” tasks that stimulate conjecture and application of guessing. In ATD terms one can say the task should be “closed” so as to give a *controlled* vector from the (FE), the moment of first encounter, to (EX) and (T), and also a (somewhat) predictable outcome during the following discourse, which usually would concern the establishing of the technological-theoretical environment (TT). The task should also be “open”, by giving the student a chance to make individual choices during the exploration (EX), and later give ample material for discussion, so as to invigorate the didactic process.

Souma means that, starting from standard tasks in the ordinary textbooks of mathematics, the teacher can modify parts of the tasks or change the way of stating them as in the example we saw. If the tasks presented during a sequence of lessons, are carefully constructed, it lead to conjectures, new problems and methods that, in ATD terms, productively connects the local MO’s covered with more global ones and inspire to technological and theoretical discourses on higher-level MOs. The insistence on open-endedness of the task is common with the “open approach method” (Nohda, 1991), which is another proposed variant of POLS. The open approach method is used and analysed by Japanese educators (Hino, 2007). Open-ended problems often take the form of formulating a mathematical model for some phenomenon that lead to multitude of problem formulations, techniques and solutions. The intent is to let students develop and express different approaches and let them reflect on their own ideas by seeking to grasp those of their peers (Miyakawa & Winsløw, 2009). Souma (Personal Communication, 2010) judges the open-approach method as something that cannot be used in everyday school mathematics. POLS lessons applying too ambitious open-ended problems might be isolated from ordinary lessons that, for instance, aim to train students’ basic mathematical skills, but Souma (1987) acknowledge this type of projects at the end of a course. Nohda also notifies that “We do the teaching with the open-approach once a month as a rule” (Nohda, 1991, p. 34). Bosch et al. (2007) have discussed the danger with open-ended activities, which are introduced at school without any connection to a specific content or discipline. They state that this type of didactic

technology suffers the risk of causing the construction of very punctual mathematical organisations, since this is what students are trained to study.

If we return to the lesson template and the example, the teacher should let students who have different types of solutions present their problem in class. The teacher then leads the class to discuss the reason behind each method and have all students determine which of the techniques they have used and why. This is the didactic technique of *whole class discussion* of solutions, which PSO has in common with POLS in general. The discussion of alternative solutions gives an opportunity to introduce, establish and reinforce technological and theoretical components of the MO studied, like in this case, the expansion of the square, the rule of the conjugate and the different use of variables. The primary motivation is to steer the didactic process into (TT), where new methods and techniques are approved. Solutions and motivations given by the students are sometimes unexpected and may make more sense for other students. The class discussion also serves the purpose of increasing the participation in the didactic process.

After this, Souma recommends that the students have an opportunity to reflect upon the mathematical theory. The teacher can point out what they have learned by having a student read out from the textbooks explanations of the theory relevant for the lessons. During this *theoretical reflection*, the teacher can steer the didactic process towards, say, (I) institutionalisation or (EV) evaluation. Souma states firmly (1997) that studies in mathematics should be organised and based on a well-written textbook that gives a clear explanation of the mathematical definitions and theories. The classroom discourse is only one form of the study process, the study of mathematics will always entail individual studies and individual problem solving inside and outside school. Moreover, the textbook allows the students to recognize and get familiar with the theory, which the textbooks usually explain in more full detail. In other words, the *textbook technology* is proposed, for the purpose of further covering of all six moments.

### ***a mathematical problem oriented class in Japan***

The following episode illustrates a mathematical problem oriented lesson where the teacher practices the PSO approach. This study take place during a lesson study in grade eight at a lower secondary school affiliated to the School of Education in Asahikawa, Japan, 2009. The teacher is a former Masters student of Souma. The number of students in this class is 40. The lesson is about how to solve a system of linear equations and is the third lesson on this topic. The students have already studied the addition method by solving linear equations obtained from word problems with an everyday life character. The lesson plan was written and distributed by the teacher to us observers beforehand. Posing the mathematical tasks and problems presented during the lesson is common with the POLS based lesson plans, but distinct to PSO is, that it is always written “students possible conjectures”



and “students possible solutions”, so that teachers always prepare different didactic responses depending on which act students take (Souma, personal communication, 2010).

In the guidebook of Japanese national curriculum standards “The curriculum guidelines” (2008) for mathematics for Japanese secondary school, a system of linear equations with two variables is described (p. 90) as follows: “Solving a linear equation with two unknowns is to make clear that this can be done by using a method that eliminates one of the two variables and then solve equations with one unknown, which is a method students already know”. Thus, the didactic transposition of the praxeology “System of linear equations” to the knowledge to be taught in class (Chevallard, 1985 in Bosch & Gascón, 2006), focuses here on the technique of elimination; reducing the pair of two variables equations to one equation with one unknown. Techniques and technological terms present are substitution, row operations, isolation, coefficients, variables, etc. which are collected from the theoretical base of “Elementary algebra”.

### **The lesson**

As the first step, the teacher shows the problem by verbally reading out a system of linear equations;  $\{7x + 3y = 30, x - 5y = 26\}$  and the students are asked to copy this in writing. He asks: “There are two boys, Taro and Jiro, who both solved this problem. Taro said, “I eliminate  $x$ ”. Jiro said, “I eliminate  $x$  as well”. Their answers were the same, but their methods of the solutions were different. Today’s task is to consider how they solved the problem differently”. The teacher does not show the techniques; the students must consider the possible techniques, which obviously is not only one.

The teacher gives them a few minutes (“individual thinking activity” –according to the lesson plan) and encourages them to find as many solutions as possible. He states in lesson plan that this is especially meant for the gifted students who find solutions quickly. The teacher picks up two students who have obtained different techniques and lets those two students write their solutions on the blackboard. The teacher asks the class how many of them used the technique one of the two students has used. The students raise the hands and it is 37 of them. The teacher asks what is the name of this technique and gets the answer “the addition method” which the class already learned at the previous lessons. The teacher asks the class how this technique works. A student answers “Change the coefficient to the same and erase one of the variable”. The student who has written the solution on the black board explain her reasoning how she has “changed the coefficient”. She says, “ $x$ ’s coefficient must be changed, so I multiplied it by 7”. The teacher responds, –“OK, you multiplied by 7 and got the same coefficient for all the  $x$ :s”. He changes his voice tone a little and then asks “And then, (looks around the class) *what can you do with the  $x$ ?*” Several students respond, “We can *eliminate* the  $x$ ”.

They later discuss the other solution technique called the “substitution method”. He inquires again how many of the students came up with an example of that technique (17; –many of them used both methods), and asks for the name of the technique, and then lets the students explain how the technique works. (Some students might already have learned about the technique at “Juku” – a private school offering special classes held on weekends and after regular school hours.) The teacher later asks if there are any students who found variants of the addition method, with an intention to let the students be aware to variation of techniques of the addition method. One student presents his solution by multiplying with  $-1/7$  to  $7x + 3y = 30$ , instead of multiplying by  $-7$  to  $x - 5y = 26$ . This presentation awakes a big discussion in the class if it is not a bit too complicated. The teacher concludes the discussion by encouraging the student with: “But it worked? Didn’t it?”

After the class has had this look at the two different techniques, the teacher lets one student read out loud a passage from the chapter in the textbook, explaining the substitution method. The students work out three to five textbook problems using the substitution method from the book. Afterwards, the teacher asks the class “In which types of problem do you use the addition method and in which types do you use the substitution method?” He lets the students write down their reasoning. The students are then encouraged to *create* several examples of problems they think fit each technique and different proposals are then later discussed.

### **ATD analysis of the lesson**

The purpose of this lesson is to introduce the substitution method and compare it with it to the addition method and to show that both methods reduce the system to the one-variable one-equation case. In his lesson plan, the teacher writes that “The aim of the task” is to “make the students *find out* that there is an another method than addition method through mentioning that two boys use different methods”. He asks how to reconstruct the solution of two boys, instead of asking them “Solve this system of linear equations using the substitution method”. This is an instance of the Souma’s guessing technique, since all students are assumed to be able to use the addition method and students are requested to make proposals rather than final answers. This is also an example of an “open–closed” task; with alternative solutions, but a limited number of possible outcomes. As intended, the task steer the didactic process from (FE) to (EX) and (TT), since it is about finding a new technique, where (TT) is mainly covered during the whole class discussion. The task will also entail (T), technical work, since the students should solve the system with the chosen method. The teacher stimulates *participation* by having all students report which method they have followed. In the discourse, the teacher takes care to make the students use the correct technological terms, like “addition method”, “substitution method”, and the use of “eliminate” rather than “erase”. Much of the same holds for the final task when they are asked to construct problems that are suitable for each method. As proposed by Souma, reflection on theory is carried out

when one student reads out loud from the textbook. This steers the didactic process to the moment of (I), so that the class now verifies what they have done during the lesson. More (T) is covered when the students work on problems in the textbook.

## DISCUSSION AND CONCLUSION

One can summarise Souma's approach as one firmly grounded in the POLS tradition. PSO, in particular, focuses on how to start up the didactic process using the guessing technique by adding the elements of conjecture, construction and choice from the start, stimulate students' curiosity to tackle with the mathematical tasks. Souma argues for the didactic techniques of presenting problems followed by whole-class discussion, theoretical reflection and the use of textbooks. The main difference with POLS in general is *guessing* and that Souma stress the need for *open-closedness* when it comes to task construction. Souma holds that a DO based on open-ended activities needs to be established as a long-term *didactical contract* (Brousseau, 1998) and that it should be used as the regular lesson structure.

In this paper I *describe* PSO using the analytical framework of ATD and also use ATD motivate the didactical techniques: By using the *guessing* technique, the teacher allows *all* the students in the class to *participate*. *Open-closed* tasks, *textbook* and *theoretical reflection* are techniques with the dual purpose of both *invigorating* and *control* the didactic process. Some qualitative predicates, like "participation", "activity" and "invigorate", regarding the didactic process were introduced to cover the psychological/cognitive motivations, which are implicit and explicit in PSO. ATD is of course a theory with a wide scope and is also a "radical" theory that problematises the content of mathematical curricula, and, in this respect, Souma's insistence on following a textbook and the focus on closedness is perhaps a contradictory "conservative" trait. But, on the whole, I think that most normative implications of ATD are in line with PSO and with POLS in general. Fundamentally, by attacking problems and following up by whole-class discussion, one links praxis with logos.

## REFERENCES

- Barbé, J., Bosch, M., Espinoza, L. & Gascón, J. (2005). Didactic restrictions on teachers' practice -the case of limits of functions at Spanish high schools. *Educational Studies in Mathematics*, 59(1-3), 235-268.
- Bosch, M. & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, 58, 51-63.
- Bosch, M., Gascón, J. & J. Rodríguez, E. (2007). An anthropological approach to metacognition: The "Study and research courses". Pitta-Oantazi, D. & Philippou, G. (Eds.), *Proceedings of the CERME 5* (pp. 1798-1807). Cyprus: Cyprus University Editions.
- Brousseau, G. (1998). *Théorie des situations didactiques*. Paris: La pensée sauvage.

- Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In Bosch, M. (Ed.), *Proceedings of the CERME 4* (pp. 21-30). Barcelona: Universitat Ramon Llull Editions.
- Dagens Nyheter. Tala om matematik (Ledare). Retrieved September 13, 2010, from <http://www.dn.se/ledare/huvudledare/tala-om-matematik-1.965901>
- Dewey, J. (1910). *How we think*. Boston: Heath.
- Garcia, F., Gascón, J., Higuera, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *Zentralblatt für Didaktik der Mathematik*, 38(3), 226-246.
- Hiebert, J., Stigler, J. & Manaster, A. (1999). Mathematical features of lessons in the TIMSS Video Study. *Zentralblatt für Didaktik der Mathematik*, 31(6), 196-201.
- Hino, K. (2007). Toward the problem-centred classroom: Trends in mathematical problem solving in Japan. *ZDM: The International Journal on Mathematics Education*, 39, 503-514.
- Kunimine, S. & Koseki, K. (1999). *Yoi jugyo" no souzou to tenkai* [To Create and developing "Good lessons"]. Tokyo: Meijitoshō.
- Miyakawa, T., & Winslow, C. (2009). Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics*, 72(2), 199-218.
- Ministry of education, culture, sports, science and technology (2008). *Chugakko gakushu shido yoryo kaisetsu, sugaku-hen* [Commentary to the curriculum standards for lower secondary school, mathematics]. Tokyo: Kyoiku Shuppan.
- Nohda, N. (1991). Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematics problem solving. *Zentralblatt für Didaktik der Mathematik*, 23(2), 32-37.
- Polya, G. (Second edition, 1957). *How to solve it*. Princeton, New Jersey: Princeton University Press.
- Souma, K. (1987). Chuugakkouni okeru "Sougou sugaku" e no ayumi [Trying synthetic mathematics in lower secondary school]. *Journal of Japan Society of Mathematical Education*, 69(11), 302-309.
- Souma, K. (1995) *Yosou wo toriireta Suugaku jugyou no kaizen* (Improvement of the lessons of mathematics with guessing]. Tokyo: Meijitoshō.
- Souma, K. (1997) *Sugakuka Mondaikaiketu no jugyou* [The problem solving oriented approach -The subject of mathematics]. Tokyo: Meijitoshō.
- Tsubota, K. (Ed.), (2007) *Sugakuno Katsuyouryoku wo Sodateru Jugyo*. [Lessons, which foster practical mathematical knowledge]. Tokyo: Kobun Shoin.

# **LESSON STUDY AS A PROCESS FOR PROFESSIONAL DEVELOPMENT: WORKING WITH TEACHERS TO EFFECT SIGNIFICANT AND SUSTAINED CHANGES IN PRACTICE**

Jenni Back and Marie Joubert

University of Plymouth and Bristol University

*This paper draws on data from lesson study initiatives within the Researching Effective Continuing Professional Development (CPD) on Mathematics Education (RECME) (Joubert, Back, DeGeest, Hirst, & Sutherland, 2009) and subsequent work with lesson study groups through the Centre for Innovation in Mathematics Teaching at the University of Plymouth. We explore how teachers engage with the process of lesson study and to identify features of involvement which lead to significant changes in practice that may be sustained over the longer term.*

Keywords: lesson study, continuing professional development, teacher change, knowledge for teaching

## **INTRODUCTION**

This paper is concerned with a model of professional development known as ‘lesson study’. It draws on data from lesson study initiatives within the Researching Effective Continuing Professional Development (CPD) on Mathematics Education (RECME) and subsequent work with lesson study groups through the Centre for Innovation in Mathematics Teaching (CIMT) at the University of Plymouth.

Lesson study, as with other models of professional development, aims to develop teachers’ professional knowledge, and hence to improve (change) teachers’ classroom practice. Improved student learning is sometimes used as an indicator of improved classroom practice. The paper uses these indicators to explore the professional development of teachers involved in lesson study initiatives. The research questions considered are: What evidence is there of professional development of teachers of mathematics who engaged in lesson study? How do teachers engage with the process of lesson study? What features of this engagement may contribute to significant and sustainable changes in their professional knowledge as it is revealed in practice?

## **BACKGROUND**

Professional development for teachers of mathematics typically aims to develop aspects of teachers’ mathematical knowledge for teaching, which is generally agreed to include: knowledge about mathematics, knowledge about ways of teaching mathematics and knowledge about the ways in which learners engage with and make sense of mathematics (Joubert & Sutherland, 2008).

Lesson study, which pays attention to all these aspects of mathematical knowledge is a form of CPD based on Japanese models of professional development to a greater or lesser extent (Lewis, 1995) (Fernandez & Yoshida, 2004) (Burghes & Robinson, 2010), which particularly emphasises student learning. The process of lesson study involves a group of teachers in collaboratively planning a lesson called the ‘study lesson’ or ‘research lesson’ and one of the teachers teaches this lesson. It is videoed or observed by the whole team, with a particular emphasis on the student responses to the lesson, after which the group meet to discuss the video or observation. The lesson is then developed on the basis of the students’ responses to it and re-taught to a different group of students. This may then be repeated or a different lesson may be developed. Generally speaking, lesson study groups involve up to six teachers with a minimum of three, but this may vary considerably.

This constitutes the basic pattern of the approach but there are many variations on it both in Japan and in other contexts, which engage with the principles underpinning lesson study to a greater or lesser extent. Engaging with these principles is not straightforward or easy, and learning and change tends to be slow. As James Stigler says in his foreword to Fernandez and Yoshida’s book:

A superficial implementation of lesson study is not likely to have any positive impact on the learning of teachers and students and, given our impatient political climate, a lack of immediate results may result in lesson study being declared a failure before it is even understood in any deep sense.

## **METHODOLOGY AND METHODS**

This study draws on our research into three lesson study initiatives from the RECME project (Joubert, et al., 2009) and two from CIMT. We were interested in the influence of the lesson study initiative on the teachers and their commitment to it. This suggested the need for rich and deep data, and hence a qualitative methodology using interviews and observations. In all five cases we observed CPD meetings and observed lessons taught by the teachers involved. We also interviewed some of the teachers involved. The data were collected in the form of field notes, video and audio recordings. To address the question of the influence of the lesson study initiative on the teachers, data was analysed within a framework of teacher learning in mathematical knowledge for teaching, changes in practice and improved student learning. For the question of teachers’ commitment to the ideas of lesson study, data have been analysed using an approach based on grounded theory (Sinclair & Coulthard, 1975). In many cases the data have been fed back to the participating teachers and their comments used to co-construct accounts of the professional development and its consequences for them professionally. Evidence for professional change is accepted as reported by the teachers involved although some triangulation was possible through observations of the study lessons and other lessons taught by the teachers involved.

## FINDINGS

This section reports first on the influence of the lesson study initiative on teachers' professional development (learning, changes in practice and improved student learning). It goes on to report on teachers' commitment to the principles of lesson study.

### Teacher learning

**Knowledge of mathematics:** In lesson study, focusing on the details of one lesson and the approach to teaching a specific topic tends to lead to the group addressing mathematical ideas in great detail. In fact the topic for the lesson is often chosen because it is problematic to teach or the teachers find it hard to teach. One example from our studies involved a group of primary school teachers. One of the teachers said:

T 1: I think also having the time to research particularly something in maths was good because time and time again I'm finding I need to ask whether my subject knowledge is really that hot on that area of maths. So having read all the research and bringing it all together you get that clearer vision, don't you?

She went on to say that being a participant in the lesson study group:

T 1: raises your awareness about the importance of mathematical subject knowledge and relevant terminology and how much you should be using and what you should be using and why – it means that instead of making some glib comment about something that you might think is mathematically correct you would take the time to make sure

The leader of this group expressed the view that the lesson study helped to develop a different approach to subject knowledge from that which he had held previously:

T 2: it's sort of your mentality and your attitude to subject knowledge. It's not that you need to fill your head with vast amounts of knowledge for every subject, it's that your attitude to it is slightly different, as you work with the children in it. It's thinking well maybe we need to look into that a bit more and not seeing it as a bank of stuff that you have to get into children's heads. I think that's a huge shift.

From these comments it seems that the development of teachers' subject knowledge occurs through the consideration of the mathematics that they intend to teach. We observed this group of teachers talking about their developing understanding of the concepts involved in fractions as well as the different processes which they hoped to tackle with their students. They talked about the distinction between continuous and discrete quantities and offered examples of their students' in a task involving finding fractions of quantities. It seems that this understanding was new to them and was gleaned from their reading of professional and research literature such as (Halletta, Nunes, & Bryant, 2010; Thompson, 2005; Williams & Shuard, 1970). The detailed attention paid to the learning opportunities to be offered to the students seemed to force these teachers to pay close attention to the ideas involved and we suggest that

this led to the development of their subject knowledge in a way that was strongly practical and related to their pedagogy.

**Knowledge of ways of teaching mathematics:** As teachers consider the topic of the lesson that is their focus, they consider different approaches to teaching the topic and resources that they might use to support their teaching. In one secondary school lesson study, previous approaches to teaching had been dominated by textbooks. For the study lesson they developed resources based on approaches they had discovered through researching their topic of factorising quadratic expressions. These approaches were based on principles of ‘active learning’ originating from a professional development course developed for college tutors (Swan, 2005) which is widely available to schools in England. The task that they chose to adapt for their lesson involved sorting and matching different quadratic expressions with their factors.

One primary school group focused on finding fractions of quantities and they intended to give the children different experiences of finding fractions of quantities that were both discrete and continuous. They had long discussions about how they should do this and were concerned that the tasks should be creative and realistic. They wanted to offer the children open learning settings and they discussed the nature of the learning opportunities involved in the study lesson in detail:

T 2: I think the issue of open versus closed is quite interesting as well because actually that was quite a closed activity. There was an outcome, they had to do the picnic. It was more or less set up for them. It was quite actually, quite closed. That issue of deciding you know how closed or open activities should be and deciding what kind of an activity it is because I think sometimes as teachers we think they should be exploring but if we are clear about what kind of activity it is then it shouldn't be a problem

**Knowledge of ways in which students make sense of mathematics:** The detailed focus on the specifics of the teaching of one topic tends to lead the teachers into a lot of research into the topic and associated misconceptions and problems with teaching it. In one primary group, the articles found by their leader were used by them to inform the decisions they made about the content of the lesson. This led them to focus on the need to make sure the vocabulary they and the children were using was correct:

T 3: I think what we were saying earlier about using maths vocabulary and making sure that children are at ease with that and they use it confidently. I think I would put more of an emphasis on that from having had been involved in this process I've seen how important that is.

Another teacher in the group was struck by the ways in which the lesson they developed was applied to a practical problem and made use of a range of different units of measure:



## Working Group 17

T 1: For me making sure that we had a range of ... um, that we looked at grams, that we looked at millilitres and we looked at centimetres and so we were trying to draw lots of areas of maths together. ...

T 2: It sort of fits in with the research that showed that they need lots of different representations and practical experiences of concepts and the language has to come out of the experiences rather than just being told.

The focus on considering how students make sense of mathematics is captured in the following comment from one of the teachers:

T 2: There's a level when I'm reading about maths, there's a level at which I just don't understand it and there is a level where I think is it just me that I can't understand it because I should be able to understand it only I can't. Now children are getting to that level at almost a very basic level with even counting – they're the same level as me when I read a book that has complicated formulae umm so it's sort of empathising with anybody who doesn't know anything at whatever level – children are in the same position as us trying to teach fractions.

This shows the teacher making connections with his own understandings of mathematics and the things he struggles with and seeing how his feelings of being out of his depth might relate to those of the children he is teaching. This empathy and understanding illustrates how the lesson study process has deepened his understanding of how his students might make sense of mathematics.

A teacher from one secondary group found that the process of acting as the camera man and making the video of the study lesson enabled him to focus on the ways in which the students were making sense of the mathematics. He felt this was subtly different from other observations that he had done as the leader of the mathematics department in this school. We got a sense that he had an understanding of the potential of the students to engage in mathematical activity, as evidenced, for example, by the following comment:

T 4: I think what the lesson study work has made me realise is that it is quite deep and impacting on a very subtle basis. Obviously it is interesting filming a lesson that you know you can be taking. I think with the emphasis of the lesson study being on the girls and filming the girls - it felt very different from being the teacher going round

### **Changes in classroom practice**

Many teachers claimed that their general practice has been transformed by their involvement in the practice of lesson study. As one of the primary teachers said of the impact of lesson study on her teaching:

T 3: I think it definitely makes you look at other aspects of maths not just fractions and it has made me think every time I've planned lessons since like 'how could I do this differently this time?' And I look into different ways of exploring a lesson that I've done before and again maybe looking in to research based on that concept.

This teacher's comments indicate that she sees research related to teaching specific mathematics topics as a possible source of information about how she should approach teaching them. The group of primary school teachers to whom this teacher belongs spoke about all of the aspects of professional development that have been identified as important (Joubert, et al., 2009): knowledge of mathematics, knowledge of ways of teaching it and knowledge of the ways in which students make sense of mathematics but it was their students' learning to which they paid the most attention. For them it was this focus on the learning of their students that was the motivating factor as they continued their work in their group and constituted a change in their focus away from their teaching and onto the learning opportunities they offered their students. We would suggest that this constitutes a significant change in their approaches to their practice.

Similar evidence was found in the data gathered from one of the secondary groups. In talking about the impact of involvement in the lesson study process on his teaching more generally, one of the teachers commented:

T 4: I feel more confident in the sense, not of my knowledge base, but more confident in front of the class, putting across more excitement and enthusiasm... I think it has chivvied me even further along the road of getting the girls involved more, of taking a slightly more background role.

This also illustrates the focus of the lesson study on the learners and shows a teacher who is willing to pass over more control of the learning to his students.

### **Student learning**

The observations of the study lesson by the whole group of teachers have the effect of focusing the observers' attention on the students' responses to the learning opportunities that the lesson offers. This is a development of the process of engaging with the ways the students make sense of the mathematics. The central thrust of a study lesson is often to make the students' learning visible to the teacher and observers and this is accomplished through choosing tasks that facilitate this and encourage students to express their mathematical ideas both orally and on paper. Once this has been planned the observations can focus on noting the students' learning. The teachers in the primary group were struck by how observing the learning in this way altered their perceptions of the lesson:

T 1: The thing with observing which is really helpful is that you kind of put yourself in the position of the child whereas when you're teaching – yes, you're thinking about their learning but you've also got to think about what you're doing and how you're sort of delivering it. With observing you listen to the teacher as if you were that child and so you really see how they are learning.

T 2: So it's enabled you to draw out what's happened that you might not have seen as a teacher. Knowing that children might be having really rich discussions even though you

## Working Group 17

haven't heard them as a teacher is really useful to know. It gives you the confidence to think that you don't need to direct everything.

The second comment draws attention to the additional information that the team of observers were able to gather from watching the lesson: they had been able to comment on all the talk of all the different working groups within the class. This would not be possible with one teacher in the classroom and it had shown them that the mathematics in the talk of all the groups was sound and that the discussion was about the task.

These comments from the teachers were supported by observations of the study lessons that they developed as part of their work. In these lessons the students were actively engaged in solving mathematical problems related to the aims of the lesson and their conversations illustrated their thinking about the mathematics involved rather than echoing a procedure that their teachers had demonstrated. In this the study lessons did begin to make the learning of the students more visible.

### **Engaging with lesson study**

We described above how lesson study is organised and emphasised that it is not always easy or straightforward. It involves a significant time commitment on the part of the teachers involved, which in turn implies that, without the support of their schools, they will not be able to engage fully with the process. Our sample includes three initiatives which demonstrate the difference this commitment seems to make.

In the first, the teachers came from three different secondary schools and their work was carried out in their own time after school. They had little support or recognition of their efforts from senior managers in any of their schools and it was difficult for them to arrange time within the school day to collaborate and teach together or observe each others' classes. They planned a lesson together, but then each taught a part of it and none of them was able to observe students' responses while a colleague taught. Although the teachers reported that the collaborative planning of the lesson had been of value, we suggest that their experience could have been more valuable if they had had the opportunity to observe the students more closely. The point is that they did not seem to understand that lesson study involves more than collaborative planning.

In a second example, a group of teachers working under the strong leadership of the head of the mathematics department has been engaged in the process of lesson study over a period of several academic years. The head of department was well informed about the Japanese lesson study model and he showed us a shelf of books about lesson study he referred to in discussions with us and the members of his department. He said that he encouraged the group to follow the model very closely and this was apparent in the Open House which the department put on showcasing two study lessons and their analysis for a large audience. Each member of the department is part of a lesson study group and the focus of the work for the whole department is

linked with a theme that applies to all those involved. Senior management including the head teacher support the process of lesson study and the teachers are allowed time to engage with the process and to observe one of the study lessons. Our observation of one of their lesson study meetings suggests that the teachers are used to the idea of attending to all three aspects of mathematical knowledge for teaching and that they were fluent in talking about expected and actual student responses to the study lessons. It seems that the time these teachers put into the lesson study initiative, and the commitment they have to the ideas behind lesson study, have resulted in significant professional learning.

In terms of changes in classroom practice, it seems that there has been significant and sustained change amongst the majority of members of the department. The head of department reported that most teachers took a more 'open' approach in their teaching and encouraged their students to discuss their mathematical thinking more. One of the teachers told us that much about the way she teaches has been influenced by the lesson study. She said that she has learnt to let go of control and to let all the students have a voice.

T1: I now let the students come up to the board and make contributions. I am much more aware of what I do in the classroom.

She reported that she is also more of the possible approaches students might take, and she takes these into account when she is planning. Our observations of one of her lessons confirms that she encourages her students to discuss mathematical ideas and it seems that she provides an environment in which they feel able to put forward their ideas.

It is not surprising, given the institutional support for the initiative, that the second of the examples here appears to have been more successful than the first in terms of teacher learning and change (it would be worrying if it had not been). What is perhaps more interesting, however, is the teacher commitment to the concepts underpinning lesson study in the second example which emphasise paying attention to all three aspects of mathematical knowledge for teaching as well as planning a lesson collaboratively.

A third example demonstrates an intermediate level of engagement. In this case Teacher 2 (T 2) was the head teacher in the school and leader of the group. He was an avid reader of research into education and particularly about children's learning. He had an egalitarian approach to the leadership and management of his school describing himself as the 'lead learner' in all documentation rather than the head teacher. He had identified lesson study as an approach to professional development amongst his staff that respected their mutual need to develop practice that supported both their individual professional learning journeys and the learning of their students. He gave the rest of the team opportunities to voice their concerns and ideas and appeared not to dominate the group. He saw his role as a facilitator of the

process in that he was able to organise time for meetings and observations but he also expected and got from the participants considerable investment of their free time in the process and most meetings were held after school. This institutional support for the process of involvement in lesson study contributed to the engagement that the participants were able to make and sustain.

## CONCLUSIONS

This paper explored teachers' reported learning and changes in practice during and after their engagement in lesson study initiatives. It provided evidence of their learning in terms of three aspects of mathematical knowledge for teaching. It seems that there was learning in all three areas, and we suggest that lesson study is particularly effective in encouraging learning in all three aspects. The paper discussed changes in classroom practice and again it seems that changes in classroom practice were evident and can be seen as effective because they were in line with the changes promoted by the initiatives of professional development. There was evidence of improved student learning and this suggests that the ultimate goal of the professional development was achieved.

However, it seems that not all teachers had equally positive experiences. We examined the ways in which the lesson study initiatives were approached, and suggested that if the teachers' schools are committed to supporting the teachers in taking part in lesson study and if teachers are committed to the ideas underpinning lesson study, then it is likely that the teachers' learning and changes in classroom practice will be deeper and more sustained.

Lesson study focuses on teachers identifying for themselves key issues and objectives that relate to their own context and which are important to them. As a group of colleagues they have an understanding of each other's practice and can move together, in directions that they feel inclined to try, to deal with their own problems. As one of the primary teachers observed:

T 1: it's that traditional CPD you know that we go to and have some sort of, whether its in house or out of house, and we sit and listen and we have things thrown at us and we have initiative this and initiative that. And this is a much more productive form of CPD, it's much more real and it will have a greater impact on learning and teaching.

This conveys a sense that lesson study for this teacher is a real process grounded in her practice as a teacher, something that she has control over and not something that is 'done to her' by some expert outsider with questionable understanding of her working context.

Lesson study is a subtle, complex and difficult process as these accounts suggest. However for those who engage with it at a deep level and develop an understanding of the process, it offers strong support for professional change and learning in all the key areas. Our data gathered from a number of different groups of teachers involved

in lesson study indicates that changes in practice which the teachers claim to be sustained and significant only occur once they have grasped the key focus of lesson study on students' responses. This focus shifts their reflections on their teaching to a consideration of the learning opportunities they are offering the students and the students' responses to those learning opportunities. It pushes them to create study lessons that make the learning of the students visible to the observers. This is a subtle and difficult concept that necessitates engagement at a deep level with the process of lesson study. We suggest that it is only through sustained involvement that the teachers will develop a deeper understanding of these processes involved in this complex approach to professional development and of the way in which it has the potential to support and sustain their professional development. This deep involvement requires the participants to understand the 'spirit' of lesson study rather than limiting their involvement to the 'letter', or the superficial form, of this way of working. Lesson study involves more than groups of teachers planning lessons together and requires consideration of the mathematics that is the focus of the lesson, the potential ways of teaching it and the ways in which the students are likely to respond to those ways of teaching. Most importantly it involves paying attention to the learning of the students and for teachers to do this they need to create lessons in which that student learning can be observed. As Stigler says: 'The devil (and God!) are in the detail' (p.x) (Fernandez & Yoshida, 2004).

## REFERENCES

- Burghes, D., & Robinson, D. (2010). *Lesson Study: enhancing mathematics teaching and learning*. Reading: CfBT.
- Fernandez, C., & Yoshida, M. (2004). *Lesson Study: A Japanese approach to improving mathematics teaching and learning*. Mahwah, New Jersey: Lawrence Erlbaum Associates.
- Halletta, D., Nunes, T., & Bryant, P. (2010). Individual Differences in Conceptual and Procedural Knowledge when Learning Fractions. *Journal of Educational Psychology*, 102(2), 395 - 406.
- Joubert, M., Back, J., DeGeest, E., Hirst, C., & Sutherland, R. (2009). *Final Report: Researching Effective CPD in Mathematics Education (RECME)*. London: National Centre for Excellence in the Teaching of Mathematics.
- Joubert, M., & Sutherland, R. (2008). *A perspective on the literature: CPD for teachers of mathematics*. Sheffield: National Centre for Excellence in the Teaching of Mathematics.
- Lewis, C. (1995). *Educating Hearts and Minds: Reflections on Japanese pre-school and elementary education*. Cambridge: Cambridge University Press.
- Sinclair, J. M. H., & Coulthard, M. (1975). *Towards an Analysis of Discourse: The English used by Teachers and Pupils*. Oxford: Oxford University Press.
- Swan, M. (2005). *Improving Learning in Mathematics: Challenges and strategies*. London: DfES, Standards Unit
- Thompson, I. (Ed.). (2005). *Issues in Teaching Numeracy in Primary Schools*. Milton Keynes: Open University Press.
- Williams, E., & Shuard, H. (1970). *Primary Mathematics Today: towards the 21st Century*. Harlow, Essex: Longmans.

# AN ATTEMPT AT DEFINING TEACHERS' MATHEMATICS THROUGH RESEARCH ON MATHEMATICS AT WORK

Nadine Bednarz & Jérôme Proulx

Université du Québec à Montréal, Canada

*Through recognizing mathematics teachers as professionals who use mathematics in their workplace, this paper traces a parallel between teachers' professional mathematical and the characteristics of the mathematics used in workplaces found in studies of professionals. This parallel is developed through the five characteristics highlighted in Noss (2002). We use this parallel to help develop further on our conceptualisation about teachers' use of mathematical knowledge in their practice.*

There is general recognition in our community of the professional finality of teacher education. One main aim of pre-service teacher education is to develop teachers' capacity to act as professionals. This finality has consequences for all the components of teacher education, in particular the mathematical dimension of their training. It therefore leads to the question of the relevant mathematics needed for the professional practice of teaching mathematics in schools. Our research interest starts from this preoccupation of teachers' professional mathematics (Proulx & Bednarz, 2009), as well as a constant questioning of the presence of a gap between the mathematical experiences lived by teachers through their teacher education and the mathematical practices teacher enact in their classroom (see our review in Proulx & Bednarz, 2008). This questioning led us progressively to the necessity of better understanding the specificity of these mathematical knowings of teachers, so as to be able to think of and develop teacher education approaches better aligned with this professional mathematics in order to prepare them for the demands of their work.

It seemed natural, in that perspective, to be interested in the research studies conducted on the mathematics used in practice by different professional groups; what is often termed "mathematics at work." If teachers are seen as professionals, thus the characteristics of the mathematics used in their practice could be informed by this important corpus of research (see e.g. Noss & Hoyles, 1996; Pozzi, Noss & Hoyles, 1998; Noss, 2002; Noss, Bakker, Hoyles & Kent, 2007). As we began reading this body of literature, we were stunned by the parallel we could trace between their results and analysis and our own data and theorizations on teachers' mathematical practices. We perceived this as an opportunity to continue advancing on our conceptualization of the nature of knowing and using mathematics in teaching (see e.g. Bednarz & Proulx, 2009). This paper is thus centered on exploring this parallel, in an attempt to define better teachers' professional mathematical practices through Noss et al. studies on "mathematics at work." In particular, we draw on Noss's (2002) conceptualization, developed as a result of studies conducted among professional groups in different settings (e.g., engineers, bankers, nurses). We thus document, *for each of the 5 characteristics of this conceptualization*, the parallel we

see with the mathematics used by teachers in practice – supported by illustrations from our data.

## **BRIEF METHODOLOGICAL CONSIDERATIONS**

This parallel is documented through combining two sources of data. The first source comes from collaborative research studies in which teachers and researchers meet to elaborate teaching situations. These studies enable us to describe and analyze the knowledge teachers deploy when designing classroom situations. The second source comes from an in-service research project, in which 3 groups of 8-10 teachers participate, centered on the exploration of mathematics situations *related to teachers' everyday teaching practice*. The exploration of these tasks help gain access to the nature of the mathematical knowledge they mobilise in the action of teaching.

## **DEVELOPING THE PARALLEL THROUGH NOSS' 5 CHARACTERISTICS**

A preliminary remark is necessary before we engage in this parallel. Teachers are a particular professional group, whose profession is to teach mathematics to students. They are in that sense different from other professional groups such as bankers, engineers or nurses whose profession is not directly associated to mathematics. Mathematics teachers are, then, *doubly* associated to mathematics: (1) they teach *mathematics* to students and (2) they enact a certain sort of mathematics in their professional practice (their profession is to teach mathematics and they have developed professional mathematical knowings to do so). Thus, teachers' mathematics at work (our object of analysis) must not be confused and restricted to the mathematics they teach (a certain content linked to the curriculum). The content of school mathematics is seen to act as a structuring resource, in Jean Lave's sense, of this (mathematical) practice of teachers – e.g., in the same way that a drug acts as a structuring resource for the nurses' mathematical practices (the knowledge of the drug structures the mathematical reasoning in the activity of drug's administration to patients) (e.g., Pozzi et al., 1998). But, even if the content teachers teach acts as a structuring resource, the mathematical knowledge enacted within this practice, as we will see in this paper, is considerably more sophisticated and different than the one encountered by their students (the mathematics linked to the curriculum).

The context of teachers' practice is a complex one, both mathematically and socially. As the employees of the bank in Noss et al. study (1996) are concerned with making profits for the bank and benefiting for themselves, rather than learning mathematics for its own sake, schoolteachers are concerned with students' learning. They are concerned with their students' understandings of the mathematics they teach and with their students' success in this learning; teachers are not (necessarily) interested in their own personal learning of mathematics in itself. This finality gives a particular color to the mathematics at work in their practice, as we will see through the documentation of the five characteristics conceptualized in Noss (2002).



**Characteristic 1. A fragmentation of the knowledge structure of the workplace**

Noss et al. (1996), in an attempt to understand the banker's ways of thinking about quantitative data, put in light the fragmented knowledge structure that characterizes the practice of investment banking: "We encountered departments specializing in the finest detail on one financial instrument, sharing a common wall but no common language with another – essentially similar – department" (Noss, 2002, p. 51). This fragmentation shaped the way bankers saw graphs – conceived as a display of numbers and data in relation to work activities (and not, e.g., as a functional relationship or an indication of a trend about data that permits prediction). This fragmentation of the knowledge structure of the workplace characterizes also in a certain way the one of teachers, e.g. from one grade-level to another or from one order to another (primary to secondary, secondary to post-secondary). Even if practitioners do share a common language for speaking about their practice and referring to it, the knowledge structure of the workplace (by level, by order) shapes their mathematics in use, as we see in the following excerpt.

This excerpt comes from our professional development project (with secondary teachers) and concerns an important part of mathematics-in-use in teaching, that of symbolic notations used in mathematics teaching. After having explored a number of tasks around fractions, a discussion is initiated about an assessment task recently given to grade-8 students where the expression  $\frac{(2x+1)}{4}$  played a role. In this discussion, teachers elaborate on their expectations for their students' answers.

Mary (grade-8): I would like to see the division symbol ( $\div$ ) because I work around the division of polynomial expressions by constants. Hence, I expect my students to do the following, using the division symbol:  

$$\frac{(2x+1)}{4} = (2x+1) \div 4 = (2x \div 4) + (1 \div 4)$$
 . [She will later say: "I don't want to see  $x \div 2$  in a result, but  $x/2$ ." And, as a result of her interaction with other teachers who questioned her on this idea, she will reply: "well, except if I have  $\frac{x}{2} \div \frac{x}{3}$  because for students there is the possibility of a confusion with the place where the division stands (e.g.,  $\frac{\frac{2}{3}}{\frac{1}{4}}$  is  $\frac{2}{3} \div \frac{1}{4}$  and not  $2 \div 3 \div 1 \div 4$ )"].

Clara (grade-8 and 10): I would like my students to write  $\frac{1}{2}x + \frac{1}{4}$ , because I want to prepare my students for the usual notation used for linear functions.

Cathy (grade-7): I would also in that case use the division symbol (even if I don't teach algebra), because my objective is that my students work with operations on natural numbers and their properties like the distributive law:  $(2 \times 51 + 1) \div 4$ .

Sandra (grade-9): I prefer they continue simplifying this expression from  $\frac{2x}{4} + \frac{1}{4}$  to  $\frac{x}{2} + \frac{1}{4}$  because I want to make visible the rate of change in this expression.

Jerry (grade-10 and 11): I write it  $2(x + \frac{1}{2})$  to draw out parameters and the transformations associated to them [Robert (grade-10 and 11) agrees].

This discussion highlights the subtle ideas engaged by teachers concerning symbolic notations, rooted in the demands of their work activity: e.g., for preventing possible confusion with notation in students; for expressing the results in line with the finality of their work in their classroom (note that both grade-8 teachers did not share the same expectations); for preparing students for the upcoming years and topics. We can see how teachers' mathematical knowledge in use, here about algebraic notations, is finely tuned to their professional activity and its pragmatic demands: it is shaped by the fragmentation of the knowledge structure of the school. A certain rationality guides their choices, putting in evidence different meanings underlying this notation and a diversity of interpretations related to the level where they teach: division of algebraic expressions (the fraction notation has to be seen as a quotient, a division); general expression associated to linear function; seeing the variation rate through the notation; seeing the parameters through the notation and transformations associated.

### **Characteristic 2. The role of artefacts and tools in mathematics at work**

In their ethnographic studies on nurses, Noss et al. show the central role of artefacts in the workplace settings. Artefacts such as notational systems, physical tools, work protocols, etc., act as structuring resources in an ongoing dialectic of producing and being produced by activity in a certain social practice.

For example, in one study on a hospital ward, we found that a seemingly straightforward artefact like a fluid balance chart, contained within it the crystallized activity of the hospital community, shaping in complex – but unnoticed – ways the actions and discourse of those using it. (Noss, 2002, p. 52)

Our data also show the importance of artefacts in the work of teachers. For example, symbolic notations, algorithms and modes of representation to which teachers often refer in their teaching illustrate how these artefacts act as structuring resources in their practice. If, for nurses the fluid balance chart is embedded in the routine of a work protocol for taking care of patients, for teachers a symbolic notation is embedded in their routine of teaching mathematics to students and giving meaning to concepts. These artefacts are so embedded in professional routines that in most cases their underlying structure is hidden: nurses take for granted the explanatory power of the chart, precisely because it is part of their routines; teachers often take for granted the explanatory power of representations, notations or algorithms in their teaching, precisely because it is part of their routine. To understand the complex, but often less visible parts of decision-making related to these artefacts, Pozzi et al. (1998) have noticed that those become more visible through situations involving what they call “breakdowns” in the normal habits of practice, where these oft-hidden meanings are questioned by other practitioners that use or perceive the role of these artefacts in different ways. In that perspective, breakdown situations have a central role to help explicit the mathematics in use within these artefacts, where the models underpinning the artefacts can rise to the surface and become open for observation

by researchers; Pozzi et al. even suggest to provoke situations of breakdown to attain these goals. We can see the potential of such breakdown situations in the next excerpt concerning algebraic notations, where the interactions between teachers around routine practices at each grade level rises to the surface. The notations act here as what Pozzi et al. call a “disputed territory,” which serves both as a breakdown in routines for each teacher and as an opportunity for us as researchers to understand the algebraic notation used.

The symbolic notation is the focus of teachers’ discourse about their practice in this excerpt coming from the professional development research project with secondary teachers. The discussion emerged from an exploration of different students’ solutions to a problem of trigonometry (drawing a point on the trigonometric circle, finding which point is nearer to  $\frac{\pi}{6}$ ). The link that the grade-8 teacher wanted to do with her teaching to prepare the future work of students on trigonometry illustrates the way she works with the circle. She explains that she writes “ $2 \cdot r \cdot \pi$ ” during the process of elaboration of the formula of the circumference of the circle, to help student see the origin of the formula. However at the end of the process, only “ $2 \cdot \pi \cdot r$ ” will be acceptable (a certain written notation corresponding to the convention accepted). She will do the same for  $1 \cdot x$ , acceptable during the process of mathematization of an algebraic problem, but not at the end (the answer has to be written as  $x$ ). In the work on the trigonometric circle, the grade-11 teacher will accept a notation like “ $2n \cdot \pi$ ” to show the repetition on the circle of  $\pi$  “ $n$  times” and to help visualize the point associated. But the other teacher (grade-10) reformulates it in “ $2 \pi n$ ” to put the constant before (like the accepted convention). This teacher will also point to the non coherence in the way it is written and explains that the algebraic expressions “ $2n \cdot \pi$ ” means, normally, “ $2n^\pi$ ” to be coherent with the way we usually say and write those expressions (like with “ $pi-r-two = \pi^2$ ” or “ $x-two = x^2$ ”).

This discussion shows the invisible relationships buried in these artefacts (and often taken for granted) that a breakdown from routine practices contributed to explicit (in this case the need for teachers to understand the notation in use by other teachers). Teachers had to make sense of what they used, and why, through these discussions. We can also see how these artefacts are shaped by the professional context of the teacher (e.g., teaching at a certain level, making sense of a formula, visualizing the positioning of points on the trigonometric circle) within a broader social system (e.g., answering to certain mathematical school conventions, taking account of the coherence), as well as this practice shapes in return this way of working out these “conventions” by offering new ways of making sense of them and using them.

### **Characteristic 3. The anchoring of mathematical meanings in practice**

The different ethnographic studies conducted by Noss and his team point to the situated nature of the mathematical knowledge of professionals, rooted in the professional activity and context of work. For example, in the case of nurses, their findings show that the different strategies used in relation to drug’s administration would, in the literature on proportional reasoning, be described as lacking meaning.

## Working Group 17

However, a fine-grained analysis shows this not to be the case, as the various strategies “was anchored in an intimate knowledge of the drug itself, as well as in the properties of familiar packaging constraints of prescribed doses. The knowledge was mutually constituted and expressed both mathematical relation and culturally-shared situational noise” (Noss, 2002, p. 54). The “situational noise” of a situation is an integral part of professional knowledge at work. In a previous paper (Bednarz & Proulx, 2009), we have brought forth similar findings for mathematics teachers illustrating the anchoring of mathematical meanings in practice produced by the coordination of multiple dimensions (mathematical, didactical, pedagogical, institutional). The “noise of the teaching situations” (e.g. engaging students in the given problem, showing the value of various solutions and not only of the answer, showing how students find it, reacting to some solutions, etc.) appeared to be a key element of teacher’s mathematical knowledge. As a way of illustrating this situated nature of mathematical knowledge rooted in context, we use the following vignettes, taken from a collaborative research study centered on the elaboration of teaching situations (Saboya, 2010). We join Nadia and her students in the following two vignettes as they work with different expressions.

<b>Vignette 1: expression</b> $\frac{10^5 + 10^8 + 10}{10^2}$	<b>Vignette 2: expression</b> $\frac{10^4 + 10^5}{10^2 + 10^3}$
<p><b>Laura:</b> I am not sure but what I would do ... it is like 10 to the 5, so 5 minus 2 then 8 minus 2 then 10 ... 10 to the 1 minus 2 then after that it would ... then I can't put them together because these are additions, because the law on exponents does not work, so it will be 10 to the 3 plus 10 to the 6 plus 10 to the minus 1.</p> <p><b>Marc:</b> Is it possible to write <math>10^{13}</math>?</p> <p><b>Students:</b> No because it is a “plus.”</p> <p><b>Nadia:</b> You see there is a plus, and so we say “there is a ‘plus’ which ruins everything.” This ‘plus’ stops me from putting everything together.</p> <p><b>Brad:</b> Instead of 1/10 could we write “-10”?</p> <p><b>Mary:</b> Does it mean that it is equal to <math>\frac{10^5}{10^2} + \frac{10^8}{10^2} + \frac{10}{10^2}</math>?</p> <p><b>Nadia:</b> When I have something like that (she shows the second part of the expression, under the line), when I have a sum like that it is as if I have split the fraction in three. So, it is the same principle: <math>\frac{10^5 + 10^8 + 10}{10^2} = \frac{10^5}{10^2} + \frac{10^8}{10^2} + \frac{10}{10^2}</math></p> <p><b>Brad:</b> Instead of 1/10 could we write “-10”?</p> <p><b>Nadia:</b> See, some say it's -10 or minus something. This is why I don't want negative exponents because you make errors and it doesn't work.</p>	<p><b>Nadia:</b> <math>\frac{10^4 + 10^5}{10^2 + 10^3}</math>, we cannot do anything with this, we leave it like this.</p> <p><b>Joe:</b> Why? Can't we separate those?</p> <p><b>Nadia:</b> No we can't separate this.</p> <p><b>Joe:</b> Why not?</p> <p><b>Nadia:</b> if you separate it, it gives <math>\frac{10^4}{10^2} + \frac{10^5}{10^3}</math>. It would mean that you add fractions. And do we have the right to add numerators and denominators?</p> <p><b>Joe:</b> No.</p> <p><b>Nadia:</b> Ok, no, but when you do as you say, this is what you do.</p> <p><b>Julie:</b> Couldn't we just do <math>\frac{10^4}{10^2}</math>?</p> <p><b>Nadia:</b> No. What do you do when you separate them like this <math>\frac{10^4}{10^2} + \frac{10^5}{10^3}</math>? Is that what you've done?</p> <p><b>Julie:</b> Yes.</p> <p><b>Nadia:</b> This does not work because you are adding two fractions and you add numerators together and denominators together. You cannot do that.</p>

As for nurses, where strategies used to calculate drug administration to patients were anchored in an intimate knowledge of the drug itself and varied in relation to the type of drug and the packaging constraints (Pozzi et al., 1998), the previous extracts show similar results in relation to teachers' context. Strategies used in action by Nadia to manipulate algebraic expressions with exponents and simplify them is anchored in an

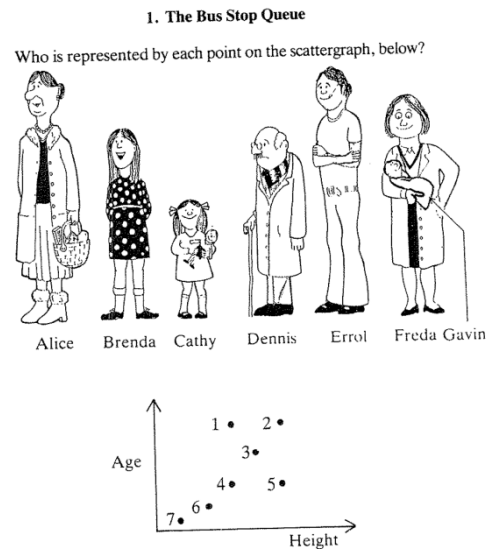
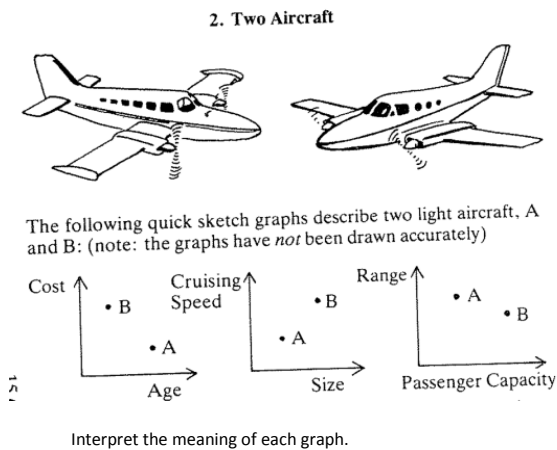
intimate knowledge of the expression itself and of different types of expressions in relation to students' difficulties (e.g., expressions with additions or not, additions at the denominator, negative exponents). This knowledge is interwoven with her anticipation of her students' difficulties, her sensibility to their errors in such expressions, her intention to prevent these errors, etc. The fact that she does not pursue the simplification in the second case ( ) – even if it is possible – is related to this professional mathematical knowledge intertwined in the context of practice (and unfolded on the spot). The way she mathematically works with the expressions is related to her teaching intentions (prevent errors, working on exponents, using exponents laws, seeing the parallel with fractions), and those shape the way she acts, so that the factorization of a common factor at the numerator and denominator will not be used by students, or that she will not encourage it.

#### **Characteristic 4. Qualitative restructuring of mathematical knowledge**

In their studies on nurses and on engineers, Noss and colleagues highlight that a transformation of mathematical knowledge happens within the professional activity at work. For example, a certain meaning of average and variation emerged in nurses' thinking on the ward, in the manipulation and interpretation of quantitative data and relationships related to patients. For a nurse, it is the variation that is crucial, and not the notion of average of a population; the average is considered if it is individually mediated (it is the individual who is the focus of her care). This restructuring and transformation of knowledge in practice is also true for engineers, where they have develop alternative ways of making sense of concepts (e.g., load path); in manners that differ from the "usual" or "formal" way of approaching this concept in traditional mathematics/physics. This transformation of mathematical knowledge *in situ* provides to the engineers a way of thinking more relevant to allow judgments about the validity of the quantitative analysis of the structure.

The following excerpt, taken from our professional development project with secondary teachers, highlights similar findings. It concerns an exploration of different graphs and the anticipation of what their students could answer to them, as well as the relevance of giving these sorts of tasks. These tasks were given to teachers (taken from Shell Centre for Mathematical Education, University of Nottingham, 1985).

## Working Group 17



The debate on the value of this type of graphical representation is provoked by Mary (grade-8 teacher): “What is the relevance of this type of graph if there is no relation between the magnitudes considered on the two axes? It goes against what we try to develop in students, that is, the idea of independent variable, dependant variable and a relation between the two.” Then, Mary appeared ambivalent about the relevance of using this type of graph: “On one hand, I find these graphs interesting because they force a lecture of the axes by the students, which is one of the difficulties they have and that I observed when they work with graphs. They simply do not read the axes. But, on the other hand, I am not very comfortable with the idea of presenting a graph where there is no relation between the two magnitudes involved on the axes”. This argumentation will be supported by other teachers, like Clara (grade-8 and 10 teacher): “What is the relevance of it if there is no relation between variables? A graph is interesting for seeing if there is a correlation between two variables (e.g., in statistics, to study the behavior of a series of data and its tendency, its correlation) or if it corresponds to a functional relationship. If there is not one and only one value that corresponds to another specific value of the independent variable, we do not have a function, so it is not interesting, since we can’t speak of inverse or reciprocal functions, for example.” Through the discussion, it became clear that all teachers agreed with the fact that in these cases there was no real relevance to use a graph; they raised, e.g., the idea of placing the information available in a table.

This interaction clearly shows a conceptualization of graphs transformed by the professional activity of the teacher, as it was the case for the bankers in the Noss and Hoyle’s (1996) study. For bankers, the graphs were seen as a display of data, as a picture of numbers and as a quick and easy way to display data, and *not* as a medium for expressing relationships (between a certain quantity and time, e.g.). They regarded the shape of the graph as determined by its representational characteristics (number of elements to be displayed, availability of color, target of audience, frequency of variables, scale of axes, etc.) and not in terms of the underlying relationships between those data. On the opposite, for these teachers, the graph was

not seen or accepted as being a simple display of data or of numbers. The central role played by functions in the curriculum they teach influenced their conceptualization of graphs in a very specific way. Teachers regarded graphs as necessarily determining relations between two variables. These two contrasted examples of teachers and bankers concerning the usage/utility of graphs shows how knowledge is transformed within the professional practice.

### **Characteristic 5. The situativity of abstraction**

As we have seen, mathematical knowledge at work is situated. This situativity of knowledge doesn't mean that this knowledge in action is only local. A variety of the observations made on nurses in Noss et al.'s (2002) study reveal, through the different strategies used by nurses to calculate some drug administration to patients, a kind of invariant through a nurses' sense of the quantity of concentration, taking the form of a co-variation constant between the mass and the volume in the drug solution; this invariant is not linked to a particular patient, nor a specific drug's administration. Thus, nurses engage in abstraction, developing abstracted knowledge grounded in their practice. But this abstraction remains to some extent *situated* in that it retains crucial elements of the setting in the ways it is conceptualized. The authors use the expression "situated abstraction" to describe this kind of abstraction at work. It attempts to describe how a conceptualization of mathematical knowledge can be at the same time abstract *and* situated. Finely tuned to their situative constructive development, to their use in professional practice, mathematical invariants are abstracted within that community of practice. The notion of "situated abstraction" confronts the usual way of conceiving abstraction in the mathematics education community, as it is normally thought of as something extracted from a set of situations, essentially de-contextualized. The noise of the context is then perceived in this conception of abstraction as an obstacle to this abstraction: there is a necessity to shift away from the context and from the situation where it was extracted from. The notion of situated abstraction questions this conception, that is, it questions the fact that mathematical abstractions must be separated from the context of their construction or application. On the contrary, situated abstraction suggests that what can be seen as "noise" in the traditional view of abstraction is in fact central to it in a professional context. However, as Noss (2002) explains, this notion of situated abstraction needs to be additionally documented, as it is now more conceived as representing an hypothesis than a finding *per se*.

Our observations through the various sessions with the different groups of teachers show that some aspects of situated abstraction are taking form in teachers' sense of mathematical concepts. For example, an aspect we have explored in depth in Bednarz and Proulx (2010) is the notion of referent in regard to fractions (e.g.  $\frac{1}{2}$  of something is different from  $\frac{1}{2}$  of another thing). In this case, a conceptualization of fractions in relation to a certain referent emerges within the explorations in the sessions, interwoven with their professional practice concerning issues like

considering their students' possible answers, the possibility to consider or not the relativity of this referent in their teaching, managing this idea in the daily teaching with 27 children, etc.). This notion of referent was also afterwards re-invested in other sessions on other topics (e.g. decimals numbers, area, volume, division and problem solving). This invariant is thus highlighted in various conceptual situations for various concepts, showing that teachers have abstracted within this exploration a specific mathematical knowledge. This abstracted knowledge remains however situated, embedded within teaching and learning contexts for making sense of it: it carries issues of classroom context, of viability of bringing about the referent, of difficulties for students to understand, of their easiness for managing this mathematical openness of the answer, of deception toward their previous teaching or textbooks that did not insist on this powerful idea of referent, of the potential of this idea for students' reasoning and sense making, etc. Those are all "present" in the abstractions teachers develop about the specific notions worked on. However, in the same way that Noss (2002) stresses it, we still are in need of probing further on this issue of situated abstraction in order to get a better and more robust grasp of its meaning and the role it plays in teachers' professional mathematics.

## DISCUSSION AND CLOSING REMARKS

This parallel made with these characteristics of teachers' mathematics at work questions and even confronts the usual mathematical preparation of teachers in universities, particularly in regard to the situated nature of this mathematical knowledge and the situated nature of abstraction. Issues like the "noise" of the practice situation is, as we have seen, an essential part of this mathematical knowledge at work for teachers, as it shapes teachers' knowledge as well as the abstractions they draw and how they "restructure" their mathematical understandings in relation to these practices. It therefore calls for a very specific mathematical preparation, precisely in relation to artefacts and their often-hidden meanings in routines, to the de-fragmentation of the workplace, and to the central issue of situativity of professional knowledge – so as to better prepare teachers to act mathematically as professionals in their everyday practice. As Noss (2002) points, the lack of robust connection between both worlds of "mathematics at work" and "mathematical preparation at university" appears to be a central element to take into account for the mathematical preparation of professionals.

[...] the majority of structural engineers do not "use mathematics" of any sophistication in their professional careers [...]: Once you've left university you don't use the maths you learnt there, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers in this firm, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again" (p. 54).



In that sense, this conceptualization of teachers' mathematical knowledge helps to think about the elaboration of approaches for the professional development of teachers, ones better aligned with the "mathematics at work" of the teaching practice.

## References

- Bednarz, N., & Proulx, J. (2009). Knowing and using mathematics in teaching: Conceptual and epistemological clarifications. *For the Learning of Mathematics*, 29(3), 11-17.
- Bednarz, N., & Proulx, J. (2010). Processus de recherche-formation et développement professionnel des enseignants de mathématiques. *Éducation et formation*, e-293, 21-36.
- Noss, R. (2002). Mathematical epistemologies at work. *Proc. 26<sup>th</sup> Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 1, pp. 47-63). PME.
- Noss, R., Bakker, A., Hoyles, C., & Kent, P. (2007). Situated graphs as workplace knowledge. *Educational Studies in Mathematics*, 65, 367-384.
- Noss, R., & Hoyles, C. (1996). The visibility of meanings: modeling the mathematics of banking. *International Journal of Computers for Mathematical Learning*, 1(1), 3-31.
- Pozzi, S., Noss, R., & Hoyles, C. (1998). Tools in practice, mathematics in use. *Educational Studies in Mathematics*, 36, 105-122
- Proulx, J. & Bednarz, N. (2008). *The mathematical preparation of secondary mathematics schoolteachers*. Paper presented at ICME-11 in TSG-29. Monterey, Mexico.
- Proulx, J., & Bednarz, N. (2009). Resources used and "activated" by teachers when making sense of mathematical situations. *Proceedings of PME-33* (vol. 4, pp. 417-424). PME.
- Saboya, M. (2010). *Élaboration et analyse d'une intervention didactique co-construite entre chercheur et enseignant*. Ph.D dissertation. Université du Québec à Montréal, Canada.

# **ADOPTING AN INQUIRY APPROACH TO TEACHING PRACTICE: THE CASE OF A PRIMARY SCHOOL TEACHER**

Claire Vaugelade Berg

University of Agder, Norway

*The research reported in this paper is situated within a developmental research project called Teaching Better Mathematics where the collaboration between teachers and researchers is based on the idea of inquiry. The teaching practice of an experienced primary school teacher participating in the project is considered and elements of her practice are conceptualised using the theoretical construct of an inquiry cycle. The analysis shows that the teacher engaged with inquiry at different levels, into the mathematics and into her teaching practice. Possible implications for teacher education are suggested.*

## **INTRODUCTION**

The origin of inquiry (Dewey, 1905) can be traced back through the work of Polya in mathematics (Polya, 1945) and his ideas have been developed further during the 1970s and 1980s with an international movement in mathematics learning and teaching aimed at promoting problem-solving and conjecturing in mathematics classrooms (Mason, Burton, & Stacey, 1982; Schoenfeld, 1985). Furthermore, during the last decades, research has been focused on the role played by teachers in their teaching practice in relation to pupils' learning. More specifically, it seems that there is a consensus concerning a possible link between development in teaching and improvement of pupils' mathematics learning in classrooms. In this research report, my aim is to present how professional development for in-service teachers is organised in a developmental research project *Teaching Better Mathematics* (TBM) at the University of Agder (UiA), where the idea of inquiry is fundamental.

This paper addresses the link between theory and practice by referring to the theoretical constructs of teaching cycle, inquiry cycle and developmental research and by exemplifying the idea of an inquiry cycle through the analysis of a primary school teacher's teaching practice. In other words, through this paper I am in a position to link theory, teaching and practice and to address the following research question: what does it mean to follow an inquiry approach to teaching? The structure of this paper is as follows: First I present central aspects of the TBM project and emphasise the theoretical constructs of inquiry, teaching cycle and inquiry cycle, and developmental research. Then I turn to an example from a primary school teacher's practice and show how the theoretical constructs introduced previously are useful in conceptualising different aspects of the teacher's teaching practice. I conclude by discussing possible development of and implications for developmental research for teacher education and especially the importance of the inquiry cycle in the development of mathematics teaching and learning.

## CENTRAL ASPECTS OF THE TBM PROJECT

### Co-learning agreement with teachers

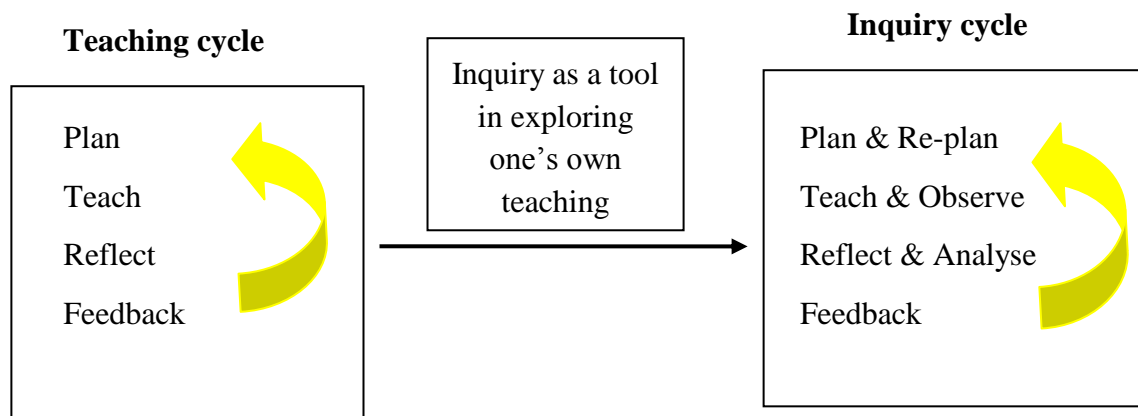
The aims of the TBM project are as follow: first to promote better understanding and competency in mathematics for pupils in schools, and second to explore better teaching approaches. These aspects are visible in the title of the project depending on the emphasis - Teaching *Better Mathematics* or *Teaching Better Mathematics*. In the project, we collaborate with in-service teachers from 4 kindergarten, 6 primary and lower secondary schools and 3 upper secondary schools. Our collaboration with teachers is organised around workshops and school visits. In the project, we see teachers and didacticians as working together as co-learners (Wagner, 1997). This implies that both teachers and didacticians are engaged in action and reflection, and by working together, each have the opportunity to develop further understandings of the world of the other and of his/her own world. Through our collaboration with teachers, we, as didacticians, might learn about the teachers' teaching practice in schools and, at the same time, about our own research practice. Central to the project activity is the design of mathematical tasks, by the didacticians at the university, as a means for engaging in mathematics. This engagement takes place between teachers and didacticians during the workshops and between teachers and pupils in the classrooms. Teachers often take mathematical tasks, which were presented during the workshops and adapt them to their own teaching practice (Berg, 2010a, 2010b).

### Theoretical basis of the TBM project

A fundamental aspect of the TBM project is the idea of inquiry. Inquiry is understood as asking questions and recognising problems, seeking for answers and solutions, and at the same time wondering, exploring and investigating the activity we are engaged in, while looking critically at what we do and what we find (Jaworski & Fuglestad, 2010). In our project we see inquiry at three different levels: at the first level, inquiry into mathematical tasks in relation to pupils' mathematical learning in classrooms; at the second level, inquiry into the developmental process of planning for the classroom and exploring ways of developing better learning environments for pupils in mathematics; and at the third level, inquiry into the research process of systematically exploring the developmental processes involved in the two previous levels (Jaworski, 2007). While the second and the third levels address the collaboration between teachers and didacticians/researchers, the first level addresses the collaboration in classrooms between a teacher and his/her pupils. Due to limitations of space, I focus on just one level, inquiry into the developmental process of planning for the classroom and exploring ways of developing better learning environments for pupils in mathematics. As mentioned earlier, we see our collaboration with teachers as co-learning and we aim at creating and developing a *community of inquiry* where inquiry is used both as a tool in all practices, the teachers' and ours, but also as a way of being, indicating a willingness to becoming or taking the role of an inquirer (Jaworski, 2007). The idea of *community of inquiry*

derives from Wenger's (1998) *community of practice* where the practice refers to the activities in which we, teachers and didacticists, are engaged, and are mainly learning, teaching and didacting. In a community of inquiry, we use inquiry as a tool in all aspects of our practice and aim at developing "inquiry as a way of being" in practice (Jaworski, 2004).

In order to make explicit the processes involved in inquiring into the developmental process of planning for the classroom and exploring ways of developing better learning environments for pupils in mathematics, and to conduct fine grain analysis, I introduce the theoretical constructs of *teaching cycle* and *inquiry cycle* (Jaworski, 2007). A *teaching cycle* is an analytical tool we use to conceptualise teachers' practice: that is, how they usually plan tasks for the classroom, use the designed tasks in class, reflect on his/her own experiences, and then feed back into future planning. However, in a community of inquiry, we are not necessarily satisfied with the normal or usual state of teaching practice and, therefore, our aim in the project is to promote an *inquiry cycle* within which the teachers start to question their practice, explore possibilities and thereby adopt an inquiry approach to their own practice. An inquiry cycle differs from a teaching cycle since it consists of a teacher's planning and re-planning tasks for the classroom, using and observing the designed tasks in class, reflecting and analysing on his/her own experiences, and then feeding back into future planning (Figure 1). Thereby, teachers engage in the process of researching their own teaching practice.



**Figure 1: From a teaching cycle to an inquiry cycle**

## METHODOLOGICAL CONSIDERATIONS AND RESEARCH SETTING

Within the TBM project we follow a developmental research approach. This implies that we, as didacticists, engage in studying, documenting and researching the development of the teachers' practice and, at the same time, our research activity contributes to its development (Goodchild, 2008). Central to developmental research is a *research cycle* and a *development cycle*. The research cycle refers to a cycle between global theories and local theories and, in the TBM project, global theory

refers to community of practice (Wenger, 1998), while local theory refers to community of inquiry (Jaworski, 2007). Furthermore, a development cycle consists of a cyclical process between thought experiment and practical experiment. In other words, the notion of *thought experiment* refers to the preparation of the workshops where we collaborate with teachers, while *practical experiment* refers to the actual realisation of these. Feedback from participants informs the next step of thought experiment. Concerning the development of teachers' teaching practice, I consider that by introducing the idea of *inquiry cycle*, as presented above (see Figure 1), I am in a position to develop and elaborate on the idea of *teaching cycle* further since the dimension of inquiry implies that a teacher is not only engaged with teaching, but teaching *and observing*, and not only reflecting, but reflecting *and analysing*. This theoretical construct is exemplified in the next section.

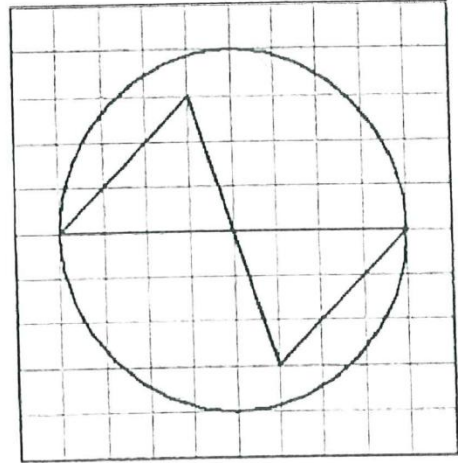
### **The research setting**

The data presented in this paper was collected during a teaching period at a primary school in December 2008. In addition I conducted semi-structured interviews with the teacher both before and after the teaching period. I was invited by the teacher to follow her in her classroom since she was inspired by a particular task that had been introduced during a workshop, which was organised a week earlier. Usually the workshops are organised according to the following pattern: first teachers are invited to present their own reflections concerning the previous workshop, the way they implemented tasks in their own teaching practice and thereby sharing with others their experiences and inviting other participants to comment on their presentations. Next is a plenary presentation, usually by one of the didacticians, on the chosen mathematical theme. After a break, all participants divide into groups, according to the level they teach, and engage with mathematical tasks. Finally, all participants meet again and each group presents the results of their investigation. The aim of the group activity is to offer opportunities to teachers and didacticians to work together and to promote further discussions concerning the use of inquiry-based tasks in classrooms. In addition, the teachers are encouraged to take ideas and tasks presented during the workshop, and to modify and adapt these for their own classes. In the next section, I present the way an experienced primary school teacher modified and adapted a task, which was presented during the workshop. In addition, it is possible to follow how she engaged in teaching and observing her class, and later in reflecting on and analysing the teaching period.

### **A PRIMARY TEACHER'S INQUIRY CYCLE**

The theme of the workshop was "Communication in mathematics: To ask good questions". The team of didacticians at UiA had decided to present the T-shirt task as a means to engage and explore what communication in mathematics might mean. The T-shirt task was previously introduced during a seminar at UiA concerning theoretical perspectives in mathematics teaching and learning through an article

concerning socio-mathematical norms (Tatsis & Koleza, 2008). The task was designed as an imaginary phone call where one person had to explain to another the design of a logo to be reproduced on a T-shirt (Figure 2). In this context the nature of the questions the other person may ask, as a means to reproduce the logo of the T-shirt in an accurate way, was important. Following from this workshop two experienced teachers contacted me, one from a primary school and one from a lower secondary school. Due to limitations of space here, I focus on just one implementation of the T-shirt task by the primary school teacher (Kari) in grade 6. I address implementation in lower secondary school in another article (Berg, 2010a).



### Thought experiment: Interview before the teaching period

A week after the workshop I followed Kari in her class (6. grade) and could observe how she modified and implemented the T-shirt task. During the semi-structured interview before classroom observation she explained to me that she had modified and simplified the logo in order to start with an easier figure, a logo with only positive x-coordinates (see Figure 3). In addition, she explained that she introduced the x- and y-axes as a means to facilitate the pupils' engagement with the task (originally the T-shirt logo was represented on a grid, without specification of the axes, see Figure 2). Analysis of the data presented in the next sections shows evidence of the teacher's reflections before teaching period, the way she taught and observed during the teaching period and her reflection and analysis immediately after the lesson.

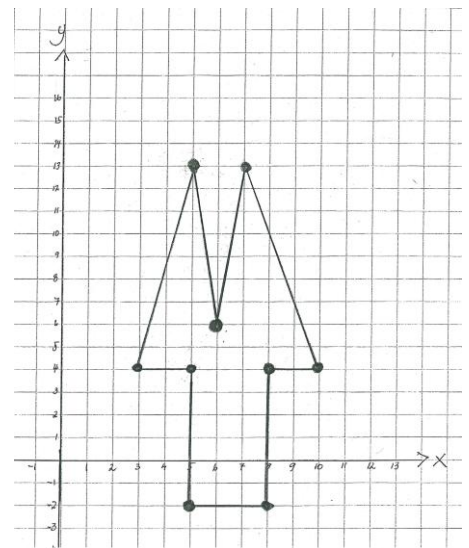


Figure 3

During the interview I also asked Kari the rationale for choosing and adapting the T-shirt task in her class. She explained her choice as follows:

"Yes, it [the task] captured me, and then, when I started to think that it could be about coordinate system, then, then I thought that this is a task I will use.... I relate it [the task] to my teaching and to what I do on that grade.... Pupils will need to use their mathematical language, they can talk about circles, triangles, and several concepts I would like them to have."

My interpretation of Kari's explanation is that during the workshop where the T-shirt task was introduced (see Figure 2), she engaged in inquiry at two levels. First inquiry

in mathematics as she explored the mathematics task “it [the task] captured me”, then inquiry in planning for her classroom since, from her knowledge of the task, she could envisage how to modify and adapt the task for her pupils and how to link the task to her current teaching. She refers to the coordinate system and imagines how the T-shirt task could be modified as a means for pupils to engage further with the coordinate system and with mathematical language and terminology. It seems that her aim with this task is that pupils will have the opportunity to use the mathematical notions of circle, and triangle while engaging with the task. I consider that Kari’s explanation illustrates the essence of the “Plan & Re-plan” phase as she was in a position to envision how the teaching-learning process could proceed in her class.

### **Practical experiment including inquiry: Teaching and observing**

The class was organised as following: small groups of two or three pupils sitting together and one of them having the logo on a sheet of paper in the front of them, holding the sheet of paper vertically and describing the logo to the others in the group. The dialogue between two boys (Peter and Jon) refers to the description of the logo as in Figure 3. The dialogue starts when Peter has finished indicating the coordinates of all nine points in the logo (see Figure 3). Jon has drawn all points on his sheet of paper and the issue at stake concerns how to join all points in order to have exactly the same logo as Peter.

Peter: and then line [he starts pointing to Jon’s sheet of paper], down, yes, down [looking at Jon’s drawing and following what his friend is doing]

Teacher [going around in the class and commenting on what Peter just said]: now you have to imagine that you are talking by phone, and then do you know what you are drawing down to?

Peter [not paying attention to what the teacher just said]: and up, yes, on the side [following what his friend is doing], hmm

The teacher leaves the two boys and move to another group

Jon: do I need to go down to this? [pointing to the point 8, -2]

Peter: no, hmm, not now

Jon: just look at what is here [pointing to Peter’s sheet of paper with the logo] so you can see how it should look like

Peter: no, this is wrong, hmm, go to ... the right

Teacher [coming back to the group]: but if you are talking on the phone, can you see where the lines will go? Is there another way to tell?

Peter: yes, ..., hmm, ..., take a line from [the point] 8, 4 to [the point] 8, -2

The teacher is leaving the two boys

Jon: yes, and ...

## Working Group 17

Peter: and then from 8, -2 to 5, -2

Jon: yes, like this

Teacher [talking to the whole class with a loud voice]: let me have a little time now, I can hear many similar cases. If you are talking on the phone can you then know if the line should go up, or down, forward or something else?

One pupil: no

Teacher: then how can you explain this [the logo], think about it a little, and try.

Peter was able to indicate to his friend Jon the position of all points using correctly the coordinates of each point. The issue at stake, when the dialogue starts, is related to how to connect all these points together such as the logo on Jon's sheet of paper will be exactly the same as on Peter's sheet of paper. From the video-tape it is possible to see how Peter started to point at Jon's sheet of paper without finishing his gesture. Here it seems that the boys struggled with keeping the imagined context of the task, a phone call, and chose to use indexical expressions as "down". Of course, the meaning of this adverb is clear if one knows the context, in the sense of "down" from which point to which point. It seems that this is the reason why Peter was pointing to Jon's sheet of paper in order to indicate from which point Jon needed to draw a segment down to another point. Kari, who was close to Peter and Jon's group, observed and commented on Peter's explanation and tried to remind the boys of the context of the task within which it is not possible to use adverbs such as "down". It seems that Peter did not pay attention to Kari's interruption as he continued to use adverbs as "up". My interpretation of the fact that Kari left Peter and Jon's group is that she became aware of the boys' way of describing the logo, using indexical expressions, and that she wanted to observe other groups in order to see if they also used similar expressions. The dialogue between Peter and Jon shows that they continued to use pointing and indexical expressions such as "down to this" and "go to the right". It is possible to see a shift when Kari came back and, as she recalled the context of the task, she challenged the boys "can you see where the lines will go?" and encouraged them to find "another way to tell". Now it seems that the boys were in a position to follow her advice as Peter, after a pause, said to Jon "take a line from [the point] 8, 4 to [the point] 8, -2". Compared to his previous utterances, this claim makes sense in a phone call situation. Then Kari chose to interrupt all groups and shared with the whole class her observations "I can hear many similar cases" and, as a result, she emphasised the lack of meaning of indexical expressions in the context of a phone call. I understand her last utterance "then how can you explain this [the logo], think about it a little, and try" as an invitation to inquire in the mathematics and the use of coordinate in order to keep the context of the task and to draw the logo as represented in Figure 3. This implies that, as a result of her observations in class, Kari decided to interrupt the different groups in class and to emphasise the context of the task, a phone call, as a means to avoid indexical expressions and to encourage



pupils to use the coordinates of the points. I consider that this episode offers evidence of how Kari was able to teach *and observe* during the teaching period.

### **Inquiry cycle: Reflecting and analysing**

Immediately after the classroom observation, I had the opportunity to interview Kari. She started by commenting on her pupils' activities:

I was very impressed by what they were doing at the beginning; I was very impressed by that. I think they were very clever, clever and reliable concerning coordinate system. I did not expect it would have gone so nice. I was afraid it would not go so easy.

From Kari's utterance it is possible to follow how she was both reflecting and analysing after the teaching period. She started by recognising the pupils' ability in using the coordinate system. My interpretation is that she was referring to the first part of the drawing of the logo which consisted of placing the different points on a grid using the coordinates of each point. She characterised the pupils' activities as "clever" and "reliable" and it seems that she did not expect that they would be able to reproduce and identify the different points so quickly. However, evidence from the dialogue both with Peter and Jon and with the whole class shows that, during the teaching period, she had to make some comments on the way the pupils engage with the task in order to avoid the use of indexical expressions. Kari commented on this aspect later during the interview:

If I should do it [the T-shirt task] again, just the same [task], then I would have organised them [the pupils] in a way that they could not see each other. Because it was exactly what I said, what can you tell by the phone, I asked them a question about that. By the phone can you say that the line should go down there? Because you can't, and then they must start using the coordinates.

Here Kari developed further both her reflections and her analysis of the teaching situation. She focused on the situation which appeared in class, where pupils used indexical expressions as a means to draw the shape of the logo. She remembered drawing the pupil's awareness on the context of the task "because it was exactly what I said" and asking them "what can you tell in a phone call?" where the "what" was directly oriented to the kind of information and questions one can ask in the context of a phone call. The argument was elaborated further in the last part of her utterance as she emphasised that "during a phone call, can you say that the line should go down there?" The answer to that question was that the particular context of the task implied that one needed to use the coordinate system, not only for identifying the position of the points on a grid, but also for being able to draw segments joining the different points accurately such that the logo will be identical to that in Figure 3. As a result of her reflections and analysis, Kari was in a position to envisage a modified teaching – learning path. She mentioned, right in the beginning of her utterance, that if she were to have the same activity again, she would organise the class differently in order to avoid the possibility that pupils could see each other

and thereby help each other through gestures and indexical expressions. I consider that these two utterances provide evidence of how Kari was able to reflect *and analyse* during the interview after the teaching period.

## DISCUSSION

In the beginning of this article I introduced and explained the theoretical constructs of inquiry, teaching cycle and inquiry cycle. My aim, by presenting an example taken from classroom observations in a primary school, was to illustrate how these observations could be conceptualised using the introduced theoretical framework. In addition I consider that the result of data analysis emphasises an important aspect of developmental research which is the didacticians' activity as engaging in researching, studying and documenting the development of the teacher's practice, but also, at the same time, how our research activity contributes to its development. This aspect was visible since I had the opportunity to follow Kari both in her class observing the lesson, and to conduct interviews before and after the lesson. It is in the process of asking questions, trying to find justifications, and comparing learning outcomes together, following a co-learning agreement, that the teacher had opportunities for engaging in the development of her teaching practice. Furthermore the fundamental role of inquiry enabled her to move from a teaching cycle to an inquiry cycle where both observation during teaching and analysis of her own reflections were crucial. Through the analysis it is possible to follow how Kari engaged with inquiry both in the mathematics and in her teaching practice. These aspects were visible as she modified the original T-shirt task, introducing the axes, and choosing a different figure with only positive coordinates. In addition she observed how pupils used indexical expressions such as "up", "down" and "down to this" and thereby did not engage with coordinates both as a means to identify the position of the point on a coordinate system and to make connection between the different points. Her reflections and analysis showed that a change in the organisation of the class is needed in order to keep the imagined context of the T-shirt task. I argue that by following this research approach I am in a position to establish a link between theory, practice and teaching. This recognition begs the following questions: what can we learn from this research that would be relevant for teacher education? And how can the idea of inquiry be introduced in teacher education? In the TBM project teachers and researchers work together as co-learners. A central question would be: what kind of collaboration between teacher students and teacher educators could foster an inquiry approach both to mathematics and to the teacher students' future practice? I consider these issues as fundamental since, in my opinion, one of the aims of teacher education is to encourage teacher students to develop awareness of possible tensions and limitations of their future teaching practice and thereby to support them to look critically at their professional practice.

## REFERENCES

- Berg, C. V. (2010a). Le projet TBM: un exemple de modalité de collaboration entre chercheurs et praticiens en Norvège. In S. Kahn, M. Hersant & D. O. Ravachol (Eds.), *Recherches en Education*, Hors Série n<sup>o</sup> 1, 130-146.
- Berg, C. V. (2010b). Designing and implementing mathematical tasks: the T-shirt task. *Research in Mathematics Education*, 12 (2), 151-152.
- Dewey, J. (1905). *How we think*. Mineola, New York: Dover Publishers.
- Goodchild, S. (2008). A quest for “good” research. In B. Jaworski & T. Wood (Eds.), *International handbook on mathematics teacher education: Vol. 4. The mathematics teacher educator as a developing professional: Individuals, teams, communities and networks* (pp. 201-220). Rotterdam, The Netherlands: Sense Publishers.
- Gravemeijer, K. (1994). *Developing realistic mathematics education*. Utrecht, The Netherlands: Freudenthal Institute.
- Jaworski, B. (2004). Grappling with complexity: co-learning in inquiry communities in mathematics teaching development. In M. Johnsen Høines and A. B. Fuglestad (Eds.), *Proceedings of the PME28* (pp. 1-17-1-36). Bergen, Norway: Bergen University College.
- Jaworski, B. (2007). Theoretical perspectives as a basis for research in LCM and ICTML. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild & B. Grevholm (Eds.), *Læringsfellesskap i matematikk – learning communities in mathematics*. Bergen: Caspar Forlag.
- Jaworski, B. & Fuglestad, A. B. (2010). Developing mathematics teaching through inquiry : a response to Skovsmose and Säljö. *Nordic Studies in Mathematics Education*, 15 (1), 79-96.
- Mason, J., Burton, L. & Stacey, K. (1982). *Thinking mathematically*. London: Addison-Wesley.
- Polya, G. (1945). *How to solve it*. New Jersey: Princeton University Press.
- Schoenfeld, A. H. (1985). *Mathematical problem solving*. NY: Academic Press.
- Tatsis, K. and Koleza, E. (2008). Social and socio-mathematical norms in collaborative problem solving. *European Journal of Teacher Education*, 31, 1, 89-100.
- Wagner, J. (1997). The unavoidable intervention of educational research: a framework for reconsidering researcher-practitioner cooperation. *Educational Researcher*, 26 (7), 13-22.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge University Press.

# **MATHEMATICAL INVESTIGATIONS IN THE CLASSROOM: A CONTEXT FOR THE DEVELOPMENT OF PROFESSIONAL KNOWLEDGE OF MATHEMATICS TEACHERS**

Ana Paula Canavarro

Universidade de Évora, Portugal

Mónica Patrício

E.B.I. André de Resende, Évora, Portugal

*This article reports part of a study developed with the purpose of understanding professional knowledge of teachers involved in the development of investigative tasks in the classroom. The study adopted the interpretative paradigm and elaborated a case study of a first-grade teacher. This teacher worked regularly in collaboration with one of the researchers, selecting, planning and developing investigative tasks with her students in the classroom, and also reflecting on her teaching practice with investigations. The analysis of the data allows us to identify several aspects in which the professional knowledge of the teacher was deepened and broadened. It is also possible to identify the main factors contributing to the development of the teacher's professional knowledge.*

Keywords: Professional knowledge; investigative tasks; teacher practice and reflection

## **PURPOSE OF THE STUDY**

Investigative tasks promote a rich mathematical experience for the students and provide them with a significant intellectual challenge (NCTM, 2000). These tasks have an open nature and can be approached in very different ways, depending on the work of the solver. This kind of task can contribute to providing students with the mathematical learning experiences recommended by curricular orientations of many countries. In Portugal, they are explicitly suggested in the National Curriculum of Mathematics and in the recent syllabus (2007) of Mathematics for Basic Education.

However, investigative tasks can be very complex and their development in the classroom represents a serious challenge for many mathematics teachers (Oliveira *et al.*, 1999). It is important to understand the professional knowledge of teachers involved in the development of investigative tasks in the classroom, namely for first-grade teachers, since the majority of Portuguese teachers of this grade are not familiar with them. This was the main purpose of the study reported in this article. We investigated the practice of a teacher who participated in a collaborative context with the researcher (second author). Our purpose was to identify the components of professional knowledge she applied to develop investigative tasks in the classroom, and also to identify the influences of that work in the development of her professional knowledge.

## THEORETICAL FRAMEWORK

Teachers have particular knowledge they apply in their day to day practices of teaching mathematics (Schön, 1992; Ponte & Chapman, 2006; Sowder, 2007). This professional knowledge is multifaceted and can be organized into different components (Elbaz, 1983; Sowder, 2007). According to Ponte (1995) and Canavarro (2004), there are four interrelated components of professional knowledge that assists the teacher directly when he/she prepares and conducts mathematical lessons: knowledge of mathematics, knowledge of students and their learning processes, knowledge of curriculum and knowledge of instructional process.

The knowledge of mathematics includes knowledge of specific topics, an overview of mathematics, as a science and as a school discipline, the perspective on its nature and its relation to reality (Ponte, 1995). This is essential knowledge for teaching. Having poor mathematical knowledge can inhibit the action of the teacher in the classroom, particularly the mathematical tasks that he/she presents to the students (Leikin & Levav-Waynberg, 2007).

Knowledge of students and their learning processes consists of the knowledge of characteristics of the students, aspects that motivate their learning, and how they develop and acquire knowledge from the learning situations (Canavarro, 2004).

Curricular knowledge includes knowledge of the purpose and guidelines of the curriculum, specific knowledge of the contents of the syllabus that the teacher teaches and of the syllabus of the years before and after, and also includes knowledge about the approaches, strategies and materials proposed (Ponte, 1995).

Finally, the knowledge of instructional process refers to the knowledge directly related to the organization and concretization of classroom practice. It refers to the phases of planning, conducting and evaluating teaching and learning (Canavarro, 2003). This knowledge has a decisive influence on the kind of teaching that the teacher puts into practice (Sowder, 2007).

Research in mathematics education has been explaining how the professional knowledge of the teacher influences the way he/she develops investigative tasks with his/her students (Oliveira *et al.*, 1999). This knowledge is reflected in two aspects of the teacher's work during the development of investigative tasks, which are mutually dependent: the mathematical aspects concerning the investigative task in question – when a teacher assesses the mathematical scope of the task, when he/she is involved in mathematical reasoning in front of the students, when he/she establishes connections between the knowledge present in the task and other mathematical concepts; and the didactical aspects that are fundamental to accomplish the objectives of the investigative activity. These are present when the teacher creates or selects the task, when he/she plans the lesson, when he/she conducts its development, when he/she has to handle unexpected situations of uncertainty, when

he/she has to understand the thought processes of different students and to promote collective reflection of the class (Ponte *et al.*, 1999).

Oliveira *et al.* (1999) stress that the selection and planning of investigative tasks is a complex activity because it requires from the teacher a broad knowledge about their students, their knowledge, interests and abilities. It also requires a solid knowledge of curriculum objectives, materials and resources they have available. The conduct of the class with investigative tasks is also very important, requiring many roles of the teacher (to challenge students, to evaluate their progress, to reason mathematically, to recall or provide relevant information, to support the progress of students and promote their reflection). In addition, the teacher should be aware of the type of interactions he/she has with students in order to promote an investigative ambience (to encourage the discussion of different opinions, the critical sense, reflection and debate). The teacher should also provide, and teach students how to use, technological tools that help them in investigative work (Oliveira *et al.* 1999; Ponte *et al.*, 1999).

The professional knowledge of the teacher is a dynamic knowledge, evolving from teaching practice and the reflection of the teacher about his/her practice. This reflection can be encouraged in the context of the school teacher, through collaborations of the teacher in complicity with colleagues. These collaborations are especially interesting when teachers experiment with the development of new and challenging tasks or new ways of working with students in the classroom (Sowder, 2007).

## **METHODOLOGY AND CONTEXT OF THE STUDY**

As previously explained, this study is intended to contribute to the understanding of the professional knowledge of the teachers involved in the development of investigative tasks in the classroom (Patrício, 2010). The researchers adopted the interpretative paradigm and a qualitative approach, and elaborated case studies of teachers. The option for this design was because the study was intended to provide an explanation of a well-delimited phenomena, embedded in context, for which we seek thick description from the standpoint of the subjects being investigated (Erickson, 1986; Merriam, 1991).

The researchers decided to consider a teacher of the first grade because the practice of these teachers is still poorly documented with regard to the development of investigative tasks in classroom. The teacher should be an experienced one and denote propensity for innovation and investment – these conditions offer some guarantee of considerable professional knowledge and a willingness to embark on a new experience. This is the case for the first-grade teacher Petra, who is the focus of this paper. She is a teacher of 40 years, teaching a fourth-grade class (9 year-old students). Petra had never developed investigative tasks in their lessons. The total inexperience of the teacher with these tasks recommended the collaboration with the

## Working Group 17

one of the researchers (the second author). That collaboration created conditions to operationalise this study, allowing more reliable and complete data collection, and representing an opportunity for professional development for the teacher.

The collaborative work took about two months. Petra had the opportunity to know and solve several investigative tasks and to discuss their potential with the researcher. The tasks were open and they could be developed in a more or less complete ways by the students. In the next paragraph, we give an example of one investigative task that Petra selected to work on with her students:

*Dividing by 11, 111, ...*

Look at the figures of the non integer part of the decimal representation of the following fractions:

$3/11$                        $9/11$                        $18/11$                        $47/11$                        $52/11$                        $125/11$

Can you find any pattern in it?

Is it possible to estimate what the figures are of the non-integer part of the decimal representation of any fraction with denominator 11? Please explain.

And what happens if the denominator is 111? Write your conclusions.

You can also investigate with 1111...

In this collaborative work, Petra also responded to the challenge of the researcher to develop investigative tasks with her students in the classroom. She carried out five two-hour lessons with different investigative tasks, one per lesson.

The role of the researcher was to provide a collection of investigative tasks for the teacher, to support her in selection and in the planning of the lessons, and to promote her reflection on teaching practices. This collaboration emphasized discussion of the plans of the lessons that the teacher elaborated to each class with investigative tasks. The plans were organized according to a script provided by the researcher. The script considers three phases to the development of the lessons: a first phase of presentation of the task to the students; a second phase of autonomous work by the students, individually or in group; and a third phase of a collective discussion of the work done by the students. The script also includes the provision of a list of questions to ask the students in order to promote their mathematical thinking while performing the task.

In the classroom, when the teacher was developing the investigative tasks with her students, the researcher took the role of a non-participant observer and collected data. The teacher's classroom practice was particularly relevant for data collection, considering its different moments (preparation, conduction and reflection). After each lesson with investigative classes, the researcher and the teacher always spent some time reflecting on the teaching practices. They focused on mathematical episodes that took place during the lesson, on the mathematical learning of the

students, and they also tried to identify factors of influence in the way the tasks were developed.

The techniques and instruments used for collecting data were those generally recommended (Merriam, 1991): the interviews (the researcher conducted two formal long interviews at the beginning and at the end of the study; five informal interviews during the preparation of the investigative tasks developed in classroom; five informal interviews at the end of the lessons with investigative tasks – all fully transcribed); the observation of the teacher classroom practice (the researcher observed five classes with the investigative tasks, all audio-recorded and fully transcribed); the documental analysis (the researcher analysed all the lesson plans elaborated by the teacher).

## **THE TEACHER AND THE INVESTIGATIVE TASKS – RESULTS OF THE STUDY**

Petra accepted with great enthusiasm, dedication and responsibility the invitation to participate in the study because she anticipated that she could learn and develop herself professionally. She was teaching a fourth grade class (9 year-old students).

Petra had never developed investigative tasks in her lessons, neither knew their characteristics. From the beginning, she revealed a great appreciation for the nature of the investigative tasks and the work they provide for students in the classroom. This is very consistent with what she valued in the teaching of mathematics: “Mathematics, for me, is to teach to think, is developing the reasoning, is awakening to the problems.”

Her lesson plans were very detailed. She worked hard trying to explore all the mathematical possibilities she could imagine. For each task, she elaborated an extended list of questions she could pose to her students to help them to progress. She explained that she wanted to take full advantage of the potential of the tasks and also to reduce the degree of unpredictability of the work with the students.

The teacher practice with investigative tasks was developed according to the three phases previously described. Her main purpose was to promote the development of students mathematical reasoning, but she tried to establish connections with the concepts emerging from the tasks. For example, for the task *Dividing by 11, 111...*, she wanted the students “to develop confidence in exploring mathematics in an autonomous way; to explore and use patterns on division by 11; to learn about periodic decimal representations.”

In the post-lesson interviews and in the final interview of overall reflection, Petra identified several aspects that she considered gains in her experience with the investigative tasks. In the following sections, we synthesize these aspects, organized by the four components of professional knowledge that we adopted, despite the fact that some of them could be included in more than one component.



## **Mathematical knowledge**

Petra acknowledges having learned mathematical knowledge. When planning for the lessons, she forced herself to get into deep mathematical exploration of the tasks and she involved herself in investigative work.

I spent the weekends searching... and studying for the mathematical knowledge to tackle the tasks... And then I interrogate myself: "what can you investigate more on this?" And I get deeper but I never find everything, of course! (...)

This work revealed some "gaps" in her mathematical knowledge. She consulted some books, colleagues and the researcher trying to overcome the difficulties:

I learned a lot of mathematical knowledge, is true... Because I had to search for things I didn't know at all and others I was not sure... I also asked other colleagues, I asked you ... Because sometimes I had some doubts... the investigation of the rational numbers, for example. I had never worked with infinite decimals, with periodic decimals, my decimals were always finite! And exploring their period, hum... when I was solving the task myself with the calculator, I was not sure about the numbers it was showing me... that's when I called you!"

Petra also experienced some difficulties with a task in the domain of probability. She considered that her knowledge was too informal to explore the task correctly. She had to learn how to count all the possible cases in a complex situation – for example, how to differentiate between 1-6 and 6-1 when throwing two dice.

## **Curricular knowledge**

Petra said that she has extended her repertoire of the types of tasks that she considers appropriate to work with students in the context of current curriculum guidelines. She confessed that she always liked open tasks and she was a great enthusiast of mathematical problems. But after this experience, she came to privilege the investigative tasks:

Now I'm a fan of investigations, seriously! I am fan! I think they have enormous potential... This year was the unknown, it was my first experience with this kind of task... I also like problems very much, but the investigative tasks are more challenging and have more potential for the learning of the students.

Petra said that the investigative tasks contribute to several aspects of mathematical competence that students should develop:

Because they are complex in nature, mathematical investigations provide a greater intellectual challenge and promote the development of more complex capacities of students, as mathematical reasoning, mathematical communication and critical thinking.

Petra also stressed that the investigative tasks, instead of constituting something marginal to the school curriculum, helped to promote compliance with the mathematics syllabus in an interesting way:

## Working Group 17

With the investigative tasks you facilitate the achievement of the curriculum and of the mathematics syllabus, one thing is inseparable from the other... I think it only helps! You're not compromising anything... You are addressing mathematics in a much more interesting way and this is how it is expected to be addressed.

### **Students and learning process knowledge**

Petra considered that investigative tasks have great potential to motivate the students to the learning of mathematics because they provide contexts for meaningful learning. The teacher also recognised that students may have an important role as creators of knowledge when developing investigative tasks:

And I think the investigative tasks have great power to motivate the students to mathematics – they like, they get involved and the knowledge that comes from this involvement is great. I'm sure they made significant learning. And they are creating knowledge, they are learning - and this is great!

### **Instructional process knowledge**

The planning of investigative tasks allowed Petra to acquire a new vision about what makes a lesson plan effective: to consider other aspects besides the content, such as ideas of how to conduct the class or of the good mathematical questions to pose to the students. Petra concluded that a careful plan is essential to explore and conduct the lesson in order to take full advantage of all the potential of the tasks:

I learned so much! Look, I learned to plan properly, because I think this is the right way (...) It was ideal that we could always do so but it is too time consuming... When you have a major concern in planning – in the choice of tasks, the choice of materials, the good questions to make them think, the knowledge that could result from the task... – it is obvious that the development of the task is much better and the results can be very positive.

The experience with the investigative tasks in the classroom allowed Petra to rethink various aspects of her practice, although the teacher was used to adopting methodologies that value the role of students and was familiar with students working in groups. The idea that it is very important to limit the support that the teacher provides to students while they are working on the tasks was strengthened. During the developmental phase of investigative tasks, Petra began to avoid giving answers and/or validating the assumptions of the students – otherwise she would prevent the development of mathematical communication and reasoning:

But... I think I can do it better, I still have the tendency to give them the answer... I need to still my tongue... I can not tell them everything, I just can tell a small part of what I am used to...

The ideas of Petra, related to the final lesson phase of discussion, also seem to have changed. The teacher began to attribute more importance to discussion and she reviewed her role in its conduct. She decided to try not to summarize, during that

phase, as each student disclosed the work they had done. Instead, she became aware of the importance of promoting the discussion and reflection of all students:

You know another thing I have come to notice? Is that when they pass to the communication phase, they reveal some lack of interest... this is because I validate all the assumptions that they ask me during the work group... So, when they present their work to the class, they are already quite sure of what they say... and the others also know that everything is ok, so... there is no discussion among the groups! The next task, I would like to do something different: I will not validate anything. I will give them the task, I will ask them to do all the work and after that the class will validate... or not... I want to bring the discussion to class. They will present and after... "So, what do others think about this? Is it right, is it wrong? Why?"

### **THE DEVELOPMENT OF PROFESSIONAL KNOWLEDGE – FINAL CONCLUSIONS**

This study shows how professional knowledge of a first-grade teacher was revealed in the development of investigative tasks in the classroom. It also shows the contributions that the teacher took through this experience to deepen and broaden her professional knowledge in various components.

It is noteworthy that this experience involved the teacher in the search for a greater understanding and deepening of her mathematical knowledge. The exploration of the investigative tasks was quite demanding for the teacher, because she wanted to be prepared to deal with the multiple assumptions that students could propose and that do not always appear so obvious to the teacher. The teacher's mathematical preparation is crucial to the development of his/her teaching practices (Leikin & Levav-Waynberg, 2007). And, the more demanding these practices are, the more complete and deep the mathematical knowledge needs to be.

This study also emphasizes that this experience provided the teacher the opportunity for reflection on some aspects of her knowledge of the instructional process. On the one hand, the teacher recognized the value of careful planning, referring to it as crucial when exploring open tasks and promoting the discussion of students in the mathematics classroom (Franke, Kazemi & Battey, 2007). On the other hand, the teacher acknowledged the advantage of introducing changes in the way she conducted the class, either at the phase of the development of the tasks by the students, or at the final phase of collective discussion. She considered it to be necessary to moderate the intervention of the groups in order to achieve more shared discussions and challenge the students to validate genuine conjectures and conclusions of other colleagues (Oliveira *et al.*, 1999).

Three main factors contributed to the development of teacher professional knowledge. One was the contact and work with challenging and powerful mathematical tasks with great potential for the mathematical experience of the teacher herself and of her students, with consequences in the way she usually

conducted the classroom (Stein & Smith, 1998). Another factor was the teacher's perception of the acceptance of the investigative tasks by the students, and of the mathematical experience and mathematical learning that they revealed – a factor of great influence in the professional development of teachers (Canavarro & Rocha, 2008; Sowder, 2007). A third factor was the focus on the teacher's teaching practice provided by the opportunity of collaborative work with the researcher, who introduced the tasks, discussed the plans of the lessons, supported implementation in the classroom and encouraged reflection on practice, based on the analysis of the episodes from the classroom (Canavarro & Rocha, 2008; Sowder, 2007).

## REFERENCES

- Canavarro, A. P. (2004). *Práticas de ensino da Matemática: duas professoras, dois currículos*. Lisboa: APM.
- Canavarro, A. P., & Rocha, I. (2008). Desenvolvimento profissional de professores de Matemática: Mais valias e desafios do Programa de Formação Contínua em Matemática nos distritos de Évora e Leiria. In R. Luengo, B. Gómez, M. Camacho & L. Blanco (Orgs.), *Investigación en educación matemática XII* (pp. 577-586). Badajoz: SEIEM.
- Elbaz, F. (1983). *Teacher thinking: A study of practical knowledge*. London: Croom Helm.
- Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock, *Handbook of research on teaching* (pp. 119-161). New York: MacMillan.
- Franke, M., Kazemi, E., & Battey, D. S. (2007). Understanding teaching and classroom practice in mathematics. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 1, pp. 225-256). Charlotte, NC: Information Age Publishing.
- Leikin, R. & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies of Mathematics*, 66, 349-371.
- Merriam, S. B. (1991). *Case study research in education. A qualitative approach*. S.Francisco: Jossey-Bass Publishers.
- NCTM (2000). *Principles and Standards for School Mathematics*. Reston: NCTM.
- Oliveira, H., Ponte, J. P., Santos, L., & Brunheira, L. (1999). Os professores e as actividades de investigação. In P. Abrantes, J. P. Ponte, H. Fonseca, & L. Brunheira (Eds.), *Investigações matemáticas na aula e no currículo* (pp. 97-110). Lisboa: APM e Projecto MPT.
- Patrício, M. (2010). *O conhecimento profissional do professor e as investigações matemáticas na sala de aula: um estudo nos 1º e 2º ciclos do ensino básico* (tese de mestrado da Universidade de Évora). Lisboa: APM.

- Ponte, J. P. (1995). Saberes profissionais, renovação curricular e prática lectiva. In L. Blanco & V. Mellado (Eds.), *La formación del profesorado de ciencias y matemática en España y Portugal* (pp. 187-202). Badajoz: Universidad de Extremadura.
- Ponte, J. P., & Chapman, O. (2006). Mathematics teacher's knowledge and practices. In A. Gutiérrez & P. Boero (Orgs), *Handbook of Research on the Psychology of Mathematics Education: past, present and future* (pp. 461-464). Rotterdam: Sense.
- Ponte, J. P., Oliveira, H., Brunheira, L., Varandas, J. M., & Ferreira, C. (1999). O trabalho do professor numa aula de investigação matemática. *Quadrante*, 7(2), 41-70.
- Shön, D. A. (1983). *The reflective practitioner: How professionals think in action*. Aldershop Hants: Averbury.
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (Vol. I, pp. 157-223).Charlotte: Information Age Publishing.
- Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School*, 3(4), 268-275.

# WHY DO SOME FRENCH TEACHERS PROPOSE « *PROBLÈMES OUVERTS* » IN MATHEMATICS TO THEIR PUPILS IN PRIMARY SCHOOL ?

Christine Choquet,

IUFM des Pays de La Loire, Ph.D Cren, University of Nantes, France.



*“Problèmes ouverts” in France (Arsac, Germain & Mante, 1988), literally “open problems”, they can be defined as non classical problems which aim is to develop capacities of research and argumentation. This paper analyses, using a case study, why Xavier, a mathematics teacher in primary school, proposes such a problem and how he organizes his session. It reports also a study on the knowledge which is approached during this session. Our results show that, even if the knowledge can be specified, it is still difficult to conclude on the real learning of pupils. According to us, these results become concrete subjects of reflection during teachers’ training.*

Key-words : teacher’s activity, mathematics, open problem, primary school.

## INTRODUCTION

Some teachers in France, in primary school, choose to propose “problèmes ouverts” (Arsac, Germain & Mante, 1988) to their pupils in their mathematics’ teaching. These non classical problems would like to develop capacities of research and to discover proofs in mathematics. Different researchers in France have worked on this subject and the studies are not ended. Differences persist, concerning the knowledge boarded in these problems or concerning the real utility for the pupils to learn to resolve them in the classroom. Arsac & al. (1988) propose a characterization of « problèmes ouverts » and insist on the implementation of these in the classroom. Glaeser (1973) proposes a classification of statements to help teachers to make the best choices for their class. Douaire (1999) shows that these problems could be the occasion to teach different forms of reasoning. Hersant (2008) proves that, when pupils study open problems, they work on mathematic objects but they study also methodologic knowledge linked to problem solving. Houdement (2009) establishes a classification of open problems, in order to explain that, if the teachers think about the choice of the problems which they propose, their pupils could develop different capacities of reasoning and also capacities to argue about their results. In our work, using a qualitative case study, we analyse the place which is reserved to these problems, in primary school. We search to understand how and why a teacher uses this type of problems and also to determine the impact of such problems on the short-term learning of pupils. In this paper, with the analysis of one session, we show the choices made by the teacher : why does he propose this problem and how does he

decide to organize the session ? We think also about the knowledge which is boarded : what could be learnt by pupils during this session, with the chosen problem ?

## **THEORETICAL FRAMEWORK**

To answer our questions concerning the choices that the teacher makes for his/her class when he/she proposes an open problem and concerning the pupil's learning during the observed session, we use elements of the theoretical framework "double approach" (Rogalski & Robert, 2008). First, this theoretical framework suggests to consider the teacher in his/her class, with objectives of learning for the pupils and to study his/her practice in connection with the knowledge approached on the session. Secondly, this model considers teaching as a professional exercise, a job trying to reach professional purposes, with specific constraints. Like Rogalski and Robert, we take into account all the elements which constitute the practice of the teacher and we analyze these elements according to five dimensions (cognitive, mediative, social, personal and institutional). Both dimensions, cognitive and mediative, allow to characterize the practice of the teacher during his/her work of preparation before a session and during the effective progress of the session. The social and institutional dimensions allow to define the internal and external constraints of the class. With the personal dimension, we can obtain more precise answers concerning the teacher (for example, personal competencies or personal representations of mathematical notions and their teaching).

## **METHODOLOGY**

### **Methodology of data collection**

Xavier is a forty-year-old teacher, he has been teaching for about fifteen years and this is his fifth year in the school where the observation takes place, with classes of 9-10-year-old children ("CM2" in France). This class level is the last one in the primary school ; after this, pupils enter the secondary school ("*collège*" in France). We take this example in this paper on the one hand because Xavier has experience and is not a beginner in the concerned class level, because we have been observing him for three years (2007-2010), which allows us to have access to numerous information concerning him. On the other hand, we choose this school level because the problems which can be proposed to the pupils, seem to us to be more interesting for our study concerning the knowledge which may be learned.

The observation is led in a natural way, in the sense that we do not intervene either in the choice of the problem, or in the preparation of the session. The session is observed, filmed and transcribed. The exchanges between the pupils, and between the pupils and the teacher in working groups are recorded, to have as much information as possible on these interactions. The pupil's written tasks (drafts, posters) are collected. Conversations at the beginning of our study, then before and

after the session are also transcribed to have the teacher's opinion on what the pupils are learning during the session.

### **Methodology of data analysis**

In the following paragraph, we will present an *a priori* analysis of the problem chosen by the teacher. We check if the problem is really a “problème ouvert” using back the characterization of Arsac, Mante & Germain (1988). We look for all the possible procedures to solve it and we determine which procedures can be used by 9-10-year-old pupils. To answer the questions concerning the knowledge which the pupils build or will have been able to build, we determine those which can be the object of synthesis, the object of institutionalization <sup>[1]</sup>. Concerning the practice of the teacher and the questions about his choices, we inform one by one the five dimensions to obtain more and more precise answers.

## **OUR HYPOTHESES ABOUT THE TEACHER**

### **Why does he propose open problems ?**

First, one hypothesis is that the official instructions in France (2008) insist on the fact that the situations of exploration, discovery or research on problems have to “*develop the pleasure of searching, the reasoning, the imagination, the capacities of abstraction, the rigor and the precision*”. Other hypotheses, according to the work which were already made on the subject (Douaire, 1999 ; Hersant & Thomas, 2008 ; Houdement, 2009), can explain the reasons for the teacher's choice. And with our analysis of different open problems, we wondered which objective of learning could have a teacher with this class level. It can be a question of reinvesting simple mathematics notions. It can be also a question of discovering and learning methodologic notions useful to resolve problems (Hersant & Thomas, 2008). The teacher can have the objective to explain pupils that they can use an experimental reasoning, he can also have the objective to go farther than this experimental reasoning, to begin to use a deductive reasoning to validate their answers (Douaire, 1999 ; Houdement, 2009). Another objective put in evidence by Houdement (2009) and also by our analyses, is to teach pupils how to check their results. Finally, the proposition of an open problem during a session of mathematics can be the opportunity to develop more interdisciplinary knowledge (for example to read a poster and to expose results in front of the class, Douaire, 1999).

### **Hypotheses on which problems are chosen, in which resources ?**

Here we do not develop the analysis made on these resources, we only specify that we are trying in particular to determine the legitimacy of the authors in the educational environment, to explain why a teacher uses such or such type of resources. According to our analysis, we conclude that the reasonable and available resources for a teacher in France are different textbooks, (*Capmaths, Euromaths, Ermel ...*),



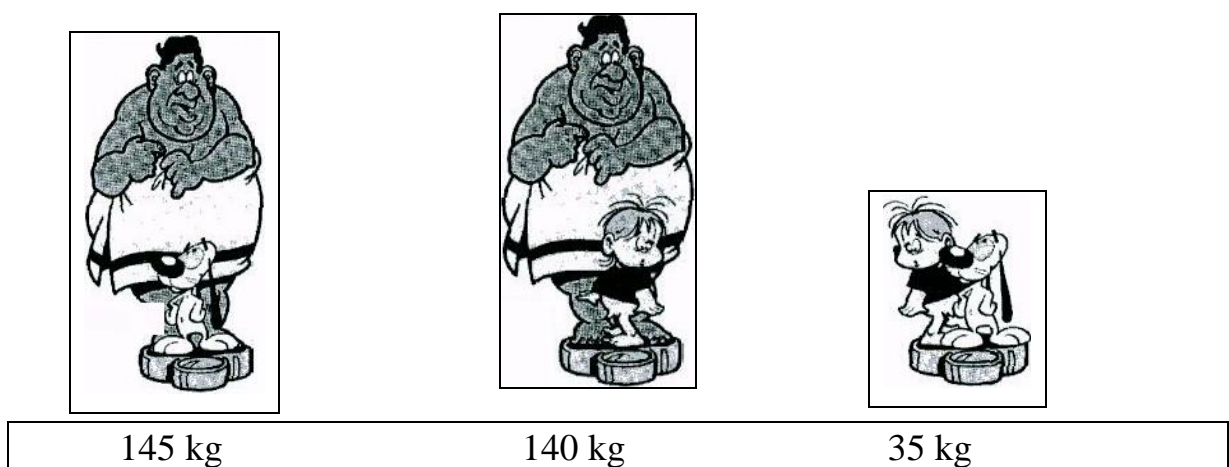
pedagogical reviews (*Grand N*) but also web sites and pre-service or in-service teachers' training.

### Hypotheses on the implementation in the classroom

In the official instructions (2008), the educational freedom of the teacher is quoted. Concerning a progress or concerning the number of open problems that the teachers have to propose along the school year, nothing is specified explicitly either in the official instructions, or in the various textbooks. However, considering Xavier's age, we suppose that he worked with a document linked to the last official programs (2003) and we make the hypothesis that he keeps in memory the model of implementation which was presented in this document and that Xavier's choices in this domain are inspired by it.

### AN A PRIORI ANALYSIS OF THE PROBLEM

To check a part of these hypotheses and to answer our questions, we have chosen to present the analysis of one among six sessions which Xavier proposed during the school year. The problem, which aim is to determine the weight of the dog, the child and the man, is the following one :



En utilisant les informations données par ces trois dessins, détermine combien pèsent le gros Dédé, le petit Francis et le chien Boudin.

### Why is this problem a real “problème ouvert” ?

According to the characterization of Arsac & al. (1988), this problem is for Xavier's pupils a open problem. The statement of the problem is short, in the form of a drawing and an only question using a simple vocabulary for 9-10-year-old children. The situation has a concrete, daily character. It can be extracted from the everyday life. This problem can be understood by the whole class, every pupil can appropriate the situation quickly and can understand why he/she is asked to search from the first individual readings. Furthermore, the necessary mathematical

knowledge being elementary, every pupil can easily begin a research. The problem is nevertheless substantial. The answer is not immediate and asks for a research from all pupils in this classroom. Cheeking the result seems also possible for the pupils whatever the moment of their research. As soon as a solution appears, they can confirm it or not.

### **The different procedures**

To resolve this problem, five procedures exist. A procedure, noted A, consists in putting in equation the problem and in resolving a system of three simple equations with three unknowns. Another procedure, B, without using the algebraic domain, consists in adding the weight of two scales and, in subtracting the weight of the third. The third procedure, C, consists in calculating the sum of the weight of the three scales, in dividing by two then in subtracting the weight registered on one of the scales. A procedure, D, is based on essays and adjustments, on experimentations. The fifth procedure, E, takes into account a five-kilogram difference between the first two scales, which allow to end after some essays and adjustments. The procedures A and B are not possible by pupils of this class level. The procedures C, D and E are possible.

### **Knowledge in this problem for 9-10-year-old pupils**

When we determined the knowledge which can be the object of institutionalization, in this school level ("CM2" in France), the resolution of this problem especially appeals to the notion of addition, subtraction or division by two. This mathematical knowledge is not new for these pupils, this problem can be the opportunity to reuse it.

During this session, we suppose that a pupil can discover two types of reasoning : one is experimental (using essays and adjustments, corresponding to the procedure D) and the other one is deductive (a reflection on the numbers present in the statement helps to deduce elements allowing to end more quickly, corresponding to the procedures C and E). A pupil of primary school can learn also that at any time of his/her research, he/she is able to check his/her result by himself/herself (by some simple calculations and coherently with the statement's data), without the teacher's help.

### **IN XAVIER'S CLASSROOM**

Throughout the analysis of the session, we inform the five dimensions of the theoretical framework of the "double approach". The mediative and cognitive dimensions allow us to reconstitute the progress of the session proposed to the pupils. With the institutional and social dimensions, we clarify the constraints linked to mathematics teaching in primary school. The personal dimension completes the results, it allows us to answer questions concerning Xavier's representation of mathematics teaching and it can explain some of his choices.

## Xavier's choices

The analysis of this session and conversations with Xavier on this problem before and after the session, explain why he goes on proposing to his pupils this type of problems. He thus responds to the official instructions, he also convinces us that : “[...] *to make mathematics in primary school especially then in secondary school, means to try to solve problems.*”. The problem presented here is an answer to his personal representation of mathematics and representation of mathematics teaching. With such problems, Xavier wants to prepare pupils for their entry to secondary school, by teaching them to search alone or in small groups, without the intervention of the teacher. Furthermore, with previous official instructions (2003), Xavier kept in mind that the problems have to allow pupils “*to take some pleasure in searching*”. He thus chooses this funny, atypical problem which is susceptible to develop the child's curiosity and to lead them to search. The resources which he uses, illustrate this aspect. Xavier does not use either textbooks or pedagogical reviews to find problems. He just consults Web sites (discovered with the keywords like “problems to search, CM2” or “mathematical problems, CM2”) and chooses generally the most playful statements.

## Implementation in the class

To precise the analysis of the session, we share it in 3 different steps ; every step corresponds to a different task asked the pupils, by the teacher.

During step 1, Xavier gives briefly some instructions (2 min). He asks the pupils to read the statement and to search individually for an answer. For a couple of minutes, the teacher pushes the resolution of the problem to become the responsibility of each of the pupils. In this case, we observe that the process of devolution<sup>[2]</sup> on the individual level is successful.

During the longer step 2 (30 min), pupils search in small groups (four pupils) and have to draft a poster giving their answer and their explanations. The teacher is standing back, he does not intervene in the groups.

Step 3 (10 min) is dedicated to the presentation of the procedures and to the synthesis.

## The pupil's activity

We also study the pupil's productions to obtain elements concerning the teacher's practice. Indeed, as C. Orange (2005) puts it, “*The productions are considered as revealing of the pupil's intellectual activity,[...] I mean “productions” in the broad sense : diverse linguistic productions, gestures, material productions [...]*”. Their personal work is going to be more easily approachable through their various productions ; it deals for our study, with linguistic productions (the explanations for example within a small group), intermediate individual productions (considered here as drafts) and final productions (here, a poster by a group of four pupils). The fact of wondering about the intellectual activity of the pupils is to have informations on the

teacher's practice (Orange, 2005). It's possible then, to establish links between this practice and the pupils' learning.

Resuming the poster of each of five groups (numbered from 1 to 5) and the individual productions of each pupil, we redraw the history of these posters ; this gives us the most likely possible idea of the reasoning which took place in every group. Having to identify the procedures of resolution, we try to organize them into a hierarchy. We find :

- Procedures which are partially mathematics (noted PMv) : a five-kilogram difference is found between the weight of the child and the dog. After chosen which one of the dog or the child is the heaviest, a subtraction gives the man's weight (groups 2 and 5).
- Procedures which touch especially other domain of rationality, the everyday life for example (noted PVm) : evaluation of a weight, essays and adaptations (groups 1, 3, 4).

A heterogeneousness of procedures appears with a majority of more or less effective PVm. Group 2 mobilizes a PMv. These pupils don't have difficulties with mathematics. They justify the five-kilogram difference mathematically but then, to justify that Francis is the heaviest, they appeal to their everyday life. Group 5 also uses a PMv, which is different from the previous one. They reflect on the numbers, the decompositions of the numbers which they find (for example, a pupil said : "35, it's 20 and 15 or 25 and 10 [...]"), which allows them to advance in their reasoning. Group 3 uses a PVm instead. Even if they find the five-kilogram difference, they do not justify it clearly and appeal then to the everyday life : "*Big person may be weigh about 100 kilograms*". This assertion directs their essays and they find the answer rather quickly. Groups 1 and 4 use a more or less effective PVm. They don't remark the five kilograms difference, they try some weights for the child, the man or the dog inspired by their everyday life (between 5 and 30 kilograms), then they just check the hypothesis made on Francis and the dog. For group 1, this check is enough but not for group 4 which does not succeed in resolving the problem correctly.

### **Interactions between the teacher and the pupils.**

We study how these interactions are evolving to understand better how the responsibility of the learning during this session is distributed.

During step 2, Xavier is standing back, the pupils can count only on their small group to solve the problem, to exchange on their researches and their possible difficulties. Nobody asks Xavier questions about the problem and he stays behind his desk, "*to avoid the questions*" (he says). This attitude is commented by G. Brousseau (1986) who writes that the teacher "*has to, by his/her attitude, convince the children of his/her neutrality [...] so that they give up asking for the information and the help which they need, to count only on themselves*". Xavier knows, by experience that if he moves in the class, he would have difficulty not to answer a pupil and that his answer could guide too much this pupil.

Xavier comes back to lead the class when step 3 begins, the step which is dedicated to the presentations of the results and the procedures. Every group exposes the answer to the class. When the answer is correct, the teacher asks the other pupils if they understood the procedure (it's resumed if needed). When the answer is erroneous, Xavier turns to the class to ask : "*Comments ? Is it correct ?*". By this attitude and this questioning, he does not leave the pupils to have the responsibility to validate or not the results that are proposed by the others, he is in fact just the person who shows what is correct or not.

## DISCUSSION AND CONCLUSION

The analysis of this session allows us to clarify the choices made by Xavier when he proposes a "problème ouvert" in his class. We obtain elements to understand these choices but we think that there are certain points which must be still studied.

First, we have to continue to fathom out what pupils could learn during a session like this one and how they can learn it. We can think that Xavier's synthesis is not complete, it does not completely return on reasoning which enables to solve a problem and to check the result. So, we wonder about the capacities of the pupils to reuse the methods which they have discovered. French researchers (Robert & Robinet, 1996 ; Sarrazy, 1997) show themselves the ambiguity of the situation because of their difference of points of view. Robert & Robinet would maybe take the opportunity of this problem to specify the pupils methods and tools to solve them better : "*It's on the occasion of an activity that we can hope to teach knowledge [...] Teacher tries to offer pupils the opportunities to build and to put on a part of this knowledge.*". However, Sarrazy thinks that even if the teacher gives the pupils these methods, these tools to solve a problem, the pupils will not necessarily know how to reuse them in another situation of research. "*[...] the conditions of the use of rules and the field of application cannot be defined before, that's why it is erroneous to believe that the pupil can control himself/herself the relevance of the use he/she makes of these rules.*". So, like Sarrazy, we can also suppose that, even if the synthesis is very detailed, pupils will not know necessarily how to reuse this knowledge to solve new open problems.

Secondly, the model of the "double approach" is an effective tool for our study, to describe and understand the observed phenomena. With the study of the cognitive and mediative dimensions but also with the institutional and social dimensions, we obtain answers to our questions. However, throughout our analysis, it seems that the personal dimension holds a particular place in this kind of activity linked to open problems. This last dimension gives us important elements to explain Xavier's choices. We note that the personal dimension has a real influence on the cognitive and mediative dimensions. Indeed, the choice of the statement, of the implementation in the class, the choices made by Xavier concerning the synthesis are largely linked to Xavier's personal representation of mathematics teaching. We continue to think

about the influence of the personal dimension on the cognitive and mediative dimensions to understand better teachers' practices and also in the objective to improve teachers' training.

## NOTES

1. Brousseau characterizes the process of institutionalization by the fact that "*there must be somebody outside (the teacher) to come and time the activities (of the pupils) and identify those who have an interest [...]*", Brousseau G. (2004), *Théorie des situations didactiques*, La pensée sauvage, p.282.
2. By the process of devolution, G. Brousseau wants to notice the fact that "*the teacher would like the pupil to feel like holding the answer only by himself*", Brousseau G. (2004), *Théorie des situations didactiques*, La Pensée sauvage, p.303.

## REFERENCES

- Arsac, G., Germain, G., Mante, M. (1988). *Problème ouvert et situation problème*, IREM de Lyon, 117.
- Arsac, G., Mante, M. (2007). *Les pratiques du problème ouvert*, IREM de Lyon, SCEREN-CRDP, Académie de Lyon, 198.
- Brousseau, G. (2004). *Théorie des situations didactiques*, Editions La Pensée Sauvage, 395.
- Douaire, J., Hubert, C. (1999). *Vrai ? Faux , On en débat ! De l'argumentation vers la preuve en mathématiques au cycle 3*, INRP, 208.
- Hersant, M. (2008). Problèmes pour chercher : des conduites de classes spécifiques, *Grand N*, 81, 57-75.
- Hersant, M., Thomas, Y. (2008). Quels savoirs dans les problèmes pour chercher ? Le cas de problèmes d'optimisation au cycle 3, *Actes du XXXV<sup>e</sup> colloque des Professeurs et des Formateurs de Mathématiques chargés de la Formation des Maîtres*, Bombannes, 2-4 juin 2008.
- Hersant, M. (2008). "Problèmes pour chercher" : experience, possible and necessity in pupils' reasoning, *Symposium on the Occasion of the 100<sup>th</sup> Anniversary of ICMI*, Rome, 5-8 March 2008, URL <http://www.unige.ch/math/EnsMath/Rome2008/partWG1.html>
- Houdement, C. (2009). Une place pour les problèmes pour chercher, *Annales de didactique et de sciences cognitives*, IREM de Strasbourg, 14, 31-59.
- Orange, C. (2005). Une forme d'analyse des pratiques didactiques : l'analyse centrée sur les productions des élèves dans leur diversité, *L'analyse des pratiques en questions*, IUFM des Pays de La Loire, 43-50.
- Rogalski, J., Robert A. (2008), *La classe de mathématiques : activité des élèves et pratiques des enseignants*, coordonné par Vandebrouck, F., Octares éditions, 9- 68.
- Robert, A., Robinet, J. (1996), Prise en compte du méta en didactique des mathématiques, *Recherches en Didactiques des Mathématiques*, 16, n°2, 145-176.
- Sarrazy, B. (1997), Sens et situations : une mise en question de l'enseignement des stratégies métacognitives en mathématiques, *Recherches en Didactiques des Mathématiques*, 17, n°2, 135-166.

# THE NEED TO MAKE ‘BOUNDARY OBJECTS’ MEANINGFUL: A LEARNING OUTCOME FROM LESSON STUDY RESEARCH

Dolores Corcoran

St Patrick’s College, Dublin City University

*In this paper I examine how three student teachers used ‘ready prepared’ lesson plans as a teaching resource. These might be considered ‘boundary objects’ since they were developed in one practice, for possible use in others. The paper draws on a yearlong study of student teachers that opted to take a course called Learning to Teach Mathematics Using Lesson Study. The students and myself as course tutor and researcher sought to develop a research-based approach to designing and implementing problem-based mathematics lessons. Findings were that ‘boundary objects’ were not always used as intended and successful participation in the practice of lesson study focusing on children’s responses to the mathematics to be taught depended on negotiating shared meanings for specific curricular objectives and teaching materials.*

Key words: boundary objects; community of practice; lesson study; student teachers

## INTRODUCTION

As public interest in mathematics teaching and learning has increased over the past twenty years, so has the output of commercially produced textbooks and teaching resources increased. Of course, the availability of a comprehensive variety of research-based materials to assist in teaching mathematics is to be welcomed. However, the phenomenal communication explosion arising from global access to the internet has resulted in a proliferation of ‘off the-shelf’ mathematics teaching resources, some of which deserve to be used with caution. There is a propensity - particularly among student and novice teachers - to seek out novel and easily accessible teaching aids that they proceed to use in their classrooms without question. This tendency has been encouraged by the electronic publication by some government bodies of such plans (National Numeracy Strategy, 2006). Despite the perception that such materials should be ‘teacher-proof’, their adoption by some teachers can be problematic. Difficulties in implementation are sometimes attributed to deficits in teacher knowledge either of mathematics subject matter or of pedagogical content knowledge (Ball and Cohen, 1996). The study reported here sought to research how third year student teachers would use off-the-shelf resources while engaging in lesson study.

## LESSON STUDY

The practice of Japanese lesson study has been gaining acceptance, as a means of developing the teaching of mathematics, particularly in the USA (Fernandez, 2005), the UK and Ireland (Corcoran and Pepperell, 2011) since the practice was identified

by Stigler and Hiebert (1999) as a possible contributory factor in the mathematics achievement of Japanese students where lesson study is integral to the education system in primary and secondary schools and in teacher preparation (Isoda, 2007). In 2006, Lewis, Perry and Murata appealed for further research into how the demonstrable improvements in instruction brought about by participation in lesson study might be explained. They themselves offered two conjectures. One, that participation in lesson study generates “refined lesson plans”; that is, that the *products* - the “actionable artefacts” - created by the lesson study team are the most important contribution, and two, that participation in the *process* of lesson study enhances teachers’ pedagogical content knowledge, teachers’ commitment and community and teaching-learning resources (ibid., p. 287). These scholars appeared to favour conjecture two and suggested that conjecture one might be dismissed as ‘polishing the stone’ a fruitless exercise that did not warrant sustained pursuit. However, there is a resonance with conjecture one to be found in the “teaching-learning resources” element of improved pathways to mathematics teaching through lesson study. Further research among US teachers participating in lesson study to enhance the teaching of mathematics confirmed findings that conjecture two was the more valid *raison d’être* for lesson study where the emphasis on *process* rather than *product* was facilitated by four major changes: (1) increased use of reflection and feedback loops; (2) refinement of protocols and tools; (3) increased use of external knowledge sources; and (4) increased focus on student thinking (Perry and Lewis, 2009). Yet Japanese proponents of lesson study continue to publish detailed annotated lesson plans and regard a sustained and focused period of study of these and other resource materials as an essential element of lesson study (Takahashi, Watanabe and Yoshida, 2006). It appears from this study that sustained engagement in the process of collaborative negotiation of shared meanings is necessary to optimise use of lesson materials and resources.

## THE LESSON STUDY PROCESS

Lesson study is a cyclical approach to improving teaching through collaborative planning of a ‘research lesson’, which one person teaches, while observed by others who then review the lesson. It is premised on the Confucian saying that “seeing something once is better than hearing about it one hundred times” (Yoshida, 2005). Its ultimate purpose is to gain new ideas about teaching and learning based on a better understanding of children’s thinking so the observation of actual research lessons is at the core of the lesson study process. The lesson study cycle however encompasses much more than studying children’s responses while observing a research lesson. It requires time dedicated to intensive *kyozai kenkyu*, - a process in which teachers collaboratively investigate all aspects of the content to be taught and instructional materials available – and to *jyugyo kentuikai* – the post lesson review session (Takahashi, Watanabe, Yoshida and Wang-Iverson, 2005). In this paper, I will briefly outline the lesson study research process as practised by six student



teachers during two cycles. Then I will draw on data which indicate the different uses made of a particular ‘boundary object’ - a unit lesson plan adapted from the English National Numeracy Strategy on-line resources (NNS, 2006) - by Irish student teachers –Teresa and Finola - in the research lesson they each taught concurrently during lesson study cycle one. Next, I will refer briefly to a second boundary object, a recommended lesson plan (Burns, 1987), which was adopted and taught by one of the students, Ethna, in lesson study cycle two. Ethna’s treatment of this ‘ready made’ lesson will be discussed, together with findings from another lesson study group of US teachers who adopted the same lesson as a ‘research lesson’ (Fernandez, 2005). First, I outline what I mean by boundary objects.

## **THEORETICAL PERSPECTIVE**

The term “boundary object” was coined by Star and Griesemer (1989, p. 388) to describe:

objects which are both plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites ... They may be abstract or concrete ... The creation and management of boundary objects is key in developing and maintaining coherence across intersecting social worlds.

The notion of boundary objects has found particular resonance in computer science where “design continues in usage” (Rabardel and Bourmaud, 2003, p. 666), and “the conceptualisation of instruments [is] an activity distributed between designers and users” (Rabardel and Waern, 2003, p. 643). In such a context “interpretative flexibility” serves a recognised and useful purpose (Ruthven, Hennessy and Deaney, 2008). However, in mathematics education particularly, the fidelity or otherwise in interpretation of textbooks, designed in one community for use by another community has been problematised (Haggarty and Pepin; 2002). In researching communities of practice, Wenger (1998) uses the construct of ‘reification’ to explain the collaborative process of giving ‘thingness’ to inanimate or intangible aspects of practice, which imbues them with negotiable meaning within the practice and renders them portable. Because the products of reification within one practice may be transported to other practices they become “boundary objects”. By such a definition, the mathematics curriculum, mathematics textbooks and ancillary teaching resources all constitute boundary objects, the potential value and actual meaning of which have to be negotiated afresh by the community where they are being used. According to Wenger (*ibid.*) participation in the practice is synonymous with negotiating shared meanings for each object used within the practice. In this study, which presents an account of pre-service teachers engaging in lesson study, the interplay between participation in the practice of teaching problem-based mathematics lessons and the reification therein of certain understandings of mathematics and of mathematics teaching resulted in a growing respect for boundary objects and their potential value in teaching mathematics.

## THE STUDY

This research project – tier three of a larger study into the development of mathematical knowledge for teaching (Corcoran, 2008) – involved forming a community of practice dedicated to learning how to teach primary mathematics from a reform perspective. The differing ways in which a mathematics teachers' community of practice can develop has been reported at CERME 6 (Reinup, 2009) and the community of practice comprising six third year Bachelor of Education students and myself as course tutor and researcher seemed to exhibit the necessary qualities she outlines. We adopted Japanese lesson study as a model for teacher development and three cycles of lesson study were pursued over the course of one academic year. In each lesson study cycle, student teacher participants prepared two mathematics lessons collaboratively. For the research lessons the students divided into two groups and went to different schools. Then, one student taught a research lesson, observed by two other students who studied children's mathematical responses during the lesson. Each research lesson was video-recorded. I was present at one of the research lessons during each lesson study cycle. The research lesson experience was followed by a reflective meeting to analyse the lessons using the Knowledge Quartet (KQ) framework as a means of focusing on different dimensions of the mathematical content of the lesson (Rowland, Huckstep and Thwaites, 2005). Together with serving as an analytic framework for looking at mathematics teaching along its four defining dimensions of *foundation*, *transformation connection* and *contingency*, the language labels of the eighteen contributory codes for the KQ became part of the “shared repertoire of ways of doing things” (Wenger, 1998, pp. 82-84) within the community and helped to focus the student teachers' engagement with the problem-based mathematics teaching enterprise. The potential of lesson study as a means of bridging the gap between planning and pupil learning outcomes was discussed at the first meeting and the possibilities for developing teaching inherent in observing pupils' responses were mooted. I was party to the lesson study process both as researcher and as course tutor/knowledgeable other and was also engaged in negotiating meanings for these roles (Corcoran, 2011).

## THE RESEARCH LESSONS IN CYCLE ONE

During the initial preparation meeting of cycle one, the students opted to teach lessons from the Measures strand of the Irish primary curriculum. They chose the strand unit *weight*, because they perceived the teaching of ‘weight’ as difficult. The lesson study group had two three-hour meetings before this lesson was taught and various realistic contexts for teaching weight were discussed. The fact that 9-10 year old children ‘should’ and were already likely to know something about standard units of measure and reference to the curriculum objectives led students to adopt a somewhat ambiguous goal for the lesson; “that children would learn about the weight of a kilogramme”. After much exploratory discussion a context for engaging children in the mathematics of weight was suggested and discussed; the directive that

carry-on luggage on a Ryanair flight must not exceed 10kg. Students discussed their goals in approaching the lessons and the learning outcomes they hoped for their pupils, and how these could be recognised. Much discourse focused on the logistics of organising a lesson - time to be spent on pair work, group work and the time and place for teacher talk. These might be termed generic pedagogic concerns. The students organised their lesson plans according to the usual recommended 'introduction', 'development' and 'closure' phases. In one research lesson, Treasa introduced a problem: 'Would you be able to take your school bag on a Ryanair flight where there is a restriction of 10kg on carry-on luggage?' Children, in groups of six, were asked to compare and order their school bags by hefting them from hand to hand and to record their findings on a scale, - a straight line with six equally spaced marks - which Treasa drew on the board.

In the other research lesson, the problem was phrased as: 'How heavy is your schoolbag in relation to your body weight?' As a conclusion to the investigation Finola drew a scale, going from 0 to 3.5kg, on the board and invited children to tell her where to place their school bags on this scale according to their weight. As she talked, Finola began to draw a horizontal line across the middle of the board. This she calibrated with whole numbers from 0 to 3 then a mark midway between 3 and 4 to denote 3.5 and finally 4. She knew Tom's bag was heaviest at 4 kg and recorded his name over the four. She left intervals of less than 30cm in length between the whole numbers. If she had planned to record the weights of schoolbags and names of every one in the class it might have been better to have made the intervals much bigger, but Finola and her lesson study colleagues had not anticipated the complexity of the mathematical task the lesson involved. Finola's hand hovered indecisively between 0 and 1 on the scale and she finally marked a point half way between them as 100g with Sean's name underneath. Quickly she prompted the Joe, "... and yours was 500g?" At this stage Finola rubbed out the 100g and put in 500g, then rubbed out the 500g and put a child's name under the 1kg mark.

Child: One kilo.

Child: Mine weighed four grammes.

Finola: Four grammes? Really?

Finola decided to conclude the recording of school bag weights there, by saying:

Finola: Ye're all good. You've all got safe schoolbags. They're not going to damage your back at all are they? I don't think so. Now ...

While there were many similarities in the two research lessons, there were many differences in how they turned out. Some of the lesson study group's critical reflection on the two lessons focused on where each of the student teachers had modelled the recording of children's details on a scale. Treasa was confident in her teaching that calibrations should be equally spaced to represent the weights of the six school bags per group. She appeared to convey certainty and control in her handling

of the lesson but later expressed doubts about what she had done. In contrast to Treasa's lesson and a tighter approach to planning for teaching, Finola's lesson and approach appears loose, fluid and more dialectical. Finola also had a particular representation of a scale in mind and she was nonplussed when the messy, uncertainties of children's findings were presented. Obviously she was expecting neat, manageable round numbers of kilogrammes and half kilogrammes, instead children offered 900g, 500g 100g, 3g and 4g and her lesson could be challenged along both *connection* and *contingency* dimensions, of the KQ (Rowland, Huckstep and Thwaites, 2005) not because she didn't know the relationship of grammes to kilogrammes but because she couldn't think where they would fit on her scale.

### **Recording on Scales Revisited**

During the post-lesson reflective session on Treasa's research lesson of comparing weights of school bags, the lesson study group did the activity again themselves using their handbags and the purpose of the calibrations on the scale immediately became questionable. This ordering of bags could be matched to the scale but not to the marks where she as teacher had proposed putting the names of children who owned them. As the student teachers began to realise, this was an ordinal scale only and they had no way of knowing at what intervals the school bags should be marked. It was possible that two or more bags could be mapped to the same point on the scale. With regard to Finola's scale, the group explored the possibility of extending it by asking the children to imagine the space between 0 and 1 to be a much longer line and drawing that underneath and calibrating it in hundreds of grammes. It was an exciting discovery for these student teachers to think that they could do exactly the same with space between 0 and 100g and calibrate that space in tens of grammes. This notion of there being no such thing as an exact measure appeared new to the students and they began to think in terms of the arbitrariness of units and instruments, and the pedagogical purpose of the recommended use of a scale for recording weights.

### **ETHNA'S RESEARCH LESSON IN CYCLE TWO**

In lesson study cycle two, the complexity of the boundary object was greater although it was a much more explicit and detailed pre-determined lesson plan that was used. It was the teacher of the research lesson, Ethna, who had strongly advocated the use of a particularly detailed lesson plan she had chosen. Ethna's lesson study group of two other students agreed an already tested lesson script published by Marilyn Burns (1987) and took pains to set the scene and involve the children in hands-on activity manipulating the fraction pieces. Ethna and her two colleagues brought paper cut outs of cookies and had prepared worksheets for the children and enlarged versions to hang on the board as she explained the tasks. Children were to manually divide four, then five, three and seven paper 'cookies' among four children. The lesson concluded with Ethna simply telling children the

meaning of the symbols for a half and a quarter. She drew circles to represent 'wholes' on the board and shaded in the appropriate fractional pieces, finally asking the children to copy these representations into their copybooks.

Burns (1987, p. 40) describes the mathematical potential in the summarizing of the lesson and Ethna's careful preparation of worksheets ensured that she too had provided material for mathematical discussion with each of her three groups. But at the end of her lesson, Ethna chose to revert to an older, more traditional form of teaching which involved children in copying teacher's work and learning mathematics by looking, listening and remembering. Instead of following through with facilitating children's learning about fractions by discussing their solutions to the activities she had so carefully prepared Ethna closed the lesson by returning to routines of teaching mathematics she may have experienced as a child (Cooney, Shaely and Arvold (1998). It may be she felt 'safer' as teacher in this role or it may be that she felt the mathematics could be more 'safely' transmitted by this mode of teaching. When pushed for time Ethna 'forgot' to ask important *connection* questions but 'remembered' to rely on routines. It appeared that Ethna had persuaded her colleagues that this was a suitable lesson, who had explored and resourced its use without question, but when engaged in the actual teaching of the lesson she ignored the fact that by using Burns' lesson she had challenged the children to reason mathematically about fraction pieces, a task they had completed with considerable success. Instead, Ethna treated the Burns' lesson as a boundary object with little meaning for her practice beyond 'going through the paces'. She finished the research lesson with an impromptu and unplanned return to 'teaching by telling'. The pedagogical meaning of the boundary object had not yet been successfully negotiated nor integrated into the lesson study practice.

## DISCUSSION

A critical approach to the sparseness of the curriculum guidelines and the class textbooks in terms of important mathematics emerged from lesson study cycle one, and coupled with detailed observation of the children's talk and actions began to be transformative of practice. The idea of the scale had been imported from the National Numeracy Strategy Unit Plan 4, without heeding the significance of differing calibrations of scales on OHT4.3 (NNS, 2006). This document represented a boundary object (Wenger, 1998, p. 107) for the community, and could be said to represent other mathematical material to which the student teachers were exposed but which they did not necessarily make meaningful. Lesson study advocates recommend that teacher participants actually do the mathematics together that they propose to use in the research lesson. Our lesson study community had done so but in a cursory manner that had not accessed mathematics involved in sufficient depth. The NNS unit plan suggestion remained a boundary object to these student teachers until they had studied its use and negotiated its meanings in relation to the two applications of the problem of recording weights of children's school bags in their

actual research lessons. Much of the work of the lesson study community of practice over the following weeks was to critically align their own experiences and beliefs about mathematics teaching with the skeletal list of teaching objectives in the primary curriculum and to flesh out the resultant lessons from available resources in order to maximise the children's learning. In the light of these two lessons, it appears that suggesting contexts and providing resource materials is not support enough for student teachers. Alignment with the aims of the enterprise of learning to teach mathematics requires development of a deeper understanding of the mathematics to be taught, which in these instances were rooted in studying the actual teaching of research lessons.

The same 'off-the-shelf' lesson (Burns, 1987) which Ethna's group chose to teach as a research lesson has also been used by a lesson study group in the US where that group's learning from the process has been documented and studied in relation to two research questions: a) what opportunities to learn about mathematics for teaching does lesson study offer and b) to what extent can participants take advantage of these opportunities when they often bring limited subject matter knowledge to the situation (Fernandez, 2005). Unlike Treasa, Finola and Ethna, the teachers in Fernandez study were practising teachers with some years experience and her work shows evidence of opportunities for learning pedagogical content knowledge afforded to them by engaging with, discussing and adapting Burns lesson for teaching as a research lesson. She argues that the act of sustained and focused engagement in studying and interrogating the pedagogical implications of the lesson itself, despite their limited subject matter knowledge, acted "as a vehicle for teachers to learn about content in a way that directly feeds into their understanding of how best to teach this content and an incentive to learn more" (ibid., p. 282). Similarly, the student teachers in my study benefited from adopting, trialling, interrogating and adapting boundary objects and participation in this *process* resulted over time in meaningful learning of mathematics *in* and *for* teaching.

## CONCLUSION

I have chosen the three cases of Treasa, Finola and Ethna because the manner in which these student teachers (and practising teachers in the US) used the boundary objects in the examples cited above might be attributed to their limited knowledge of the potential for mathematical thinking inherent in the objects, but such a conclusion is not helpful, nor does it tell the full story. Subsequent collaborative work - during which the lesson study participants teased out the possible mathematical meanings, children's likely and actual responses to the mathematics and the teachers' explicit teaching and learning objectives - contributed in each case to increased mathematical subject matter knowledge, more focused and refined pedagogical content knowledge and a desire to go on learning through further engagement in lesson study. I conclude that engagement in a sustained, communal and situated, research-oriented approach to mathematics teaching is necessary if the uses of the many valuable resources

available are to be optimised. This finding has implications for the interpretation and implementation of curricula, of prescribed textbooks and ancillary resources and for the continuing professional development of teachers. It could give rise to “intelligent teachers using intelligent curricula intelligently” (Russell, 1997). By recognising ‘new’ resource materials as boundary objects, deeper understandings of mathematics, of curriculum and of teaching become a focus for negotiation of meaning as participants in lesson study engage with their initial interpretations, enact them, then renegotiate and refine them as “a work in progress” (Wenger, 1998, p.158). In this manner, learning to teach mathematics from a reform perspective involves a state of “constant becoming” (ibid, p. 154) and the practice of lesson study itself becomes the curriculum, where both conjectures concerning the power of lesson study to improve the teaching of mathematics complement each other.

## REFERENCES

- Ball, D., & Cohen, D. (1996). Reform by the book: What is – or might be – the role of curriculum materials in teacher learning and instructional reform. *Educational Researcher*, 25(9), 6–8, 14.
- Burns, M. (1987) Fractions with “Cookies,” in *A collection of math lessons: from grades 3 through 6*, Sausalito, CA: Math Solutions, pp. 37-43.
- Cooney, T., Shaely, B., & Arvold B. (1998). Conceptualising belief structures in PSMT, *Journal for Research in Mathematics Education*, 29 (3) pp. 306-333.
- Corcoran, D. (2008). Developing mathematical knowledge for teaching: a three-tiered study of Irish pre-service primary teachers. PhD thesis (University of Cambridge).
- Corcoran, D. & Pepperell, S. (2011). Learning to Teach Mathematics Using Lesson Study. In T. Rowland, K. Ruthven (Eds) *Mathematical Knowledge in Teaching*. NY: Springer.
- Corcoran, D. (2011). Learning from Lesson Study: Power Distribution in a Community of Practice. In L. Hart, A. Alston & A. Murata (Eds) *Lesson-study Research and Practice in Mathematics Education: Learning Together*. New York: Springer.
- Fernandez, C. (2005). Lesson Study: A Means for Elementary Teachers to Develop the Knowledge of Mathematics Needed for Reform-Minded Teaching? *Mathematical Thinking and Learning*, 7: 4, pp. 265-289.
- Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal*, 28(4), pp. 567–590.
- Isoda, M. (2007). Where did Lesson study Begin, and How Far Did it Come? In M. Isoda, M. Stephens, Y. Ohara and T. Miyakawa (Eds) *Japanese Lesson Study in Mathematics: Its Impact, Diversity and Potential for Educational Improvement*. Singapore: World Scientific, pp. 8-15.
- Lewis, C., Perry, R., & Murata, A. (2006). How Should Research Contribute to Instructional Improvement? The Case of Lesson Study, *Educational Researcher*, 35(3), pp.3-14.

## Working Group 17

- Perry, R., & Lewis, C. (2009). What is successful adaptation of lesson study in the US? *Journal of Educational Change*. Vol. 10 No. 4, pp. 365-391.
- National Numeracy Strategy (NNS) (2006) Unit Plans Year Four, Measures, available online at: [http://www.standards.dfes.gov.uk/primary/teachingresources/mathematics/nns\\_unit\\_plans/year4/Y\\$T%20Unit4measures/nns\\_unitplan050703y4t2unit4.pdf](http://www.standards.dfes.gov.uk/primary/teachingresources/mathematics/nns_unit_plans/year4/Y$T%20Unit4measures/nns_unitplan050703y4t2unit4.pdf). (Accessed 22 June 2008).
- Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers*, 15, pp.665–691.
- Rabardel, P., & Waern, Y. (2003). From artefact to instrument. *Interacting with Computers*, 15, pp. 641–645.
- Reinup, R. (2009). Developing of Mathematics Teachers' Community: five groups, five different ways. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello, (Eds) *Proceedings of the CERME 6*, Lyon, France: Institute National de Recherche Pedagogique. pp.1831-1840.
- Russell, S. (1997). The Role of Curriculum in Teacher Development. In S. Friel & G. Bright (Eds) *Reflecting on Our Work: NSF Teacher Enhancement in K-6 Mathematics*. Lanham, MD: University Press of America. pp. 247-254.
- Ruthven, K., Hennessy, S., & Deane, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education*. 51 (1), pp.297-317.
- Rowland, T., Huckstep, P., & Thwaites, A. (2005). 'Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi'. *Journal of Mathematics Teacher Education*, 8, pp. 255-281.
- Star, S., & Griesemer, J. (1989) Institutional Ecology, 'Translations' and Boundary Objects: Amateurs and Professionals in Berkeley's Museum of Vertebrate Zoology, 1907-39, *Social Studies of Science*, 19, pp. 387-420.
- Stigler J. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
- Takahashi, A., Watanabe, T., Yoshida, M., Wang-Iverson, P. (2005) Improving Content and Pedagogical Knowledge through Kyozaikenyu, in P. Wang-Iverson, M. Yoshida (Eds) *Building Our Understanding of Lesson Study*, Philadelphia, PA: Research for Better School. pp. 101-110.
- Takahashi, A., Watanabe, T. & Yoshida, M. (2006). Developing good mathematics teaching practice through lesson study: a U. S. perspective. Available online at: [http://e-archives.cried.tsukuba.ac.jp/data/doc/pdf/2007/11/APEC-HRD-03-2006-PhaseI\\_Progress\\_Report.pdf#page=141](http://e-archives.cried.tsukuba.ac.jp/data/doc/pdf/2007/11/APEC-HRD-03-2006-PhaseI_Progress_Report.pdf#page=141)
- Wenger, E. (1998). *Communities of Practice*. UK: Cambridge University Press.
- Yoshida, M. (2005). An Overview of Lesson Study. In P. Wang-Iverson and M. Yoshida (Eds) *Building Our Understanding of Lesson Study*. Philadelphia, PA: Research for Better Schools. pp. 3-14.



# **ANALYSIS OF THE TEACHER'S ROLE IN AN APPROACH TO ALGEBRA AS A TOOL FOR THINKING: PROBLEMS POINTED OUT DURING LABORATORIAL ACTIVITIES WITH PERSPECTIVE TEACHERS**

Annalisa Cusi, Nicolina A. Malara

Università di Modena e Reggio Emilia

*In this work we present an activity we carried out with perspective teachers (PTs) during a brief training course aimed at providing them with theoretical and methodological tools useful for the analysis of class processes concerning the development of reasoning through algebraic language. After an outline of the theoretical framework we introduced during the course, we will deal with the problem of the use of theory in the analysis of class processes, highlighting the difficulties faced by PTs.*

Keywords: teacher education; algebraic thinking; analysis of class processes; theoretical tools; reflection on practice.

## **TEACHERS' PROFESSIONAL DEVELOPMENT: THE SIDE OF THE ACTION IN THE CLASS**

For reshaping teachers professionalism several scholars stress the importance of a critical reflection by teachers on their own activity in the classroom (Mason 2002; Jaworski 2003). Mason, in particular, claims that the skill of consciously grasping things comes from constant practice, going beyond what happens in the classroom, and recommends the creation of suitable social practices in which teachers might talk-about and share their experience. Also Jaworski (2004) stresses the effectiveness of communities of inquiry, constituted by teachers and researchers, emphasizing how teachers' participation in these groups helps them develop their individual identity through reflective inquiry.

Our research model is framed in these conceptions, but it also stems from the Italian model of research for innovation, which units both an innovation in teaching and a promotion of teachers' professional development. According to this model the interaction between researchers and teachers plays an important role in the training processes in which teachers are involved before and during the experimentation of innovative didactical paths. The key-idea is that research and practice develop in a dialectical process: theoretical results produced by researchers are supported by teachers' practice and evolve through it (Malara&Zan 2002).

In our work, on the side of teachers' professional development, we study difficulties and effects of practices involving collective reflection, identifying categories of behaviour that may be productive for students' conscious learning (Cusi&Malara 2009). Our research experience with teachers made us aware of the difficulties they

meet in both designing and implementing socio-constructive teaching. Therefore, we set up and experimented instruments and methods to empower their way of managing whole-class discussions (Malara 2008). Our report takes place in this frame and, precisely, it concerns with the analysis of the role played by the teacher during activities aimed at a renewal in the teaching of algebra, in a perspective that will be outlined in the following paragraph.

## **THE DIDACTIC OF ALGEBRA: OUR MODEL AND THE ROLE OF THE TEACHER**

Our vision of the didactic of algebra has developed in a framework in which algebraic language, conceived as a fundamental tool in modelling and in the development of reasoning, is the key-element. Many research studies support an approach to the teaching of algebra aimed at helping students develop an awareness about the role played by algebraic language and the importance of studying it (see for instance Arcavi 1994, Arzarello et Al. 2001, Kieran 2004). Many of them stress the need of devoting more time to activities for which algebra is used as an effective tool but which are not exclusive to algebra (global/meta-level activities according to Kieran's distinction). Referring to the problems related to this particular approach to algebra, Arzarello *et Al.* (2001) stressed that an awareness of the power of algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* [1] and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of interpretative processes. Moreover, Boero (2001) argues that *anticipation* [2] is a key-element in producing thought through processes of transformation.

Since we agree with Wheeler's idea (1996) that activities of proof construction through algebraic language could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene", these kind of activities play a central role in our approach to the teaching of algebra. Therefore we planned and implemented an introductory path to proof in elementary number theory, to be inserted, in coordination with syntactical activities, in the math curriculum of classes of the first biennium of secondary school (grades 9-10). In our experimentations we were able to highlight the difficulties faced by teachers in making their students develop both the fundamental competences for the constructions of proofs through algebraic language and an awareness of the role played by algebraic language during these kind of activities. Therefore we decided to focus on the crucial role played by the teacher during the educational process. Our hypothesis is that teacher's attitudes and behaviours in the class are decisive in fostering (or inhibiting) students' construction of the competences which are necessary for the development of reasoning through algebraic language.

Our research framework about the teaching and learning processes is based on these three fundamental ideas: (1) thanks to the interaction with adults or with more expert peers, the students can activate internal learning processes which help them achieve a higher level of mental development (Vygotsky 1978); (2) one of the main aims of teaching should be fostering, through activities performed in social contexts, a real awareness of the learning process, focussing on the meaning of the actions which are performed in the class (Leont'ev 1977); (3) in order to foster a meaningful learning it is necessary to give students “the opportunity to observe, engage in, and invent or discover expert strategies in context” (model of the *cognitive apprenticeship*, Collins, Brown and Newman, 1989). Giving this opportunity is possible if the teacher is able to bring cognitive and metacognitive tacit processes into the open, trying to make thinking visible. We were inspired by the idea of a teacher who is able to activate in his/her students behavioural processes which are similar to the ones he/she activates in order to identify effective strategies for problem solving. Therefore we decided to focus on one of the possible roles that a teacher could play in the class: the role of model, which is particularly significant especially in the context of activities of proof construction through algebraic language, central in our project. Our research studies made us develop the idea of defining the theoretical construct of *teacher who poses him/herself as a model of aware and effective attitudes and behaviours for students* (Cusi&Malara 2009). In order to make the features of this construct clearly explicit, we will analyze a class discussion, proposed during a laboratorial activity which will be discussed in this report.

## THE ANALYSIS OF A CLASS DISCUSSION AND THE CONSTRUCT OF TEACHER AS A MODEL OF AWARE AND EFFECTIVE ATTITUDES AND BEHAVIOURS

The following discussion refers to the second phase of our introductory path to proof through algebraic language. The class (10 grade) has already faced activities of translation from verbal to algebraic language and vice-versa. The problem posed to students is the following: “*how can we justify that, if  $n$  is an odd number,  $n^2$  is an odd number too?*”. In this particular phase, the teacher aims at making students understand the limits of a verbal justification and at guiding them to a conscious use of algebraic language, showing them how to face these kind of problems. During the initial phase of the discussion, two students propose to formalize the hypothesis of this implication through the equality  $n=2x+1$ . The following excerpt refers to the course of the activity.

1. T [3]: (*addressing A, one of the two students who propose the formalization  $n=2x+1$* )  
How can we convince ourselves that if  $n=2x+1$ , then  $n^2$  is odd?
2. A: Because an odd number to the second power gives an odd number.
3. T: How can we see this?

## Working Group 17

4. A: Because odd times odd is odd!
5. T: So here there is the concept of multiplication. (*Addressing the class*) They say: if I multiply an odd times an odd, where do I find factor 2?
6. B: I don't find it.
7. T: So, it is odd.
8. T: And you, Z, how can you see it?
9. Z: Squaring an even number, you get an even. Adding 1 to an even, you get an odd.
10. T: Hold on. Here I read  $n^2$ . Why are you saying "I add 1"?
11. Z:  $2x+1$  squared gives an odd number because:  $2x$  squared is  $4x^2$ , then there is plus 1.
12. T:  $(2x)^2$  is  $4x^2$ .
13. T: You say  $(2x)^2=4x^2$ .  $(2x+1)^2$  is  $4x^2+1$  ?
14. Chorus : No !
15. T: Let's get back to what Z says. I can't say that  $(2x+1)^2$  is  $4x^2+1$ . But if I want to convince you that  $(2x+1)^2$  is odd, what can I do?
16. O: Let's solve it! (T writes  $(2x+1)^2=4x^2+4x+1$ )
17. T: Now there is "+1" ... This quantity here is the problem (*points to  $4x^2+4x$* ).
18. P: Let's make the total: we take out  $4x$ .
19. T: Do we really need to take out  $4x$ ?
20. O: It's enough to take out 2.
21. T: Why 2?
22. O: Because then we can highlight an even number, plus 1. (T writes  $2(2x^2+2x)+1$ )
23. Z: But  $4x^2+4x$  is the same as  $2(2x^2+2x)$ !
24. T: Yes, it's the same thing.
25. Z: Ah, I see why! Because taking out 2 you see you get an even. [4]

Let us analyze this discussion from the point of view of both the different roles played by T and the students-teacher interaction, trying to highlight: (1) weaknesses and strengths of the discussion, with reference to the application of conceptual

frames and anticipating thoughts and the coordination between different frames; (2) the role played by the teacher as a “stimulus” to foster an approach to algebra as a tool for thinking, and as a “model” and “guide” in the construction of reasoning. The excerpt can be broken down in three distinct moments: (1) phase of verbal argumentation (lines 1-7); (2) towards a formalization of the property (lines 8-14); (3) proof of the property and reflection upon the importance of choosing a certain representation (lines 15-25).

In the first phase of the discussion A enacts the frame “factorization of a number” to make explicit to the class the justification at the basis of her answer (line 4). Despite her attempt to formalise the answer, A only proposes a purely verbal argumentation. The teacher immediately sets herself in the same frame as the pupil and repeats the reasoning proposed by A to the rest of the class, pointing out the relationship between the fact that 2 is not in the factorisation of  $n$  and the fact that  $n^2$  is odd (lines 5 and 7). Through the metaphorical question “where do I find factor 2?”, T reminds that 2 is not a factor in the multiplicative representation of an odd number. Though T seems to pose him/herself only on the operative level, neglecting the metacognitive one (there seems to be a lack of an aloud reflection), this particular arithmetical knowledge was already well-established in the class, therefore it can be an implicit assumption in the development of reasoning. T’s third statement (line 7), which reinforces A’s assertion, seems to block a discussion about the need of a formal proof of the property. Actually, because of the particular moment in the class activity (recollection of students’ different point of views), T refrains from intervening in order to pose him/herself as a listener. This fact becomes clear when T invites an other student (Z) to express her reasoning (line 8).

Z’s intervention (line 9) is immediately taken by T as an opportunity to introduce the class to a justification of the property based on algebraic formalization. Z, in fact, refers to the additive representation of odd numbers to justify her answer, trying to co-ordinate the frames “even/odd” and “polynomials” while she is trying to ‘mentally’ manipulate the expression  $(2x+1)^2$ . Although Z activates a good anticipating thought (she grasps the idea that the objective is to transform the expression until it gets to the form “an even number plus 1”), she faces some difficulties at the level of syntactical transformations, probably because she tries to proceed only verbally. This is a moment in which T must try to foster in students an harmonic balance between semantic and syntactic aspects. When Z makes an evident mistake in calculating the square of a binomial, the teacher poses him/herself as a reflective guide, echoing the student in the form of a question asked to the whole class (lines 12 and 13). Once he/she has amended Z’s mistake, T underlines the objective of the syntactic manipulations carried out (line 15) and asks the class to suggest him/her how to proceed. In this case, T is playing a double role: investigating subject, putting to the class the question of researching a path suitable to reach the prearranged objectives, and activator of anticipating thoughts, clarifying

the aim of the activity in order to foster the activation of the “even-odd” frame and the research of the correct syntactical treatment to be performed.

Following O (line 16), T gets to construct the expression  $4x^2+4x+1$ . At this point, the teacher decides to guide the activity, playing the role of an investigating subject. She actually remarks that “+1”, Z had mentioned, is in the determined expression, but she points at the remaining binomial  $4x^2+4x$  as a “problem to be solved” (line 17). In this way, she lets the class guide the activity, although she remains the point of reference for the discussion. Through this technique, the teacher again acts as an activator of anticipating thoughts. After P shows he has not enacted a correct anticipating thought (line 18), T echoes P’s proposal, sending it back to the class as a question (line 19). At this point, O enacts the correct anticipating thought, suggesting that 2 might be taken out (line 20). T asks her to justify her idea, so that she can make what she has activated explicit to the whole class. The comment by Z (line 23) shows that the pupil has not interpreted the objective of the manipulation within the frame “even-odd”: she actually shows she has not understood the sense of taking out a factor 2 from  $4x^2$  and  $4x$ . T decides to echo her (line 24), simply repeating that the pupil’s statement ( $4x^2+4x$  is the same as  $2(2x^2+2x)$ ) is right. At that moment Z realises that taking out 2 is a way to make explicit the fact that the expression  $4x^2+4x$  is even (line 25). It is important to stress that T always tries not to impose the moments devoted to reflection: her way of repeating students’ assertion, also if they are erroneous, and of sending back students’ question to the whole class is a clear methodology aimed at stimulating students’ development of reflective attitudes and metacognitive acts.

This analysis can help the reader clarify some of the definitory elements of the construct of *teacher who poses him/herself as a model of aware and effective attitudes and behaviours* (TMAEAB). This kind of teacher must: (a) be able to play the role of an *investigating subject*, stimulating in students an attitude of research on the problem being studied, and acting as an *integral part* of the class in the research work being activated; (b) be able to play the role of a *practical/strategic guide*, *sharing* (rather than transmitting) knowledge with students, and of a *reflective guide in identifying effective practical/strategic models* during class activities; (c) be aware of his/her responsibility in maintaining a *harmonized balance between semantic and syntactic aspects* during the collective construction of thought processes through algebraic language; (d) try to *stimulate and provoke* the enactment of fundamental skills for the development of thought processes through algebraic language, playing the role of both an *activator of interpretative processes* and an *activator of anticipating thoughts*; (e) stimulate and provoke *meta-level attitudes*, acting both as an *activator of reflective attitudes* and as an *activator of meta-cognitive acts*.

## LABORATORIAL ACTIVITIES WITH PERSPECTIVE TEACHERS

The activities we refer to in this paper involved a group of 10 new graduates

particularly motivated, who still have not worked in school and, waiting for new rules about teacher training courses from the Ministry of Education, expressly have asked to the Science Faculty to organize a brief course propaedeutic to teacher training. As we stressed previously, the training paths for teachers that we usually propose are characterized by a constant dialectic between theoretical aspects and didactical implementation. In the moment we had to work with PTs who still have not had the opportunity to enter in the classes, neither as observers, we faced the problem of a lack in this dialectical relationship between theory and practice. Therefore our main aim became that of giving them theoretical and methodological tools to learn how to interpret their future actions in the classes. The methodology we adopted during this course is strictly connected with this particular situation, but it is in tune with the framework we outlined before. In fact, the activities we performed with PTs can be considered preparatory to the critical reflection they will have to do when, as teachers, they will have to analytically examine their actions to improve the effects of their practice. The course (20 hours) was subdivided into 5 sessions. The activities started with a session devoted to the presentation of: (1) our theoretical framework for the didactic of algebra; (2) the different activities about this theme we realized in the classes, highlighting in particular the experimental path for the construction of proofs through algebraic language (carried out with 9-10 grades students). During the following sessions PTs were involved (sometimes individually, sometimes in groups) in activities of reflection about class practices: we proposed them to analyse excerpts of both class and small groups discussions (produced during our experimentations). Every activity of reflection was followed by a collective discussion aimed both at activating a comparison between PTs and at introducing, not in a transmissive way, theoretical issues about methodological aspects of teaching-learning processes, in order to gradually outline our framework.

In order to investigate the incidence of PTs' learning of theoretical aspects in their capability of analyzing the role of the teacher during class processes involving the development of reasoning through algebraic language and to highlight the problematical aspects connected with their use of theory in performing this kind of analysis, we devote the last part of this paper on a particular activity of reflection, individually faced by PTs, characterized by the analysis of the previous discussion (see paragraph 3). In the following we will present the task proposed to PTs and our qualitative analysis of their answers. The aim of our analysis is to highlight the difficulties faced PTs: (a) in referring to the theoretical constructs to analyse T's attitudes and behaviours during the discussion and (b) in contextualizing T's actions in tune with the particular didactical moment that the class is living.

## **PROBLEMATICAL ASPECTS HIGHLIGHTED IN PTS' REFLECTIONS**

We asked PTs to highlight: (1) weaknesses and strengths of the discussion, with reference to the activation of conceptual frames, coordination between different

frames and activation of anticipating thoughts; (2) the moments in which T plays the role of a TMAEAB; (3) the moments in which T's approach differs from the approach which characterizes a TMAEAB; (4) the (positive and/or negative) effects of T's work on students during the discussion. The analysis carried out by PTs are prevalently line-by-line. Only one PT was able to rationalize local observations in an objective and argued frame about T's attitudes and behaviours. In this paragraph we will focus, in particular, on the problematical aspects highlighted by our study of PTs' protocols, referring to three main aspects: (1) appropriation of the theoretical constructs of reference and their use in performing the analysis of the discussion; (2) interpretation of T's actions with reference to the context (didactical project, particular didactical moment); (3) highlighting of the interrelation between T's behaviours and students' behaviours.

In regard to (1), the examined protocols can be subdivided, referring in particular to the TMAEAB construct, into the following four categories: (a) the PT has interiorized the theoretical construct and he/she is able to refer to it in a pertinent way; (b) the PT recognizes, within the class process, typical components of a TMAEAB, but he/she is not able to identify the specific actions which characterize the highlighted components; (c) the PT only partially recognizes the components which characterize a TMAEAB and he/she does not conduct a punctual analysis of the discussion, making sometimes improper references to theoretical constructs; (d) the PT proposes a naïve analysis of the class process, without referring to the theoretical constructs or referring to them in an improper way.

Most of the protocols belong to the categories (b) and (c): few PTs were able to always correctly refer to the theoretical aspects and appropriately use the specific terminology. Because of space limitations, we present here only some reflections belonging to categories (c) and (d) because they better reveal the difficulties met by PTs in interiorizing the theoretical constructs. For example, many PTs did not realize that the rhetorical question proposed by T in line 5 is aimed at making A's reasoning (line 4) explicit in order to stimulate a moment of reflection. R, for example, observes: *"When T asks 'where do I find factor 2?', he/she plays the role of a prompter, inhibiting the anticipating thoughts that could have arisen from students' reflections"*. This reflection testifies a widespread approach used by PTs. They, in fact, often do not analyze T's actions in the context, with reference to what the class already knows and to the particular moment in the didactical path. Their inability of contextualizing the discussion makes PTs interpret as a 'didactical mistake' the fact that T considers obvious that 2 is a factor in the multiplicative representation of an even number. Indeed T's attitude is understandable: aiming at focussing students' attention on the way of developing reasoning through algebraic language, he/she prefers not to re-propose, as a problem, syntactic aspects that most of the students already control.

A similar observation can be done referring to T's choice of quickly performing,



without involving students, the syntactic transformation in line 16. The inability of contextualizing this action makes, for example, M assert: *“The teacher is the one who correctly writes the equality  $(2x^2+1)^2 = \dots$ ; again he/she is does not make his/her students reflect on the meaning of algebraic expressions and on the equivalence between the expressions at the two sides of this equality”*.

As regards to (1), other difficulties are related to a lack in recognizing, referring to the particular context, some typical features of a TMAEAB. Some PTs, for example, consider as negative T's approach in line 15 because they are not able to recognize that he/she is playing the role of investigating subject. R, for example, states: *“T's attitude of ‘I want to convince you’ clashes against the meaning of the proving activity”*. An other assertion proposed by T that was not correctly interpreted by many PTs was the one in line 17. Instead of recognizing that T is trying to play the double role of investigating subject and activator of anticipating thoughts, some Pts declare that T is posing him/herself as a mere prompter: *“T suggests the frame to refer to and the quantity on which they have to operate. Therefore he/she is not playing the role of an activator of anticipating thoughts”*(S).

Our study has also pointed out that PTs sometimes propose conflicting interpretations of some micro-actions performed by T. Referring to line 19, for example, we highlighted a contraposition between comments that consider T's approach positive, stressing that T aims at activating a moment of reflection on the meaning of the syntactic transformation to be carried out, and comments that look at T's approach as completely negative, since *“T immediately interrupts the student's proposal”* (D). PTs who propose negative comments to line 19 do not understand that students' bewilderment can be justified because this is one of the first proving activities that the class is facing.

Referring finally to (3), we can observe that only few PTs have tried to highlight the effects of T's action on his/her students. In particular, those who tried to highlight an interrelation between T's actions and students' actions only propose very concise global comments.

## BRIEF FINAL REMARKS

The comments we presented in the previous paragraph highlight the difficulties faced by PTs both in using theoretical tools to analyse class processes and, in particular, in proposing an analysis of T's actions which takes the particular didactical moment that the class is living into account. Many PTs, in fact, showed to be non completely aware that conducting a balanced lesson on this topic means making students autonomously operate, but also guiding them toward a meaningful learning of how to 'reason' through algebraic language. We think that the limitation of the period of work with these PTs can justify their incomplete assimilation of the theoretical constructs they studied. At the same time, the complete lack of teaching experiences in their careers can be considered an important reason of their difficulties in correctly

contextualizing T's actions in tune with the theoretical constructs of reference. However we believe that 'clashing' with these kind of problems could represent for PTs the beginning of a process which could lead to a real professional development. This is testified by the fact that, beyond our evaluation of their protocols, during the following discussions with PTs, they turned out to be very interested in these kind of studies and to really need to go on with the analysis of class processes.

## NOTES

1. Conceptual frame is defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem".
2. Anticipating is defined as "imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process".
3. Here, A, B, O, Z, P and G indicate 5 pupils involved in the discussion while T is for the teacher. Chorus means that the sentence was uttered by a group of pupils in the class.
4. The discussion ends with a further moment of reflection upon the importance of choosing a representation.

## REFERENCES

- Arcavi, A. (1994). Symbol sense: informal sense-making in formal mathematics. *For the Learning of Mathematics*, 14(3), 24–35.
- Arzarello, F., Bazzini, L., & Chiappini, G. (2001). A model for analyzing algebraic thinking. In R. Sutherland R. et Al. (Eds.), *Perspectives on School Algebra* (pp. 61-81). Kluwer: The Netherlands.
- Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving. In R. Sutherland et Al. (Eds.), *Perspectives on School Algebra* (pp. 99-119). Kluwer: The Netherlands.
- Collins, A., Brown, J.S. & Newman, S.E. (1989). Cognitive Apprenticeship: Teaching the Crafts of Reading, Writing and Mathematics! In L.B. Resnick (Ed.), *Knowing, Learning, and Instruction: Essays in Honour of Robert Glaser* (pp. 453-494). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Cusi, A. and Malara, N.A. (2009). The Role of the Teacher in developing Proof Activities by means of Algebraic Language. In M. Tzekaki et Al., *Proceedings of PME 33*, vol.2 (pp. 361-368). Thessaloniki (Greece).
- Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54, 249-282.
- Jaworski, B. (2004). Grappling with Complexity: Co-learning in Inquiry Communities in Mathematics Teaching Development. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of PME 28*, vol.1 (pp. 17-36). Bergen, Norway.

### Working Group 17

- Kieran, C. (2004). The core of algebra: reflections on its main activities. In Stacey, K. et Al., *The future of the teaching and learning of algebra. The 12th ICMI study* (pp. 21-34). Kluwer: The Netherlands.
- Malara, N.A. (2008). Methods and tools to promote a socio-constructive approach to mathematics teaching in teachers. In Czarnocha, B. (Ed.), *Handbook of Teaching Research*. University of Rzeszów press, Rzeszów, 89-102.
- Malara, N.A. and Zan, R. (2002). The Problematic Relationship between Theory and Practice. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 553-580). LEA, Mahwah, NJ.
- Mason, J. (2002). *Researching Your Own Practice: the Discipline of Noticing*. The Falmer Press: London.
- Leont'ev, A.N. (1977). *Activity, Consciousness and Personality*. Englewood Cliffs: Prentice-Hall.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher mental processes*. Cambridge, MA: Harvard University Press.
- Wheeler, D. (1996). Backwards and Forwards: Reflections on different approaches to algebra. In Bednarz, N. & Al. (Eds), *Approaches to algebra. Perspectives for research and teaching* (pp. 317-325). Netherlands: Kluwer Publishers.

# THE IMPACT OF TEACHING MENTAL CALCULATION STRATEGIES TO PRIMARY PGCE STUDENTS

Sue Davis

University of Leicester

*Following a previous study involving five primary Post Graduate Certificate of Education (PGCE) students, which showed that there was short term impact resulting from an intervention (Davis, 2009), I revisited three of these ex-students to consider whether there had been any continued impact in the intervening two years. Using a case study approach I will show how these teachers, with very different experiences in school, have each built upon the skills, techniques and knowledge learned in the initial intervention sessions. I will also consider how this supports or contradicts the findings of other researchers studying the impact of Teacher Education.*

Key words: Impact, Mental calculation, Primary, PGCE, Teacher.

## INTRODUCTION

One of my strongest memories from primary school is that of completing a mental mathematics test every Friday afternoon, in preparation for the 11+ examination (selection for the local grammar school) during the final year. We were taught no strategies, other than learning multiplication tables by rote. Questions were fired at us and we had to calculate the answers ‘in our heads’ as the Dutch call this type of calculation (Thompson, 1999). For me, this merely meant picturing the formal written calculation and carrying out the formal method, working as quickly as possible, jotting down first the unit part of the answer, then the tens etc. It was not until many years later, as I trained to teach, that I realised that there was more to ‘mental calculations’ than this. The fact that I could choose from a range of strategies and use my knowledge of the number system to calculate much more efficiently was enlightening. Shortly after this, mental calculation became an important feature of the English primary mathematics curriculum when the National Numeracy Strategy was introduced (DfEE, 1999) and strategies were expected to be explicitly taught and regularly rehearsed and discussed. The recent ‘Williams Review’ (DCSF, 2008) has called for a refocusing on oral and mental mathematics in order to particularly benefit under-attaining groups of children.

Two years ago I conducted a study with five Initial Teacher Education (ITE) students (Davis, 2009) to identify what mental calculation strategies they possessed, how confident they were to teach mental strategies to children and whether my teaching of a range of strategies could increase their confidence of teaching this topic. This study involved an intervention which included teaching them a range of mental calculation strategies, offering opportunities for the students to compare these to their own mental methods for different calculations (e.g.  $483 + 89$ ;  $58 - 34$ ) across

all four operations (+, -,  $\times$ ,  $\div$ ). This taught session was audio recorded (and later transcribed) and questionnaires were completed before and after. Five weeks later, following the students' final weeks of Teaching Experience, a further questionnaire and short intervention session (based again on activities involving discussion of calculation strategies) provided further data. The results showed that whilst these students knew only a limited range of strategies before my intervention, they became much more confident afterwards and for two of them there was a clear impact on their teaching in school. These students from my study have now successfully completed their first two years of teaching and three of them were willing to allow me to investigate whether there had been any longer term effect of my intervention.

## LITERATURE REVIEW

I have considered the literature surrounding the impact of both ITE and Professional Development (PD) courses for teachers as both have the same aim: to develop the teaching skills of the participants and therefore to enhance the learning of their pupils.

A key aspect of successful professional development is the style or nature of the activities. Both Darling-Hammond (1999) and Elliott et al. (2009) found that discussion amongst participants is central to successful PD which is, of course, the basis of many leading theories of education, particularly that of social constructionism. A successful teacher of mathematics has to develop a deep and flexible understanding of subject knowledge and be able to make connections with other aspects of the subject (Darling-Hammond, 1999). Elliott et al., (2009) develop the idea further by specifying that it is Specialised Content Knowledge which has a strong impact on teaching skills. This literature seems to suggest that my intervention should have had a positive impact in the classroom.

However, Larose, Grenon, Morin and Hasni (2009) have discovered that practice observed by students in school often overrides the information given during the university part of ITE. Students have also been found to lose their enthusiasm for their own learning during their Newly Qualified Teacher (NQT) year and tend to focus on behaviour management rather than subject knowledge (Haggarty, Postlethwaite, Diment, Ellins, 2009). Students in their first year of teaching do not appear to gain from the type of PD experiences that Darling-Hammond and Elliott et al. advocate.

The other consideration for my research is whether mental calculation strategies should be taught at all. Thompson (1999; 2000) advocates the teaching of some frequently used strategies for addition and subtraction so that children can develop flexible methods of calculation. Murphy, however, found that explicitly teaching an addition strategy did not mean that the children used this when faced with calculations that would benefit from its use (2004). Torbeyns, De Smedt, Stassens, Ghesquière and Verschaffel's research supports these findings (2009). They found

that there appears to be no difference in whether a particular subtraction strategy is used, whether or not children have been taught it. Threlfal goes a step further, by arguing that there is no need to teach particular strategies, but instead it is essential to develop a stronger understanding of the number system (2002). If we are to enable children to use flexibility in their approach to mental calculation I do not see these as mutually exclusive. Teaching some strategies may indeed support the understanding of the number system, and vice versa.

## METHODOLOGY

The main question I wished to answer was: Is there any evidence of long term impact of the teaching of mental calculation strategies to primary Post Graduate Certificate of Education (PGCE) students? These are students who train to teach on a one year programme having already completed an undergraduate degree in any discipline, so may not have received any mathematics teaching for at least five years.

In addition, there were also a number of subsidiary questions which would support me in answering the main question: Did the students teach any mental calculation strategies during their NQT year? Did the students only teach mental calculation strategies when it was already on their school's planning? Is there any evidence that the children are using the strategies taught? Has the confidence of the students been affected by their teaching of mental strategies?

Each teacher held a particular interest for me and I wished to gain a 'rich picture with 'thick description'' (Thomas 2009, p.116) of them. Therefore the flexibility of a case study approach enabled me to study all three teachers in detail. I wished to develop a detailed 'holistic understanding' (Baxter & Jack 2008, p.554) of these new teachers' experiences throughout their first years of teaching. I had no hypothesis at this stage of what the results of gathering this data would show, which also fits in with the idea of a case study approach.

As with any case study, there is no possibility of drawing any generalisations from this research. This does not mean, though, that I cannot use any information that I gather to improve my own practice. As Baxter and Jack report, a case study 'can inform professional practice' (2008, p.544). A particular strength of this research was to build upon the close collaboration and relationship that exists between me as the researcher and the teachers I was researching. In some methodologies this relationship might be seen as a disadvantage, and might be considered a barrier to the authenticity of the data collected. Using this case study approach, though, it can be seen as an advantage as this is the very reason that I am selecting this particular sample. I was very careful to ensure that the teachers understood I wanted to know what really happened, rather than them hoping not to offend me by telling me that my intervention had an impact on their teaching if it did not.

In order to gain as much information as possible for my 'finished story' (Thomas 2009, p.115) and to corroborate the evidence I collected, I used a range of data

collection tools. Using a variety of data sources ensured that I was not exploring ‘through one lens, but rather a variety of lenses’ (Baxter & Jack 2008, p.544) and this is where the case study approach enabled me to build up a particular picture of the teachers which might not have been possible with other methodologies.

I began my study by designing questionnaires. The questions were ‘precise and non-leading, that neither assume nor presume’ (Castle 2010, p.67) and were a combination of closed questions and open questions to allow for comments about their teaching of mental calculations during their first years of teaching (Cohen, Manion & Morrison, 2000). I included some questions about confidence and competence in teaching mental mathematics strategies from the original questionnaires, completed by this group of students nearly two years ago, in order that I could make comparisons. For this reason I used a five-point Likert scale, as that was used in the original study.

In order to establish whether Haggarty et al.’s (2009) findings regarding a strong focus on behaviour management overriding other aspects of teaching were true for my sample, I also included a section where they ordered a range of five statements about areas they felt they developed most during their first year of teaching. These statements included both behaviour management and subject knowledge. Finally, the questionnaires included three open ended questions about the impact of the PGCE year as a whole on their development in their first year of teaching.

Following an initial analysis of the questionnaires I conducted semi-structured interviews with each teacher, which were recorded and transcribed. In particular, this method of data collection enabled me to prompt the teachers to reflect on any impact during their NQT year of my mental calculation strategy intervention and provided an opportunity for them to consider whether any teaching of strategies had any impact on the mental maths skills of the children. Planning and assessment records to support these opinions were collected where available.

## **CASE STUDIES**

### **Ellie**

‘Ellie’ was a mature student who taught across the primary age group in one school as a regular ‘supply’ teacher for a year, before spending two terms in a Year 5/6 class. She has recently returned to the first school to resume regular supply cover, again covering Years 1-6 (ages 5-11).

The data shows that Ellie has taught a range of mental mathematics strategies to children across this age group. In particular, teaching doubling and halving with all age groups was clearly very important to her, as this was mentioned for the first time less than ninety seconds into the interview. Although she initially taught this strategy to Key Stage 1 children (ages 5-7) Ellie also teaches it regularly to 9 to 11 year olds,

focusing on how this can support a range of other calculations; for example calculating percentages, equivalent fractions or simply using it to divide by 4 easily.

Ellie also teaches rounding and approximating to Year 3 and 4 pupils, to support their mental mathematics, and using Interactive Teaching Programmes (ITPs) (DfE, 2010) and physical resources, such as bead strings, to support the children in understanding and visualising the number system.

Ellie is very keen to encourage children to discuss their mental methods whenever appropriate and this was a topic she frequently returned to during the interview. She believes this is a direct result of my intervention sessions. Her passion for this crucial element of teaching mental strategies can be seen in the following extract:

Ellie: I often, I do, I do ask them about methods a lot now because I remember sitting with you and we all had so many different methods and it might be that there's children in that classroom that just have not worked out a really easy method but by swapping methods, sort of do it this way, have you got a different way so you've counted on, you've rounded, and y'know, added bits on and taken bits off.....

Later in the interview Ellie returned to this topic when she reflected on the fact that until her PGCE course she generally had just one way of tackling any set mathematical problem. The structure of the intervention enabled her to realise that there were other ways of approaching mental calculation and this realisation has altered her own way of teaching, ensuring she teaches a range of methods. She says she has 'a better awareness, which is probably why I now ask the children, 'how did you do that?' and I think oh yeah, 'cause \*\*\* (names one of the students) did it differently to the way I did it.'

In contrast to Haggarty et al. (2009), who found that Newly Qualified Teachers (NQTs) did not develop subject knowledge but focused merely on classroom and behaviour management, Ellie had made a conscious effort to continue to develop her subject knowledge throughout her first two years of teaching, attending three mathematics courses in this time. Indeed, improving subject knowledge in core subjects was the area she had most developed in her NQT year.

Whilst Larose et al. (2009) found that practice in school overrode the practice learned during the university part of ITE, it is clear that Ellie had reflected on all of her university based mathematics sessions, including my intervention sessions. Her maths file, containing notes from all the taught sessions, has been in constant use throughout the intervening two years. As she says, 'I know that I have dipped in and out of that file a lot, for resources and, and for bits and pieces ..... no doubt about it.'

Ellie agrees with the findings of Murphy (2004) and Torbeyns et al. (2009), that learning a range of strategies does not necessarily mean that children will use them. However, she also believes that revisiting and rehearsing constantly is a crucial part of children's learning and she found herself going, 'over it and over it and over it'.



She has found that if children are exposed to these strategies regularly, and are encouraged to talk about them regularly, they do begin to select an appropriate strategy more independently.

### **Donna**

‘Donna’ completed her NQT year in a year 1 class (ages 5-6) and has spent the subsequent year teaching Foundation Stage, Year 1 and Year 2 (ages 4-7) children in the same school, where she is now employed to cover other teachers’ classes during their time for other commitments. This has meant that Donna has been in the unusual position of teaching some of the children this year who she taught throughout last year, which has given her the opportunity to see how the children have developed their skills and to judge the longer term impact of her teaching.

Donna has taught and practised a range of strategies to support mental calculation, the first that she mentioned (after only 53 seconds) being doubling and halving. Like Ellie, she was able to explain why she believes this to be an important skill, and she began this with Year 1 children by using counters and other resources to support their learning. As she says, ‘in Key Stage 1 its probably one of the first .... non-counting operations .... you do mentally’. Some of these very young children were already able to recall doubles up to 20 by the age of 6 and this year she has seen how they are able to use these skills to support other aspects of mental mathematics. Many others were still learning one to one correspondence, though, and basic counting skills, so by no means were all children able to calculate mentally even in Year 2.

It is interesting to note that Donna encourages the development of mathematical skills across many other areas of the curriculum, and sees this as an important part of their learning. Indeed, this approach is advocated by many leading educationalists, and forms one of the main recommendations of the recent ‘Cambridge Primary Review’ (Alexander, 2010). Donna particularly focuses on these skills in Physical Education (PE) lessons:

Donna: But in PE and in games we used a lot of .. things like counting in twos to score, rather than always counting one point, y’know for getting one thing back in a race or something. Counting in different numbers or thinking about .. playing games with beanbags. How many more beanbags do you need to make 10? ---- and having teams racing against each other.... and I found PE was a really good way of, .. kind of incorporating maths...

It was clear from interviewing Donna that she was working in a team who were all keen to encourage children to develop mental skills, combined with a strong level of independence of thought. This made me wonder whether my two intervention sessions had had any impact on her teaching, or whether, as suggested by Larose et al. (2009), practice observed by students (and presumably NQTs) in school overrides

the information given during the university part of teacher training. Donna acknowledged that she has learned a lot from these new colleagues, and, for that matter, she has always used doubling to support her mathematics since her own school days, but she makes it very clear that despite the two sessions lasting less than two hours in total, they have had a significant impact on her teaching:

Donna: DEFINITELY, definitely ..... probably even more so this year because I've been working more with year 2....trying to get the children to explain to me the different strategies and then explain why they would choose a particular one and why it was better, and picking out the features. It really stuck with me that when we were talking, how useful that was, so I've done that quite a lot with the children in the class. Definitely.

Tutor: And was that something you might have picked up anyway on your TE placement?

Donna: I might have done but I think that really rammed it home, that..., how important it was for the children to find out for themselves which ones work for them and which ones are..are more efficient than, than others, erm and that unless you get them to think about what they're doing and why they're doing it they might not, you know, I think I was just more overt with it than maybe I would have been anyway.

Donna indicated that she had received no professional development in mathematics since completing her PGCE, and I wondered if this might support Haggarty et al.'s view (2009), that behaviour management becomes the main focus for NQTs, with a lack of enthusiasm for development of their own subject knowledge. From the initial questionnaire it was clear that Donna felt that behaviour management was the skill she developed most in her NQT year, but further discussion during the interview revealed that this was because she had two children displaying such challenging behaviour that both the children and Donna received regular specialist support from the local authority throughout the year. This did not mean that Donna had a lack of enthusiasm for developing her own subject knowledge; in fact, she attended three courses and an NQT conference, all of which heavily focused on subject knowledge, albeit not mathematical subject knowledge. Donna would really appreciate the opportunity to receive further training in mathematics, and made it clear that it was not through choice that the courses she attended had different foci.

From her responses in the questionnaire it is clear that Donna is very confident about her own knowledge of mental calculation strategies and she has increased her confidence in her own ability to teach these strategies to children since she was a PGCE student. This was supported by her discussion of the impact of her teaching on

the children, which was particularly evident when working with one group of children for both years of her teaching career.

### **Belinda**

‘Belinda’ has spent her first two years in a Foundation Stage class (ages 4-5), although the PGCE course she completed was a Key Stage 1 and Key Stage 2 course (ages 5-11). Children are divided by achievement into maths groups and she teaches the higher achievers.

Once again, teaching doubles was one of the first strategies mentioned by Belinda (less than two minutes into the interview); although despite mentioning this strategy at various points during the interview it is interesting to note that she did not mention the corresponding halves at all. However, the children learn doubles up to 20, using resources such as counters, pencil marks or fingers to initially support this, before they develop as ‘known facts’.

Belinda has learned an enormous amount from her colleagues, and readily admits that most of her mathematics teaching is based on their advice and experience rather than on her learning from her PGCE course, which, in contrast to Donna and Ellie, supports the findings of Larose et al (2009). She has not looked at her mathematics file since the day she left the course. Having said this, she still believes that the two intervention sessions have had some impact on her first years of teaching. In particular, these sessions made her realise that ‘everybody visualises things differently and they see things very differently and they have to use different ways to work it out’. She thinks this, combined with her colleagues’ advice, ensures that she offers children a range of choices throughout their mathematical learning, despite this not being a strategy included in the published scheme that the school generally work from.

When considering developing subject knowledge compared to developing behaviour management skills, Belinda has attended a number of courses during her first two years of teaching and these have all been based on subject knowledge. Bearing in mind that she is teaching in an age phase for which she was not trained, this is probably unsurprising. The courses have taught her knowledge of the Foundation Stage Curriculum and ways to assess children of this age. Behaviour management is something which Belinda feels she has made least progress in as it has not been a problem for her.

### **CONCLUSION**

Considering the main aim of my research, to establish whether or not there is any long term impact of the teaching of mental calculation strategies to PGCE students, there clearly was for all three of these teachers. In particular, all had a strong belief that an ability to quickly mentally double and halve numbers is crucial to mental calculation. I would certainly agree with this and this was a message given

throughout my intervention sessions. It can clearly support multiplication and division by four or eight, but it can also support multiplication by 5 (multiply by ten and halve); division by 5 (double then divide by 10); multiplication by 20 (double and multiply by 10); division by 20 (divide by 10 and halve) and can similarly support multiplication by 50, 25 and so forth. Percentages can also be worked out mentally using the knowledge of doubles and halves; for example to find 25% of a number just halve and halve again; to find 15% just find 10%, halve it and add the two numbers together. Similarly, equivalent fractions can be found by doubling or halving both numerator and denominator.

Whilst this has been a very small scale study and I therefore cannot possibly draw any general conclusions from my data, I have succeeded in answering all of my subsidiary questions. All three teachers taught mental calculation strategies; they taught these even when it was not on the school's planning; they all believed that not only could the children use the strategies, but some children could also select from a range of strategies, even at the age of five. Confidence was a different issue altogether, as only one student had increased her level of confidence in teaching these strategies, and this seems to be entirely down to the individual experiences of the students.

So why might these teachers have been different to those researched by Larose et al (2009) and Haggarty et al. (2009)? Why have they built on the teaching they received on their PGCE course and why have they continued to develop their own subject knowledge? As suggested by Elliott et al. (2009), I am convinced that this is as a result of the type of teaching they received. If suitable consideration is given to the content and delivery of ITE taught sessions, a small amount of input can have long term impact on their teaching. For maximum impact, the subject knowledge related to mental calculation needs to be taught in an environment where students, teachers or children can discuss their methods freely with one another.

## REFERENCES

- Alexander, R. (Ed.). (2010). *Children, their world, their education. Final Report and Recommendations of the Cambridge Primary Review*. Routledge: Oxford.
- Baxter, P., & Jack, S. (2008). Qualitative case study methodology: Study design and implementation for novice researchers. *The Qualitative Report*, 13(4), 544-559.
- Castle, K. (2010). *Study Skills for your Masters in Teaching and Learning*. Exeter: Learning Matters.
- Cohen, L., Manion, L. & Morrison, K. (2000). *Research Methods in Education*. London: Routledge Falmer.
- Darling-Hammond, L. (1999). *Teacher Learning That Supports Student Learning: What Teachers Need to Know*. Retrieved from Edutopia at <http://www.edutopia.org/teacher-learning-supports-student-learning>

- Davis, S. (2009). A study of primary student teachers' mental calculation strategies. *Proceedings of the British Society for Research into Learning Mathematics*, 29(2), 25-28.
- DCSF. (2008). *Independent Review of Mathematics Teaching in Early Years Settings and Primary Schools*. Nottingham: DCSF.
- DfEE. (1999). *The National Numeracy Strategy Framework for Teaching Mathematics*. Sudbury: DfEE.
- DfE. (2010). Interactive Teaching Programs, retrieved from <http://nationalstrategies.standards.dcsf.gov.uk/search/primary/results/nav:49909> accessed 25/08/10.
- Elliott, R., Kazemi, E., Lesseig, K., Mumme, J., Carroll C. & Kelley-Petersen M. (2009). Conceptualizing the work of leading mathematical tasks in professional development. *Journal of Teacher Education*, 60(4), 364-379.
- Haggarty, L., Postlethwaite, K., Diment, K. & Ellins J. (2009). *An examination of beginning teacher learning during the induction year*. Paper presented at the British Educational Research Association Annual Conference, University of Manchester. Paper retrieved from <http://www.leeds.ac.uk/educol/documents/184292.pdf>
- Larose, F., Grenon, V., Morin, M-P. & Hasni, A. (2009). The impact of preservice field training sessions on the probability of future teachers using ICT in school. *European Journal of Teacher Education*, 32 (3), 289 – 303.
- Murphy, C. (2004). How do children come to use a taught mental calculation strategy? *Educational Studies in Mathematics*, 56 (1), 3-18.
- Thomas, G. (2009). *How to do your Research Project*. London: Sage Publications.
- Thompson, I. (1999). Mental calculation strategies for addition and subtraction - Part 1, *Mathematics in School*, 28 (5), 22-25.
- Thompson, I. (2000). Mental calculation strategies for addition and subtraction - Part 2, *Mathematics in School*, 29 (1), 24-26.
- Threlfall, J. (2002). Flexible mental calculation, *Educational Studies in Mathematics*, 50 (1), 29-47.
- Torbeyns, J., De Smedt, B., Stassens, N., Ghesquière, P. & Verschaffel, L. (2009). Solving subtraction problems by means of indirect addition, *Mathematical Thinking and Learning*, 11, 79–91.

# RE-DEFINING HCK TO APPROACH TRANSITION

Sáinz Fernández, Lourdes Figueiras, Jordi Deulofeu, Mario Martínez

Universitat Autònoma de Barcelona

*Mathematics learning is a continuous process in which students face some abrupt episodes involving many changes of different natures. This work is focused on one of those episodes, transition from primary to secondary school, and targets teachers and their mathematical knowledge. By including expert teachers' views about transition and characterising the mathematical knowledge that they need to smooth transition processes, we aim to highlight their importance in the continuity in mathematics education. The concept of Mathematical Knowledge for Teaching (MKT) developed by Ball among others (Ball, Thames & Phelps, 2008) and, within this framework, the construct Horizon Content Knowledge (HCK) emerge as our theoretical response to the knowledge for teaching mathematics in a continuous way, particularly relevant during transition to secondary school. The enrichment of the idea of HCK and its expression in the teaching practice intends to develop a theoretical tool to approach transition from teachers' mathematical knowledge perspective.*

Keywords: Transition; Continuity; Mathematical Knowledge for Teaching; Horizon Content Knowledge.

## INTRODUCTION

Mathematics learning is a continuous process in which students face some abrupt episodes which involve many changes of different nature that derive in a variety of alterations in their educational path. This work is focused on one of those episodes, transition from primary to secondary mathematics, a compulsory transition for students that involves many external changes and targets teachers' mathematical knowledge.

An extensive survey let us detect that previous research on transition from primary to secondary school has been mostly focused on its effects in students' academic attainment (McGee, Ward, Gibbons & Harlow, 2003) whereas a focus on teachers and their perspectives has been mostly absent. Considering specifically transition in mathematics, the following questions arise immediately: which characteristics describe it? Looking at previous research foci of attention and conclusions the mathematical content emerges as a determinant factor: the step from arithmetic to algebra (Boulton-Lewis et al., 1997; Boulton-Lewis, Cooper, Atweh, Pillay & Wilss, 1998; Cooper et al., 1997; Flores, 2002; Gonzales & Ruiz Lopez, 2003), the learning of integer numbers (Gallardo, 2002; Pujol, 2006) or the development of the need of proofs in geometry (Berthelot & Salin, 2000-2001; Sdrolas & Triandafillidis, 2008) and other fields appear as particular well-known problems embedded in the teaching and learning of mathematics that involve transition to secondary school. Since

teachers shape the access of students to this content, we consider their role in transition as crucial.

In conclusion, albeit transitions are processes with a broad range of effects and consequences in the whole students' educational experience, we believe that it is necessary to investigate the specificity of mathematics in transition and particularly, the role of the teacher and his/her professional knowledge during this process. More specifically, the objectives of our investigation are:

O1. Find out mathematics teachers' views about transition.

O2. Characterise the mathematical knowledge that teachers need to smooth transition processes.

With regard to O1, we believe that teachers' opinions and practical everyday experience on transition must be taken into consideration in order to broaden our understanding of transition in mathematics, since they are the only professionals that experience the process of transition along with the students in the classroom. By involving the views of *expert* mathematics teachers we want to enlarge the multiple perspectives from which transition to secondary school can be considered and thus gain a better comprehension of it. Experts' responses will constitute the justification of the appropriateness of the theoretical perspective adopted in O2.

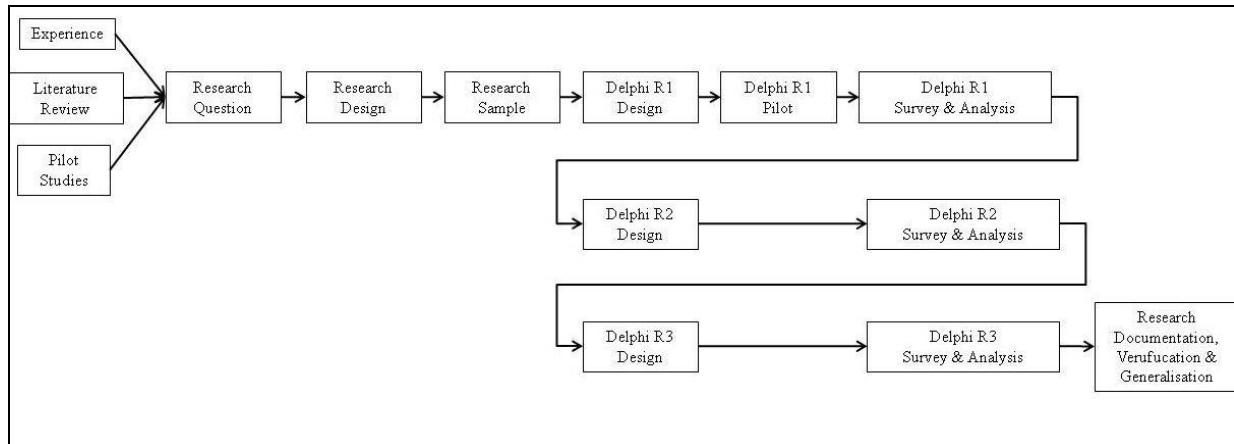
In order to accomplish O2 we require a solid theoretical frame that suits our purpose of focusing on the role of the teacher and, particularly, on the relationship of his or her mathematical knowledge and practice with primary to secondary transition. The concept of Mathematical Knowledge for Teaching (MKT) (Ball, Thames & Phelps, 2008; Hill, Rowan & Ball, 2005; Hill et al., 2008) and its division in different domains and sub-domains appears as a suitable framework for this part of our investigation. Within this theory, the construct Horizon Content Knowledge (HCK) emerges as our theoretical response to the knowledge for teaching mathematics in a continuous way, particularly relevant during transition to secondary school. Hence, our research focus in this part of the investigation is the enrichment of the idea of HCK in order to use it as a theoretical tool to approach transition.

## **EXPERTS' VIEWS**

We aim to find out experts' opinions about transition to secondary mathematics regarding the following questions: which elements affect in a determinant way the transition to secondary mathematics? And which of these refer to the teaching practice? Moreover, not only we want to collect judgements of individual experts but we also want to offer a scenario of group decision, where participants are capable of interact anonymously and reach a consensus. In the following, we will consider an *expert* a mathematics teacher that has taught in both levels and has shown a particular concern about transition issues.

In order to create an experts' group discussion we use the classical Delphi method, which allows us to create a scenario of group decision where experts feel free to

express their opinions and interact anonymously in a very efficient way, ensuring that the information obtained is of “good quality” and reliable (Plans & León, 2010). The versatility of the Delphi method seems particularly appropriate for our research, since it includes open and closed questions as well as qualitative and quantitative analysis. Figure 1 shows the steps included in a typical three round classical Delphi method (Skulmoski, Hartman & Krahn, 2007, p3) which guides the design of this part of our work.



**Figure 1: Classical three round Delphi method (Skulmoski, Hartman & Krahn, 2007, p3)**

### Delphi study

Our own teaching experience led to the problem of transition to secondary school and a previous survey detected a theoretical gap in transition research focused on teachers. Our uncertainty involves expert teachers' opinions on the elements that affect transition and particularly those related to the teaching practice in both levels.

In the study 15 experts answered our open questionnaire, which included three questions: What factors affect students' mathematical learning during transition to secondary school? Which qualities should primary teachers have in order to smooth students' transition to secondary mathematics? Which qualities should secondary mathematics teachers have in order to smooth students' transition from primary school? Among the factors that affect student' learning they identified methodological issues and the influence of teachers' initial training as especially remarkable. For example, scarce use of manipulatives or group work in secondary school, and no balance between teachers' pedagogical and mathematical knowledge. Experts pointed out that in order to manage transition, teachers must have a set of skills, attitudes and knowledge related to general education such as preparing primary students for a greater autonomy, valuing diversity, managing the interaction, using active and constructive methodologies or encouraging participation. We can infer from data that they refer to those skills from a theoretical framework that conceives a pure pedagogical knowledge. Experts also refer to a specific mathematical knowledge for teaching, which, in terms of Ball, Thames and Phelps



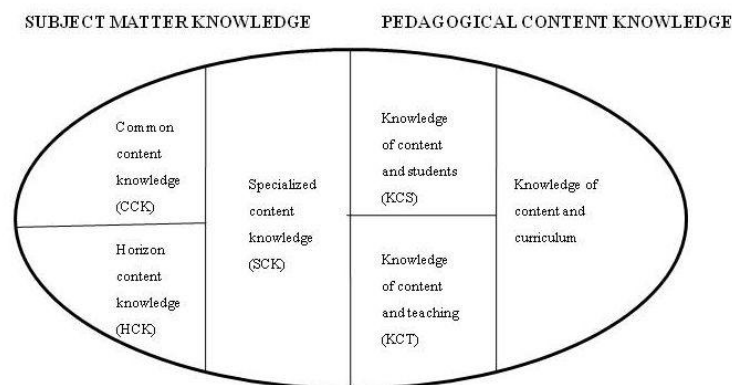
(2008) we call Mathematical Knowledge for Teaching. For example, they refer to manage with different degree of mathematical rigor, know the mathematics taught at both levels and have a global vision of the contents. Some kind of experience connecting the two levels appears essential, as the need to know about the previous and/or forthcoming stages. Following Ball et al. (op cit.), we find that experts refer to characteristics that describe the long-term perspective embedded in the idea of Horizon Content Knowledge. This way, experts' responses become our way in and justify the second part of our investigation, in which the framework of the Horizon Content Knowledge appears crucial to approach teachers' role in transition.

## MKT AND HCK

In this part of the investigation we explore the concepts of MKT and HCK in order to clarify the latter's place in the diagram and introduce a different approach to the MKT's organisation above that concludes with the refinement and placement of HCK in this framework and thus, with the inclusion of the notion of continuity in MKT's theory.

### Mathematical Knowledge for Teaching overview

Ball, Thames and Phelps (2008) distinguish two domains within the MKT, namely Pedagogical Content Knowledge and Subject Matter Knowledge (see Figure 2). These are not independent from each other, but it is their combination which defines the knowledge needed for teaching mathematics.

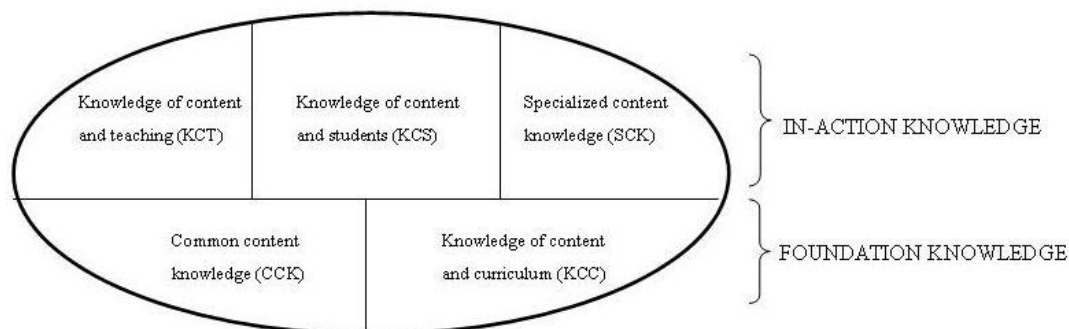


**Figure 2: Categories of Mathematical Knowledge for teaching (Ball, Thames & Phelps, 2008, p.403)**

Pedagogical content knowledge is subdivided attending three foci of attention within the teaching practice: students, methodology and curriculum. Knowledge of content and students (KCS) involves students' expected difficulties, questions, motivations, etc. and teacher's preparation and responses to those. Knowledge of content and teaching (KCT) concerns methodology issues such as the design of the sequence of a topic or the use of appropriate tasks, representations and examples, etc. Finally,

knowledge of content and curriculum (KCC) is the curricular knowledge needed for teaching. This includes not only the knowledge of the mathematical topics that are included in a particular curriculum, but also the specific moments when they have to be taught and how they are developed in the educational path. Subject matter knowledge is sub-divided in three categories: common content knowledge (CCK), which is the mathematical knowledge that is common to other professions and specialised content knowledge (SCK), which is the specific mathematical knowledge needed for the teaching practice (Ball, Thames & Phelps, 2008).

Before centring our attention on the last sub-domain, the HCK, we detect that KCS, KCT and SCK arise and are expressed only during the teaching practice in mathematics or in the observation of other's teaching practice, while the KCC and the CCK are not necessarily linked to the teaching practice. We highlight this observation by considering KCC and CCK as *foundation* knowledge and KCS, KCT and SCK as having an *in-action* nature. It is important to remark here that the word *foundation* denotes our idea of this theoretical knowledge being the basis and it is not related to the more complex construct of Foundation knowledge included in the Knowledge Quartet of Mathematical Knowledge in Teaching (MKiT) framework (Rowland, Huckstep & Thwaites, 2005). Figure 3 shows our interpretation of the categories of MKT from this approach, not considering yet HCK.



**Figure 3: Categories of Mathematical Knowledge for teaching without the HCK**

### **Horizon Content Knowledge**

Our main interest is focused on the Horizon Content Knowledge (HCK), provisionally included within subject matter knowledge. About HCK the authors say

We are not sure whether this category is part of subject matter knowledge or whether it may run across the other categories. (Ball, Thames & Phelps, 2008, p.403)

HCK refers to the general awareness of the previous and the forthcoming, and requires an overview of students' mathematical education so that it can be applied to the mathematics taught in the classroom (Ball, Thames & Phelps, 2008). Teachers' consciousness of the past and the future within their subject is actually very closely related to continuity in mathematics education and thus, our view of HCK comprises

this teacher's longitudinal perspective required for continuity. However, this longitudinal view that we understand as HCK encompasses a complex combination of pedagogical and mathematical knowledge, skills and experience that must be clarified in order to successfully approach transition issues in mathematics from this framework.

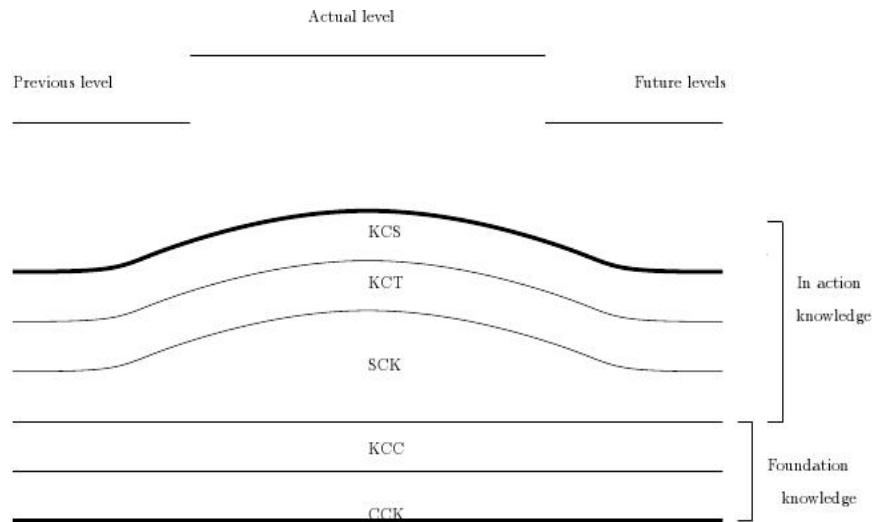
Firstly, despite the fact that the HCK may be related to the knowledge of the curriculum (KCC), it is independent from the curriculum itself. HCK is not only an awareness of how mathematical topics are related over the span of mathematics included in the curriculum but also refers to the global knowledge of the evolution of the mathematical content and the relationship among its different areas needed for the teaching practice. This general knowledge does not depend on the curriculum context and it is different to the curriculum awareness that a teacher must have in order to teach the appropriate topics at a particular grade. In other words, a teacher could have good level on KCC but fail on approaching this knowledge from a *long-term* perspective.

Secondly, HCK influences the KCS, the KCT and the SCK. For example, HCK must include the ability of the teacher to find out students' previous mathematical ideas and to prepare them for the future. This ability involves KCS (knowledge of previous, current and future students' difficulties, misconceptions or questions) and KCT (different ways students might have seen that represent the same idea or types of tasks that facilitate students' learning in the future). Also, the SCK of a teacher for a particular grade depends on whether that teacher currently teaches in that level. If so, his/her SCK for that grade will be obviously greater than for the other grades. The inclusion of HCK in the framework implies the extension of the SCK to those topics that may really have an effect in what students are learning at the moment or to those future topics for which a teacher is setting up the basis.

Thirdly, HCK has a different nature than the other sub-domains since it does not seem sensible to find the presence of HCK in a particular teaching situation if there is a previous absence of KCS, KCT or SCK. In fact, these are the required bases that allow the posterior gradual inclusion of HCK in the professional practice. From this perspective HCK is not another category in the diagram but it adds a more *sophisticated* (continuous) perspective to the teaching practice. For instance, we would not expect to observe a teacher recognising previous misconceptions in a particular topic, dealing with students' difficulties or conveying a prospective mathematical view of the future if that teacher does not know the topic and its methodological issues at first.

The previous ideas lead us to consider HCK, not as another sub-domain of MKT, but as a mathematical knowledge that actually *shapes* the MKT from a continuous mathematical education point of view since it must be present in every *in-action* category in order to attend transition. Figure 4 shows our interpretation of the framework with the inclusion of the HCK. The idea of *opening* the previous diagram

(Figure 3) and *shaping* it when including the HCK intends to indicate: a) the difference in the expression of each *in-action* category in the teaching practice with or without HCK; b) the connection with past and future mathematical levels, particularly important from a transition (or a continuous) point of view and c) the fact that HCK has a different nature than the rest of categories since it does not appear in the diagram but its presence modifies the teaching practice.



**Figure 4: HCK shapes MKT and outlines its nature**

### Characterisation of HCK

At this point, our consequent aim now is to characterise more specifically our idea of HCK. Hill et al (2008) highlight the need of clarifying how teachers' knowledge affects classroom instruction by carrying out an investigation in which the relationship between teachers' MKT and the quality of their practice is analysed. We follow this idea for the particular case of refining the construct of HCK. Since we do not consider HCK as a theoretical cluster itself but a type of knowledge that shapes the *in-action* knowledge needed for teaching and also because our purpose is to obtain a useful tool for future research on transition, we adopt a practical perspective from which the following question emerges immediately: how does HCK get expressed in teaching practice?

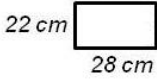
In order to obtain an answer to this question, a series of non-participant observations were carried out. Mathematics lessons in the last year of primary school as well as the first year of secondary school were observed during three months. Participating schools were city centre comprehensive schools that were chosen in order to cover the different possibilities in transition: schools with transitional programs and links among primary and secondary teachers, schools with transitional programs but no communication among teachers and schools without transitional programs. The observation and analysis of these mathematics lessons allowed us to identify classroom episodes in which students' and teacher's interventions offered a good potential in terms of achieving a more specific characterisation of the HCK. We

identified the episodes in which there is a clear opportunity for the teacher to express the HCK. Despite the fact that, on the contrary, its absence or weak presence is evident in the real situations observed, these episodes have proved to be very rich as objects of our investigation, since they let us reflect about the different possibilities of reaction that the teacher could have. Hence, the following question arose: what could have happened, had the HCK been present?

The episode shown in Figure 5 exemplifies these ideas by targeting, in this case, the well documented misconception between area and perimeter.

The teacher poses this problem from the textbook:

Calculate the perimeter and the area of the following rectangle:



The teacher starts solving the problem on the board:  $2 \cdot 22 + 2 \cdot 28 = 100$

The following dialogue occurs:

Student1: Why 100?

Teacher: To calculate the perimeter you do this operation and it gives 100

Student2: It's like giving 3 from the 28 to the 22 and you get four sides of 25. Si it's 100.

Teacher: Ok. And now, how do you calculate the area?

Student2: 25 times 25

**Figure 5: Example of a real classroom situation in which HCK may get expressed**

Two critical moments can be observed in this episode. The first one occurs when Student1 asks the reason of obtaining 100. The second one arises from Student2's misconception of considering that two shapes with the same perimeter should also have the same area.

Student1's question about the result points out at the operation that the teacher has used to calculate the perimeter of the rectangle: if the perimeter is the sum of all sides, why do we double them and add the results in this case? With the aim of clarifying Student1's confusion, the teacher could show how both calculations are equivalent. Moreover, the teacher could make the *connection* with algebra and generalise by reminding/showing that  $a+a+b+b=2a+2b$ . We observe that the ability of the teacher to make this connection is a pre-requisite for this possibility to occur. In this case, the connection required refers to the concept of perimeter itself and its relationship with algebra. We generally name this level of connection as *intraconceptual connection*, since it is in the essence of the particular mathematical concept.

Student2's ingenious proposal of transforming the rectangle into a square to calculate the perimeter in a more simple way offers an excellent opportunity to attend the common (and broadly studied by research) misconception that follows in the

episode: two shapes with equal perimeter have also the same area. One way the teacher could proceed would be to return the mistake to the student or the classroom by posing a series of questions that target the origin of Student2's mistake: do two rectangles with the same perimeter have always the same area? With the help of the teacher (questions, counterexamples, etc) and/or the rest of the class, the responsibility of correcting the mistake would be returned to the student or the class. As in the previous critical moment, it is necessary that the teacher makes a connection between the notions of area and perimeter in order to attend the misconception arisen. This is the next level of connection since it links two mathematical concepts and thus, we regard it as an *interconceptual connection*. Moreover, with an eye on the future and in order to strength the basis for the future learning of more complex geometry concepts, the teacher could also extend these ideas to other 2D shapes or even 3D solids by comparing surface and volume. This *temporal connection* with future (or past) topics of students' mathematics educational path is the third level of connection required for the teacher to attend these types of situations with a continuity perspective.

## FINAL REMARKS

This work intends to move towards a better understanding of continuity in mathematics education. HCK emerges as a noteworthy construct within MKT framework that allows the researcher to approach continuity questions in mathematical education by looking at teachers' professional knowledge. Identifying real practice situations in which an opportunity of attending continuity in the classroom is missed, leads to the detection of the connections required for the potential expression of the HCK. The future systematic investigation of HCK's expression in the teaching practice and its consequences on transition shows a path to the potential inclusion of this mathematical knowledge as part of teacher training programs designed to smooth transition. Moreover, proposing examples like the one described in this work and discussing them in teacher training programs could be a future way in to attend continuity in mathematics education.

## NOTE

This study is conducted under the auspices of Ministerio de Ciencia e Innovación (grant EDU2009-07298).

## REFERENCES

- Ball, D.L., & Hill, H. (2009). R and D: The curious - and crucial - case of mathematical knowledge for teaching. *Phi Delta Kappan*, 91(2), 68-71.
- Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59 (5), 389-407.
- Berthelot, R. & Salin, M.H (2000-2001) L'enseignement de la géométrie au début du collège. Comment concevoir le passage de la géométrie de constat à la géométrie déductive? *Petit x*, 56, 5-34.

## Working Group 17

- Boulton-Lewis, G.; Cooper, T.; Atweh, B.; Pillay, H.; Wilss, L.; Mutch, S. (1997). Transition from arithmetic to algebra: A cognitive perspective. *21st Annual Conference of the International Group for Psychology of Mathematics Education*, 2, pp. 178-185. Lahiti, Finland.
- Boulton-Lewis, G.; Cooper, T.; Atweh, B.; Pillay, H. & Wilss, L. (1998). Arithmetic, pre algebra and algebra: A model of transition. *Mathematics Education Research Group of Australasia*, Gold Coast.
- Cooper, T.; Boulton-Lewis, G.; Atweh, B.; Pillay, H.; Wilss, L. & Mutch, S. (1997). Transition from arithmetic to algebra: Initial understanding of equals, operation and variables. *21st Annual Conference of the International Group for Psychology of Mathematics Education*, 2, pp. 82-89. Lahiti, Finland.
- Flores, A. (2002). Geometric Representations in the Transition from Arithmetic to Algebra. In Fernando Hitt (Ed), *Representations and Mathematics Visualization*. (pp. 9-29). México, DF. Departamento de Matemática Educativa del Cinvestav-IPN.
- Gallardo, A. (2002) The extension of the natural number domain to the integers in the transition from arithmetic to algebra. *Educational Studies in Mathematics*, 49, 171-192.
- Gonzales, F.E. & Ruiz Lopez, F. (2003) Las centenas cuadrículadas: un material matemáticamente potente para ilustrar el tránsito de la aritmética al álgebra. *SUMA: Revista sobre la enseñanza y el aprendizaje de las matemáticas*, 42, 47-59.
- Hill, H.C., Rowan, B. & Ball, D.L. (2005) Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42 (2), 371-406.
- Hill, H. C.; Blunk, M.; Charalambous, C.; Lewis, J.; Phelps, G.; Sleep, L. & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430-511.
- McGee, C., Ward, R., Gibbons, J & Harlow, A. (2004) *Transition to Secondary School: A Literature Review*. Hamilton: The University of Waikato.
- Plans, B & León, O. (2010) ¿Cómo debe ser el doctorando ideal en Psicología? [http://www.uam.es/personal\\_pdi/psicologia/orfelio/doctorando\\_ideal.pdf](http://www.uam.es/personal_pdi/psicologia/orfelio/doctorando_ideal.pdf)
- Pujol, R. (2006) Diagnóstico sobre la disposición del alumnado para aprender matemáticas a través de la resolución de problemas. PhD Thesis. Universitat Autònoma de Barcelona.
- Rowland, T., Huckstep, P. & Thwaites, A. (2005) Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education* 8(3), 255-281.
- Sdrolias, K.A. & Triandafillidis, T.A. (2008) The transition to secondary school geometry: can there be a “chain of school mathematics”? *Educational Studies in Mathematics*, 67(2), 159-169.
- Skulmoski, G. J., Hartman, F. T. & Krahn, J. (2007). The Delphi method for graduate research. *Journal of Information Technology Education*, 6.

# CONCEPTIONS AND PRACTICES OF MATHEMATICAL COMMUNICATION

António Guerreiro<sup>1</sup> and Lurdes Serrazina<sup>2</sup>

<sup>1</sup>School of Education and Communication, University of Algarve (Portugal) and

<sup>2</sup>School of Education, Instituto Politécnico de Lisboa (Portugal)

*This paper reports the evolution of a primary school teacher's conceptions and practices of mathematical communication, during a study supported by a collaborative work among three teachers and the first co-author. It is part of a large research which intends to study the evolution of the conceptions and practices of mathematical communication and interaction in the primary school classroom. This paper focuses on the need for training teachers in mathematical communication and mathematical knowledge for teaching.*

Key-words: mathematical communication; conceptions about communication; teaching practices; mathematical knowledge for teaching; collaborative work

## INTRODUCTION

The curriculum guidelines on mathematical communication emphasises the representation of mathematical ideas in the speaking, the writing and the reading *of* and *about* mathematics, in abstract and symbolic language and in the social interactions in classroom. The value given to student-student and student-teacher interactions in constructing a significant mathematics learning (ME, 2001, 2007, NCTM, 1991, 2007) hinges on the notion of communication as social interaction, as opposed to the traditional view of communication as a process of transmission of information and knowledge.

The first co-author of this paper conducted a research about the evolution of conceptions and practices of mathematical communication, during a collaborative work focused on the analysis and the reflection of the teacher's communicative practices in the classroom, with three primary school teachers. This research was supervised by the second co-author.

The research problem was specified the following questions: (i) How do the teacher's conceptions concerning mathematical communication, in the classroom, evolve? (ii) How do the teacher's communication practices value the students' acquisition of mathematical knowledge? (iii) How are teacher-students communication and interaction patterns related to the negotiation of mathematical meanings? (iv) What is the relation between the moments of reflection about the teacher's communication practices in the classroom and the evolution of such practices?

In this paper we present a theoretical synthesis about mathematical communication framed in theories of communication, emphasizing the role of mathematical



communication as social interaction in classroom. We present a summary of methodological options, where we stress the methodological challenges of the collaborative work among three primary school teachers and the first co-author.

This paper proposes to explore the role of teachers in the development of mathematical communication and of the student-student and student-teacher interactions. It further intends to highlight the evolution of the teacher's conceptions and communicative practices in the mathematical classroom. We seek to reflect on the increment of mathematical communication in class and the teacher's mathematical knowledge for teaching (Ball, 2003). It results from the work undertaken with one of the teachers who participated in the study – Laura.

## MATHEMATICAL COMMUNICATION

The curriculum guidelines for mathematics teaching advocate mathematical communication as a process of social interaction, which differs from others views of communication as transmission of information and knowledge. The greater value given to the role of dialogue and the sharing of information are opposed to a more traditional form of communication based on an one-way process. Brendefur and Frykholm (2000) refer to the existence of *unidirectional communication* and of *contributive communication* in the teacher-oriented classroom discourse, where the student limits himself to attend the class with few interventions. It exists in opposition to the *reflective communication* and the *instructive communication*, where the students use mathematical conversations to share their ideas, strategies, and solutions with peers and teachers.

Interactions between teacher and students generate interaction patterns (Godino & Linares 2000; Wood 1994, 1998), which can be typical either of a teaching process based on memorization, questioning and reproduction, grounded on the teacher's mathematical knowledge – patterns of reciting, funneling and focusing –, or of a learning process grounded on the personal contribution of the learner, making room for an evaluation and reflection on the mathematical activities and on mathematical knowledge constructed in the classroom – patterns of extraction and discussion. From a communication standpoint, the role of teacher and students acquire substantially different meanings, in line with the conceptions and practices of mathematics teaching (Thompson, 1984, 1992).

Mathematical communication as a process of social interaction results from the sharing of meanings constructed and reconstructed by individuals, where the subject identifies himself/herself with the other, and at the same time, expresses and affirms his/her singularity (Belchior 2003). The communicative function is to create and maintain understanding through the negotiation and reconstruction of meaning between individuals (Godino & Linares, 2000; Yackel, 2000). This communicative action is characterized as a process in which the subject affirms his/her view of world and understands the points of view of the others (Habermas, 2004, 2006).

Mathematical learning on the part of the subjects arises from the interactions between the individual and the culture (Sierpinska, 1998), including the interactions between students and the teacher. Mathematical learning results from the students-teacher-mathematics interactions, where the discourse is understood as a language in action, thinking with words with the others and for the others (Godino & Llinares, 2000; Sierpinska, 1998). The practices of communication, based on the process of interaction between the subjects, seem to be linked with a mathematical learning based on the value given to the students' mathematical ideas and meanings (Thompson, 1984, 1992).

## **METODOLOGICAL OPTIONS**

The background investigation for this study fits into a qualitative methodology (Bogdan & Biklen, 1994), which adopts the interpretative paradigm and follows the design of a case study (Stake, 1994; Yin, 1989). Three teachers from the same primary school – Alexandra, Carolina and Laura – participated in this study, in a context of collaborative work with the first co-author, the point of which being the reflection on their professional practices concerning mathematical communication, for two year (from December 2006 to February 2009). The selection of the teachers resulted from the first author acquaintance with them and their willingness to work on the mathematical communication in the context of the primary school classroom.

The data collection consisted of initial and final audio-taped interviews with the teachers, audio-taped descriptions of the collaborative meetings between the first co-author and the teachers (both collectively and individually), and audio and video-taped classroom reports. The multiplicity of instruments for data collection did not depend on the triangulation of data, but on the need to clarify, supplement the meaning of information and identify different ways of seeing the events (Stake, 2000).

The data analysis was organized in case studies: the characterization of the collaborative work between the first co-author and three primary school teachers, and the evolution of the teacher's mathematical communication conceptions and practices. The procedures for data analysis involve various stages until the construction of the interpretative text of the case. The data were reduced and transcribed to expressive episodes, incorporating the complexity of the phenomena and contexts, which allowed reconstructing the experiences of participants (Goetz & LeCompte, 1984).

## **CONCEPTIONS AND PRACTICES**

### **Conceptions**

Laura's conceptions about mathematical communication seem to have evolved from the perspective of mathematical communication as transmission of information and knowledge – "They are small and we are transmitting a lot of things" [2006

december \_ initial interview] – to mathematical communication as social interaction, in an attempt to reach a better understanding between all the students involved – "When they [students] do not understand anything, I always say «please, ask us, don't keep the questions to yourselves»" [2009 february \_ final interview].

Laura integrated in her conceptions about mathematical communications and mathematical teaching and learning aspects related to social interaction like the students' knowledge and their cognitive ability to learn:

I let them think. (...) They would come to me and I would say «It is wrong». (...) I would say it was wrong and I wouldn't be expecting their thinking. Not anymore. It's different. I was expecting: «So, clear it up for me. So, why did you do this? How was it?»

[2009 february \_ final interview]

Laura acknowledged the close proximity that existed between the processes of interaction in classroom and the mathematical learning. She had given priority to the students' personal knowledge and the moments of communicative interaction. This modification in the teacher's practices seems to have caused a change in the discourse and in the teacher's conceptions, in such a way that the students were now seen as communicative partners in mathematical teaching and learning. This recognition of the student as a knower was also reflected in her personal and professional knowledge for mathematical teaching. The existence of these conceptions seems to confirm the dichotomy between the transmission of information and the construction of knowledge through social interaction.

## Practices

*Kind of communication.* At the beginning of the collaborative work, the teacher's communicative practices in classroom valued the contribution of students – *contributive communication* – with short interventions, used by the teacher (Laura) to assess their knowledge:

Laura: Today, we will go to do an activity about shapes. Do you remember? We talked about squares, more...

Students: Triangles, Rectangles...

Laura: Squares, Triangles, Rectangles and...

Students: Circles.

[2007 june \_ lesson \_ first year of primary school]

Laura has begun to encourage the students' participation, particularly in discourse in mathematics classroom, and the interactions between the students in the discussion of different perspectives of mathematical resolutions. In the problem of the River Crossing [1], the teacher opted to begin the discussion with a solution that was incongruent with the conditions of the problem.

The student Monica presented the solution of her group, writing:

## Working Group 17

Little Johnny takes the rabbit in the boat. Little Johnny takes the cabbage in his lap and the dog on one side, and they go on their way

While the student was writing on the board, some students were waiting with their hands up, as a sign that they wanted to question their colleague.

Teacher: There are hands up.

[2008 march \_ lesson \_ second year of primary school]

The teacher alerted Monica to the questions of her colleagues and she ended her presentation and chose one of the other students to ask her a question. Following the conclusion about the impossibility of more than two *passengers* in the boat, one of the members of this same group – Tiago – presented a new proposal for the solution, writing:

First goes the dog [the students became agitated because they consider that what their colleagues wrote was wrong]. Second goes the cabbage. And last goes the rabbit.

Gonçalo, observing the solution written by Tiago, said:

Gonçalo: I know what's wrong.

Teacher: So go up there Gonçalo. Go to the blackboard and say what's wrong.

Gonçalo went up to the blackboard and explained his reasons to Tiago.

Teacher: Tiago, stay there to defend yourself.

[2008 march \_ lesson \_ second year of primary school]

The comments of the teacher were intended to promote the interaction between the students – “*There are hands up*” – and to encourage the justification of student's reasons – “*Stay there to defend yourself*”. This attitude of this teacher promoted a greater interaction between the students in the classroom. The exploration of the incomplete or incorrect resolutions by the teacher resulted from the collaborative work. Usually, Laura would say: «This is wrong. What do you think? What is it saying there?» [2008 april \_ collaborative meeting with teacher]

The participation of the students in the classroom discourse went beyond the routines of *contributive* communication. The emergence of *reflexive* communication resulted from the value given to the students' mathematical ideas and knowledge:

Laura: What do you make of these two tables? (Each table was paved with equal squares but different from one table to the other)

Marcia: The biggest pave more and the smallest pave less.

Laura: Stop, what did you say?

Marcia: The biggest pave more and the smallest pave less. The biggest are less used and the smallest are more used.

[2008 april \_ lesson \_ second year of primary school]

The increase in mathematical communication resulted in the need for training in mathematics knowledge for teaching as a way to support and to value the students' mathematical ideas and strategies:

If I don't understand how can I explain? (...) My fear is what will happen if one day I will face a situation I do not know and I think «So what now, how do I use this? What should I do? ». (...) This is true, sometimes I think about this. [2008 april \_ collaborative meeting with teacher]

The collaborative work seems to contribute to the evolution of mathematical communication as social interaction in the primary school classroom and to the teachers' awareness that they should become more involved in getting more mathematical knowledge.

*Patterns of interaction.* The centrality of the lesson around the teacher's mathematical knowledge seems to discourage the student's questioning about his own learning. This kind of approach seems to result in the existence of the pattern of reciting, characterised by questions related to the checking of knowledge addressed to the students. Collective or individual answers are validated by the teacher as she repeats the answer:

Teacher: Why do you say this is a square?

Students: It has four equal sides.

Teacher: It has four equal sides.

[2007 june \_ lesson \_ first year of primary school]

A strategy often used by the teacher aiming to overcome the students difficulties is asking simple and straightforward questions that will direct them to the intended answer or to the resolution of the proposed task, generating an interaction characterised by funnel and focus patterns. At different key moments in the lesson, Laura resorts to this sort of strategy right after wrong solutions have been given. In one of the lessons we observed, the students were supposed to write consecutive natural numbers starting at a given number. Miguel, one of the students, went to the blackboard to fill in the gaps with the predecessor and the successor of 300. Faced with the student's difficulty in determining the predecessor of 300, Laura pointed him towards the solution by means of a dialogue that could be said to be characteristic of the funnel pattern:

Teacher: What is the one before that? Is it 300?

Miguel: No.

Teacher: What is it? What is the order that we should follow? What is it?

[The Student remains silent.]

Teacher: Is the number that comes before that one three hundred and something? Is it?

Miguel: No.

Teacher: So what is it? If it isn't three hundred, what is the hundred before three hundred?

## Working Group 17

[Student falls back into silence.]

Teacher: What is the hundred before three?

Miguel: Two.

Teacher: So that other number has to begin with...

Miguel: Two hundred.

[2008 february \_ lesson \_ second year of primary school]

Laura feels powerless to change this sort of patterns because she believes that she lacks enough knowledge and creativity to take her students on a different path: 'Many times, when I'm trying to help them out, I'm already providing the answer unwittingly. Sometimes I also lack some imagination to fetch other examples, to pursue different paths' [2008 april \_collaborative meeting with the teacher]. Such lack of imagination and creativity, along with a superficial mathematical knowledge may explain traditional interaction patters based on questions of knowledge validation and focusing on procedures.

However, the *discussion* pattern emerged with the increasing number of interactions among the students and between the teacher and them. In the presentation of resolution strategies concerning a problem on height [2], Laura helped the students to clarify their presentation, following a scheme similar to the pattern of discussion:

Student: 180 is her height standing on the stool and 45. And now we subtracted 45 from 180.

Teacher: Which is...?

Student: Which is her height standing on the stool.

Teacher: 45?

Student: No, 180.

Teacher: So 45 refers to what?

Student: To the stool.

Teacher: So what do we have to do to figure out her height only?

Student: We have to subtract 45 from 180

[2008 may \_ lesson \_ second year of primary school]

Traditional interaction patterns, such as reciting, are now practically nonexistent, despite the teacher's omnipresence. The increase in the number of interventions of the students and a growing command of mathematical discourse resulted in the emergence of patterns other than funneling and focusing, such as patterns of discussion, where the students themselves assumed the explanation of resolutions and strategies, with the help of the teacher in the clarification of procedures and learning processes.

## SOME FINAL CONSIDERATIONS

Laura's conceptions and practices about mathematical communication evolved from the transmission of information and knowledge to mathematical communication as social interaction. Mathematical knowledge was socially constructed in the

classroom as a result of the value given to mathematical ideas, strategies and knowledge of the students.

The increase of social interaction in the classroom among students and among students and teacher contributed to intensify the role of Laura in encouraging the participation of her students in collective talk in the classroom. It also contributed to increase the interaction between students, either mediated by teachers or not and to value the students' mathematical knowledge and individual strategies.

The interactions between the students played a major role in the presentation and discussion of strategies and results of mathematical tasks. The interactions between students were intentionally fostered by the teacher to encourage students in the collective discourse of the classroom. These interactions originated a growing respect among students as they shared their mathematical knowledge and ideas.

Social interaction appears to lead to a change from a mathematical communication centered on the teacher to a mathematical communication centered in the classroom – students and teacher – based on the singular knowledge of each student as a structural component of mathematics learning. In this perspective, the mathematical communication conceptions and practices of the teacher seem to adjust to mathematical communication as social interaction, resulting in value being given to the students' knowledge and strategies: all of this due to reflective communication based on inquiring questions and the discussion in the mathematics classroom.

## **NOTE**

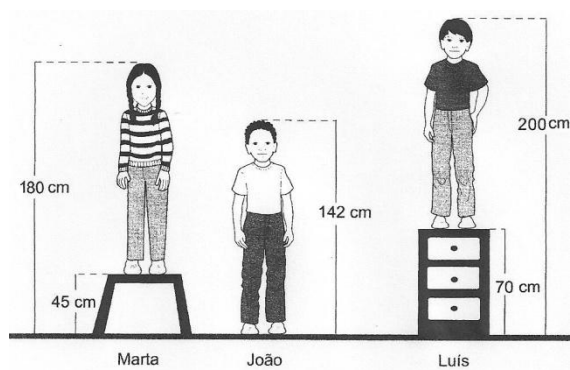
[1] River Crossing - The hunting dog, the rabbit and the cabbage

Little Johnny was crossing a dry, unshaded field on the way to his grandfather's house. He was taking with him a hunting dog to go with his grandfather on the hunt, a jack rabbit for his grandmother to put in her rabbit hutch with a pretty female rabbit and a nice cabbage for lunch.

All along the way, the dog wanted to eat the rabbit and the rabbit to eat the cabbage. Little Johnny had to be very careful as he walked along to avoid anything going wrong. After a while Johnny came to a river he had to cross.

In order to cross the river there was a small boat which he could use, but it was so small that he could only take with him one passenger at a time: the dog or the rabbit or the cabbage. He could never leave the dog alone with the rabbit or the rabbit alone with the cabbage, so how can he get all of them across without any problem? You are going to help to resolve this problem.

[2] Luís and his two friends were playing heights as you can see in this picture. Each of the three friends has a different height.



Considering only the measurements in the picture, write the name of each of the three friends, from the shortest to the tallest.

## REFERENCES

- Ball, D. (2003). *What mathematical knowledge is needed for teaching mathematics?* Paper presented at the U.S. Department of Education, Secretary's Mathematics Summit, Washington, DC, February 6, 2003 (File PDF)
- Belchior, F. (2003). Pedagogia, comunicação e existência. *Revista Portuguesa de Pedagogia*, ano 37, nº 3, 197-230.
- Bogdan, R. & Biklen, S. (1994). *Investigação Qualitativa em Educação*. Porto: Porto Editora.
- Brendefur, J. & Frykholm, J. (2000). Promoting Mathematical Communication in the Classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education*, 3, 125-153.
- Godino, J., & Llinares, S. (2000). El Interaccionismo simbólico en educación matemática. *Revista Educación Matemática*, Vol. 12, nº 1, 70-92.
- Goetz, J. & LeCompte, M (1984). *Ethnography and qualitative design in educational research*. Orlando: Academic Press, Inc.
- Habermas, J. (2004). *Pensamento pós-metafísico*. Coimbra: Almedina.
- Habermas, J. (2006). *Técnica e ciência como "ideologia"*. Lisboa: Edições 70. (primitive edition in German, 1968)
- ME (2001). *Currículo nacional do Ensino Básico – Competências Essenciais*. Lisboa: Departamento de Educação Básica/Ministério da Educação.
- ME (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Direcção-Geral de Inovação e de Desenvolvimento Curricular/Ministério da Educação.
- NCTM (1991). *Normas para o currículo e a avaliação em Matemática escolar*. Lisboa: Associação de Professores de Matemática e Instituto de Inovação Educacional (primitive edition in English, 1989).



- NCTM (2007). *Princípios e Normas para a Matemática escolar*. Lisboa: Associação de Professores de Matemática (primitive edition in English, 1989, 2000).
- Sierpinska, A. (1998). Three epistemologies, three views of classroom communication: Constructivism, sociocultural approaches, interactionism. In H. Steinbring, M. G. B. Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 30-62). Reston, VA: NCTM.
- Stake, R. (1994). Case studies. In. Dezin, N. & Lincoln, Y. (Eds.) *Handbook of qualitative research*. Londres: Sage.
- Stake, R. (2000). Case studies. In. Dezin, N. & Lincoln, Y. (Eds.) *Handbook of qualitative research. Second Edition* (pp. 435-454). Londres: Sage.
- Thompson, A. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 5(2) 105-127.
- Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 127-146). New York, NY: Macmillan.
- Yackel, E. (2000). *Creating a Mathematics Classroom Environment that Fosters the Development of Mathematical Argumentation*. [On-line] Available in <http://www.nku.edu/~sheffield/eyackel.html>.
- Yin, R. (1989). *Case study research: Design and methods*. Londres: Sage Publications.
- Wood, T. (1994). Patterns of Interaction and The Culture of Mathematics Classrooms in S.. Lerman (Ed.). *Cultural Perspectives on the Mathematics Classroom* (pp. 149-168). Dordrecht: KAP.
- Wood, T. (1998). Alternative Patterns of Communication in Mathematics Classes: Funneling or Focusing? in H. Steinbring, M. Bussi & A. Sierpinska (Eds.). *Language and Communication in the Mathematics Classroom* (pp. 167-178). Reston, V. A.: NCTM.

# LESSON STUDY IN TEACHER EDUCATION: A TOOL TO ESTABLISH A LEARNING COMMUNITY

Guðný Helga Gunnarsdóttir and Guðbjörg Pálsdóttir

University of Iceland, School of Education

*This paper presents a study on the use of lesson study in teacher education. Teacher education in Iceland has been under constant development and the authors of this paper have taken part in developing mathematics teacher education based on recent research on teacher education. The study shows that lesson study can create a learning community where teacher students can develop their professional language and collaborative competence which are considered to be important issues in teacher education. Lesson study also helps teacher students to put focus on pupils learning and content when planning lessons together. Lesson study with teacher students seems to be a good way for them to learn to teach and to develop as professionals.*

Key words: lesson study, mathematics teacher education, learning community

## INTRODUCTION

Mathematics teacher education in Iceland has been changing and developing during the last decades. From 1971 teacher education for teachers in compulsory school (grades 1-10) has been a three year B. Ed. degree<sup>85</sup>. The structure has varied but teacher students have always specialized in one or two subjects. From 2007 the B. Ed. degree has consisted of 80 ECTS<sup>86</sup> in pedagogy and didactics, 80 ECTS in specialization and 20 ECTS for studies of own choice (Stefnumótun Kennaraháskóla Íslands 2005-2010). In their specialization the mathematics teacher students study mathematics and mathematics education. In some courses they study either mathematics or mathematics education while in others the study is combined. For their teaching practice (12 weeks) the teacher students have a home-school.

The authors of this paper have taught different mathematics education courses for more than 20 years and have taken part in developing the studies in cooperation with colleagues. In choosing a structure and content for our mathematics education courses we have put an emphasis on creating a learning community among the teacher students. We play an active role in this community as participants in discussions and as group leaders or experts. The teacher students are to be active and bring in their knowledge, views, and thoughts about mathematics education. We have chosen to introduce our students to lesson study (Lewis, 2002) to establish

---

<sup>85</sup> In 2011 a five years M. Ed. degree will be required to qualify as compulsory school teacher.

<sup>86</sup> European Credit Transfer System – A full academic year of studies is 60 ECTS

learning communities within the frame of the courses. This is based on the belief that the creation of learning communities in teacher education it gives the students good learning opportunities for developing a professional language and a collaboration competency. Our aim is also to introduce a professional learning strategy to our students they can use in different contexts when they enter the teaching profession.

In this study we explore the effectiveness of using lesson study to develop a learning community within our courses. Our research questions are:

- How does the learning community develop during the lesson study process?
- What characterizes the learning community?

## **THEORETICAL BACKGROUND**

Mathematics teacher education has become an important field of study among researchers in mathematics education during the last 20 years. The increased interest was marked by the publication of the first issue of the Journal of Mathematics Teacher Education in 1998 and the publication of the first International Handbook of Mathematics Teacher Education in 2008 (Krainer & Llinares, 2010). The collaboration of teachers, teacher educators and pupils seems to be a central issue in many of the research studies. Team work, learning communities, networks and design research are often suggested as ways to help teachers to tackle and meet the complexity of teaching mathematics and to support them in their lifelong learning process. (Jaworski, 2005, 2006, 2007; Krainer, 2003; Wood, 2002; Wood & Berry, 2003).

Many researchers have tried to identify a knowledge base for mathematics teaching or important competencies for mathematics teachers. A professional learning competency is considered to be an important factor and teacher students need to learn to create professional learning opportunities during their pre-service teacher education. They also have to learn to collaborate with others and to create learning communities that will support their learning as teachers (Grevholm, 2006; Hiebert, Morris, & Glass, 2003). Teacher students have to realize that they are only starting to create a knowledge base and that they are entering a profession where lifelong learning is essential.

Hiebert, Morris, & Glass (2003) describe learning environments prospective teachers must learn to create in order to sustain their own and other teachers learning. In these environments teacher students need to learn to learn from their teaching in collaboration with others. They mention lesson study as an example of such a learning environment.

According to Darling-Hammond (1998) the best learning environment for teacher students is when they are given the opportunity to teach, study and reflect in collaboration with others and by looking closely at the pupils and their work and share what they see with others. This requires a close connection to schools and

teaching practices. Teacher students need both opportunities to try out their ideas in practice and support to reflect upon and interpret practice (Darling-Hammond, 1998).

Professional learning communities seem to play an important role in supporting teachers in continuously improving their teaching and sustaining their professional learning. Trends in teachers' professional development show increased attention to professional learning strategies grounded in classroom practice. (Fernandez, 2002; Loucks-Horsley, Stiles, Mundry, Hewson, & Love, 2010). Professional learning communities where teachers share understandings about the nature of good teaching and work together on planning and improving teaching seem to provide particularly conducive settings for learning to teach (Hammerness, Darling-Hammond, & Bransford, 2005).

Lesson study is often referred to as an example of a professional development strategy that creates a learning environment in which teachers engage in learning with their peers (Lewis & Perry, 2009). It is also a strategy that aims to achieve all of the four outcomes that according to Loucks-Horsley and her colleagues characterise effective professional development. They are; enhancing teachers' knowledge, enhancing quality teaching, developing leadership capacity, and building professional learning communities (Loucks-Horsley, et al., 2010)

Lesson study is mentioned as an example of a pedagogy for preparing teachers for teaching as a lifelong learning process (Hammerness & Darling-Hammond, 2005). Several research studies report on successful use of lesson study with teacher students (Burroughs & Luebeck, 2010; Tsui & Law, 2007) and teacher educators have shared their experiences of using lesson study with teacher students in journals like NCTM's Mathematics Teacher.

### **The Lesson Study Process**

The main idea of lesson study is that a group of teachers, develop a teaching plan for one lesson. The structure of the lesson, the role of the teacher and pupils learning are in focus. The lesson study process can be described as a cycle, a process where the group of teachers repeatedly goes through the phases, discussion, goal-setting and planning, research lesson.

In the first phase, the group has to discuss the aim and the content of the lesson.. Then the participants explore the content, both what it means to acquire understanding of the content it and how it can be approached in teaching. The participants are in control, and make decisions about the process. The participants often deepen their knowledge of the content and possible teaching approaches. They communicate, do research, work together, make decisions, plan teaching, and experience the advantages of participating in a learning community. The research lesson is taught by one of the participants and the other participants are present and take notes. Some times an outside expert is invited to observe the research lesson

and take part in the post lesson discussions. On basis of observations during the research lessons og the post lessons discussions the lesson plan is revised and the lesson taught again for a similar group. The cycle can be repeated several times (Lewis, 2002)..

In organizing our courses we have focused on establishing a learning community with our teacher students. A learning community where teacher educators and teacher students share ideas, discuss and work together on planning good teaching. Such an environment gives opportunity to develop professional language and collaborative skills. By engaging in the lesson study process the teacher students take part in creating a learning community where the focus is on the pupils learning of a specific content.

### METHOD

This research-study focuses on two 10 ECTS courses in mathematics education. The teacher students attend them in their second year and are at the same time attending courses on geometry and number theory. Teaching practice in grades 7-10 is an important part of the mathematics education courses. In the fall they follow the course: *Mathematics teaching and learning in lower secondary school* where they are introduced to mathematics education as a field of study. They learn about lesson study, discuss and study various resources for lesson study (<http://www.lessonresearch.net/>),. Lesson study is introduced as a tool to use in a professional development. The focus in on how being a part of a learning community gives teachers the supporting environment to develop their teaching collaboratively. In the spring term the teacher students attend the course: *Mathematics teaching and learning for all*. Then they try out the ideas of lesson study. All the students, around 15, work as one lesson study group. They start with discussing the aim and the content of the lesson. They focus on what could be relevant for 8<sup>th</sup> and 9<sup>th</sup> grade in their home-school and use the curriculum guide and their own analysis and interest in their considerations. The teacher students plan the lesson in collaboration with their teacher educators and make use of different literature and teaching materials as well as their own experiences. In the making of the lesson plan they use a four-column lesson plan (Matthews, Hlas, & Finken, 2009). The columns are: Steps of the lesson: Learning activities and key questions (1), expected pupils reactions and responses (2), teacher's response to pupils reactions/things to remember (3) and goals and method(s) of evaluation (4).

Some of the teacher students are distance learners and all students share a digital learning environment (Blackboard). The students on campus work together in class and records of the work in progress are shared with the distance students in Blackboard. The distance students contribute to the process by communicating their ideas, thinking and reflections the same way. Three or four weeks are used for the preparation of the lesson. Then the teaching plan is introduced to the practice-

teachers in the home-schools. They can make comments to the lesson plan and share their experiences with the group. Groups are created for teaching and observing research lessons. In each groups there is a teacher educator, practice teachers and teacher students from two home schools if possible. The groups adjust the lesson plan to the actual situations in the school and make practical decisions regarding the research lesson. All groups teach the lesson two times. After the research lesson there is a short evaluation meeting. When all groups have taught the lesson once the teacher students and teacher educators meet at campus to discuss the experience and make revision of the lesson plan. The reasons for different adjustments of the lesson plan in each home school are also discussed. When the teacher students are back at the university time is taken to discuss the lesson study project and this way of planning and collaborating as a tool for professional growth.

The lesson study process reported on in this study has been conducted with two groups of teacher students in total 30 students. Both groups developed lesson plans around prime numbers. The students were attending a course in number theory and the teaching of numbers and number theory was a topic in the math education course. The overarching aim for the teacher students was to find ways to make the mathematics teaching and learning interesting and fun for the pupils. They also found it challenging to make this particular topic, prime numbers, interesting and purposeful for their pupils. The first group developed a lesson plan around the question: What use can be made of prime factorization? They developed six different tasks that could be worked on in groups and then discussed in the whole class. The second group developed a game where the pupils were supposed to find out whether a number was a prime number or a composite number and how they could argue for their categorization.

### **DATA**

In this research-study the data was gathered during the school-years, 2008-2010. This study builds on prior studies on teaching teacher students similar courses. There the focus was on how research on mathematics teacher educations has influenced the structure, content and teaching approach in the courses (Gunnarsdóttir, Kristinsdóttir, & Pálsdóttir, 2008; Gunnarsdóttir & Pálsdóttir, 2010). The data in this study consists of two lesson plans, teacher educators' notes, an interview with two teacher students, audiotapes from planning meetings, correspondence between the teacher students at campus and the distance students, notes from evaluation meetings, a video tape from the evaluation meeting at campus and notes from discussion after the teaching practice period. We also have an audiotape from the oral presentation of the final course assignment where the teacher student presented their ideas about ideal mathematics teaching.

The data was analysed with our goal for using lesson study, to create learning communities, in mind. When reading and listening to the data our attention was

drawn to four main themes that are all important in a learning community. The four main themes; professional language, collaboration competence, focus on pupils learning and teacher students elaboration of the content are evident in all our data. The development of a professional language and collaborative competency are often mentioned as the main goals for establishing learning communities (Hammerness, et al., 2005; Jaworski, 2007). A learning community based on a lesson study process puts focus on pupils learning and mathematical content. We will provide some examples of how these four themes emerge from the data.

### **Professional language**

In the beginning of the lesson study process the teacher students found it hard to understand how you could use several weeks to plan one lesson. It took one session to discuss the approach and decide on content. But when they started to work on the content in more details they realized that there were many things to consider and many different ways to go. They felt the need to understand each other's ideas and they needed to be more precise in their use of language and had to support their arguments by referring to prior readings and common course literature in mathematics education. They also had to refine their shared understanding of concepts from general didactics and mathematics. The discussions became longer and more intense. They were developing their ideas in collaboration and trying to consider the teaching of the content from all points of view. Because of the distance students a part of the dialog was in written format and that demanded a more precise use of professional language. All the teacher students were going to teach the lesson at some point so they felt the need to understand and agree with the lesson plan. Despite of the assumed shared understanding written in the lesson plan the students realized that the lesson turned out differently in the schools when they met after the first round of teaching. The teacher students' use of professional language developed considerably during the process, they used more professional concepts, their discussions lasted longer and they referred more often to literature connecting theory and practice. In her final assignment Elsa refers to the book *Adding it up* and that she wants to establish communities of learners in her ideal school. When she introduces her idea to her fellow students they elaborate on the idea and discuss it in a professional way with reference to their shared experiences during the lesson study process.

### **Collaborative competence**

When using lesson study the challenge is to develop an effective way to teach a certain content or concept with some long term educational goals in mind. This is done in collaboration and participants have to present their ideas, discuss them and take joint responsibility for the planning of the lesson. They have to reason with each other and build on their previous experiences. They come to realize how important it is to collaborate, think and plan together. There are many things to consider when

planning a lesson and when it is done in collaboration more details are discussed and from different points of view.

Kristin: It was fun and rewarding to plan this together. We all thought we understood the plan the same way but it turned out differently.

It also encourages and gives the teacher students opportunity to try out things they otherwise would hesitate to do. They experience what to collaborate about in a learning community and what teacher collaboration can mean. The teacher students' collaborative competence strengthens during the lesson study process. It is evident they have more to discuss and they experience the benefit of having colleagues and common experiences to relate to. They feel it is important for their future development as mathematics teachers to be a part of a learning community and that they themselves have a responsibility in creating such a community. In their final assignment almost all the teacher students see themselves as a part of learning community planning and reflecting on teaching.

### **Focus on pupils' learning**

In the lesson study process the teacher students discuss possible teaching approaches. In choosing an approach they use their knowledge of pupils, their situations and different learning needs. The teacher students try finding an approach that many pupils will find appealing and is at the same time challenging to themselves. In the lesson plan they focus on what the pupils should do but they often fail to anticipate pupils' responses and how they as teachers should react to them. During the teaching of the lesson there is always at least one teacher student in the role of an observer. The teaching approach is well known to the observers and therefore it does not become the main object of study even though they have an interest in how the teaching develops. Instead the observers focus more on the pupils and their reactions and learning. In the evaluation meeting the teacher students often refer to particular pupils responses and become fascinated of how the different pupils deal with the content. Changes in the lesson plan are based on their analysis of the pupils' responses. Their main focus is on making the lesson a positive learning experience for the pupils. The teacher students feel that by choosing approaches like games and group work they succeed in creating good conditions for pupils learning. During the preparation of the lesson the teacher students express their ideas on what is important for the pupils' learning.

Elsa: We need to make use of their interests.

Anna: They have to be able move around.

Karen: It has to be fun and it should be hands on.

Berta: Pupils show more interest when they can work together and decide what to do.

When discussing their future teaching the teacher students point out the importance of getting to know the pupils ideas about mathematics and mathematics learning and base their teaching on that knowledge.



### **Focus on mathematical content**

The choice of content is influenced by what the teacher students are learning in mathematics courses. They have some prior knowledge of prime numbers, rules for finding prime factors and whether a number is a prime or not and rules for divisibility of numbers. But there are some shortcomings in their understanding that become obvious during their initial discussions. They make use of each other's knowledge, make connections and refine their understanding through common discussions. They become more aware of what it means to understand prime numbers and composite numbers.

Tina: The pupils know about prime numbers and ways to find them. They have also worked with divisibility, square roots and composite numbers. We need to discuss with them how to use this when finding out whether a number is a prime number or composite number.

Through the discussion the teacher students also better understand the importance of a good content knowledge for a teacher. They experience that good content knowledge gives you more power and flexibility when designing a lesson process. During the process all the teacher students become better in expressing their knowledge and for many their basic understanding of the content improves a lot. They realize that there are many things to consider if you want to teach this content with understanding.

### **CONCLUSION**

During the lesson study process the teacher students have developed their professional identity. They have realized the complexity of teaching but they have also become more eager and stronger in dealing with the complexity. In the beginning the teacher students focused on the teaching plan as a platform for collaboration. But during the process they realized that they needed to collaborate more closely regarding other aspects of the teaching and the learning of the content. They also felt how rewarding these collaborations were. They felt the need to verbalize their ideas and thinking, ask into others thinking and reflect on that. More and more teacher students became active and they became more able to dwell on things and discuss them in details. Both the distance students and the students on campus became more willing to collaborate and share ideas and they developed a community of learners where everyone had a voice. It was noticeable in the discussions that the teacher students were also ready to take risks and could provide some strong arguments for their ideas. They became more convinced that they could learn a lot from others and by trying out things in collaboration with others.

Trust and openness characterized the learning communities that were established. The fact that it was only a short term commitment and the teacher students had to take part in the project as a part of their study influenced their participation. Because it was not a long term commitment it did not involve much risk for them. It became a

positive experience that they can use and learn from when they enter into teaching. But trying to establish a learning community in school can become much more risky because it is more long term and there are other expectations and power structures in place. Because the teacher educators were a part of the learning community the teacher students did not have to take full responsibility for the process or the outcome of the process, the lesson plan.

The lesson study process is an important venue to connect theory and practice and it also gives the teacher educators an opportunity to enter the practice field more in the role of a partner than an evaluator. The focus was more on making the learning opportunities for the pupils learning better than on the teacher students' performance. Lesson study has a potential for changing the focus in teaching practice and puts the collaboration between the teacher educators, teacher students and practice teachers in a different context. Many teacher educators are trying to use and develop lesson study with teacher students and here there are many opportunities for further research and development. It is a future challenge to establish learning communities with all three partners, teacher students, teacher educators and practice teachers that could strengthen professional learning communities in schools

### REFERENCES

- Burroughs, E. A., & Luebeck, J. L. (2010). Pre-service Teachers in Mathematics Lesson Study. *The Montana Mathematics Enthusiast*, 7(2&3), 391-400.
- Darling-Hammond, L. (1998). Teacher Learning That Supports Student Learning. *Educational Leadership*, 55(5), 6-11.
- Fernandez, C. (2002). Learning from Japanese Approaches to Professional Development: The Case of Lesson Study. *Journal of Teacher Education*, 53(5), 393-405.
- Grevholm, B. (2006). Matematikdidaktikens möjligheter i en forskningsbaserat lärarutbildning. In S. Ongstad (Ed.), *Fag og didaktik i lærerutdanning. Kunnskap i grenseland* Oslo: Universitetsforlaget.
- Gunnarsdóttir, G. H., Kristinsdóttir, J. V., & Pálsdóttir, G. (2008). Theory and practice in mathematics teacher education: Building a learning community. In T. Jóhannsdóttir & K. Jónsdóttir (Eds.), *Relationen mellem læreruddannelsen og skoleudviklingen. Kongressrapport til den 10. nordiske læreruddannelses kongres*. Reykjavík: University of Iceland, School of Education.
- Gunnarsdóttir, G. H., & Pálsdóttir, G. (2010). Mathematics Teacher Education at Iceland University of Education. In B. Sriraman, C. Bergsten, S. Goodchild, G. Pálsdóttir, B. D. Søndergaard & L. Haapasalo (Eds.), *The First Sourcebook on Nordic Research in Mathematics Education* (pp. 467-477). Charlotte, NC.: Information Age Publishing.
- Hammerness, K., & Darling-Hammond, L. (2005). The Design of Teacher Education Programs In L. Darling-Hammond & J. Bransford (Eds.), *Preparing Teachers for a Changing World: What Teachers Should Learn and Be Able to Do* (pp. 390 - 442). San Francisco, CA: Jossey Bass.

## Working Group 17

- Hammerness, K., Darling-Hammond, L., & Bransford, J. (2005). How Teachers Learn and Develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing Teachers for a Changing World: What Teachers Should Learn and Be Able to Do* (pp. 358-389). San Francisco, CA: Jossey Bass.
- Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to Learn to Teach: An "Experiment" Model for Teaching and Teacher Preparations in Mathematics. *Journal of Mathematics Teacher Education*, 6(3), 201-222.
- Jaworski, B. (2005). New Beginnings and Recurring Themes. *Journal of Mathematics Teacher Education*, 8, 1-3.
- Jaworski, B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. . *Journal of Mathematics Teacher Education*, 9, 187-211.
- Jaworski, B. (2007). Learning Communities in Mathematics: Research and Development in Mathematics Teaching and Learning. In C. Bergsten, B. Grevholm, H. S. Måsøval & F. Rønning (Eds.), *Relating Practice and Research in Mathematics Education. Proceedings of Norma 05, Fourth Nordic Conference on Mathematics Education*. Trondheim: Tapir Academic Press.
- Krainer, K. (2003). Teams, communities & networks. *Journal of Mathematics Teacher Education*, 6, 93-105.
- Krainer, K., & Llinares, S. (2010). Mathematics Teacher Education. In P. Peterson, E. Baker & B. McGaw (Eds.), *International Encyclopedia of Education* (Vol. 7, pp. 702-705). Oxford: Elsevier.
- Lewis, C. C. (2002). *Lesson Study: A Handbook of Teacher-Led Instructional Change* Philadelphia, PA: RBS Reasearch for Better Schools, Inc.
- Lewis, C. C., & Perry, R. R. (2009). Improving mathematics instruction through lesson study: a theoretical model and North American case. *Journal of Mathematics Teacher Education*, 12, 285-304.
- Loucks-Horsley, S., Stiles, K. E., Mundry, S., Hewson, P. W., & Love, N. (2010). *Designing Professional Development for Teachers of Science and Mathematics* (3rd ed.). Thousand Oaks, CA: Corwin.
- Matthews, M. E., Hlas, C. S., & Finken, T. M. (2009). Using Lesson Study and Four-Column Lesson Planning with Preservice Teachers. *Mathematics Teacher* 102(7).
- Tsui, A. B. M., & Law, D. Y. K. (2007). Learning as boundary-crossing in school-university partnership. *Teaching and Teacher Education*, 23, 1289-1301.
- Wood, T. (2002). Demand for complexity and shophistication: Generating and sharing knowledge about teaching. *Journal of Mathematics Teacher Education*, 5, 201-2003.
- Wood, T., & Berry, B. (2003). What does "design reseach" offer mathematics teacher education. *Journal of Mathematics Teacher Education*, 6, 195-199. .

# THE USE OF THE EMPTY NUMBER LINE TO DEVELOP A PROGRAMME OF MENTAL MATHEMATICS FOR PRIMARY TRAINEE TEACHERS

Gwen Ineson

Brunel University, UK

*In England and Wales, mental mathematics has been emphasised in the primary school curriculum for over a decade but little has been done to prepare trainee teachers faced with the task of delivering this curriculum. The Williams Review (2008) recommends that this is an area which should receive 'careful attention' on initial teacher training programmes. Design-based research sought to use conceptual pedagogy to develop an effective programme for trainee teachers to develop their mental mathematics knowledge for teaching. This paper describes the use of the empty number line within the intervention programme for 129 trainee teachers on a one year post graduate training programme.*

**Key words:** Mental mathematics, subject knowledge, conceptual pedagogy

## INTRODUCTION

The mathematics curriculum in England and Wales has changed dramatically over the last ten years since the introduction of the National Numeracy Strategy (NNS) in 1998. Pupils in primary schools are expected to be competent in adding and subtracting any pair of two digit numbers before they are taught formal written calculation strategies. As a result, mental calculation has attained a dominant position in the primary curriculum and teacher education programmes in the UK have required adaptation to reflect these changes.

### Teachers' Subject knowledge

Until the 1970s, little research had been carried out to determine what sort of mathematics teaching made for effective pupil progress (Ball, 1990). However, during the next few decades, mathematics education theorists and researchers began to consider the implications of Shulman's (1986) general categories of subject knowledge for the specification of the mathematics knowledge requirements for teaching and the construct was appropriated and refined for the purpose of mathematics research and practice.

In attending to mathematics content for teaching, some researchers concentrated on subject matter knowledge and delved deeply into the nature and purpose of the discipline to illuminate the task of ascertaining the mathematics knowledge required of teachers of mathematics (Goulding, Rowland & Barber, 2002). Further research was inspired by respect for the discipline of mathematics. It reflected research studies conducted in the USA, for example Ma (1999) who was specifically concerned with discipline-specific knowledge for teaching mathematics in its content

(substantive and syntactical) and pedagogical forms, and in the UK, for example Askew, Brown, Rhodes, Wiliam, & Johnson (1997), McNamara, Jaworski, Rowland, Hodgen, and Prestage (2002) and Rowland and Turner (2008) who sought to relate subject knowledge to effective teaching.

Ma's work (1999) explored the relationship between discipline-specific knowledge for teaching and pedagogy. Her results showed that American teachers had limited knowledge of mathematics and that their knowledge was often faulty. These findings were not new to American researchers (Ball, 1990) but what was novel was the comparison with Chinese teachers. This and other similar examples led Ma (1999) to contend that Chinese teachers identified and encouraged the specific conceptual understanding which was of importance in covering these types of problems. She described it as a profound understanding of fundamental mathematics (PUFM) – teachers' recognition, within their teaching of:

- the connectedness of the simple but powerful ideas at the core of mathematics;
- the capacity of these ideas to sustain multiple perspectives and flexibility;
- the location of the ideas within a coherent and holistic disciplinary structure.

Crucially, within this specification, there was no differentiation between conceptual subject matter knowledge and pedagogical content (or specialist) knowledge. The conceptual foundation for PUFM was teachers' sense of the connectedness of mathematics ideas; in pedagogical terms it meant that they formed knowledge packages around a central core. Conceptually, teachers had an awareness of the importance of simple but powerful ideas, which were especially stressed and developed through pedagogy. Conceptually, teachers were able to entertain multiple perspectives; pedagogically this meant that, as teachers, they could analyze their advantages and disadvantages and lead students to a flexible understanding. The coherence of their teaching over time reflected teachers' conceptual sense of the primary mathematics curriculum as a whole. Pedagogically, it meant that they could exploit what pupils had already studied and create foundations for what was to follow.

### **Mental mathematics**

A new curriculum for mathematics was established in England and Wales in 1998 (DfEE, 1998) in order to raise mathematical achievement (Brown, Askew, Baker, Denvir & Millet, 1998). This new curriculum – the National Numeracy Strategy (NNS) emphasised the development of mental calculation, which was in line with research in mathematics education in the Netherlands, where mental calculation and, in particular, informal strategies, had been a key component of the Realistic Mathematics Education (RME) programme.

The hundred-square, commonly found in primary classrooms in England and Wales, and used in the Netherlands prior to the introduction of the Realistic Mathematics

Education programme, was seen as restricting, particularly in terms of mental calculation. The Empty Number Line (ENL) was introduced as a model in schools in the Netherlands, which helped pupils visualise the 'quantity value' of numbers. It replaced the practice of partitioning using base ten that encouraged pupils to concentrate on the 'column value' (Thompson, 1999) of numbers. Beishuizen (1999) emphasised the benefits of the approach taken in the Netherlands in terms of general mathematics competence:

'dealing with whole numbers supports pupils' understanding and insight into number and number operations much more than the early introduction of vertical algorithms dealing with isolated digits' (p.159).

Furthermore, Klein, Beishuizen and Treffers (1998) suggested that other benefits accrued from the action of replacing the hundred-square with the ENL, such as 'enhancing the flexibility of mental strategies' (p.427).

### **Mental Mathematics and Teachers**

The new mathematics curriculum left university teacher educators charged with the need to make provision within their courses and programmes for opportunities to promote and assess the development of trainee teachers' mental mathematics teaching competence.

So what were the knowledge requirements of teachers training to teach mental mathematics in primary schools? What did an understanding of mental mathematics mean for a primary trainee teacher charged with developing pupils' 'with the head' (Beishuizen, 1997) use of mental mathematics, as opposed to the 'in the head' notion of rote learning? These problems were formulated into the following research questions: What are the most effective interventions to enable trainee teachers to enhance their mental mathematics subject knowledge for teaching and how can these be implemented during the university-based element in a one-year postgraduate primary teacher education programme?

### **THE PROJECT**

The project involved the formulation of an innovative intervention programme to develop mental mathematics for teaching. Design-based research, through various diverse methods, involves accumulate a body of evidence that supports and enriches the theoretical principles underpinning a specific intervention and leads to the refinement of the intervention in situ. In their definition of design studies Wang and Hannafin (2005) argue that design-based studies provided access to:

...a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development and implementation (p. 6).

The purpose of design-based research is the development of contextually sensitive theories in real-world settings and for these reasons, they offer a methodological

option which met the specific needs of this study manifested in its research question and underlying commitment to theory development.

### **Theoretical principles for designing the intervention**

The design study was predicated on the belief that, in the context of teacher education, a knowledge-framed curriculum was an appropriate object for the study and development of trainee teachers' mental mathematics subject knowledge for teaching. Furthermore, the knowledge-framed curriculum was driven by two theoretical principles, one discipline specific, relating to the conceptual foundations of mental mathematics and the other context specific, relating to the embedding of the conceptual within teaching.

The first principle was that the conceptual foundations of mental mathematics knowledge for teaching implied a commitment to working with numbers as wholes, an understanding of the reciprocal nature of mathematical operations as well as a strategic approach to calculation.

The second theoretical principle was that, in order to extend and realise these conceptual features within their teaching, trainee teachers required access, through conceptual pedagogy - the integration of conceptual and pedagogic knowledge into a bigger whole - to a connected, flexible, coherent approach to subject knowledge for teaching. In other words, the course needed to make provision for the development of what Ma (1999) described as profound knowledge.

The design of curricular activities in the study was based on design principles created by the contextualisation of these theoretical principles within the context of teacher education. Activities were designed to:

- Connect the three conceptual features – whole numbers, interchangeable operations and strategic approaches; provide the means to generate data for review of the effectiveness of the activities by inviting trainees to model pupil behaviour
- Focus on the simple and powerful ideas of mathematics (such as number structure, the laws of arithmetic, principles of counting, equations); re-visit and provide opportunities for their identification
- Nurture flexibility through comparisons by trainees, individually and in groups, with others' perspectives, with their own previously held views and habits and through the application of mathematical ideas within discussions of 'best' strategies
- Reinforce the coherence of mental mathematics in relation to the primary curriculum

The process of design of activities and materials for use with a cohort of 129 primary trainee teachers during a year-long programme of teacher education was based on

these principles. Micro-analyses of trainee teachers' interactions with the designed activities - individually, through group work within the university and in teaching placement contexts – led to review and modification of activities while offering a theoretically-informed commentary on their effectiveness.

Two strategies were adopted to improve validity and reliability: the use of data collection methods that captured the complexity of trainees' interactions with materials and activities in as full a range of contexts as possible; and the creation of a partnership between a small group of trainees and the researcher as a means to challenge tacitly held assumptions and to establish consensus.

### **The intervention**

During the programme, interventions, both planned and reactive, gave recognition to the importance of the principles outlined above. Each of them permeated the mathematics programme's interventions, and specific activities were also planned to target particular areas.

The principles underlying the simultaneous presentation of the four number operations (addition, subtraction, multiplication and division) were derived from the requirements of mental mathematics – the reciprocity of the operations – and of conceptual pedagogy – in this case, connectedness and coherence.

Materials were prepared to highlight the way in which the empty number line could be used in teaching to model and, therefore connect, each of the four number operations. The intention during the initial stages of the session was to provide a stimulus for discussion within small groups in the form of: an explanation of the structure of the empty number line; a brief description of its introduction and use in the Netherlands; and an account of the way in which it was incorporated into the mathematics curriculum in the UK. At this stage group activity would involve the further exploration of the concept and the framing of questions for whole group discussion.

A variety of problems were chosen for use during the next stages of the session, with the specific aim of connecting the four operations. Once again, trainees were to be asked to participate in a group activity and to use the empty number line to find solutions to the problems. Pre-designed materials included the use of the empty number line to tackle, for example, division problems through the repeated subtraction of 'chunks'. Finally trainees were to be asked, in a group activity, to reflect on the appropriateness of the empty number line in the context of their own preparation for teaching and to reveal their thinking.

During the year, qualitative data on trainees' responses were gathered continuously during normal teaching sessions. This was achieved by creating opportunities for trainees to record the detail of the processes used individually and during group discussions and, where appropriate, to justify their responses in writing or verbally, in the whole group public arena.



## Working Group 17

Each mathematics session in the programme was planned to focus on progressing trainees' mental mathematics knowledge for teaching by providing access, through conceptual pedagogy, to a flexible, connected and coherent approach to subject knowledge for teaching. This involved a commitment to working with whole numbers, the relationship between mathematical operations and a strategic approach to calculation over a number of weeks and environments (university and school placement) during the one-year programme.

Following the ENL activities described above, some trainees queried the effectiveness of the use of the empty number line. They were directed to research about alternative approaches to partitioning when calculating and critiques of these approaches (Thompson, 1999). After considering the research, trainees in six seminar groups, each of between 25 – 30 trainees, discussed the possible effects of focusing on one approach rather than the other. Initially, three groups reported in the feedback sessions that some members could not understand the need for such a tool because they had learnt strategies for each of the operations which 'worked for them'. One group reported that this tool could become too much of a prop for pupils, encouraging counting in ones. For another group, the notion of using a number line was alien, as the experience of group members, prior to the course had tended to focus on a more formal approach to calculation. One member, commented that 'the empty number line is an interesting concept, but perhaps one of the most challenging methods for me to grasp'. It also became apparent that some trainees were developing an algorithmic approach when using the empty number line. During workshop sessions where trainees were looking at subtraction by counting back or counting on, one trainee commented 'I'm confused, which numbers do I write on the line?' The response by other trainees was that, 'you put the largest number on the right, then the other number at the other end'.

Such data led to the adaption of the intervention programme. Specific examples were introduced which allowed trainees to consider whether counting on or counting back was more appropriate. Trainees commented that specific examples also made them realise the importance of stopping to consider the numbers involved, which enabled them to assess the most appropriate calculation approach. The intervention was adapted further to provide opportunities for trainees to make use of individual whiteboards (similar to those used in primary schools) in order to do rough jottings using the empty number line. One trainee noted that the process of actually using the empty number line for her own calculations had enabled her to become more 'flexible and confident in the use of the empty number line for teaching'.

The comment made, by one trainee, that she had found the empty number line to be a 'revelation' in the classroom, generated overwhelming agreement. The following comments are representative of the group's views about the usefulness of the empty number line:

Trainee 1: The empty number line is a very important visual resource for pupils.

## Working Group 17

Trainee 2: I agree... that drawing jumps on the line works well as a natural way of keeping track and recording mental solution steps.

Trainee 1 described a scenario from her classroom, where a group of pupils were using the rounding and adjusting strategy. The example was  $98 + 137$ . The pupils rounded the 98 to 100, but were then unsure whether to add or subtract the 2 which had been rounded. She demonstrated to the group how she had used the empty number line to encourage the pupils to consider the whole numbers involved. She reported that the pupils went on to complete the examples successfully, and claimed that although the pupils did not continue to draw the empty number line, they said that they were visualising it. Trainee 1 was challenged about whether the method had become an algorithm, but she was confident that the empty number line was used in quite a different way, as a means for visualising the relative sizes of the numbers.

At this stage the opportunity to broaden the cohort's discussion was used to compare the whole numbers approach with alternative approaches in the form of formal written strategies or algorithms. A particular example was selected from an audit which had been used at the beginning of the year ( $199 + 174$ ). Data from this audit revealed that over half of the cohort had used a formal written strategy at the beginning of the year. A number of the cohort described this as an unacceptable use of a formal written strategy. A quick paper survey was undertaken to establish how many of the cohort agreed with this 'unacceptable' use of the formal strategy and it was found that 93% of the group (106 trainees) agreed with this description for this particular item. Some trainees reported that they had not considered an alternative strategy at the beginning of the programme because they were using what they described as their security blanket - those algorithms which were quick to administer and gave reliable results. One group admitted that its members had not considered the numbers involved before diving in to carry out a procedure. A few trainees from other groups said that they were reluctant to move away from the strategies with which they felt comfortable. They could not appreciate why pupils should be introduced to so many different strategies as they felt this could cause confusion. All other groups, however, concluded that while formal algorithms may be effective and, ultimately, be the most efficient strategy, the 'stop and think' approach would encourage thinking about the numbers involved, before making a strategy choice and that this was of greater importance. This reference to a 'stop and think' approach was re-visited throughout the programme in order for trainees to consider alternative approaches.

The empty number line was further developed during discussion regarding problems involving decimals, where trainees reported that they made use of the empty number line for these types of problems purely as a visual image, to consider the relative sizes of the numbers involved. In this context, one trainee noted that the empty number line had helped her to appreciate the structure of the number system. She wondered why she had been 'so concerned about working with decimals, because if

you just think about them on the number line, they are just normal numbers, but zoomed in'. The commitment to focusing on numbers as wholes was an indication of the success of the use of the ENL as a model for developing mental mathematics for teaching.

## CONCLUSIONS

As noted above, some trainees were initially sceptical about the value of the empty number line, but for many, this perception changed over the course of the intervention. After spending time in the classroom, trainees began to change their views as they saw the potential benefit of this approach. One trainee observed, during school experience, what was seen as the benefit of planning activities to emphasise connectedness using the ENL 'these children were slowly beginning to make some creative connections between numbers that they could use to help them solve a variety of calculations.'

Although most trainees concluded that a flexible approach to calculation was what they would strive for in their teaching, this commitment was not always evident in their own calculations. The following is one trainee's explanation of her initial response to  $199 + 174$ , where she had used a formal written method:

'When I look down I think 199 and I know automatically that you should see this as 200.... But when it is written down like that it's not, to me, it's not obvious that it would relate to 200....I definitely only see the digits.'

During review the data which had been collected throughout the programme was referred to and used to provide some insight into these contrasting experiences. A minority of trainees had routinely reported during the first workshops that they found the standard formal written methods efficient and straight forward to apply, and that it was difficult to understand why any other methods were taught in school. When invited to reflect on their experiences during seminars and in schools, there was general agreement that it felt as if they were 'learning backwards'.

Such comments indicate the potential of conceptual pedagogy for change. Even in a generally negative example there is evidence that a trainee was enabled to analyse and identify a lack of coherence as the source of the insecurity of her knowledge of mental mathematics:

'See this is the problem that I have got, I've got all these little facts and they are all over the place. So that, so somehow I know that obviously three quarters is 0.75 and then you times it by 10 and it's 75, I've got all these useless, they are not useless, but, well, they are useless unless I can apply them.....See that's the thing, I've got all these jumbled up facts and I know all these silly little things, that that needs to be that but I don't know why, there is no sense to it.'

This clearly showed that there had been insufficient time for this particular trainee to come to terms with her early experience of mental mathematics, which was

sufficiently strong to resist the coherence promoted in the programme. However, the statement shows that the trainee does appreciate the need for integration and for meaning in the knowledge possessed by teachers of mental mathematics.

Others had been able to use experience of the programme to come to terms with their own knowledge base. One trainee explained how the programme had affected her mathematics knowledge:

'Like most students I would instinctively choose the column method and (abbreviated) long division for these problems, but I would have great trouble teaching them and this brought home the requirements that I must in a sense *ignore* my own knowledge and capabilities.'

This comment is yet further evidence of the recognition that teachers have a different requirement in terms of mathematics knowledge. It is not sufficient to know mathematics, what is significant is the way in which mathematics is known for teaching. These types of comments suggested that, even if they were not fully successful in terms of their own facility with mental mathematics, trainees were able to appreciate the significance of the different approach to mental mathematics that is required of teachers.

Although some trainees had not found a way to resolve their deficiencies in their own mental mathematics through engagement in the programme, for the overwhelming majority of participants, who claimed that they now recognised the negative and lasting effects of embedded routines and practices learnt while they were at school, the situation was acceptable, since they argued that they could operate differently in their teaching. Further speculation about the intellectual significance of the way trainees dealt with the dissonance created by their own development would not be warranted on the basis of this study's data. However, the study has provided legitimate grounds for seeking answers to questions about the limits, durability and architecture of this way of holding knowledge.

At this stage it is possible to conclude that the evolving behaviours of trainees fuelled the development and refinement of an intervention that was based on conceptual pedagogy as applied within the context of mental mathematics knowledge for teaching. Successful outcomes include the illumination and exemplification of existing theoretical constructs such as coherence and connectedness over a range of mental mathematics content involving the use of the empty number line.

## REFERENCES

Askew, M., Brown, M., Rhodes, V., Wiliam D & Johnson, D. (1997). *Effective Teachers of Numeracy*. London: King's College London, for the Teacher Training Agency.

Ball, D. L. (1990). Prospective Elementary and Secondary Teachers' Understanding of Division. *Journal for Research in Mathematics Education*, 21(2), 132-144.

- Beishuizen, M. (1997). Mental arithmetic: mental recall or mental strategies? *Mathematics Teaching*, 160, 16–19.
- Beishuizen, M. (1999). The empty number line as a new model, in Thompson, I. (Ed.) *Issues in teaching and numeracy in primary schools* (pp.157–168). Buckingham, Open University Press.
- Brown, M., Askew, M., Baker, D., Denvir, H. & Millet, A. (1998). Is the National Numeracy Strategy research based? *Research Review*, 46 (4), 362–385.
- Davis, S. (2009). Oral and Mental Mathematics. *Mathematics Teaching*, 125, 45-48.
- Department for Education and Employment. (DfEE). (1998). *The NNS framework for teaching mathematics from reception to year 6*. London. DfEE.
- Goulding, M., Rowland, T. & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics, *British Educational Research Journal*. 28 (5), 689-704.
- Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic Versus Gradual program design. *Journal for Research in Mathematics Education*, 29 (4), 443-464.
- Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.
- McNamara, O., Jaworski, B., Rowland, T., Hodgen, J., & Prestage, S. (2002). *Developing Mathematics Teaching and Teachers: A Research Monograph*. [www.maths-ed.org.uk/mathsteachdev/pdf/mdevpref.pdf](http://www.maths-ed.org.uk/mathsteachdev/pdf/mdevpref.pdf).
- Rowland, T. & Turner, F. (2008). Issues for Further Research. Paper Presented at *Seminar 6: Formulating a Research Agenda on Mathematical Knowledge in Teaching*. Nuffield Seminar Series.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15 (2), 4-14.
- Thompson, I. (1999). Implications of research on mental calculation for the teaching of place value, *Curriculum*, 20, (3), 185-191.
- Wang, F., & Hannafin, M. J. (2005). Design-based research and technology-enhanced learning environments. *Educational Technology Research and Development*, 53 (4), 5-23.

# **READINGS OF THE MATHEMATICAL MEANING SHAPED IN THE CLASSROOM: EXPLOITING DIFFERENT LENSES**

Maria Kaldrimidou, Haralambos Sakonidis, Marianna Tzekaki

University of Ioannina, Democritus University of Thrace, Aristotle University of Thessaloniki

*Research literature indicates that the mathematical knowledge interactively constructed in the classroom is closely related to teachers' management of the epistemological features of mathematics. In this study, we focus on teacher-students' classroom interactions, adopting a learning and a teaching perspective to examine the mathematical status of the knowledge emerging in the context of two primary teachers' lessons. The results indicate the limits of these two perspectives and the need for multiple lenses in order to adequately capture the complexity of this knowledge.*

**Keywords:** mathematical meaning, epistemological features, teaching practices

## **INTRODUCTION**

Several studies attempted to examine the 'quality' of the mathematical knowledge shaped in the classroom (e.g., Steinbring 1998). The theoretical frameworks adopted in these studies allowed the identification of limitations and deficiencies in teachers' classroom practices but do not appear to have provided powerful lenses for judging the actual nature of the mathematical meaning constructed by the pupils (Kaldrimidou et al. 2008). In the present paper we adopt a learning and a predominately didactical perspective to pursue further the search for analytical tools that enable us to effectively examine the epistemological status of the mathematical knowledge interactively constructed in the classroom.

## **THEORETICAL ISSUES**

Recent developments suggest that socio-cultural factors cannot be ignored in teaching and learning mathematics, thus directing attention to the adult-learner interactions taking place in the classroom (Lerman, 2006). Two distinct patterns of these interactions have been indicated early on by Wood (1994): funnelling (pupils are provided with leading questions which aim at guiding them to a pre-determined answer) and focusing (students' attention is drawn to the critical aspects of the problem through questioning and summarizing what is taken as shared knowledge).

Anghileri (2006), in an attempt to "identify a hierarchy of interactions which relate to teaching practices that can enhance mathematics learning" (p. 33), re-visited the

metaphor of ‘scaffolding’ introduced by Wood et al. (1976) and suggested the notion of three levels scaffold. At level 1, environmental provisions of tasks and resources enable learning to take place without the direct intervention of the teacher. Such provisions include artefacts (e.g., manipulatives) and classroom organization (e.g., peer collaboration), sequencing and pacing events, structuring tasks, etc. The subsequent two levels are related to teacher interactions that are increasingly directed to enriching the support of the mathematical learning.

Level 2 concerns three types of direct scaffolding interactions between teacher and students related specifically to mathematics: showing and explaining the ideas to be learned (limited attention to pupils’ contributions), reviewing and restructuring (responsiveness to the learner). Reviewing is about encouraging students’ reflections and concerns five types of interaction: getting students to look, touch and verbalize, getting students to explain and justify, interpreting students’ actions and talk, using prompting and probing questions and parallel modelling (solving auxiliary tasks). Restructuring, on the other hand, is about altering students’ understandings and regards interactions like providing meaningful contexts to abstract situations, simplifying the problem, rephrasing students’ talk and negotiating meaning.

At level 3, interactions address developing conceptual thinking through specialized processes such as generalization, extrapolation and abstraction. Interactions at this level include making connections, developing a range of representational tools and generating conceptual discourse (initiating reflective shifts and accentuating mathematically valued ways of thinking).

Anghileri’s notion of the three levels scaffold appears to offer a challenging socio-cultural lens to explore the mathematical meaning constructed as a result of classroom interactions, allowing for students’ creativity and teachers’ responsiveness to individuals to be taken into account in the shaping of this meaning.

The quality of the mathematical meaning constructed in the classroom came under scrutiny also in studies inquiring into the relationship between the mathematical knowledge for teaching (MKT - the subject matter knowledge both common to diverse professions and supporting teaching) and the mathematical quality of instruction (MQI – determined by features characterizing the rigor and the richness of the mathematics of the lesson, e.g., the mathematical justifications exploited).

Early studies in this area identified certain deficits in the mathematics teaching: significant mathematical inaccuracies, e.g., definitions missing key elements, as well as mathematically poor teaching patterns, e.g., the use of material that does not support further development (e.g., Cohen 1990). Follow-up studies focused on the practices of teachers involved in reform initiatives, indicating their tendency to adopt practices enriching the mathematics available to the learner: open-ended questioning, careful work with definitions, linking representations, probing for students explanations, and so on (e.g., Lampert 2001).

Hill et al. (2008) lately attempted to carefully test the relationship between MKT and MQI. To this purpose, they pointed out to five determinants of the mathematical quality of instruction: mathematical errors (computational, conceptual, representational or other), responding to students appropriately or inappropriately (addressing successfully or unsuccessfully their mathematical utterances or misunderstandings), connecting classroom practice to mathematics (relating the work done to significant mathematical ideas), richness of mathematics (effective use of multiple representations, provision of mathematical explanation and justification, explicitness about mathematical practices like proof and reasoning) and mathematical language (coherent and accurate language to present mathematical ideas).

The approach suggested by Hill et al. (2008) allows for concretely defined features of the mathematical knowledge to be traced in specific mathematics teaching practices, thus constituting a promising candidate for examining the mathematical knowledge shaped in the classroom.

On the basis of the above, it could be argued, Anghileri (2006) and Hill et al. (2008) offer two challenging but rather distinct frameworks for inquiring into the mathematical outcome of the teacher-learners interactions in the classroom. In both perspectives, these interactions are seen in the context of the instructional practices employed by the teachers. However, the status of the mathematical outcome is examined through the learning process from within Anghileri's framework and via the instructional process from within Hill and colleagues' perspective.

In previous studies we adopted successively socio-constructivist, interactionist and epistemological perspectives to examine the mathematical meaning shaped in the classroom, concluding that they all offer partial readings of its nature (Kaldrimidou, et al. 2008). Moving forward, in the present study we exploit comparatively the lenses provided by Anghileri and Hill et al. in yet another search of analytical tools that allow for a satisfactory identification of the epistemological status of the meaning constructed in the mathematics classroom.

## THE STUDY

The aim of the study was to explore the degree to which each of the two perspectives allow for the identification of a) the mathematical status of the meaning shaped in the classroom and b) the task management efficiency (i.e., how students' productions were dealt with) in relation to a).

Two primary teachers, a young male (A) and a more experienced female (B), both professionally well developed, participated in the study. The teachers were asked to teach a lesson on the properties of symmetry designed for 5<sup>th</sup> grade students. The central task required from the students to work in groups and to decide whether eight pairs of triangles were symmetrical or not and why. The students were expected to eventually note that, in order for the triangles to be symmetrical, they should be



equal, to the same distance from the axis and reverted. The teachers discussed the task and issues related to its management with one of the researchers thoroughly. In particular, they were encouraged to let students experiment and to support them in their search of the properties of symmetrical figures, intervening only when it was absolutely necessary.

The teaching session devoted to the task by each of the two teachers was audio-taped, transcribed and carefully examined, in search for episodes that would exemplify and substantiate the categories suggested by Anghileri (2006) and Hill et al. (2008).

*(a) Results I: Anghileri's framework*

The analysis of the data showed that, in general, the two lessons were very much alike, sharing a central characteristic: the students were allowed to work on their own and to develop their own ideas, but they were not encouraged to present them synthetically to the class, thus missing opportunities for negotiation of meaning and connecting ideas, that is, for systematizing the properties of symmetrical shapes.

More specifically, the students in teacher A's class came up with all the required criteria simultaneously, while in teacher B's class each group ended up with a different criterion of symmetry. The quotes below provide an idea of how teachers A and B attempted to prompt their students' work respectively (Extract 1):

Teacher A: We must find something, so that we can say whether a shape is symmetrical to another or not. The easy solution is folding, but we cannot always fold ... If it is drawn on the blackboard, how will you tell me whether it is symmetrical? We need to find ways that will allow us, when we see two shapes, to be able to say whether they are symmetrical about an axis or not.

Student in teacher's B class: In order for the two shapes to be symmetrical, they have to have the same shape ...

Teacher B: Fine! Is it only then? Have you observed something else? Look at the cases one by one. See if you agree that, indeed, there is symmetry .... What things should be happening, what should be in effect?

The two teachers' scaffolding practices indicate that they mainly function at level 2 (Restructuring & Reviewing). Initially, they appear to adopt mostly *reviewing* strategies, as they often invite students to verbalize or explain (Extract 2):

Teacher A: What does the (symmetry) axis do to the two shapes? ... What is the first thing we ended up with? The first result in relation to the two shapes? ... What do we mean when we say that it makes them the same?

Teacher B: Fine, and what is the problem in the first case, why there is no symmetry? ... Does it matter where the (symmetry) axis is or what matters is where the

## Working Group 17

two shapes are? ...Since what matters is where the (symmetry) axis is, where should this line be then, in order to have symmetry?

Also, the two teachers tend to often *prompt* children via questioning (Extract 3):

Teacher A: The one (shape) should look right and the other left? ...Can it also be something else?... Initially, are they (symmetrical) or not? Why they are not? Which are the three elements that we mentioned?

Teacher B: Before we get to the shapes, have you something to complete in the axis? ...  
... Since we agree for case 5, let's move to case 6. What happens here?

The restructuring practices followed by the two teachers are limited and mainly related to *rephrasing* occasionally students' thoughts (Extract 4):

Student A<sub>A</sub>: The fourth case is not (a symmetry case), because, first of all, they are different shapes ...

Teacher A: It is not, Tasos says, because they are different shapes, one is small and one is large, so it goes. And the other two? Are they the same?

Student B<sub>B</sub>: It should be the same...

Teacher B: The same shape! There should be the same shape on both sides of the axis! Do you agree with this?

Possibly the most interesting but not frequent scaffolding strategies are those bearing characteristics similar to the ones present in *negotiation of meaning* situations (Extract 5):

Student C<sub>A</sub>: One (shape) should look to the right and the other to the left ...

Teacher A: The one should look to the right and the other to the left? Can it also be something else?

Student C<sub>A</sub>: Yes, one of the two should look to the opposite side

Teacher A: It is not only right-left. I have above-below, northern-southern, eastern-western ...

Student C<sub>A</sub>: The one should look at the opposite side compared to the other

Teacher A: That is, when someone wishes to go somewhere, how would you explain to him? Are you going to only tell him right-left? How do we call all this?

Student D<sub>B</sub>: The one shape shouldn't be like the other, because they are opposite to one another

Teacher B: They should be the same, but not exactly the same, they should be opposite to one another! I would like you to explain this "opposite" to me ...What is the meaning of this "opposite"?

Student E<sub>B</sub>: They should be equal but from the other edge, the edges should be opposite to one another.

## Working Group 17

Student F<sub>B</sub>: They should be opposite to one another, that is, they shouldn't be...

Teacher B: I heard a word before, what does this "opposite" you are trying to explain to me mean?

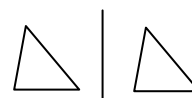
Despite their common way of dealing with pupils' ideas, some distinct differences are also noticeable. In particular, the language employed by teacher A to express mathematical meanings appears to maintain a certain level of accuracy, while teacher B is more tolerant to the adoption of a vernacular, everyday language to this purpose.

Also, despite the fact that both teachers' practices remained at level 2, teacher A tended to be more exploratory than teacher B in utilizing pupils' ideas. For example, he attempted to exploit a counter-example related to representational aspects, initiated by one student's suggestion (Extract 6):

Student G<sub>A</sub>: Two shapes, in order to be symmetrical, they have to be exactly the same.

Teacher A: (He draws on the board). Are these symmetrical?

Student G<sub>A</sub>: (Tries to explain) ... the triangle should be like this  
(he rotates)



Teacher A: We rotate it (he rotates the right-hand shape).  
This one?



Student G<sub>A</sub>: Ah, no! (they carry on looking for a way to say it)

Thus, teacher A tried to summarize the symmetry criteria identified by the pupils within a new representational framework, unlike teacher B, who maintained a rather funneling approach until the end.

Based on the above, it could be argued that Anghileri's approach appears to offer a useful reading of teachers' exploitation of their pupils' ideas but does not help in making sense of the mathematical meaning emerging in the classroom. In particular, it provides a rather limited tracing of the students' approach to the task and also of their actual understanding of the properties of symmetrical shapes.

### (b) *Results II: Hill and colleagues' lenses*

Unlike Anghileri's approach, which makes possible the exploration of the mathematical meaning constructed in the classroom from within a learning perspective, the framework by Hill et al. (2008) turns the lenses on the teacher and his/her teaching actions, trying to identify how these actions affect the final mathematical outcome.

To this direction, the analysis showed that both teachers made very few *mathematical errors*. In fact, the only ones identified were related to mathematical justifications and explanations provided by them and, on the whole, concerned no or incomplete explanations. For example, in the extract below, the term 'symmetry' is incompletely described by teacher A, whereas in an incomplete, if not problematic,

## Working Group 17

justification is provided with respect to the meaning of each of the words ‘next’, ‘opposite’ and ‘same’ and their interrelationship by teacher B (Extract 7):

Teacher A: What does the symmetry axis do to the two shapes?

Student H<sub>A</sub>: It makes them the same

Teacher A: When we say it makes them the same, what do we mean?

Student H<sub>A</sub>: Symmetrical

Teacher A: Symmetrical! Our triangles can be similar but not symmetrical. It is another thing to be ‘similar’ and another to be ‘symmetrical’! Symmetrical are two things separated by this axis of symmetry, the way it is on the one side, it is also on the other side.

Teacher B: Finally, the shapes are next or opposite to one another? I want you to tell me a little better what do you mean by ‘opposite’.

Student I<sub>B</sub>: The axis of symmetry separates the shapes

Student J<sub>B</sub>: You can say that they are both opposite and next to one another

Teacher B: Mmmm... They are the same

As for *connecting classroom practice to mathematics*, teacher A appears to occasionally try to accentuate mathematical elements in the work carried out in the classroom by linking this work with mathematical ideas and procedures. Thus, in the introduction of the lesson, he explicitly stresses the necessity of identifying criteria for deciding whether two shapes are symmetrical or not (see Extract 1).

Closing the session, teacher A brings forward the most critical question, that is, whether the three criteria should be fulfilled at the same time, in order for the plane shapes to be symmetrical (Extract 8):

Teacher A: Fine! K<sub>A</sub> says that we know that these shapes are not symmetrical, because they are different! Therefore, in order to examine if two shapes are symmetrical, we need to examine all three criteria or only one?... I must check all three or only one of the criteria, as your schoolmate argued?

Student K<sub>A</sub>: If one of the criteria does not apply, you do not need to look for the rest.

However, this behavior of teacher A is not stable, as he occasionally slips to contradictory actions. In particular, while he resorts to the constructivist exploitation of counter-examples, in order to help students identify the criteria for symmetry, he simultaneously deters them from using folding to realize its presence (providing practical but not mathematical reasons for this). Folding might not be a mathematical procedure, but it is the only available to the students at the time being, in order to process this mathematically significant checking. Thus, students are left with no choice but to judge holistically by visual means (see Extract 1).

There is not much connecting of classroom practice to mathematics in teacher B’s lesson. The only action that could be seen as contributing to this direction is the

## Working Group 17

emphasis he places upon checking, allowing the use of folding but without explicit reference to the mathematical necessity of it.

On the whole, both teachers appear to be effectively *responding to pupils* by encouraging them to reflect on their sayings. There are very few cases where this is done inappropriately (Extract 9 – see also extract 1):

Teacher A: You say that the two triangles, when joint, should make up one what ...?

Student L<sub>A</sub>: When you will join the two shapes, you should have either a square or a rectangle or a rhombus.

Teacher A: Two shapes are symmetrical about an axis, when ...? We need to bring together these three things, eh? 'Separate', 'share' and 'two symmetrical shapes' indicate different things, eh? Forget folding! How should it be?

Teacher B: What do you mean by this 'opposite', because I do not understand. Can you explain using more mathematical terms?

Student K<sub>B</sub>: They look each other ... The two 'points/noses' of the shapes are different, but is like when two people look at each other...

Teacher B: So, which points should be opposite to each other?

Both teaching sessions appear weak with respect to the *richness of the mathematics* and, in particular, of the mathematical explanations and justifications. Even teacher A, who makes explicit references to practices and ways of reasoning does not invoke mathematical necessity or does not explain why.

Finally, with respect to the *mathematical language*, both teachers tend to question the meaning of the mathematical terms used by the pupils, but the descriptions provided remain often incomplete. Also, terms with double meaning are utilized without clarifications and self-evident notions and relationships are presented in confusing ways (e.g., below, the equality of shapes turns to equality of sizes, a necessary but not sufficient condition for the former).

Teacher A: We have two shapes; the distance from the axis should be the same! ... So, the first is that the distance of my shapes must be the same, I do this. Are they symmetrical now?



Student L<sub>A</sub>: They are not, because they have different size.

Teacher B: They should be reverse ... the same sides of the shape should be the one opposite to the other...(A little later on, talking about the axis of symmetry)

Teacher B: Mmm... what should be to equal distance from the axis? When we say 'in the middle', what should be equal?

Student M<sub>B</sub>: The shapes?

Teacher B: The shapes should be what, equal with respect to what?

Student M<sub>B</sub>: Size.

Based on the above, Hill et al. perspective allows for the identification of some but not of other significant aspects of the two teachers' teaching actions closely related to the quality of the emerging mathematical knowledge. For example, teacher A pays attention to the use of mathematically 'legitimate' processes, thus reinforcing the mathematically unacceptable visual means of judging whether two shapes are symmetrical. Teacher B, on the other hand, encourages the exploitation of not mathematically 'legitimate' processes (folding), attributing to checking special mathematical value. These contradictions in the two teachers teaching practices are poorly addressed by this particular perspective. The same is true for issues related to the epistemological nature of the enacted argumentation. For example, in both classes, the development of a definition turns to applying the definition (symmetry), giving rise to a distorted meaning for symmetry (e.g., that the line joining a point and its symmetrical should be vertical to the axis of symmetry is neither addressed nor identified).

## CONCLUDING REMARKS

The classroom management of the same task carried out by different teachers offers a fruitful occasion to examine the emerging mathematical meaning. This is because the fixed mathematical content allows for the identification of the impact of specific learning and teaching issues on the mathematical outcome. To this direction, the present study adopted two distinct analytical tools, one focusing on the learning and the other on the instructional management of two teachers, in order to examine the quality of the mathematical knowledge given rise in their classrooms.

The two approaches utilized highlighted different aspects of the teaching process. Anghileri's perspective essentially indicated the ways in which the teachers interacted with their students and dealt with their ideas, revealing a moderate effort to exploit these ideas in supporting the mathematics learning. On the other hand, the lenses offered by Hill and her colleagues related the two teachers' management of their pupils thinking with only certain mathematical features. An especially interesting issue within this context was the use of language, one of the differentiating characteristics of the two lessons.

Bearing in mind the central research question of the study, that is, what is the mathematical meaning shaped by the two teachers' actions in the classroom, as they deal with the same task, it appears that both approaches provide some useful insights into this shaping, without though allowing a satisfactory access to the status of the mathematics constructed by the pupils. In particular, Anghileri's analytical lenses showed that both teachers dealt with pupils' ideas in a funneling mode, with the exception of teacher A, who was functioning predominately at level 2, moving only occasionally to level 3. Hill et al perspective, on the other hand, indicated how concrete aspects of the two teachers practices affected particular features of the mathematics emerging in the classroom.

The analysis of the two teachers' transcripts identified some common characteristics of the two lessons in relation to the mathematical errors and the management of the students' thinking carried out by the teachers. However, some differences concerning the connection of the teaching practices to the corresponding mathematical concepts were also apparent. Thus, both perspectives seem to constitute valuable tools for analyzing mathematics teaching as far as the orientation followed by the management of the tasks in the classroom is concerned. This is an issue of great interest for mathematics education in general and for mathematics teachers' education in particular, because, as Wood (1994) argues, passing from funneling (dominating traditional teaching) to focusing (characterizing constructivist teaching) requires a clear understanding of the essential changes in process.

Beyond this, however, the question of what mathematical meaning is constructed in the classroom is an extremely complex issue requiring the combination of approaches in order to be captured.

## REFERENCES

- Anghileri, J.(2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, 9, 33-52.
- Cohen, D.K. (1990). A revolution in one classroom: The case of Mrs Oublier. *Educational Evaluation and Policy Analysis*, 12, 311–330.
- Hill, H.C., Blunk, M. L., Charalambous, C.Y., Lewis, J.M., Phelps, G.C., Sleep, L., & Ball, D.L.(2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction*, 26(4), 430 - 511.
- Kaldrimidou, M., Sakonidis, H., & Tzekaki, M. (2008). Comparative readings of the nature of the mathematical knowledge under construction in the classroom, *ZDM – the International Journal on Mathematics Education*, 40, 235 – 245.
- Lampert, M. (2001). Teaching problems. New Haven, CT: Yale University Press.
- Lerman, S. (2006). Socio-cultural research in PME. In A. Gutierrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education* (pp. 347-366). Rotterdam/ Taipei: Sense Publishers.
- Steinbring, H. (1998). Epistemological constraints of mathematical knowledge in social learning settings. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain* (pp. 513-526). Dordrecht: Kluwer.
- Wood, T. (1994). Patterns of interaction and the culture of mathematics classrooms. In S. Lerman (Ed.), *Cultural perspectives on the mathematics classroom*. Dordrecht: Kluwer.
- Wood, D., Bruner, J. & Ross, G. (1976). The role of tutoring in problem solving. *Journal of Child Psychology and Psychiatry*, 17, 89–100.

# THE NATURE OF PRESERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

Hulya Kilic

Yeditepe University

*The focus of this paper is to present some of the findings emerged from a study investigating the development of preservice mathematics teachers' pedagogical content knowledge (PCK) in a methods course and its associated field experience. Six preservice teachers participated in the study and the data were collected in the forms of observations, interviews and written documents. The analysis of data revealed that preservice teachers' knowledge of subject matter was influential on the development of their PCK.*

**Keywords:** pedagogical content knowledge, preservice, mathematics, secondary

## INTRODUCTION

The major goal of teaching is to enhance students' understanding and learning. Teachers need to be equipped with various knowledge and skills to establish and maintain effective teaching environments that enable them to achieve that goal. Shulman (1986) used the term pedagogical content knowledge (PCK) to name a special knowledge base that involves interweaving such various knowledge and skills. He defined PCK as "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). He stated that PCK includes teachers' knowledge about specific topics that might be easy or difficult for students and possible conceptions or misconceptions that student might have related to the topic.

Many scholars accept PCK as a distinct knowledge domain for teachers but there is no single definition of PCK due to ambiguity of what constitutes PCK (e.g., Ball, Thames, & Phelps, 2008). Because PCK is perceived as knowledge of how to teach a particular subject matter (An, Kulm, & Wu, 2004), viewing PCK as the integration of content and pedagogy would not address all requirements needed for effective teaching. Teachers not only need to possess knowledge of subject matter and pedagogy but also they need to know about students, curriculum, educational goals, and instructional materials to promote students' understanding as well as to achieve learning goals identified in the curriculum. Therefore, some scholars (e.g., An, Kulm, & Wu, 2004; Marks, 1990) accept knowledge of subject matter, knowledge of pedagogy, knowledge of students, and knowledge of curriculum are the components of PCK. Teachers need to know characteristics and needs of a particular group of students, and their conceptions and misconceptions about a particular topic that will be taught. They also need to know the arrangement of the topics covered in a particular grade level and how to use curriculum materials to achieve the learning goals identified in the written curriculum.



Pedagogical content knowledge is assumed to be developed as teachers gain more experience in teaching because it is directly related to the act of teaching (Borko & Putnam, 1996). However, studies of preservice mathematics teachers' knowledge and skills related to teaching have revealed that methods courses and field experiences are likely to contribute to the development of PCK to some extent (e.g., Tirosh, 2000). Although there is no widely accepted standardized instrument specifically developed to measure teachers' PCK or the development of their PCK, researchers could learn about the nature of teachers' PCK by using different methods such as classroom observations, structured interviews, questionnaires, and journals (e.g., An, Kulm, & Wu, 2004). A methods course for mathematics teachers could be designed in a way that preservice teachers would have various opportunities such as analyzing students' error, developing a task, and microteaching to improve their PCK (e.g., Ball, 1988). Therefore, I aimed to investigate what components of preservice secondary mathematics teachers' PCK developed in a methods course and its associated field experiences.

## **THEORETICAL FRAMEWORK**

Based on my review of the literature of which I could only discuss very limited part of it above, I accepted that PCK involved four aspects of knowledge; subject matter, pedagogy, students, and curriculum, and there exists reciprocal relationships between them. In my definition of PCK, knowledge of subject matter refers to both teachers' procedural knowledge and conceptual understanding of mathematics. Knowledge of pedagogy refers to teachers' ability to choose appropriate tasks, examples and representations for a particular group of students and their repertoire of teaching strategies. Knowledge of students involves teachers' knowledge of students' conceptions, misconceptions, and possible difficulties about a particular topic and their ability to diagnose and eliminate such misconceptions and difficulties effectively. Finally, knowledge of curriculum includes knowledge of learning goals for different grade levels and knowledge of instructional materials. Although Ball et al. (2008) separate knowledge of subject matter from PCK in their model of Mathematical Knowledge for Teaching (MKT), they accept that teachers should know the content to teach, to help the students and to understand and apply the curriculum. Therefore, I accept knowledge of subject matter as one of the aspects of PCK.

## **METHODOLOGY**

I conducted a qualitative study to investigate what aspects of secondary mathematics preservice teachers' PCK developed in a methods course and its associated field experience in fall 2008 at a large public university in the southeastern U.S. I wanted to understand the variety and the extent of the issues discussed in these courses and how preservice teachers could benefit from those discussions and field experiences.

There were 29 preservice teachers taking both courses. I administered a questionnaire at the beginning of the semester to learn how they perceived their level of PCK. The questionnaire consisted of 13 items; eight of them were multiple-choice, one was Likert-type and four were short-answer question. Based on their overall scores, I formed a representative group consisting of six preservice teachers varying degree of perceived level of PCK (low, medium, high).

I was a participant-observer in all class sessions in both classes and I took some notes and collected any written documents given in the courses. I also conducted three interviews with each participant during the semester. At the beginning of the interviews, I asked them to reflect on the issues discussed in the methods and the field experience courses and how they contributed to each aspect of their PCK. Then I gave them some content-specific questions to understand the nature of their PCK. Although the methods course and its associated field experiences were not designed with an intention of developing preservice teachers' PCK, in each session, the preservice teachers were discussing how to teach a particular mathematical concept that was determined by the instructor. Therefore, I used my field notes to prepare content-specific tasks that I asked during the interviews. I also wanted the preservice teachers to reflect on their field experiences. I looked at the students' assignments to gain a better understanding of the course topics and students' thoughts and reflections about those topics. During the last interview, I gave them a shortened version of questionnaire to see how they perceived their knowledge levels at the end of the semester. Furthermore, I asked them to make an overall evaluation of the methods and field experience course in terms of their gains from these courses. Then, I transcribed all interviews and coded them according to the PCK framework developed for this study. I compared the answers to similar types of questions to determine the similarities and differences between the explanations and also to detect any change, if there was one, in their knowledge level of that particular knowledge domain. I discussed my decisions about each participant's responses to the interview questions with a member of faculty from the mathematics education department and we agreed on almost all of them.

## RESULTS

I identified four salient features of the nature and the development of preservice teachers' PCK: 1) knowledge of subject matter is a crucial aspect of PCK and influences the quality of the other aspects of PCK, 2) the course practices and field experiences raised preservice teachers' awareness of some issues of teaching and learning mathematics; however, they were not able to apply this knowledge, 3) the preservice teachers benefited from the course practices and field experiences to varying degrees, and 4) the preservice teachers generally overestimated the level of their knowledge of each aspect of PCK. Because of space limitation, I will not discuss the findings in detail, but give examples from interview data to support only the first bullet.

## Choice of Teaching Activities, Tasks, and Examples

The preservice teachers' knowledge of pedagogy was investigated in terms of their choices of tasks, examples, and teaching activities and their repertoire of teaching strategies. The findings revealed that the preservice teachers' choices of teaching activities, tasks, and examples depended on their views of teaching and learning mathematics. Collectively, the preservice teachers viewed mathematics as the set of rules, procedures, and facts. When asked to teach a particular topic, they mostly stated mathematical facts and described how to carry out the procedures or apply a rule. Given a set of examples and asked to place them in the order in which they would solve them, some of the preservice teachers looked at their surface features such as the number of terms involved in a given equation or number of steps to solve that equation rather than paying attention to how the examples would facilitate students' understanding. For instance, given the task shown in Figure 1, four of the preservice teachers preferred to begin with the fourth example because it seems easier. One of them reported that she would begin with the second one because she would get a linear equation when cross multiplying the terms and it is easier to solve a linear equation. The other one would begin with the third example because students are familiar with adding fractions with unlike denominators so they would solve it easily.

Similarly, during the second interview I asked them put given linear equations in an order to teach how to graph linear equations. Two of the preservice teachers preferred to begin with line  $y=5$  because it is horizontal line and there is nothing to think about. Although drawing horizontal line is easier than drawing the graph of  $y=x+5$ , students usually fail to recognize that  $y=5$  is a line not a point. Therefore, teachers should make sure that students know the difference between a line and a point.

In which order would you like to use the equations to introduce rational equations?			
$\frac{2}{x(x-2)} = \frac{1}{x-2}$	$\frac{1}{x-4} = \frac{2}{3x+1}$	$\frac{5}{x} + \frac{x}{x-1} = 1$	$\frac{3}{x+1} = \frac{x}{2}$

**Figure 1. The solving rational equations task.**

## Identification of Source of Students' Difficulties and Errors

The nature of preservice teachers' mathematical knowledge and their views about mathematics reflected on how they would help students when the students had misconceptions about or difficulties in understanding of a particular topic. The preservice teachers thought that the students fail in mathematics because they do not know the mathematical rules or procedures or they apply them wrongly. They did not state how flaws in students' conceptual understanding would likely lead to failure in

generating a correct solution. Therefore, when they were asked to address students' errors they inclined to tell how to apply the rules or carry out the algorithms correctly. They did not attempt to justify the reasoning underlying those rules or algorithms.

During the third interview, I asked the preservice teachers how they could help a student who made a mistake when solving inequalities such that the student did not change the direction of the inequality after dividing the coefficient of the  $x$  term by a negative number. All participants stated that they would tell the student that when dividing by a negative number you need to flip the inequality sign. To convince the student that the answer was incorrect they would ask her to check the reasonableness of the result by assigning a value from the solution set to  $x$ . Furthermore, all of them were aware of that there was a mathematical explanation for why they need to change the inequality sign; however except one preservice teacher they failed to state it clearly.

### **Conceptual Understanding of Mathematics**

The preservice teachers' answers to the content-specific questions revealed that their knowledge of subject matter is mostly procedural and they did not know the conceptual foundations of some topics such as ellipses, polynomial equations, permutation and combination. For instance, none of the preservice teachers were able to define what ellipse is. They knew what it looks like but they had no idea what is used for in mathematics. Two of them told that you could form an ellipse by combining two parabolas. Although an ellipse could be visualized as a combination of two parabolas, it is a mathematically invalid argument. They also failed to remember the expressions for  $P(n,r)$  or  $C(n,r)$  even though they knew that permutation refers to ordering objects while combination is finding different combinations of given objects.

The weakness of preservice teachers' conceptual knowledge of mathematics was evident when answering other type of content-specific tasks such as identification of the source of students' errors. Because they did not know the reasoning for flipping the inequality sign when dividing or multiplying inequality with a negative number, they would tell their students that it is a rule. Furthermore, some of them said that they had not been taught some of the concepts such as ellipses in depth in high school nor studied on them in the college. Therefore, they did not have any idea about them.

### **DISCUSSION**

The findings of this study supported the findings that PCK involves various knowledge and skills which are highly interrelated to each other (e.g., Even & Tirosh, 1995). A teacher should possess in-depth knowledge of subject matter, have a rich repertoire of teaching strategies, and be able to critically select tasks, examples, representations, and instructional materials to promote students'

understanding of a particular topic, and to diagnose and eliminate students' errors and misconceptions effectively. Moreover, among the other components of PCK, knowledge of subject matter needs specific attention because a teacher should have strong knowledge of the subject matter s/he would teach in order to be able to develop effective teaching strategies that are appropriate for a particular group of students, to choose appropriate tasks for them, and to identify the reasons underlying their errors and address them effectively.

The findings of this study supported the fact that preservice teachers lack knowledge of pedagogy (e.g., Ball, 1990) and knowledge of students (e.g., Morris, Hiebert, & Spitzer, 2009). For instance, when the preservice teachers were asked to order given examples of linear equations, some of them preferred to start with " $y=5$ " because they thought that it was the simplest one. They disregarded the fact that some students might fail to distinguish between a line and a point, therefore " $y=5$ " might not be easily understood by students as they assumed to be so. As indicated in the solving inequality task, the preservice teachers perceive teaching as telling the rules, showing students how to use them, and then having students practice them (e.g., Kinach, 2002). Furthermore, preservice teachers' pedagogical decisions are influenced by their knowledge of subject matter (e.g., Borko & Putnam, 1996). When they were not sure about the reasoning underlying the algorithms as in the case of solving inequalities, they just preferred telling rules.

Although the purpose of this study was to investigate the development of preservice teachers' PCK in a methods course and its associated field experience, I was not able to detect improvement in their PCK, because their knowledge of subject matter was the overriding determinant of their success in answering the questions. I used various tasks involved secondary school mathematics content for the questionnaire and interviews, and I used different items in each interview. This was problematic because if the preservice teachers did not have a strong conceptual understanding of the subject matter involved in an item, the item revealed their knowledge of subject matter rather than another aspect of their PCK. For instance, at the beginning of the semester some of the preservice teachers were able to provide conceptual foundations underlying mathematical facts because they knew the subject matter involved, but at the end of the semester they performed poorly on a similar item involving different subject matter because they did not know much about it.

## REFERENCES

- An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education*, 7, 145–172.
- Ball, D. L. (1988). Unlearning to teach mathematics. *For the Learning of Mathematics*, 8(1), 40–48.

- Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90, 449–466.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389–407.
- Borko, H., & Putnam, R. T. (1996). Learning to teach. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 673–708). New York: Macmillan.
- Even, R., & Tirosh, D. (1995). Subject-matter knowledge and knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics*, 29, 1–20.
- Kinach, B. M. (2002). A cognitive strategy for developing pedagogical knowledge in the secondary mathematics methods course: Toward a model of effective practice. *Teaching and Teacher Education*, 18, 51–71.
- Marks, R. (1990). Pedagogical content knowledge: From a mathematical case to a modified conception. *Journal of Teacher Education*, 41(3), 3–11.
- Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, 40, 491–529.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4–14.
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, 31, 5–25.

# LITERACY IN MATHEMATICS – A CHALLENGE FOR TEACHERS IN THEIR WORK WITH PUPILS

Bodil Kleve

Oslo University College

*This paper focuses upon the importance of aspects of teachers' mathematical knowledge in their work in making mathematics available for pupils. As theoretical background, Bruner's theory about two modes of thought, syntagmatic and paradigmatic, is presented. The Knowledge Quartet is used as an analytical tool in analysing transcripts from classroom observations before discussing the episodes from a literacy perspective. Finally, suggestions are made to put more weight on mediating tools, rather than on basic skills, in the learning process.*

**Keywords:** Teacher Knowledge in Mathematics, Mathematics Literacy, Knowledge Quartet, Paradigmatic and Syntagmatic Thinking

## BACKGROUND

Our project, "The Didactic Challenge of New Literacies in School and Teacher Education", focuses on literacy both in classrooms and teacher education from the perspective of didactics in several subjects. In Kleve & Penne (2010), which was a joint presentation in two subjects, Norwegian and mathematics, we explored the expression "Knowledge is made, not found" (Olson, 2001, p. 104), and discussed how meaning is constructed. Drawing on Vygotsky and particularly on Bruner's distinction between syntagmatic and paradigmatic modes of thinking (Bruner, 1986), we suggested some didactical perspectives that we claim have been obscured and ignored in recent years' focus on basic skills. Also, in Norway, student-centred pedagogy with the slogan 'responsibility for own learning' has been a focus over time. Thus, teachers' expertise and knowledge have been taken less into account.

In this paper, I will follow this approach and discuss the importance of the two modes of thought in teaching and learning mathematics, and how a teacher Cecilie, took her pupils with her in using these modes of thought in the mathematics classroom. To emphasise the importance of teachers' expertise and knowledge, I have used the Knowledge Quartet (Rowland, Huckstep, & Thwaites, 2005) as an analytical tool to find out how aspects of Cecilie's mathematical knowledge surfaced in her work in taking her pupils with her in using the two modes of thought in mathematics. The importance of these aspects of a teacher's knowledge is discussed and implications for teacher education are suggested.

The Knowledge Quartet has four broad dimensions; Foundation, Transformation, Connection and Contingency. *Foundation* is the mathematical knowledge the teacher has gained through his/her own education, it is knowledge possessed, which can inform pedagogical choices and strategies. It is the reservoir of pedagogical content

knowledge you draw from in planning and carrying out a lesson and thus informs pedagogical choices and strategies. *Transformation* focuses on the teacher's capacity to transform his or her foundational knowledge into forms that can help someone else to learn it. It is about examples and representations the teacher chooses to use. The third category, *Connection*, binds together distinct parts of the mathematics and concerns the coherence in the teacher's planning of lessons and teaching over time and also coherence across single lessons. *Contingency* is the category that concerns situations in mathematics classrooms that are impossible for the teacher to plan for; the teacher's ability to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas are important classroom events within this category.

## TWO MODES OF THOUGHT

Vygotsky made a strict distinction between two levels of mental functioning:

The lower level he characterized as natural, biological, causal and shared with nonhuman animals. The higher mental processes he portrayed as representational, sociohistorical, linguistic, voluntary, conscious and distinctively human (Olson 2001, p. 107).

As a development of Vygotsky's distinction between "lower" or natural mental functions and "higher" or cultural functions, Bruner (1986) introduced the *paradigmatic mode* and *syntagmatic mode* as "two modes of cognitive functioning, two modes of thought, each providing distinctive ways of ordering experience, of constructing reality" (p. 11). The syntagmatic mode of thought is basically related to linear time, and has primarily a narrative character. It is told from a subjective point of view and does not require truth, but rather a general probability or verisimilitude, referring to life in culture. This everyday thinking communicates an experienced world, and communicating in this mode means that the narrative structure is the most pervasive cognitive structure. It is merely based on tacit knowledge and acquired as part of communication in the family and in our everyday life, and thus slips easily into the mind. We all live in an "obvious" discursive world and we are linked together in communities through common discourses. Syntagmatic thinking is useful and necessary in our lives. From a didactical perspective however, this way of thinking can lead us only into obvious and tautological reasoning. We therefore need something more which can make us more conscious of other opportunities than what is obvious.

The paradigmatic mode of thought is different. The mind is more resistant to this way of thinking, reasoning about universal aspects of meaning and conclusions valid beyond time and context. Bruner (1986) writes:

[It] attempts to fulfil the ideal of a formal, mathematical system of description and explanation. It employs categorization or conceptualization and the operations by which categories are established, instantiated, idealized, and related one to the other to form a system (p. 12).



The paradigmatic mode deals in general cases and uses procedures in testing for empirical truth. Consistency and non-contradiction are required.

According to Bruner, the narrative structure acts as a mediating tool in the syntagmatic mode whereas in the paradigmatic mode concepts and metaphors are important mediating tools.

For mathematics, this means that exercising skills and methods (procedural knowledge) takes place in a syntagmatic mode of thinking, whereas generalisations, reasoning beyond time and context and building concepts (conceptual knowledge) takes place in a paradigmatic mode of thinking.

Bruner emphasised that the two modes of thought will never act “alone”. We all use them both continuously, however to different extents depending on the context. Science has often started out as hypotheses in everyday discourse and then been developed scientifically through a paradigmatic effort, a perspective which actively and consciously strives to transcend the narrative structure in the syntagmatic discourse. We therefore need continuous interaction between the two modes of thought to understand both everyday life and science. Bruner (1986) put it this way:

We all know by now that many scientific and mathematical hypotheses start their lives as little stories or metaphors, but they can reach scientific maturity by a process of conversion into verifiability, formal and empirical, and their power at maturity does not rest on their dramatic origins (p. 12).

As an example of the importance of both ways of thinking in the process of constructing concepts, Olson (2001) used a child’s learning to represent nothing with something, a zero. He claimed that concepts cannot pass smoothly from culture to mind, and, therefore, for a child to compare what is in his consciousness, “no cats”, with what is in the culture, “the zero”, is problematic. Olson suggested that both the knowledge of absence and the knowledge of a sequence of numerals (learned by rote?) have to be available in the child’s consciousness.

Learning then, consists of applying the memorized sequence of the numerals to the prior knowledge of absence. In so doing the child is not merely making explicit the known but forming a concept applicable to all sorts of nothings (Olson, 2001, p. 113).

According to Bruner, knowledge is made and not found, even for scientists. Olson (2001) put it this way: “Children, like adults, make what they find” (p.113). For teaching this means that teaching defines the problem space from which knowledge is constructed by the children and constructivism makes competent teaching significant. Competent teachers are needed to determine what children have found from which knowledge can be made rather than to “cover” a settled curriculum.

Mediational means are central in Vygotsky’s (1986) theory. The task for the competent or expert teacher is to make sure that pupils get access to the language or mediational means that can open up new aspects of the world, and develop their

thought. But how does transformation from mediational means to inner thoughts take place, given that children do not find new knowledge, they make it? Discussing this question, Olson (2001) elaborated how knowledge construction takes place:

Knowledge is constructed I have suggested, when the learner can take one set of concepts and use them as a model for thinking about some other sets of events. My example was seeing [ ] nothing [no cats] in terms of a numeral [zero]. The basic mode is metaphor, abduction, narrative construal, inference to the best representations rather than things. These representations, accumulated archivally in maps, charts, books and computer programs provide many of these most important models (p.113).

Thus importance is put on metaphors; through the metaphor, zero became a part of the child's thinking. According to Olson, it is easier to understand Vygotsky through metaphor theory.

In her book, *Thinking as Communication*, Sfard (2010) discussed the same aspects from Bruner's theory and she put weight on discourse and metaphors in science and everyday life:

Jerome Bruner describes the transition from a metaphor to its operationalized, "scientific" version in a beautifully metaphorical way. After stating that metaphors are "Crutches to help us get up the abstract mountain," The author notes: Once up, we throw them away (even hide them) in favor of a formal logically consistent theory that (with luck) can be stated in mathematical or near-mathematical terms (p.41).

How do teachers 're-present' their mathematical knowledge to pupils in terms of examples, demonstrations, illustrations, activities and questions? What crutches are the pupils given which they can throw away? In the next section of this paper, *I will present examples from mathematics classrooms and discuss the importance of aspects of the teacher's mathematical knowledge, with emphasis on the transformation aspect, in encouraging pupils to think in both modes of thought, syntagmatic and paradigmatic*. For pupils to learn to think in a paradigmatic mode, to form new mathematical concepts on which they can perform operations, their interaction with more knowledgeable peers (the teacher), who can use the concept in a paradigmatic mode, is of decisive importance.

This stresses the importance of both modes of thinking, syntagmatic and paradigmatic, in pupils' learning process. In our curriculum, LK06 (Kunnskapsdepartementet, 2006) "basic skills" are emphasised. From a literacy perspective, it is even more important for the pupils to learn how to use mediating tools to understand and develop new insight and to form new mathematical concepts. They need tools of the mind to extend mental abilities. In this work, aspects of the teacher's mathematical knowledge play a crucial role.

## EXAMPLES FROM THE MATHEMATICS CLASSROOM

I will now present extracts from two mathematics lessons in 10<sup>th</sup> grade with a teacher, here called Cecilie. The data are taken from Kleve (2007), a study in which I investigated how mathematics teachers in lower secondary school implemented a curriculum reform in Norway. Four mathematics teachers were observed over a period of three months, and the mathematics lessons were audio recorded and transcribed. The two episodes, which I present below, are selected because they illustrate how shifts between syntagmatic and paradigmatic modes of thought took place in Cecilie's lessons.

### Episode 1

Cecilie had drawn a right-angled triangle on the board and asked the pupils to have their calculators ready:

- 1    Cecilie:      What kind of a triangle do we have there, Mikkel?
- 2    Mikkel:      Right-angled triangle
- 3    Cecilie:      Then we know the lengths of the two sides. [ ]. How can I find the third side, Leif?
- 4    Leif:          You have to use Pythagoras
- 5    Cecilie:      Yes, have to use Pythagoras. Let us try to do that with this triangle. If we call this side for x, Leif?
- 6    Leif:          Must take  $x^2 = 3.6^2 + 4.8^2$   
(Cecilie wrote it on the board)
- 7    Cecilie:      Yes, let us calculate that. Three point six squared is?
- 8    studs:        Twelve point ninety six
- 9    Cecilie:      Four point eight squared is?
- 10   pupils:       twenty-three point o four
- 11   Cecilie:      Twenty-three point o four (wrote it on the board). The sum of these numbers is?
- 12   Pupil:        Thirty-six
- 13   Cecilie:      It is thirty-six
- 14   Pupil:        It makes six
- 15   Cecilie:      Yes okay it became six long. This was lots of calculations. If we look at the numbers here, we could have simplified it. Is it like, here I have added one point two, and if I add another one point two I'll get the third side? *Is that a rule which always works?* Let us take another example. New triangle (She drew a new triangle on the board with sides like 7.5 and 10). If that is seven point five and that is ten, will that one (the hypotenuse) be twelve point five? Can you check if it works?
- 16   Baard:        Yes
- 17   Cecilie:      That worked as well. Your exploratory task is now: Does it always work? Does it work for any length?

Later in the lesson, when the pupils had found counterexamples to the teacher's "rule", that in a right angled triangle, you can add the difference between the two smaller sides to get the hypotenuse, Cecilie said:

If it had been that easy in all cases, we wouldn't have had this rule (Pythagoras' sentence). Then I'd tricked you to calculate a lot. Next question is then: why does it work on these sides? What is special with the numbers here? Why does it work with my examples? Take a look at the lengths of the sides.

Towards the end of the lesson, when they had worked out that the ratio between the smaller sides had to be  $\frac{3}{4}$ , they investigated further. They went on to Pythagorean Triples, taking 3-4-5 as a starting point and Cecilie presented Euclid's formula  $p^2 - q^2$ ,  $2pq$ ,  $p^2 + q^2$ , which makes Pythagorean triples when  $p$  and  $q$  are whole numbers, and makes the special 3-4-5 triple when  $p=2$  and  $q=1$ .

Before discussing this episode from a literacy perspective, I will investigate aspects in the KQ of the teacher's mathematical knowledge which became evident in the episode and especially when Cecilie took her pupils with her in what I see as a shift from a syntagmatic to a paradigmatic mode of thought. Since there were no contingent moments in this episode, (which may be because she was not open to it), I will focus on the three other aspects, Foundation, Transformation and Connection of the KQ.

I will start with the teacher's *foundational knowledge*, the knowledge from which she chose examples, illustrations and kinds of questions asked. First of all, Cecilie demonstrated knowledge of Pythagoras' Theorem and the sentence to be investigated: 'In right angled triangles where the ratio between the two smaller sides is  $\frac{3}{4}$ , you can take the difference between the two smaller sides and add to the largest to get the length of the hypotenuse'. Also the teacher's knowledge of Euclid's formula was demonstrated. This shows that the teacher in this case had the foundational knowledge required for the topic of this lesson.

How did Cecilie make this knowledge accessible for her pupils? A feature of the *transformation aspect* of Cecilie's knowledge was that she related what was new to something well known. She started off with a right-angled triangle and Pythagoras' Theorem. That way she used for the pupils a well-known theorem to explore a new sentence. From her knowledge of the sentence to be proved (which was part of her foundational knowledge), Cecilie chose examples which made the sentence true. Also, when introducing Euclid's Formula, she used the, for the pupils, now familiar 3-4-5 triple to exemplify. To incorporate the pupils in this episode, she asked questions for them to answer.

In this episode, there was a *connection* from the start through calculating the hypotenuse in one triangle then in another triangle, then finding a counterexample to explore in which cases the sentence was true. Cecilie then made the link to Euclid's formula, which shows that she was in a position that made her able to make

connections between Pythagoras' Theorem, the sentence to be explored and Euclid's formula.

Looking at this episode from a literacy perspective, we will see that the course of the lesson was in a syntagmatic mode until turn 15. Throughout turns 1-14 the teacher had funnelled the pupils through calculations, stage by stage, using a well-known theorem (Pythagoras). Pythagoras' Theorem was used in calculating the hypotenuse in a triangle with the shorter sides being 3.6 and 4.8, and in another triangle with sides like 7.5 and 10. Then, in turn 15, she shifted to: "Is that a rule which always works?" And the question was restated in turn 17: "Does it always work? Does it work for any length?" All of a sudden they were challenged with the word *why*. And she gave them a hint in saying: "Take a look at the lengths of the sides". One strategy was to find a counterexample, which is a well-known way of proving that something is not true. The next stage was to investigate in which cases the "rule" worked. That way, the teacher took her pupils with her towards a paradigmatic mode of thinking which, in mathematics, incorporates generalisations. In which cases does it always work? To prove that something works, it is not sufficient to find many, thousands, examples with numbers in which it works. Using algebra is a way of proving, generalising and reasoning beyond time and context.

In recent curricula in Norway, the algebra content is reduced, and more weight is put on "basic skills". Thus mathematics in school has moved towards a more linear discourse or syntagmatic mode. Less weight has been put on vertical generalisations and paradigmatic thinking. In turn 17 above, "does it always work?" the pupils were challenged into another mode of thought, the paradigmatic, from the one in which they had so far been working. Until now, they had only carried out simple calculations on their calculators and answered the teacher's closed questions. The examples the teacher used, the questions she asked, and the connections she made acted as mediating tools for the pupils.

## Episode 2

Let us take a look at a similar episode from a mathematics lesson with the same teacher two weeks later. The starting point was a task from a test. Like in episode 1, there was a sudden break in discourse, or a shift in mode of thinking, from syntagmatic to paradigmatic. The generalisation question was in the text. However, on the test, the pupils had worked out the task with concrete examples. The teacher took those examples as a starting point and thus offered the pupils mediating tools to solve the task which was: "The length of a rectangle is increased by 15% and the breadth is reduced by 20%. How many percent does the area of the rectangle change?" Thus, the relation to something known for the pupils, a reified object, was established, before a similar break or shift in discourse (like in episode 1, turn 17) took place.

- 1 Cecilie: The length in a rectangle is increased by 15% and the breadth is reduced by 20% how many percent does the area of the rectangle

## Working Group 17

change? And the way everybody who answered that task did it, was that you chose a rectangle. Let us take this rectangle in which the length is 20 and the breadth is 5 (she drew it on the board). How big is the area of the rectangle?

- 2 pupils: Hundred
- 3 Cecilie: It is hundred. The area is hundred. And then the task was: The length is increased by 15%, how much will the new length be?
- 4 Leif: Twenty-three
- 5 Cecilie: Very good, Leif, very good mental calculation. The new length becomes twenty-three, and when the breadth is reduced by 20 %, what is the new breadth then?
- 6 Baard: Four
- 7 Cecilie: Yes, and what is the area?
- 8 Baard: It is ninety-two, right? It's a guess.
- 9 Cecilie: Right. It is not a guess, it is a mental calculation. How many percent is the area reduced?
- 10 Baard: Eight percent
- 11 Cecilie: Yes, it is. If it was hundred percent earlier, then it is ninety-two percent now, an eight percent reduction. Then the question is: are you sure it is applicable for other rectangles as well? This was for one special rectangle.

Also, here, I will discuss aspects of the KQ before studying the episode from a literacy perspective. In this episode, Cecilie demonstrated that she knew that length 20 and breadth 5 would make the area 100, that the change in area and the change in percent then would be the same. Also, 15% of 20 and 20% of 5 giving two whole numbers (3 and 1), which again gave them two whole numbers with which to calculate further, is worth noticing. This knowledge, which is a feature of the foundation aspect of the teacher's mathematical knowledge, informed her choice of measures in the example in this episode. Her *choice* of using a 100 rectangle was informed by her foundational knowledge, which included that the change in area and percentage change would have the same value. This demonstrates the transformation aspect of her knowledge. Also taking the pupils' answers to the task as a starting point, which had been choosing a concrete rectangle, demonstrates how she transformed or "re-presented" her knowledge of percentage change to be available for the pupils. She offered the pupils a 100 rectangle as a mediational means to percentage change. Thus a relation within mathematics was used, which demonstrated the KQ's connection aspect of Cecilie's knowledge.

In this episode, like in episode 1, they started in a syntagmatic mode, working with concrete examples, and as in episode 1 (is that a rule which always works?), we can see a shift in discourse, a shift in thinking: *Is it applicable for other rectangles?* A move to the paradigmatic mode of thought was initiated. After this episode, later in the lesson, the teacher emphasised that, on the test, it had not been sufficient to show the percentage change for concrete rectangles. She required a way to find out if it

applied to all rectangles. A shift to a paradigmatic mode was taken. In mathematics, generalisations play a crucial role in this mode, and use of algebra is a way of carrying out generalisations. After the episode presented above, a pupil suggested using algebra:  $1.15a \cdot 0.8b = 0.92a \cdot b$ , which is  $0.08 = 8\%$  change related to  $a \cdot b$ . That way they had proved that the change was 8% for all rectangles.

In this lesson, Cecilie referred to the lesson two weeks earlier in which they concluded that  $\left(\frac{4}{3}a\right)^2 + \left(\frac{3}{4}b\right)^2 = \left(\frac{5}{3}a\right)^2$ . This elucidates another feature of the connection aspect of Cecilie's mathematical knowledge. There was a connection, a relation, between the two lessons, a relation the teacher made the pupils aware of. Such connections can act as mediational means in pupils' learning, and are important in strengthening their paradigmatic thinking.

## DISCUSSION

In this paper, I have shown how Cecilie, in two lessons, took concrete examples as a starting point, and the examples she used acted as mediational means for the pupils to generalize and to move from a syntagmatic mode to a paradigmatic mode of thinking. I have used the Knowledge Quartet as an analytical tool and suggest that aspects of teachers' knowledge, with emphasis on transformation and connection are of crucial importance in the work with pupils in both modes, syntagmatic and paradigmatic, of thinking. In episode 1, the well known Pythagoras' Theorem was used as a crutch to find lengths of sides. In episode 2, a 100-rectangle was used as crutches to find percentage change. This implies that there is "something" needed which can serve as mediational means to make pupils more conscious for other opportunities than the obvious. Based on how they had solved the task on test, I suggest that it was obvious for the pupils how big the percentage change was in concrete rectangles. In solving the task on the test, they had been in a syntagmatic mode of thinking. However, for the pupils it was not obvious that the percentage change was the same for all rectangles; that the change was valid beyond concrete rectangles.

In earlier research, I have discussed how pupils' difficulties, which surfaced in contingent moments, in the conceptual understanding of fractions greater than one can be traced back to the transformation aspect of the teacher's mathematical knowledge (Kleve, 2009, 2010). From a literacy perspective, I suggest that the examples and illustrations the teacher used in that lesson was more of a hindrance than a help for the students to think in a paradigmatic mode. They failed in serving as mediational means between concrete conceptions of something more than one and improper fractions. In Kleve (2009b), I discussed the teacher's difficulties in illustrating improper fraction, and suggested that the focus on fractions as part of a whole, the "easiest" way for pupils to understand a proper fraction, also acted as a

hindrance in illustrating an improper fraction. They were stuck in the syntagmatic mode of thinking.

Based on this, I suggest that aspects of teachers' mathematical knowledge are crucial factors and therefore important to focus upon in teacher education. What are student teachers' choices of examples and illustrations and questions informed by? How do questions, examples and connections they choose influence pupils' thinking and learning in mathematics? For teachers, it is of great importance to be conscious of the two modes of thinking and consequently not let that work be overshadowed by "responsibility for own learning" and focus on basic skills.

## REFERENCES

- Bruner, J. (1986). *Actual Minds, Possible Words*. Cambridge, London: Harvard University Press.
- Kleve, B. (2007). *Mathematics teachers' interpretation of the curriculum reform, L97, in Norway*. Doctoral Thesis, Høgskolen i Agder, Kristiansand.
- Kleve, B. (2009). *Aspects of a teacher's mathematical knowledge on a lesson on fractions*. Paper presented at the British Society for Research into Learning Mathematics, Loughborough.
- Kleve, B. (2010). Contingent Moments in a lesson on fractions. [Current Report]. *Research in Mathematics Education*, 12(2), 157-158.
- Kleve, B., & Penne, S. (2010). *Literacy - en fagdidaktisk utfordring. Et eksempel med utgangspunkt i norsk og matematikk*. Paper presented at the Fou i Praksis, May 10-11, 2010.
- Kunnskapsdepartementet (2006). *Læreplanverket for Kunnskapsløftet: Midlertidig utgave*. Oslo: Utdanningsdirektoratet.
- Olson, D. R. (2001). Education, the Bridge from Culture to Mind. In D. Bakhurst & S. G. Shanker (Eds.), *Jerome Bruner Language, Culture, Self*. London: Sage Publications.
- Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary Teachers' Mathematics Subject Knowledge: The Knowledge Quartet and the Case of Naomi. *Journal of Mathematics Teacher Education*, 8(3), 255-281.
- Sfard, A. (2010). *Thinking as Communicating. Human Development, the Growth of Discourses, and Mathematizing*. Cambridge: Cambridge University Press.
- Vygotsky, L. S. (1986). *Thought and language*. Cambridge, Mass.: MIT Press.



# HELPING IN-SERVICE TEACHERS ANALYZE AND CONSTRUCT MATHEMATICAL TASKS ACCORDING TO THEIR COGNITIVE DEMAND

Eugenia Koleza, Christos Markopoulos, Stella Nika

Department of Primary Education, University of Patras, Greece

*The focus of this study is on the investigation of seven teachers' conceptions concerning the cognitive demands of the mathematical task. This study is part of an ongoing research on elementary teachers' professional development concerning their ability in designing goal oriented activities for mathematics teaching. Through our research, we tried to investigate precise components of teachers' Mathematical Knowledge for Teaching and Pedagogical Content Knowledge. Teachers collaborated as members of a Community of Practice in order to design, comment and reevaluate mathematical activities, concerning mainly their cognitive demand.*

**Keywords:** mathematical task, cognitive demand, professional development

## INTRODUCTION

In our ongoing research we investigate ways to support elementary pre service and practicing teachers in developing a much deeper understanding of mathematics curriculum via problem posing activities.

Research indicates that success of any educational reform depends mainly on teachers. Teachers use textbooks in different ways. Some tend to follow the textbook almost as a script for instruction (Remillard, 1992), whereas others adapt textbook activities and instructional suggestions to the needs of their classroom (Stake & Easley 1978). Researchers, point on these different attitudes drawing the distinction between designed and enacted curricula (Ball & Cohen, 1996). Many teachers function as designers of curricula that are enacted in their classrooms. Nevertheless, traditional use of textbook and teacher-directed approaches dominate (Jaworski & Gellert, 2003), because of a number of socio-cultural issues relating to classroom culture, teachers working individually, textbooks' structure in discrete lessons, teaching for exams and grading, teachers personal epistemologies, and teachers lack of knowledge. Designing a mathematical unit of study, teachers must first clearly understand the interrelationships of the various ideas within the unit, and second be able to choose or to construct learning activities in order to help students see and appreciate these connections.

In order for teachers to be able to act creatively concerning curriculum, they need support on a conceptual and on an attitude level. To change the way they teach mathematics, teachers must have opportunities to learn mathematical content and pedagogy in new ways, and believe to their capacity to implement the changes.

Concerning knowledge teachers need to teach mathematics, recent research has specified Shulman's categories (1987) of content knowledge and pedagogical content knowledge subdividing them into *common content knowledge* and *specialized content knowledge*, on the one hand, and knowledge of content and students and *knowledge of content and teaching*, on the other (Hill and Ball, 2004; Ball, Thames & Phelps, 2008). Ma (1999) in her study of the differences between U.S and Chinese teachers pointed to four aspects of knowledge-for-teaching. These are:

- knowledge of basic mathematical ideas
- the ability to make connections between these ideas
- the capacity to create and use multiple representations of these ideas in teaching
- deep knowledge of the curriculum continuum.

Second and third aspect of Ma's knowledge taxonomy, is similar to specialized content knowledge (SCK). *"Perhaps of most interest to us is evidence of the second category — specialized content knowledge. Like pedagogical content knowledge it is closely related to practice, but unlike pedagogical content knowledge it does not require additional knowledge of students or teaching. It is distinctly mathematical knowledge, but is not necessarily mathematical knowledge familiar to mathematicians"* (Ball et al. 2008, p.394). This kind of knowledge is of special interest if we want teachers not only to apply but also to be able to choose or construct mathematical learning tasks with high cognitive demand. By cognitive demand we mean the kind and level of thinking required of students in order to successfully engage with and solve the task. For example, tasks as Martha's Carpeting Task and Fencing Task (Stein et al., 2000) may help students think of fractions, decimals, and percents as different but equivalent representations of rational numbers, but are tasks of different cognitive load. At the heart of SCK lies the skills and knowledge required to unpack, to "decompress" a mathematical concept or skill into its sub concepts. And "decompression" of mathematical knowledge is a prerequisite of specifying and formulating curricular goals and designing corresponding learning activities (Ball et al., 2008; Hill et al., 2008).

## OUR RESEARCH

Our research, deals with the professional development of elementary and middle-schoolteachers. The whole project foresee two phases: Design/Teachers' Formation, and Implementation. In the present paper we are referring to the first one.

Having in mind that the effectiveness of a lesson depends significantly on the care with which the lesson plan is prepared, during first phase we designed a seminar in order to support elementary teachers on formulating instructional goals and on assessing and constructing mathematical tasks in terms of their cognitive demands.

Planning phase is the most important moment of instruction, because it is during this phase that “*teachers make decisions that affect instruction dramatically. They decide what to teach, how they are going to teach, how to organize the classroom, what routines to use, and how to adapt instruction for individuals*” (Fennema & Franke, 1992, p. 156).

The main question guiding our seminar’s planning was: *How can we help teachers improve their capacity to plan (and enact) lessons that support students’ learning?*

Four main competences we anticipated that teachers acquire through seminar. These competences are referred as important for teachers’ professional development by a number of researchers (Stigler & Hiebert, 1997):

- Designing and assessing mathematical tasks of different cognitive demands
- Connecting mathematical tasks with learning goals
- Generating questions that could be asked to promote student thinking during the lesson, and considering the kinds of guidance that could be given to students who showed one or another types of misconception in their thinking
- Anticipating solutions, thoughts, and responses that students might develop as they struggle with the problem

Researchers (Stein et al. 1996; Stein et al., 2000) distinguished four categories of tasks related to cognitive demand: memorization tasks, procedures without connections, procedures with connections, and doing mathematics. They characterized the first two categories as Math Tasks of Low Level Cognitive Demand (LLCD), whereas the last two as Math Tasks of High Level Cognitive Demand (HLCD). The kind of tasks teachers use, largely define what students learn (Hiebert & Wearne, 1993). But, between designing mathematics tasks and implementing them in the classroom intervene many parameters. Further research on cognitive demand of mathematical tasks has shown that mathematical tasks alone do not guarantee students’ learning because teachers often may not implement challenging tasks as they were intended. Stein et al. (1996) found that only half of HLCD tasks are treated as such in the classroom. Investigation of parameters that influence teachers’ decision to maintain or change the cognitive demand of an activity was one of the main goals of the second phase of our research.

In this paper we present instances from the seminar. More precisely, we comment teachers’ difficulties in assessing mathematical tasks, focusing on the investigation of their conceptions concerning the cognitive demands of mathematical tasks

## **METHODOLOGY**

Eight in service elementary teachers (2 men and 6 women) participated in the course. Helen, Marianna, and Martha have 20, 22 and 23 years teaching experience in primary schools, Vaso and Despina have 14 and 16 years teaching experience and

Maria, Nikolas and Dionisis have 7 years teaching experience each.

Teachers participated on a volunteer basis, and thus, this is not a random sample and may not be representative of the population of in service elementary teachers. Nevertheless, as mentioned before we tried to engage teachers from three different groups concerning their teaching experience. Although all were graduated from a University Department of Primary Education they were not familiar with the ‘philosophy’ of new mathematics textbooks. For example discussing during seminar about tasks involving critical thinking, most of them considered that the request for justification is just enough to raise the cognitive load of a task even in the case that the task is an algorithmic routine one. i.e. *“Are the fractions  $1/2$  and  $5/10$  equivalent? Justify your answer.”*

For seminar’s design, we used (lightly modified) the 4-I Model (Teacher-Innovator Model) (Yeap 2006), a teacher -development model for good practices. The model comprises four stages: Ignoring, Imitating, Integrating and Internalizing.

At **Level 0 (Ignoring)** during three approximately two-hours meetings we presented and discussed with the team of teachers three theoretical Frameworks concerning Instructional Design: The “Understanding by Design” (Wiggins & Tighe, 2005), The “Learning by Design” (Kalantzis & Cope, 2005) and “The Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development Ways of Knowing in Science Series” (Stein et al, 2000).

Wiggins & Tighe (2005) describe an approach of designing the teaching unit, focusing on Understanding. They have developed a theory, presenting a multifaceted view of what makes up a mature understanding, the «six facets of understanding»: explanation, interpretation, application, perspective, empathy, and self-knowledge.

Kalantzis & Cope (2005) describe eight «knowledge processes» which represent a range of different ways of making knowledge. Each knowledge process means something different in the structuring of the learning activities. These knowledge processes are: Experiencing the known—or reflecting on our own experiences, interests and perspectives. Experiencing the new—or observation of the unfamiliar, immersion in new situations, reading and recording new facts and data. Conceptualising by naming,—or developing categories and defining terms. Conceptualising with theory—or making generalisations and putting the key terms together into theories. Analysing functionally—or analysing logical connections, cause and effect, structure and function. Analysing critically—or evaluating critically your own and other people’s perspectives, interests and motives. Applying appropriately—or applying insights to real-world situations and testing their validity. Applying creatively—or making an intervention in the world that is truly innovative and creative and that brings to bear your life’s interests, experiences and aspirations.

Neither of these two Frameworks is especially designed for mathematics classroom.

Nevertheless, we considered them as important knowledge source for the seminar, because of their emphasis on teaching and learning by understanding.

**Level 1 (Imitating):** “The Implementing Standards-based Mathematics Instruction”, framework was the main tool of our formatting process. More precisely, after presenting and discussing with teachers the Task Analysis Guide (Stein et al, 2000), we spent one three-hours meeting in commenting and sorting mathematical tasks as of their cognitive demand. At the end, we gave them for reflection the following activity: Choose five activities from your class textbook (or from any other mathematics textbook you wish): 2 of LLCD and 3 of HLCD. For each one of these activities, try to answer the following questions:

1. **Do the activity.** What are all the ways the task can be solved?
2. **What is the cognitive demand of the activity?** (Choose one of five categories of the Task Analysis Guide). Justify your response: In what kind of thinking processes does the activity engage students?
3. **Identify the mathematical goal(s) of the activity.**
4. **What mathematical ideas does the activity develop?**
5. What misconceptions might students have? What errors might students make?
6. **What are the possible difficulties** students may be confronted? Could you anticipate their possible errors?
7. What questions will you ask to focus their thinking? **Identify specific questions through which you could activate students thinking process** concerning the activity (especially in case they are stuck)
8. (In case you think it is necessary) **How you would modify this textbook activity?**
9. **Design, you yourself, an activity of HLCD with the same mathematical goal.**
10. **Write anything else you think as important about this activity**

They had at their disposition one week to reflect and react.

At **Level 2** (Commenting/Integrating) teachers’ reaction to previous questions became the object of a team-discussion during one three-hours meeting. During this phase we tried to apply in our community of practice the six elements of a “Learning by Inquiry Process” as it is described in (Grevholm, 2009), i.e. Teachers are encouraged to ask questions (Questioning), to investigate each others ideas and collect information in order to reformulate their own ideas (Investigation). In this way new knowledge is created (Creation). Teachers, as members of the community discuss the new knowledge (Discussion), and reflect on their old knowledge and practices (Reflection) Discussion and reflections leads to wondering, which raises new questions (Wondering).

**At Level 3** (Internalizing) we asked teachers to design a teaching unit following the preceding theoretical framework.

In what follows, we focus on the teachers' responses to the given activity (Level 1) as well as their interaction trying to justify their choices within the team (Level 2). The data consists of the three-hour meeting's transcribed audio, the researchers' field notes and the teachers' written responses to the given tasks. During interaction, each teacher in his turn spoke and argued for every question or subject that emerged during the talks, explaining and illustrating his point of view. One could interrupt the flow of the conversation to add something, to express a different or contrasting view or to ask for a clarification.

## ANALYSIS

Through the analysis, a number of interesting issues concerning the way that the teachers conceive the mathematical task and its cognitive load emerged.

### Defining the cognitive demand of a mathematical task

The distinction between mathematical tasks of Low Level Cognitive Demand (LLCD) and High Level Cognitive Demand seemed to be a difficult activity for the teachers. We observed that often there were disagreements about the characterization of an active as a LLCD or a HLCD.

Most of them considered that the discrimination of the level of the cognitive demand of a task depends mainly on the complexity of the involved arithmetical operations. The case of Vaso, who tried to justify her decision of a HLCD task, was representative of the teachers' tension to focus mainly on the arithmetical operations that the task involves. She argues that *"...division between large numbers is very difficult for them (students)"*. Particularly, she proposed the following task:

**(Task 1)** *In a summer camp there are 60 scouts. The scouts are divided into groups of 15 people and form circles to play games.*

- a) How many circles are going to form? The scouts will form ..... circles.*
- b) In each circle there are 3 guides. How many guides are all together?*
- c) How many people are together scouts and guides?*

Vaso argues that the fact that there is a number of arithmetical operations that students are asked to carried out transforms the task to HLCD. On the contrary, Maria, another member of the team, considers this particular task as a LLCD arguing that *"....the solution path is predefined. There is no doubt about the solution approach."*

Teachers' inadequacy in providing all the possible solutions of the mathematical task prompted them to underestimate its "cognitive load". An illustrative example was the case of Despina who considered that a task involving percents is a LLCD one, as it can be solved applying the specific algorithmic routine that students have already taught. She focused only on one possible solution, ignoring a number of interesting approaches, which involve proportions, rates, fractions, and even the use of the

number line representations. In particular, the task she proposed as an LLCD was the following:

**(Task 2)** *The director of a movie theatre, notice that the usual number of the audience on Mondays is about 70 persons. In order to increase the number of the audience he announced that every Monday for each one of the first 45 tickets a movie poster will be provided free of charge. Next Monday he calculated that the 45 persons who won the movie poster were the 60 percent of the audience.*

*a) Find out how many persons watch the movie that Monday.*

*b) The cost of each poster is 2€ and the profit of each ticket is 6€. Comparing the total income of that particular Monday with the income of the preceding Mondays, does it worth continuing this particular promotion for the next Mondays?*

Through the interaction with the members of the team Despina figured out that, in fact, there are a number of interesting approaches to the task. Moreover, the members of the team realized that the level of the cognitive load of a task could be raised by the investigation-during teaching- of a range of possible solutions. In other words, they concluded that the “openness” of a task is a factor that possibly defines the level of its cognitive load. Another factor they also considered was that the placement of the task in the teaching sequence. For example, they argued, the summer scouts camp could be a HLCD one, if it had been used as an introductory to the concept of division. Marianna, referring to the specific textbook, asserted: “actually this is the first mathematical problem involving division that students confronted in the textbook of the third grade”.

### **Modifying the level of the cognitive load**

Teachers’ responses to the inquiry of finding ways of “raising” the cognitive load of a task were initially limited to the creation of more difficult/complex arithmetical operations. For example, Maria suggested in the case of the summer scouts camp task to modify the number of the scouts or the number of people in a group so as to provide a division with a remainder. Nikolas transformed that task by involving a reverse arithmetical operation. An interesting case was Martha who argued that giving students –through task 2- a general rule of finding percents applicable to a great number of similar tasks, could make task 2 a HLCD one. Actually, she combined cognitive demand and range of applicability. Martha suggested that in order to rise the cognitive demand of task “*I should ask students to explore all the possible solutions, present them in class and justify which one is the best*”. Nikolas questioned the term “*best*” for a solution. He argued that it is appropriate to define the criteria under which students will choose a solution. He claimed that the “*easiest*” solution is eventually the “*best*” for the students.

The use of multiple representations and connections between different mathematical concepts in order that the task evolves into a “*more advanced mode*” was also an

idea that teachers discussed. For example, Despina suggested asking students represent the data of the “movie theatre task” in an empty number line and make estimations about the possible solutions. In her opinion, such an intervention may encourage students to make connections between the different representations of the quantities and possibly conceive the interrelations between percents, ratio and proportions and fractions.

### **DISCUSSION - Concluding Remarks**

The project created opportunities for in-service teachers to learn and participate with their textbooks in professional development and provided them with opportunities to change their notions of learning and teaching mathematics. The project also offered the opportunity for collaboration between primary school teachers and university researchers, under the main goal to develop knowledge and practice in the teaching and learning of mathematics, so that teachers in schools have better teaching experiences and achieve better conceptual understandings of mathematics with their students.

The analysis of our data provides interesting issues concerning teachers’ ability to characterize mathematical tasks according to their cognitive load as well as their efficiency to reconstruct/redesign tasks, raising their cognitive demands. In almost all cases, the mathematical content of the tasks as well as their cognitive load were not so obvious for the teachers who restricted themselves to school practices reproducing “well known” techniques. For the teachers we worked with during this project, cognitive demand of a task is not independent of the activity’s goal as it is described in textbook’s instruction. More specifically, the position of the task in the unit is a parameter that influences its cognitive demand. A task that is placed in the start of the instruction in order to introduce the students (for example) to the concept of division can be characterized as high -level demand task. The same task could be considered as a low-level task demanding practice if it is placed in the end of a lesson plan.

Concerning the “raising” of the cognitive level of a task, teachers initially confronted difficulties. Their first suggestions were limited to the creation of more difficult/complex arithmetical operations, modifying an arithmetic/numerical element (data) of the problem. But during the progress of the meeting and with the growing discussion and the interactions within the team, teachers orientated towards more effective ways to manage and organize the task. I.e. they referred to the exploration, presentation and discussion of all possible solutions of a task, the use of multiple representations and connections between different mathematical concepts such as the use of a blank number line and estimations of the solutions.

The goal in this research was not to achieve complete agreement between the team but to provide teachers the opportunity to participate in a thoughtful analysis of the tasks, to emerge their shared interest for discussing the characteristics of the tasks



and to raise the level of discussion among participants toward a deeper analysis of the sorting of the tasks. The point was to encourage teachers to dig beneath the surface in determining the level of thinking required to complete a task, based on the point of researchers (Stein, Smith, Henningsen, & Silver, 2000) who claim that when teachers take the opportunity to analyze the tasks, they become more alert to the potential for slippage between intentions and actions in their teaching.

Specific issues generated lively discussion on topics such as the difference between “level of cognitive demand” and “difficulty” of a task, the factors associated with the decline or maintenance of the level of cognitive demands of mathematical tasks during the implementation phase in the classroom, and construction of goal-oriented high-level mathematical tasks.

Hopefully, the results of this study will provide insight into some substantial issues that form the instruction of mathematics. Understanding the cognitive demand level of the mathematical tasks and the ways to “rise” this level will have potential benefits for teachers in acquiring the competencies needed for a better design and implementation of an effective instruction.

## REFERENCES

- Ball, D.L., & Cohen, D.K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher*, 25(9), 6-8, 14
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59, 389–407
- Fennema, E., & Franke, M. L. (1992). Teachers’ knowledge and its impact. In D. A. Groews (Ed.), *Handbook of research on mathematics teaching and learning* (147–164). New York: Macmillan.
- Grevholm, B. (2009). Teaching better mathematics-A developmental research study. In M Lepik (Ed.), *Teaching Mathematics: Retrospective and Perspectives Proceedings of the 10th International Conference*, Tallinn University, May 14-16, 169-176.
- Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students’ learning in second-grade arithmetic. *American Educational Research Journal*, 30, 393-435.
- Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26, 430–511.

- Hill, H. & Ball, D. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, 35(5), 330-351.
- Jaworski, B. & Gellert, U. (2003). Educating new mathematics teachers: Integrating theory and practice, and the roles of practicing teachers. In A. Bishop, M.A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education (Part Two)* (829-875). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Kalantzis, M., Cope, B., and the Learning by Design Project Group. (2005). *Learning by Design*. Melbourne: Victorian Schools Innovation Commission in association with Common Ground Publishing.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*, Lawrence Erlbaum Associates, Mahwah, NJ.
- Remillard, J. (1992). Teaching mathematics for understanding: A fifth-grade teacher's interpretation of mathematics policy. *The Elementary School Journal*, 93, 179-183.
- Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, 35(5), 352-388.
- Shulman, L. (1987). 'Knowledge and teaching: Foundations of the new reform'. *Harvard Educational Review*, 57, 1-22.
- Stake, R., & Easley, J. (1978). *Case studies in science education*. Urbana: University of Illinois.
- Stein, M. K., Grover, B. W., & Henningsen, M. A. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, 33, 455-488.
- Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards based mathematics instruction: A casebook for professional development*. Reston: NCTM.
- Stigler, J. W., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan*, 79(1), 14-21.
- Wiggins, G. & McTighe, J. (2005). *Understanding by Design* (2nd ed.). Alexandria, VA: ASCD. (Original work published 2001)
- Yeap, B. H. (2006). Good practices in mathematics teaching and teacher development. *Tsukuba Journal of Educational Study in Mathematics* 25, 233-236.

# **PROFESSIONAL KNOWLEDGE RELATED TO BIG IDEAS IN MATHEMATICS – AN EMPIRICAL STUDY WITH PRE-SERVICE TEACHERS**

Sebastian Kuntze<sup>1</sup>, Stephen Lerman<sup>2</sup>, Bernard Murphy<sup>3</sup>, Elke Kurz-Milcke<sup>1</sup>, Hans-Stefan Siller<sup>4</sup>, Peter Winbourne<sup>2</sup>

<sup>1</sup>Ludwigsburg University of Education, Germany; <sup>2</sup>London South Bank University;

<sup>3</sup>Mathematics in Education and Industry, U.K.; <sup>4</sup>University of Klagenfurt, Austria

*Professional knowledge of mathematics teachers related to big ideas in mathematics and mathematics instruction can enhance teachers' competencies of designing rich learning opportunities in the mathematics classroom. Responding to a need of empirical research on professional knowledge connected with big ideas in mathematics, this study presents results of a test administered to more than 100 German pre-service teachers. The results indicate that the pre-service teachers often were unable to discern big ideas behind mathematical contents and to link elements of content matter according to these big ideas. The results call for an emphasis in teacher education not only on a solid content matter knowledge base, but also on overarching concepts and meta-mathematical ideas.*

*Keywords: Professional knowledge, big ideas, connecting PCK/CK components*

## **INTRODUCTION**

There is a consensus that designing rich learning opportunities in the mathematics classroom can be supported by an emphasis on big ideas associated with mathematics and mathematics instruction. Awareness of such big ideas requires professional knowledge of teachers, especially in the areas of content knowledge (CK) and pedagogical content knowledge (PCK). However, empirical research on professional knowledge connected with big ideas in mathematics is scarce, even though evidence is needed, e. g. for designing teacher education programs on an empirical base.

Consequently, this study concentrates on components of professional knowledge linked to big ideas. A test and questionnaire instrument has been developed in the EU-funded teacher education and research project ABCmaths ("Awareness of Big Ideas in Mathematics Classrooms") for this purpose and has been used in a first approach to assess the knowledge of 117 pre-service teachers. The results show that the pre-service teachers were often not able to link mathematical contents according to selected big ideas and to communicate about these ideas. There is hence a need of supporting reflective competencies of teachers with respect to mathematics-related overarching concepts like big ideas.

In the following first section we give an overview on the theoretical background, presenting both a working definition of big ideas and an outline of the theoretical

model of professional knowledge components used in this study. The second section leads to the research interest of the study. After having given information about research design and sample in the third section, we report on results in the fourth section of this paper. The fifth section concludes with a short discussion of the evidence and of its implications.

## **THEORETICAL BACKGROUND**

### **Big ideas related to mathematics and mathematics instruction**

Providing rich learning opportunities in the mathematics classroom is essential for instructional quality and the development of competencies of students. Many researchers have emphasised the role of overarching concepts or fundamental ideas in mathematics and its teaching and learning for creating conceptually rich learning opportunities (e.g. Schweiger, 1992, 2006; Bishop, 1988). Supporting these goals, the aims of national standards in many countries (e.g. Office of Qualifications and Examinations Regulation, 2002; KMK, 2003; NCTM, 2000) can be associated with such implicitly shared underlying “big ideas”. According to the working definition of *big ideas* in the teacher education project ABCmaths, *big ideas associated with mathematics in the classroom* anchor, link and constitute mathematical knowledge in contexts (within maths, the curriculum and/or beyond maths) and foster making sense of and communicating this knowledge in a more general way. Following a pragmatic approach, which aims above all at encouraging teachers’ reflection on overarching concepts in mathematics and on their potential for learning, four important aspects have been collected that may help to identify big ideas (ABCmaths team, in preparation). Big ideas can be characterised as:

- Ideas that should have a high mathematics-related potential of encouraging learning with understanding of conceptual knowledge (including orientation, linking and anchoring of knowledge)
- Ideas that should have a high relevance for building up meta-knowledge about mathematics as a science (adapted to the target group of learners) including knowledge necessary for interdisciplinary comparisons
- Ideas that should support abilities of communicating meaningfully about mathematics and providing mathematics-related arguments
- Ideas that should also encourage reflection processes of teachers connected with designing rich and cognitively activating learning opportunities as well as with accompanying and supporting learning processes of students.

These aspects can be seen as a pragmatic answer to a partly divergent discussion of multiple approaches in the area of big ideas: In the German-speaking discussion for example, the diversity of the notions of “fundamental ideas” (e.g. Schweiger, 1982, 2006), “central ideas” (Schreiber, 1983), “universal ideas” (Schreiber, 1983), “core ideas” (Gallin & Ruf, 1993), “leading ideas” (e.g. Vollrath, 1978) and “basic

ideas/basic conceptions” (“Grundvorstellungen”, e.g. v. Hofe, 1995) give a heterogeneous picture and call for an integrative and sufficiently open perspective like suggested by the criteria given above.

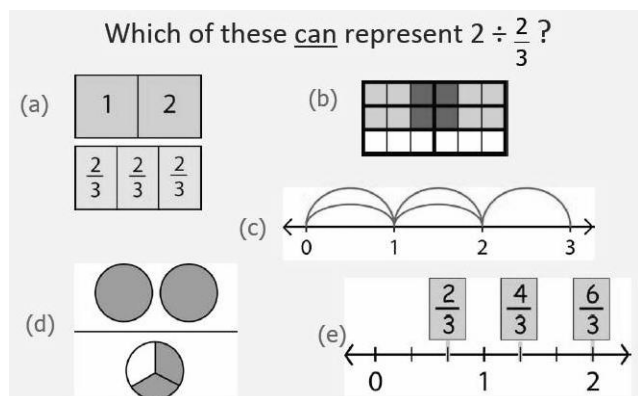
Three examples of big ideas are the following:

- *Using multiple representations*: This big idea reflects strategies used and required in many mathematical domains that have to do with the use of different ways of representing mathematical facts or concepts as well as with changing representations and linking them.
- *Giving arguments/proving*: Mathematics as a science can be characterised by the forms of argumentation and proof used in this discipline (cf. Heinze & Reiss, 2003). As a consequence, argumentation plays an important role in all domains of mathematics.
- *Dealing with infinity*: This big idea highlights the significance of exploring phenomena linked with infinity in mathematics, strategies of maintaining generality in thoughts in order to include an arbitrary, often infinite number of cases as well as patterns and structures that bear infinity often in themselves.

*Big ideas associated with pedagogical content knowledge (PCK)* support teachers when designing rich conceptual learning opportunities for encouraging mathematical thinking. More detailed considerations about these big ideas and examples are provided in (ABCmaths team, in preparation). The big ideas “using multiple representations” and “giving arguments/proving” appear both in the domains of CK, i.e. as mathematics-related big ideas, and PCK.

### Professional knowledge related to big ideas

Professional knowledge about big ideas is not only “horizon knowledge” (Ball, Thames, & Phelps, 2008; Ball & Bass, 2009) – it has strong links to different areas of CK and PCK (Shulman, 1986). For example, using multiple representations requires CK in various domains. This can be seen e.g. when looking at a sample task published by the group of D. Ball (s. Ball & Bass, 2009) which relates to professional knowledge about the idea of multiple representations, clearly requires CK and which focuses on knowledge relevant for teaching in the mathematics classroom (Figure 1).

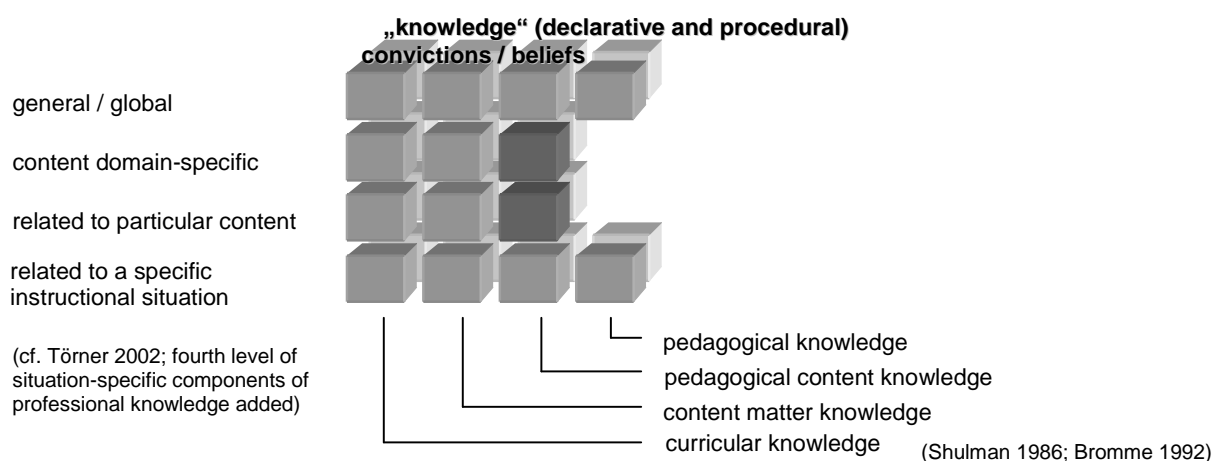


**Fig. 1: Sample item in Ball & Bass (2009)**

Hence, knowledge about big ideas is central for many components of professional knowledge and potentially links different components. For this reason, the teachers’ awareness of big ideas in mathematics and mathematics instruction appears to be

crucial not only for designing learning opportunities, but also for the ongoing professional learning of teachers.

As a base of reference for this study, we refer to a framework for the description of components of professional knowledge (shown in Figure 2) that includes instruction-related views held by individual mathematics teachers (Kuntze, submitted; Kuntze & Kurz-Milcke, 2010). This model of professional knowledge integrates the spectrum between knowledge on the one hand and convictions/beliefs on the other, because a dichotomous theoretical distinction between knowledge and beliefs is impossible (cf. Pajares, 1992). Hence, both are considered as being contained in the notion of professional knowledge. The distinction between different domains of professional knowledge as suggested by Shulman (1986) appears in the vertical columns (see Ball, Thames, & Phelps, 2008, for the possibility of further refinement into domains). Taking into account that individual professional knowledge is often organised in an episodic structure (Leinhardt & Greeno, 1986; Bromme, 1992), we consider furthermore different levels of globality (Törner, 2002; Lerman, 1990; Kuntze & Reiss, 2005). Besides global components of professional teacher knowledge, e.g., a general cognitive constructivist view of mathematics teaching and learning (Staub & Stern, 2002), content domain-related components are considered relevant, e.g., views linked with content domains like geometry or decimals. Further, studies by Biza, Nardi and Zachariades (2007) as well as by Kuntze (accepted) focus on views of teachers related to concrete tasks, hence are specific to a particular content. Finally, views of teachers concerning (videotaped) instructional situations have also been included in empirical studies (e.g. Lerman, 1990; Kuntze & Reiss, 2005). Professional knowledge related to big ideas is relevant for different levels of globality, as they are important for mathematics and mathematics instruction as a whole but also relevant for many specific contents and instructional situations.



**Figure 2: Model for components of professional knowledge (Kuntze, submitted)**

Together these components of professional knowledge are likely to influence instruction and hence learning opportunities for students in mathematics classrooms.

Thus, evidence about teachers' professional knowledge can help to design teacher education programmes with the aim of helping teachers to improve their teaching.

## RESEARCH INTEREST

Against this theoretical background, the project ABCmaths ("Awareness of Big Ideas in Mathematics Classrooms", [www.abcmaths.net](http://www.abcmaths.net)) aims at investigating professional knowledge of mathematics teachers related to big ideas as those introduced above and at its further development through both pre-service and in-service teacher professionalisation programmes at the level of CK and PCK. For this purpose it is necessary to explore the status quo of professional knowledge in this area in so-called analysis of needs studies. Moreover, new test and questionnaire instruments had to be developed and piloted.

This paper presents an analysis of needs study with pre-service teachers. Further research in ABCmaths will include in-service teachers and evaluation research on effects of teacher professionalisation programmes. In order to get insights into the professional knowledge of pre-service teachers, we focus on the three mathematics-related big ideas introduced above, namely "using multiple representations", "giving arguments/proving", and "dealing with infinity". At later stages of ABCmaths, the research will be extended to further big ideas. Hence, this study aims at providing evidence for the following research question: *What professional knowledge related to the big ideas "using multiple representations", "giving arguments/proving" and "dealing with infinity" do German pre-service teachers have?*

More explicitly, the study focuses on professional knowledge about big ideas related to linking contents and examples of subject matter with big ideas, as well as analysing them against the background of specific big ideas. This component of professional knowledge is likely to play a key role for competencies of designing rich learning opportunities in the classroom and it requires also mathematics-related meta-knowledge about specific big ideas.

## SAMPLE AND METHODS

In order to find out about the research question above, a test was administered to 117 German pre-service teachers (78 female, 35 male, 4 without data) before the beginning of a university course. The pre-service teachers had a mean age of 22.33 years ( $SD = 3.56$  years) and had been studying on average for 2.19 semesters ( $SD = 1.12$ ). 61 pre-service teachers were preparing for being teachers in primary schools, 35 in secondary schools for lower-attaining students, and 15 for working in schools for students with special needs.

Corresponding to the aspects emphasised in the previous section, the test focused especially on analysing contents and perceiving links across contents according to big ideas. The test instrument concentrated on the ideas, "dealing with infinity", "giving arguments/proving", and "using multiple representations".

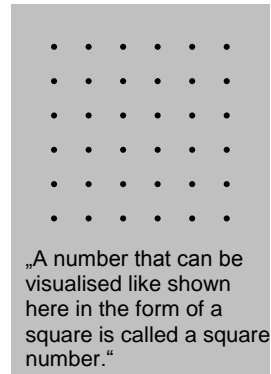
## Working Group 17

A sample task related to the idea “using multiple representations” is shown in Figure 3. In this task, the pre-service teachers were given an example of two representations of the definition of square numbers and asked to give other examples of mathematical contents where definitions can be based on different representations.

On the right, there is a graphical representation of the definition of “square number”. This representation affords an additional access compared to the symbolic definition (“if  $q=n^2$  for a positive integer  $n$ , the  $q$  is called a square number”).

Can you think of other mathematical concepts, for which a symbolic definition can complement with a non-symbolic representation in a similar way?

Please give as many examples as possible.



**Figure 3: Sample task related to the idea “multiple representations” (task 6)**

The answers of the pre-service teachers were collected in an open format. There were two tasks related to the big idea of “dealing with infinity”, and three tasks for the ideas “using multiple representations” and “giving arguments/proving”, respectively. The pre-service teachers were given as much time as they liked to devote to answering the test.

For gaining an overview of the quality of the answers of the 117 pre-service teachers, a top-down coding method was used. The coding categories concentrated on the aspects of existence of a codable answer, the quality and transfer level related to the examples provided, and the embedding of these examples. For an easier understanding, more details about the codes are reported together with the corresponding results in the following section.

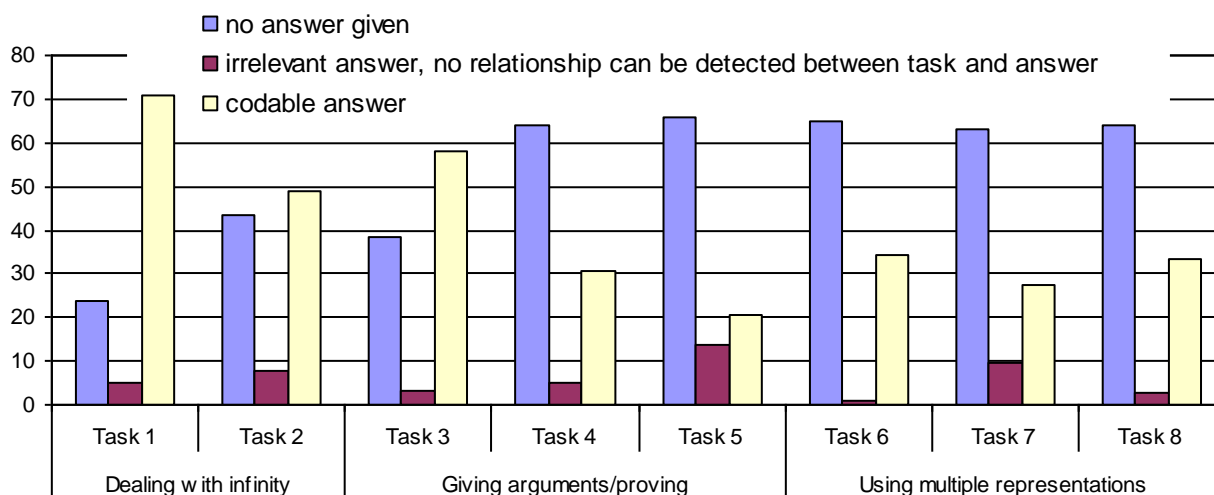
## RESULTS

In order to get an overview of the number of codably answered tasks, an initial coding assigned the answers to the categories “no answer given to the task”, “irrelevant answer given, i.e. no detectable semantic relationship between answer and the task” and “codable answer with respect to quality codes”. The frequencies of codes for this initial coding are displayed in Figure 4. An over-all observation is that the frequencies of codable answers were low: For almost all tasks, more than a third up to about two thirds of the pre-service teachers could not give an answer at all.

For task 6, which is the sample task given in Figure 3, we give a more detailed picture of the findings, in order to provide more in-depth information about the professional knowledge of the pre-service teachers.



## Working Group 17



**Figure 4: Frequencies of answers to tasks**

In the task, the participants were asked to provide other examples related to multiple representations. In a corresponding coding of the quality of these examples provided, 65.8% of the answers were classified in the code “no example given”, 17.1% of the answers were coded as “ ‘peripheral’ relationship of example with big idea visible, but substantial gaps or disruptions / inadequate character of the example given” (an example of an answer classified in this code is presented in Fig. 4), 10.3% of the pre-service teachers gave one adequate example, 6.0% provided two adequate examples and 0.9% gave more than two adequate examples.

An additional coding relating to the quality of the answers focused on their *transfer level*. As an example was given in the task, the coding distinguished whether the pre-service teachers gave adequate examples in other content domains which were not ‘close’ to the given example – an indicator whether teachers are able to link contents according to big ideas across content domains. In case of more than one example provided in the answer, the highest category was coded. Out of the 17.2% of answers with at least one adequate example, a majority of 89.5% gave at least one adequate example from another content domain, which is 15.4% of all pre-service teachers.

Finally, the level of embedding or argumentation linked with the examples given was coded for the 17.2% of answers with at least one adequate example. Out of these answers, 55% did not have any embedding or reflecting comments on the examples provided, whereas in the remaining 45% of the answers, embedding comments for the examples were given. The category of “adequate argumentational embeddings/justifications/analysing comments e.g. about how the example fits to the big idea” turned out to be hypothetical for this sample of pre-service teachers, as 0% of the answers fulfilled this criterion.

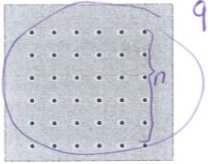
Figure 5 shows some sample answers. The first answer was coded as an appropriate example close to the given example, as the given example is just modified. In the

second example, possibilities of representing fractions graphically or as partitions of a cake are used as an example, which was still coded as an appropriate example, even though the example seems neither completely developed, nor linked or embedded by explanatory terms.

The last answer in Figure 5 was coded not to be an appropriate answer, because the representations can not replace each other but one might be used as an illustration or specification only, even if a correct and complete diagram of the geometrical situation had been given. Beyond the findings related to the coding, all examples are relatively short and seem not to correspond to high skills of mathematics-related communication.

Rechts ist eine graphische Darstellung der Definition von "Quadratzahl" gegeben. Diese Darstellung bietet im Vergleich zur symbolischen Definition („wenn  $q=n^2$  für eine natürliche Zahl  $n$ , dann heißt  $q$  Quadratzahl“) einen zusätzlichen Zugang.


Fallen Ihnen andere mathematische Begriffe ein, bei denen die symbolische Definition in ähnlicher Weise einer nicht-symbolischen Darstellung gegenübergestellt werden kann?  
Bitte schreiben Sie möglichst viele Beispiele auf.



„Eine Zahl, die wie hier gezeigt in Form eines Quadrats veranschaulicht werden kann, heißt Quadratzahl.“

---

• Des selbe Beispiel im Dreidimensionalen mit  $q=n^3$




“The same example in three dimensions”

---

“Fractions, e.g. slices of a cake, a football ground etc.  
(1 cake for 4 persons)”

Bruchzahlen, z.B. Kuchenstücke; Fußballfeld etc.




(1 Kuchen für 4 Personen)

---

“Pythagorean theorem”

- Satz des Pythagoras

$$a^2 + b^2 = c^2$$


**Figure 5: Sample answers**

## DISCUSSION AND CONCLUSIONS

The results suggest that pre-service teachers were not able to give a lot of examples for mathematical contents linked to the big ideas considered in the test instrument. As answers referring to relatively simple mathematical contents were possible, this can be interpreted as a lack of awareness of big ideas in the professional knowledge of the pre-service teachers.

The results seem to suggest that the tasks were too difficult for the pre-service teachers. Indeed, the tasks were also designed to be used with in-service teachers, which raises research questions about the role of instructional experience for professional knowledge related to big ideas. However, from a theoretical viewpoint, the tasks focus on knowledge that teachers really should have – ideally already as a consequence of their school mathematics experience, which should have provided them with a base of examples and networking knowledge linking these contents to ideas. The results however indicate that there is a need for professional development in this area, potentially both for a content knowledge base of examples and networking skills or reflection knowledge related to big ideas that can organise knowledge and help to develop rich learning opportunities.

Further attention should be devoted to research questions concerning the knowledge of in-service teachers, possibilities of developing professional knowledge by teacher education programs, extensions to other big ideas, and the role of culture.

## ACKNOWLEDGEMENTS

The project ABCmaths is funded with support from the European Commission (503215-LLP-1-2009-1-DE-COMENIUS-CMP). This publication reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

We acknowledge the contribution of Martin Rauscher, who has been working in the project ABCmaths as a student assistant and who has helped in the coding process of the data presented in this paper. Moreover, we acknowledge the cooperation of Karl-Josef Fuchs, Anke Wagner, Claudia Wörn, Christiane Vogl, and Michael Schneider who collaborate in the ACBmaths project team.

## REFERENCES

- Ball, D., & Bass, H. (2009). *With an Eye on the Mathematical Horizon: Knowing Mathematics for Teaching to Learners' Mathematical Futures*. [keynote address at the 43rd Jahrestagung für Didaktik der Mathematik held in Oldenburg, Germany, March 1 – 4, 2009, [http://www-personal.umich.edu/~dball/presentations/032309\\_oldenburg.pdf](http://www-personal.umich.edu/~dball/presentations/032309_oldenburg.pdf)].
- Ball, D., Thames, L., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, 19, 179–191.
- Biza, I., Nardi, E., & Zachariades, T. (2007). Using Tasks to Explore Teacher Knowledge in Situation-Specific Contexts. *Journal of Mathematics Teacher Education*, 10, 301-309.
- Bromme, R. (1992). *Der Lehrer als Experte. Zur Psychologie des professionellen Wissens*. [The teacher as an expert. On the psychology of professional knowledge]. Bern: Huber.
- Gallin, P., & Ruf, U. (1993). Sprache und Mathematik in der Schule. Ein Bericht aus der Praxis. *Journal für Mathematik-Didaktik*, 12(1), 3-33.
- Heinze, A., & Reiss, K. (2003). *Reasoning and Proof: Methodological Knowledge as a Component of Proof Competence*. Proceedings of CERME 3, Bellaria, Italy. [[www.lettredelapreuve.it/CERME3Papers/Heinze-paper1.pdf](http://www.lettredelapreuve.it/CERME3Papers/Heinze-paper1.pdf)]
- v. Hofe, R. (1995). *Grundvorstellungen mathematischer Inhalte*. Heidelberg Berlin Oxford: Spektrum Akademischer Verlag.
- Kultusministerkonferenz (KMK). (2003). *Bildungsstandards im Fach Mathematik für den Mittleren Schulabschluss*. [<http://www.kmk.org/>].
- Kuntze, S. (accepted). In-service and prospective teachers' views about modelling tasks in the mathematics classroom – Results of a quantitative empirical study. [Proceedings book of ICTMA 14].

- Kuntze, S., & Reiss, K. (2005). Situation-specific and generalized components of professional knowledge of mathematics teachers. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conf. of the Int. Group for the Psych. of Mathematics Education (PME)*, Vol. 3 (pp. 225-232). Melbourne: PME.
- Kuntze, S., & Kurz-Milcke, E. (2010). Content domain-related and task-specific views of mathematics teachers in the area of statistics. In Pinto, F.M. & Kawasaki, T.F. (Eds.), *Proc. of the 34<sup>th</sup> Conf. of the Int. Group for the Psych. of Math. Education*, Vol. 3 (pp. 193-200) Belo Horizonte, Brazil: PME.
- Leinhardt, G., & Greeno, J. (1986). The cognitive skill of teaching. *Journal of Educational Psychology*, 78, 75-95.
- Lerman, S. (1990). Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics. *British Educational Research Journal*, 16 (1), 53-61.
- National Council of Teachers of Mathematics (NCTM). (Ed.). (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
- Office of Qualifications and Examinations Regulation (2002). *GCE Adv. Subs. (AS) / Adv. (A) Level Specifications. Subject Criteria for Mathematics*. [<http://www.ofqual.gov.uk/files/2002-12-gce-maths-subject-criteria.pdf>].
- Pajares, F.M. (1992). Teachers' Beliefs and Educational Research: Cleaning Up a Messy Construct. *Review of Educational Research*, 62(3), 307-332.
- Schreiber, A. (1983). Bemerkungen zur Rolle universeller Ideen im mathematischen Denken. *Mathematica didactica*, 6, 65-76.
- Schweiger, F. (1992). Fundamentale Ideen. Eine geistesgeschichtliche Studie zur Mathematikdidaktik. *Journal für Mathematik-Didaktik*, 13, 199-214.
- Schweiger, F. (2006). Fundamental Ideas: A Bridge between Mathematics and Mathematical Education. In J. Maaß & W. Schlöglmann (Eds.), *New Math. Educ. Research and Practice* (pp. 63-73). Rotterdam/Taipei: Sense Publishers.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Staub, F., & Stern, E. (2002). The nature of teacher's pedagogical content beliefs matters for students' achievement gains: quasi-experimental evidence from elementary mathematics. *Journal of Educ. Psychology*, 94 (2), 344-355.
- Törner, G. (2002). Mathematical Beliefs. In G. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 73-94). Dordrecht: Kluwer.
- Vollrath, H.-J. (1978). Rettet die Ideen! *MNU*, 31(8), 449-455.

# THE ROLE OF VIDEO-BASED EXPERIENCES IN THE TEACHER EDUCATION OF PRE-SERVICE MATHEMATICS TEACHERS

Miriam Liston\* and Olivia Gill†

\*National Centre for Excellence in Mathematics and Science Teaching and Learning

† Department of Mathematics and Statistics, University of Limerick

*This paper will report on the role of video-based experiences and reflection among pre-service secondary school mathematics teachers as the drive towards a more constructivist approach to teaching and learning mathematics strengthens in Ireland through the introduction of a new initiative called 'Project Maths'. This study uses the critical analysis and reflection of video recordings to examine pre-service mathematics teachers' subject content knowledge. 29 pre-service teachers were video-taped teaching a mathematics class to mature students at the University of Limerick (UL) from January to May 2010. Rowland's (2008) Knowledge Quartet was the framework used for reflection of this teaching. The findings are discussed and implications for mathematics teacher education are highlighted.*

**Keywords:** Mathematics Teacher Education, Pre-Service Mathematics Teachers, Video-Based Experiences, Subject Content Knowledge and Pedagogical Content Knowledge

## INTRODUCTION

The low uptake of Higher Level<sup>1</sup> mathematics (16% of total cohort in 2011) and the large failure rate of mathematics at Leaving Certificate<sup>2</sup> has highlighted the need for reform of mathematics education in Ireland. A move towards addressing the teaching and learning of mathematics is currently being implemented under a new initiative called 'Project Maths' which will see much greater emphasis placed on student understanding of mathematics concepts and applications.

However, there is also need for change to occur in the training of our mathematics teachers. Recent PhD work by the first author (ML) suggests that first year pre-service mathematics teachers (studying Physical Education and Mathematics at UL) have a fragmented, disjointed view of mathematics and the approaches that they adopt to learning mathematics are mainly of a procedural nature. There has been similar research internationally portraying the poor conceptual understanding that often exists among pre-service mathematics teachers (e.g. Nicol, 1999). This study was undertaken in order to link the research to a current teacher education programme by critically assessing, reflecting and most importantly, developing pre-service mathematics teachers' awareness of their subject content knowledge, as well as their pedagogical content knowledge, through the use of video-based experiences. The study is predominantly exploratory in nature since it aims to identify and create

awareness among pre-service mathematics teachers' in relation to their subject and pedagogical content knowledge. Although not analysed in this study, it is also hoped that in light of their Knowledge Quartet (KQ) training outlined in the methodology below, pre-service teachers will develop their knowledge in preparation for their video-based teaching experience and indeed after this experience. Shulman (1986) clarified subject matter knowledge as knowledge of the content of the discipline such as facts and concepts. He described pedagogical content knowledge as the manner in which a teacher can represent the subject in a way that others can comprehend and an understanding of what makes the learning of the subject easy or difficult.

There has been much research carried out on the use of video-based experiences in the training of prospective teachers as a tool to critically observe and reflect on their own teaching and the teaching of others. Alsawaie and Alghazo (2010) concluded from their study that video-based experiences increased pre-service teachers' knowledge about problems in practice, developed their sensitivity toward student learning and lead to them to think in depth about efficient instructional strategies. In addition, Maher (2008) claims that it provides students with an opportunity to reflect and review theirs, and others, mathematics teaching, helping them to become aware of their practice as well as assisting them to grow in their pedagogical content knowledge.

For these reasons, one focus of the paper is on pre-service teachers' reflection of their own teaching. The authors also believe that the use of video-based experiences in mathematics teacher education pedagogy classes are beneficial to teacher educators by enabling them to observe and evaluate their students' subject and pedagogical content knowledge. Therefore, this study also focuses on the researchers' critical observation and reflection of the pre-service teachers. Due to constraints to the length of this paper, as well as the need for further analysis of the data collected, pre-service teachers' critical observation and reflection of their peers is not reported on here.

## **THEORETICAL FRAMEWORK**

The theoretical framework employed in this study for investigating pre-service mathematics teachers' subject content knowledge and to a lesser extent, their pedagogical content knowledge, is now discussed.

The Knowledge Quartet (KQ) devised by Rowland, Huckstep & Thwaites (2005) was the framework upon which this study was conducted. Rowland and his colleagues created this framework for the observation and review of mathematics teaching. It consists of four units: foundation; transformation; connection and contingency. Each unit is subdivided into smaller sub codes of which there are 17 in total. The framework used in this study is a slight adaptation to Rowland et al.'s (2005) KQ since one or two of the sub codes are not identical to the original version e.g. "depth of mathematical knowledge" was not a code in the original KQ. Rowland

(2008) describes foundation as trainees' knowledge, beliefs and understanding acquired in preparation for their role in the classroom. Transformation concerns "knowledge-in-action as demonstrated both in the planning to teach and in the act of teaching itself" (Rowland, 2008, p.289). Connection, the third category, links together choices and decisions for the more discrete parts of mathematical content. It includes making connections between concepts and procedures as well as sequencing of subject matter. The final category, contingency, incorporates the pre-service teachers' ability to respond to students' ideas and think on one's feet. Table 1 below summarises the Knowledge Quartet framework adapted from Rowland and his colleagues for use in this study.

Foundation	Transformation	Connection	Contingency
<ul style="list-style-type: none"> <li>- Depth of mathematical knowledge.</li> <li>- Use of terminology</li> <li>- Use of textbooks</li> <li>- Reliance on procedures</li> </ul>	<ul style="list-style-type: none"> <li>-How the mathematics is communicated to the learner (the difference between someone who knows mathematics and someone who knows how to teach mathematics</li> <li>- Example Choice (real-life examples, other subject areas etc.)</li> <li>- Analogy</li> <li>- Demonstration</li> <li>- Representation</li> <li>- Illustrations</li> </ul>	<ul style="list-style-type: none"> <li>- Making connections between mathematical concepts</li> <li>- Making connections between mathematical procedures</li> <li>- Sequencing of subject matter (order in which the mathematic concepts are taught)</li> <li>- Anticipation of complexity (knowledge/awareness of areas which students will find difficult)</li> </ul>	<ul style="list-style-type: none"> <li>- Ability to think one one's feet</li> <li>- Response to unexpected</li> <li>- Deviate from lesson plan if advantageous</li> </ul>

**Table 1: The Knowledge Quartet (adapted from Rowland, 2008 and Rowland et al., 2005)**

## METHODOLOGY

### Research Design

The research was carried out in two main stages. In stage one both authors separately watched and compared three extracts from a TIMSS video study and analysed them according to the KQ identifying aspects of each of the four units that impacted on the lesson. Following this pre-service teachers were provided with KQ training in the form of two workshops. The first workshop included a 20 minute lecture and discussion of the KQ. They then observed and analysed one extract from the TIMSS video study (selected by both authors) in the same manner as that previously done by the authors. This was done individually firstly, following by paired and whole class discussion on findings. Finally, the pre-service teachers were provided with a copy of the researchers' analysis to compare and contrast their own analysis with. The pre-service teachers had one further workshop where they again observed and reflected



on a 30 minute extract from another TIMSS video and in pairs/groups they discussed what they had observed and reported on based on the KQ units and sub codes.

In the second stage of this research, the pre-service mathematics teachers were split into pairs for teaching a 50 minute support tutorial (25 minutes each) to “Access” mature students. As the name suggests, these tutorials support the work done in lectures and in the students’ regular tutorials provided by the University. Each pre-service teacher was required to teach a support tutorial and reflect on it once provided with the DVD of the lesson. They were also required to attend two other lessons observing and reflecting on their peers’ teaching although this is not reported on in this paper as explained earlier. The KQ is the basis for all observation and reflection. Pre-service teachers were provided with a copy of the KQ table and a critical observation and reflection report form devised by the authors for this analysis (available from the first author). Prior to teaching, each pre-service teacher was provided with a tutorial sheet from the lecturer of the Access Mathematics or Access Statistics course with specific questions to follow. The first author also attended all support tutorials and observed and reflected on each pre-service teaching session.

### **Research Sample, Data Collection and Data Analysis**

The sample included 29 pre-service secondary school mathematics teachers studying Physical Education and Mathematics at UL. Access to the sample was not a problem since their mathematics pedagogy module from January 2010 to May 2010 was taught by the second author (OL). Olivia Gill is also manager of the Mathematics Learning Centre in UL and assigns tutors for all support tutorials. This provided us with the sample to teach – mature students studying an access certificate course designed to refresh students’ skills in areas such as basic mathematics and statistics. Many of these mature students have not studied any form of mathematics in a number of years and are returning to education in the form of this Access course which provides a means of entry to third level undergraduate courses in the future.

Prior to data collection, consent was obtained from all mature students and pre-service teachers for recording the lessons. The participants were informed that all data was anonymous and that it would be stored securely for the authors’ use only. Four pre-service teachers taught per week, two at the mathematics support tutorial and two at the statistics support tutorial with exception of one pre-service teacher who taught a full lesson on his own due to odd numbers. All lessons were recorded and DVDs developed of each lesson. Each pre-service teacher was given the DVD of their teaching only. The authors also had a copy of the DVD of each lesson.

This paper reports firstly on the researchers’ analysis of the 29 pre-service teachers’ teaching. The first author attended each lesson and at a later date she again reflected on all video-recordings in more detail. Again, the KQ and the codes designed by Rowland et al. (2005) was the framework for this analysis. On completion of this stage of analysis, the authors determined the main themes or findings that emerged



from the data under each of the four units. The findings coming from the data were compared by the two researchers for consistency.

The pre-service teachers' own critical observation and reflection reports are also analysed and discussed in this paper. These reports were completed in the same manner as the authors in that each pre-service teacher watched the DVD of their own teaching and reflected on it by identifying aspects of each of the four units of the KQ that impacted on their lesson. The reports were then submitted to the first author. One pre-service teacher failed to submit his report so analysis is based on 28 pre-service teachers. Again, the first author determined the main themes/findings emerging from the pre-service teachers' reports in terms of pre-service teachers' awareness, or otherwise, of aspects of the KQ that impacted in any way on their lesson.

## FINDINGS AND DISCUSSION

The findings and discussion in terms of both the researchers' critical observation and reflection of the pre-service teachers and in terms of the pre-service teachers' critical observation and reflection on their own teaching are now presented and discussed under the four main categories of the KQ; foundation, transformation, connection and contingency. The KQ sub codes as in Table 1 are in italics for clarity.

### Researchers' critical observation and reflection of the pre-service teachers

#### Foundation

The *depth of mathematical knowledge* demonstrated by the pre-service teachers was poor with only six teachers displaying a good depth of knowledge. In general, the pre-service teachers *relied on procedural knowledge* (19 out of 29), described by Skemp (1978) as instrumental understanding or knowing the 'how' rather than knowing the 'why'. For example, one pre-service teacher did not relate to students' method for calculating the median as it differed from her method. She was confused stating that

"5.5 is the median because it is the right middle number. I don't know why the formula isn't working".

Many explanations were also focussed at a procedural level. When explaining the basic laws of probability one teacher reinforced the idea that students should multiply when they see the word 'and' and add when they see the word 'or'. Mason & Spence (1999) stress the failings of rehearsal and practice of techniques. While a mixture of both procedural and conceptual understanding is important, an overreliance on procedural understanding can be damaging. A concluding remark from one teacher was that

"If you can learn off the formulas you'll be fine".

This may have been the manner in which this pre-service was taught himself in school. Developing subject matter knowledge is essential for these pre-service teachers since improvements in particular kinds of subject matter competence contribute to better analysis of practice thus improve teaching (Hiebert, Morris, Berk & Jansen, 2007).

In addition, 25 out of the 29 pre-service teachers frequently used *poor mathematical language* or failed to introduce *terminologies* in their teaching.

### Transformation

*Communicating the mathematics* to the learner is one of the sub codes under this category. 8 out of the 29 pre-service teachers were categorised as good transformers or communicators of knowledge, 14 categorised as poor transformers of knowledge and 7 as average. The pre-service teachers were categorised according to the way in which the teacher transformed his or her own meanings and descriptions of the content. An example of where knowledge was transformed effectively through *representation* and indeed by using a *real-life example* was where a pre-service teacher used a You Tube video of a car overtaking another car to introduce the concepts of velocity and acceleration. While the first author is not suggesting that real world contexts are the only effective way to transform the mathematics to the learner, she is in agreement with researchers such as De Lange (1996) that some real world connections develop students' understanding of mathematical concepts.

While pre-service teachers were provided with specific questions to follow, they were given scope and encouraged to provide their own examples and vary their teaching strategies throughout the lesson. 5 out of the 29 pre-service teachers used *real-life examples* more than once in the lesson, while 7 out of the 29 made one attempt to put some concepts in context, and the remaining 17 used no real-life contexts. Boaler (1994) explains that teaching in context motivates students and builds their confidence and interest in mathematics so long as a realistic view of mathematics is given which makes sense both in the classroom and in the real world. There was some varied use of *demonstrations and analogies* but at times these were incorporated in the lesson to little effect e.g. one teacher used the interactive software package 'GeoGebra' to introduce the meaning of differentiation but she struggled to explain what was happening on the diagram. The importance of foundation knowledge is again to the fore here.

### Connection

The main finding to emerge from this category in terms of *making connections between concepts or procedures* was that many of the pre-service teachers lacked the knowledge to do just that. In a number of lessons, in particular the statistics lessons, no link was made between the answer obtained and the actual concept involved. Ball, Lubinski & Mewborn (2001, p. 433) report on a lack of understanding of the mathematical knowledge necessary to teach well.

*Sequencing of subject matter* is another sub code within this category but as students were provided with tutorial sheets prior to the lesson the code was less relevant to this study. It was interesting to note however, that only four of the 29 pre-service teachers re-ordered the sequence of exercises or topics according to what they felt would most benefit the students' learning. There were eight incidents observed where the pre-service teachers *anticipated difficulties* their students may have with a particular concept.

### Contingency

The final category in the KQ is contingency which is determined by the pre-service teachers' *ability to think on his or her feet, respond to the students and deviate from the lesson* where he or she feels it would be beneficial for one or all learners. These three codes are discussed together since they are very much interlinked. The main findings emerging from this category were that most pre-service teachers (21 out of the 29) appeared to lack the content knowledge necessary to interpret students' questions and misconceptions and to confidently deviate from the lesson plan to explain such misconceptions. For example, in a lesson on integration a number of students were confused when the pre-service teacher removed the integral sign when he had not in fact integrated yet. The pre-service teacher was unaware and struggled to correct his error. The importance of subject content knowledge has been reported many times throughout this paper.

There was however, some evidence of good *responses and ability to deviate from lesson plan* where appropriate and beneficial for students (6 out of the 29).

### **Pre-service teachers' critical observation and reflection on their own teaching**

#### Foundation

There was a mix of awareness among the pre-service teachers when reflecting on their foundation knowledge. 10 of the 28 pre-service teachers recognised their strengths or weaknesses in terms of *depth of mathematical knowledge*, use of *mathematical terminology* and *reliance on procedures*. One pre-service teacher who was categorised by the authors as having poor foundation knowledge recognises this to be the case and reflected that he

“Doubts own knowledge of the content while explaining (solve  $64^{\frac{2}{3}}$ ) to the class, thus creating confusion for the pupils”.

A further 9 pre-service teachers displayed a poor ability to critically reflect on the aspects of foundation failing to recognise a *reliance on procedures* and unaware of their poor content knowledge. One pre-service teacher stated that she had a ‘comprehensive knowledge of the topic’ despite the fact that she struggled to calculate the median. This would suggest that her beliefs about her knowledge do not match her actual content knowledge. Sullivan (2008) talks about the importance of teachers understanding the relevant mathematics needed to appreciate the work that

their students are expected to do. The remaining four pre-service teachers submitted vague reflection reports suggesting that perhaps their interpretation and understanding of the KQ was not as the authors would have hoped.

### Transformation

The authors categorised 11 of the 28 pre-service teachers as demonstrating good critical reflection skills in terms of transformation of knowledge. These pre-service teachers were very much aware of the way in which the knowledge was *communicated* be it in an effective way or otherwise. One pre-service teacher recognises the effective transformation of Pythagoras' Theorem through use of YouTube video while also noting that her explanation of angles of elevation was unclear. Similar findings were noted in the authors' analysis. There was also some poor understanding of transformation (9 out of the 28 pre-service teachers). The pre-service teachers whose reports highlighted both good and poor critical awareness and reflection were categorised as mixed transformation reflections. One such pre-service teacher reflected on the many effective aspects to his lesson but also maintained that he used pair and group work to encourage peer learning. This was not evident to the authors and perhaps the pre-service teachers need more guidance in terms of how to implement group and pair work effectively.

### Connection

This category was the most poorly reflected on by the pre-service teachers. 16 out of the 28 demonstrated very poor critical analysis and reflection here and it seemed they were unaware of what 'connection' means. Another possible reason for the poor reflection was the focus of some pre-service teachers on procedural knowledge. According to one student he had linked and made *connections between the mathematical procedures* because he had "*Reiterated to pupils the importance of showing all workings in case you make a mistake*".

The importance of teachers' beliefs comes into play again here and the author is in agreement with Grootenboer's (2008) suggestion that there is a strong case for considering the development of reform of prospective teacher's beliefs.

### Contingency

There were once again varied critical reflections for this category with 11 out of the 28 pre-service teachers demonstrating good critical awareness of where contingency was in action or where it could have been improved upon. 8 of these 11 pre-service teachers admitted that they lacked the content knowledge to *deviate from their lesson plan* or respond effectively to students' questions. According to Shriki (2010), many teachers do not possess the abilities needed to foster their students' creativity in mathematics, mostly due to lack of prior experience or proper college preparation. In the authors' reflections, it was noted that foundation knowledge seemed to impact on the pre-service teachers' contingency. Of the 9 pre-service teachers who reflected

poorly on contingency, 6 were not aware that insufficient foundation or content knowledge hindered their ability to *respond to the unexpected*.

## CONCLUSIONS

The above findings offer an insight into pre-service teachers' subject and pedagogical content knowledge from the researchers' perspective and from the pre-service teachers' own perspective. Some findings suggest that pre-service mathematics teachers may not have sufficient subject matter knowledge to alter their teaching strategies and ultimately teach for understanding. There is an onus on mathematics teacher educators to develop pre-service teachers' ability to move away from traditional approaches to teaching and create an awareness of the benefits of doing so. Video-based experiences are one way of doing that and this is one of the main reasons as to why the data was firstly analysed from the researchers' perspective. It offers a challenge to teacher educators to create awareness among pre-service teachers of the need to develop their subject and pedagogical content knowledge which was the focus of this study. From their own reflections and the findings reported above, it is clear that there is a mixed awareness among the pre-service teachers about the need for them to develop these essential tools for effective mathematics teaching. This offers a major challenge for teacher educators.

Video-based experiences also provide trainee teachers with the opportunity to develop critical reflection skills in terms of their own teaching which is why this study included analysis from the pre-service teachers' perspective. Evidence from the paper highlights the further need to enhance pre-service teachers' reflective skills and develop their awareness of how to do so effectively. The process of mathematics teachers' reflecting on teaching situations is described by Garcia, Sanchez & Escudero (2006, p. 2) as "an important process providing information that contributes to our understanding of their professional knowledge".

## NOTES

1. There are three levels of mathematics in the Irish examination system with the highest level referred to as Higher.
2. The Leaving Certificate, commonly referred to as the Leaving Cert., is the final course in the Irish secondary school system and culminates with the Leaving Certificate Examination.

## REFERENCES

- Alsawaie, O.N. and Alghazo, I.M. (2010). The effect of video-based approach on prospective teachers' ability to analyze mathematics teaching. *Journal of Mathematics Teacher Education*, 13 (3), 223-241.
- Ball, D.L., Lubienski, S., and Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of Research on Teaching* (4th ed.) (pp. 433-456).

New York: Macmillan.

- Boaler, J. (1994). When do girls prefer football to fashion? An analysis of female underachievement in relation to 'realistic' mathematics contexts. *British Educational Research Journal*, 20(5), 551-564.
- De Lange, J. (1996). Using and applying mathematics in education. In: A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 49–97). Boston: Kluwer Academic Publishers.
- Garcia, M, Sanchez, V and Escudero, I. (2006). Learning through reflection in mathematics teacher education. *Educational Studies in Mathematics*, 64(1), 1-17.
- Grootenboer, P. (2008). Mathematical belief change in prospective primary teachers. *Journal of Mathematics Teacher Education*, 11 (6), 479-497.
- Hiebert, J., Morris, A.K., Berk, D., & Jansen (2007). Preparing to learn from teaching. *Journal of Teacher Education*, 58 (1), 47-61.
- Maher (2008) 'Video recordings as pedagogical tools in mathematics teacher education'. In D. Tirosh & T. Wood (Eds.), *International Handbook of Mathematics Teacher Education: Vol.2 Tools and Processes in Mathematics Teacher Education* (pp.65-83). Rotterdam: Sense Publishers.
- Mason, J. and Spence, M. (1999) 'Beyond mere knowledge of mathematics: the importance of knowing-to act in the moment'. *Educational Studies in Mathematics*, 38 (1-3), 135-161.
- Nicol, C. (1999) 'Learning to teach mathematics: questioning, listening, and responding'. *Educational Studies in Mathematics*, 37(1), 45-66.
- Rowland, T. (2008). Researching teachers' mathematics disciplinary knowledge. In P. Sullivan and T. Wood (Eds.) *International handbook of mathematics teacher education: Vol.1. Knowledge and beliefs in mathematics teaching and teaching development* (pp. 273-298). Rotterdam, the Netherlands: Sense Publishers.
- Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8(3), 255-281.
- Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shriki (2010). Working like real mathematicians: developing prospective teachers' awareness of mathematical creativity through generating new concepts. *Educational Studies in Mathematics*, 73(2), 159-179.
- Skemp, R.R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, 26(3), 9-15.
- Sullivan, P. (2008). Education for the knowledge to teach mathematics: it all has to come together. *Journal of Mathematics Teacher Education*, 11(6), 431-433.

# MULTICOMMENTED TRANSCRIPTS METHODOLOGY AS AN EDUCATIONAL TOOL FOR TEACHERS INVOLVED IN CONSTRUCTIVE DIDACTICAL PROJECTS IN EARLY ALGEBRA

Nicolina A. Malara – Giancarlo Navarra

University of Modena and Reggio Emilia (Italy)

*This paper presents results regarding the Multicommented Transcripts Methodology (MTM) we have enacted to promote in teachers of primary school and secondary school awareness of their own ways of being in the class and to guide them in managing mathematical discussions. After a brief overview of the theoretical framework and the methodological structure of MTM, two multicommented classroom-based episodes are proposed, with the aim of highlighting the MTM educational potentialities. Some final remarks concerning the formative, cultural, educational and methodological principles of MTM are made.*

Keywords: Early algebra, Metacognition, Multicommented Transcripts Methodology, Reflective Teaching, Teacher Education

## INTRODUCTION

Early algebra is proving to be an appropriate approach to algebra for 5 to 14 years-old pupils, which allows them to achieve a better control over the meaning of the algebraic objects as well as of their generative processes. This achievement may occur in different ways depending on the age of exposure to early algebra, either in the first years of primary school or after several years of traditional teaching. In the first case, arithmetic should be *introduced* in a *pre-algebraic* perspective, whereas in the second one it should be *revisited from a relational and structural point of view* overcoming the traditional focus on algorithms execution. This entails a re-framing of teaching in the arithmetic-algebraic area requiring a greater attention to the construction of algebraic language as an instrument for representing relations and properties. This change of perspective leads teachers to revise their own knowledge, beliefs, attitudes, working styles.

The MTM, on which we report here, was born for this aim. It developed within our ArAl Project, which involves in-service teachers in long term educational process. In the project the teachers deal with basic theoretical issues in early algebra together with the development of teaching sequences across the school grades: from algebraic generational activities to meta level activities (Kieran 1996), such as modeling and proof 1. This led us to design ways and tools to study the behavior of teachers involved in our project and engaged in early algebra teaching sequences, with the aim to lead them to reflect on their actions in the classroom and understand how these may be improved. (Malara & Navarra 2009, Cusi & Al. and related references).

## SOME THEORETICAL INDICATIONS

Several studies highlight how teacher's knowledge, beliefs, emotions and attitudes are intertwined and determinant components of teaching and learning processes (see Malara & Zan 2008 and related references). They underline that the study of these relationships is crucial to provide teachers with useful suggestions in their professional development. In this respect, the analysis of interactive and discursive practices, the awareness of the variables that influence the classroom process and self-observation during action are fundamental. The value of the teacher's *critical reflection* is a well known fact in the achievement and empowerment of the above mentioned skills (see for example Mason 2002, Jaworski 2003). In Jaworski's view, the essence of the reflective practice consists of making explicit teaching approaches and processes, so that they become the object of a detailed critical examination. She promotes the usefulness of *communities of inquiry in teaching*, discussion groups composed of teachers and researchers, in which the teacher has the opportunity to develop a specific identity.

Our teacher training model follows these conceptions and modalities. But it represents the outcome of research and training practices developed in Italy since the 1970's. Our hypothesis is that by critically reflecting on socio-constructive teaching/learning processes, the teacher is led to become aware of the different roles he/she is supposed to play in the classroom, of the best modalities to interpret them and can also get useful suggestions about how to behave in the classroom. Moreover, it is crucial for teachers to approach research results that can be useful for practice and become aware of the importance of studying them for their own professional development.

The focus of our research is on the analysis of classroom-based processes that develop along teaching sequences planned with the teacher. These studies aim at: showing teachers the micro-situations which compose a process and the higher or lower effectiveness of the micro-decisions made; favoring the achievement of control over their own behavior and communication styles, as well as noticing the impact the latter have on pupils' behavior and learning. More in general they aim at gathering both theoretical and practical tools for pre-service and distance teacher training.

## THE METHODOLOGY

In our project teachers are involved for at least two years in training activities. After planning and implementing lesson units together with the researchers, the teachers carefully record some lessons they choose, transcribe them according to a predefined format, add details coming from the notes they have taken during the lessons (gestures, expressions, etc.) and include reflections and comments. The teachers then engage themselves in a network exchange of e-mails with their mentors and sometimes with other teachers. The exchange consists mainly of reflections and comments regarding the classroom transcripts, through which the mentor infers the teachers' interpretation of their theoretical frame and the developed cultural values,



as well as their progressive harmonization with the background and the previous attitudes. This is the core of MTM.

There are two different ways of implementing MTM. One way takes place in a university environment as part of a national or international research and training programs. It involves a small number of researchers and teachers who strictly interact with specific research proposals. The other way is implemented in schools, from different Italian regions and organized in networks, by teachers who take part into the ArAl Project. It is characterized by a few meetings with researchers and teachers and several long distance interactions, conducted via e-mail. A high number of teachers are involved (in the year 2010 nearly 150), organized in small groups of work. Each group is coordinated by a researcher, who plays the fundamental role of E-tutor. This latter type of intervention is mainly aimed at training although with important spin-offs for research.

As first step, teachers are required of including in their transcripts of class session, either positive or negative comments concerning mathematical issues or critical points in the development of the discussion, possibly attaching some class material. The transcript of each session is sent by e-mail to the E-tutor, who makes comments and sends it to other teachers and researchers involved for further comments. Each of them can intervene again in the cycle with further meta-comments. So, the multicommented transcripts (MT) reify. They become an important object of study for the teachers through sharing with colleagues within the school and during meetings with the E-tutors. They are also published in the schools websites, in some cases included in [www.aralweb.unimore.it](http://www.aralweb.unimore.it) as ‘good practices’. In the following we wish to highlight their educational potentialities through some excerpts.

## ANALYSIS OF CLASSROOM EPISODES: TRANSCRIPTS AND COMMENTS

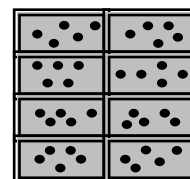
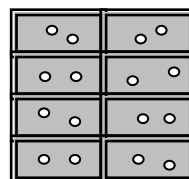
Here, two MT excerpts, which document both the interactions among the actors and the variety of the faced themes, are presented. The order of their presentation is: (a) context; (b) transcript of session; (c) comments. In the comments, the words written in *Italic* indicate key elements of the ArAl Project theoretical framework, which are described in its Glossary 2 (some examples can be seen in <http://www.aralweb.unimore.it/on-line/Home/ArAlProject/Glossary.html>).

### Episode 1 (year 3 primary, 8-9 years old)

The teacher is participating in the project for the second year and she is working on the distributive law, already discovered by the class in simpler cases.

She shows two boxes, divided in eight parts, containing two types of marbles, as shown above. She

says they belong to Marina’s collection who has organized them in a very orderly



## Working Group 17

way. Then the teacher formulates the task: “Represent the situation in mathematical language so that Brioshi 3 may find the total number of marbles in the two boxes”. Pupils work and their proposals are written on the blackboard. Many of them formulate more than one proposal.

Andreina, Danilo, Francesca, Martina	$16 \times 40 = n$ $40 \times 16 = n$ $n = 16 \times 40$ $n = 40 \times 16$
Andrea	$(5 \times 2) + (4 \times 2) = n$ $n = (5 \times 2) + (4 \times 2)$
Maria	$2 \times 8 = n$ $5 \times 8 = n$ $n = 2 \times 8 + 5 \times 8$
Bruno	$(2 \times 8) + (5 \times 8) = n$
Melania	$4 \times 2 + 4 \times 5$
Sara, Elena, Giovanna	$5 \times 8 = n$ $2 \times 8 = n$ $n = 5 \times 8$ $n = 2 \times 8$
Francesco	$(2 \times 8) + (5 \times 8) = n$
Chiara	$2 \times 2 + 5 \times 2 = n$ $n = (2 \times 8) + (5 \times 8)$ $n = 4 + 10$

Teacher: Good! Now, as usual, let's open up the discussion. [Comm 1]

Andreina: Teacher, we were wrong because 16 is not repeated 40 times.

Teacher: Explain it better.

Andreina: I understood that we didn't have to multiply red marbles and green marbles, but rather put them together.

Teacher: What do you mean by 'put together', try to use mathematical language better.

Andreina: United...

Teacher: Do you know a more suitable term to explain what Andreina means?

Francesco: You must add.

Teacher: Yes, this sounds clearer... Any other remark?

Melania: In my opinion the translations made by Andreina's group are *opaque*.

Teacher: What do you mean?

Bruno: They are *opaque* because they have already found the number of marbles.

Chiara: It was not up to us to find 16 and 40, but rather write the translation to be sent to Brioshi. They have nearly solved the problem.

Bruno: It's true, they found the *product and not the process*.

Teacher: What do you think about Andrea's representation?

Andrea: Miss, I got wrong too... erase, erase.

Teacher: Hold on, explain what you wrote (*Andrea can't explain*).

Melania: I also realize that I forgot to write something. I wrote  $4 \times 2$  and  $4 \times 5$  because I saw separate columns, but now I understood that my representation is not complete, I must add ' $\times 2$ '.

Teacher Tell me which changes I should make.

Melania:  $4 \times 2 \times 2 + 4 \times 5 \times 2 = n$ .

Francesco: But she wrote like me... like Bruno... and like Maria Giovanna, because  $4 \times 2$  equals 8.

Andrea: Melania factorized eight!

The expression by Francesco is chosen to be sent out to Brioshi:  $n = 2 \times 8 + 5 \times 8$ .  
[Comm 2] [Comm 3]

1. Comment by the E-tutor: Regardless how correct the expressions are, the pupils show they use the *letter* as indicator of a number to be found. It is a naive and sometimes not pure use of the letter, like in the cases of Maria, Sara, Giovanna, Elena and Chiara, where the same letter stands for different quantities. Pupils should be led to reflect upon this from the beginning.

2. Comment by the E-tutor: The transcript shows how the class is familiar with *mathematical discussion*. Also, it shows the good argumentative skills of the pupils and the fact that they draw on important theoretical constructs such as the distinction between *opaque* and *transparent* representations and between *process* and *product* of a calculation. It would be appropriate not to overlook a collective investigation on expressions like Melania's, which are incorrect but revealing her initial vision of the situation. How do they get to decide that Francesco's expression should be sent out to Brioshi?

3. Comment by the coordinator: The iconic representation proposed by the teacher for translation into mathematical language is problematic and deserves reflection. Its negative influences can be detected in the translations made by Andrea or Melania. Many pupils use the (correct) representations  $2 \times 8$  e  $5 \times 8$ . But the numerical representation consistent with the given representation is:  $2 \times 4 + 2 \times 4 + 5 \times 4 + 5 \times 4$ . Moreover, reading by rows, one may be led to the representation  $(2 + 2 + 5 + 5) \times 4$ , changeable into  $(2 \times 2 + 5 \times 2) \times 4$ , for the meaning of multiplication as repeated addition, an expression which can be, in turn, modified into  $(2 + 5) \times 2 \times 4$ , for the distributive law they just met. The latter expression permits a link with representations 'perceived' by many pupils:  $(2 + 5) \times 8$  e  $2 \times 8 + 5 \times 8$ . These steps are certainly very sophisticated for pupils aged 8-9 and require control over parentheses and the property itself. The teacher should appropriately make hypotheses about the possible interpretations induced by an iconic representation and constructs a discussion sketch for each of them, in case some pupils propose it or even to show how a representation can be viewed in more different ways. A general point must be highlighted: the need to favor the interpretation of paraphrases in mathematical language contributing to the construction of meaningful skills in pupils.

**Episode 2 (grade 9, pupils aged 11-12)**

A general representation of the sequence 4, 11, 18, 25, 32, ... is sought for. The table reported below is constructed on the blackboard to identify the link between place number and term of the sequence.

Place [Comm 4]	Term	Rule
1	4	$7 \times 2 - 10$
2	11	$7 \times 3 - 10$
3	18	$7 \times 4 - 10$
...	...	...
8	53	?

Pupils find various rules, but they are focused on one in this case. The teacher is encouraging reflection upon relationships between numbers of the first and the third columns.

Teacher: You have discovered that “the place equals the term preceding the second factor of the multiplication”. So, for instance, what is the rule at place 8?  
[Comm 5]

Serena: You must do  $7 \times 9 - 10$ .

Teacher: Do we all agree? [Comm 6]

Many: Yes.

Teacher: And is it 53? [Comm 7A] [Comm 7B]

Many: Yes.

4. Comment by the E-tutor: ‘place’ is used instead of ‘place number’. It is appropriate to be precise not to induce pupils to identify quality and quantity.

5 Comment by the E-tutor: Pay attention to the abbreviations (see Comm 4). The fact that pupils know the meaning of ‘preceding’ in Italian is not enough for a *translation* in algebraic language. They must learn to express ‘preceding’ in relation to the number that follows. If they are able to paraphrase it only with ‘that precedes’ (as in the transcript), they get stuck, because these *paraphrases* are *opaque*. They should rather be induced to make explicit the link between the two numbers and express the preceding number as a function of the subsequent. Natural language supports the achievement of this expression. In a grade 8 class, for instance, the proposal made by a pupil: ‘the preceding number is always one unit smaller than the number that follows’ turned out to be very effective and decisive for translation.

6. Comment by the E-tutor: There should be a *shared didactical contract* according to which monosyllabic answers are not acceptable. They do not help the teacher understand how the topic has actually been understood and do not help classmates either. Questions that require only ‘Yes’ or ‘No’ answers are not productive as well. Pupils do not *argue*,

they only answer the teacher's questions and let her guide them towards her objective. Requests like 'Explain what you mean' make the pupil himself define the objective of his reasoning, and construct the related explanation. In these cases, the teacher should only evaluate the quality of the discussion, *sorting out the argumentative traffic* by intervening mainly on the methodological plane, thus favoring the *social construction of knowledge* through *negotiation, sharing* and *stabilization* of meanings. A teacher should become aware that control over natural *language* and implementation of *social practices* are basic elements for the understanding of mathematical language and therefore of mathematical concepts.

7A. Comment by the teacher: At this point (I refer to Serena's 'You must do' and to my 'it is', but this also occurs elsewhere in the transcript), I realize how imprecise my language is. I could have said "Does  $7 \times 9 - 10$  'represent' number 53?" or "Does it 'correspond' to 53?"

7B. Comment linked to the previous one by the E-tutor: Ok, right. But it is not only a matter of language, I think this reveals rooted attitudes which reflect hidden convictions. Very often teachers' algorithmic approach is 'dominant' (referring to: operations, result, calculate, solve, 'how much is', 'it is', ...), the relational one is 'recessive' (mainly focusing on: *relations, structure, representation, ...*). These activities in an early algebra environment aim to induce teachers to reflect on this point.

## REFLECTION ON EPISODES AND COMMENTS

Comments in the transcripts are valuable for training in several aspects. Some of the most meaningful are reported below.

**Socio-linguistic aspects.** We underlined how important linguistic aspects are in the construction of mathematical knowledge and how central they are in mathematical discussions. Pre-requisite for teachers to be able to make the discussion a shared instrument for the class is that they acquire many skills: to create a good context for interaction, to enact socio-mathematical norms that lead to compare different solutions, evaluate if a solution is acceptable or of a good quality, to steer the direction of the discussion in the different phases, to involve pupils in meta cognitive acts and so on.

**The relationship theory-practice.** Another aspect emerging from comments is reference to the theoretical framework and to the glossary of the project not only shared by researchers and teachers but also – with appropriate adjustments – by teachers and pupils. Sharing is extremely important in both cases, because teachers and above all pupils are enabled to understand how aspects apparently far from mathematics, such as: 1) competence in using languages, mainly natural language, and control of their semantics and syntax; 2) being able to translate from one language to another; 3) difference between representing and solving a problem situation; 4) distinction between process and product; 5) recognizing the meanings of the equal sign; ... *are the foundations of a meaningful construction of mathematical knowledge.*

**Mathematical aspects: an example, the conquest of the letter.** In the first episode the letter is used in a very naïve way, in the second one it represents a high-level goal (in the next step the variable ‘n’ is introduced as place number and to represent  $7 \times (n-1) - 10$  by  $t_n$  as the n-th term of the sequence). *Algebraic babbling* (a theoretical constructs of the ArAl project, which compares modalities of construction of algebraic language to those of construction of natural language) emerges throughout exploration, discovery, conjectures, failed attempts which entail the introduction and use of the letter with various meanings (generic number, unknown, variable). Through transcripts and the analysis of comments, teachers become aware that the main difficulty for pupils is *to get to understand that a letter can represent a number*. It is an epistemological jump, fundamental for algebraic thinking, which may become a block if the pupil is not guided enough.

The use of comments in the training process aims to make the teacher sensitive to basic general issues, such as: are students aware they are *communicating* through mathematical language? What kind of relationship do they have with the *semantics* and *syntax* of mathematical language? Which environment (situation, contest) can improve algebraic thinking? How can one detect the awareness of ‘algebraic content’ in pupils’ sentences, intuitions, proposals, representations? These kinds of questions make teachers’ reflections profound, meaningful and productive.

## CONCLUDING REMARKS

MTM appears to be an effective instrument in teacher’s training processes involving mathematics.

This methodology has an important pre-requisite: a trusting relationship between teachers and researchers. Moreover, when the teacher edits a transcript, he puts himself on a different level. He detaches himself from the activity he was part of and critically reads what happened in the classroom. His class is no longer his class. His transcript is no longer narrative, it acquires scientific aspect and becomes a training instrument. Comments may bring her/his misconceptions to the surface, touching sensitive points. Many teachers immediately realize that the comments are valuable and accept the remarks. Others see their competence jeopardized; they feel uncomfortable and refuse to accept that their transcripts may become public. Others tend to ‘watch and wait’, they need time to familiarize themselves with the methodology and be convinced before using it. These different types of behavioral patterns are monitored by researchers, who are always trying to make teachers understand that MTM is meaningful and productive only if participants engage in the project with open and sincerely committed minds.

## Notes

1. For an overview of the project and related bibliography, see the site [www.aralweb.unimore.it](http://www.aralweb.unimore.it).
2. Glossary terms are more than 100 and belong to several categories: theoretical constructs, which are both original or coming from previous studies of mathematics education, terms relating to linguistic or

psychological aspects. These are interconnected in a network of references, which allows the teacher to build a reticulum of knowledge that led him/her gradually to a new vision of the arithmetical-algebraic area and its teaching.

3. Brioshi is a metaphor from the ArAl Project. He is a virtual Japanese student exchanging messages in mathematical language with pupils. His acknowledged skill in this area, encourages pupils to check the correctness of the mathematical expressions to be sent out to him.

### REFERENCES

- Cusi, A., Malara, N.A., Navarra G. (in press). Early Algebra: Theoretical Issues and Educational Strategies for Bringing the Teachers to Promote a Linguistic and Metacognitive approach to it. In Cai. J. (Ed.), *The Development of Algebraic Thinking: Cognitive, Curricular, and Instructional Perspectives*, Springer.
- Kieran, C. (1996). The changing face of school algebra. In Alsina C. et al. (Eds.), *8<sup>th</sup> International Congress on Mathematics Education: selected lectures*. Siviglia, S.A.E.M. Thales, 271-290.
- Jaworski, B. (2003). Research practice into/influencing mathematics teaching and learning development: towards a theoretical framework based on co-learning partnerships. *Educational Studies in Mathematics*, 54, 249-282.
- Malara, N.A. (2008). Methods and tools to promote a socio-constructive approach to mathematics teaching in teachers. In Czarnocha, B. (Ed.), *Handbook of Teaching Research*. University of Rzeszów press, Rzeszów, 89-102.
- Malara, N.A., Navarra, G. (2009). The analysis of classroom-based processes as a key task in teacher training for the approach to early algebra. In Clarke, B. Grevholm, B., Millman, R. (Eds.) *Tasks in Primary Mathematics Teacher Education*, 235-262. Berlin: Springer.
- Malara, N.A., Zan, R. (2008). The Complex Interplay between Theory and Practice: Reflections and Examples. In L. English (Ed.), *Handbook of International Research in Mathematics Education- II Edition*, 539-564. New York: Routledge.
- Mason, J. (2002). *Researching Your Own Practice: the Discipline of Noticing*. London: The Falmer Press.

# LABORATORY ACTIVITIES IN TEACHER TRAINING

Francesca Martignone

Math Department of the University of Modena and Reggio Emilia, Italy

*The Laboratory of Mathematical Machines of Modena, supported by the “Regione Emilia Romagna” (Italy), invested the most recent mathematics education researches in the teacher training project MMLab-ER. The training course was unique both in methodology and focus: teachers joined laboratory activities with mathematical machines analysing the interactions (between peer, experts and also with tools) and the cognitive processes involved. The paper presents examples of these activities where teachers construct, and then analyse, different resolution strategies carried out during ruler and compass constructions.*

**Keywords:** *Mathematical laboratory, teacher training, resolution processes.*

## INTRODUCTION

The Project "Laboratory of Mathematical Machines for Emilia-Romagna" (MMLab-ER) [1] aims at facilitating the implementation of a laboratory approach in the teaching and learning of mathematics. The first step of the Project was the set up of a network of math laboratories distributed among five cities in Emilia Romagna region (Italy), followed by the training of in-service teachers (primary, secondary and high school) on laboratory activities with special tools, such as the mathematical machines: reconstructions of tools belonging to the historical phenomenology of mathematics from ancient Greece to 20<sup>th</sup> century (i.e. curve drawers, pantographs and mechanical calculators) [2]. The training course started with laboratory activities on ruler and compass constructions (the compass is one of the oldest and well known mathematical machines) and continued by introducing other curve drawers and pantographs for geometrical transformations used in history both for mathematical purposes and also for practical purposes. In all these activities it is highlighted how, through appropriate tasks, the mathematical machines laboratory activity can be a suitable environment to develop crucial aspects in the teaching and learning of mathematics: for example the exploration processes, the production and comparison of conjectures and argumentations.

This paper presents some examples of teacher training activities in which teacher educators focus the attention on important learning goals, such as the development of *adaptive reasoning*: “the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments” (Kilpatrick, 2001, p. 107).

## THEORETICAL FRAMEWORK

MMLab-ER Project is grounded on a laboratory idea that is well expressed by this metaphor: “We can imagine the laboratory environment as a Renaissance workshop,



in which the apprentices learned by doing, seeing, imitating, communicating with each other, in a word: practicing” (Bartolini Bussi et al., 2004, p. 2) [3]. For this reason, the math laboratory should not be conceived only as a physical space in which teaching practices based on the use of specific technologies are developed, but rather as a teaching methodology. This laboratory idea, linked to the tradition of the European cultural history and highlighted by ICMI (International Commission on Mathematical Instruction) since the last century, is suggested by the Italian Commission for the Teaching of Mathematics in the “Mathematics curriculum for the citizen” [4]. During the training, teachers were involved in this type of mathematics laboratory. The hands-on tools used and analysed in these activities are the mathematical machines.

The MMLab-ER Project is based on the experience gained from MMLab in the laboratory activities carried out in school: in particular researches on epistemological and educational aspects involved in activities with mathematical machines (Bartolini Bussi, 2000; Bartolini Bussi & Maschietto, 2006; Maschietto & Martignone, 2008). In recent years the MMLab research analyzed also the cognitive aspects involved during mathematical machines activities. In particular Martignone & Antonini (2009) studied the interaction between a subject and a mathematical machine using Rabardel theory (Rabardel, 1995). According to Rabardel, an instrument is defined as a hybrid entity made up of both artefact-type components and schematic components that are called utilization schemes. Following these ideas, during the interaction with a machine, we have identified artefact exploration processes and different (utilization) schemes carried out to solve a specific task. This study focused our attention on the importance of analysing these aspects and it was also useful to project tasks on the machines explorations dealing with the analysis of artefact components (how is done) and the genesis and development of utilization schemes (how do you use it? What is done?).

The theoretical framework, used to describe and interpret the different phases of laboratory activities with artefact and the role of the teacher, is the construct of semiotic mediation introduced by Mariotti & Bartolini (2008). In this framework

the teacher's main roles are the following: to construct suitable tasks; to create the condition for polyphony, eliciting the polysemic feature of the artefact; to guide the transformation of situated “texts” (signs) into mathematical “texts”. In this way the teacher mediates mathematical meanings, using the artefact as a tool of semiotic mediation. (Bartolini Bussi, 2009, p.125)

The Project has provided the opportunity to rethink on these existing researches bringing some innovations. For example, the study of the potential that the laboratory activities with mathematical machines can offer in the genesis and development of students’ exploratory and argumentative processes. Our attention to the study of these processes is grounded into different studies: e.g. the researches collected in *Theorems in school: From History, Epistemology and Cognition to*

*Classroom Practice* (Boero, 2007) and the study of exploration processes during problem solving activities (Martignone 2007).

The training project has also taken into account international researches in the field of teacher education, many of which were presented in *The International Handbook of Mathematics Teacher Education* [5]. In particular, we are in agreement with Watson & Sullivan (2008) research because we designed *tasks for teachers* that focused on important aspects of mathematical activity “to provide insight in to the nature of mathematical activity” (Watson & Sullivan, 2008, p. 110) and we worked with teachers who faced these tasks reflecting on what they are doing and on what way they could make something similar for their students (*classroom tasks*).

“We use classroom tasks to refer to questions, situations and instructions that teachers might use when teaching students and task for teachers to include the mathematical prompts many of which may be classroom tasks, that are used as part of teacher learning” (Watson & Sullivan, 2008, p. 109)

## **LABORATORY ACTIVITIES WITH TEACHERS**

### **Teacher training features**

The MMLab-ER involved primary, secondary and high school math teachers [6]. For this reason, it was an important opportunity to foster a dialogue and a discussion between teachers from different school levels. Teachers could share ideas and thoughts about the role of the teacher and the different cultural aspects and contents that emerged from the laboratory experiences with mathematical machines. The training course, designed and managed by the author, exploited this opportunity by offering activities that could be an inspiration to teachers from different types of schools, with the common goal of the acquisition of laboratory methodology, the development of attention on exploration and argumentation processes and on relative verbalization.

One of the purposes of MMLab-ER training was to give room for the dynamic discussion and comparison of solution strategies among peers and experts. In particular we asked to explain the procedures of geometrical constructions in order to understand their roots, motivations and development not only related to the mathematical contents involved, but also to the use of tools. Therefore, the focus is on the analysis of own and others' problem solving processes.

It is important to stress that MMLab-ER training did not wish to give pre-packed worksheets for classroom activities. The working session aimed at providing ideas and guidelines for possible teaching experiments, which were designed during the course according to the needs and the goals of teachers.

Summarizing, the peculiarities of MMLab-ER teacher training are related to:

- The course methodology: during training sessions the teachers are placed, with the obvious differences, in learning situations “acting as students”; they are divided in working groups and joined the discussions orchestrated by a teacher educator. In these activities the tasks are open and the different possible solution strategies are described and discussed with all other colleagues who belong to different schools and grade levels.
- The exploration and use of special hands-on tools coming from the history of mathematics and from everyday life.
- The choice to focus on the verbalization and comparison of problem solving strategies (analysing roots, choices, procedures and arguments of the resolutions).

This article presents an experience carried out during the MMLab teacher training, from individual tasks to collective discussions in which teachers described and analyzed different ruler and compass constructions. It will also shed light on the role of the teacher educator who manages the collective discussion with teachers through specific techniques, such as asking to explain, summarizing and highlighting the implicits of resolutions strategies.

### **An example: ruler and compass constructions**

The first mathematical machines used in the MMLab-ER training is the most known: the compass. We wanted to reassess the importance of ruler and compass constructions in mathematics teaching-learning: the compass, in fact, although widely used for practical purposes (e.g. in technical education), is not often analyzed in its foundational role in the mathematics culture (we just think to Euclid's Elements).

Following Rabardel theory, teachers analysed the compass. At first, the physical object with its specific characteristics, and then the utilization schemes that develop under specific tasks, for example to draw circles and the measurement transfer. This first activity set the protocol of exploration that will be the basis of each machines exploration. After analyzing the artefact components and its utilization schemes (answering to the questions: *how is it done? What it does?*), teachers studied the role of structure and movements in order to justify the machine functioning (*why it does that?*). Obviously, in this particular case, the exploration was carried out very rapidly because the compass is already well known by the teachers. The analysis of instrument finished with a problem solving activity guided by the open question: “*What if it does change...?*”. Teachers explored the possible changes of the compass (e.g. with equal or different rots and with extensions) and the existing different types of compasses (e.g. plane compass and the blackboard compass).

After this instrument analysis the teachers faced this crucial task: *To construct an isosceles triangle using ruler and compass.*

From the point of view of contents, the choice of the isosceles triangle was made because it is a figure whose definition and properties are known since primary school and therefore suitable to all course participants and reproducible in different classes (with obvious adjustment). Moreover, the request is deliberately left open (we do not ask: to construct an isosceles triangle given the sides) in order to encourage the genesis of different construction strategies. This activity, carried out by all teachers involved in the Project (about a hundred divided into five provinces) was proposed to focus teachers attention on the following aspects: how and why the same final product (in this case, the isosceles triangle) can have different constructions; the importance of analyzing the theoretical and practical reasons grounding the different choices; the role of artefact components and utilization schemes analysis in the planning and development of the resolutions.

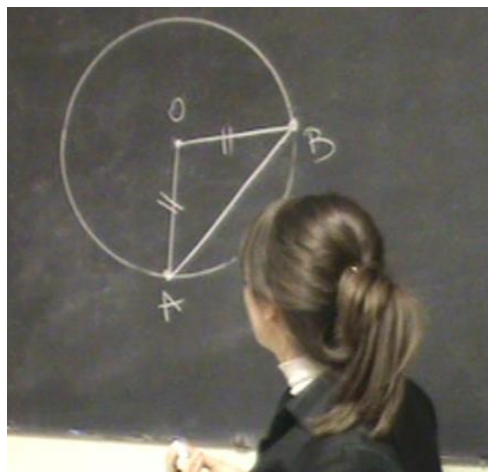
Teachers faced the task individually and discussed the possible solutions in small groups and then collectively. This methodology should foster the verbalization of their processes and argumentations. The teacher educator orchestrated the collective discussion using different techniques: calling teacher to play his/her construction, asking to dictate the procedure highlighting the implicits, comparing different constructions and giving suggestions for other possible constructions.

Below we show and analyse some excerpts selected from a collective discussion in which are presented different constructions of the isosceles triangle. The discussion started after the working group session.

Teacher educator: Who can explain to me his construction? Always step by step so we can reproduce it.

Teacher A: I was lazy and I have only drawn a circle with radius at will, then I connected the circle center to two points belonging to the circumference.

The teacher educator performs the procedure on the blackboard (fig. 1).



**Figure 1: Solution A**

Teacher A: I almost chose the sloped side and then I constructed the triangle.

## Working Group 17

Teacher educator: Okay, first you gave me a procedure and now, do you want to justify it?  
The procedure is correct: I found an isosceles triangle. I know that it is just an isosceles triangle because ...

Teacher A: Because the points I have chosen are on the circumference.

Teacher B: Yes, they are the circumference rays and so they are equal.

Teacher A: But now ... I'm asking myself this question: I said at will, but I had to be careful...

Teacher C: They should not be on opposite sides.

Teacher D: They should not belong to the diameter.

Teacher educator: This is an additional step: asking if...

Teacher A: Then, perhaps in order to be more precise I should have said that I excluded the points diametrically opposite.

Teacher educator: Yes, I should not take B aligned with O and A. The exploration of the limit cases is important. With a dynamic geometry software this exploration could be facilitated [...]

As we can see in this excerpt, the teacher educator orchestrates the discussion asking questions, highlighting the limits and peculiarities of the suggested procedures, raising ideas, and summarizing.

Now we show a second excerpt in which a teacher describes another construction and, as before, the teacher educator reproduce that on the blackboard.

Teacher E: I draw a segment, I open the compass at will and I draw a circumference pointing on the two extremes.

Teacher F: But with the same opening.

Teacher E: Yes, maintaining the same opening.

The teacher educator opens the compass less than half of the segment (selected as triangle base) and she draws the two circumferences while the classroom starts to rumormur.

Teacher E: Not in this way.

Teacher educator: You told me "as I will"...

Teacher E: Larger.

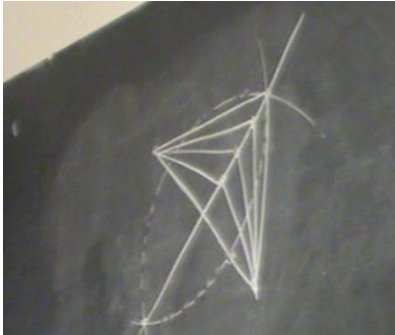
Teacher groups: More than half.

Teacher educator: Ok, why? In the meantime, let us ask our self, what we said before: my construction starts from a definition or a property of the isosceles triangle. What property would you use?

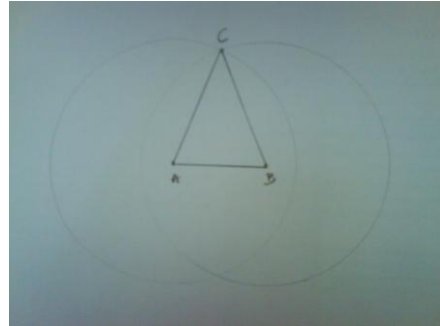
Teacher E: I wanted to use the properties of segment axis.

In this excerpt we notice the attention and the effort involved in highlighting the implicits in teachers' procedures and in the roots of constructions.

The discussion continues analyzing other constructions (see figure 2-3).



**Figure 2: Solution B: a construction that uses the segment axis**



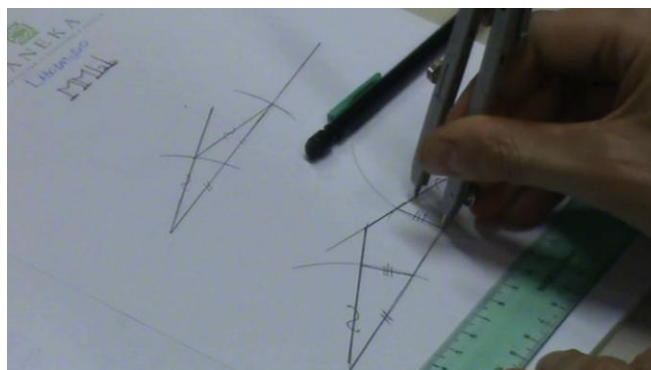
**Figure 3: Solution C: a construction of a triangle given three sides when two sides are equal**

In these collective discussions the teachers see and listen to procedures (made by colleagues or by teacher educator), by asking questions and focusing more on understanding the reasons behind the single steps in relation to mathematical concepts involved and the role of compass. This work leads teachers to reflect on the formulation of procedures, the limit cases, the roots of the constructions and the argumentations that support them.

The teacher educator notices that nobody has used the isosceles triangle propriety of “having two equal angles” and therefore she suggests this task: *To construct an isosceles triangle given one of the equal angles.*

This task is interesting because not all the teachers remember the procedure and, for this reason, the activity is seen as a challenge that they have to face by collaborating and discussing in small groups.

Here is a brief excerpt from a discussion that followed the explanation of the procedures carried out by a teacher (teacher 1) who made the construction shown in fig.4.



**Figure 4: Solution D: a construction of isosceles triangle given one of the equal angles**

The teacher 1 constructs an isosceles triangle cutting the angle with a circle and then re-constructs it as we can see in fig.4.

Teacher 2: But the problem is that...we said it is equal to this (*he indicates the isosceles triangle sides identified by the circumference that cuts the angles sides*). This is equal to this, so these two triangles have equal sides, so they are isosceles.

Teacher educator: The fact is that you have chosen how to cut the angle with the circumference.

Teacher 2: But in this way they are isosceles, did we construct an isosceles triangle using isosceles triangles?

Teacher educator: Yes, they are isosceles triangles, but it could be done also by using the construction of a scalene triangle. The crucial step is when you construct this arc (*she points out the line segment that connects the points of intersection between the circumference and the angle sides*).

Teacher 1: As a matter of fact, I constructed two congruent triangles, so now I have the correspondent angles congruent; the isosceles triangles are made in few steps.

Teacher 2: It is true, what interested me was to construct an angle congruent to another and in order to do that it is sufficient to construct two congruent triangles. Naturally for the construction of an isosceles triangle I have to put them in this way (*he points that out*), if they do not mirror, it does not work.

In the excerpt we see how the construction steps are analyzed and discussed with a teacher who wants some clarification about the key elements that ensure the construction validity.

The whole activity on the isosceles triangle construction (individual task followed by collective discussion) highlights how, even in this simple task, we can use the laboratory approach in order to develop the analysis and the comparison of different solutions, underlining the importance of verbalization and explanation of choices (theoretical knowledge, practices, etc.) and procedures. From the point of view of mathematical contents, teachers realize that even starting from the same definition of isosceles triangle (a triangle that has two equal sides) there are different constructions and cognitive processes involved: i.e. the solutions A and C.

The final discussion dealt with the constructions frequency because more than half of the teachers made constructions like in solutions B and C, few like the solution A, and almost none like the solution D. Teachers thought about these choices and concluded that the first two strategies (B and C) are more frequent because they are used both in technique drawing (*“when you learn the construction of triangles, you do so”*) and when you draw on squared paper (*“these are constructions that we, and our students, usually made using the squared paper: the segment axis are simple and*

*fast to do*"); while construction D has not occurred because is not used frequently (*"it has not come to my mind because I never do it"*). It should be noted that all these reflections were been useful to teachers in the subsequent phase of teaching experiments design, in particular for the a priori analysis, for the tasks choice and for the management of group discussion.

## CONCLUDING REMARKS

The methodology and tasks carried out during the MMLab training course aimed at developing the teacher's attention to other ways of thinking and awareness of their own resolution processes. The development of this type of competence is not only useful for teachers to analyze the possible solutions of their students (the peculiarities in their thoughts and the possible mistakes or misconceptions), but it opens up the horizons on important mathematics features which are not reduced to symbols manipulation or reproduction of proofs studied on books. The reflection on these processes and on the role of theoretical and practical knowledge was one of the most important guide lines in the laboratory activities designed by trained teachers and developed in teaching experiments during the two years of the Project. Even if the analysis of project results is only at the beginning, because the project ends this year, we already have found a correspondence between the guide lines of the training and the teaching experiments carried out by trained teachers. In these teaching experiments, teachers, fostering individual production and critical observations, have always asked their students to verbalize their solutions, by discussing and sharing their knowledge.

## NOTES

1. The MMLab-ER is a two-year project (2008-2010) founded by "Regione Emilia Romagna" and coordinated by M. G. Bartolini Bussi and M. Maschietto. The responsible of the teacher training design and development are F.Martignone and R. Garuti.
2. For information about the MMLab mathematical machines: [www.mmlab.unimore.it](http://www.mmlab.unimore.it).
3. <http://www.icme-organisers.dk/dg20/italians.pdf>
4. "Matematica per il cittadino": <http://umi.dm.unibo.it/scuola--99.html>
5. *The International Handbook of Mathematics Teacher Education*, Terry Wood (Ed.) Sense Publishers, Rotterdam/Taipei, 2008.
6. The training lasted 28 hours (distributed in seven meeting in each of the five provinces) and was managed by expert teachers or researchers who joined the MMLab group.

## REFERENCES

Bartolini Bussi, M. G. (2000). Ancient Instruments in the Mathematics Classroom. In Fauvel J., & van Maanen J. (Eds), *History in Mathematics Education: The ICMI Study* (pp. 343-351). Dordrecht: Kluwer Ac. Publishers.



- Bartolini Bussi, M.G. (2009) In search for theories: polyphony, polysemy and semiotic mediation in the mathematics classroom. In Tzekaki, M., Kaldrimidou, M., & Sakonidis, H. (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*: Vol.2. (pp. 121-128). Thessaloniki, Greece: PME.
- Bartolini Bussi, M. G., Chiappini, G, Paola, D., Reggiani, M & Robutti, O. Learning Mathematics with tools, *Proceedings of ICME10, DG20 Current Problems and Challenges in Upper Secondary*, Copenhagen, <http://www.icme-organisers.dk/dg20/italians.pdf>
- Bartolini Bussi M.G., & Mariotti M.A. (2008). Semiotic Mediation in the Mathematics Classroom: Artefacts and Signs after a Vygotskian Perspective. In: L. English, M. Bartolini, G. Jones, R. Lesh, B. Sriraman, D. Tirosh (Eds). *Handbook of International research in Mathematics education* (2nd edition) (pp. 746-783). New York: Routledge Taylor & Francis Group.
- Bartolini Bussi M. G. & Maschietto M. (2006). *Macchine Matematiche: dalla storia alla scuola*, Milano: Springer.
- Boero, P. (Eds.). (2007). *Theorems in school: From History, Epistemology and Cognition to Classroom Practice*. Rotterdam: Sense Publisher.
- Kilpatrick, J. (2001). Understanding mathematical literacy: the contribution of research. *Educational Studies in Mathematics*, 47, 101-116.
- Martignone, F. (2007). Mathematical background and problem solving: how does knowledge influence mental dynamics in Game Theory Problems. *Proceedings of the Fifth Conference of European Research in Mathematics Education* (pp. 2340-2348). Larnaca.
- Martignone, F., Antonini, S. (2009), Exploring the mathematical machines for geometrical transformations: a cognitive analysis. In Tzekaki, M., Kaldrimidou, M., & Sakonidis, H. (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*: Vol. 4 (pp. 105-112). Thessaloniki, Greece: PME.
- Maschietto, M., & Martignone, F. (2008). Activities with the Mathematical Machines: Pantographs and curve drawers. *Proceedings of the 5th European Summer University On The History And Epistemology In Mathematics Education*. (pp. 285-296). Prague: Univerzita Karlova, Vydavatelsky Press.
- Rabardel, P. (1995). *Les hommes et les technologies - Approche cognitive des instruments contemporains*. Paris: A. Colin.
- Watson, A. & Sullivan, P. (2008). Teachers learning about tasks and lessons. In D. Tirosh & T. Wood (Eds.), *Tools and resources in mathematics teacher education* (pp. 109-135). Rotterdam, Netherlands: Sense Publishers.

# PLANNING TEACHING ACTIVITY WITHIN A CONTINUOUS TRAINING PROGRAM

Cristina Martins<sup>1</sup> and Leonor Santos<sup>2</sup>

<sup>1</sup>School of Education, Polytechnic Institute of Bragança (Portugal) and <sup>2</sup>Institute of Education, University of Lisbon (Portugal)

*The present text is part of a more comprehensive work purporting to study the professional development of primary school teachers within the program for continuous training in mathematics (PCTM). We will present a cross analysis of the case studies of teachers Dora and Aida, focusing on the meaning they confer upon the process of planning a teaching activity, tasks planned and planning development within PCTM. Aida bestows great importance on this process. Dora's participation in PCTM widened her horizon to the necessity of undergoing it.*

Key-words: teacher planning, practice, teacher training

## FOREWORD

This text is based on a work whose main purpose is to study the professional development of in-service primary school teachers within the program for continuous training in mathematics (PCTM). We specifically wish to ascertain how didactic knowledge relating to teachers planning develops through participating in PCTM. I.e.: what meaning do they confer upon planning? How do they plan? What sorts of tasks do they select?

We consider that professional development is conceived as a permanent, continuous and intentional process, aiming at improving professional knowledge, teaching practices and reflection thereupon, thus contributing towards better students' learning in Mathematics (Guskey, 2002; Sowder, 2007). Participation in training programs is taken to be a tool for professional development (Guskey, 2002; Wu, 1999). One of the goals of PCTM is "to foster the undertaking of curriculum development experiences in mathematics which contemplate class planning, class direction and reflection by the teachers involved, supported by peers and coaches" (Serrazina *et al.* 2005, p. 3), thus foreseeing that in PCTM there will be an intentional investment in the aforementioned components. Thus, this program aims at providing, throughout the academic year, a site for experimentation and joint reflection, between teacher-trainer and trainees, so that one can reflect upon practices and use them to develop a sustained knowledge, which takes into account the characteristics of the students it addresses (students aged 6 to 10).

The activities to be developed within this program, take the form of:

— Group training sessions (GTS), biweekly joint sessions for planning and reflection upon activities associated with the teaching practice, involving the teacher trainer and a group of teachers voluntarily enrolled in PCTM.

— Classroom supervision sessions (CSS), sessions for the development of classroom curriculum activities corresponding to conducting practices materializing the planning detailed in the joint sessions and respective discussion, involving the teacher trainee and the trainer in her role as supervisor;

— A joint work session for the development of other enlivening actions with the teachers.

As far as evaluation is concerned, the elaboration of a portfolio reflecting the professional development resulting from the training is proposed.

In this text we wish to present Aida and Dora's vision of the planning process, trying to perform a cross analysis of these two cases on the basis of the following categories: (i) meaning/importance of planning; (ii) tasks planned under PCTM; (iii) weight of planning undertaken under PCTM; and (IV) collaborative work in performing planning.

## **THEORETICAL FRAMEWORK**

Teaching practice is a key component of a teacher's professional life. Teaching Mathematics, regardless of level, involves students, teachers, administrators and schools in contexts which change on a daily basis making the creation of a formula, "a kind of guide", or even of a set of practices teachers can adopt, difficult (Franke, Kazemi, & Battey, 2007). Within teacher's action three basic stages are usually considered, concerning teaching practice: pre-active, interactive and post-active phase. (Canavarro, 2003; Clark and Peterson, 1986, Santos, 2001; Vale, 2000). Classroom practice begins with planning, this being the phase where the teacher identifies content, materials and teaching methods necessary for the practice. For Yinger and Hendricks-Lee (1995) teachers have, on one hand, daily and yearly, the responsibility of selecting and conceiving learning experiences based on course content and, on the other, must be prepared to take the utmost profit from non-planned teaching opportunities which might arise in the course of educational interactions, and being able to achieve these purposes thus demanding preparation.

Pacheco (2001) sees planning "as a practical activity allowing the organization and contextualization of didactic action taking place at classroom level" (p. 104), presenting two main functions, one being to clarify what one wants to perform in the classroom and the other to predict and modify forecasts, throughout the process in agreement with the didactic situation (Pacheco, 2001). Thus, "the act of planning presents itself as a specific and essential teacher's competence which allows him/her to configure, by means of a mental or written plan, the several didactic elements used as a basis to structure the teaching learning process, providing a reduction of incertitude or insecurity" (Pacheco, 2001, p. 105). Even when written plans are produced, they represent only a small part of the true planning that has been taking place in the teacher's mind (Arends, 2007, p. 100).

According to Yinger and Hendricks-Lee (1995), to be prepared for interactions as dynamic as the ones that take place during class seems to be less a question of prediction and control and more a question of preparation and response ability. Specifically, the teacher must anticipate difficulties and student's resolution procedures, foresee how to monitor them and sequence the possible interventions and connections that might be established (Stein, Engle, Smith, & Hughes, 2008). Within this framework, planning producing more generic charts, more flexible and activity-based, will be more useful than the one producing strictly specified and goal-oriented plans. Likewise, plans will become more useful if not conceived to be implemented as means for interaction control, but rather as framing tools purveying a starting point for educational interactions (Yinger & Hendricks-Lee, 1995).

The selection of tasks to submit to students is envisioned as the main point for planning the teaching/learning process (Fernandes, 2006; Ponte, 2005). It is up to the teacher to be responsible for their preparation and direction, taking three concerns under account: mathematical content, students and their learning paths. It is also important to understand how students and teachers *deal* with the diversity of existing tasks, namely concerning evaluation of work undertaken, progress achieved and difficulties to be faced, and cognitive and metacognitive processes and strategies associated with each range of tasks submitted to them (Fernandes, 2006).

The possibility of performing teaching practice collaborative planning work allows capitalizing energies, to provide extra support, to multiply perspectives, to enrich reflection (Serrazina *et al.*, 2006). Collaborative learning as a professional development strategy (Marcelo, 2002) involves group-oriented formative processes, causing not only that learning activities be performed with others in interactional context but also that goals and results of such learning also present a collaborative aspect. PCTM is presented as a privileged means to perform this task (Serrazina *et al.*, 2006). In the context of teacher's professional development Joubert, Back, De Geest, Hirst and Sutherland (2010) state that there are different models but most of them aim at providing opportunities for teachers to become involved in learning and change processes. They suggest, however, that different teachers, influenced by their work contexts and personal motivation, beliefs, theories and experiences, will perceive different opportunities, and such perceptions may change in the course of time.

## INVESTIGATION METHODOLOGY

The undergoing investigation is qualitative/interpretative in nature (Teddlie & Tashakkori, 2003), using case study techniques (Stake, 2007).

The wider study has considered three primary school teachers, Aida, Dora and Sara, belonging to the same training group, who enrolled voluntarily in PCTM. Selection criteria were number of teaching years and academic training. In this text we will only focus on Aida and Dora.

Aida is about 45 years old, and has 25 years experience as a teacher. She underwent one year in PCTM. As academic training she has a primary school teacher's degree (three years course), a specialized course of superior studies in the French teaching area (license degree) and a master's degree in History of Education. Mathematics has always been one of her favorite subjects, although her post-college training has been unconnected with this area. From PCTM she hopes to become up-to-date and try that "children might learn to see mathematics in a different way, as a subject that can be interesting" [Aida, portfolio].

Dora has less than 10 years teaching experience and is about 40 years old. Dora underwent two years of PCTM. She has a license degree (four year course) in Basic Teaching, specialized in visual and technological education, enabling her to teach in primary school. Her life as a student is determined by mathematical failure and a conflictive relationship with it. Dora expects from PCTM "to be able to overcome myths" [Dora, first interview] she has felt since childhood and to learn.

Data gathering started in the academic year 2006/2007 and took place for two consecutive years, through semi-structured interviews (namely one initial, one half-way, one final, one two years after completion of the program and four post-observation interviews conducted with Aida who only took one year of training, the same number with Dora plus 5 post-observation interviews in the second year of training) participant observation of the Group training sessions and of the Classroom supervision sessions, and documental analysis of materials produced by the teachers and field notes and accounts of the teacher-trainer.

Following an interpretative paradigm an analysis of information started at the end of the training year, consisting in organization and interpretation of data, according to *à posteriori* defined categories, taking into account the problem under consideration, theoretical presuppositions and empirical work undertaken. The interpretative paradigm subscribes to a relativistic perspective of reality "envisaging the real lived world as a construction of social actors who, at each moment and place, construct the social meaning of events and phenomena and reinterpret the past" (Santos, 2001, p. 186).

In this text, according to the categories developed, different topics were found characterizing these teachers' vision about the process of planning teaching activity.

## **AIDA AND DORA: PLANNING TEACHING PRACTICE**

### **Meaning and importance of planning**

To Aida, planning teaching activities is a key aspect of the teaching/learning process, assertively stating that she has always planned, specifically noting the prediction relevance of the aspects planning includes:

I really don't know how people manage to teach a course they have not planned beforehand. Although there might be one day when one does not have the time, but

everything is lined-up in one's mind. As rule I have everything actually written down. [Aida, initial interview].

In her final interview she mentions class preparation, seeing it as a privileged moment to consider the several aspects present in a class, an essential condition for its success:

It is essential because when we are preparing the course we reflect about class content, about the material we are going to bring, about what students know and about how to do it. So, I think one has to go this way. We must prepare every class for it to succeed, although improvisation is important, but only up to a point. I think preparation is essential for every class to take place as we want it. [Aida, final interview]

Dora clearly associates planning task to writing, which she claims not to like: "It is true, I don't like it, I never did" [Dora, initial interview]. She claims she usually prefers to mentally plan tasks, and only organize some ideas on paper, which are often changed when she enters the classroom:

I always think about what to do, but I don't write every step down, every step I must take, that is, I put down some topics for my own guidance, it's my way of organizing myself. Of course I prepare materials, the files I am going to use, and I bring it with me like that. I just get there and I don't follow a script, never, never. [Dora, initial interview]

In her final interview, she talks about planning teaching activities during PCTM clearly reinforcing the importance of the selection of tasks to put forward: "After having had the idea I prepared the material, oh yes... . That part is what's more important. For instances I made up the problem I was going to bring with me. And some of the material I used. If its correctly made up, why not using it, right?" [Dora, final interview].

At the end of the first year of PCTM she assumed the importance of planning, namely referring to planning investigation activities (speech 1) as well as undertaking planning in general (speech 2):

1. I prepared classes, that's obvious, it's just that I did it lightly, and not anymore, I go down to details more. Even because regularities taught me, I must pay more attention to that part, I must be rigorous when I prepare classes, to know whether or nor I am prepared for the answers and questions of students. [Dora, final interview]

2. Useful to know what we are going to do, to have a sequence, to know the steps we have to take, to consider what we are going to do, what must be done, the objectives, to know all those steps. [Dora, 2<sup>nd</sup> post-observation interview]

Although Aida has always acknowledged planning as an essential part of the teacher's teaching practice, associating it to a written record, Dora became, through participation in PCTM, more sensitive to its necessity, distancing herself from the idea of something exclusively mental and mainly associated with selecting tasks to face the students with.

## Tasks planned in PCTM

Ever since the beginning of their training Aida and Dora have decided to experiment in tasks that were not usual in their teaching practice which caused them, for instances, to gain a new vision on the meaning and importance given to problem solving. Although Aida considered that problem solving was part of her classroom strategies, she had seldom presented problems that could not be solved directly through the use of an algorithm and whose resolution was possible using different strategies. This aspect is remarkable in her choice to include this task in the portfolio:

The choice of this task to become a part of the portfolio is due to the fact that it was precisely the first and, as such, was surrounded by larger expectations. Another reason which has defined my option is also related to its content. Introducing a “different” kind of problem from the ones you usually face the students with, which constitutes a challenge regarding my professional practice. [Aida’s portfolio, justification for inclusion of the first task]

Dora also mentions that before the training she already engaged in problem solving, but that she considered them to be different from the problems she currently uses, upon which the student can use different solving strategies:

But they were not problems just like these, sort of games that cause mental reasoning to develop, that make them think, that can be practical, useful for their daily life, less routine. And they do them very gladly and with much more willingness than the others. The others are much more a mathematical task. That is an operation! Is it a sum, a subtraction, or a division? It’s always the same thing. And in these problems they don’t see it, they see them as game, a challenge. [Dora, 1<sup>st</sup> post-observation interview]

Thus, for both of them, the tasks planned within PCTM have constituted a challenge in setting up a new meaning.

## Elaboration of planning within PCTM

**Resorting to planning developed within PCTM.** Aida followed planning worked out within the training group, but always adding a personal touch to it. For instances, in the first task undertaken, Aida thought about presenting the students with a problem similar to the one worked out in the group, but adapted to a context fitting the Christmas season, stating regarding this in the 4<sup>th</sup> GTS: “I am going to make the experience” and she went on “I’m thinking about introducing precisely this one for a 4<sup>th</sup> grade, not the spider, but with Santa Claus, giving it another development”. [Aida, transcript of 4<sup>th</sup> GTS]

While experimenting with classroom tasks, Dora based herself upon the planning “discussed in the group” but “afterwards did not follow them to the letter” [Dora, final interview]. According to herself, this option is connected to her own way of being; “Besides, if I get stuck to a piece of paper I am less spontaneous, I don’t feel

like myself. I like to improvise, ...” [Dora, final interview]. However, when she talks about not fulfilling the first planning undertaken within PCTM she also grounds her justification in the students:

The fact that I did not lecture everything I had planned to, was my own option, and because, as I already mentioned, the students had perfectly understood the mechanisms for solution. Thus, to continue would only be useful for the students to apply the already mechanized procedure. Besides, I did what I ought to have done, we must not stick to what is written down, in this case in the planning, but, instead, in under what conditions is the activity carried out, what the students needs are, and, unquestionably, their mental availability. This is a factor that has a great weight in my way of living the profession. [Dora’s portfolio, 1<sup>st</sup> written reflect]

Thus, while Aida when elaborating her planning adapts it to her students, Dora adapts it to herself, although having the students under consideration.

**The role of collaborative work.** Aida has always been careful to prepare her planning in writing and in detail, and to discuss it within the work group. “We went on to Aida’s planning, who presented in detail the task she intends to explore, an activity of mathematical investigation based upon a multiplication double entry table” [report of the 7<sup>th</sup> GTS]. In planning classroom tasks she not only profited from and requested collaboration from colleagues, as she gave her own opinion about other people’s work. In the 4<sup>th</sup> Group Training Session she questioned a colleague about how on long it had taken to perform a problem solving task: “Is is a 3<sup>rd</sup> or a 4<sup>th</sup> grade?, How long, more or less, has the task taken?” [transcript from the 4<sup>th</sup> GTS]. In the same session, faced with the suggestion of a problem put forward by a member of the group for experimentation in a classroom, she mentioned: “It seems to me even more complicated [than the one with the spider]” [report of the 4<sup>th</sup> GTS].

She also never refrained herself from asking the supervisor for clarification which could help her to sort her ideas out and, simultaneously, in planning tasks:

I was thinking about, this time, a mathematical investigation activity starting from a double entry multiplying table, correct me if I am not using the appropriate terminology, (...) And since we are in a clarifying mood, regarding the 2 multiplying table, is it more correct to say that  $1 \times 2$ ,  $2 \times 2$ ,  $3 \times 2$ ,  $4 \times 2$ , or, instead,  $2 \times 1$ ,  $2 \times 2$ ,  $3 \times 2$ ,  $4 \times 2$ , as we were taught. [transcript of the 7<sup>th</sup> GTS]

In her portfolio, she alludes to and includes the material given by the supervisor in the group training sessions “since these constitute an essential support for the choice and programming of the task”, including “[Internet] research carried out during preparation phase, because I consider that the whole process of teaching and learning presupposes some kind of investigation”, indication of bibliographic research carried out, as “it also turns out to be a legitimation of some of the options taken” [Aida’s portfolio, justification for the inclusion of the material used in the preparation of the first task].



Dora also considers the participation of the other elements of the group and the supervisor in carrying out planning. Thus in the 7<sup>th</sup> Group Training Session Dora shares some of her concerns regarding time management:

We proceed to Dora's planning, who had previously exchanged some ideas with me through e-mail. Her task consist in finding regularities in the 2, 5 and 9 multiplying tables, with the students grouped in pairs.

Dora: Do you think I can use up the whole time just with this?

Researcher: I think so, if you let them air their discoveries.

Dora: Aida has got so much here! (...) [Report of the 7th GTS]

Dora mentions in particular the importance of the supervisor in preparing the tasks to experiment. Specifically about the second task undertaken, she makes it clear that she needed "a lot of help from you, a lot of guidance", which leads her to conclude: "but I also learned from this, it's not just that you helped me, you helped me out with doing the planning and you taught me how to do it" [Dora, 2<sup>nd</sup> post-observation interview] As a matter of fact, Dora recognized that planning a mathematical investigation activity for the first time constituted a controlled risk, as she was participating in the program and had the support of the supervisor, and also recognized that student's activities and their learning from undertaking this task, overcame all other factors. "I risked it because I had the support of the supervisor, in case it would be needed, and as I liked the theme, I thought it was interesting for students, as they could perform several explorations and, lead them to be interested, in a more attentive way, in the numbers and in the investigation" [Dora's portfolio, justification of the choice of the 2<sup>nd</sup> task].

To Aida, collaborative work, either with the whole group either with the supervisor, was used in a double perspective, for her own support and for that of the others. It was seem as a multidirectional collaboration. For Dora instead, this working context, relies more upon the supervisor, and in a one way direction. It is a one directional relationship, to learn.

## FINAL CONSIDERATIONS

The starting point of these two teachers is clearly different, so it would be expectable that their professional development during PCTM would also be different. Aida starts out from two kinds of expectations for the training, one connected to teaching, one connected to learning, helping out her students, Dora, on the other hand, takes teaching as her strong bet, to learn and to gain confidence in an area with which she has always entertained a conflictive relationship.

Aida identifies the different aspect present in planning, contents, tasks and materials, methodologies and what students already know (Canavarro, 2003; Clark and Peterson, 1986, Santos, 2001), while at first Dora concentrates herself on the materials, seeming that the associated objectives are not determinant for classroom

work orientation. For Aida, one needs to clarify in order to be able to modify (Pacheco, 2001), Dora instead, who stresses the importance of change within the class, does not associate it with a thorough preparation, but rather to her whim of the moment, resorting to the unforeseen. With the course of the training, Dora begins to mention the importance of planning to ascertain the sequence of the task in accordance with the objectives defined and to be prepared to answer students (Stein, Engle, Smith, & Hughes, 2008) and address unforeseen situations of the teaching practice to use them for the better advantage (Yinger & Hendricks-Lee, 1995), recognizing the advantage of resorting to a written record.

The planning undertaken in group during training constitutes the starting point for both teachers. However, once again, one can see that while Aida values from the very beginning the learning component, adapting it to her students, Dora centers herself upon teaching, adjusting it in the classroom to herself, also taking the students under account. The different levels of professional development that both teachers present are also shown in the way they profit from collaborative work undertaken during training, as an asset to all elements involved (Hargreaves, 1994), or as a context for personal learning. However, the tasks carried out during training have constituted a context for professional development for both teachers.

With the analysis of a training program with specific and innovating characteristics we aim to provide answers which contribute to improve initial and continuous teacher training. However, from the evidence gathered from these two cases, the question remains to determine up to what point the same training format can apply to such different teachers. In which way can one foster professional development of teachers starting up from different levels? How to ensure the sustainability of this program?

## REFERENCES

- Arends, R. I. (1995). *Aprender a ensinar*. Lisboa: Mcgraw-Hill.
- Canavarro, A. P. (2003). *Práticas de ensino da Matemática: duas professoras, dois currículos* (Phd thesis). Lisboa: DEFCUL.
- Clark, C., & Peterson, P. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 255-296). New York: Macmillan.
- Fernandes, D. (2006). Para uma teoria da avaliação formativa. *Revista Portuguesa de Educação*, 19(2), 21-50.
- Franke, M., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 225-256). Charlotte: Information Age Publishing Inc.
- Guskey, T. R. (2002). Professional development and teacher change. *Teachers and teaching: Theory and practice*, 8(3/4), 381-391.
- Hargreaves, A. (1994). *Changing teachers changing times*. London: Cassel PLI.

- Joubert, M., Back, J., De Geest, E., Hirst, C., & Sutherland, R. (2010). Professional development for teachers of mathematics: opportunities and change. *Proceedings of the CERME 6* (pp. 1761-1770). Lyon: Institute National de Recherche Pédagogique.
- Marcelo, C. (2002). La formación inicial y permanente de los educadores. In Consejo Escolar del Estado, Los educadores en la sociedad del siglo XXI, (pp. 161-194). Madrid: Ministerio de Educación, Cultura y Deporte. Retrieved from <http://prometeo.us.es/idea/mie/pub/marcelo/COnsejo%20escolar.pdf> on the 04/03/07
- Pacheco, J. A. (2001). *Currículo: Teoria e prática* (Coleção Ciências da Educação). Porto: Porto Editora.
- Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Ed.), *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: Associação de Professores de Matemática.
- Santos, L. (2001). *A prática lectiva como actividade de resolução de problemas: Um estudo com três professoras do ensino secundário* (Phd thesis). Lisboa: Associação de Professores de Matemática, Coleção Teses.
- Serrazina, L., Canavarro, A., Guerreiro, A., Rocha, I., Portela, J., & Gouveia, M. J. (2006). *Programa de Formação Contínua em Matemática para Professores do 1.º Ciclo*. (unpublished paper)
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (1.<sup>a</sup> ed., Vol. I, pp. 157-223). Charlotte: Information Age Publishing.
- Stake, R. E. (2007). *A arte da investigação com estudos de caso*. Lisboa: Fundação Calouste Gulbenkian.
- Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, 10, 313–340.
- Teddlie, C., & Tashakkori, A. (2003). Major issues and controversies in the use of mixed methods in the social and behavioral sciences. In A. Tashakkori & C. Teddlie (Eds.), *Handbook of mixed methods in social and behavioral research* (pp. 3-50). Thousand Oaks: Sage Publication.
- Vale, I. (2000). *Didáctica da Matemática e formação inicial de professores num contexto de resolução de problemas e de materiais manipuláveis* (Phd thesis). Lisboa: Associação de Professores de Matemática, Coleção Teses.
- Wu, H. (1999). Professional development of mathematics teachers. *American Mathematical Society*, 46(5), 535-542.
- Yinger, R. J., & Hendricks-Lee, M. S. (1995): Teacher Planning. In L. W. Anderson (Ed.), *International encyclopedia of teaching and teacher education* (2nd ed., pp. 188-192). Kidlington, Oxford, UK: Elsevier Science Ltd.

# MATHEMATICS PROBLEM SOLVING PROFESSIONAL LEARNING THROUGH COLLABORATIVE ACTION RESEARCH

Joyce Mgombelo & Kamini Jaipal-Jamani

Faculty of Education, Brock University, Canada

*This paper draws from a study that aimed at finding out how teachers' collaborative action research in the Teachers Learning Together (TLT) project promotes professional development in the area of mathematics amongst elementary school teachers. The paper focuses on an analysis of teachers' experiences of teaching and learning of problem solving in a collaborative action research. Qualitative research approach was used to conduct case studies of three teacher-researcher teams. In the two cases analysed in this paper, collaborative action research provided opportunities for teachers to learn about problem solving process, creating an environment that fosters students' learning through problem solving and mathematics knowledge for teaching*

**Keywords:** professional development; problem solving; collaborative action research; mathematics knowledge for teaching

## INTRODUCTION

Mathematics education reform movements reflected in documents such as the Ontario Mathematics Curriculum (2005), NCTM Principles and Standards, (2000) call for mathematics teachers and educators to look at different ways of teaching and learning mathematics. One of the elements of these reforms is a focus on problem solving as a key component of effective mathematics teaching and learning. For example, according to the National Council of Teachers of Mathematics (NCTM, 2000), "solving problems is not only a goal of learning mathematics but also a major means of doing so" (p. 52). Recently mathematics education researchers have recognized the importance of mathematics teachers' understandings of the underlying assumptions and theories of teaching and learning of problem solving as reflected in these documents and their practical applications. As such, researchers have argued for the importance of professional development for teachers to enable them to enact, in their classrooms, the teaching and learning of problem solving.

Traditionally teachers' professional development models comprised of one-day training workshops provided by experts who come and go. Researchers have argued that this type of professional development does not result in a change of teachers' practices (Joyce & Showers, 1995). Researchers also contend that professional development that is close to the classroom, collaborative, content focused, and relies on expertise and lived experiences of participating teachers have the potential to affect teaching practices (Kratzer & Teplin, 2007).

This paper draws from a study that aimed at finding out how teachers' collaborative action research in the Teachers Learning Together (TLT) project promotes professional development in the area of mathematics amongst elementary school teachers (Mgombelo & Jaipal, 2010). The study was guided by the following questions:

- In what ways does collaborative action research, when used as professional development, influence teaching practice?
- In what ways does participation in teacher-designed action research inform teachers' understanding of elementary school mathematics and elementary mathematics teaching and learning?

The paper focuses on an analysis of teaching and learning of problem solving in a collaborative action research. TLT is a professional development initiative of The Elementary Teachers Federation of Ontario (ETFO). ETFO invited and provided support for teacher teams from the same school or in similar roles at different schools to come together to conduct action research projects relevant to their specific professional needs, circumstances and interests. For the 2008 school year, the focus of professional development was on mathematics. Over the course of the year, teams of teacher-researchers were supported by university facilitators as they conducted their own research projects at their schools. As part of this project, ETFO contracted university facilitators to facilitate the professional development and conduct three case studies of some of the teacher teams with whom they were working. In this paper we describe some of the findings from two case studies that we conducted -- these teacher teams investigated problem solving in their action research projects.

## **THEORETICAL FRAMEWORK**

The research was framed by the following ideas: teachers develop knowledge of teaching and learning mathematics in practice by engaging in activities that promote collaboration, reflection and experimentation such as collaborative action research; effective mathematics teaching requires a sound knowledge of mathematics for teaching; and teachers need to learn how to teach through problem solving.

### **Collaborative Action Research**

Action research derives its roots from the work of the German social psychologist Kurt Lewin (Carson, 1992). Lewin was concerned about the gap that existed between theories about society and the dynamics of social practice. In the past two decades action research has emerged as a significant form of research into practice. In education, in-service and pre-service teachers are being engaged in action research as part of professional development and as part of educational reform efforts (Feldman, 1996). Following Lewin, various manifestations of action research have developed. Action research can be individual or collaborative. Collaborative action research can be defined as collaborations between teachers and outsiders, such as university

researchers (Savoie-Zajc & Descamps-Bednarz, 2007) or collaborations among teachers (Feldman, 1996). In the TLT project, teachers were involved in a collaborative action research. Feldman (1996) defines collaborative action research as research that involves practitioners working together to take actions within their situations in order to improve their practice and to come to a better understanding of that practice. The words “collaborative”, “research” and “action” in collaborative action research might be conceived as:

- Collaborative: group of teachers working together
- Research: systematic, critical inquiry made public
- Action: Understanding the teaching and learning system requires taking action within the system and paying close attention to the results of taking those actions

In their paper “Complexity Science and Educational Action research: toward a pragmatics of transformation” Davis and Sumara (2005) notice a similarity between sensibilities promoted by collaborative action research and those promoted by complexity science. Both collaborative action research and complexity science are concerned with “what one might do to bring together the self interests of autonomous agents into grander collective possibilities” (p. 454). One way that complexity science addresses this question is by elaborating on conditions that are necessary for bringing together a collective learning system. These conditions include making sure that there is redundancy among the participants in a team. Redundancy, understood as commonalities among participants in a team, is necessary to ensure a transition from a collection of me’s to a collective of us (Davis & Simmt, 2003). Another condition that is necessary for the emergence of a collective learning system involves the presence of diversity among the members of the team. Diversity allows novelty; it is the source of a collective’s flexible response—its intelligence.

### **Mathematics Knowledge for Teaching**

The question of the relationship between knowledge of subject matter and knowledge of teaching has been a central concern in teacher education. In practice any mathematics professional development program has to contend with the question of how to integrate knowledge of subject matter (mathematics) and knowledge of teaching. An initial characterization of this integration comes from Shulman’s (1987) work on pedagogical content knowledge. Shulman defined pedagogical content knowledge as a particular form of knowledge that embodies the aspects of content most germane to its teachability. Recently, Ball and Bass (2002) elaborated on pedagogical content knowledge and used the term “mathematics knowledge for teaching” to capture the complex relationship between mathematics content knowledge and teaching. Ball and Bass suggest that any inquiry into teachers’ knowledge of mathematics should begin by analyzing the work of teaching and questioning what mathematics matters for this kind of work; they maintain a

delineation between knowing mathematics and knowing mathematics that is useful for teaching. In addition to knowing mathematics content, knowing mathematics that is useful for teaching involves specialized content knowledge that is not pedagogy but includes for example knowing how to represent fractions and decimals with diagrams.

### **Teaching through Problem Solving**

There is consensus in mathematics education about the importance of problem solving and its potential for increasing students' engagement with mathematics. Consequently, teaching through problem solving has gained prominence in mathematics education over the last two decades. From the student perspective, the experience is one of "learning through problem solving," and NCTM (2000) states that:

Instructional programs should enable all students to build new mathematical knowledge through problem solving; solve problems that arise in mathematics and in other contexts; apply and adapt a variety of appropriate strategies to solve problems; and monitor and reflect on the process of mathematical problem solving. [p. 52]

Although problem solving is regarded as a beneficial approach to teaching mathematics, teachers however, have often found effective teaching of mathematics problem solving to be a major challenge. For example in Finland, Pehkonen (2007) found that teachers who believe problem solving is beneficial still fail to implement it in within their classrooms.

### **METHODOLOGY**

We used qualitative research methodology to conduct case studies of three teacher-researcher teams, focusing on the teachers' learning as they engaged in their own action research activities. Overall, the two Mathematics educators acted as facilitators for nine teacher-researcher teams as part of the Teachers Learning Together project. ETFO held a symposium over two days in August 2008 where participants were introduced to the Teachers Learning Together project and action research methodology. From the nine teacher teams that were assigned to us by ETFO, we selected three groups to be the focus of our case studies. Selection of teams was based on the following aspects:

- The diversity provided by the three teacher-team projects selected according to their home school districts, the grade levels addressed, mathematic topics addressed, the teachers' levels of teaching experience, and their experience with research
- The potential of the each team's plan to address our research questions
- The alignment of the each teacher team's action research project with the Ontario Elementary Mathematics Curriculum (2005).

## **Data Collection methods**

Data sources were:

- 1) Copies of teacher-created artefacts relating to the focus of the teachers' research such as lesson plans, teaching materials, and teachers' post-lesson reflections in journals: These were collected by the teachers on our behalf throughout the school year
- 2) Teacher-created action research project reports: The teachers' final action research reports described and presented findings of how their action research had impacted their professional practice.
- 3) Transcriptions and field notes of teacher-researcher team meetings (selectively audio-recorded).
- 4) Our own researcher journals: Each researcher kept a written record of her own reflections relating to the research project over its duration.
- 5) Transcriptions of one focus group interview per team at the end of the project, audio recorded

## **DESCRIPTION OF TWO CASES**

### **Case 1: Student Questioning in a Problem Solving Context**

This teacher team of four junior division (grade 4-6) teachers were mathematics facilitators from four different schools with a history of engaging in collaborative professional development projects initiated by the Ministry of Education and the District school boards. At the time of the TLT project, the four teachers taught one grade 4/5 class, one grade 5 class and two grade 6 classes respectively, located in two rural and two urban schools. Three schools were K-8 schools and one was a grade 4-8 school. Their research question emerged from observations of students learning during their two-year professional learning in the District School board project that had focussed on teaching through problem solving. Teachers observed that students were not asking mathematically relevant questions and engaging in productive dialogue that furthered their mathematical understandings. They therefore wanted to make math more "visible" to students. For this TLT project, the teachers used instructional strategies that included math congress and gallery walk (Fosnot, 2007; Fosnot & Dolk, 2001; 2002). These instructional strategies involve students in dialogue, conversation, discussion, and questioning of mathematical solutions presented by their peers. After discussions with each other and the researchers, the group narrowed their focus to the role of students' questioning and dialogue during problem solving contexts. Their research question was: How can we enhance productive questioning and dialogue through the implementation of targeted instruction so that classroom conversations make the mathematics more accessible to all students? The goal of their study was to identify strategies that would promote



effective student questioning and dialogue during problem solving to foster mathematical reasoning for all students.

After conducting an extensive literature review, the team observed that most studies on mathematical questioning focused primarily on teacher questioning. The limited literature in the area of student questioning in mathematics meant that the team would have to explore a number of targeted strategies designed to engage students in rich dialogue and effective questioning (by teachers and students), to find out if such strategies made the mathematics accessible to all students. Since they were already familiar with a number of strategies learned in the district school board project, they decided to try out the following strategies: Math Congress, (Fosnot, 2007), Gallery Walk / Post-it-notes, Think/Pair/Share, Modeling, Anchor Charts, Annotation of Key Mathematical Ideas / Student Comments / Questions, Summarizing / Generalizing, Wait Time / Pacing. Their intent was to try out these strategies, observe students, reflect as a group, and adjust the strategy.

## **Case 2: Students Communicating their Problem Solving Strategies through Bansho**

Participants in this case consisted of four Grade 3 teachers from four different schools and one special assignment teacher in mathematics. Similar to the first team, all five teachers had participated as a group in their school boards' professional development initiative, prior to the TLT project. The team had been working on problem solving as the focus on the project. They observed that their students had problems communicating their computational strategies during problem solving. Upon further discussion and dialogue with each other and the researchers, they therefore decided to focus on students' communication of problem solving strategies. Their research question was: "How can we increase our students' ability to communicate their problem solving strategies more effectively by using the strategy of *bansho* in the classroom?" They decided to implement *bansho*, a Japanese strategy where students' problem solving strategies, serving as a public record, is displayed on a board. The teacher organizes the board writing so that the progression of ideas in the entire lesson builds logically and is captured for the duration of the lesson so that students may use the record for their note taking. The public record, or the *bansho*, is the class's collective thinking. The mathematical strand they chose was number sense and operations.

The team initiated the implementation by meeting to familiarize themselves with the mathematical landscape of learning for multiplication (Fosnot, 2007). This was crucial to understanding students' thinking strategies in problem solving. The teachers selected four different problems that involved multiplication concepts. The implementation was organized in a cycle of co-planning and co-teaching and was repeated 4 times. In each cycle, a *bansho* lesson was co-planned for half a day, one classroom was selected for two teachers to co-teach the lesson, three other classes were taught by the respective class teachers independently, and all teachers collected

student work in the form of the *banshos* for a post –lesson debriefing on a second half day. During the post lesson debriefing, the team spent time collaboratively dialoguing about their implementation and students’ learning – specifically discussing the *bansho* strategy and how students’ communicated their learning through the strategy. Based on these post lesson debriefings, they refined how to implement the *bansho* strategy.

## FINDINGS

Through a qualitative analysis of data and informed by theoretical framework we identified a number of themes that characterise the experiences of teachers in collaborative action research. In what follows we present some of the themes which resulted from the analysis of two cases of teachers’ experiences about teaching and learning of problem solving in a collaborative action research.

### **Relation between collaborative action research and problem solving processes**

For the teachers the collaborative nature of the action research process enabled them to experience learning environment similar to an environment that promotes learning through problem solving. In other words as teachers engaged in their own learning, they were empowered to foster the same learning environment for their students. For example one teacher felt that her experience in the project empowered her to be able to create a similar collaborative learning environment in her own classrooms to support students to take ownership and risks in learning.

It really is almost a mirror of exactly what is going on here [action research group]. When you get support, and when you get respect, and when you allow people to sort of figure out and talk to each other and find their way, then you get happier learners. You get more committed learners, you get more enthusiastic learners. You get learners who are willing to take risks, which we all are now. It’s almost like this is the mini version and then we take it back to our classroom and we create it.

Another teacher felt that her own learning process that she was engaged in the TLT action research project reflected the learning process that *bansho* strategy for teaching problem solving was promoting for her students. Her awareness of her own learning enabled her to support her students’ learning through problem-solving.

I might say the *bansho* does for the students what the math TLT does for me this year. It validates my opinions, it listens to me, it allows me to show it. The other kind of P.D. we get, is more like the old textbook: “okay, today we’re going to look at pages 1 to 10. Here’s the lesson. I want you to do questions 3 and 4. Take 5 and 6 home for homework. Tomorrow we’ll take it up.” ...whereas, the math TLT we’ve been doing this year is a *bansho*. My opinions are validated. I’m looking at other people’s strategies.

## **Impact on Students' Learning through Problem Solving**

In a couple of instances, teachers noted that their professional learning in the project had positive effects on their students' learning. For instance, one teacher expressed how the targeted instruction on developing questioning skills during collaborative problem solving gave students an awareness of the value of learning by listening to different students' ideas, and engendered respect in students for other students' ideas.

I think one of the biggest things for me is having the kid that is traditionally smart in math, see a child that he or she would assume is in the lower end of the class, giving a strategy, presenting a strategy, explain their strategy...and then that really sharp kid goes "Ah! I never really thought of it like that!" That opens their eyes to the potential of the other kids in the class, which sometimes they just...ignore...

Teachers in the first case reported progress in students' mathematical questioning and dialoguing during problem solving. For instance, teachers described students' growth in mathematical thinking in terms of students' use of mathematical language in their questioning of peers' solutions of problems.

As the year progressed we noticed an improvement in the facility with which students used mathematical language. Comments such as: "How did the open array help you?", "What are the dimensions of the long rectangle?", "Are those equations part of the Venn diagrams? And "Can it work so that the area stays the same and the perimeter change?" show this development.

## **Learning Mathematics Knowledge for Teaching**

It seems that collaborative action research project provided opportunities for teachers to learn mathematics knowledge for teaching. One teacher noted how she was better able to use math resources and know what to look for in the resources in ways that supported her growth in mathematics knowledge for teaching.

My math shelf is constantly [used]...I am constantly taking things off, putting things on, looking things up, making notes, looking for references, figuring out the math, looking at my math dictionary, looking at all my professional resources... and to say, "okay, I want to do this lesson, but what are the strategies that might come out of this problem solving? What might I see on my *bansho*?" It [action research] certainly, it has been more useful as a professional learning tool than many others.

Teachers identified specific aspects of their projects that had an impact on growth of their knowledge of mathematics for teaching. For example teachers reported how mathematics they focused on in the *bansho* strategy was also enhanced through reflection on student's work.

## Working Group 17

I think we were all aware of it [*bansho*] before, but having the opportunity to chew it up the way that we did, the depth of our understanding certainly has increased.

By identifying the strategies the students used in our *banshos*, we also looked for the ways that those strategies were connected with respect to the [mathematics] Big Idea. This enriched our pedagogical content knowledge and sometimes even our own understanding of mathematics.

## CONCLUSIONS

In the two cases analysed, collaborative action research engaged teachers in ways that enabled them to learn about problem solving process and how to create an environment that fosters students' learning through problem solving. Teachers' participation in the project led to growth of their mathematics knowledge for teaching. Collaborative action research spurred teachers to do their own research into math topics and use a variety of resources to support their planning and teaching. Collaborative analysis of students' work and teaching strategies promoted reflection on how different problem solving strategies supported growth of student mathematical thinking and increased teacher's understanding of how to teach through problem solving.

In conclusion it appears that collaborative action research can lead to the generation of new mathematics knowledge and understanding of problem solving for teachers. This has significance for mathematics professional developers and educational reformers. Professional development for teachers should foster a collaborative environment in which teacher can experiment with new ideas in theory and practice.

## REFERENCES

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241-247.
- Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.
- Ball, D., & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. Plenary lecture at the 26th Annual Meeting of the Canadian Mathematics Educators Study Group, Kingston, ON, May.
- Ball, D.L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group*, (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

- Carson, T. R. (1992). Understanding curriculum implementation: scenes from a conversation. *JCT: An Interdisciplinary Journal of Curriculum Studies*. 9(3), 71-96.
- Davis, B. & Simmt, E. (2003). Understanding Learning Systems: Mathematics Education and Complexity Science. *Journal for Research in Mathematics Education*, 34, 2, 137-167
- Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of mathematics that teachers (need to) know. *Educational Studies in Mathematics*, 61(3), 293- 319.
- Davis, B. & Sumara, D. (2005). Complexity science and educational action research: Towards a pragmatics of transformation. *Educational Action Research*, 13, 3, 453 – 464
- Feldman, A. (1996). Enhancing the practice of physics teachers: Mechanisms for the generation and sharing of knowledge and understanding in collaborative action research. *Journal of Research in Science Teaching*, 33(5), 513-540.
- Fosnot, C. T. (2007) Investigating Number Sense, Addition, and Subtraction. In the series, *Contexts for Learning Mathematics*. Portsmouth, NH: Heinemann.
- Fosnot, C. T., & Dolk, M. (2001). *Young Mathematicians at Work: Constructing Number Sense, Addition, and Subtraction*. Portsmouth, NH: Heinemann.
- Fosnot, C. T., & Dolk, M. (2002). *Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents*. Portsmouth, NH: Heinemann.
- Gladwell, M. (2008). *Outlier: The story of success*. New York, Little Brown and Company
- Joyce, B., & Showers, B. (1995). *Student achievement through staff development: Fundamentals of school renewal* (Second ed.). White Plains, NY: Longman.
- Kratzer, C. & Teplin, A. (2007). From Lesson Study to Lesson Link®: Classroom-Based Professional Development paper presented at the April 2007 Annual Meeting of the American Educational Research Association, Chicago, IL, April 9-13, 2007.
- Ministry of Education, Ontario. *The Ontario Curriculum Grades 1-8 Mathematics*, (2005). Available at: <http://www.edu.gov.on.ca/eng/curriculum/elementary/>
- Mgombelo, J., & Jaipal, K. (2010). Mathematics professional learning through collaborative action research: Three case studies. 49 pp. Prepared for the Elementary Teachers Federation of Ontario
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston: VA: NCTM

### Working Group 17

- Pehkonen, E. (2007). Problem solving in mathematics education in Finland. Unpublished manuscript, University of Helsinki at Finland. Retrieved on 8 March 2010 from <http://www.unige.ch/math/EnsMath/Rome2008/ALL/Papers/PEHKON.pdf>
- Savoie-Zajc, L. & Descamps-Bednarz, N. (2007). Action research and collaborative research: their specific contributions to professional development. *Educational Action Research* 15, 4, 577–596
- Shulman, D. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22
- Van de Walle, J. A. & Lovin, L. H. (2006). *Teaching Student-Centered Mathematics: Grade 5-8*. Boston: Pearson Education, Inc.

# A STUDY OF THE DIFFERENCES BETWEEN THE SURFACE AND THE DEEP STRUCTURES OF MATH LESSONS

Edyta Nowińska

The Adam Mickiewicz University of Poznań

*There is a consensus that designing rich learning opportunities in the mathematics classes and thus the improvement of the students' mathematical knowledge and skills can be supported by the application of findings about instructional practice. This application is a challenging task for both researchers and teachers. In a study conducted in Germany it was possible to investigate the reasons for the lack of positive effects of the application of theoretical findings about instructional practice on students' mathematical behavior. The results of the investigation and their consequences will be presented in the paper.*

**Key-words:** deep structure, surface structure, teacher's activity, students' activity in the classroom

## INTRODUCTION

International studies like TIMSS and PISA show that students face considerable difficulties in reaching well-founded mathematical knowledge and in using it autonomously. Such difficulties point to the need of developing effective instructional methods that have a potential to modify the teaching and learning processes in the classroom and to improve the students' mathematical achievement. Numerous initiatives have been taken in many countries to innovate mathematical classroom practice with new instructional methods (Kramarki & Mevarech, 2003; Depaepe et al., 2010; Gärtner, 2008). In the discussion about the quality of mathematical instruction the role of metacognition (e.g. NCTM 2000) and cognitive activation (e.g. Lipowsky, 2009) are stressed. From the researchers' point of view, it is obvious that those new instructional methods should encourage students to engage with the learning content and thus to develop elaborated knowledge basis. For an effective implementation of any instructional methods, it is important to distinguish between visible and invisible students' activities. For example, if an instructional method involving a cooperative learning form is performed, it can be expected that the student-student interaction around matters of teaching and learning also increases and that the students articulate their thinking, explain their mathematical reasoning and learn critical thinking. In doing that, the diversity in the students' prior knowledge will be used for an internal cognitive process of knowledge construction, carried out by an individual learner. These are some of the factors that make learning in cooperative groups promising (Slavin, 1980). The fact that the students are seated in cooperative working groups does not mean that they are actually working cooperatively and that the cooperative group has an impact on their cognitive activity and on their learning outcomes.

However, the literature on mathematical education change shows that it is difficult to transfer theoretical findings in the instructional practice and thus to reach the aimed positive effects in students' mathematical knowledge and skills. Depaepe et al. (2010) provide one example which highlights this difficulty: A group of researchers conducted a 7-month video-based study in two sixth-grade classes focusing on the teachers' metacognitive and heuristic approaches to problem solving. The researchers investigated the extent to which two teachers focused on metacognitive and heuristics skills in their teaching of mathematical problem solving. They found a positive relationship between the students' spontaneous application of heuristics to solve non-routine word problems and the teachers' references to these skills in their problem-solving lessons. However, increased application of heuristics did not result in students' better performance on those non-routine word problems. It can be supposed that the students didn't use the heuristics for a better understanding of the given problems and their solution ways were incorrect. Depaepe et al. (2010) try to explain the findings, but the explanation is difficult. It needs a more critical and qualitative perspective to appropriately interpret the relationship between the teacher's instructional methods and the students' learning processes and learning outcomes. The results of the study emphasized that it must be questioned and investigated to what extent metacognitive and heuristic skills were mainly done exclusively by the teachers and to what extent their students also internalized these new activities and skills.

The study of Depaepe et al. points to the need for the explanation of the lack of positive effects of the application of some new instructional methods designed to improve the students' metacognitive and heuristic skills and thus their problem-solving skills. Another promising study of a new instructional method (i.e. increasing the use of challenging and cognitively activating mathematical tasks) but without the expected positive effect on students' competencies (i.e. reflection, critical thinking, cooperation, and communication) was conducted by Gärtner (2008). Gärtner supposes that the expected positive effects regarding the students' competencies cannot be achieved within a year. He supposes that more time is needed to gain these positive effects, but there is no precise explanation how they can be achieved later. Moreover, Gärtner investigated classroom practice using teacher and student questionnaires. And he did not verify the results through direct observation.

In my PHD research (Nowińska, 2010) conducted in Germany it was possible to investigate the lack of the positive effects of a project to improve the students' mathematical knowledge. To explain the lack of the effectiveness of the project, I investigated the instructional practice, particularly the teacher and students' activities and the relationship between them, when new instructional methods were being practiced. An in-deep transcript analysis leads to the assumption that mainly the visible structure of the instructional practice itself (visible teachers' and students'



activities) has changed. Consequently, the cognitive potential of the new teaching content, challenging tasks and classroom discourse could not be used effectively by the students. The achieved explanations for the unexpected lack of positive effects of the project will be presented in this article.

In the first section to follow, I give an overview of the background of my study, presenting both the idea of the project to improve the students' mathematical knowledge and skills and the dataset and research question. The second section (Theoretical background) leads to key theoretical ideas used in my study. Information about the result of the investigation will be given in the third section. The fourth section concludes with a short discussion of the results.

## **A PROJECT TO IMPROVE MATHEMATICAL INSTRUCTION**

### **Project idea**

The aim of the project that I analyzed in my PHD research was the application of a learning environment that makes it possible (for students) to reach a deeper understanding of mathematical concepts and a well-founded mathematical knowledge. The project was conducted in Germany, in two six-grade classes (13 years old students, Realschüler). The project was based on some key ideas of *Cognitive Mathematics Education (CME)* (Cohors-Fresenborg & Kaune, 2005). *CME* uses the description of the mathematical knowledge of pupils via the concepts 'frame' and 'procedure', which were introduced to cognitively-oriented mathematics education by Davis and McKnight (1979). *CME* puts the construction of a cognitive mathematical operating system as a priori goal of mathematics lessons. Its most important elements are the 'function frame' and the 'contract frame' with suitable, attached procedures. Both frames use the frame 'formal representation of intuitive knowledge'. The development of the function frame, in students' cognitive mathematical operating systems was the main aim of the learning environment designed for the two six-grade classes in the project that I analyzed. The students involved in the project should develop a certain concept of function with several variables. The function concept makes the organization of the mathematical knowledge in students' minds and the connection between its parts possible. This hinders fragmentation of the knowledge. Conceptual understanding of the mathematical tool 'function' forms the basis for formalizing the intuitively existing knowledge (i.e. when dealing with complex mathematical tasks), for working out the formal aspects of this knowledge and thus for creating more transparency and understanding functional dependence. It makes it possible to work on complex mathematical problems. The conceptual understanding of 'function' constitutes the basis to improve the learners' skills to expose, explain and present functional dependence. For those skills the expressions *thinking in functions* or *functional thinking* will be used. They include an excellent handling of mathematical formalism of functions of several variables as well as the concept 'variable'. In Kaune (1995),

the role of thinking in functions and their formal representations is explained in a more detailed way. The improvement of those students' skills should be supported by the new learning environment with different external representations of the mathematical tool 'function' and by new formats of tasks, exercises and mathematical problems which should be solved by using functions.

What is more, the change in mathematical instruction should be supported by a change of classroom culture. The classroom culture should encourage students to cognitive, metacognitive and discursive activities (Cohors-Fresenborg & Kaune, 2007a, b). The key features of the classroom instruction complement one another to the concept of 'cognitive activation', as explained in Lipowsky (2009, p. 529): "In cognitively activating instruction, the teacher stimulates the students to disclose, explain, share, and compare their thoughts, concepts, and solution methods by presenting them with challenging tasks, cognitive conflicts, and differing ideas, positions, interpretations, and solutions. The likelihood of cognitive activation increases when the teacher calls students' attention to connections between different concepts and ideas, when students reflect on their learning and the underlying ideas, and when the teacher links new content with prior knowledge. Conversely, the likelihood of cognitive activation decreases when (...) the teacher merely expects students to apply known procedures".

There are numerous visible or easy to implement "elements" of the new instructional concept (mathematical tasks, content, some features in teacher behavior). Anyway, the main focus when using or practicing these elements should be on the students' cognitive, metacognitive and discourse activities that are main factors of knowledge acquisition in constructivist view of learning.

The intended changes in the instructional practice were challenging for the teacher involved in the project. Two teachers from one school decided to take part in the project. The teachers had diverse possibilities to improve their content and pedagogical-content knowledge in order to be able to realize the project. Supportive coaching and meetings with the researchers leading the project were realized before and during the project.

### **Dataset and research question**

The dataset includes a video recording of 46 lessons, students' worksheets and written answers to control tasks. The way in which teachers actually put the project idea into practice was investigated through an in-deep analysis of the transcriptions of the lessons. The main focus of the investigation was on the way how the teachers involved students in using the function concept and formal representation and to solve the assigned tasks.

Students' written answers to the tasks designed for the lessons and for control tests during the school year were carefully scrutinised by looking for accuracy of the

application of the function concept and for their argumentations and explanations given to the designed tasks and questions.

The main research problem was to work out the explanations for the unexpected lack of positive effects of the change of the instructional method that had been achieved. A quantitative analysis with a pretest-posttest design at the end of the school year showed no project effect: The students of the project group ( $N = 29$ ) and a control group ( $N = 111$ ) have developed – concerning their mathematical achievements – almost identically over one school year. Because of this result the videotaped instructional practice had to be investigated intensively to work out some explanation for that unexpected project result.

## THEORETICAL BACKGROUND

In order to work out an explanation for the unexpected project result, a qualitative in-deep analysis of transcription of the lessons was conducted. Thereby, the attention was directed to the students' use of learning opportunities and of the new instructional features. For that purpose, the surface and the deep structures of teaching and learning practice were investigated. The distinction between the surface and the deep structures of teaching and learning practice was introduced by Oser and Baeriswyl (2001). The surface structure means the situation the teacher sets for students (learning topics, methods, social forms, teaching styles), routines and regulation in the teacher's and students' activities observed in the situation and all teacher's activities planned to initiate learning processes and to provide learning opportunities. Observing the surface structure helps to assess the classroom instruction regarding its cognitive potential for meaningful learning processes by students. The deep structure means the quality of the initiated learning processes and, in particular, the internal learning sequences, or operations that students follow to appropriate knowledge, develop socially, solve problems and acquire skills (Oser & Baeriswyl, 2001, p.1041). Any investigation of the deep structure needs to take a closer look at the interplay between the presented mathematical content and its use, at the mediation by the teacher between the mathematical content and at the students' activities and their cognitive, metacognitive and discourse behavior. According to Cobrun (2003, p. 4), the investigation of changes which go beyond the surface structures and procedures of a classroom instruction is important if the success of the implementation of new instructional methods should be evaluated.

The deep structure of teaching and learning practice was investigated by using *the system for categorizing metacognitive and discourse activities during stepwise controlled argumentation in mathematics lessons (CMDA)* (Cohors-Fresenborg & Kaune 2007a,b). When analyzing a transcript with *CMDA*, metacognitive and discourse activities of teachers and students are made transparent in the transcript and thus easy for further analyzing of patterns in practicing these activities in student-student and teacher-student interactions. Using *CMDA* helps to assess if the

instructional features of ‘cognitive activation’ are internalized by a teacher and also by students and to what extent students make use of challenging teachers’ questions, demanding mathematical tasks or classroom discussion for their learning processes.

## RESULTS

The first results of the qualitative analysis of instructional practice were promising: the behavior of the participating teachers gave evidence of the intended change in the instructional method. The teachers asked the students to explain misconceptions and errors in argumentations and in internal representations of mathematical terms and concepts. They leaded their students to check and assess arguments of other students, to explain functional dependencies, to discuss and assess different approaches to mathematical tasks and also to listen carefully in order to have discussions with other students. In so doing, the teachers provide learning opportunities for students. They encouraged students to practice metacognitive activities (like monitoring or reflection) and to engage cognitively in classrooms discussions.

The revealed changes in the teachers’ behavior and in the implemented teaching contents pertain to the survey structure of the teaching and learning practice. They show its cognitive potential to provide meaningful learning opportunities for students. Another positive result of the project was that the learners were able to handle complex modeling tasks and to formalize complex functional dependencies. To investigate how students use the cognitive potential for their own learning activities and how teachers promote these activities, the deep structure of the teaching and learning activities in the classroom was analyzed. The results of the analysis lead to the assumption that exclusively the surface structure of classroom practice has changed as desired. Consequently, the cognitive potential showed to the outside in the visible structure of the teaching and learning practice was only seldom used by the student. The teachers organized what is visible (the surface structure of learning) and neglected to consider if the students actually reflect on what, how and why they are doing when using functions or on benefit they have by using functions. That can be an explanation for the lack of positive learning effects and an explanation for the unexpected results of changes in instructional practice. This phenomenon will be illustrated with the following example.

### Transcript, example

Numerous demanding mathematical tasks were designed to be implemented in the classroom practice. When solving these tasks, students’ skills to think in functions and their cognitive activities by using the function concept when precisely stating and formalizing the tasks should form the center of the teacher’s attention. Those activities play an important role in problem-solving and building a mental representation of a given situation.

In the following part of a complex mathematical task a functional dependency between the total price for the holiday in an apartment on the north beach and two

variables should be formalized. It is the first task in which the notation of functional equation with two variables has to be written, so students' explanations, argumentations and interpretation of the situation described in the task should be intensively supported by a teacher. These activities make the learning situation when discussing this task particularly challenging and they should be actually realized by students. This mathematical task as a part of the surface structure presents a cognitive potential of the instructional situation.

By the investigation of the deep structure, attention should be focused on the mediation by the teacher between the mathematical task and the students' activities to understand and solve the task.

Mr. and Mrs. Schulte, their two adult children and their dog, are planning a 14-days holiday trip to the island Borkum. They have to choose between two offers. The first offer states the following:

"Apartments on the north beach": Price per week 504 €, costs for the final cleaning 100 €, pet one-off 55 €. Write down a functional equation that can be used to calculate the total price for the holiday in an apartment on the north beach:

$x_1$ : number of days

$x_2$ : number of pets taken with you.

An analysis of a classroom discussion when discussing and solving this task revealed that there are no reflection activities on functional dependency described in this task and on the mathematical tool 'function' which has to be used to formalize this dependency. One student suggested the following equation as a solution for this part of this task:  $p(x_1, x_2) = 504 \cdot \frac{x_1}{7} + 55 \cdot x_2 + 100$ . However, the student did not explain how he got his functional equation and why this equation is correct. He gives no interpretation of the situation described in this task. As long as the functional equation is formulated, the cognitive potential of this task is not used effectively in the learning situation. The following transcription shows the further classroom situation.

- |    |          |  |
|----|----------|--|
| 1  | Teacher: | Now, I need somebody who can explain the price function in his own words.                |
| 3  | Student: | The whole function?  |
| 4  | Teacher: | Yes, I'll point at it and you (...) So, the first part stands for?                       |
| 5  |          | [L. pointing at 504 of the functional term.]   |
| 6  | Student: | Well, the first part, so 504 stands for what they have to pay for... er... for one week. |
| 8  |          | [L. pointing at $x_1/7$ of the functional term.]   |
| 9  | Student: | The $x_1$ divided by seven means that they have to pay it for seven days.                |
| 11 | Teacher: | Yes.   |
| 12 |          | [L. pointing at the functional term $55 \cdot x_2$ .]                                    |

## Working Group 17

- 13 Student: And the 55 times  $x_2$  means er, that they have to pay 55 Euros for the pet, for one pet.
- 15 [L. nods and points to the functional term 100.]
- 16 Student: 100 Euros, that is the final cleaning, that occurs only once.
- 18 Teacher: Ok. So you have it, the function.

The teacher asks a student to explain the functional equation (line 1). This demand is a part of the surface structure of the instructional situation and has a great cognitive potential. It suggests that the teacher tries to encourage the students to engage cognitively by working on this task.

Now, the deep structure of the situation is to investigate and thereby the focus is on directing the students' activities. Using *CMDA*, the first teacher's intervention (lines 1-2) is to be understood as a demand for the reflection on the suggested formal representation. Nevertheless, there is no correct answer from the student trying to explain the functional equation. No explanation of this equation can be understood as a reflection on the formal representation and on its semantics or syntax. The student gives only fragmented information about the function term and does not take the function name and the arguments into consideration. The student focuses on the formal representation but when interpreting it, he switches to some information about the Schulte family: "*they* have to pay" (lines 6, 9, 13). His remarks on variables  $x_1$  and  $x_2$  are not clear and not correct. The teacher, however, accepts the student's answers by saying "Ok. So you have it, the function".

The transcript shows that neither the cognitive potential of the task nor the cognitive potential of the teacher's demand led to the expected students' activities. Such situation does not contribute to foster students' skills to *think in function* because there is no possibility to learn how to create more transparency and understanding of functional dependencies and how this dependency can be represented precisely. The teacher leaves no room for students to wonder how to tackle the formal representation for the given task and how to get the solution. Consequently, cognitive activities that meaningfully constitute the use of a function concept cannot be internalized by the students.

There is one more interesting aspect in this classroom situation. The teacher directs students' attention by pointing their finger at selected parts of the functional equation (lines 4, 8, 12, 15). The teacher's intention to react in this way is difficult to interpret. This reaction and the teacher's comment in line 18 lead to the assumption that the teacher has difficulties to use his knowledge about the new instructional methods to support and control students' learning activities. His attention is concentrated more on the surface structure of the teaching and learning situation than on the deep structure. This classroom situation shows that positive changes in the surface instructional structure do not implicate that also the deep structure changes as desired. The presence of specific tasks or the teacher's challenging questions is not a guarantee for the intended change of the ways teachers engage students in using

these tasks and questions and support the internal learning operations that children should follow to gain knowledge or to acquire skills. The students were able to use function to solve the designed mathematical task. Therefore, their knowledge about functional dependency and their skills to explain and formalize functional dependencies were not sustainable. The lack of the intended change in the deep structure can be an explanation for that unexpected result of the project.

## DISCUSSION

Our analysis of the surface and the deep structure of lesson suggests that teachers' attention in class was focused on immediate outcomes, that is, on obtaining solutions to given tasks and using functions as mere procedures for calculations rather than on the long-term goal – on developing students' functional thinking. With the teacher's support, students were able to solve even very complex mathematical problems using functions with several variables, but they did not seem to be given the opportunity to internalize the acquired skills, to control their activities and to (re-)organize their knowledge. Teachers' expectations for their students included application of well-known procedures, but not reflection on them. Our findings stress the importance of the qualitative analysis of the deep structure of the instructional practice when the practice is changing through the application of the theoretical findings about effective teaching and learning approaches. Our results are similar to those stressed by Robert and Rogalski (2005), achieved by using their methodology with a twofold analysis of the classroom session dynamic. The findings stress the need to pay more attention to the ways teachers manage the relationship between students and mathematical content, in a 'normal' lesson and also while changing the teaching and learning practice to improve the students' mathematical knowledge and skills.

## REFERENCES

- Coburn, C . (2003). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher*, 32(6), pp. 3–12.
- Cohors-Fresenborg, E. & Kaune, C. (2005). The Metaphor "Contracts to deal with Numbers" as a Structuring Tool in Algebra. In Bosch, M. (Ed.) *Proceedings of CERME 4*, pp. 300-310.
- Cohors-Fresenborg, E. & Kaune, C. (2007a). Modelling Classroom Discussions and Categorising Discursive and Metacognitive Activities. In *Proceedings of CERME 5*, pp. 1180-1189.
- Cohors-Fresenborg, E. & Kaune, C. (2007b). *Kategoriensystem für metakognitive Aktivitäten beim schrittweise kontrollierten Argumentieren im Mathematikunterricht, 2. überarbeitete Auflage*. Arbeitsbericht Nr. 44. Osnabrück: Forschungsinstitut für Mathematikdidaktik.

- Davis, R.B. and McKnight, C. (1979). 'Modelling the processes of mathematical thinking', *Journal of Children's Mathematical Behaviour* 2, 91-113.
- Depaepe, F., De Corte, E. & Verschaffel, L. (2010). Teachers' metacognitive and heuristic approaches to word problem solving: Analysis and impact on students' beliefs and performance. In *ZDM: The International Journal on Mathematics Education*, 42, pp. 205-218.
- Gärtner, H. (2008). Einführung neuer Lehrpläne mit Hilfe eines videobasierten Qualitätszirkels – Unterrichtsmonitoring. In Berkemeyer, N. et al. (Eds.): *Unterrichtsentwicklung in Netzwerken: Konzeptionen, Befunde, Perspektiven*. Münster: Waxmann, pp. 141–150.
- Kaune, C. (1995): Der Funktionsbegriff als ein Fundament für den gymnasialen athematikunterricht der Sekundarstufe I, S. 66–76. In Steiner, H.-G. & Vollrath, H.-J. (Hg.): *Neue problem- und praxisbezogene Forschungsansätze*. Köln: Aulis.
- Kramarski, B. & Mevarech, Z. R. (2003). Enhancing Mathematical Reasoning in the Classroom: The Effects of Cooperative Learning and Metacognitive Training. *American Educational Research Journal*, 40(1), pp. 281-310.
- Lipowsky et al. (2009). Quality of geometry instruction and its short-term impact on students' understanding of the Pythagorean Theorem. *Learning and Instruction*, 19(6), pp. 527–537.
- National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA.
- Nowińska, E. (2010). *Kognitionsorientiertes Lehren – Analyse eines Implementationsprojektes zur Einführung des Funktionsbegriffs*. Osnabrück: Forschungsinstitut für Mathematikdidaktik.
- Oser, F. & Baeriswyl, F. J. (2001). Choreographies of Teaching: Bridging Instruction to Learning. In: V. Richardson (Ed.): *Handbook of research on teaching*. Washington D.C.: American Educational Research Association, pp. 1031–1065.
- Robert, A. & Rogalski, J. (2005): A Cross-Analysis of the Mathematics Teacher's Activity. An Example in a French 10th-Grade Class. *Educational Studies in Mathematics*, 59, pp. 269-298
- Slavin, R. (1980). Cooperative learning. *Review of Educational Research*, 50, pp. 315-342.
- Sommer, N. (2008). Lernzuwachs und kognitive Aufgabenanforderungen – Untersuchungen am PISA-E-2000 Datensatz. *Journal für Mathematik-Didaktik* 29 (1), pp. 3–19.



# TEACHERS MANAGING THE CURRICULUM IN THE CONTEXT OF THE MATHEMATICS' SUBJECT GROUP

Cláudia Canha Nunes

Fernando Pessoa School, Lisbon

João Pedro da Ponte

Institute of Education, University of Lisbon

*This study addresses the teacher daily practice. The goal is to understand, from the teachers' perspective, the challenges and difficulties that they face when attempting to involve the pupils in mathematics learning. We are particularly interested in issues that arise when teachers assume curriculum management decisions in the context of the school mathematics department. The methodology is qualitative and interpretive, with case studies. The results indicate that curriculum management supported by the collaborative context creates tensions when a teacher makes decisions that diverge from those assumed collectively and also between the collaborative group with an innovative approach to teaching and teacher professional identity.*

Key-words: Curriculum management, mathematics teacher, mathematics subject group.

## INTRODUCTION

A key element of teachers' professional practice is the way he/she interprets and manages the curriculum, taking into account the students' characteristics and the conditions and resources of the school. This study draws on several fields of knowledge: teachers' professional knowledge and identity, curriculum management in mathematics, and collaboration and leadership in school context. It strives to understand the practice of collaborative curriculum management in the context of a school mathematics department. Particularly, we address two questions: (i) How teachers conduct curriculum management, in this context, as they attempt to diversify students' learning experiences? (ii) What is the potential of collaborative work around curriculum management in the development of a professional culture at the school?

## CURRICULUM MANAGEMENT AND TEACHERS IDENTITY

It is usual to distinguish different curriculum levels – e.g., the prescribed (or formal) curriculum of official documents, the available curriculum mediated by school textbooks, the curriculum planned (or shaped) by the teacher, the curriculum in action enacted by the teacher in the classroom, the curriculum learned by the students, and the curriculum evaluated, for example, through national examinations

(Gimeno, 1989; Stein, Remillard & Smith, 2007). Curriculum management refers to the actions of the teacher that contribute to the construction of the curriculum in the classroom (Gimeno, 1989; Ponte, 2005). For the teacher, the focus of the curriculum management process is students' learning, and it is according to this (at least in theory) that he/she takes all the necessary decisions. Therefore, as Ponte (2005) suggests, curriculum management has to do, essentially, with the way the teacher interprets and shapes the curriculum, on two levels: a *macro* level, concerning the overall planning of teaching for an extended period, and a *micro* level, corresponding to the teaching process in the classroom. The teacher makes decisions selecting tasks, strategies, and materials appropriate to the objectives and purposes of mathematics teaching, taking into account his/her students and working conditions. The teacher adjusts the curriculum as he/she evaluates and periodically reflects on his/her professional practices. Nowadays, as a curriculum manager, the teacher faces new challenges: the modern society poses constantly new demands on schools, the student population assumes a cultural diversity never seen before, and curriculum orientations proposes a major change on the role of the teacher, from a "deliverer" of knowledge, to that of a facilitator of learning.

To teach well, the teacher must know teaching techniques, the content of what is taught, the students, and the school context (Shulman, 1986). But, fundamentally, the teacher teaches what he/she is (Elbaz, 1983; Ponte & Chapman, 2008). The teacher's professional identity is an aspect of his/her social identity, which presupposes the existence of a community providing ways to think and act which constitute collective values (Dubar, 2002). Ponte and Chapman (2008) indicate that in the construction of a professional identity, the teacher takes the culture, values and norms of the professional group, but also has the possibility of influencing and thus contributing to the change of the group, mobilizing his/her cultural background and personal experience. Moreover, the socialization process of the teacher at the school and in his/her mathematics department is often complex, sometimes inhibiting the experimentation of new ideas, the involvement in curriculum innovation projects and the establishment of personal relationships and sharing of experiences. This process makes a real the duality that often exists between the model of professional culture devised by the teacher and the reality that the teacher faces in daily practice (Ponte & Oliveira, 2002). In order to schools experience a significant development, the most important element to address is the teachers' professional involvement and collective work (Nunes & Ponte, 2010). These critical dimensions may help understand how teachers develop their work, includes how they manage the mathematics curriculum.

## METHODOLOGY

This study follows a qualitative approach (Erickson, 1986), with a case study design (Stake, 1994; Yin, 1989). The study involves a group of 14 mathematics teachers of a secondary school with 12-18 years old students. The mathematics teachers have an extensive experience of working collaboratively and, in recent years, they developed

various projects at the school. Most of these projects emerged from the need that they felt to improve their practice and to help students to overcome their difficulties. During the school year 2007/08 the teachers of the mathematics department developed the project “Investigations, proof and problem solving tasks in textbooks and in curriculum management”, involving all classes from grades 7 to 12. This project aims to diversify tasks in the mathematics classroom, in order to encourage the students’ in learning mathematics.

This study considers the group of teachers involved in the project and within that group, focuses particularly on three teachers: Ana, the coordinator of the mathematics department, Matilde, a teacher that arrived recently to school and to the department, and Simon a teacher at the school for 28 years. These cases provide several contrasts that enable understanding the relationships between professional knowledge and curriculum management, as well as regarding collaboration and leadership at the school. In this article, we present a small glimpse of the cases of the three teachers.

Collection of data was done during the school year 2007/08 and includes participant observation (Jorgensen, 1989) of the teachers’ working sessions and of two classes, with record of field notes in a research journal, two interviews with each of the three teachers selected for case studies, and collection of documents (Adler & Adler, 1994; Patton, 2002; Yin, 1989). According to the research plan, data analysis began simultaneously with data collection, to identify the need for further collection of data. The second level of data analysis involves the development of categories focused on professional knowledge, curriculum management, collaboration, and leadership that may help in noticing interesting relations. The third level of analysis seeks to explain the meaning of the data, in order to provide contributions to the understanding of the phenomenon under study (Merriam, 1988).

## **RESULTS**

### **The group working sessions**

This group of teachers holds a session every week, for three hours, working collaboratively in curriculum management. Usually, they start altogether collectively but to work in a specific school level they splits in subgroups. Collectively,

The group shares their practices experiences, plan and prepares tasks and assessment instruments, defines classroom strategies and questions to help students in their difficulties, and later the group reflects on students work, in particularly in the tasks related to the group project that are realised in all grades [Research journal on Group Working Sessions-GWS].

In subgroups of two or three teachers,

They plan teaching units, using first the textbook and other curriculum materials, particularly the official curriculum documents; construct assessment

tests; and discuss specific issues related to the school level involved [Research journal on GWS].

Also the purpose of these subgroups is,

To plan lessons and regulate practice, defining common tasks to be developed and classroom strategies with different classes, sharing experiences and difficulties teachers with colleagues [Research journal on 10<sup>th</sup> grade subgroup work].

During the working sessions all teachers are invited to participate and have the opportunity to share and express their point of view. All work is developed in a supportive environment and there are friendly interpersonal relationships among all teachers. This dynamic is focused on a particular concern of this group of teachers – developing and practicing an effective mathematics teaching that may guarantee students' success in their learning: "We want them prepared for the future, especially those who want to access to higher education" [Ana Interview-INT, 25/10/2007].

Some of these teachers, being aware of the educational changes, strive to be always updating their professional knowledge [Researchers' journal, GWS]. This may explain their frequent participation in professional meetings outside the school. Also, this may clarify why these teachers are invited to share what they learn from those meetings by providing of a workshop based on what they learned for the other colleagues of the group. As an example, a teacher has attended an in-service training course in calculators and he organized a workshop for his colleagues on this topic [Research journal, GWS 2].

There are different statuses and roles assigned to each group member. Since the last ten years, one norm of this group is that there is always a teacher responsible to lead the group of teachers that are teaching classes from grades 7 to 9. One of the teachers, Simon, explains:

This leader is chosen from the group of teachers that belong to this school for more than 15 years. He/she has to teach grades 7 to 9 and has to be replaced every three years. When none of us is there [leading] things go wrong! There are problems with parents, some of the curriculum topics are not achieved... [Simon INT, 16/10/2007].

Simon's words suggest that the group seeks to take into account the expectations of students and parents. This may explain why the teachers manage the curriculum and build the assessment tools in group or subgroups. Doing so, they strive to harmonize them with the views of all teachers and to support the decisions of each teacher about their own students' learning and assessment.

Also, this seems to be a way the group found to support the younger teachers, who are usually responsible for teaching middle school grades, as this allows a stronger regulation of the teaching-learning process to assure the quality of students learning

in these grades. In contrast, older teachers are responsible for teaching grades 10 to 12, but every three years they have to teach one or two classes from grades 7 to 9, keeping in close contact with all the issues of all school grade curriculum goals and eventual changes.

### **The professional experience of the three teachers**

Matilde has 11 years of experience teaching mathematics classes from grades 7 to 12. She is in this school since 2006. She has already worked in seven different schools performing roles as a mathematics teacher and as a class director. She has a degree in mathematics teaching. She says that, outside the school, all her time is dedicated to her family.

Ana is 39 years old. She has been a teacher for 12 years and she has a master's degree in mathematics. Her capacities and work are recognised by all mathematics teachers by the school community. The results get by her students in the national exam are well known and the number of her students who go into university contribute to this social recognition. At the beginning of the school year of 2007/08, she was elected, by her colleagues, the head of the mathematics department. Her colleagues mention that "Ana's first reaction was panic because she felt that she was not prepared to face the challenges and she did not know all the tasks involved in this job" [Research journal, 11/09/2007].

Simon is a teacher with 28 years of experience teaching mathematics classes from grades 7 to 12. Throughout his career he played several roles in his school such as deputy head teacher, in-service teacher education coordinator, department coordinator, and project coordinator (of mathematics projects and of other school projects). He is an in-service teacher educator in professional development courses and belongs to several working groups in and outside his school. Because of his professional experience and the initiatives he promotes in the group, Simon is recognized by his colleagues as the unquestionable leader of the group. This academic year he has only grade 12 classes.

Matilde, Ana and Simon are in different stages of their careers and have a relationship with the school and the mathematics group marked by their personal and professional trajectories.

### **Managing the curriculum: Individual dimension**

Concerning the individual work of the teachers in managing the curriculum, Matilde shows little identification with the perspective of the group. The voices some concern in the decisions that she makes in the classroom when using more open tasks; Ana shows confidence in taking decisions in relation to the work to develop with students; and Simon shows how to articulate the work with the textbook with working on open tasks, diversifying the tasks proposed to students.

Matilde teaches grades 7 and 8. The way she works with open tasks in the classroom, contradicts the view of the group in relation to mathematics' teaching and learning:

*Matilde* – I think that I often influence the students' reasoning while they are solving a problem. I say what they should do! (...) I do this because I have to go on!

*Simon* – That exactly what you should never do!

*Sebastian* – We have to control our selves! We have to go step by step, questioning the students but wait for their answer. They have to think by themselves. Today you spend some time more, but you will gain tomorrow because your student have learned and achieved the goal. We can help you! [GWS, 20/11/2007].

In this discussion it is possible to understand that Matildes' decision contradicts the perspectives of the group about the way of manage student work with open tasks. It is also perceptive that Sebastian, the leader of the subgroup grade 7 and 8, tries to give Matilde some clues to help her future working, sustained in his large teaching experience (26 years).

Ana teaches grade 12. She follows the planning done in the subgroup and in the classroom she uses the textbook as a central resource. She offers to the students the tasks that are prepared in the context of the grade 12 Subgroup:

I do something that I already did a few years ago, it is not new. I issue students a challenge and I want them to write anything about these issues. They write funny mathematics' reports. [Ana INT, 26/09/2007]

Simon teaches grade 12. He follows the planning done in the subgroup and, in the classroom he also uses the textbook as a central resource. He also offers his students the same tasks as Ana but he proposes solving them in two phases. He argues that classroom work must be focused on the student. The first approach is always the textbook. [GWS, 11/09/2007].

To learn, students have to like what they are doing, and so what I like most is that they solve their own problems. First, I would like them to be able to read a problem and not turn their arms down, not get discouraged, therefore grasping the problem. (...) Achieving that with my classes is to get weapons to grasp and solve the problems which arise. [Simon INT, 16/10/2007]

In summary, the evidence shows us that Matilde has some difficulties to manage students' difficulties while they solve more open tasks and decides to provide them the answers. Simon tries to help his students to be autonomous assuming that students have to reflect on their own work and mistakes to learn.

### **The three teachers and the mathematics subject group**

In this section we present some of the dimensions of the relationship between the teachers and the group. Matilde does not identify herself with the culture of the group, Ana shows some embarrassment related to her role as head teacher, and Simon appreciates the dynamics and the work that the group develops. As the natural leader of the group, he nurtures his relationship with his colleagues using curriculum management as a focal activity.

Matilde recognizes that she is an outsider regarding the group:

I feel [quite outdated] by people who are here in school for longer than me. (...) I never felt this, but [now] I feel, because I think they [the other teachers] search for professional development training and I do not. [Matilde INT, 15/01/2008]

But at the end of the study, Matilde recognises that she has learned a lot with the teachers of this group:

In this group I felt that we should invest: I saw happy teachers even when difficulties emerge. We felt supported by my colleagues and still have a lot to learn to empower my practice [Matilde Final Reflection - FR, 14/07/2008].

As a mathematics teacher, Ana needs to share her work with her colleagues. As subject leader, she considers this to be a “special” group, where all the work is planned in collaboration and where there is a strong reflexive attitude, all members sharing with each other their own practices and experiences:

This is a special group. We work together for a long time. I hope it will stay like this! This one is the first [school] where the teachers of the mathematics department work in collaboration sharing all the tasks. [Ana INT, 25/10/2007]

I've been a coordinator, in a school much smaller than this one at the beginning of my career. (...) Working with colleagues is sometimes difficult. I have a bit of fear in this task, but I try to be ready. [Ana INT, 08/05/2008]

Ana refers to Simon as the catalyst element of the working processes developed inside the mathematics' subject group, and, in the different projects developed by the group. She refers to him,

He has his own beliefs about how professional and school culture should be – with strong collaborative work and a continuous development and learning attitude – and the entire group follows his vision [Research journal].

Furthermore, and besides the fact of being the subject leader, Ana decided to share the coordination of the school project. She looks at collaboration as a natural working situation of the members of the subject group and as a tool for help, support, and sharing:

I am always telling what happens in my classes and I like to know my colleagues' experiences, so I can have different opinions on my decisions and classroom strategies [Ana INT, 08/05/2008].

Decisions about assessment provide an interesting episode concerning the relationship of Simon and the group. In fact, the other grade 12 teachers felt that the students should do assessment tasks just in one phase. That is what Ana and Diogo indicate:

*Ana* – I think that if the task is to assess the students' learning then it has to be done individually. (...) I do not agree to give a second chance, because there are students with private tutoring and already know the task and many of them can provide ready-made answers.

*Simon* – I think that they perform much better in a second stage. And I do not agree with you [Ana] that the reason is that they have external help and they already know the task.

*Diogo* – I agree with Ana. In addition, if it counts for assessment, we have to do all in the same way, so that some [students] benefit and others do not. [GWS, 20/11/2007]

However, Simon decided to use a different strategy and gave a second chance to his students to improve their first response to the task, once corrected and commented. This decision was discussed in the following working session, as Simon announced his decision and suggested the group to analyse and reflect on the performance of his students in both phases. There were some negative reactions, especially from Ana and Diogo who have disagreed with Simons' decision [GWS, 4/12/2007]. The issue was taken up later at meetings in which the group built tasks and discussed how to implement them in the classroom [GWS, 15/01/2008; 19/02/2008; 8/04/2008; 6/05/2008]. As a result, some other members of the group began to use Simons' strategy. In particular, at the end of the study Diogo admitted that this strategy can help students improve their learning, as he has verified with his own classes [GWS and FR, 14/07/2008].

Simon says that the discussions that the group has done in the project working sessions have been very "interesting" for him. In particular, he stresses the construction of open tasks and the definition of criteria to assess and reflect on the results of students:

The construction of tasks of proofs, problems and explorations and investigations and their implementation in the classroom, the discussions we had in the group working sessions, has always been very enriching, and the exchange of ideas and clarification of points were a highlight of this project. (...) Discussions on the grading of the students' work and their achievements to give them feedback were undoubtedly very important aspects for my learning.



The contributions of all colleagues made me to reflect on my practice in these aspects, questioning what we did and discovering ideas and suggestions perfectly workable in practice in the future. [Simon FR, 14/07/2008]

From Simons' words it is evident the importance of teachers' reflect on their own practice and have opportunities to share experiences and doubts about their practice. These are key elements for teachers' professional development and also for the sustainability of a collaborative work culture in school context.

### CONCLUSION

Ana, Matilde and Simon are teachers in different phases of their careers, from the same mathematics department that work collaboratively. The results of this study show that the curriculum management made in the context of a collaborative group and the various initiatives of the group in developing innovative practices that involve the development of exploratory tasks are significant changes in educational practice and enable the sustainability of a culture of collaboration (Nunes & Ponte, 2010). There are situations that generate conflicts in the group, especially when most participants support some decision and some individual practices diverge from that, as illustrated by the case of Matilde and Simon.

But this dynamic and working context seem to motivate the involvement of the teachers in teaching and learning. In particular, such dynamic appears to support the professional development the teachers and their capacity to accept new challenges. For example, Ana feels that she still has much to learn. But the context, in which she develops her profession, in particular the subject group she leads, represents an advantage for her. As one of the youngest members of the group, when she was elected subject leader, she chose to have a quiet and learning attitude. Every time she needs help she asks her colleagues and she can count on them for collaboration and experience, mostly from Simon.

One important conclusion that we draw from this analysis is that Simon, the natural leader of the group, bases most of his relationship with his colleagues in the activity of curriculum management. The professional practice of these teachers, supported by this working environment, shows that current curriculum orientations may be implemented not just at an individual or small group level, but by a whole school mathematics department.

Finally, the group culture and collaborative work seems to help in the gradual socialization process of new elements such as Matilde, while fostering a climate of confidence conducive to sharing experiences and difficulties (as showed in the discussion involving Matilde, Simon and Sebastian in the group working sessions), essential elements for the construction of teachers' professional identity development (Ponte & Chapman, 2008). Also, these group culture and collaborative work in the context of a project that involve all the teachers of the mathematics department of a

school shows some important elements for an effective way of promoting sustainable mathematics teachers' professional development (Zehetmeier & Krainer, 2011).

From this study new issues emerge for future research, namely: How teacher's practices and curriculum management influence students' learning of mathematics? What conditions are necessary in schools, and more widely in the social context, so that this kind of collective curriculum management takes place, very much in line with current curriculum orientations? How can we create an effective network between professional development programmes and school mathematic departments in able to promote sustainable collaborative culture and professional development?

## REFERENCES

- Adler, P., & Adler, P. (1994). Observational techniques. In N. Denzin & Y. Lincoln (Eds.), *Handbook of qualitative research* (pp. 377-391). Thousand Oaks, CA: Sage.
- Dubar, C. (2002). *La crisis de las identidades: La interpretación de una mutación*. Barcelona: Bellaterra.
- Elbaz, F. (1983). *Teacher thinking: A study of practical knowledge* (1st ed.). London & New York: Croom Helm & Nichols.
- Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York, NY: Macmillan.
- Gimeno, J. (1989). *El curriculum: Una reflexión sobre la práctica*. Madrid: Morata.
- Jorgensen, D. (1989). *Participant observation*. Newbury Park, CA: Sage.
- Merriam, S. B. (1988). *Case study research in education: A qualitative approach*. S. Francisco, CA: Jossey Bass.
- Nunes, C. C., & Ponte, J. P. (2010). A gestão curricular no contexto de um grupo de professores de Matemática. In F. Nogueira, A. Oliveira, A. Baptista e D. Nova, *Desafios teóricos e metodológicos I ENJIE – Encontro nacional de jovens investigadores em educação* (pp. 276-280). Aveiro: CIDTFF – Universidade de Aveiro.
- Patton, M. (2002). *Qualitative research & evaluation methods*. London: Sage.
- Ponte, J. P. (2005). Gestão curricular em Matemática. In GTI (Ed.) *O professor e o desenvolvimento curricular* (pp. 11-34). Lisboa: APM.
- Ponte, J. P., & Chapman, O. (2008). Preservice mathematics teachers' knowledge and development. In L. D. English (Ed.), *Handbook of international research in mathematics education*, (pp. 223-261). New York, NY: Routledge.
- Ponte, J. P., & Oliveira, H. (2002). Remar contra a maré: A construção do conhecimento e da identidade profissional na formação inicial. *Revista da Educação*, 11(2), 145-163.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

## Working Group 17

- Stake, R. E. (1994). Case studies. In N. Denzin & Y. Lincoln (Eds.), *Handbook of qualitative research* (pp. 236-247). Thousand Oaks, CA: Sage.
- Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning* (vol. I, pp. 319-369). Charlotte, NC: Information Age.
- Yin, R. K. (1989). *Case study research: Design and methods*. Newbury Park, CA: Sage.
- Zehetmeier, S., & Krainer, K. (2011). How to promote sustainable professional development? In CERME 7 conference, *WG 17B - From a study of teaching to issues in teacher education* (<http://www.cerme7.univ.rzeszow.pl/index.php?id=wg17b>).

# PROSPECTIVE MATHEMATICS TEACHERS' NOTICING OF CLASSROOM PRACTICE THROUGH CRITICAL EVENTS

Despina Potari\*, Giorgos Psycharis\*, Eirini Kouletsi\* and Maria Diamantis\*\*

\*University of Athens, Greece, \*\* Southern Connecticut State University, USA

*This paper reports on a study conducted to explore prospective mathematics teachers' reflections on teaching practice at the secondary level through noticing key aspects of classroom interactions. The study used critical incidents taken from everyday classroom situations as a means to make the act of noticing more concrete. The participating prospective teachers were engaged in a number of different activities including observing, designing and teaching. The results indicate a progression of prospective teachers' noticing of classroom practice and development of teaching awareness marked by shifts in analysing and interpreting classroom events.*

Key words: reflection, noticing, critical incidents, teaching awareness

## INTRODUCTION

In this paper we aim to tackle the theory-practice problem in mathematics teacher education (cf. Mason, 2002, Jaworski, 2006) by exploring prospective teachers' reflection on teaching practice through noticing key aspects of everyday classroom situations. Our approach involves the use of critical incidents (Goodell, 2006) as the means by which noticing -and thus reflection on teaching practice- is facilitated to emerge. Our theoretical position towards reflection is based on Jaworski's (1998) interpretation of Dewey's definition of reflective thinking, i.e. "firstly, a recognition of questions to address, identifying some perplexity, making some aspects of teaching problematic; and, secondly, through some processes of enquiry, to seek solutions, or resolutions to, or new ways of understanding, the problems identified." (ibid., p. 7). In resonance with a number of current research approaches (c.f., Scherer & Steinbring, 2006, Jansen & Spitzer, 2009) we see noticing as an activity involving description, analysis and interpretation of teaching practice, thus creating a framework for reflection. A number of research approaches have indicated a number of difficulties that prospective teachers face while engaged in reflection on classroom interactions (eg. collecting evidence about students' learning as well as developing interpretative analysis of classroom instruction (Morris, 2006)). A recent focus on the prospective teachers' reflections on critical incidents taken from classroom situations (Goodell, 2006) supports the idea that critical incidents can be a powerful tool towards promoting prospective teachers' reflective practices to develop. In our study, we used critical incidents as means to engage prospective teachers in reflecting on teaching practice so that they could learn to attend to their students' thinking, interpret classroom phenomena and start to develop ideas of alternative teaching actions. This paper demonstrates how this approach can facilitate

the progression of prospective teachers' noticing of classroom practice and development of teaching awareness marked by shifts in interpreting classroom events.

## **THEORETICAL FRAMEWORK**

Two main bodies of research informed our study: the first, concerns reflective thinking through *noticing* of classroom interactions in teacher education, and the second, *critical incidents* of classroom practice as a concrete basis for reflection and interpretation. In his elaboration of the idea of noticing in teachers' professional development, Mason (2002) pointed out the importance of teachers' attention on the students' learning processes as well as to the teachers' self-observation practices in the classroom. In this view, *noticing* has been related to systematic reflection on acts or issues, leading to shifts in the structure of attention and, through this, to different levels of awareness both in mathematics and in mathematics teaching. In the research reported by Scherer and Steinbring (2006), noticing of students' learning processes was at the core of the joint reflection of teachers and researchers. The analysis suggests that a critical step towards a positive change of teaching activity consists in moving the dominating focus of mathematical interaction in teaching from the teacher to the learning students.

In the domain of preservice teacher education existing research studies suggest that it may be possible to help prospective teachers engage in reflective thinking through noticing so as to enhance their ability to focus on key aspects of teaching practice. Morris (2006) reported that under the condition to form hypothesis about the sources of students' difficulties in a videotaped mathematics lesson, prospective teachers appeared to be able to develop claims and conjectures about the connection between specific instructional activities and students' mathematical understanding. Spitzer et al. (2011) reported that a rather short classroom intervention (2 lessons) involving joint reflection and discussion on written classroom transcripts provided by the researchers, produced substantial improvement in prospective teachers' ability to identify and analyze evidence of student understanding. Similar findings were also reported by research studies in which prospective teachers were engaged in reflecting on their own teaching. The results revealed that reflective activities served as transition mechanisms that promoted prospective teachers' awareness of the need to monitor student understanding during the lesson (Artzt, 1999) and develop hypothesis when interpreting how their teaching affects their students' learning (Jansen & Spitzer, 2009).

Along with a focus on prospective teachers' reflective practices, researchers have been concerned about the introduction of sufficient structures for making the act of inquiry into teaching practice more concrete. A particular example of a structured framework for reflection on classroom episodes are *critical incidents*, i.e. everyday classroom events which are significant for the teachers, make them question their

practice and seem to provide an entry for their better understanding of teaching-learning situations (Hole & McEntee, 1999). Recent research focus on the use of critical incidents in pre-service teacher education (Goodell, 2006) supports the idea that critical incidents can be considered as a means to facilitate prospective teachers' productive engagement in noticing teaching events and critically reflecting on them. In resonance with this approach, in this study we were interested to stimulate prospective teachers' noticing through critical events and see if and how the developing process of selecting and reflecting on critical incidents in different contexts (i.e. observing, designing and teaching) might promote changes in the prospective teachers' stance towards analysing and interpreting classroom events.

## **METHODOLOGY**

### **Context of the study and participants**

The research took place in the context of a 16-week mathematics education undergraduate course (taught in one semester by the first author, mentioned as teacher educator in this paper) at the University of Athens in Greece. The philosophy of the course was to link theory-driven instruction on the teaching and learning of mathematics at the secondary level with realisation of mathematics teaching in real classroom settings. The aim was to engage prospective students in critically consideration of aspects of mathematics teaching as they emerge from the complexity of teaching practice in schools. Every second week (for the entire semester) prospective teachers were asked to participate in a number of field activities such as to observe other teachers' course in cooperating schools, to conduct a didactical intervention in one group of students and to design and implement lessons in the classroom. Each week following the field activities-week included a 3-hour class session taking place at the University. Instructional practice in this session aimed to support prospective teachers' reflective activities on their recent field experience and to link emergent issues with existing mathematics education research. The 22 prospective teachers (9 males, 13 females) who served as participants in this study were divided in pairs and carried out collaboratively the field activities under the supervision of 8 experienced secondary mathematics teachers who served as mentors. Apart from the teacher educator, the research team consisted of two more researchers of mathematics education (the second and fourth author) and an experienced teacher (the third author) who acted as mentor-researcher in the study. Enrolling in the course, prospective teachers had a background of undertaking at least four other mathematics education courses as a part of their teacher education program at the University. Most of them also parallel to their university studies were helping school students on a private base with their mathematical homework.

### **Research design and data sources**

Reflection on teaching practice through noticing of classroom events in the framework of critical incidents was the foundation of our research design. Critical

incidents were considered as a methodological tool for triggering prospective teachers' reflection on teaching practice. In the first class meeting, prospective teachers were introduced to the idea of critical incidents mostly through examples and also by reference to teacher education research. In the next class meeting and in the first half of the third, the groups of prospective teachers completed tasks that required them to identify why some classroom incidents "provided" by the teacher educator (e.g. transcripts of lessons or videotaped teaching episodes) could be considered as critical, discuss the features of those incidents, and finally analyze and interpret them. These incidents were considered as critical by the researchers as they could indicate an important teaching and learning moment (e.g. an unexpected student's response, an "effective" classroom interaction etc.) In the second half of the third class meeting and in all the subsequent ones, the groups of prospective teachers were asked to select and present in the next class session a critical incident that represents an unexpected situation that they had experienced during their fieldwork activities (i.e. observation of lessons, didactical interventions, design and implementation of lesson plans). It was expected that these presentations and the subsequent class discussions would provide a fruitful terrain for studying the development and evolution of prospective teachers' noticing through critical incidents in different contexts. All class sessions (8 in total) were video recorded. The data for this study was conducted over the entire semester, and consisted of: (a) prospective teachers' personal portfolios including their written accounts of critical incidents and material related to the design, implementation and presentation of their field activities in the classroom (e.g. worksheets, lesson plans, presentation files); (b) video recordings of all class sessions at the University; (c) audio-recordings of interviews with some of the prospective teachers regarding their field activities, and (d) researchers' field notes. For the analysis verbatim transcriptions of all recordings were made. The analysis presented in this paper is based only on the video transcripts of the class sessions.

### **Data analysis**

In this study we worked broadly through a grounded theory approach (Strauss & Corbin, 1998) as our goal was rather exploratory. The unit of analysis was the episode, defined as an extract of interactions performed in a continuous period of time around a particular issue. The episodes which are the main means of presenting and discussing the data were selected (a) to involve prospective teachers' interactions on student learning according to an unexpected teaching event and (b) to represent indications of emerging shifts in prospective teachers' noticing of classroom interactions involved in the episode.

### **RESULTS**

The prospective teachers identified a variety of critical incidents throughout the activities of the course. The issues that were addressed by them concerned mainly

students' difficulties; classroom management; curriculum and textbooks, overall educational and social context. In this paper, we report findings indicating the shifts in analysis and interpretations of students' learning that occurred during the university class meetings.

### **Noticing students' learning in teaching examples provided by the researchers**

In the initial class sessions the prospective teachers' comments and interpretations of critical incidents were mostly related to students' misconceptions and to ineffective teacher's strategies. They appeared to attribute these problems either exclusively to students' responsibility or to wider social factors that framed teaching and learning. Moreover, these factors seemed to have been considered in a fragmented way despite of the teacher educator's attempts to encourage prospective teachers to see teaching and learning at its complexity. We provide below some evidence of some of the issues described above by referring to an extract from the second classroom meeting. The teacher educator presented a task related to students' proofs for the statement: "The sum of two consecutive odd numbers is divisible by four". (taken from Boero & Guala, 2008). One 14- year- old student provided the following response:

"By making some trials like for instance,  $3+5$ ,  $15+17$ ,  $31+33$  I realise that I always get sums made by the first odd number and by the same odd number increased by two, thus I get the double of an odd number plus two. This result is divisible by four because the sum of two equal odd numbers would be (alone) an even number divisible only by two, but if I add two I get the consecutive even number, which is divisible by four because even numbers follow each other with the rule that if one is divisible only by two, the following one is divisible by four (like: 2, 4; 6, 8; 22, 24; etc) because the multiples of four are four units far from each other" (ibid, p.238).

Initially, prospective teachers considered student's reasoning empirical:

"The student makes an attempt to generalise but constructs some rules that hold for small numbers but then he concludes arbitrary that this is true for all." (Lefteris, 2<sup>nd</sup> class meeting)

Later on in their attempts to develop a better understanding of the student's thinking they started to consider much deeper issues such as the symbols' use in a mathematical proof, what constitutes a mathematical proof, and the distance between curriculum demands and students' mathematical understanding. The following example indicates prospective teachers' initially rather narrow perspective about the nature of mathematical proof:

"It is like another example we had seen in the previous lesson where the student could not use the symbols. Although this student seems to understand what the answer is and how more or less to get it, this is not a mathematical proof... it does not have operations and relations." (Ioanna, 2<sup>nd</sup> class meeting)

Although in this part of the discussion the teacher educator's intervention was minimal, the prospective teachers started to express different opinions from their



initial ones indicating their appreciation of student's reasoning and their efforts to provide an adequate justification (2<sup>nd</sup> meeting):

Spirithoula: We need to remember that this student is only in the 9<sup>th</sup> grade and he does not have yet the experience to write the even number in the form of  $2k$  and the odd in the form of  $2k+1$ , so that to construct an accurate mathematical proof. I think that for a student of that age the whole thinking was very good.

Adriana: I would also agree, let's not forget that it took ages to develop the formalism ...

Oliver: Diofantus also did not use algebraic symbols...

Adriana: How can a student of that age construct such a proof?

Although the class seemed to come to a consensus, one prospective teacher reminded the audience that the mathematics curriculum suggested a formal approach to problems of that kind. This creates some tensions again:

"I completely disagree. It is very good that a student does something different from what he has been taught. Following a problem solving method mechanically is not good for his future mathematical development. It is more important to encourage him to make explorations." (Oliver, 2<sup>nd</sup> meeting)

Through the above comment, Oliver brought to the foreground the critical role of exploration in students' mathematical development in the long term.

Prospective teachers' attention to students' thinking in this context seemed to have made a number of shifts in the ways they analysed students' understanding. In particular, they started to recognise students' reasoning beyond the formality of the symbols and to view it as an integral part of students' mathematical future development.

### **Noticing students' learning in classroom observations**

In the third, fourth and fifth meeting the prospective teachers commented on critical incidents which they had noticed in the classroom observations. A variety of issues emerged in the discussions such as conceptual and procedural learning, students' difficulties to make connections between different representations, the relation between the nature of teacher's questions and students' answers, curriculum and wider social issues and their impact on learning and teaching, the role of students' prior knowledge in learning and the effectiveness of specific teaching examples and tasks (e.g. the use of paradoxes, the connection among different content areas). In the discussions a central issue was the construction of mathematical meaning. Initially the focus was on students' mistakes that were due to lack of understanding. However, during the discussion the prospective teachers started to link this phenomenon to teachers' choices (examples, tasks, questions) and to research findings. We will try to indicate these shifts by using examples from our data.

A number of critical incidents that the prospective teachers presented were related to the fact that the students often apply a method to solve a mathematical task without understanding the underlying properties. Some examples were the transformation of a fraction to an equivalent one, the solution of a first degree equation, arithmetic or algebraic computations. For example, one critical incident reported by a prospective teacher was about a classroom interaction between two students concerning the transformation of the fraction  $\frac{7}{5}$  to its equivalent with 30 as denominator. The first student completed the transformation by multiplying both terms of the fraction by 6. Then the second student wondered why he did not use a faster common technique based on the use of the appropriate factor that is “kept” in a place over the nominator. The prospective teacher interpreted the phenomenon by considering this technique as a “picture” in the student’s mind which might provide a barrier to conceptual understanding:

“The second student seems to have clear in his mind a picture without knowing why this method works, the essence of the method.” (Kostas, 3<sup>rd</sup> class meeting)

In a subsequent stage of the discussion, the teacher educator attempted to move the class attention on how to deal effectively with the situation in order to make the meaning of this specific technique transparent to the student. Kostas stated that he would ask him to reflect on his actions “What are you actually doing?” “In what way your approach is different from your peer’s?”. Another prospective teacher recalled from his fieldwork observation how another teacher managed a similar situation. Instead of stressing the rule “change side, change sign”, commonly used, in solving algebraic equations, he emphasized the properties involved in the solution process. The prospective teacher found this approach original as it was beyond his own experiences:

“There was not the method of moving it to the other side and change its sign but the teacher was emphasizing that we do the inverse operations. I find this approach very different, more advanced.” (Lefteris, 3<sup>rd</sup> class meeting)

In a subsequent class meeting, the prospective teachers themselves started to build connections between learning, teaching and research. They had been asked to find and read a research paper that would help them to develop adequate explanations of the fact that students often do things at an operational level without deeper understanding of the underlying properties. In terms of students’ learning they managed to give deeper interpretations by realising the meaning of the variable, the double meaning of the equal sign and the transition that the students needed to make from arithmetic to algebra. In terms of teaching they identified tasks such as a tree diagram that could help students to understand the priority of operations and use it for solving equations or they talked about the emphasis needed to be given on algebraic structures in arithmetic. Some typical comments were:

“The students need to understand that a variable is an element of a set, something like this.” (Ioanna, 5<sup>th</sup> class meeting)

“We read about a tree diagram that helps students to read and see how the algebraic relation is structured and it also uses a computer program to represent it.” (Lefteris, 5<sup>th</sup> class meeting)

By summarising, in this context the prospective teachers extended their own examples about mathematics teaching and learning and started to reconsider and evaluate the effectiveness of some of the teaching approaches they experienced as school students. We also noticed deeper interpretations of students’ mathematical contributions by relating them to the research findings. Finally, they started to focus on the role of teaching practices to the development of learning and identify fine elements of teaching.

### **Noticing students’ learning in prospective teachers’ teaching**

In the last three class meetings the prospective teachers presented critical incidents from their own teaching. The critical incidents were related to students’ difficulties or unexpected responses, to the appropriateness of the designed tasks, to epistemological aspects and to classroom management. Almost all the prospective teachers participated in the discussion by presenting and justifying their critical incidents as well as by challenging their peers’ interpretations and claims. In this phase, the main part of the class discussions was based on the prospective teachers’ interactions. Some of the main issues that emerged were: a teacher’s difficulty to notice students’ learning; the problem of time; the connection among different representations; the difference between procedural and conceptual understanding; the management of students’ different mathematical backgrounds and interests; the difficulty to design a mathematically challenging task consistent with students’ cognitive and affective needs; and the epistemological characteristics of geometry.

In terms of students’ learning the prospective teachers’ interpretations focused more on the students’ strategies and thinking processes rather than on their difficulties and errors. Moreover, they often seemed to overtly recognise the critical role of tasks in challenging students’ mathematical thinking. We are giving below some examples from prospective teachers’ reflections.

In the 7<sup>th</sup> class meeting, Katerina talked about what she learned from her 8<sup>th</sup> grade students while working on a task she had designed for comparing the areas of three irregular polygonal areas:

“We wanted to see how the students were thinking while they were dividing the areas to regular shapes. We let the students to work on their own. I had expected them to develop three or four different strategies but when I analysed them afterwards I discovered that they were twelve!...What I have understood is that when you let the students to work on themselves they have a lot of different ideas. We can also see how they are thinking... All the students in the class had done something.” (Katerina, 7<sup>th</sup> class meeting)

Katerina recognised that students' thinking can be very powerful through the analysis of their strategies. Moreover, she acknowledged the importance to provide space to all students to think mathematically during the teaching. In the discussion that followed the other prospective teachers also commented on the added pedagogical value of students' multiple solutions of a mathematical task:

"I think that it has to do with the nature of the tasks. A very specific task does not allow for multiple solutions and answers. So, I do not have to ask questions that have as answer "yes" or "no". We need to ask why." (Spirithoula, 7<sup>th</sup> class meeting)

In the last class meeting Aggeliki and Maria presented a critical incident from their teaching in a 9<sup>th</sup> grade class. Their teaching goal was for the students to make sense of the algebraic formula  $(a+b)^2 = a^2 + 2ab + b^2$  through a geometrical task they had developed. One student who was engaged in calculating areas in the geometrical context he recalled the formula without connecting it to the problem. The two prospective teachers did not expect this response and they interpreted that the student did not make any connection to the problem but he only recalled the relation without understanding:

"I expected to hear that the area of the total land was the sum of the four rooms and he gave me the algebraic formula." (Aggeliki, 8<sup>th</sup> class meeting)

"I used to do this when I was at school. The teacher was telling me something and when I did not know it I was giving him the formula I knew." (Maria, 8<sup>th</sup> class meeting)

In their attempt to interpret student's approach the prospective teachers were trying to go more deeply to student's thinking process. For example Lefteris mentioned the fact that the student worked at the operational level and could not see the relation structurally:

"The student says that the solution of this relation is... he does not see the equivalence of the two quantities, he only sees that he will expand the  $(a+b)^2$  and he will find the result. He has acted only procedurally." (Lefteris, 8<sup>th</sup> class meeting)

Overall, by reflecting on their actual teaching the prospective teachers seemed to focus on key aspects of student's learning and to relate it to features of the tasks (e.g. openness, kind of representations).

## CONCLUDING REMARKS

Our purpose in this paper was to illustrate a particular approach to encourage and study prospective teachers' reflection on teaching practice by noticing key aspects of classroom interactions through critical incidents. The results indicate a progression of prospective teachers' noticing of classroom practice marked by shifts in the analysis and interpretation of critical incidents. An initial analysis of students' thinking at a surface level has gradually been moving towards considering salient features of the learning process. Towards the last class sessions prospective teachers seemed to be able to make connections between students' learning with particular

aspects of teaching practice. Finally, this process seemed to be carried out through the integration of different sources of knowledge such as prospective teachers' tacit knowledge about teaching from their experiences as students and private tutors and the academic knowledge they were developing at the University course.

## REFERENCES

- Artzt, A. F. (1999). A structure to enable preservice teachers of mathematics to reflect on their teaching. *Journal of Mathematics Teacher Education*, 2, 143–166.
- Boero, P. & Guala, E. (2008). Development of mathematical knowledge and beliefs of teachers. In P. Sullivan and T. Wood (eds.), *Knowledge and Beliefs in Mathematics Teaching and Teaching Development* (pp. 223-244). Rotterdam: Sense Publishers.
- Jaworski B. (2006). Theory and practice in mathematics teaching development: Critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187-211.
- Goodell, J. E. (2006). Using critical incident reflections: A self-study as a mathematics teacher educator. *Journal of Mathematics Teacher Education*, 9, 221-248.
- Hole, S., & McEntee, G. H. (1999). Reflection is at the heart of practice. *Educational Leadership*, 56, 34–37.
- Jansen, A. & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education*, 12, 133–151.
- Jaworski, B. (1998). Mathematics teacher research: process, practice and the development of teaching. *Journal of Mathematics Teacher Education*, 1, 3-31.
- Mason, J. (2002). *Researching Your own Practice: The Discipline of Noticing*. London: Routledge Falmer.
- Morris, A. (2006). Assessing pre-service teachers' skills for analyzing teaching. *Journal of Mathematics Teacher Education*, 9(5), 471–505.
- Scherer, P. & Steinbring, H. (2006). Noticing children's learning processes – teachers jointly reflect on their own classroom interaction for improving mathematics teaching. *Journal of Mathematics Teacher Education*, 9, 157–185.
- Spitzer, S. M., Phelps, C. M., Beyers, J. E. R., Johnson, D. Y. & Sieminski, E. M. (2011). Developing prospective elementary teachers' abilities to identify evidence of student mathematical achievement. *Journal of Mathematics Teacher Education*, 14(1), 67-87.
- Strauss, A. & Corbin, J. (1998). *Basics of Qualitative Research*, Sage Publications.

# CURRICULAR CHANGES IN PREPARATION OF FUTURE TEACHERS – FINANCIAL MATHEMATICS COURSE

Michaela Regecová, Mária Slavíčková

Department of Algebra, Geometry and Didactics of Mathematics  
Faculty of Mathematics, Physics and Informatics  
Comenius University in Bratislava, Slovakia

*The paper deals with problem of introduction of new mathematics content into secondary school curricula and its influence on changes in curricula for future teachers of mathematics. We are focused on development of content knowledge. We choose area of (everyday) financial mathematics because of the new school reform in year 2008 and the basic financial mathematics is obligatory part of the mathematics education at secondary schools in Slovakia. In our research we observed changes in students' content knowledge from financial mathematics after following the course of financial mathematics for future mathematics teachers. The results imply that their knowledge is not sufficient and therefore there is need to give more courses/lecture of financial mathematics in preparation of future teachers focused on basic financial notions and mechanisms.*

**Key words:** financial mathematics, financial literacy, school reform, preparation of future mathematics' teachers

## INTRODUCTION

The importance of financial education has increased in recent years, in particular as a result of developments in the financial market and demographic, economic and strategic changes. Currently, consumers have better access to a whole range of credit and saving structures that differ with regard to fees, interest rates, maturity or other parameters.

“Financial education is the process by which financial consumers/investors improve their understanding of financial products and concepts and, through information, instruction and/or objective advice, develop the skills and concepts and, through information, instruction and/or objective advice, develop the skills and confidence to become aware of (financial) risks and opportunities, to make informed choices, to know where to go for help, and to take other effective actions to improve their financial well-being and protection.” *OECD (2003)*

Ministry of Education of Slovak Republic in compliance with OECD Recommendations How to Improve Financial Literacy (OECD, 2003) prepared National standard of financial literacy (Ministerstvo školstva SR & Ministerstvo financií SR, 2008) in October 2008. The main goal was start with financial education at schools according with new State educational programs (SEP) within new school reform.

New school reform started in September 2008 at primary and secondary schools. This reform innovated content of many subjects, financial education was integrated mostly into mathematics education. Implementation of this reform unfortunately did not count with necessary changes on the universities that prepare future mathematics teachers. The effect of this situation is that universities continue with preparation of future teachers that are not appropriately prepared for teaching according the new State educational programs.

Therefore we suggested study material from financial mathematics (FM) for students – future mathematics teachers. These students were just graduated from secondary schools; they followed the old system of education – it means without the FM course and other changes that are part of the new school reform from 2008.

The aim of our research was to find out if the study material prepared for teaching FM improves knowledge of the students in the 1<sup>st</sup> year of study on university to be teachers of mathematics. To verify suitability and propriety of suggested course of financial mathematics as a part of the subject Didactical seminar 2 from the standpoint of students understanding of basic terms and mechanism of FM we suggested research in this area.

The research question was if the students' knowledge from financial mathematics after attending the FM course is on the level of institutionalization, it means if they are able to apply obtained knowledge from FM.

Following formulated research question we observed students' ability to apply knowledge from FM in solving the real life problems from financial area that could improve students' attitudes towards mathematics education (Vankúš & Kubicová, 2010).

## THEORETICAL FRAMEWORK

We can identify at least three general approaches in didactics of mathematics: cognitive, social and epistemological. Epistemological approach of the research is focus on using of the mathematical knowledge and its spread into educational institutions (Kohanová, 2008). Therefore we choose *Theory of didactic situations* (TDS) for the analysis of students' solutions in our research.

Brousseau (1998) defines the *didactic situation* as a situation for which is possible to describe the social intention of acquirement of student's knowledge. This situation is realized in system called the *didactic system* (didactic triangle) that is composed from three subsystems: learner (student), learning (teacher), information (knowledge) and from relations between them. In didactical situation are subjects confronted with prepared milieu. The *didactic milieu* (Brousseau, 1990) is the basic notion of Theory of didactic situation. Following the Piaget's theory the milieu is source of contradictions and non-steady states of learner (subject) by process of adaptation (by Brousseau (1986) it is assimilation and accommodation). All knowledge has its specific environment in which students are confronted with.

The didactic situation can be described as a game between a (person in the role or position of the) teacher and the student-milieu system. Every game has its rules and strategies. The rules and strategies of the game between the teacher and the student-milieu system, which are specific of the knowledge taught, are called the „*didactic contract*“. The rules of the didactic contract are not explicit and they can be slightly different from classroom to classroom, culture to culture, and they can even change in the history of a single classroom with the same teacher and the same students. (Sierpinska, 2003b)

Analysis of our research was carried out following knowledge building by TDS: action, formulation, validation, and institutionalization according Sierpinska (2003a).

The results followed from research focused on level of financial literacy of university students described in Regecová & Slavičková (2010) indicate that students' knowledge from financial mathematics are mostly on the level of action or formulation. In presented research we were interested also in ability of student to estimate the result of problem. As was written in Brisudová & Slavičková (2006) students are not able to make a guess and their calculation always depend on calculator. The ability of guessing is missing on mathematics lessons.

## METHODOLOGY

Following the model of design-based research (DBR) (Wood & Berry, 2003) and in accord with our research question we developed a proposal of proper course of financial mathematics (FM) for student of the 1<sup>st</sup> year on Faculty Mathematics, Physics and Informatics (FMPI) and Faculty of Natural Sciences (FNS) to be a teachers of mathematics. This curricular change should permit to the students' teachers to be obtaining necessary knowledge from the FM both for usage in real life and notably in their future teacher of mathematics career on secondary school.

The main idea of State Educational Program (SEP) (Ministerstvo školstva SR, 2008) is to know use knowledge from financial mathematics in real life and on secondary school mathematics level. Therefore we tried to suggest tasks, problems and examples in accord with SEP. There were no Slovak text-books for secondary schools that can be used on the lessons corresponding with New Curricular Reform (NCR) in that time. When we were preparing the course of FM for future teachers on our university, we use the text-book from Czech Republic (Odvárko, 2005), in which the educational program focused on FM is very close to the program determine by Slovak SEP. The other reason were similar trends in both countries (from historical, linguistic and socio-cultural point of view) and similar school reform in Czech Republic that was realised in 2007 (one year sooner than in Slovakia). From this textbook we pick up tasks that were in accord with NCR and we change these tasks a little bit (change the names of persons and institutions to made it more attractive).



Because of not enough space in subject Didactical seminar 2 on bachelor degree of the teacher education program on FMPI and NFS (2 lessons) we pick up 42 tasks and divide them into 2 thematically independent parts. The 1<sup>st</sup> part was focus on bank accounts, fix deposit, bankbook and different type of save up products. The 2<sup>nd</sup> one was focus on different type of loans, leasing, and credit. (Regecová, 2009)

Integration of the FM course into 2 lessons in winter 2008 allowed us to start with teaching in summer term 2009. Because of many tasks need a bigger calculating, or more difficult formulas, we realize to teach both - with using of ICT and common way (chalk and blackboard). We used MS Excel and free software GeoGebra.

Before we started with the experimental teaching of FM we prepare didactical test. The test had 4 tasks chosen to cover important areas from financial mathematics taught on secondary school level (2 tasks were focused on saving products and 2 on loans). Therefore content and construct validity of this test was high.

We tested sample of 25 students at university level in the 1<sup>st</sup> year of study to be a teacher of mathematics (2 groups of students). Pre-test was written in the same week in both groups. Students in the groups were taught by different teachers (one group taught Michaela, the second group taught Maria) and they did not know that they will be involved to the experiment. The teaching/learning process started one week after the pre-test. The innovation of teaching was not only in adding a new topic into the existing subject. The approach to the problem was different. We started to use ICT not only as demonstration tool, but students also had computers to made their own simulation, calculation and modeling, students had possibility to use ICT to made up their home works etc. Using of the ICT was common, individual work of the students was necessary, home works could be prepared by using of ICT.

According to teaching/learning process we would like to specify that every student has one task to solve one week before the lesson. Therefore process of action and formulation of hypothesis was realized at home environment of every student. There was possibility to discuss students' hypothesis among schoolmates whole week. Validation of students' hypothesis started during the presentation of their solutions on the lesson. There was discussion among students themselves and among students and teacher. Institutionalization of new knowledge was up to the teacher.

During the teaching we were doing also observation of the behavior of the students, their attitude and opinion on these changes. It of course differs from student to student. Mostly girls were discouraged because of computers.

Posttest was written approximately 2 weeks after the finishing of the topic financial mathematics. We offer analysis of two chosen tasks from the experiment in this paper.

## RESEARCH AND RESULTS

Before the testing of the students we write analysis a priori of all 4 tasks. We point out the aim, terminology and possible models of solving the tasks. In general we can write that the aim of the test was:

- find out which model for solving chooses students to solve standard tasks from financial mathematics, it means, if they know terminology and models of financial mathematics,
- if students are able to guess and count the better alternative, if they know mechanism of loan interest or credit and if they know influence of loan interest or credit on profitability of the loan.

Necessary terminology which had to understand for solving the task was bank account, annual interest, interest rate, interest tax, credit, loan, mortgage, principal, repayment plan.

Models used in solving the test differ according to the aim of the tasks. We assumed also incorrect solving strategies/models. We can divide the strategies into two main categories: M1 – financial mathematics approach, M2 – non-financial mathematics approach.

Detailed a priori analysis of students' solutions is described in Regecová & Slavičková (2010). In this article we would like to focus on analysis of student knowledge improvement and errors in their solutions.

The results from the first part of the research imply that students do not have proper knowledge from FM. The comparing their result before the curricular changes and after is described bellow. We compare level of knowledge of each student before and after the course of financial mathematics.

### Task 1

We would like to set up new bank account and deposit 100 EUR (3 012,60 SKK ) every month starting at the beginning of the year. The deposit will always be made at the beginning of the month. We know that “our bank“ offers a 2.3% fixed interest rate (the interest rate will not change throughout the year). The interest rate is calculated and deposited *only once (every quarter* in post-test) to our account, and always at the end of the calendar year. How many EUR would have in the account by December 31<sup>th</sup> with a 2.3% interest rate? How many EUR would we save by the end of the year if we would put 100 EUR every month to our piggy bank?

### Pre-test:

There were 23 students who used model M2 to solve the task 1; it means the model without the tools of FM in the pre-test. None of these students achieved the correct answer. Most often mistakes were: 12 students did not think of tax 19 % (missing relation with real life), 2 students made error in the order of number magnitude (missing the guess of possible result), incorrect interest rate (1 student), and not

proper tax (1 student). One student solved this task correctly (model M1) and one student did not solve this task.

### Post-test:

All 25 students used model M1 in the post-test, it means after the course of FM. All of them tried to apply their knowledge from FM in solving the task. But correct answer gets only 4 students. The most often error was that students did not think of periodicity of the deposit – 8 students did not count with every month deposit 100 EUR. Similar, 4 students did not think on quarter period, but months', 5 students did not use the proper interest rate (they use months', quarters' instead of the years'), 2 students think of quarter period but in next quarter count only with interest, not whole amount of deposit from the 1<sup>st</sup> quarter, 1 student think in quarters, but do not think that in the second quarter we have also amount from the 1<sup>st</sup> one, in the 3<sup>rd</sup> from the 2<sup>nd</sup> one etc. The positive result is that every student count with 19% tax.

### Comparing pre-test and posttest:

From the results of pre-test and post-test in each students (25) we can see, that all of them tried to use these knowledge to solve the task after the attending the financial mathematics course. The one of possible reasons could be didactical contract (Sierpinska, 2003b) the effort of the students to apply obtained knowledge by any way. But the higher level of the task 1 in post-test comparing to pre-test could be the reason that only 4 students get correct answer to the question. In general we can say that students after the financial mathematics course are able to apply the methods and tools of FM but with some problems. Their knowledge of FM was not institutionalized during courses of FM.

We demonstrate progress in using of solving method by one chosen student in the Table 1 even if presented solutions are not correct.

<b>Pre-test</b>
Sum of deposits: $12 \cdot 100 = 1\,200$ EUR
Counting with tax 19 %: Tax $2.3 \cdot 0.19 = 0.44$ %
Interest rate after taxation: $2.3 - 0.44 = 1.86$ %
Value of the interest add to the account: $1\,200 \cdot 0.0186 = 22.356$ EUR
At the end of year: $1200 + 22.356 = 1222.356$ EUR
Piggy bank money: $12 \cdot 100 = 1200$ EUR
<b>Post- test</b>
Sum of deposit and interest (in EUR) in three months separately, so in one quarter:
January: $100 + 100 \cdot \frac{3}{12} \cdot \frac{2,3}{100} \cdot 0.81 = 100.466$ , February: $100 + 100 \cdot \frac{2}{12} \cdot \frac{2,3}{100} \cdot 0.81 = 100.311$ ,
March: $100 + 100 \cdot \frac{1}{12} \cdot \frac{2,3}{100} \cdot 0.81 = 100.155$ , ..., July: $100 + 100 \cdot \frac{6}{12} \cdot \frac{2,3}{100} \cdot 0.81 = 100.932$ , ...
Total sum: $12 \cdot 100 + 100 \cdot \frac{2,3}{100} \cdot 0.81 \cdot \left( \frac{1}{12} + \frac{2}{12} + \frac{3}{12} + \frac{4}{12} + \dots + \frac{12}{12} \right) = 1212.11$
Piggy bank money: $12 \cdot 100 = 1200$ EUR

**Table 1: Example of student solution in pre-test and post-test.**

**Task 2**

A brokerage firm received a loan for 99 000 EUR with a 13% interest rate over 3 years. According to the contract with the bank, they will start the payments one year from the origination of the loan. Payment is yearly and the bank collects interest once a year. How much would the yearly payment be? Round your answer to the nearest cent.

**Pre-test:**

There were 4 students who tried to solve this task in model M1 but because of the weak knowledge of FM only 1 got a correct answer in the pre-test. 10 students did not start to solve this task, other students tried to save the fairness in payments and divide the amount into 3 equal parts and use interest rate to get an answer, or they start with counting on the interest rate and the result divide by 3, or 12, or 36. The obtained number they declared as an annuity. The counting was mostly illogical what implies that students did not understand the term annuity and there is again didactical contract present – students tried to count “something”, because it was expected.

**Post-test:**

In the post-test 23 students used M1 model, but 6 of them obtained correct answer. The rest 2 students did not start to solve this task. The most often errors were: using the formula instead of the logical procedure (13 students), but many students did not use this formula in correct way. One student tried to solve the task by both models, but without any success. Two students who used the formula correct were not sure with the answer, for just in case they multiply this result by 3. Three students used months' interest rate period instead years'.

During the schooling we tried to do not use formula, we tried to lead the students to understand the mechanism of different financial products. The fact that students use formula even that we forbid it implies that students were not able to use mechanism and obtained knowledge was formal. Their knowledge of FM was deformed and not institutionalized during courses of FM.

**Comparing pre-test and posttest:**

Analysis of pre-test and post-test in every students pointing that after the attending FM course most of the students tried to solve the task by mechanism and knowledge of FM. This fact is necessary to highlight mostly in 10 students who in pre-test did not try to solve this task. In total we can say that students had an effort to use FM tools and mechanism in solving this task, but as we say before, only some of them get correct answer.

We demonstrate progress in using of solving method by one chosen student in the Table 2 even if presented solutions are not correct.

**Pre-test**

Interest paid in total:  $99000 \cdot 0.13 = 12870$  EUR

Interest paid in one month:  $12870 : 12 = 1072.5$  EUR

Annuity will be 1072.5 EUR.

**Post-test**

The end of:	Annuity	Interest	Principal	Debt
	–	–	–	99 000
the 1 <sup>st</sup> year	x	12 870	$x - 12\,870$	$99\,000 - (x - 12\,870)$
the 2 <sup>nd</sup> year	x	$99\,000 - (x - 12\,870)$	$x - [99\,000 - (x - 12\,870)] = y$	$99\,000 - (x - 12\,870) - y = z$
the 3 <sup>rd</sup> year	x	z	$x - z$	0

$$99000 - x - 12870 + [99000 - x - 12870] - x - 0.13 \cdot 99000 - x - 12870 = 99000$$

$$x = 62370 \text{ EUR, but if I use formula, then } a = \frac{99000 \cdot 0.13 \cdot 1}{1 - 1 + 0.13 \cdot 1^{-3}} = 41928.70 \text{ EUR.}$$

**Table 2: Example of student solution in pre-test and post-test.**

**CONCLUSION**

The results of our research show that knowledge of the students (graduated secondary school students) from FM is not on sufficient level. We did not noticed progress in solving the tasks by students after attending the FM courses in duration 2 hours. Future teachers are only able to apply their knowledge to problem solving in a superficial way relying on recalling formulate or remembering procedures. One of the possible reasons why deformation of educational process happened is the lack of the teaching hours. Therefore the students were not able to shift their knowledge on level of validation and institutionalization and they obtain only formal knowledge. That indicates that it is necessary to pay attention to this topic in more detailed way and in bigger time donation. The results also indicate that information from the media and everyday life are not sufficient. Therefore the students are not prepared for it and need to learn more at secondary and university/higher school level from this very important area of the life. As we can see, the one of the aim of the teaching mathematics (following NCR) “*develop the analytical thinking and ability to solve the task in real life*” was not reached – students are able to memorize, but they do not understand, they cannot make their own construction; they need to be lead by teacher/tutor. The other ability – make a guess is still not on adequate level.

In our study we focused on development of content knowledge but there should be balance between developing future teachers’ content knowledge and their pedagogic knowledge. The question is, what approaches on the part of teacher educators can help future teachers to develop all aspects of their knowledge for teaching

(knowledge of math, ways of teaching math and ways in which students learn math) at the same time?

## REFERENCES

- Brisudová, Z., Slavíčková, M. (2006). Problems with using calculators on mathematical lessons, *Acta Didactica Universitatis Comenianae - Mathematics*, Issue 6, Comenius University, Bratislava 2006, ISBN 80-223-2138-9, p.19-31
- Brousseau, G. (1986). Fondaments et méthodes de la didactique des mathématiques. *Recherches en Didactique des Mathématiques*. Grenoble, La Pensée sauvage
- Brousseau, G. (1990). Le contrat didactique: le milieu. *Recherches en Didactique des Mathématiques*, Vol 9/3, Grenoble, La Pensée sauvage
- Brousseau, G. (1998). *Théorie des situations didactiques*. Grenoble, La Pensée sauvage
- Kohanová, I. (2008). *Teoretické rámce vo výskumoch v didaktike matematiky*, 2. Zborník príspevkov štipendistov z projektu JPD 3 BA 2005/1-043, Bratislava: Knižničné a edičné centrum FMFI UK, 2008
- Ministerstvo školstva SR (2008). Štátny vzdelávací program. Retrieved November 29, 2010, from <http://www.minedu.sk/index.php?lang=sk&rootId=2319>
- Ministerstvo školstva SR, Ministerstvo financií SR (2008). *Národný štandard finančnej gramotnosti verzia 1.0*. Retrieved July 24, 2010, from [http://www.mpc-edu.sk/library/files/narodny\\_standard.pdf](http://www.mpc-edu.sk/library/files/narodny_standard.pdf)
- OECD (2003). *OECD Project on Financial Education and its International Network on Financial Education*. Retrieved July 24, 2010, from <http://www.financial-education.org/dataoecd/8/28/44409678.pdf>
- Odvárko, O. (2005). *Úlohy z finančí matematiky pro střední školy*, 1.vydanie, Prometheus 2005 Praha, ISBN: 80-7196-303-8, pp. 198
- Regecová, M. (2009). *Návrh kurzu finančnej matematiky pre študentov odboru učiteľstvo akademických predmetov: prieskum finančnej gramotnosti študentov*. Vzdelávanie - veda - spoločnosť: sekcia matematické vedy (CD ROM), Komárno : Univerzita J. Selyeho, ISBN 978-80-89234-84-4, S. 62-81
- Regecová, M., Slavíčková, M. (2010). Financial literacy of graduated students. *Acta Didactica Universitatis Comenianae – Mathematics*, Issue 10, Comenius University in Bratislava, Bratislava, ISBN 978-80-223-2904-0, pp. 121-147
- Sierpinska, A. (2003a). *Introduction to the concept of didactic situation and the distinction between situations of action, formulation and validation, Lecture 1*. Retrieved August 16, 2010, from <http://annasierpinska.wkrib.com/pdf/TDSLecture%203.pdf>
- Sierpinska, A. (2003b). *The notion of 'didactic contract', Lecture 3*. Retrieved August 16, 2010, from <http://annasierpinska.wkrib.com/pdf/TDSLecture%203.pdf>
- Vankúš, P., Kubicová, E. (2010). Postoje žiakov 5. a 9. Ročníka ZŠ k matematike, *Acta Mathematica*, Vol. 13, UKF Nitra, Nitra
- Wood, T., Berry, B. (2003): What does “design research” offer mathematics teacher education? In *Journal of Mathematics Teacher Education*, 6(3), pp. 195-199

# KNOWING MATHEMATICS AS A TEACHER

C. Miguel Ribeiro; José Carrillo

Research Centre for Spatial and Organizational Dynamics (CIEO), University of Algarve, Portugal; University of Huelva, Spain

*Drawing on the conceptualization of Mathematical Knowledge for Teaching (MKT) this paper considers the knowledge underlying the practice of Maria, a primary school teacher, as evidenced in an episode where she attempts to explain the reason for representing the concept of area with superscript:  $dm^2$ . Through analysis of situations identified as revealing critical gaps in her MKT, we discuss the particularities of the mathematical knowledge required to allow the pupils to achieve a full understanding of the content in question.*

Keywords: Mathematical Knowledge for Teaching, teachers' practices, primary school.

## INTRODUCTION

The way we approach mathematics and its teaching is intrinsically linked to our knowledge of each topic we are to cover. This knowledge necessarily influences the nature of the tasks we set and how we implement them in the classroom, in particular with respect to regulating the mathematical demands involved (Charalambous, 2008). The teacher's knowledge allows a wealth of factors to be taken into account when devising tasks (in terms of both design and delivery) and contributes to the effective construction of mathematical knowledge on the part of the students. It follows, then, that research into the mathematical knowledge involved in the teaching process is of great importance. We follow the line taken by Stylianides and Stylianides (2010) in which mathematics knowledge is considered an applied knowledge, specifically linked to the nature of education and the "problems" associated with the task of teaching.

Only by being in possession of a solid knowledge of mathematics for teaching (and the mathematics underlying this) can it be possible for us, as teachers, to develop a practice that encourages sustained learning by students, enabling them to create networks of concepts and fruitfully navigate between them (e.g. Ribeiro (accepted)). This paper conceptualises mathematical knowledge according to the proposals of the research group led by Ball (e.g. Ball, Thames and Phelps (2008) and Hill, Rowan and Ball (2005)), and follows their concept of Mathematical Knowledge for Teaching (MKT).

In order to be able to seek improvement in practice and likewise in teacher training, it is essential to know the areas of mathematical knowledge in which teachers find themselves most deficient. Fuller knowledge of these areas, and the situations in which they can be brought to light in the classroom, would (one hopes) lead to a re-

structuring of training programs to focus on these areas, and to teachers becoming more active and reflective professionals, better informed of their own MKT.

This paper considers several aspects/sub-domains of MKT brought to the fore in the practice of Maria (a teacher in the ‘1<sup>st</sup> Cycle’<sup>87</sup>), during an episode concerning an attempted explanation as to why the unit of area one square decimetre is written as  $1\text{dm}^2$  (i.e., the reason for the ‘2’ in superscript). We chiefly discuss the gaps this incident highlights and the potential impact on the pupils’ learning, both immediate and future. This analysis and discussion aims to achieve a broader understanding of the possible factors leading to these failures, so that we can devise strategies for intervening and supplying the missing knowledge.

## THEORETICAL FRAMEWORK

The work of Shulman and his associates (e.g. Shulman (1986) and Wilson, Shulman and Richert (1987)) has provided various perspectives and conceptualisations of the professional knowledge of mathematics teachers. Three of these conceptualisations which have proved to be influential are Mathematics for Teaching (MfT) (Simmt & Davis, 2006), the Knowledge Quartet (KQ) (Rowland, Huckstep & Thwaites, 2005) and Mathematical Knowledge for Teaching (MKT) (Ball et al. 2008). These have in common that they start from practice and take as the key focus the mathematical knowledge that teachers need to perform their teaching duties, although each work takes a different approach.

In MfT, Brent Davis and colleagues (e.g. Davis and Renert (2009) and Davis and Simmt (2006)) focus on the teacher from a theoretical perspective, and bring a systemic interpretation to the complexity of practice, with a view to understanding how teachers learn. The KQ, developed by Tim Rowland and associates, constitutes a theoretical framework grounded in practice and developed inductively (e.g. Rowland, Huckstep & Thwaites, 2005; Rowland, Thwaites & Huckstep, 2003). Their work focuses on “what the teacher actually knows and what he believes and how opportunities to enhance knowledge can be identified” (Rowland et al., 2005, p. 257). The model jointly considers teachers’ knowledge and beliefs about the teaching process. As the name implies, the KQ is composed of four distinct dimensions: *foundation*, *transformation*, *connection* and *contingency*. The group led by Deborah Ball regards the professional knowledge that teachers should possess as the knowledge of mathematics required to develop the various tasks involved in the act of teaching students (tasks of teaching), which it denominates MKT. Once again theory is very much rooted in practice (Ball & Bass, 2003), with their

---

<sup>87</sup> Education in Portugal is divided into four cycles: the 1st Cycle covers the first four years at school (pupils aged between 6 and 9), with the class teacher covering all the curricular areas; the 2nd Cycle stretches over just two years (5th and 6th years of primary) with pupils aged from 10 to 11; the 3rd and 4th Cycles, at Secondary level, each cover three years and complete the 12 years of compulsory education.



multidimensional model of the mathematical knowledge involved in the task of teaching emerging from the analysis performed and relationships obtained.

For the analysis of practice presented here, and consideration of the mathematical knowledge involved (applied mathematics), we chose to employ the conceptualization of MKT, with its various sub-domains. The choice of this conceptualization over the others mentioned above (MfT or KQ), or indeed any other, was due to the fact that our aim was to identify, from observed practice, what knowledge the teachers were deploying at each specific moment, and hence the system for making this identification played a key role<sup>88</sup>. In addition, this conceptualisation attributed a very specific orientation to the teacher's mathematical knowledge, placing emphasis on the mathematical reasoning they are immersed in during the course of their work (understood as more than correcting mistakes and evaluating students' answers). While the KQ focuses, originally, on pre-service teachers' knowledge in action, and MfT deals with the mathematical domain mainly, the MKT embraces the focus of knowledge in action from the perspective of the mathematical knowledge teachers' show in relation to teaching.

Ball and her associates introduced the notion of MKT, a specific type of mathematical knowledge in relation to teaching which is not necessarily picked up or increased through the attendance on mathematics courses of a purely scientific bent. This mathematical knowledge must be related to the kind of mathematics they are going to teach, and concerns the ability to make the content comprehensible to their students, being aware of the students' mathematical background, and knowing how to relate previous knowledge to the learning opportunities to come (Hill et al., 2005). In their words MKT can be defined as

*the mathematical knowledge used to carry out the work of teaching mathematics. Examples of this "work of teaching" include explaining terms and concepts to students, interpreting students' statements and solutions, judging and correcting textbooks treatments of particular topics, using representations accurately in the classroom, and providing students with examples of mathematical concepts, algorithms and proofs. (p. 373)*

In the multidimensional model which has been developed by the group, Content Knowledge and Pedagogical Content Knowledge (following Shulman) are each divided into three sub-domains. They consider Content Knowledge to be formed by Common Content Knowledge (CCK), i.e., typical 'schoolboy' mathematics,

---

<sup>88</sup> This process of identification should not be understood as referring to a limited number of dimensions, but should be understood in a broader perspective that takes into account future research projects involving teachers and trainee teachers in which one of the points of focus is the mathematical knowledge that they have/ require to teach different contents in order to improve their own performance and the type and focus of the training available to mathematics teachers (current or future).

Specialised Content Knowledge (SCK) and Horizon Content Knowledge (HCK). Pedagogical Content Knowledge they likewise divide into three types, each a variant of Knowledge of Content and: Teaching (KCT), Student (KCS), and the Curriculum (KCC). Teachers should have a specific professional knowledge, so that in addition to knowing ‘how to do’, considered in CCK, they should also have a knowledge of ‘how to teach to do’ (how to teach to understand), which corresponds to SCK. SCK is seen as the knowledge required by the teacher who genuinely wishes their students to understand what they do and not merely run blindly through a set of given procedures. This component is not limited to knowledge regarding procedures, but has a broader meaning which includes the necessary concepts. The teachers should also be familiar with the way that each the various mathematical topics relates to the others and the way in which the learning of a particular topic develops as one moves up the school (HCK). In relation to the situation presented below (cf. the section ‘Applying the analysis to practice’), knowing how to write the area of measurement  $1\text{dm}^2$  is considered CCK, whilst understanding and being able to explain why the unit is written in this particular way corresponds to SCK (ie, the reason for the superscripted ‘2’ in the written representation – what it means and represents within the mathematical domain).

In addition to knowledge of content, teachers should also have a thorough knowledge of the curriculum and pedagogy. Knowledge of Content and Teaching (KCT) corresponds to the type of knowledge which the teacher draws on in order to organise the different ways the students explore mathematical contents, such as determining the sequencing of tasks, choosing examples, and selecting the most appropriate representations for each situation. Regarding Knowledge of Content and Students (KCS), Ball et al. (2008) relates it to the need for the teacher to anticipate what the students are likely to think, their difficulties and motivations as well as listening to and interpreting their comments. The teacher must be aware of the students’ capacity to understand in such a way that it could allow him/her to go further in deepening the students’ knowledge. With respect to Knowledge of Content and Curriculum (KCC), the authors agree entirely with Shulman (1986, p. 10) that teachers should have a complete picture of the diversity of programs for teaching certain subjects and topics at a particular level/year group, and a variety of educational materials they can draw on. They should also be able to recognise the varying circumstances which suggest the adoption of one approach over another. In general terms, their curricular knowledge should be what can be termed both vertical and horizontal in its scope.

## **DESCRIPTION OF THE STUDY AND CHOICES MADE**

This paper is part of a broader study (involving two primary teachers) focusing on teachers’ professional development in which the MKT is one of the dimensions under consideration. To facilitate the identification of this knowledge we chose a case study approach combined with a qualitative methodology. Here, we shall analyse and

discuss various aspects of the MKT that was in evidence on the part of a teacher (Maria) teaching year 4 (9 year-olds). Maria was a teacher of 18 years' experience and had been with her class since their first year. Amongst several volunteers to work on such project<sup>89</sup>, Maria revealed a clear will to improve her practice, not being "afraid" to recognize gaps in her knowledge, intending to learn to be able to fulfil them.

Data collection consisted of audio and video recordings of the teacher in action, supplemented by informational talks before and after each lesson (to gather lesson previews – lesson image – and to clarify certain inferences). All the audio recordings were transcribed and complemented with the information from the video, which allowed us to capture the greater part of the teacher's actions and her interactions with the pupils (Brophy, 2004). The transcripts were then used to divide each lesson into episodes (associated with the immediate goals of the teacher). The identification of each of the sub-domains of MKT was then made for each of these episodes. Such identification allows an in-depth analysis of the teachers practice.

Our primary interest has been in instances where a lack of knowledge on the part of the teacher becomes evident, as these represent an opportunity to learn (Hiebert & Grouws, 2007). We explore possible causes for such occurrences and consider various means of supplying the lacking knowledge (as part of teacher training).

## APPLYING THE ANALYSIS TO PRACTICE

Here we present a brief contextualisation, description, and subsequent analysis, of an episode previously considered by Maria as part of her lesson image. Prior to this episode the students had performed various activities for measuring the area of a surface (table top) using triangles, rectangles and squares with a ratio of 2 and 4 between them (there were not enough units available to cover the surface because Maria intended to use the task to remind students of the rectangular model of multiplication). The pupils measured the sides of one of these squares after which the teacher told them that "because it had these special measurements", it was known as a square decimetre.

### Lines      Transcription

878      (T writes on board:  $1dm^2$ )

879      T    So why is it called a square decimetre?

880      (Points to what she had written on the board) It's a square decimetre.

881      S    A 'two'?!

---

<sup>89</sup> These volunteers were part of a group of 14 teachers participating in a National (Portuguese) Program for Continuous Training in Mathematics for Primary School Teachers (PCTM). For more information concerning such PCTM see Ribeiro and Martins (2009).

- 882 T (S holds up one of the squares with sides of a decimetre)  
 883 That is, this is a geometric shape is a decimetre ... is a square with  
 884 what length sides?  
 885 S A decimetre.  
 886 T With sides of a decimetre.

Illustration 1 – Transcript of an episode were the teacher aims to present how to write  $1dm^2$  (T: Teacher; S: student)

Later in the lesson a second student again questioned the teacher about the reason for that "way of writing it", suggesting that it would be better to put a square "up there" (instead of a '2'). Maria answered as before, adding that they were always "written" in this way. This environment informs us about some of the sub-domains of the MKT, as well as some gaps in these components, but also provides access to information regarding beliefs about school mathematics. (Following Climent (2005), Maria reveals beliefs concerning such school mathematics to coincide precisely with what the textbook says; it is precise and polished, and has the objective of equipping pupils with the basic skills for everyday life.)

Concerning subject matter knowledge, Maria demonstrates that she knows how to write a square decimetre ( $1dm^2$ ), an item of knowledge pertaining to CCK. But despite having given thought to the situation previously, she had not anticipated the question raised by the student (line 881) because she was not in possession herself of the knowledge that might have made her ponder the convention – beyond assimilating it as one more "rule". It reveals an understanding of the contents themselves, but without questioning their origin and mathematical explanation.

Maria reveals a lack of SCK in that she was unable to explain why units of area are written with a '2' in superscript, conflating the exponential notation (for powers) with the geometric shape for measuring area (a square of 1dm). She showed that she lacked the knowledge that would have allowed her to give a mathematically valid explanation for the role the symbol takes in this position.<sup>90</sup> This association of the two notations (with natural base and exponent) might result in students assuming, later in their school life when they cover the topic (on the curriculum for the 2nd cycle (Ponte, Serrazina, Guimarães, Breda, Guimarães Sousa Menezes, Martins & Oliveira, 2007)) that powers correspond to analogous concepts/representations, impeding a full understanding of this subject content.

Such gaps in terms of MKT frequently prevented her, among other things, from providing appropriate and rich mathematical explanations in order to make the

---

<sup>90</sup> This lack of knowledge was acknowledged by Maria in the discussions after class, saying that she had *never questioned herself about that, she supposed it was just another math convention.*

contents comprehensible to students. She provided examples which did not allow the mathematical exploration of doubts raised by the students – it revealed not knowing herself the reason for the use of the ‘2’, so that she was unable to distinguish its use for dimensions from that of exponent, which could lead to students then assuming that  $dm^2$  is obtained by performing  $dm \times dm$  (as if it were two variables).

These shortcomings led us to reflect on what mathematics underpins a mathematically adequate and comprehensible explanation for students in year 4. The explanation of the role/meaning of the superscript ‘2’ would perhaps be much simpler (from a mathematical perspective) if the question were posed by students in higher year groups to their mathematics teacher. This leads us to agree with Rowland’s (2009) observation that much elementary mathematics teaching is ‘difficult’ compared with teaching in the secondary grades and beyond, because the very concepts being taught lie somewhere beneath our conscious awareness, and our ability to pedagogically analyse in useful ways. It is therefore also difficult to provide a more detailed specification of the distinct aspects of mathematical knowledge that could be included in the specific and necessary knowledge required by the teacher (in terms of knowledge content such as PCK).

### **SOME FINAL NOTES**

The analysis we undertook allowed us to discern some aspects of the teacher's MKT and, from there, look to different approaches and perspectives for facilitating improvement (with respect to this teacher in particular, but also to teachers generally during the course of their training). The identification of sub-domains in the conceptualization of MKT served as a starting point for a fuller discussion of the mathematics involved in this specific situation, and/or what should underlie this for optimal learning – this was one of the reasons why we adopted this conceptualization and not the MfT or the KQ. Hence, our approach to this teacher's knowledge took a general form, without intending to suggest the supremacy of one dimension over another, but rather considering (clearly) all sub-domains interconnected and interdependent.

The MKT revealed by Maria, as with the shortcomings identified, are applicable only to each individual situation and cannot be extrapolated to other contexts (Delaney, Ball, Hill, Schilling & Zopf, 2008). However, this information (and knowledge of gaps) allows us to focus attention on mathematically critical hypothetical situations for teachers (current or future) which could lead to poor (or even inaccurate) learning by the students and hence limit the opportunities for learning provided in the classroom (Hiebert & Grouws, 2007). Even in those situations in which Maria shows a lack of knowledge, the constituents of the dimensions of the MKT are in harmony (in the sense understood by Potari and Jaworski (2002)) even though these cases correspond to instances in which the students' learning may not reflect correct

mathematical constructions/concepts, being based on somewhat incorrect assumptions on the part of the teacher.

The analysis and study of this teacher's MKT (through her practice) aimed chiefly to provide a richer understanding of certain critical features which emerge from practice, as a means of contributing to the improvement of this practice, and not merely to identify potential gaps. So that in future studies the analysis does not remain at the level of identification/attribution (or is applied simply to highlight teachers' lack of "pure" mathematical knowledge, which in this conceptualisation concerns only CCK), it is necessary that the practice and specific situations identified be subsequently discussed with the teachers, so that they can reflect on them and convert them to mathematically correct ones (considering the various dimensions involved).

Initial and in-service training could also benefit from this system of analysis as a bridge between theory and practice, hence promoting a dialogue based on a common language and a shared understanding.

We consider it essential that training should be based on practice and in particular on any gaps in knowledge which come to light in this practice, with the aim of enabling teachers to respond "with understanding" to all situations that come up in their day-to-day work. This point of departure is important, too, as we believe that learning is circular – we teach what we are, what we know, and how we know it. Each of us learns by facing situations designed to stimulate this learning. So despite being constantly faced with different situations demanding our immediate response, it is only through "outside" help that it is possible to focus attention on what we don't know (some gap in our knowledge brought to light) and we are able to learn.

### **Acknowledgements:**

This paper has been partially supported by the Portuguese Foundation for Science and Technology (FCT).

This paper forms part of the research project "*Mathematics knowledge for teaching with respect to problem solving and reasoning*" (EDU2009-09789), Directorate General for Research and Management of National Plan I + D + i. Ministry of Science and Innovation of Spain.

### **REFERENCES**

- Ball, D. L. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.
- Ball, D. L., Thames, M. H. & Phelps, G. (2008). Content knowledge for teaching: what makes it special? *Journal of Teacher Education*, 59(5), 389-407.

- Charalambous, C. Y. (2008). Mathematical knowledge for teaching and the unfolding of tasks in mathematics lessons: Integrating two lines of research. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds.), *Proceedings of the 32<sup>nd</sup> Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281-288). Morelia, Mexico: PME.
- Climent, N. (2005). *El desarrollo profesional del maestro de Primaria respecto de la enseñanza de la matemática. Un estudio de caso*. (Michigan: Proquest Michigan University. [www.proquest.co.uk](http://www.proquest.co.uk)).
- Davis, B. & Renert, M. (2009). Mathematics-for-teaching as shared dynamic participation. *For the learning of mathematics*, 29(3), 37-43.
- Davis, B. & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, 61, 293-319.
- Delaney, S., Ball, D. L., Hill, H. C., Schilling, S. G. & Zopf, D. (2008). Mathematical Knowledge for teaching: adapting U. S. measures for use in Ireland. *Journal of Mathematics Teacher Education*, 11(3), 171-197.
- Hiebert, J. & Grouws, D. (2007). The effects of classroom mathematics teaching on students' learning. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 371-404): NCTM: Information Age Publishing.
- Hill, H. C., Rowan, B. & Ball, D. L. (2005). Effects of teachers' mathematics knowledge for teaching on student achievement. *American Education Research Journal*, 42(2), 371-406.
- Ponte, J. P., Serrazina, L., Guimarães, H., Breda, A., Guimarães, F., Sousa, H., Menezes, L., Martins, M. E. & Oliveira, P. (2007). *Programa de Matemática do Ensino Básico*. Lisboa: ME – DGIDC.
- Potari, D. & Jaworski, B. (2002). Tackling complexity in mathematics teaching development: using the teaching triad as a tool for reflection and analysis. *Journal for Research in Mathematics Education*, 5, 351-380.
- Ribeiro, C. M. (accepted). Uma abordagem aos números decimais e suas operações no primeiro ciclo. A importância de uma "eficaz navegação" entre representações. *Educação e Pesquisa*.
- Ribeiro, C. M. & Martins, C. (2009). The impact of a professional development program in teacher's practice. In J. Novotná & H. Moraová (Eds.), *International Symposium Elementary Maths Teaching (SEMT 09)* (pp. 212-219). Prague, Czech Republic: Charles University, Faculty of Education
- Rowland, T. (2009). Foundation knowledge for teaching: contrasting elementary and secondary mathematics In V. D. Guerrier, S. Soury-Lavergne & F. Arzarello

(Eds.), *Proceedings of the 6th Conference of the European Society for Research in Mathematics Education (CERME6)* (pp. 1841-1849). Lyon, France.

- Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, 8, 255-281.
- Rowland, T., Thwaites, A. & Huckstep, P. (2003). *Novices' choice of examples in the teaching of elementary mathematics*. Paper presented at the The Mathematics Education into the 21st Century Project, The Decidable and the Undecidable in Mathematics Education, Brno, Czech Republic.
- Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15 (2), 4-14.
- Stylianides, G. J. & Stylianides, A. J. (2010). Mathematics for teaching: A form of applied mathematics. *Teaching and Teacher Education*, 26, 161-172.
- Wilson, S., Shulman, L. & Richert, A. (1987). 150 different ways of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers thinking* (pp. 104-124). London: Cassel.



# SECONDARY MATHEMATICS TEACHERS' CONTENT KNOWLEDGE: THE CASE OF HEIDI

Tim Rowland, Libby Jared and Anne Thwaites

University of Cambridge, UK

*The Knowledge Quartet (KQ) is a theoretical framework for the analysis and development of mathematics teaching. It focuses attention on classroom situations when the teacher's knowledge of mathematics and of mathematics-related professional knowledge comes to the fore. This focus for analysis and reflection is a stimulus to the enhancement of teacher knowledge and the improvement of teaching. The KQ has been developed in empirical research since 2002 in the context of elementary mathematics teaching: the purpose of this paper is to demonstrate the application of the KQ in a secondary school mathematics context.*

**Keywords:** teacher knowledge, secondary, novice teacher, Knowledge Quartet

## INTRODUCTION

A programme of research at the University of Cambridge (SKIMA: subject knowledge in mathematics) from 2002 to the present has investigated the mathematics content knowledge of novice teachers, and the ways that this knowledge becomes visible in planning for teaching and within classroom instruction itself. Aspects of this research programme have been reported at each CERME conference since 2003 (e.g. Huckstep, Rowland & Thwaites, 2006). A significant outcome has been the identification of a framework for the observation, analysis and development of mathematics teaching, with a focus on the contribution of the teacher's mathematical content knowledge. The framework in question, called the *Knowledge Quartet*, categorises events in mathematics lessons with particular reference to the subject matter being taught, and the mathematics-related knowledge that teachers bring to bear on their work in classrooms, as opposed to more generic features of the lesson. While Shulman's distinction between subject matter knowledge and pedagogical knowledge underpins this 'theory' of mathematics teaching, the Knowledge Quartet (KQ) is more interested in the situations in which such knowledge comes into play than in categorising the different 'kinds' of mathematics teacher knowledge. The origins of the KQ were in observations of elementary mathematics teaching, and grounded theory methodology (Glaser and Strauss, 1967), in the context of one-year graduate elementary teacher preparation.

### The Knowledge Quartet

According to the KQ, the knowledge and beliefs evidenced in mathematics teaching are conceived in four categories, or dimensions, named *foundation*, *transformation*, *connection* and *contingency*. The application of subject knowledge in the classroom always rests on *foundation* knowledge. This first category consists of knowledge and

understanding of mathematics per se and of mathematics-specific pedagogy, as well as beliefs concerning the nature of mathematics, the purposes of mathematics education, and the conditions under which students will best learn mathematics. The second category, *transformation*, concerns the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations. The third category, *connection*, includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. The final category, *contingency*, is the ability to make cogent, reasoned and well-informed responses to unanticipated and unplanned events. This conceptualisation of each of the four dimensions of the KQ is the synthesis of a set of codes which emerged from grounded analysis of the primary mathematics classroom data. Each dimension is composed of a small number of subcategories that we judged, after extended discussion, to be of the same or a similar nature. Table 1 shows the codes contributing to each of the four dimensions.

Dimension	Contributory codes
<i>Foundation:</i>	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology.
<i>Transformation:</i>	choice and use of examples; choice and use of representation; use of instructional materials; teacher demonstration (to explain a procedure).
<i>Connection:</i>	anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness.
<i>Contingency:</i>	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction.

**Table 1: The Knowledge Quartet – dimensions and contributory codes**

Further details are given e.g. in, Huckstep, Rowland & Thwaites (2006), and in Rowland, Turner, Thwaites & Huckstep (2009). We emphasise that the conceptualisation of the KQ has been refined, and the constituent codes enhanced, in an iterative response since 2003 to additional classroom data, in the process of application. This paper is best understood in the context of that process of theory evolution.

### **Rationale**

From time to time questions have arisen about the adequacy and relevance of the KQ to analyses of mathematics teaching in secondary schools, and even to subjects other than mathematics. While we could not comment on the second of these questions, we believed that it would be meaningful and productive to test the application of the KQ to mathematics teaching beyond the primary years, and began to do so systematically in 2010. We began unsure about how well the KQ might ‘fit’ secondary teaching –

and the work of trainee secondary mathematics teachers in particular – on account of certain characteristics of both the teachers and the subject matter being taught, when compared with their primary mathematics counterparts. In particular, these secondary trainees are all *specialist* mathematics teachers, with evidence of recent success in their own study of mathematics, and their teaching is supported by mathematics specialists throughout their *practicum* placements. By contrast, generalist primary teachers, who have typically specialised in the arts and humanities in their own education, often lack confidence in their own mathematical ability (e.g. Green & Ollerton, 1999). From the mathematical point of view, the subject matter under consideration in secondary classrooms becomes significantly more abstract and complex than that in the primary school. As Potari and her colleagues indicated at CERME5, “teachers’ knowledge in upper secondary or higher education has a special meaning as the mathematical knowledge becomes more multifaceted and the integration of mathematics and pedagogy is more difficult to be achieved” (Potari et al., 2007, p. 1955). The purpose of this paper is to test, and to illustrate, the application of the KQ as an analytical and developmental tool in the context of novice secondary mathematics teaching. One could expect that the secondary context could necessitate annexing additional codes to those which emerged earlier in the analysis of primary mathematics teaching (Table 1).

## **METHODS**

### **UK teacher education context**

In the UK, the majority of pre-service secondary teacher education takes place under the auspices of university education departments. Trainee mathematics teachers are required to have at least half of their undergraduate study in mathematics, and expected to have achieved at least an “upper second” class bachelor’s degree, before following a one-year, full-time course leading to a Postgraduate Certificate in Education (PGCE). In order to achieve a good theory/practice balance in the PGCE, for the last 20 years the programme has been conceived as a ‘partnership’ between the university and several collaborating schools, and two-thirds of the 36-week course is spent working in two schools under the guidance of mathematics specialist, school-based mentors.

### **Participants**

The project participants were three volunteer trainee teachers from one secondary mathematics PGCE cohort at our university. Their professional placements were in different schools, all within a half-hour commute from the university.

### **Data Collection**

The trainee participant in each school taught two ‘project’ lessons to the same class. These lessons took place in May, towards the end of the trainees’ second school-based placement, which had begun in January of the same year, so that the

participants were familiar with their schools and with the pupils in their classes. One or two members of the research team (the authors) observed and videotaped each lesson. One tripod-mounted camera, operated manually, was placed at the rear of the classroom. Sound recording was via a radio microphone worn by the trainee-teacher. Trainees were asked to provide a copy of their lesson plan, for reference in later analysis. As soon as possible after the lesson, the research team met to undertake preliminary analysis of the videotaped lesson, and to identify some key episodes in it by reference to the KQ framework. These were fragments, typically 5-10 minutes long, in which two or more of the four KQ dimensions were particularly salient, according to our preliminary analysis. Then, again with minimum delay, one team member met with the trainee to view a selection, typically two, of these episodes<sup>1</sup> from the lesson and to discuss them, in the spirit of stimulated-recall (Calderhead, 1981). These interview-discussions addressed some of the issues that had come to light in the earlier KQ-structured preliminary analysis of the lesson. An audio recording was made of this discussion, to be transcribed later. In some cases the observation, preliminary analysis and stimulated-recall interview all took place on the same day. In the case of Heidi, the trainee featured in this paper, the delay between observation and interview was nearer 20 days for both lessons, on account of her prior commitments and those of the researchers.

### **Data analysis**

The data analysis consisted mainly of fine-grained analyses of each of the lessons, both before and after the stimulated-recall interview, against the theoretical framework of the KQ. In this sense, in contrast with the earlier *SKIMA* research, analysis was primarily theory-driven as opposed to data-driven. Initially, we identified in the video-taped lessons aspects of trainees' actions in the classroom that could be construed to be informed by their mathematics content knowledge (including their pedagogical content knowledge). In addition, when possible, our interpretation of the trainees' mathematical and pedagogical purposes and intentions was further assisted by reference to their lesson plans and the post-lesson interviews. These actions were, where possible, coded in accordance with the KQ and its 20 constituent codes, thereby testing the adequacy of the theoretical framework, developed in the context of elementary mathematics teaching, in this secondary school context. Therefore, this research phase entailed 'theoretical sampling' (Glaser and Strauss, 1967), whereby the application of a theory has the potential to expose its shortcomings, laying it open to refinement, modification and possible improvement.

### **THE CASE OF HEIDI**

We come now to our case study. Heidi was, in many respects, a 'typical' secondary PGCE student, having come to the course direct from undergraduate study in Mathematics and Statistics at a well-regarded UK university. Her placement school was state-funded, for pupils aged 11-16. The school was 'comprehensive', providing

for some 1400 pupils across the attainment range. In keeping with almost all secondary schools in England, pupils were ‘setted’ by attainment in mathematics, with 10 or 11 sets in most years.

We offer Heidi’s lesson as a ‘case’ in the following sense: it is used to illustrate, and to test in the secondary context, how the KQ can be used to identify, for discussion, matters that arise from lesson observation, and to structure reflection on the lesson.

### **Heidi’s lesson**

This was Heidi’s second videotaped lesson. Her class was one of two parallel ‘top’ mathematics sets in Year 8 (pupil age 12-13), and these pupils would be expected to be successful both now and in the high-stakes public examinations in the years ahead. According to the observation notes, there were 30 pupils in the class, 17 boys and 13 girls. They were seated at tables facing an interactive white board (IWB)<sup>2</sup> located at the front of the room. Heidi stood alongside the IWB for the whole of this lesson, and the video camera was trained on her and/or the board. The objectives stated in Heidi’s lesson plan were to “Go over questions from their most recent test, and then introduce direct proportion”.

As soon as the pupils were settled at their tables, Heidi returned test papers to them, from a previous lesson, and proceeded to review selected test questions with the whole class. These questions included two on simultaneous linear equations. Several pupils offered solution methods, and these were noted on the IWB. Nearly 30 minutes of the 45-minute lesson had elapsed before Heidi moved on to the topic of direct proportion. She began by displaying images of three similar cuboids on the IWB: she explained that the cuboids were boxes, produced in the same factory, and that the dimensions were in the same proportions. The linear scale factor between the first and second cuboids was 2 [Heidi writes  $\times 2$ ], and the third was three times the linear dimensions of second [ $\times 3$ ]. Heidi identified one rectangular face, and asked what would happen to the area of this face as the dimensions increased. They calculated the areas, and three pupils made various conjectures about the relationship between them. The third of these said “I think it is that number [the linear scale factor] squared”. Heidi then introduced two straightforward direct proportion word problems, such as “6 tubes of toothpaste have a mass of 900g. What is the mass of 10 tubes?” Different approaches were offered and discussed.

### **THE KNOWLEDGE QUARTET: HEIDI’S LESSON**

Earlier, we introduced the four dimensions of the KQ, and gave a general account of the characteristics of each of them. We now offer our interpretation of some ways in which we have observed, or inferred *foundation*, *transformation*, *connection* and *contingency* (but not in that order) in Heidi’s second videotaped lesson. It will become apparent that many moments or episodes within a lesson can be understood in terms of two or more of the four dimensions. We also draw upon her lesson plan,

and upon her contributions to the post-lesson, stimulated-recall discussion with a researcher (Anne, in this case). This discussion had homed in on two fragments<sup>3</sup> of the lesson that had been selected at the preliminary analysis session a few days earlier. The first of these fragments was Heidi's review of the first of the two test questions on simultaneous equations; the second was the introduction of the proportion topic using the IWB-images of the three cuboids.

### Transformation

Heidi had little or no influence regarding the **choice of examples** (a key component of this KQ dimension) in her test review, since the test had been set by a colleague. However, the stimulated-recall interview gave an opportunity and a motive for her to reflect on the test items. There had been two questions (7 and 8) on simultaneous equations, and the related pairs of equations were

$$\text{Q7: } 2x + 3y = 16, 2x + 5y = 20 \qquad \text{Q8: } 3b - 2c = 30, 2b + 5c = 1$$

In response to an interview question, Heidi thought the sequencing appropriate. In particular (regarding Q7) she said "They could do it the way it was", seeming to refer to the fact that one variable ( $x$ ) could be eliminated by subtraction, without the need for scaling either equation. In fact, the pupils' response to Heidi's invitation to offer solution methods suggested that this opportunity was not recognised, or not welcomed. The first volunteer, Matthew, proposed multiplying the first equation by 10, and the second by 6, suggesting a desire on his part to eliminate  $y$ , not  $x$ . (We shall consider Heidi's response to this under *Contingency*). Heidi was able to explain this in her answer to Anne's question "What if the  $y$ -coefficients were the same". Heidi's first response was "That would be less difficult because they tended to want to get rid of the  $y$ . I don't know why".

In fact, in this lesson segment, when eliminating one variable by adding or subtracting two equations, Heidi reminds the class several times about a 'rule', namely: if the signs are the same, then subtract; if they are different then add. Heidi suggests, later in the interview, that the pupils tend to want to make the  $y$ -coefficients equal, as Matthew did, because their signs are *explicit* in both equations. This can be seen, in both Q7 and Q8, where the coefficient of the first variable is positive in both equations, and the sign left implicit, whereas  $+$  or  $-$  is explicit in the coefficient of the second variable. This insight of Heidi's is typical of the way that focused reflection on the disciplinary *content* of mathematics teaching, structured by the KQ, has been found to provoke valuable insights on how to improve it (Turner & Rowland, 2010). Heidi's observation is that restricting the  $x$ -coefficients to positive values (and emphasising the 'rule') has somehow imposed unintended limitations on student solution methods, with a preference for eliminating  $y$  even when "they could do it the way it was" by eliminating  $x$ .

Turning to **choice of representation**, and Heidi's introduction of the direct proportion topic, we note that we misinterpreted Heidi's use of the three cuboids in

our preliminary lesson analysis. Her lesson plan included: “Discussion point: What happens to the area of the rectangular face as the dimensions increase? What happens to the volumes of the cuboids as the dimensions increase?”. We took this to mean that she intended to investigate the relationship between linear scale factor (between similar figures) and the scale factors for area and volume. In the event she was drawn into this topic, but this had not been her intention, and the subsequent word problems make this clear. In the event, there is discussion of the area of one rectangular face of the cuboid, and how its area increases as the cuboids grow larger: there is not time to consider the volumes. When probed about her choice of context for the introduction of the direct proportion topic, Heidi said that she had chosen the cuboids because it was “a nice visual” which contrasted with the “wordy” presentation of the other problems. In fact she drew on her IWB expertise by unveiling the images of the cuboids, one by one, as if drawing back electronic curtains. She responded to appreciative ‘Wow’s from the pupils with modesty, saying “It’s not all that impressive is it?” In the interview when asked whether she agreed that she could have done the work on area comparison with rectangles, she replied “You’re absolutely right, rectangles would be enough ... but I did like my box factory”. Here we see an example of trainees’ propensity to choose representations in mathematics teaching on the basis of their superficial attractiveness at the expense of their mathematical relevance (Turner, 2008). In this instance, the preference for these ‘visuals’ took Heidi into mathematical territory for which she was not mathematically prepared (see *Contingency*).

### Contingency

Analysis of this dimension of the KQ in Heidi’s lesson intertwines with the component of *foundation* concerned with teachers’ beliefs about mathematics and mathematics teaching. Here, we begin by taking up the story of Matthew’s suggestion to solve Q7 by multiplying the first equation by 10, and the second by 6. Heidi responded to Matthew (**responding to students’ ideas**) with “Excellent, you could do that”, and talks it through (without writing on the board), saying that Matthew is trying to “make the number in front of the  $y$ , which we call the coefficient, the same”, so that both will be 30. She does point out that “You wouldn’t have to do quite as much timesing as that, quite big numbers, if you didn’t want to”, and there might be “other multiples” which could be used.

Given Heidi’s earlier comment that, with the equal  $x$ -coefficients, “they could do it the way it was”, Anne asked her why she had “run with” Matthew’s suggestion<sup>4</sup>. Heidi replied “Because it would work. You’re trying to find the lowest common denominator but it would work. Like adding fractions, it would work with any common multiple. I didn’t want him to think he was wrong”. This kind of openness to pupils’ suggestions, and ability to anticipate where they would lead, was very characteristic of Heidi’s teaching, and several examples of it can be found in our

data, although there are occasions when her prior expectations blinker her reading of events as they unfold.

Within the class discussion which followed Heidi's introduction of the three cuboids, the pupils calculated (in  $\text{cm}^2$ ) the areas of the rectangles with sides (respectively)  $2 \times 3$ ,  $4 \times 6$ ,  $12 \times 18$  (all cm) viz. 6, 24, 216 (in  $\text{cm}^2$ ). Heidi had annotated  $x^2$ ,  $x^3$ , as we noted earlier. The first pupil contribution about the relationship between the areas conjectured that cubes (unspecified) were involved. Heidi acknowledged this suggestion, but set it aside. Now, it just so happens that the third area is the cube of the first ( $6^3=216$ ). This is, admittedly, a coincidence, and we are in no position to know whether it is what the pupil had discovered. A second pupil suggested that the relationships were "timesed by 4 and timesed by 6". Heidi made it clear that she was not checking these calculations numerically ("I'm going to take your word for that"), recorded this second proposal on the IWB (writing  $x^4$  and  $x^6$ ) and said "So two times what this has been timesed by [*pointing to the linear scale factors*]. Good observations". This seemed to be the end of the matter, until a third pupil, Lucas, said "I think it is that number squared". Heidi paused, then changed the second factor to  $x^9$  and emphasised the squares.

Now, this length/area relationship in similar figures was not what Heidi had set out to teach, and it became clear at the interview that Heidi (unlike Lucas) did not know in advance about "that number squared". In the interview, the discussion proceeded:

Anne: Then you go on to areas. They give a range of options. Now, you take all these responses and give value to all of them. But this was different, in that two of these responses were not correct.

Heidi: I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority. In that case I had a bit of a brain freeze, I hadn't worked out how many times 24 goes into 216, but they're used to me putting up everything.

We see here, paradoxically, a situation – by no means the only one – in this secondary teaching data in which some subject-matter in the school curriculum lies outside the scope of the content knowledge of the trainee at that moment in time. This should come as no great surprise. For all their university education in mathematics, and their knowledge of topics such as analysis, abstract algebra and statistics, there remain facts from the secondary curriculum that they will have had no good reason to revisit since they left school. This is no disgrace, and they will have cause to remember soon enough. What is significant, however, exemplified by Heidi but more-or-less absent in our observations of primary mathematics classrooms, is a teacher with the confidence to negotiate and make sense of mathematical situations such as this (the length/area relationship) 'on the fly', as they arise.



We noted earlier Heidi's response to Matthew: "Excellent, you *could* do that", in a situation when his method differed from the one she had in mind. She used versions of this formula (praise, followed by an implied caution) on two other occasions in the lesson. In an earlier test item, it was given that £36 is 75% of some quantity ( $x$ ). Heidi had in mind solutions such as  $x = 36 \times \frac{100}{75}$ , but Adrian suggested finding a third of 36 and adding it to 36. Heidi responded "That's a perfectly acceptable way, Adrian. Yes, you *can* do it that way ...". Later, in the dog-walker proportionality problem (a dog walker walks 7 dogs in 2 days, how many dogs in 10 days; 5 days?) a pupil suggests a unitary approach (how many dogs in one day) some way into the discussion. Heidi responds "Lovely, you *can* do that", and in fact she subsequently emphasises the  $\times 3.5$  scaling. These responses suggest a tension in Heidi's mind, when **responding to students' ideas**, between acknowledging and valuing flexible, idiosyncratic solution methods and promoting standard methods that will 'work' for them now, and in the high-stakes tests to come. This tension is probably not lost on her pupils either, and is conveyed in her language, in the modal ambiguity between possibility and permission ('can' and 'could').

## Foundation

This lesson does raise a few issues about Heidi's content knowledge that might be brought to her attention, and some of them were raised in the interview. Briefly, these include: her **use of mathematical terminology**, which is either very careful and correct (e.g. 'coefficient'), or quite the opposite (e.g. 'timesing'); her lack of fluency and efficiency in mental calculation, such that she did not question the suggestion that  $6 \times 24 = 216$  herself in the cuboids situation: on occasion it appeared that she was puzzled by some of the pupils' mental calculations; thirdly, she was not aware of the length/area/volume scale-factor relationships referred to earlier.

But, after many hours spent scrutinising the recording of this lesson, and that of the post-lesson interview, our lasting impression relates to the **beliefs component** of the *Foundation* dimension. In particular Heidi's beliefs about her role in the classroom in bringing pupils' ideas and solution strategies into the light, even – as we remarked earlier – when she believed that 'her way' would, in some sense, be better. As she told Anne, "I want to take everyone's ideas on board. When you do put something on the board they correct each other rather than me being the authority". Her perception of this aspect of her role, as teacher, and the possibility of the pupils themselves contributing to pupil learning, is resonant of various constructivist and fallibilist manifestos. Balacheff, for example, advised that "[the] transfer of the responsibility for truth from teacher to pupils must occur in order to allow the construction of meaning" (1990, p. 259), and identified classroom discussion as a context in which this transfer can take place. Heidi constantly assists this 'letting go' by acknowledging pupils' suggestions, and making them available for scrutiny by

writing them on the board. Occasionally she finds herself in deep water as a consequence, but she never seems to doubt her [mathematical] ability to stay afloat.

### Connection

We identified a few events in this lesson under *connection*. For example, Heidi's introduction to direct proportionality with the cuboids seemed quite unrelated to the word problems which followed. In any case, the rather diverse objectives for the lesson were likely to make it somewhat fragmented, and we omit further analysis of this KQ dimension from the present narrative.

### CONCLUSION

The purpose of this phase of our *SKIMA* research was to test the 'fit' of the Knowledge Quartet to secondary mathematics teaching. The indicative analysis of Heidi's second lesson in this paper indicates the potential of KQ as an analytical and (potentially) developmental tool in the context of novice secondary mathematics teaching. In this lesson, there were no moments or events, in which Heidi's mathematical content knowledge became a significant and/or influential factor in the proceedings, that could not be accommodated by one or more of the four dimensions of the KQ and the existing 20 codes. There are, however, such events in the data, as a whole, that may cause us later to want to supplement the codes within existing KQ dimensions. For example, the existing four *transformation* codes might not adequately capture the kinds of extended explanations, or the imaginative task design, that we saw in some other lessons. In other cases, such as Heidi's commitment to valuing student ideas and conjectures, the difference between these data and those from our novice primary teachers is more one of degree than of kind, from a KQ perspective. In our role as mathematics teacher educators, our analysis (as researchers) of the six lessons taught by these three volunteer participants now encourages us to pilot the use of the KQ as a developmental framework for the observation and review of lessons taught by secondary PGCE trainees during their school-based placements. This, in turn, will create yet more opportunities for testing and refining the KQ in the field.

### Notes

<sup>1</sup> A DVD of the full lesson was given to the trainee soon afterwards, as a token of our appreciation, but their reflections on viewing this DVD in their own time are not part of our data.

<sup>2</sup> Interactive whiteboards, with associated projection technology, are now more-or-less universal in secondary classrooms in England.

<sup>3</sup> The two corresponding video clips were each about four and a half minutes.

<sup>4</sup> Our earlier KQ research had identified three kinds of responses to unexpected ideas and suggestions from pupils: to ignore, to acknowledge but put aside, and to acknowledge and incorporate.

## REFERENCES

- Balacheff, N. (1990) Towards a problématique for research on mathematics teaching. *Journal for Research in Mathematics Education* 21(4), 258-272.
- Calderhead, J. (1981). Stimulated recall: a method for research on teaching. *British Journal of Educational Psychology* 51(2), 211-217.
- Glaser, B. G. & Strauss, A. L. (1967) *The Discovery of Grounded Theory: Strategies for Qualitative Research*. New York: Aldine de Gruyter.
- Green, S., & Ollerton, M. (1999) Mathematics anxiety amongst primary QTS students. *Proceedings of the Conference of the British Society for Research in Learning Mathematics held at St Martin's College, Lancaster*, 43-48
- Huckstep, P., Rowland, T. & Thwaites, A. (2006). The Knowledge Quartet: Considering Chloe. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1568-1578). Barcelona, Spain: FUNDEMI IQS, Universitat Ramon Llull. (Compact Disk)
- Potari, D., Zachariades, T., Christou, C., Kyriazis, G. & Pitta-Pantazi, D. (2007). Teachers' mathematical knowledge and pedagogical practices in the teaching of derivative. In D. Pitta-Pantazi and G. Philippou (Eds.) *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 1955-2004). Nicosia, Cyprus: University of Cyprus. (Compact Disk).
- Rowland, T., Turner, F., Thwaites, A. and Huckstep, P. (2009) *Developing Primary Mathematics Teaching: Reflecting on practice with the Knowledge Quartet*. London: Sage.
- Turner, F. (2008) Beginning elementary teachers' use of representations in mathematics teaching. *Research in Mathematics Education* 10(2), 209-11.
- Turner, F. & Rowland, T. (2010) The Knowledge Quartet as an organising framework for developing and deepening teachers' mathematics knowledge. In T. Rowland & K. Ruthven (Eds.) *Mathematical Knowledge in Teaching* (pp.195-212). New York: Springer.

# **PRE-SERVICE TEACHERS LEARNING TO ASSESS MATHEMATICAL COMPETENCIES**

Norma Rubio\*, Vicenç Font\*\*, Joaquim Giménez\*\*, Uldarico Malaspina\*

\*Pontificia Universidad Católica del Perú, \*\*Universitat de Barcelona

*This study investigates the professional competence of a group of pre-service secondary-school mathematics teachers with respect to assessing the mathematical competencies set out in the PISA 2003 report. It considers only those theoretical constructs proposed by this report. The research is descriptive in nature and includes a case study. The results were: 1) pre-service teachers do not recognize the level of complexity (reproduction, connections or reflection) of the competencies needed to solve a given problem; and 2) neither do they recognize when it comes to assessing the mathematical competencies that can be inferred from the solution given to a problem.*

Key words: pre-service teachers, mathematical competencies, assessment, PISA 2003, professional competency

## **INTRODUCTION**

The competencies set out in the PISA 2003 report are inspiring competency-based curricula in several countries. An inherent problem with this type of curriculum is how to ensure that teachers have the professional competency required to assess the mathematical competencies established by such curricula. This, in turn, is related to the problem of determining the mathematical and educational knowledge which pre-service teachers need to acquire (Ball, Lubienski & Mewborn, 2001; Hill, Schilling & Ball, 2004; Sowder, 2007; Hill, Ball & Schilling, 2008; Wood, 2008; Font, Rubio, Giménez and Planas, 2009; Godino, 2009).

Assessment is the process of obtaining information that is used to make educational decisions about students, to give feedback to the student about his or her progress, strengths, and weaknesses. The professional competency about knowing how to assess the students is considered in recent literature of mathematics education for many reasons. For instance, the impact of evaluations on the work of students is very important (Romainville, 2002). Researchers, such as Brown and Coles (2000), relate this competency to the need for developing a teacher's ability to make complex decisions, and others talk about the need to develop professional formative assessment competency (Hodgen, 2007).

The present study forms part of a more general research project (EDUC2009-08120: "Assessment and development of professional competencies in mathematics and how to teach them during the initial training of secondary and high-school teachers"), which aims to determine how to ensure that pre-service secondary-school teachers develop, during their initial training, the professional competency required to assess

the mathematical competencies of pupils and their metacognitive support (Niss, 2002).

The main aim of this paper is to determine the initial competency level of pre-service secondary-school teachers with respect to assessing the mathematical competencies set out in the PISA 2003 report when using their proposed theoretical framework. As a result, the research presented here is descriptive in nature. The regional government of Catalonia (Spain) had just approved a proposal for a competency-based curriculum that differs only slightly from the PISA 2003 competencies. We therefore opted to use the PISA 2003 competencies as the benchmark for this study.

The starting hypothesis was that pre-service teachers, who have mathematical but not professional competency, will not find it easy to assess mathematical competencies purely on the basis of the information provided by PISA 2003 tests, i.e. using the constructs ‘competency clusters’ (which can be used to determine three levels of complexity: reproduction, connections and reflection) and the list of ‘competencies and their components’ set out in the PISA 2003 report (OECD, 2003).

PISA 2003 considers eight competencies (thinking and reasoning, argumentation, communication, modelling, problem posing and solving, representation, using symbolic, formal and technical language and operations, use of aids and tools), which, taken together, can be seen as constituting comprehensive mathematical competency. Each of these competencies can be possessed at different levels of mastery (OCDE, 2003). In order to assess students’ capabilities, as well as their strengths and weaknesses, PISA 2003 also uses the construct competency clusters (reproduction, connection, reflection). Such constructs had been used to build tasks for summative assessment of students’ mathematical competences. Nevertheless, such a framework does not consider formative assessment of competences as explicitly needed for several authors (Hodgen, 2007). Thus, competences manifested by the students and the degree of intensity were not indicated in a way possible to permit regulated feedback. Therefore, the teacher needs to have such artifacts in a way to promote and develop mathematical competence (Wiliam, 2007).

In our theoretical perspective, we assume that the development of teacher competence in the analysis of mathematical objects and processes activated in mathematical practices is a necessary step to develop the expertise needed to assess the mathematical competencies of students. According to the theoretical model proposed by an onto-semiotic approach (Font, Planas & Godino, 2010), the competence in the analysis of mathematical objects and processes is a component of didactical analysis for instructive processes to be developed for pre-service teachers.

## **DATA COLLECTION**

To identify the initial level of prospective teachers, we observed and analyzed two classes given in the Faculty of Mathematics of the University of Barcelona as part of the module ‘Teaching Mathematics’. Both classes addressed the topic ‘PISA 2003

competencies'. With the course tutor's agreement, the second class was modified slightly compared to what had been taught in previous academic years. The class content included an introduction to the PISA 2003 report by the tutor and individual work by students. The first class involved information about and reflection upon the 'competency clusters' (levels of complexity) and the list of 'competencies and their components' from the PISA 2003 report. Some of the problems proposed in the PISA tests were used as examples. In the second class, students responded to a questionnaire designed to assess their initial competence in assessing mathematical competencies.

The 22 students who participated in these classes were in their final year of a mathematics degree. This ensured that they would have mathematical competence but very little competence in assessing mathematical competencies, due to their lack of previous training in teaching mathematics. These final-year students from the Faculty of Mathematics were aspiring to be secondary-school teachers and were studying their first module in 'Teaching Mathematics'. The two classes took place during the third week of the academic year, which meant that students had already discussed and analyzed (during the first four classes of the term) the secondary-level curriculum drawn up by the regional government of Catalonia (Spain). As mentioned earlier, this curriculum was based on competencies that were very similar to those set out in the PISA 2003 report and the principles and standards of the NCTM.

The data collected were as follows: 1) prospective teachers' responses to a questionnaire; 2) information obtained from interviews (pre- and post-) with the tutor imparting the 'Teaching Mathematics' module; and 3) responses to a second questionnaire in which they were asked about their responses to the first one. Before giving the first questionnaire, the pre-service teachers received information about PISA 2003 constructs (the 'overarching ideas' to which the problem referred; the types of situation and context; a description of the competency clusters that serve to characterize three levels of complexity; and a list of the eight main competencies, along with a description of each one).

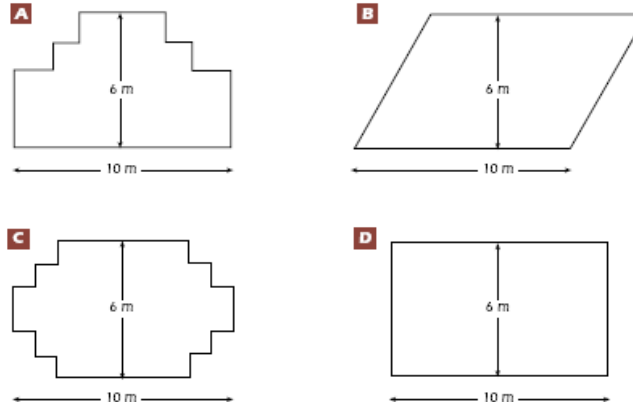
The first questionnaire was structured as follows: first let's observe several PISA problems, so called "CO2 levels" (OECD, 2009, p. 144-145) and "Internet relay chat" (OECD, 2009, p. 112), as well as a pupil's solution to a slightly modified version of the carpenter problem (Figure 1).

The modification merely involved asking for an open rather than a closed answer, in which students were also asked to justify their answer of the "carpenter" problem (OECD, 2009, p. 111), which is shown below (Figure 2). After solving the problems as secondary school students, two professional questions were proposed: (a) Classify the three problems according the type of clusters; (b) Which competences can you infer looking at the student's answers for the carpenter-adapted problem? To consolidate the data, the research team interviewed the tutor. After knowing their

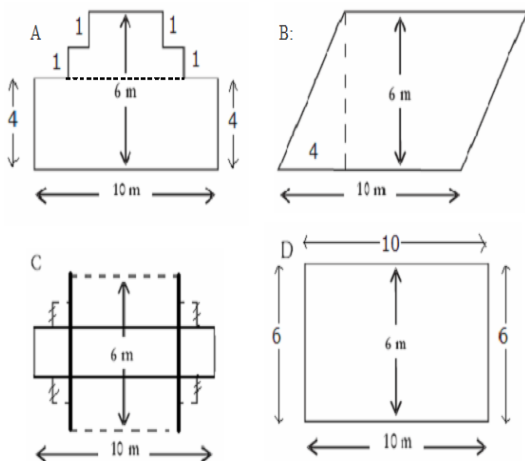
answers, in a subsequent class, a second questionnaire was proposed to the sample of students. Let us say now, as examples, only two of the questions proposed in this

**Figure 1: Statement of the carpenter problem (adapted)**

**CARPENTER (ADAPTED) PROBLEM:**  
A carpenter has 32 meters of timber and wants to make a border around a garden bed. He is considering the following designs for the garden bed. For each design, say whether the garden bed can be made with 32 meters of timber. You must answer yes or no, and explain why.



**Figure 2: Reproduction of a student's solution**



The different garden beds (A, B, C, D) that could be enclosed with 32 m of timber:

*A: First I calculated the area the perimeter of the main rectangle ( $4 \times 2 + 10 \times 2$ ), since although this figure has raised sections the horizontal lines complete the inner rectangle.*

*Then I added the four metres of vertical borders, which account for the breakdown of the rectangle as  $28 + 4 = 32$  m.*

*B: In this case it wouldn't be possible as there isn't enough for the diagonals... you would need more than the available timber to enclose  $\sqrt{6^2 + 4^2} = 8.4$  m = 7.2 m.*

*The 8.4 m added to the 10 of the length would give 37 ( $7.2 \times 2 + 10 \times 2$ ).*

*7.2 m*

*34 m*

*C: Here I'm going to apply the same system as in Figure A. In this case it would be with 2 different rectangles, so I take the longest one and add 10 m twice, and then from what's left I take the verticals, the 6 m, 32 m.*

*D: This is just adding the lengths of the sides (perimeter)  $6 + 6 + 10 + 10 = 32$  m.*

questionnaire: (a) If you remarked that one of the competences present in the solution was modelling, tell us why. (b) If you consider one of the competences (argumentation, communication) but not both of them, do you feel they are independent one from the other? Some other questions, proposed to understand students' justifications of previous answers, are not included in this text.

## ANALYSIS OF RESPONSES

The data were analyzed by means of a triangulation process. The first step involved two researchers analyzing the responses to the first questionnaire. Thus, the tutor participated in a second analysis with the research team, in order to gather his views on the initial analysis for refining the observations.

### Analysis of responses to the first questionnaire

With respect to the competency clusters, it was expected that pre-service teachers would not find these easy to apply. The analysis of their responses suggested that they did indeed have difficulties in distinguishing between levels. For example, 12 of the 22 pre-service teachers did not consider the adapted carpenter problem to be a problem of connections (even though it remains so).

Given the type of training received by students in the Faculty of Mathematics our initial hypothesis was that those students who classified the problem as one of reproduction did so basically because they considered it to be an easy problem (an exercise) and failed to put themselves in the pupil's shoes. The pre-service teachers were asked about this issue in the second questionnaire.

As regards competency in 'thinking and reasoning', it was expected that this would be mentioned in the response of all the pre-service teachers, since it would follow from the correct response of a pupil to the carpenter problem. In other words, in order to answer a problem correctly a pupil would have to be capable of 'thinking and reasoning'. The analysis revealed this to be the case, since pre-service teachers indicated the 'thinking and reasoning' competency at all three levels of complexity (reproduction, connections and reflection).

Another expected result was that the competencies 'argumentation' and 'communication' would both be present in the responses of pre-service teachers, because the solution analyzed was correctly argued and communicated. However, this expectation was not fulfilled, since a group of 11 pre-service teachers (50% of the total number of students) considered only one of these competencies. It thus appears that they did not regard these competencies as sharing a common ground. They were asked more about this issue in the second questionnaire.

It was also expected that the 'modelling' competency would appear in few responses of pre-service teachers, however this competency appeared in eight of them. Given this relatively high number of pre-service teachers who did consider this competency to be present it was decided to ask them more about the reasoning behind their response in the second questionnaire.

The 'problem-solving' competency was expected to feature in the majority of responses, unless the problem was regarded as trivial or an exercise. Of the 17 students who considered the problem on the level of connections or reflection only two failed to consider the 'problem-solving' competency. Of the five who considered



the problem on the level of reproduction, two said that the ‘problem-solving’ competency could be inferred from the pupil’s answer. These latter two students were asked about their responses in the second questionnaire.

The ‘representation’ competency was also expected to be present in almost all the pre-service teachers’ responses. However, given that a significant number of them (7) did not mention it, it was decided to ask them about the reasoning behind their response in the second questionnaire.

Given the type of training received by students in the Faculty of Mathematics, it was also expected that the competency ‘using symbolic language’ would not feature widely in the students’ responses, since the pupil’s solution they were asked to analyse contained a lot of written text in natural language and very few mathematical symbols. However, only three of the 22 students fulfilled this expectation so it was decided to ask them about the reasoning behind their responses in the second questionnaire.

The final competency ‘using aids and tools’ was not considered when evaluating the competencies that could be inferred from the pupil’s response to the carpenter problem, since only pen and paper were used to solve the problem.

#### Interview with the tutor

The tutor was in complete agreement regarding the ambiguity of the levels of complexity, and said he was not surprised that students had difficulties in applying them correctly; indeed, he added that he himself found it difficult. He also stated that the reason why some students considered the problem to be one of reproduction was because they regarded it as easy and failed to put themselves in the pupil’s shoes.

The tutor was not surprised that the ‘thinking and reasoning’ competency appeared in the response of all his students, and added that in his view it was not a very useful competency for discriminating between responses, since one could always consider it to be present, even if the answer given by a pupil was incorrect.

In contrast, the tutor was surprised that 11 of the 22 students considered that the pupil’s solution showed argumentation but not communication. He attributed this to the fact that for pre-service teachers, argumentation is associated with formalism, whereas they associate communication with oral or written expression.

The tutor agreed that the ‘modelling’ competency should not be considered as present in the solution to the carpenter problem. In his view, those students who had inferred its presence had simply responded without too much thought. He suggested asking them more about their reasoning. With respect to the ‘problem-solving’ competency the tutor’s view was that pre-service teachers did not consider this because they regarded the problem to be an exercise.

The tutor was also surprised that his students had not mentioned the competency ‘representation’, since this had come up in previous classes when discussing the secondary-school curriculum drawn up by the regional government of Catalonia.

The tutor agreed that the reason why pre-service teachers did not consider the competency ‘using symbolic language’ to be present was that the problem-solution analysed contained very few mathematical symbols. In his opinion, the profile of pre-service teachers was such that this idea was sometimes taken to extremes. i.e. if they saw a meaningless series of mathematical symbols they might say that “this is mathematics”, whereas when presented with graphs and verbal expressions they would say “this isn’t mathematics”.

The tutor was told that the present research sought to corroborate that a pre-service teacher, with good mathematical skills but very little training in mathematics education, would find it hard to assess the mathematical competencies of pupils solely on the basis of the theoretical constructs set out in the PISA 2003 report. It was also pointed out to him that one possible objection to this hypothesis could be that the same students might perform better with these theoretical tools once they had gained more experience. However, the tutor was sceptical about this latter possibility, as he considered that tools other than the theoretical constructs set out in the PISA 2003 report were required.

#### Analysis of pre-service teachers’ responses to the second questionnaire

The pre-service teachers who regarded the problem as one of reproduction did so because they considered it to be a familiar and simple problem. Examples of their responses included: “It involves calculating the perimeter of different shapes, in other words, the reiteration of a known and familiar algorithm which only requires dealing with simple expressions and formulas”; “Because for the possible garden beds you always need to calculate the perimeter, a familiar formula”; “Because he uses the perimeter formula”; “It is conceptually very simple... a very general exercise”; “The problem is simple and can be solved with routine procedures”.

The pre-service teachers who considered that only the argumentation, not the communication, competency was present stated that this was because communication implies transmitting a message or information, whereas arguing requires robust explanation. Their responses included: “Communicating means transmitting information, whereas arguing implies giving more reasons why, it’s more than just transmitting something or making it known”; “Communicating means expressing something that you think, whereas arguing involves doing something that you propose and have to resolve”; “I don’t think the two are totally independent, because when you communicate you have to argue what is communicated. However, on this occasion good argumentation is considered more important”; “I don’t think they are independent of one another, because in order to argue it is very important to communicate beforehand”; “They are independent here, because the pupil explains in

his own way how he solved the problem, but he doesn't communicate". The pre-service teacher who inferred the presence of the communication but not the argumentation competency said that in his view "communicating knows how to transmit a message whereas arguing implies that the arguments used must be robust and not contradictory".

The pre-service teachers who considered that the 'modelling' competency was present based this view on the fact that the pupil translates figures into mathematical structures. Their responses included: "The solution to the problem presented here begins with the model of the garden bed that the pupil makes in order to solve the problem"; "The problem makes you translate the figures into a mathematical structure"; "Because the pupil uses a system to find the perimeter, in other words, he makes an algorithmic model"; "Because the figures in the problem can be broken down into parts or other more simple figures and this makes it easier to solve the problem".

The two pre-service teachers who considered the problem to be one of reproduction and felt that the pupil's solution illustrated the 'problem-solving' competency gave responses such as: "You need to propose what you can use and then find the solution"; "Solving the carpenter problem obviously requires the problem-solving competency, even if the answer is a mere reproduction".

The pre-service teachers who did not consider the 'representation' competency to be present said that in this problem there was nothing to represent. Their responses included: "The representation is already given in the problem statement, you only have to make calculations"; "To calculate the perimeter, which is necessary, you don't have to interpret the different representations"; "Nothing is represented here, and nothing is drawn".

The pre-service teachers who considered that the competency 'using symbolic language' was not present in the pupil's solution did so because there is no formalism (e.g., use of symbols, variables, algebraic operations) in the solution given. Their responses included: "I think there are several sentences that could have been written mathematically"; "There are only a few common operations"; "It is not necessary to use any algebraic or formal tool to solve the problem".

final considerations

This study has highlighted a number of preconceived ideas held by pre-service mathematics teachers that derive from their previous training. For example, in determining the level of complexity they take into account, above all, their personal appreciation of the problem's difficulty. Furthermore, they consider argumentation and communication to be two competencies without common ground. Finally, they do not contemplate the competency 'using symbolic, formal and technical language and operations' when graphs and natural language, for example, are present.

The pre-service teachers do not use the theoretical constructs properly as set out in the PISA 2003 report in order to determine the level of complexity of the competencies required by a given problem. Nor do they recognize when it comes to assessing each one of the competencies that can be inferred from the response given to a problem.

There are at least two possible explanations for these findings. One would be that the pre-service teachers need more practice in using these theoretical constructs before they reach a level of competence similar to that of the experts who evaluated the PISA 2003 tests (this is to be considered in future studies). The other would be that the problem lies not in the lack of practice but, rather, in the ambiguity of the theoretical constructs used in the PISA 2003 report (for example, the different ways of understanding competency; the relationship between the notions of ‘process’ and ‘competency’, which appear as closely related and, in some cases, are even used as analogous terms; the overlap between competencies, almost all of which have a common ground, and the fact that two of them (i.e., problem-solving and modelling) are more akin to ‘meta-competencies’).

Such analysis allows us to reflect (in the wide project) about the variables influencing development of professionals in assessing competence, by means of successive observations during the pre-service training process. We found that the question that remains to be answered is what theoretical tools would be required by pre-service teachers to enable them to be aware of assessing students’ competences in a formative way. We also found that (1) to reflect about assessing realistic mathematical projects formatively, and (2) the use of suitable criteria for analysing their own practices as teachers were two crucial tools for having indicators about their developing professional competences.

### Acknowledgment

The research work reported in this paper was carried out as part of the project: “Evaluation and development of professional competencies in mathematics education for pre-service teacher training for Secondary School”, EDU 2009-08120.

### References

- Ball, D., Lubienski, S. T. & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers’ mathematical knowledge, in V. Richardson (ed.), *Handbook of Research on Teaching* (pp.433-456), Washington, D.C., EEUU. American Educational Research Association.
- Brown, L. and Coles, A. (2000). Complex decision-making in the classroom: The teacher as an intuitive practitioner. In T. Atkinson and G. Claxton (eds.), *The intuitive practitioner: On the value of not always knowing what one is doing* (pp. 165-181). Buckingham: Open University Press.

- Font, V., Planas, N. y Godino, J. D. (2010). Modelo para el análisis didáctico en educación matemática. *Infancia y Aprendizaje*, 33 (1), 89-105.
- Font, V., Rubio, N., Giménez, J. & Planas, N. (2009). Competencias profesionales en el Máster de Profesorado de Secundaria, *UNO*, 51, 9-18.
- Godino, J. D. (2009). Categorías de Análisis de los conocimientos del Profesor de Matemáticas. *Unión*, 20, 13-31.
- Hill, H.C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105, 11-30.
- Hill, H. C., Ball, D. L. & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39, 372-400.
- Hodgen, J (2007) Formative assessment. Tools for transforming school mathematics towards a dialogic practice? In D. Pitta-Pantazi and G. Philippou (edss.), *Proceedings of Fifth Congress of the European Society for Research in Mathematics Education* (pp. 1886-1895). Larnaca, Cyprus: University of Cyprus.
- Niss, M. (2002). *Mathematical competencies and the learning of mathematics: the Danish KOM Project*. IMFUFA, Denmark: Roskilde University.
- OECD (2003). *The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Paris: OCDE.
- OECD (2009). *PISA Take the Test. Sample Questions from OECD's PISA Assessments*. Paris: OECD.
- Romainville, M. (2002). *L'évaluation des acquis des étudiants dans l'enseignement universitaire. Rapport établis à la demande du Haut Conseil de l'Évaluation de l'École*. Paris : Haut Conseil de l'Évaluation de l'École.
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F. K. Lester (ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 157-223). Charlotte, NC: National Council of Teachers of Mathematics.
- William, D. (2007) Keeping learning on track : Classroom assessment and the regulation of learning. In F Lester (Ed) *Second Handbook of research on mathematics teaching and learning* (pp 1051-1089). Charlotte, NC: Information Age Publishing.
- Wood, T. (Ed.) (2008). *The international handbook of mathematics teacher education*. Rotterdam: Sense Publishers.

# **MOVING BEYOND AN EVALUATIVE TEACHING MODE: THE CASE OF DIANA**

Rosa Antónia Tomás Ferreira

University of Porto, Portugal & CMUP

*In this paper, I describe and analyse a student teacher's evolving teaching modes throughout her year-long practicum, and within a modified teacher development experiment research design. I also discuss some factors that may influence the student teacher's anticipated developmental trajectory along three teaching modes.*

Keywords: student teaching, classroom interaction, teacher development experiment, teaching modes.

## **THE TEACHING MODES AND THE CONCEPTUAL FRAMEWORK**

It is widely accepted that the quality of teacher-student classroom communication plays an important role in students' learning. In this regard, teachers' questioning, listening, and responding approaches may be seen as underpinning the core of classroom communication. In fact, these three facets of teachers' practices have been suggested to characterize their pedagogical approaches and to reflect their beliefs about mathematics and its teaching and learning. Additionally, teachers' analyses and reflections about how they question, listen, and respond to their students have been shown to foster teachers' increasing awareness of their own beliefs and practices, thus helping them in improving their own teaching (e.g., Coles, 2001; Nicol, 1999).

Research has suggested that the student teaching experience of teacher education programs does not have a significant impact on aligning student teachers' beliefs and classroom practices with current recommendations for school mathematics; furthermore, student teaching has often failed to foster prospective teachers' reflective thinking and awareness of their own practices. These are cumbersome results for mathematics teacher educators, pushing for further inquiry. Based on existing research and on theoretical developments in the field of mathematics education, I built a conceptual framework (CF) that was used to conduct a study, whose major goal was to trace and understand how the teaching modes of a group of Portuguese secondary mathematics student teachers evolved over the course of their year-long student teaching practicum (Tomás Ferreira, 2005). Following a teacher development experiment research methodology (Simon, 2000), the study was designed to provoke teacher development and to describe and interpret that development. Yet, the CF itself was investigated for its adequacy for analyzing and interpreting classroom teaching and for informing the data collection and analysis procedures.

The teaching modes are comprised of teachers' inter-related classroom questioning, listening, and responding approaches. As described in an earlier paper (Tomás

Ferreira, 2007), the CF encompasses several strands, each one of them drawn from frameworks that emerged from or were used in prior research in the field: (1) teachers' *teaching modes*, by extending Davis's (1997) framework of listening modes to include Ainley's (1988) categories of teachers' questions, as well as teachers' different forms of responding to students (e.g., Nicol, 1999); (2) teachers' dominant patterns of classroom interaction (e.g., Bauersfeld, 1992; Cobb, Wood, Yackel, & McNeal, 1992; Voigt, 1985, 1995; Wood, 1998); (3) teachers' key beliefs about mathematics and its teaching and learning (e.g., Ernest, 1989); and (4) teachers' levels of reflective thinking (e.g., van Manen, 1977; Schön, 1983). Next, I briefly explain my understanding of these constructs.

Acknowledging the lack of consensus about the notion of reflection, my understanding of teachers' reflective thinking is teachers'

intentional engagement in thinking about their classroom practices with two main goals: (a) becoming aware of their actions in the classroom and of their key beliefs about mathematics ... against the perspectives on mathematics teaching and learning envisioned by current school mathematics reform movements; and (b) using those insights to improve their teaching ... and ... students' learning (Tomás Ferreira, 2005, p. 34).

The construct of teachers' beliefs does not gather consensus either amongst the research community in mathematics education. However, teachers' beliefs about mathematics and its teaching and learning shape their classroom practices (Thompson, 1992). Ernest's (1989) model of teachers' views of mathematics as a whole, their orientations towards mathematics learning, and their models of mathematics teaching resonates with van Manen's (1977) model of teachers' reflective thinking. The patterns of classroom interaction emerge "from the permanent interaction between teacher and students, as well as among students themselves" (Bauersfeld, 1992, p. 21). Research has identified several patterns of classroom interaction; in this research, I considered six of them. Though recognizing their differences, I grouped the patterns of classroom interaction that are typically centred on the teacher and called them *traditional patterns*: the IRE pattern (e.g., Cobb et al., 1992), the funnel pattern (e.g., Voigt, 1985, Wood, 1998), the elicitation pattern (Voigt, 1985), and the direct mathematization pattern (Voigt, 1995); the patterns of classroom interaction typically centred on the students' mathematical activity were grouped into *inquiry patterns*: the focusing and the discussion patterns (e.g., Voigt, 1995; Wood, 1998). To construct each strand of the CF, I chose existing frameworks in the literature that were fairly accessible to student teachers to work with. The use of a CF was an attempt to reduce and simplify the complexity of the classroom phenomena while emphasizing some specific aspects, which were at the core of the study.

The three teaching modes – evaluative, interpretive, and generative – were not considered in isolation but sharing qualitative characteristics amongst each other. Figures 1, 2, and 3 depict the anticipated relationships amongst all four strands of the

CF for each of the teaching modes. It is important to notice that the thicker the linking lines in the figures, the stronger the relationships amongst the constructs were likely to be. No *exact* correspondences were expected to exist. Despite being instrumental in the course of this research, the conceptual framework was a focus of inquiry for its adequacy for analysing and interpreting classroom teaching, as well as for its appropriateness for guiding the whole research design (Tomás Ferreira, 2005).

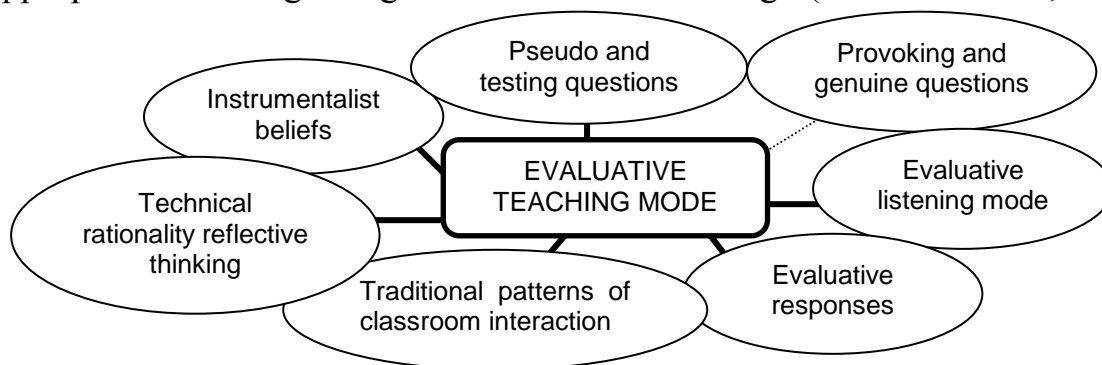


Figure 1. Evaluative teaching mode.

Some of the most distinctive characteristics of an *evaluative teaching mode* are asking students many pseudo and testing questions (Ainley, 1988) – to establish an acceptable behaviour and to find whether students answer correctly, – listening to them in an evaluative mode (Davis, 1997) – that is, looking for the *correct answers* they have in mind, – and responding to students to evaluate their answers since having students producing right answers is more valued than understanding students' underlying reasoning. Someone teaching predominantly in an evaluative teaching mode tends to closely follow lesson plans, not allowing for the *emergent* during the lessons. Communication is seen as a matter of speaking; yet, students' contributions to the classroom discourse are largely ignored. Traditional patterns of classroom interaction are the norm, such as the IRE pattern (Cobb et al., 1992) or the funnel pattern (Voigt, 1985; Wood, 1998); thus, the teacher owns the locus of authority. Typically, evaluative teachers hold instrumentalist beliefs about mathematics and its teaching and learning, seeing themselves as instructors (Ernest, 1989), and their reflective practices are very superficial, hardly ever allowing themselves to question their actions in the classroom – van Manen's (1977) technical rationality level.

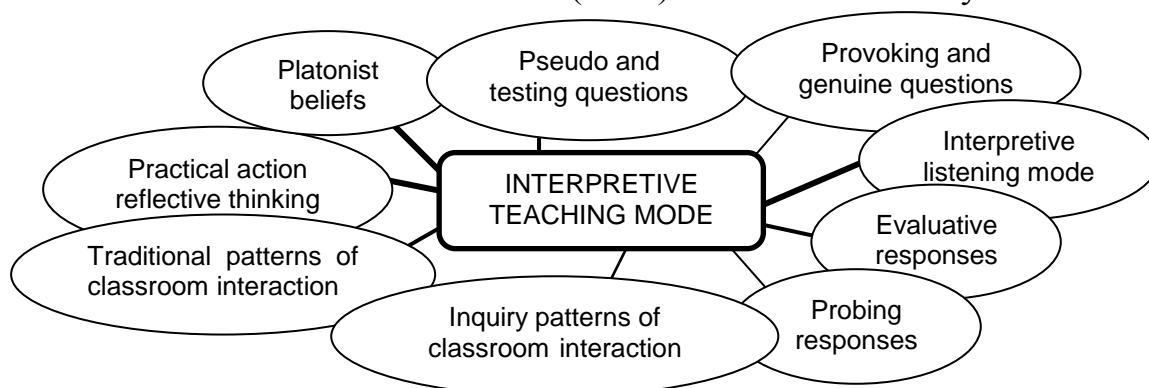


Figure 2. Interpretive teaching mode.



Teachers teaching in an *interpretive mode* tend to ask fewer testing questions than in an evaluative mode. In addition, they begin asking more genuine and provoking questions (Ainley, 1988), that is, questions that seek information and questions aimed at stimulating students' thinking and sense making. The interpretive listening mode (Davis, 1997) is now predominant, as teachers increase the opportunities for classroom interaction and discussion; yet, their responses to students are still evaluative in nature. Teaching is still much *textbook-driven*, though teachers tend to enrich classroom tasks by diversifying its nature. The “students' contributions to the discourse still do not have a significant impact on lesson unfolding, since despite some room for inquiry patterns of interaction, the traditional ones are clearly predominant” (Tomás Ferreira, 2007, p. 8). However, the locus of authority is mainly allocated to the teacher, whose Platonist typical beliefs about mathematics and its teaching and learning are related to his/her perception of self as an explainer (Ernest, 1989). Teachers teaching in an interpretive mode tend to focus their reflections on assessing the appropriateness of their teaching strategies in order to guide future practices, thus relying on van Manen's (1977) practical action level of reflective thinking. Hence, teachers predominantly reflect *on* action, in Schön's (1983) terms.

Within a *generative teaching mode*, to communicate is to participate, interpret, and negotiate meanings, involving all classroom members alike. Genuine and provoking questions (Ainley, 1988) are predominant, though there is room for other types of questions. Teachers teaching in a generative mode tend to listen to students in a hermeneutic listening mode (Davis, 1997), that is, accessing and assessing students' thinking in order to inform teaching. They tend to respond to students by stimulating discussions, probing answers, etc. Teaching is now a matter of responding in a flexible manner to constantly changing circumstances. Thus, the inquiry patterns of classroom interaction are predominant (though there is also room for other patterns), and the locus of authority is equally shared amongst all classroom members.

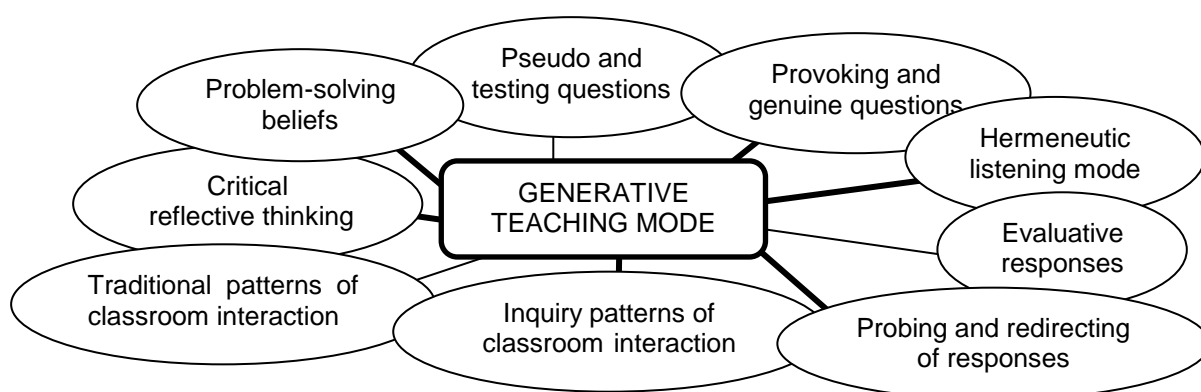


Figure 3. Generative teaching mode.

Generative teaching implies teachers' “revising of their own mathematical knowledge while exploring and constructing mathematical ideas with their students ... [and] questioning of their own practices and beliefs” (Tomás Ferreira, 2007, p. 9), perceiving themselves as facilitators of student learning. These teachers tend to hold

a problem-solving perspective on mathematics and its teaching and learning (Ernest, 1989), and to reflect at van Manen's (1977) critical level as they problematize the whole teaching in context, looking for reasons and consequences of their actions in the classroom. Hence, teachers often reflect both *in* and *on* action (Schön, 1983).

## RESEARCH DESIGN

The research reported here is part of a larger study (Tomás Ferreira, 2005) which followed an overall teacher development experiment (TDE) research design (Simon, 2000), involving a teacher-educator/researcher, myself, four Portuguese student teachers, their students, and their cooperating teacher. A TDE "can allow researchers to generate increasingly powerful schemes for thinking about the development of teachers in the context of teacher education opportunities" (Simon, 2000, p. 338). The teacher-educator/researcher "teaches courses for teachers, endeavours to understand the teachers' perspectives, identifies goals for teachers' learning, and plans course activities to promote learning in the direction of those goals" (Simon, Tzur, Heinz, Kinzel, & Smith, 2000, p. 581). The TDE design has two different components: a *whole-class teaching experiment* in the context of teacher education courses, and a *case study* approach of individual teachers in their own classrooms.

The teacher-educator/researcher engages in several interaction-reflection cycles (Simon, 2000). The *interaction phases* aim at promoting professional development based on the teacher-educator/researcher's "ideas of the current state ... [of the participant teachers' practices, and on the researcher's] current hypotheses about how development might proceed" (Simon & Tzur, 1999, p. 253). Relying on existing literature and personal experiences, and drawing on the CF presented, the participant student teachers were expected to begin their practicum enacting an evaluative teaching mode. The CF provided the hypothetical developmental trajectory I envisioned for the participants, guiding my interactions with them, as a teacher-educator, under the scope of the teaching experiment component of the whole research design. Thus, the interaction phases with the student teachers were aimed at stimulating their awareness and understanding of their own development, mainly in terms of their teaching modes and, specifically, concerning their questioning, listening, and responding approaches in the classroom.

The *reflection phases* (Simon, 2000) aim at the teacher-educator/researcher's analysis of the interactions with the participant teachers in which they engage in the context of the teaching experiment component of the overall research design. Again, using the CF, my reflections were focused on the participants' current teaching modes, as well as on the remaining three strands. Although the developmental trajectory envisioned for the participants had specific goals, the reflection phases of the interaction-reflection cycles ensured that other emergent issues would be addressed.

The context of this research was the student teaching phase of the mathematics teacher education program offered by a large public university, in a large urban area in northern Portugal. I worked with four student teachers, placed in the same school, interacting with them once or twice every week, for about six months, divided into four interaction-reflection cycles. I focused my attention on how the individual student teachers evolved in their teaching modes, and replaced the case study component of the TDE for the generation of *accounts of practice* (Simon & Tzur, 1999). Teachers' accounts of practice are characterized by the explanation of the teachers' perspectives from the researchers' perspectives, having two main goals: to describe teachers' current classroom practices, and to promote and sustain the development of those practices. In this research, the CF provided "the basis for considering current and potential development" (Simon & Tzur, 1999, p. 255) since it guided my interactions with the participants, focused my reflections about their development, and structured the generation of accounts of practice of the participant student teachers.

Each participant student teacher taught a 7<sup>th</sup> grade and an 11<sup>th</sup> grade class, sharing each class with another student teacher of the group. The sharing of classes was an innovative aspect of the placement school due to a sudden decrease in the pupil population, thus in the number of classes. The pairing of the student teachers was made by their cooperating teacher ensuring as much variability as possible, and it

... was not meant to stimulate team teaching in any aspect. Each pair of student teachers shared a class in terms that one of them was *the* teacher in charge of the class, while the other served as a secondary teacher in the classroom, as a supportive peer, not intervening in the lesson except for monitoring student behaviour ... (Tomás Ferreira, 2005, p. 137)

For each one of the four interaction-reflection cycles, data were collected through field notes of classroom observations, audiotape-recorded lessons and semi-structured interviews of distinct types (all transcribed and translated into English), and several documents such as lesson plans and written essays. The analysis was guided by the CF and had two approaches: (1) ongoing analysis occurred during and between the interaction-reflection cycles, providing foundation for planned and unplanned interactions, and allowing the modification of the participants' models of professional growth; and (2) retrospective analysis occurred after all data were collected, allowing the re-examination and reinterpretation of situations. The sharing of classrooms became a significant aspect in the generation of the accounts of practice. Next, I summarize Diana's case in terms of her evolving teaching modes, leaving the other dimensions of the CF and the analysis of the other participants to another paper.

### **THE CASE OF DIANA**

Diana always expressed her dislike for expository teaching, direct instruction. Her enthusiasm and commitment to the various activities proposed in the scope of the

teaching experiment component of the research design were probably two factors that pushed her to try to move beyond teaching in an evaluative mode from the beginning. She wanted to “give meaning to mathematics and to learning ... and to try to change the students’ heads ... It’s a huge challenge but I want to try” (October, 2002). However, over time, she experienced many tensions in her teaching modes. Diana shared a 7<sup>th</sup> grade class with Abel and an 11<sup>th</sup> grade class with Júlia. Table 1 summarizes Diana’s teaching modes throughout all four cycles of the study.

Table 1. Diana’s evolving teaching modes.

	First cycle	Second cycle	Third cycle	Fourth cycle
Teaching mode	Mostly an evaluative mode, though sometimes enacting an interpretive mode.	Mostly an evaluative mode when irritated with Abel’s role as secondary teacher or 7 <sup>th</sup> graders’ unproductive noise. Otherwise, tensions to enact an interpretive mode.	Evaluative mode with a few instances of an interpretive one when feeling more comfortable with the math and 11 <sup>th</sup> grade students collaborated in the lessons.	Evaluative mode with a few instances of an interpretive one when the students collaborated in the lessons.

Starting from the second cycle of the TDE, it became clear she was struggling to cope with Abel’s inappropriate behaviour as a secondary teacher in their shared 7<sup>th</sup> grade class. This was the most determinant factor for Diana’s tensions and difficulties in enacting in her classroom what she believed teaching should be. Abel frequently distracted the students with questions, comments, or repetitions of what Diana was explaining to the whole class. His quick movements around the classroom added a distracting factor, and his chatting with the students contributed to the widespread (unproductive) noisiness, which significantly irritated Diana:

... compare the role of Júlia in her 7<sup>th</sup> grade classroom with Abel’s ... I think he destabilizes because he gives too much confidence to the students! What am I going to do?! That is why I start getting enervated ... The students make much noise because of him and ... I start losing control ... It seems like he likes to have students almost kissing his feet ... But, I do not need that they ... venerate me. (March, 2002)

Despite her efforts to make him realize how he should act as a secondary teacher, Abel never questioned the effects of his actions on his partners’ teaching, seeming to always wanting to maintain his status as a *head* teacher. When his behaviour was not a problem, Diana did manage to engage her 7<sup>th</sup> graders in lively and fruitful discussions, using students’ input to construct mathematical meaning, stimulating students to explain their thinking to each other, and so on, thus enacting many instances of an interpretive teaching mode; otherwise, her teaching mode was mainly an evaluative one. Diana became increasingly aware of the conflicts between her actual classroom practices and what she would have liked to do with her students:

... it is very difficult for me to listen to the students in a lesson where there is noise... in a normal lesson, I am aware that ... I listen to the students much better. And ... perhaps I pose more questions when I am calmer ... I hate when there is that noise. It is a complete frustration to me ... I cannot pose questions or listen to them. I start getting irritated and I don't do anything of that! (November, 2002)

Realizing she was able to move beyond an interpretive teaching mode only seldom was a source of great frustration.

Diana fought three different kinds of battles when teaching 11<sup>th</sup> grade. On one hand, she felt much pressure from the school mathematics department to cover the curriculum and maintain a certain instructional pace gauging it in accordance with other classrooms. Diana's 11<sup>th</sup> graders showed many learning difficulties, needing more time to work on the mathematics ideas being taught than other students from other classes. The pressure to move on caused Diana frustration and a sense of helplessness. On the other hand, Diana struggled to motivate her students to engage in mathematical tasks and to share their ideas and solutions with each other. Yet, the students were always very passive and apathetic, making no efforts to surpass their difficulties. Their attitudes contrasted greatly with the dynamic and inquiry environment that Diana strove so hard to establish in that classroom:

They forget everything! Then they do nothing on their own ... They just copy things from the board! That is why these lessons are very frustrating ... Would it make a whole lot of a difference if I would go to the board and solve everything? And tell them 'You do this in this way'? ... That would be expository teaching! [sighing deeply] (January, 2003).

On the rare occasions the students were more involved in the lesson, Diana was indeed able to enact an interpretive teaching mode.

Another factor that forced Diana to enacting an evaluative teaching mode emerged during the third cycle of the study. Her insecurity regarding the topic she was teaching led her to no longer attempt to *spice up* her lessons or promote a classroom environment that contributed to her students' active participation in their own learning. Not being comfortable with her visualization skills, Diana closed the possibilities for dialogue, using rhetoric questions to structure her speech and prevent *threatening* situations, and increasing her instructional pace. Realizing the effects of her lack of sound mathematical knowledge in her teaching created another source of frustration:

I am a bit afraid of this material ... teaching something I cannot visualize very well ... I am afraid of getting mixed up ... I will continue doing more expository teaching because I do not feel at ease asking questions and all of that ... I don't know! (January, 2003)

In sum, the tensions in Diana's teaching modes throughout her entire student teaching experience were never resolved. She did not win her battle to teach in ways that resonated with her problem-solving oriented beliefs (Ernest, 1989), fighting against very different factors: if Abel's inadequate behaviour as a secondary teacher

in their shared 7<sup>th</sup> grade classroom was the determinant factor for her lack of success in teaching in an increasingly generative mode, distinct factors accounted for a great sense of frustration and helplessness when teaching 11<sup>th</sup> grade: external pressures to cover the curriculum, passive and unmotivated students, and, above all, lack of a sound mathematical knowledge to give her the sense of security she needed to teach how she wanted to teach. Nevertheless, when conditions were favourable, Diana temporarily resolved the conflicts between her espoused and enacted beliefs, evidencing clear instances of an interpretive teaching. Yet, she did not give up on pursuing her goal, postponing her attempts to enact a generative mode for further.

## FINAL REMARKS

The CF worked as a fundamental tool for conducting the teaching experiment component of the research design, especially for selecting readings for discussion and reflection. It was also instrumental for interpreting and analyzing the participants' progression regarding all its four strands, and for generating the accounts of practice. However, while the anticipated relationships between teachers' teaching modes and dominant patterns of classroom interaction, and between teachers' key beliefs and levels of reflective thinking were somewhat *confirmed*, there were inconsistencies between those two groups of teaching aspects. In addition, the CF did not show itself useful as a reflective tool for the participants themselves. More research is needed to, for example, explore and assess the influence of factors such as sharing classrooms with peers on one's own teaching modes, and to investigate the usefulness and appropriateness of the CF as a support construct to frame other professional development endeavours, in various contexts.

## ACKNOWLEDGEMENTS

Work partially funded by Grant PRAXIS XXI/BD/19656/99 from *Fundação para a Ciência e a Tecnologia* (FCT), and by Centro de Matemática da Universidade do Porto (CMUP, [www.fc.up.pt/cmup](http://www.fc.up.pt/cmup)), Portugal.

## REFERENCES

- Ainley, J. (1988). Perceptions of teachers' questioning styles. In A. Borbás (ed.), *Proceedings of the 12th Annual Meeting of the International Group for the Psychology of Mathematics Education*, Veszprém, Hungary, pp. 92 – 99.
- Bauersfeld, H. (1992). Integrating theories for mathematics education. *For the Learning of Mathematics*, 12 (2), 19 – 28.
- Coles, A. (2001). Listening: A case study of teacher change. In M. van den Heuvel Panhuizen (ed.), *Proceedings of the 25th Annual Meeting of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, vol.2, pp. 281 – 288.

- Cobb, P., Wood, T., Yackel, E., & McNeal, E. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, 29 (3), 573 – 604.
- Davis, B. (1997). Listening for differences: An evolving conception of mathematics teaching. *Journal for Research in Mathematics Education*, 28 (3), 355 – 376.
- Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (ed.), *Mathematics teaching: The state of the art*, Falmer Press, NY, pp. 249-254.
- Nicol, C. (1999). Learning to teach mathematics: Questioning, listening, and responding. *Educational Studies in Mathematics*, 37(1), 45 – 66.
- Schön, D. (1983). *The reflective practitioner: How professionals think in action*, Basic Books, Inc. Publishers, New York.
- Simon, M. (2000). Research on the development of mathematics teachers: The teacher development experiment. In A. E. Kelly & R. A. Lesh (eds.), *Handbook of research design in mathematics and science education*, Lawrence Erlbaum Associates, Mahwah, NJ, pp. 335 – 359.
- Simon, M., & Tzur, R. (1999). Explicating the teachers' perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education*, 30 (3), 252 – 264.
- Simon, M., Tzur, R., Heinz, K., Kinzel, M., & Smith, M. (2000). Characterizing a perspective underlying the practice of mathematics teachers in transition. *Journal for Research in Mathematics Education*, 31 (5), 579 – 601.
- Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (ed.), *Handbook of research in mathematics teaching and learning*, Macmillan, New York, pp. 127 – 146.
- Tomás Ferreira, R. A. (2005). *Portuguese student teachers' evolving teaching modes: A modified teacher development experiment*. Unpublished Doctoral Dissertation. Illinois State University, Normal, IL, USA.
- Tomás Ferreira, R. A. (2007). *The teaching modes: A conceptual framework for teacher education*, CERME 5<sup>th</sup> Congress of the European Society for Research in Mathematics Education, February 2007. Larnaca, Cyprus.
- Van Manen, M. (1977). Linking ways of knowing with ways of being practical. *Curriculum Inquiry*, 6 (3), 205 – 228.
- Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactique des Mathématiques*, 6(1), 69 – 118.
- Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In H. Bauersfeld & P. Cobb (eds.), *The emergence of mathematical meaning: Interaction in classroom cultures*, Lawrence Erlbaum Associates, Hillsdale, NJ, pp. 163 – 201.
- Wood, T. (1998). Alternative patterns of communication in mathematics classes: Funneling or focusing?. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpiska (eds.), *Language and communication in the mathematics classroom*, National Council of Teachers of Mathematics, Reston, VA, USA, pp. 167 – 178.

# **PRESERVICE ELEMENTARY TEACHERS' GEOMETRY CONTENT KNOWLEDGE IN METHODS COURSE**

Fatma Aslan-Tutak

Bogazici University

*The goal of this presentation is to discuss using mixed methods to study preservice teachers' (PST) geometry content knowledge. Effective geometry instruction practices in methods courses were investigated by qualitative methods to develop a protocol to enhance geometry learning of PST, then the effect of the protocol was studied by quantitative methods. The individual interviews (n=3), classroom observations and artifacts from the methods courses data yield to narrative analysis results and thematic analysis results. The following quasi-experimental (n=102) investigation to study intervention protocol which is developed from the qualitative investigation results, showed a significant change in treatment group participants' geometry content knowledge and a significant main effect of knowledge but no significant interaction between geometry content knowledge and grouping.*

Keywords: teacher knowledge, geometry content knowledge, mathematical knowledge for teaching, mixed methods

## **INTRODUCTION**

Teachers bear an important role in reform movements. "The desired learning environments can result only from knowledgeable teachers" (Putnam, et al., 1990, p. 225). Teachers' knowledge should be addressed in PST education and in professional development for in-service teachers. This study reports a two-phase research study which integrated qualitative and quantitative research methods to study elementary first PSTs' geometry learning and then their geometry content knowledge. The first phase of the study was the qualitative investigation to understand elementary school PSTs' *geometry learning* and the effective geometry learning experiences for PST. Integration of results from the study of the qualitative investigation and theoretical knowledge from the literature, the researcher *developed* a protocol for a mathematics methods course. The protocol used as the intervention for the quasi-experimental quantitative phase with purpose of improving the *geometry content knowledge* for teaching of PST.

## **REVIEW OF LITERATURE**

The most commonly accepted definition of teacher knowledge was given by Shulman (1986, 1987), who developed a cognitive model of teacher knowledge, which consists three types of teacher knowledge; content knowledge (CK), pedagogical content knowledge (PCK) and curriculum knowledge. CK refers to knowledge base of the content one is teaching, such as mathematics. PCK "... goes beyond knowledge of subject matter per se to the dimensions of subject matter



knowledge for teaching...” (p. 9). PCK is the type of knowledge that distinguishes the work of a teacher from the work of a scientist. The third knowledge, curriculum knowledge addresses effective use of curriculum materials and being familiar with other subjects that students study.

Among these knowledge types, content knowledge stands out as a point of focus for teacher education. Brown and Borko (1992) asserted that PSTs’ limited mathematical content knowledge is an obstacle for their training on pedagogical knowledge. In the mathematics education field, *mathematical knowledge for teaching* (MKT) was developed as following the Shulman’s model for teacher knowledge (Ball et al., 2008). MKT model addresses how a teacher uses mathematics for teaching (Ball, 2000). According to MKT model, there are six domains of teacher’s content knowledge which can be categorized under Shulman’s different types of knowledge (Ball, Thames & Phelps, 2008). There are three domains under subject matter knowledge: *common content knowledge* (CCK, mathematics knowledge not unique to teaching), *specialized content knowledge* (SCK, mathematics knowledge unique to teaching), and *horizon content knowledge* (knowledge of mathematics throughout the curriculum). Also, there are three domains under pedagogical content knowledge: *knowledge of content and students* (KCS, interaction of knowledge of mathematics and students’ mathematical conceptions), *knowledge of content and teaching* (KCT, interaction of knowledge of mathematics and teaching methods), and *knowledge of content and curriculum* (interaction of knowledge of mathematics and mathematics curriculum).

Many leading mathematics education researchers, Ball (2000), Rowland, Huckstep and Thwaites (2005), Usiskin (2001) discussed role of addressing content preparation of teachers in the context of teaching. There are two practices stands out in the literature to address teachers’ knowledge which are using video discussion groups (Sherin & Han, 2004) and using students’ work to analyze (Kazemi & Franke, 2004). The synthesis of the literature on these two practices shows that the practice of video discussion groups allows for deeper discussion on PCK while absence of classroom environment in students’ work allows for deeper discussion on CK by teachers (Lampert & Ball, 1998; Sherin & Han, 2004). Using students’ work to analyze what students know and what they are learning to facilitate teacher learning results in teachers’ deeper subject matter knowledge (Kazemi & Franke, 2004). Therefore, using students work in a methods course could also improve PSTs’ mathematics knowledge for teaching especially when they have no classroom connection during the methods course.

Content knowledge of teachers is important for every subject including geometry. The limited number of research projects focused on knowledge of geometry for teaching concludes that beginning teachers are not equipped with necessary CK and PCK for geometry, and it is important to address this issue in teacher education (Jones, 2000; Swafford, Jones, & Thornton, 1997). In a study of middle and

secondary school teachers' geometry content knowledge, Fostering Geometric Thinking (FGT), content activities and analysis of student work were used with in-service teachers (Driscoll, Egan, Dimatteo & Nikula, 2009). FGT study showed significant difference between control group teachers who did not receive any professional development and treatment group teachers who received 20-week long intervention. The intervention was designed to provide geometry content experiences for teachers and analysis of student work from teachers own classroom in order to address geometry content knowledge in the context of teaching.

Therefore, this study strives to investigate the following research questions in elementary school mathematics methods course from the constructivist perspective to inform PSTs' geometry preparation. The first two questions were investigated through qualitative methods in order to pursue deeper understanding of PSTs' own perception on their geometry learning and effective practices to promote their learning.

1. What is elementary school preservice teachers' understanding of geometry for elementary school?
2. What are the perceptions of elementary school preservice teachers on effective instructional strategies to promote their learning of geometry content knowledge in mathematics methods courses?

Integration of the results from the above research questions and literature on teachers' mathematics and geometry preparation led the below research questions to be investigated by quantitative methods to study PSTs' geometry content knowledge. For example the topic of geometry to focus, quadrilaterals were chosen according to the results from the qualitative investigation and suggestions from the literature.

3. Does use of geometry activities focused on quadrilaterals with analysis of student work influence elementary school preservice teachers' geometry content knowledge for teaching as measured by CKM-T?
4. Is there a difference in geometry content knowledge for teaching as measured by CKM-T between preservice teachers who are in a traditional mathematics methods course and preservice teachers who are in experimental mathematics methods course?

## SETTINGS

This study took place in elementary mathematics methods course in a large south eastern public university in the U.S. This course plays an important role in PSTs' education because it is the only mathematics methods course for elementary school PSTs. Usually, there are three sections of the course for the spring semester whereas there are four sections for the fall semester. The qualitative investigation took place during the spring semester and the following quantitative investigation was conducted during the fall semester.

## QUALITATIVE INVESTIGATION DATA SOURCE

The goal of the qualitative investigation was to understand PSTs' geometry learning and effective instructional practices to promote their learning. The results of this study informed teacher education practice to develop geometry practices for methods course to be used in the second phase (quantitative investigation). One student from each of the three sections of the elementary mathematics methods, Christiana, Emma and Liz (pseudonyms) participated in this investigation. The data collection included observations of geometry instruction in each section, and the collection of materials for the geometry instructions. Field notes were taken during the observations. The primary data source was individual interviews. The interview protocol was designed for semi-structured and open-ended narrative interviews. The narrative interviews are tailored to intrigue story telling from participants through open-ended questions or probes (Reissman, 1993).

## QUALITATIVE INVESTIGATION DATA ANALYSIS

Individuals may use narratives for meaning making or for sharing their experiences (Reissman, 1993). Furthermore, teachers may prefer to discuss their learning and their knowledge through stories (Cortazzi, 1993). According to Labov (1972) a narrative has a structure and a sequence. If a narrative is fully formed, it has six components; *abstract, orientation, complicating action, resolution, evaluation, and coda*. The structure of the narratives gives insights about how the participants perceive their experiences. In addition to structural analysis of narratives, thematic analysis (Coffey & Atkinson, 1996) was used and the whole interviews were coded. Literature supports using other analysis methods in addition to narrative analysis in order to deepen the analysis of the rich data similar to the data of this study (Lloyd, 2005; Reissman, 1993).

## QUALITATIVE INVESTIGATION FINDINGS

There were two main kinds of stories with sub headings emerged from participants' narratives: *stories as a learner* and *stories as a beginning teacher*. The thematic analysis yielded three themes from PSTs' geometry learning: *history of learning geometry, perceptions about geometry, and effective geometry instruction approaches*.

### Narrative Analysis Findings

(a) *Stories as a learner*. Even though all three participants took one of the required mathematics courses, only Liz had taken the content course before the methods course. All three participants told stories from the mathematics courses they took and they expressed that those courses were as a review of high school mathematics rather than rigorous study of mathematics topics for elementary school. The stories of Liz from the content course reflect her concerns of limited mathematics (especially in geometry) learning and the lack of the connection to her teaching career.

(b) *Stories as a beginning teacher.* The beginning teacher aspect, being able to relate college education into teaching, was briefly expressed in the narratives from mathematics courses. For example, even though Liz's priority in that the content course was to learn mathematics as a student, she had thoughts about ways to transfer the presented knowledge into her teaching. Most of the stories of all three participants as a beginning teacher took place in methods course. Only one participant (Liz) was satisfied from her learning in the methods course. The other two participants expressed their frustration as the lack of the mathematical discussions and connection between content and the teaching methods (Emma) and the misguided flow of the course by moving to the more difficult topics before discussing easier topics (Christiana).

### **Thematic Analysis Findings**

(a) *History of geometry learning.* Participants' background in geometry played an important role in their learning in college courses especially the methods course. All of them stressed the emphasis on algebraic topics in K-12 education with limited opportunities to learn geometry. Furthermore, they all perceive geometry as being different than mathematics because they have the perception of mathematics as algebraic topics.

(b) *Perceptions about geometry.* All the participants recognized the importance of visualization in geometry. Participants think geometry as a study of shapes and measurement features related to the shapes (such as area). Other important topics of geometry such as transformation were not mentioned by any of the participant. Even for the two dimensional shapes they expressed their limited knowledge in quadrilaterals. They classified topics of three dimensional shapes as difficult. Their limited experiences with geometry resulted in distorted perception of geometry.

(c) *Effective instructional approaches.* The mostly emphasized instruction approach was addressing geometry topics for elementary school in addition to the studying pedagogical aspects of those topics. Even tough, participants perceived college mathematics courses as reviews before the methods course, because those reviews did not provide desired understanding of in-depth geometry for elementary school, they were expecting content preparation from methods course too. Some instructional practices were highlighted from the data. Those practices were considered while developing the protocol.

### **QUALITATIVE INVESTIGATION DISCUSSION**

All three of the participants stressed the importance of providing discussion on content before pedagogy. The content as noted by participants is not college level geometry, but geometry that they would be teaching. Especially Emma emphasized content preparation because in spite of the effective pedagogical preparation she could not relate to the ideas. This emphasis on learning geometry for teaching is parallel with MKT model (Ball et al., 2008) in terms of knowing mathematics in the

context of teaching. The second aspect was to progress from easy to more difficult topics. Because of participants' limited knowledge of geometry, they needed to study geometry from basic topics (e.g. 2-D shapes). Furthermore, they especially stressed that they could not learn classification of quadrilaterals in spite of studying that topic in college level courses.

## **DEVELOPMENT OF GEOMETRY PROTOCOL**

The researcher developed a series of geometry activities to use in methods course for two weeks. The length of the protocol was limited because there were only two weeks for geometry topics. The topic of the activities was the classification of quadrilaterals as informed by the qualitative investigation. The first week of activities focused on content aspect of the topic whereas the second week focused on analyzing students' work in order to address geometry topics in the context of teaching. There were three groups of activities: sorting shapes, attributes of shapes, and classification of polygons. In addition to individual characteristics of the activities, the combination of them provided coherence. Participants worked individually, in pairs and small groups. The participants experienced geometry topics with visual representations. Also, the activities progressed through van Hiele geometric thinking levels. Therefore, the activities reflected suggestions from both literature and qualitative results. Kazemi and Franke (2004) suggested that the student work to be used to improve teachers' content knowledge should be challenging. In other words, the student work should show wrong student answers and misconceptions in order to intrigue teachers' discussions on mathematics topics. With this purpose, the researcher collected student work from local elementary schools with mathematically struggling students. The participants were given a protocol to study student work. The protocol was developed by suggestions from several resources (E. Kazemi, personal communication, August 17, 2008; NCTM, 2006). First in pairs, the participants discussed what the student did, what the student knew (and misconceptions), what they would ask the student in order to learn more about the student's knowledge of geometry. Then, in small groups (two pairs), participants discussed what they would do to teach these concepts to the student and how they would address the student misconceptions.

## **QUANTITATIVE INVESTIGATION DATA COLLECTION AND ANALYSIS**

There were three instructors for four sections of the methods course for the fall semester. There were one hundred and seven students enrolled and 102 of them volunteered to participate in the study. All the participants were female. Two of the sections were assigned to be the treatment and other two to be the control groups.

All the instructors were teaching geometry for two weeks during the last third of the semester. The intervention took 90 minutes (half of one class meeting) of each geometry week. The remaining half of the time of class meetings were used to discuss other geometry topics.

The instrument to measure PSTs' geometry content knowledge, Content Knowledge for Teaching Mathematics Measures (CKT-M Measures)<sup>1</sup> was developed as continuum of research on mathematics knowledge for teaching (MKT) which provides the theoretical framework for this study. The instrument aims to measure elementary school PSTs' mathematics knowledge in the context of teaching. For this current study, only the geometry section of the instrument was used. Two parallel forms of the geometry section of the test were administered as pre and post test.

Participants completed the CKT-M Measures geometry test one week before their geometry instruction. For next two weeks they received the geometry instruction and the following week they completed the post-test. Both pre and post tests were administered at the beginning of the class. In order to address the last two research questions, geometry knowledge growth of treatment group and detection of any difference of knowledge growth between treatment and control groups, two different analysis methods, repeated measures ANOVA and mixed ANOVA, were used, respectively.

## QUANTITATIVE INVESTIGATION RESULTS

In order to study the first research question, geometry knowledge growth of treatment group, repeated measures ANOVA was used. Results showed a significant change in participants' geometry content knowledge,  $F(1, 49) = 16.08, p < .001, R^2 = .25, \eta^2 = .25$ . This indicates statistically significant positive change in treatment group participants' geometry content knowledge. A mixed ANOVA method of analysis was conducted to study whether there was difference of knowledge growth between treatment and control groups. Results indicated a significant main effect of time  $F(1, 91) = 28.38, p < .001$  but there was no significant interaction between time and grouping (treatment/control),  $F(1, 91) = .21, p = .646$ . The results showed that geometry knowledge of participants was increased significantly, however the grouping did not have any effect on participants' knowledge growth. It can be concluded that even though treatment group participants' geometry content knowledge growth was significant, the difference between treatment group and control group participants' growth in geometry content knowledge was not significant.

## QUANTITATIVE INVESTIGATION DISCUSSION

The analysis of growth in treatment group can be interpreted as that use of the protocol developed from the previous studies resulted in significant increase in PSTs' geometry content knowledge. However, the control group results showed increase in PSTs who received regular instruction too. Even though treatment group participants' increase was more than the increase of control group participants, the difference was not statistically significant. However, it should be noted that the regular instruction for the control group also addressed the geometry topics from the perspective of learning mathematics in the context of teaching. During the geometry

instruction of the control group, the researcher observed the control group instruction. For further research, the control group instruction designed not to address geometry in the context of teaching may provide further information on affect of using the protocol with PSTs. Furthermore, using student work with PSTs to promote their content knowledge might not be as effective as using them with in-service teachers (e.g. Driscoll et al., 2009). In the case of in-service teachers, participants first experience teaching the materials and then analyze student work. On the other hand, in the case of PSTs, participants only experience the materials without teaching them. Therefore, this study might start the discussions on the role of actual teaching of the materials before analyzing student work.

## CONCLUSIONS

Therefore, this study provides further understanding on teacher' geometry content knowledge. It informs mathematics teacher education in three important points. For the qualitative investigation, PSTs reported that they have limited geometry knowledge as previous research studies have showed (Jones, 2000; Swafford et al., 1997). Since PSTs perception of geometry for elementary school is limited to the 2-D shapes, it may be suggested to conduct further studies on geometry content knowledge of PSTs for other geometry topics too. Lastly, use of student work in PST education may not lead to similar results as using with in-service teachers.

Endnote 1: Copyright © 2006 The Regents of the University of Michigan. For information, questions, or permission requests please contact Merrie Blunk, Learning Mathematics for Teaching, 734-615-7632. Not for reproduction or use without written consent of LMT. Measures development supported by NSF grants REC-9979873, REC- 0207649, EHR-0233456 & EHR 0335411, and by a subcontract to CPRE on Department of Education (DOE), Office of Educational Research and Improvement (OERI) award #R308A960003.

## REFERENCES

- Ball, D. L. (2000). Bridging practices: Intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education*, 51(3), 241.
- Ball, D.L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59 (5), 389-407.
- Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209–239). New York: Macmillan.
- Carpenter, T. P., Fennema,E., Peterson, P. L. & Carey, D. A. (1988). Teachers' Pedagogical Content Knowledge of Students' Problem Solving in Elementary Arithmetic. *Journal for Research in Mathematics Education*, 19(5), 385-401.
- Coffey, A., & Atkinson, P. (1996). *Making sense of qualitative data: Complementary strategies*. Thousand Oaks, CA: Sage.

- Cortazzi, M. (1993). *Narrative analysis*. London: Falmer.
- Driscoll, M., Egan, M., Dimatteo, R. W. & Nikula, J. (2009). Fostering geometric thinking in the middle grades: Professional development for teachers in grades 5-10. In T. V. Craine & R. Rubenstein (Eds) *Understanding geometry for a changing world: Seventy-first yearbook* (pp. 155-171). Reston, VA: National Council of Teachers of Mathematics.
- Franke, M.L., & Kazemi, E. (2001). Learning to teach mathematics: Developing a focus on students' mathematical thinking. *Theory into Practice*, 40, 102-109.
- Jones, K. (2000). Teacher Knowledge and professional development in geometry. *Proceedings of the British Society for Research into Learning Mathematics*, 20(3), 109-114.
- Kazemi, E., & Franke, M.L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, 7, 203-235.
- Labov, W. (1972). *Language in the inner city*. Philadelphia: University of Pennsylvania Press.
- Labov, W. (1982). Speech actions and reactions in personal narratives. In D. Tannen (Ed.), *Analyzing discourse: Text and talk* (pp. 219-247). Washington, Dc: Georgetown University Press.
- Lampert, M. & Ball, D. L. (1998). *Teaching, multimedia, and mathematics*. New York: Teachers College Press.
- Lloyd, G. M. (2005). Beliefs about the teacher's role in the mathematics classroom: One student teacher's explorations in fiction and in practice. *Journal of Mathematics Teacher Education*, 8(6), 441-467.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in china and the United States*: Lawrence Erlbaum Associates.
- National Council of Teachers of Mathematics. (2006). *Learning from NAEP: Professional Development Materials for Teachers of Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Putnam, R., Heaton, R. Prawat, R. & Remillard, J. (1992). Teaching mathematics for understanding: Discussing case studies for four fifth grade teachers. *The Elementary School Journal*, (93)2, 213-228.
- Riessman, C. K. (1993). *Narrative analysis*. Newburk Park, CA: Sage.
- Riessman, C. (2000). Analysis of personal narratives. In J. Gubrium & J. Holstein (Eds.), *Handbook of interviewing* (pp. 1-27). Newbury Park, CA: Sage.



- Rowland, T., Huckstep, P. & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi, *Journal of Mathematics Teacher Education*, 8, 255-281.
- Sherin, M. G. & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education*, 20, 163-183.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Swafford, J. O., Jones, G. A., Thornton, C. A. (1997). Increased knowledge in geometry and instructional practice. *Journal for Research in Mathematics Education*, 28(4), 467-483.
- Usiskin, Z. (2001). Teachers' mathematics: A collection of content deserving to be a field, *The Mathematics Educator*, 6 (1), 86-98.

# HOW TO PROMOTE SUSTAINABLE PROFESSIONAL DEVELOPMENT?

Stefan Zehetmeier & Konrad Krainer

University of Klagenfurt, Austrian Centre for Instructional and School Development

**ABSTRACT.** *The paper deals with the sustainable effectiveness of professional development programmes. This article links theoretical considerations with research findings a case study about a secondary mathematics teacher, who took part in the Austrian nationwide teacher professional development programme IMST<sup>2</sup> in 2002. The case study's results provide information regarding the teachers' professional growth and the sustainable effects of the professional development programme. The paper also discusses implications for the design of teacher professional development programmes.*

**Keywords:** Sustainability, professional development, effectiveness, case study, long-term impact of innovations.

## INTRODUCTION

Evaluations of long-term impact of innovations or programmes are well established in disciplines like health promotion, social medicine, or management research (e.g., Lawrence, Winn, & Jennings, 2001; Pluye, Potvin, & Denis, 2004; Scheirer, 2005). Goals and sustainable outcomes of teacher professional development programmes are of great interest, in particular for both the participating teachers and the facilitators. Despite its central importance, research on this issue is generally lacking within the educational disciplines (Datnow, 2006; Rogers, 2003).

## LITERATURE REVIEW AND THEORETICAL FRAMEWORK

In most papers that put an emphasis on the impact of teachers' professional development programmes, *teachers' learning* is the main focus (see e.g., Guskey, 2000; Lipowsky, 2004, 2010; Sowder, 2007; Zehetmeier, 2008). The major indicators for describing teachers' learning are their knowledge, beliefs, and practice. However, the situation is rather complex since each of these notions can be defined in different ways:<sup>91</sup>

*Teachers' knowledge*, for example, can be differentiated into content knowledge, pedagogical knowledge, and pedagogical content knowledge (Shulman, 1987); but it can also be regarded as knowledge about learning and teaching processes, assessment, evaluation methods, and classroom management (Ingvarson, Meiers, & Beavis, 2005); other foci are expressed by the notions of attention-based knowledge (Ainley & Luntley, 2005) or knowledge of mathematics for teaching (Ball, 1990).

---

<sup>91</sup> See more detailed in Krainer and Zehetmeier (2008).

Similarly, *teachers' beliefs* can include different aspects of beliefs about mathematics as a subject, and its teaching and learning (Leder, Pehkonen, & Törner, 2002). It includes also the participating teachers' perceived professional growth and their satisfaction (Lipowsky, 2004, 2010), their perceived efficacy (Ingvarson et al., 2005), and the teachers' opinions and values (Bromme, 1997). At the *teachers' practice* level, the focus is on various kinds of classroom activities and structures, teaching and learning strategies, methods, or contents (see e.g., Ingvarson et al., 2005).

Zehetmeier (2008) points out that the complexity of possible impact is not fully covered by this taxonomy. For example, results of an impact analysis (Zehetmeier, 2010b) in the context of an Austrian professional development project (see e.g., BMUKK, 2010; Krainer, 2008; Pegg & Krainer, 2008) show that the project made impact also on students' beliefs or other – non participating – teachers' practice. So the taxonomy of levels of impact needs to be extended. Other aspects that also could be considered are the *learning of teacher educators* and of *other relevant environments* of professional development programmes, like participating teachers' colleagues, their principals, their school, etc.

A model that comprises this rather wide range of possible levels of impact is the IPD-model (Impact of Professional Development model; Zehetmeier, 2008, 2009, 2010b, see Figure 1):

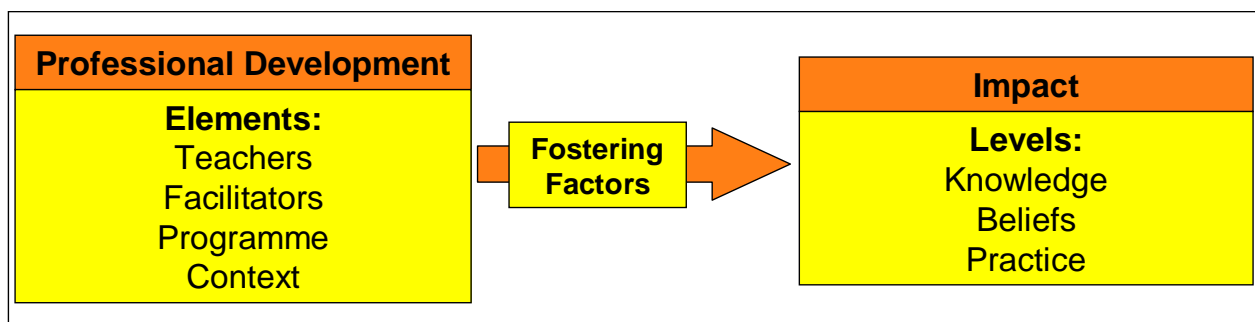


Figure 1: The IPD-model

This model uses the categories knowledge, beliefs, and practice to analyse the impact not only on the teachers' level but also on other in-school levels like pupils, colleagues, principals, or parents. Moreover, this model considers beyond-school levels to analyse the impact of professional development projects: e.g., other schools, media, policy, or scholarship (see Zehetmeier, 2010b). Besides this extended taxonomy of possible levels of impact, the IPD-model also includes an overview concerning factors fostering the impact of professional development projects (see e.g., Zehetmeier, 2010a). Moreover, the IPD-model opens the scope for various types of impact (e.g., short-term or long-term; planned or unintended) on different levels (e.g., new knowledge, changed beliefs, or new teaching practices).

## **BACKGROUND: THE AUSTRIAN IMST<sup>2</sup> PROJECT**

The initial impulse for the IMST<sup>2</sup> project (Innovations in Mathematics, Science and Technology Teaching) in Austria came from the 1995 TIMSS achievement study. In particular, the results of the Austrian high school students (grades 9 to 12 or 13) in the TIMSS advanced mathematics and physics achievement test, shocked the public. The responsible federal ministry launched the IMST research project (1998-1999) in order to analyse the situation.

The research identified a complex picture of diverse problematic influences on the status and quality of mathematics and science teaching. Mathematics education and related research was seen as poorly anchored at Austrian teacher education institutions. Subject experts dominate university teacher education, other teacher education institutions show a lack of research in mathematics education; the collaboration with educational sciences and schools is – with exception of a few cases – underdeveloped. A competence centre like those established in many other countries was not existent. Also, the overall structure (including two institutions for the education of prospective teachers that are mostly unconnected, a variety of different kinds of schools with corresponding administrative bodies in the ministry and in the institutions for the education of practising teachers, etc.) showed a picture of a “fragmentary educational system” of lone fighters with a high level of (individual) autonomy and action, however, there was little evidence of reflection and networking. The situation in science education was even worse.

The analyses mentioned above led to the four year project IMST<sup>2</sup> (2000-2004). The project focused solely on the upper secondary school level and involved the subjects, biology, chemistry, mathematics and physics. The two major tasks of IMST<sup>2</sup> were (a) the initiation, promotion, dissemination, networking and analysis of innovations in schools (and to some extent also in teacher education at university); and (b) recommendations for a support system for the quality development of mathematics, science and technology teaching.

In order to take systemic steps to overcome the “fragmentary educational system”, the approach of a “learning system” (Krainer, 2005) was taken. It adopted enhanced reflection and networking as the basic intervention strategy to initiate and promote innovations at schools.

Besides stressing the dimensions of reflection and networking, “innovation” and “work with teams” were two additional features. Innovations were not regarded as singular events that replace an ineffective practice but as continuous processes leading to a natural further development of practice. Teachers and schools defined their own starting point for innovations and were individually supported by researchers and expert teachers. The IMST<sup>2</sup> intervention built on teachers’ strengths and aimed at making their work visible (e.g., by publishing teachers’ reports on the website). Thus teachers and schools retained ownership of their innovations. Another

important feature of IMST<sup>2</sup> was the emphasis on supporting teams of teachers from a school.

Teachers' participation in IMST<sup>2</sup> was voluntary and gave them and their schools a choice among several priority programmes (e.g., "basic education" or "teaching and learning processes") according to major challenges concerning mathematics and science teaching. In general, teachers in these priority programmes were supported by mathematics and science educators and experienced teachers. The priority programmes can be regarded as small professional communities that not only supported each participant to proceed with his or her own project but also generated a deeper understanding of the critical reflection of one's own teaching, of formulating research questions, of looking for evidence based on viable data, and on methods that help to gather that data.

## METHOD

In 2005, eleven case studies (Stake, 1995; Yin, 2003) were generated to describe and explain specific aspects regarding the impact of the IMST<sup>2</sup> project (Benke, Erlacher, & Zehetmeier, 2006). In 2010, all these case studies were revisited to analyse the project's impact five years later (Zehetmeier, in preparation). For this purpose, semi-structured interviews were again conducted with the teachers who formerly took part in the IMST<sup>2</sup> project; interviews were also conducted with the teachers' respective colleagues, schools' principals, and former project facilitators.

The data gathered in 2010 was analysed according the IPD-model (Zehetmeier, 2008, see above) and contrasted with the 2005 case studies' results. This comparison allows a thorough discussion of the following questions: Which of the 2005 impact was still effective in 2010? Which were the respective fostering factors? Which impact did disappear within the last five years? Which were the factors hindering the sustainability of impact?

In this paper, exemplary results from one of these studies (the case of Barbara<sup>92</sup>) are provided to discuss the question about effective ways of promoting sustainable mathematics teachers' professional development. Thus, this study provides insight going beyond the evaluation of short-term effects of a particular professional development programme. In fact, the study analyses and discusses various levels of impact and their respective fostering factors that occur more than eight years after the programme's termination. The objective of this research is not to evaluate the respective professional development programme (which was not explicitly designed to have sustainable impact, but to support teachers for the time they are participating). Rather, the case study aims to analyse, why some impact is sustainable, while other effects disappear after the programme's termination. In other

---

<sup>92</sup> The teacher's name is a pseudonym.

words, the particular professional development programme is rather the case study's frame, not its focus.

We decided to use a case study approach for this research, because they are particularly suited for analysing the impact of innovations: "The usual survey research methods are less appropriate for the investigation of innovation consequences. [...] Case study approaches are more appropriate" (Rogers, 1995, p. 409). Similarly, Hancock and Algozzine (2006) state: "Through case studies, researchers hope to gain in-depth understanding of situations and meaning for those involved" (p. 11). The case study presented here is *historic* (Merriam, 2001), *intrinsic* (Stake, 1995) and *explaining* (Yin, 2003), since it analyses the teacher's developments over time, focuses on the particular teacher's case, and quests for her professional developments' fostering conditions.

The case study includes data from varying sources and dates to gain validity by "convergence of evidence" (Yin, 2003, p. 100): Collection of data was done during 2002 and 2006 and contained both documents (teacher's annual written project reports), archival records (first author's artefacts), and open-ended and structured interviews (with teacher, project facilitator, colleagues, and principal).

Creswell (2007) has identified eight verification procedures for qualitative studies and recommends that qualitative researchers engage in at least two in any given study. Four of these verification procedures were present in this study: *prolonged engagement*, *triangulation*, *negative case analysis*, and *rich description*: The contact with the teachers has spanned more than one year in the contexts of teaching and coursework, and the time span under research lasted for more than eight years (*prolonged engagement*). Our data came from a variety of sources (*triangulation* by convergence of evidence, see above). We refined our results with regards to disconfirming evidence until we eliminated any disagreements among the findings (*negative case analysis*). Finally, the case study provides detailed information on all persons and activities relevant for this research (*rich description*).

The data were analysed by qualitative content analysis (Mayring, 2000) in order to identify common topics, to elaborate emerging categories, and to gain deeper insight into teacher's professional growth over time.

## RESULTS

The case study's results point to different levels of impact of the IMST<sup>2</sup> project (e.g., teachers' knowledge, beliefs, or teaching practices), that endured over time, even after the project's termination. In the next sections, the 2005 case study's results are contrasted with the recent 2010 data. This allows discussing the question, which of the 2005 impact was still there in 2010. For the sake of reducing complexity, these results are presented according Zehetmeier's (2008) levels of impact (see also Figure 1, above).

### Barbara's knowledge

Barbara took part in the IMST<sup>2</sup> priority programme “teaching and learning processes” during the school-year 2001-02. Barbara integrated open learning environments into her mathematics classes. The 2005 case study showed various results concerning long-term effects, for example regarding Barbara's *knowledge*: She learned that open learning environments had positive effects on pupils' content knowledge, as well as on their self confidence. In particular, there were positive changes regarding low-performing pupils' self esteem, as well as concerning the further development of high-performing pupils' competences. Moreover, Barbara's pupils stated to have more fun and less anxiety in her mathematics lessons. This impact was sustainable: Barbara still used this knowledge in 2005 and 2010. Her expertise and her knowledge regarding open learning environments enabled her to create and implement innovative teaching methods. For example, since Barbara knows about the importance of time resources for these open settings, she is very conscious of providing enough resources in each implementation phase. The school's principal stated: “This had very positive effects on the didactics of our mathematics lessons. In particular, the open learning settings represent sustained impact” (Principal, 2005, interview).

Another impact was Barbara's *new knowledge* regarding the activities of teachers from other schools. During her participation in IMST<sup>2</sup>, she had the opportunity to network with and to learn from colleagues teaching at other schools. The teachers met in seminars, workshops, and conferences organised by the IMST<sup>2</sup> project. Moreover, all teachers wrote project reports which were distributed mutually. “This was a very useful source of ideas. I got motivation and suggestions to try similar things in my classroom” (Barbara, 2010, interview). Similarly, the principal valued these project reports as „a basis for good ideas and comparisons” (Principal, 2005, interview). This impact was not sustainable. After the projects' termination, there was no opportunity to exchange (as easy) with colleagues from other schools, nor were there any written project reports.

### Barbara's beliefs

On the level of Barbara's *beliefs*, she developed a reflective stance towards the content and the method of her teaching; she stated: “Now I see the value of letting the pupils work self-dependently and I am aware of the importance to reserve extra time for this” (Barbara, Interview, 2005). This stance was mirrored by her belief about the value of feedback: Topics like classroom atmosphere and teaching quality were discussed with her pupils on a regular basis: “Now I see the value of discussing questions of good mathematics teaching together with the pupils” (Barbara, 2005, interview). This impact was sustainable: In 2005, as well in 2010, Barbara was convinced of the importance of critically evaluating one's own teaching. “It is important to reflect on good and problematic aspects of my work” (Barbara, 2005, interview).

### Barbara's practice

Also on the level of her *practice*, the project caused some impact: Barbara actively facilitated her pupils' cooperation and communication, because: "The pupils learn much more when they work in groups autonomously and when they experience that they can solve the tasks for themselves" (Barbara, 2005, interview). This impact was sustainable: Even after the projects' termination, Barbara provided time for her pupils' open learning: "This remained: I facilitate their individual work and provide time for this. (...) I have the courage to do so" (Barbara, 2010, interview).

Another impact regarding Barbara's *practice* was the use of evaluation methods. For example, she used questionnaires in order to learn about her pupils' perspectives and used tape recordings in order to gain new insight into her pupils' competences or difficulties. This impact peaked off. Barbara still used questionnaires after the end of her participation in IMST<sup>2</sup>, but not as often as during the project phase. Tape recordings were no longer used: "During IMST<sup>2</sup>, there were smaller classes with fewer pupils, so I could manage the recordings. Now, in classes with over 30 pupils, this is not possible any more" (Barbara, 2010, interview).

### Barbara's colleagues' practice

Going beyond Barbara's individual level, there were also effects on the level of her *colleagues' practice*. For example, a system of mutual exchange and teaching observations between Barbara and her colleagues was established within the school. "Even after two years, this system of mutual classroom visitations is still in progress – without being imposed by the principal or school administration, just because we all know its value" (Barbara's colleague, Interview, 2005). In 2010, this system of mutual feedback is persisting. However, the number of participating teachers peaked off, because "now, this immediate need is no longer given. The most important and interesting things are already said" (Barbara's colleague, 2010, interview). In particular, the school's novice teachers gladly make use of this opportunity to learn from their experienced colleagues. Similarly, Barbara stated: "This peaked off. The colleagues can do it, if they want. But this opportunity is no longer used as often as in the first years after the project's termination" (Barbara, 2010, interview).

## DISCUSSION

Even though Barbara's project was not explicitly designed to have sustainable impact, the case studies' findings show that more often than not the effects could be sustained over eight years. Some other levels of impact disappeared as the professional development project was over. One explanation for this finding is that some of the respective fostering factors are tightly bound to the existence of the professional development project, while others are not.

Another highly important question, that should be considered in the conception of professional development projects, is: Which of the influencing factors can actually



be controlled or affected by the professional development project? (And which cannot?) Maybe the most important factors lie beyond the project's realm. In this case, it could also be reasonable to look out for alternative and supplementary factors that can be provided and influenced by the project itself.

To sum up: Professional development projects with the objective to cause sustainable impact should be designed by carefully considering the following questions

- Which factors are dependent from the project itself?
- Which factors are located beyond the project's realms?

For professional development projects to be sustainable, it is crucial to carefully consider the fostering and hindering factors. This implies to know these factors and to be sensible for them. Considering and facilitating these factors when designing and implementing professional development projects is one important step on the journey to effective in-service teacher professional development. The next step should be to enhance further research and evaluation to get new results regarding the relevance of these factors. These findings should be again integrated into the conception of future projects. In sum, this can lead to a virtuous circle towards the goal of effectively promoting sustainable mathematics teachers' professional development.

## REFERENCES

- Ainley, J., & Luntley, M. (2005). What teachers know: The knowledge base of classroom practice. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1410-1419). Sant Feliu de Guíxols, Spain: European Research in Mathematics Education.
- Ball, D.L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal*, 90(4), 449-466.
- Benke, G., Erlacher, W., & Zehetmeier, S. (2006). *Miniaturen zu IMST<sup>2</sup>* [Miniatures of the IMST<sup>2</sup> project]. Unpublished paper. Klagenfurt, Austria: Austrian Centre for Instructional and School Development.
- BMUKK (2010). IMST – Innovationen Machen Schulen Top [IMST – Innovations make for high-class schools]. *Austrian Education News* (Ed. BMUKK), 61, March 2010, 1-4. (Internet: <http://www.bmukk.gv.at/enfr/school/aen.xml>; last search: September 15, 2010)
- Bromme, R. (1997). Kompetenzen, Funktionen und unterrichtliches Handeln des Lehrers [Expertise, tasks and instructional practice of teachers]. In F. Weinert (Ed.), *Enzyklopädie der Psychologie. Band 3. Psychologie des Unterrichts und der Schule* (pp. 177-212). Göttingen, Germany: Hogrefe.
- Creswell, J. (2007). *Qualitative inquiry & research design. Choosing among five traditions* (2<sup>nd</sup> Edition). Thousand Oaks: Sage Publications.

- Datnow, A. (2006). Comments on Michael Fullan's, "The future of educational change: System thinkers in action". *Journal of Educational Change*, 7, 133-135.
- Guskey, T.R. (2000). *Evaluating Professional Development*. Thousand Oaks, CA: Corwin Press.
- Hancock, D., & Algozzine, B. (2006). *Doing case study research*. New York: Teachers College Press.
- Ingvarson, L., Meiers, M., & Beavis, A. (2005). Factors affecting the impact of professional development programs on teachers' knowledge, practice, student outcomes and efficacy. *Education Policy Analysis Archives*, 13(10), 1-28.
- Krainer, K. (2005). Pupils, teachers and schools as mathematics learners. In C. Kynigos (Ed.), *Mathematics education as a field of research in the knowledge society. Proceedings of the First GARME Conference* (pp. 34-51). Athens, Greece: Hellenic Letters (Pubs).
- Krainer, K. (2008). Innovations in Mathematics, Science and Technology Teaching. In J. Vincent, J. Dowsey & R. Pierce (Eds.), *Connected Maths. MAV Annual Conference 2008* (pp. 199-212). Brunswick, Vic: The Mathematical Association of Victoria (MAV).
- Krainer, K. & Zehetmeier, S. (2008). *What do we know about the sustainability of professional development programmes?* Paper presented in Discussion Group 7 at ICME 11 in Monterrey, Mexico (July 7, 2008).
- Lawrence, T., Winn, M., & Jennings, P. (2001). The temporal dynamics of institutionalization. *Academy of Management Review*, 62(4), 624-644.
- Leder, G., Pehkonen, E., & Törner, G. (2002). *Beliefs: A hidden variable in mathematics education?* Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Lipowsky, F. (2004). Was macht Fortbildungen für Lehrkräfte erfolgreich? [What makes professional development successful?]. *Die deutsche Schule*, 96, 462-479.
- Lipowsky, F. (2010). Empirische Befunde zur Wirksamkeit von Lehrerfortbildung. [Empirical results regarding the impact of teacher professional development]. In F.H. Müller, A. Eichenberger, M. Lüders, & J. Mayr (Eds.), *Lehrerinnen und Lehrer lernen. Konzepte und Befunde der Lehrerfortbildung* (pp. 51-72). Münster, Germany: Waxmann.
- Mayring, P. (2000). *Qualitative Inhaltsanalyse. Grundlagen und Techniken*. [Qualitative content analysis. Basics and techniques]. Weinheim, Germany: Deutscher Studien Verlag.
- Merriam, S. (2001). *Qualitative research and case study applications in education*. San Francisco: Jossey-Bass Publishers.

- Pegg, J., & Krainer, K. (2008). Studies on regional and national reform initiatives as a means to improve mathematics teaching and learning at scale. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education, Vol. 3: Participants in mathematics teacher education: Individuals, teams, communities and networks* (pp. 255-280). Rotterdam, The Netherlands: Sense Publishers.
- Pluye, P., Potvin, L., & Denis J. (2004). Making public health programs last: conceptualizing sustainability. *Evaluation and Program Planning*, 27, 121-133.
- Rogers, E. (1995). *Diffusion of innovations*. New York, London: Free Press.
- Rogers, E. (2003). *Diffusion of innovations (Third edition)*. New York: Free Press.
- Scheirer, M. (2005). Is sustainability possible? *American Journal of Evaluation*, 26(3), 320-347.
- Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.
- Stake, R. (1995). *The art of case study research*. Thousand Oaks: Sage Publications.
- Yin, R. (2003). *Case study research: Design and methods (3<sup>rd</sup> Edition)*. Thousand Oaks: Sage Publications.
- Zehetmeier, S. (2008). *Zur Nachhaltigkeit von Lehrer/innenfortbildung* [The sustainability of teacher professional development]. Doctoral thesis. Klagenfurt, Austria: University of Klagenfurt.
- Zehetmeier, S. (2009). Analysing the impact of teachers' research in their classrooms. In F. Spagnolo (Ed.), *Proceedings CIEAEM 61 – Montréal, Québec, Canada, July 26-31, 2009* (pp. 455-458). Palermo, Italy: Department of Mathematics, University of Palermo.
- Zehetmeier, S. (2010a). Aktionsforschung in der Lehrerfortbildung: Was bleibt? [Action research in teacher education. What remains?] In F.H. Müller, A. Eichenberger, M. Lüders, & J. Mayr (Eds.), *Lehrerinnen und Lehrer lernen. Konzepte und Befunde der Lehrerfortbildung* (pp. 197-211). Münster, Germany: Waxmann.
- Zehetmeier, S. (2010b). The sustainability of professional development. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education, January 28th – February 1st 2009, Lyon (France)* (pp. 1951-1960). Lyon, France: Institut national de recherche pédagogique.
- Zehetmeier, S. (in preparation). *Tracking the sustainable impact of professional development*.

# CONCEPTS FROM MATHEMATICS EDUCATION RESEARCH AS A TRIGGER FOR MATHEMATICS TEACHERS' REFLECTIONS

Mario Sánchez

National Polytechnic Institute, Mexico

*This article reports part of a research project that attempts to identify the elements of an online course that promote the emergence of teachers' reflections. A definition of reflection that is helpful to identify instances of reflections that appear spontaneously in an online course is used in the study. Elements of the documental approach (Gueudet & Trouche, 2009) are applied to try to establish connections between components of the design of an online course and the emergence of teachers' reflections. The main finding presented is that concepts from mathematics education research can help teachers to see their own teaching practice from a different perspective and thus stimulate reflexions in them.*

**Keywords:** Reflection, online mathematics teacher education, research literature as a tool for teacher development.

## INTRODUCTION

This paper reports part of a research project focused on identifying some of the elements in the design of an online course that promote the emergence of reflections in in-service mathematics teachers (see Sánchez, 2010a). In particular, this article addresses the following research question:

*Which non-human elements of an online course promote the emergence of mathematics teachers' reflections?*

The previous research question is located at the intersection of two sub-areas of research within the field of mathematics teacher education research, namely, *reflective thinking* and *online mathematics teacher education*. Its scientific relevance lies in trying to identify components of an online course that have the potential to encourage the emergence of teachers' reflections, which in turn are considered as an important element for the development of mathematics teachers (see for example Ticha & Hospesova, 2006).

This research was developed in an online mathematics teacher education program, aimed at in-service mathematics teachers working at different educational levels, and coming from all over Latin America<sup>1</sup>.

## CONCEPTUAL FRAMEWORK

In this section I clarify the key terms that are involved in the research question, but I also refer to the theoretical constructs that I used to address such question.

## **What is an online course?**

An online course is a course that is based on the use of the Internet. This means that the content and the activities of the course are delivered via the Internet. The participants in this type of course do not meet physically to interact and discuss. All the interaction and communication within the course are carried out by using the Internet and related communication tools such as email, discussion forums, audio and video conferencing.

## **Human and non-human elements**

Another key term is *non-human elements of an online course*. I perceive the structure and content of an online course as an amalgam of human elements and non-human elements. I use the term human elements to refer to the people who participate in an online course. In the context of this study, the human elements are the mathematics teachers and the teacher educators who are participating in an online course. When I use the term non-human elements I refer to the resources that a participant in an online course interact with, but which are intentionally provided by the teacher educator. These are resources that are part of the design of an online course. The resources can be of different nature: software, video, activities, articles, audio files, web pages. The two main characteristics of the non-human elements of an online course are: (1) they are elements that are intentionally provided by the course designer. The designer has control over them in the sense that he/she decides when and how they will appear within the course; and (2) they are elements that serve to represent and communicate mathematical and/or didactical ideas that are considered relevant to mathematics teachers' development.

I find relevant to differentiate between human elements and non-human elements of an online course, because the latter are more likely to be controlled by the designer of an online course. That is, although it could be possible to identify some of the human elements in an online course that favour the emergence of reflections (for example, attitudes or types of human interactions), such elements cannot be easily controlled and regulated within an online course.

## **Reflection**

A central theoretical construct used in this study is reflection. I think of *reflection* as a mental process by which our actions, beliefs, knowledge or feelings are consciously considered and examined. To reflect involves more than just recalling or considering something consciously. A process of reflection provides enlightenment about the actions or ideas that are being considered. A process of reflection involves a kind of "Aha! moment" in which something is discovered or revealed.

## **Comparing my definition of the concept of reflection**

A fundamental similarity between my definition of the concept of reflection and other definitions that can be found in the specialized literature is that reflection is

interpreted as a mental process in which *something* is considered or examined in a conscious way. I wrote “something” using italics because many researchers in mathematics teacher education usually interpret such “something” as the act of teaching. In other words, researchers in mathematics teacher education lay particular emphasis on the kind of reflections that are anchored in teaching practice. The widespread use of video recordings in reflection research, through which teachers are asked to analyse classroom episodes, can be considered as an evidence of the emphasis on reflection on teaching practice (see for example Stockero, 2008). The extensive use of theoretical concepts such as reflection-for action, reflection-in-action and reflection-on-action is another kind of evidence of this emphasis on teaching practice (see for example Scherer & Steinbring, 2007). However, in my interpretation of the concept of reflection not only the teaching practice can be the focus of a reflection. You can also reflect on your mathematical knowledge, on the role and application of mathematics in non-mathematical contexts or even on your own feelings and values.

An important difference between my definition of the concept of reflection and other definitions that can be found in the literature is that, in my definition, emphasis is placed on the stage of discovery or revelation (the “Aha! moment”) that a reflection can produce. I decided to include the Aha! moment in the definition of reflection on methodological grounds. This point is discussed in the methodology section of the article.

### **Documentational approach**

In order to answer the research question it was necessary to investigate the possible connections between the components of an online course and the emergence of mathematics teachers’ reflections. I used the documentational approach (Gueudet & Trouche, 2009) to investigate such connections. This theoretical approach is adequate to address the research question because it helps to study the “effects” that the different resources that a teacher interact with (books, webpages, notes, discussions with colleagues, etc.), produce in his/her practice and schemas. Thus, I used the documentational approach to try to identify the non-human elements of an online course that produced teachers’ reflections. The concepts of *instrumentation process* and *documentational orchestration* were particularly useful for the study.

In the documentational approach it is claimed that the professional development of mathematics teachers can be tracked by focusing our attention on the activities that mathematics teachers develop outside the classroom, but that influence their work within the classroom. The focus is particularly centred on *teachers’ documentation work*. That is, the interaction between the teachers and a set of elements that allows them to shape and define their work in the classroom. Expressions of such interaction are for example: to extract examples and exercises from a textbook in order to include them in their lesson plans; to analyse their students’ mathematical productions; to listen to the suggestions, ideas and experiences from colleagues; to

review the contents of a website that contains educational materials; to study a curriculum reform to be applied in their own school, etc. The set of elements with which a teacher interacts during her documentation work is called *resources*.

In the documentational approach it is argued that, when an interaction between a teacher and a set of resources takes place, a *documentational genesis* (DG) may appear. The concept of DG can be interpreted as an analogy of the concept of instrumental genesis (Trouche 2005) applied to the field of mathematics teacher education. Like the instrumental genesis, the DG is a two-way process in which the teacher appropriates and/or modify the set of resources with which she interacts (this part of the process is called *instrumentalization*), but the set of resources also shapes and influences teacher's activity and way of thinking (this part of the process is called *instrumentation*). The latter concept was used to try to establish links between the non-human elements of an online course and the emergence of reflections.

Finally, a *documentational orchestration* (DO) can be defined as the selection and arrangement of resources that a teacher educator (or a group of teacher educators) carry out with the intention of facilitating teachers' documentation work. Such documentation work is aimed at contributing to the development of teachers' professional knowledge.

## METHODOLOGY

In order to answer the research question, I designed an online course which had the scientific aim of promoting the emergence of teachers' reflections, and thus help me to study the influence of the non-human elements of the course on the emergence of such reflections. Three methodological challenges were identified at this stage: (1) to determine what non-human elements were likely to stimulate teachers' reflections (in order to include them as part of the course design); (2) if reflection is an entity that is not directly observable, how to detect a reflection in an online setting?; and (3) how to establish connections between the non-human elements of a course and the emergence of reflections?

To address the above-mentioned points (1) and (2), I conducted a literature review on the concept of reflection in mathematics teacher education research (see Sánchez, to appear). In this review I analysed, among other things, (a) what kind of methodological tools are used to detect a reflection, and (b) what type of elements or conditions have been identified as promoters of teachers' reflections. The information obtained in (a) and (b) was used as inspiration to devise a strategy to promote and identify teachers' reflections in an online setting. To try to establish the connections mentioned in point (3), I applied the concepts of documentational orchestration and instrumentation process. Next I illustrate these points.

### **Stimulating reflections in an online setting**

Several elements were identified in the literature review as promoters of reflections, but only three of them were considered in the design of the course because of their applicability in an online setting. Here I refer to the act of writing, the availability of time, and the reading of mathematics education publications.

Several researchers claim that the *act of writing* is a vehicle for reflection. For example Ponte & Santos (2005) assert: “[W]riting is a powerful way of reflecting, helping teachers to clarify ideas, to look at them from different angles, to come back and revise; the steadiness of the written word also seems to provide more depth to the ideas” (p. 123). I also found that *the relevance of time* in the emergency and the depth of a reflection has been highlighted by several researchers: For instance Sowder (2007) underlines: “[T]ime is needed for developing the ability and habit of reflection. Reflection rarely occurs when time is not a resource available to teachers” (p. 198). These two elements, the act of writing and the availability of time, were considered in the design of the course through the inclusion of asynchronous discussion forums. In this kind of forum people interact through the exchange of written messages. Here the feedback or responses to your written messages and comments are not received immediately. You can post a question in a discussion forum and get an answer some hours or even days later. The asynchronous interactions usually last several days, allowing the participants to have more time to formulate their opinions and to consider the comments and opinions expressed by the other participants. The comments and discussions expressed in the asynchronous discussion forums were the main empirical evidence used in this investigation

Researchers like Shari L. Stockero suggest that the *reading of mathematics education publications* is another activity that improves the level of reflection: “Course readings, for example, exposed the PTs [prospective teachers] to alternative ideas that allowed them to begin to think about learning mathematics in ways other than how they had learned as students. Without these readings to draw upon, the PTs may not have had the tools necessary to reflect at higher levels” (Stockero, 2008, p. 391). Thus, I decided that the structure of the online course should include some sort of writing produced within the community of mathematics education research.

### **Detecting reflections in an online setting**

In the literature review that I conducted it was found that sometimes researchers explicitly ask teachers to produce reflections. This is usually done through the application of questionnaires or through some sort of written assignment. Let me present the following quotation as an illustration of this practice:

“[T]he PTs [prospective teachers] were required to write a paper in which they reflected on their experience by analyzing how they as the teacher helped or hindered the development of students’ mathematical understanding of the problem” (Stockero, 2008, p. 378).



I think that this way of identifying reflections is somewhat artificial. I was interested in identifying reflections that could appear more spontaneously. This was one of the reasons why I decided to include the “Aha! moment” as part of my definition of reflection. My intention was to use the “Aha! moment” as an indicator that the teacher had experienced a reflection. Another reason for using the “Aha! moment” was to avoid confusing instances of reflection with instances of remembering or recalling. The “Aha! moment” indicated to me that the teacher had done more than just remembering. It indicated me that the teacher had discovered or learned something based on the explicit consideration of his/her actions or values.

### **Establishing connections between non-human elements and reflections**

To try to detect the possible connections between the emergence of reflections and the non-human elements of the online course that I designed, I did the following: Firstly, I ordered the set of non-human elements (which in terms of the theory can be called “resources”) of the course into stages. Each stage had a particular purpose and comprised a particular subset of resources. I explicitly defined the resources that each stage should contain, and the function and location of the stages within the course. I have called this sort of arrangement *documentational orchestration* (see Sánchez, 2010b).

When the course was being implemented, the concept of reflection was applied to identify teachers’ reflections within the asynchronous discussion forums. It was necessary to read and reread several times each utterance within a forum in order to become familiar with its contents. While I was trying to get familiar with the contents of a specific discussion, I also focused on locating the moments of an interaction that could be labelled as reflections, according to my own definition of the concept.

After having these two sets (the set of ordered resources and the set of teachers’ reflections), I focused on observing the instrumentation and instrumentalization processes that appeared between these two sets (Gueudet & Trouche, 2009). That is, it was studied how teachers used the resources (instrumentalization processes), but the kinds of effects that the resources produced on teachers (instrumentation processes) were also observed. When the effect produced by an instrumentation process was a reflection, then the development of such process was analysed “backwards” in order to identify the particular resource that produced it.

### **DATA ANALYSIS**

The course that was designed for this study was an in-service course on the use of technology in mathematics teaching. The didactical aim of the course was to make teachers aware of the potential changes that may occur in the mathematics classroom when the use of CAS technology is introduced.

During the course, teachers were solving different mathematical tasks (algebraic factorisations, for example), and comparing techniques based on the use of CAS and techniques based on the use of paper-and-pencil. Then they discussed their experiences obtained through these comparisons in asynchronous discussion forums.

Due to space limitations, it is not possible to describe each of the stages that constituted the course. I will only refer to the latter stage of the course in which the teachers read and discussed the paper by Lagrange (2005). The aim of this stage of the course was to make teachers to compare the experiences that they obtained when using CAS and paper-and-pencil techniques during the course, with the ideas and concepts included in the article.

Two of the main theoretical concepts included in Lagrange (2005) are the pragmatic and the epistemic value of a technique that is based on the use of technology. The *pragmatic value* of a technique refers to the efficiency and economy (of time, of effort) with which a technique helps to solve a mathematical task. For example, the pragmatic value of any CAS software may be related to the speed and efficiency with which the software performs algebraic factorisations. The *epistemic value* of a technique refers to its potential to serve as a means to understand the mathematical objects involved in the application of the technique. For instance, the epistemic value of CAS-based techniques may be related to the fact that such techniques allow a more experimental approach to elemental algebra, where students can explore several particular cases of the factorisation of an algebraic expression ( $x^n - 1$ , for example) and produce conjectures about the general factorisation of the expression.

After analysing the teachers' asynchronous discussions produced during the initial stages of the course, it became clear that many of them only acknowledged the pragmatic value of CAS techniques. In other words, teachers perceived CAS software as a tool that facilitates the execution and verification of algorithms, but not as a tool that can serve as a means for mathematical inquiry and the construction of mathematical knowledge. See for instance the following comment expressed by a teacher called Francisco<sup>3</sup>:

I agree with Rosa on the usefulness of the calculator in the sense that it saves a lot of work [...] In general, when there is a discussion on this topic I always conclude that it is important for students to first learn the methods by hand, let us say pencil and paper. [...]

However, eight days later, and after reading Lagrange's paper, this same teacher expressed the following reflection in an asynchronous forum:

Until I read Lagrange's article I only applied it [the technology], using the terminology of the article, in a pragmatic way. I even felt that without a prior knowledge the use of tools such as CAS and/or calculators did not help to generate learning, i.e., I did support the use of these tools but apparently only attaching value to their pragmatic aspect. In integral calculus I encouraged the use of these tools in all the required calculations up to derivation. In differential equations I incentivise its application in the calculation of

integrals and so on. So I was very surprised that the article emphasises the epistemic aspect of these applications. Partly he was right, because the epistemic application apparently requires planning and construction of new specific activities that do not arise naturally from the teaching with paper and pencil. I would like to conclude this contribution leaving the reflection and concern of how a methodology for applying the epistemic value should be.

## **RESULTS AND FINAL DISCUSSION**

My interpretation of the reflection mentioned in the last section is that such reflection was triggered by the interaction between Francisco and the contents of the article Lagrange (2005). The theoretical concepts contained in Lagrange's paper were the only non-human resource identified in the study as trigger for mathematics teachers' reflections. Thus, a possible answer to the research question posed at the beginning of this paper is that theoretical concepts from mathematics education research can promote the emergence of mathematics teachers' reflections.

I try to be cautious and say that it is a "possible" answer because I did not obtain more empirical evidence to confirm that the theoretical concepts contained in mathematics education articles are non-human elements that promote the emergence of reflections. The lack of more instances of reflections to support this conclusion can be caused by the definition of reflection applied in the study. Such definition is restrictive in the sense that requires the appearance of an Aha! moment. The definition for example is not appropriate for detecting reflections that are internally experienced by the individual, but which are not expressed externally by an Aha! moment.

I however claim that the answer to the research question is likely to be a result with some degree of generality. I claim this because there are other studies where it is also argued that the study of concepts and theories from mathematics education research promotes critical reflection on our own beliefs and practices as mathematics educators (see for example Even, 1999 and Tsamir, 2008). If one accepts that the theoretical concepts from mathematics education research have the potential to encourage the emergence of teachers' reflections, then a question naturally arises: what kind of theoretical concepts must be used for this purpose? Tsamir (2008) raises similar questions, without providing a specific answer. Of course these questions deserve further investigation, however, it is possible to formulate a hypothesis: I believe that the type of theoretical concepts that can help teachers to reflect on their own practice and values, must be concepts that seem applicable to them. In other words, teachers need to find some relationship or application between such concepts and their own teaching practice. Thus, it is likely that theoretical concepts with little or no relation to teachers' practice will not serve for this purpose.

## NOTES

1. More information about this educational program can be found at [www.matedu.cicata.ipn.mx](http://www.matedu.cicata.ipn.mx) (in Spanish).
2. All teachers' names are pseudonyms.

## REFERENCES

- Even, R. (1999). Integrating academic and practical knowledge in a teacher leaders' development program. *Educational Studies in Mathematics*, 38(1–3), 235 – 252. doi: [10.1023/A:1003665225190](https://doi.org/10.1023/A:1003665225190)
- Gueudet, G. & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics*, 71(3), 199–317. doi: [10.1007/s10649-008-9159-8](https://doi.org/10.1007/s10649-008-9159-8)
- Lagrange, J.B. (2005). Using symbolic calculators to study mathematics. The case of tasks and techniques. In D. Guin, K. Ruthven & L. Trouche (Eds). *The Didactical Challenge of Symbolic Calculators. Turning a Computational Device into a Mathematical Instrument* (pp. 113–135). New York: Springer. doi: [10.1007/0-387-23435-7\\_6](https://doi.org/10.1007/0-387-23435-7_6)
- Ponte, J.P. & Santos, L. (2005). A distance in-service teacher education setting focused on mathematics investigations: the role of reflection and collaboration. *Interactive Educational Multimedia*, 11, 104–126. Retrieved from <http://greav.ub.edu/iem>
- Sánchez, M. (to appear) A review of research trends in mathematics teacher education. *PNA* ([www.pna.es](http://www.pna.es)).
- Sánchez, M. (2010a). *How to stimulate rich interactions and reflections in online mathematics teacher education?* (Doctoral dissertation, Roskilde University, Denmark). Retrieved from <http://j.mp/a2iEje>
- Sánchez, M. (2010b) Orquestación documental: Herramienta para la planeación y el análisis del trabajo documental colectivo en línea. *Recherches en Didactique des Mathématiques*, 30(3), 367–397.
- Scherer, P. & Steinbring, H. (2007). Noticing children's learning processes –teachers jointly reflect on their own classroom interaction for improving mathematics teaching. *Journal of Mathematics Teacher Education*, 9(2), 157–185. doi: [10.1007/s10857-006-0004-7](https://doi.org/10.1007/s10857-006-0004-7)
- Stockero, S.L. (2008). Using a video-based curriculum to develop a reflective stance in prospective mathematics teachers. *Journal of Mathematics Teacher Education*, 11(5), 373–394. doi: [10.1007/s10857-008-9079-7](https://doi.org/10.1007/s10857-008-9079-7)
- Sowder, J.T. (2007). The mathematical education and development of teachers. In F.K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 1, pp. 157–223). Charlotte, NC: Information Age Publishing.

- Ticha, M. & Hospesova, A. (2006). Qualified pedagogical reflection as a way to improve mathematics education. *Journal of Mathematics Teacher Education*, 9(2), 129–156. doi: [10.1007/s10857-006-6893-7](https://doi.org/10.1007/s10857-006-6893-7)
- Tsamir, P. (2008). Using theories as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.), *The International Handbook of Mathematics Teacher Education Volume 2: Tools and Processes in Mathematics Teacher Education* (pp. 211 – 234). Rotterdam: Sense Publishers.
- Trouche, L. (2005). An instrumental approach to mathematics learning in symbolic calculators environments. In D. Guin, K. Ruthven & L. Trouche (Eds.), *The Didactical Challenge of Symbolic Calculators. Turning a Computational Device into a Mathematical Instrument* (pp. 137 – 162). New York: Springer. doi: [10.1007/0-387-23435-7\\_7](https://doi.org/10.1007/0-387-23435-7_7)

# TEACHER COMPETENCES PREREQUISITE TO NATURAL DIFFERENTIATION

Marie Tichá, Alena Hošpesová

*Institute of Mathematics of the Academy of Sciences of the Czech Republic,*

*University of South Bohemia České Budějovice*

*The paper describes our work with pre-service primary school teachers focusing on development of their ability to cope with heterogeneity in mathematics classroom. In our experiment students were introduced to three substantial learning environments: Way, Room, Mosaics. The students' task was to (a) pose problems supporting natural differentiation and (b) identify the substantial mathematical ideas inherent in these environments. Similarly, we worked with a group of teacher educators and researchers, in which case we focused primarily on development of properties of the concept of natural differentiation.*

Keywords: mathematics education, teacher training, subject knowledge, natural differentiation

## INTRODUCTION

It has been stressed in a variety of contexts that mathematics forms an integral part of many different spheres of life. That is why the importance of mathematics education and of achievement of functional mathematical literacy is growing. This makes considerable demands on teachers' knowledge of the subject and on their pedagogical skills and techniques, i.e. on their professional competences. There is no doubt that teachers must be helped on their way to enrich, develop and refine their professional competences. Therefore over the last years, we have searched for ways of facilitating the enhancement of teachers' professionalism (Tichá & Hošpesová, 2006).

Our research clearly shows that the quality of education greatly depends on the subject didactic competence of the teacher, i.e. on knowledge both of the content and its didactic treatment and on the application of this knowledge in school practice (Tichá & Hošpesová, 2010). This paper discusses a different area of our recent research, namely some of the aspects of coping with heterogeneity of pupils. It is another competence that teachers must master in order to create favourable conditions for the development of their pupils' educational potential.

## THEORETICAL BACKGROUND

### **Natural differences between children; the need of individualization and differentiation**

It is a generally accepted fact that there are differences in how children learn that they need different lengths of time to master the same subject matter under the same

conditions with peers and with the same teacher (e.g. Jonassen & Grabowski, 1993). That is why an increasing number of educators and teachers try to apply the principles of individualization in teaching. Individualization is closely connected to differentiation. Educators looked for and proposed various methods of external and inner differentiation which would respect the differences between pupils and their individualities, and whose objective is optimum pupils' development (Skalková, 1999).

*Natural differentiation* is one of the forms of inner differentiation. The roots of our interest in this issue go down to the solving of the just finished Comenius project "Motivation via Natural Differentiation in Mathematics" (NaDiMa). We found important sources of motivation in the works of Wittmann (1995), Scherer and Krauthausen (2010). They use such an approach to the issue of heterogeneity in teaching in which differentiation is not perceived as an obstruction but as something natural and in some aspects also beneficial and challenging.

"Usually, measures on differentiating are pre-determined by the teacher. In contrast, when using natural differentiation the topic as such enables the children to choose their own level to work on. The same learning offer for all children is one constituent characteristic of natural differentiation (the teacher being responsible for the subject-related framing). Further constituents are a holistic content-related learning offer with a sufficient level of complexity (i. e. complex tasks, e. g. in the form of a research or exploration assignment; see below), the free choice of solution strategies and manipulatives and, where applicable, free choice of the tasks as well as the form of presentation (oral or written) within the framework of social conventions (Wittmann/Müller 2004, p.15)." (Krauthausen, Scherer, 2010, authors' translation).

In addition to allowing children to work at their own level, such problems also facilitate social aspects of learning. In this approach, during the concluding discussion pupils present their solving procedures and their result and justify them. The possibility to solve the same problem on different levels is beneficial for the weaker and slower as well as for the highly talented pupils. The pupils on lower level acquire new knowledge in the final discussion and move to a more advanced level, the talented pupils are generally led to look for and apply new approaches.

### **Substantial learning environments**

If we want to enable and support natural differentiation, we must try to foster such a learning environment in which there are inherent tasks and problems that all pupils will be able to handle, albeit on various levels. Wittmann puts emphasis on the creation of *substantial learning environment* (SLE). SLE is

„a teaching/learning unit with the following properties: (1) It represents central objectives, contents and principles of teaching mathematics at a certain level. (2) It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities. (3) It is flexible and can be adapted to the

special conditions of a classroom. (4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.” (Wittmann, 1995, 365/366)

The characteristics quoted above make considerable demands on teachers who try to create such environments and teach in them. Wittmann points out that the search for, conception and creation of SLE is one of the fields where researchers’ and teachers’ objectives interlink, where not only theory and practice, but also mathematics and didactics of mathematics mingle and which is open to natural long-term, systematic cooperation of researchers and teachers.

It is not uninteresting to recall at this point that Vyšín (the prominent Czech mathematics teacher educator) already emphasized in the beginning of the 1970s’ the need for simultaneous theoretical and practical conduct of research in the didactics of mathematics. He regarded systematic cooperation with teachers and preparation of materials for development and refinement of teachers’ work to be crucial. Similarly Bell, claimed already in the middle of the 1980s’ that “The developing theory of mathematical learning and teaching must be a refinement, an extension and a deepening of practitioner knowledge, not a separate growth” (Bell., 1984, p. 109, quoted from Wittmann, 2001). There are a number of developing trends in the various forms of cooperation between researchers and teachers, e.g. action research (Benke, Hošpesová, & Tichá, 2008). For that matter the authors of this paper have been developing similar ideas for a long time.<sup>[1]</sup>

## **STUDY WITH TEACHERS AND TEACHER EDUCATORS**

### **Starting points, aims, objectives**

Most existing works on natural differentiation deals with algebra or arithmetic. That is why we tried to construct geometrical SLEs and asked whether and to what extent it is possible to use them for natural differentiation. We chose three environments, which we entitled *Way*, *Room*, and *Mosaics*. The SLEs were originally created for primary school pupils and were experimentally tested at this level (Hošpesová, Matějů, & Fantová, 2010; Tichá, 2010).

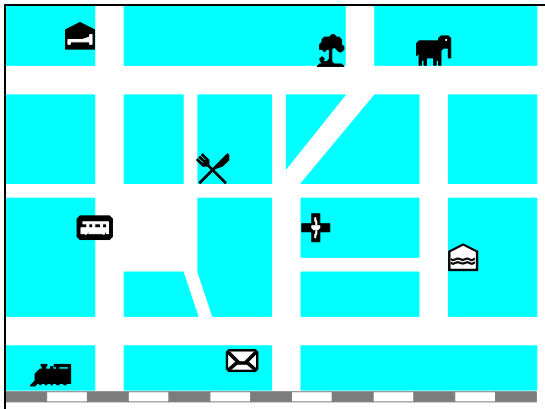
Later we used the same environments in in-service and pre-service teacher training with the aim of training both pre- and in-service teachers for work in SLE and for natural differentiation. We introduced them to the SLEs and asked them to analyze pupils’ productions and to create problems.

In a subsequent stage we also had the opportunity to study the reactions of teacher educators to these environments. We primarily focused on two areas of questions (i) which mathematical topics may be developed in the given SLE, which mathematical topics may be pursued in them, (ii) which substantial mathematical and pedagogical knowledge is prerequisite, what knowledge the teacher needs for efficient use of the environment. First and foremost it was mathematical content, potential for natural differentiation and posing a wide variety of problems that were discussed.



## Description of the created SLEs and our experience from teaching primary school pupils

The environment *Way* works with an idealized street plan (see an example in Fig. 1). In the introductory lessons pupils became familiar with the map; with the structure of a description of a way; with interpretation of instructions; with criteria that a description should meet. After that we focused on three types of problems: (a) drawing a verbally described way in the map, (b) verbal description of a mapped way, (c) pupils' individual choice of a way and its description.

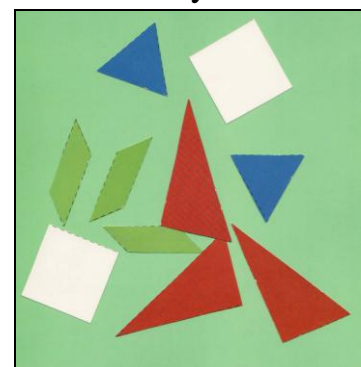


In the background of the choice of this context was our belief that development of the ability to move and orientate oneself in space contributes to development of mathematical literacy in various areas (for example construction of geometrical conceptions, modeling of real situations and use of appropriate language, introduction to various concepts and procedures and to an algorithmic approach).

**Fig. 1 Street plan**

The second environment was named *Room*. Two basic activities were used in this environment: (a) modeling of 3D space and objects in 2D and (b) placing objects in space. Pupils were given a room plan and models of furniture. They were asked to furnish the room. It was up to them how many people would inhabit the room, whether they would use all items of furniture and so on. The environment opens space especially to development of modeling, estimating, of work with scales, of the concepts of congruent shapes (symmetry, translation, rotation) and of filling up the space. The pupils' work displayed a considerable number of differences. The source of these differences often was on the social level; the differences were most often connected with different real-life experience with this activity. What was positive was the pupils' effort to find a fair layout of furniture in the room when the room was furnished for more than one person. There were differences in the ability to estimate the size of the items of furniture (estimations and measuring), the ideas of the size of the (needed) free space. There were also differences in the number of items of furniture used, the number of inhabitants etc.

In the third environment *Mosaics* we worked with different sets of geometrical shapes (see an example in Fig. 2). The goal of the designed tasks was to make pupils identify geometric figures and their properties by assembling shapes from elements of the mosaic. In a



**Fig. 2 Mosaic**

sequence of tasks, pupils successively assembled (a) the figures given in the model, (b) their own pictures, (c) all possible figures, (d) figures from 2, 3, 4,  $n$  elements of the mosaic, (e) a given figure (square, triangle etc.) from 2, 3, 4,  $n$  elements of the mosaic and (f) created their own mosaic.

### **Use of SLEs in seminars for pre-service teachers; a new perspective**

Pre-service teachers for primary school level were asked to assess all the above mentioned environments. The students' task was to explore their potential for natural differentiation and to pose questions and problems. Our former research clearly shows that problem posing is of a strong stimulating and motivational value both for pupils and for students (Tichá, 2009). That is why we decided to use a new point of view and in the following phase of our research we focused on differentiation while posing problems in a given context (a street map). We present here what our students proposed.

In the environment *Way* the students tried to pose problems enabling natural differentiation connected to the environment of the street plan. The most common problems were the following two types:

Describe the way.

Follow the shortest way to get from the hotel to the restaurant. From the restaurant, go to the bus stop and take a bus to go to the swimming pool. On the way, make a stop at the post office. Once you have reached the swimming pool, go to the zoo and from the zoo back to the hotel. How will you go? Draw it in the plan.

Where will you arrive?

We go to the right from the bus stop. On the first crossing we turn left and go straight on. On the next crossing we continue walking straight on. Then we turn left. Where do we arrive?

Very rarely we came across problems with hints of combinatorics:

Go from A to B. You can take whatever way you want. Or: Go from A to B passing C, D and E in any order.

There were differences in the posed problems, for example in the number of places that were to be passed on the way; in the number of changes of direction; in the use of "diagonal" connecting lanes (see Fig. 1) etc.

We have ascertained again that if results of pre-service teachers and pupils are compared, they turn out to be of a similar nature.

Finally we asked the students in the concluding discussion to state the mathematical concepts and methods that can be developed in the particular environment. The students suggested: orientation in space (right, left), estimation of distances (what is closer/farther), reading of a map (the meaning of symbols). They did not realize other possibilities which this SLE offers.

In SLE *Mosaics* students suggested that natural differentiation could be supported by different difficulty of the models (of some elements of the mosaic lined out in the model, only the outlines of the assembled figure given) and assembly of shapes of various difficulty. They also proposed that pupils record their results in various ways: copy the figures on blank sheets of paper, draw the figures into various grids (making it easier for them to construct the shapes). We regard this as a very becoming possibility for natural differentiation, because in geometry, construction of the solution is a substantial part of the solving process. Traditionally, geometrical problems are constructed with a pencil and ruler. Our problems were designed for primary school pupils and thus we expected that the results could be drawn and “exact drawing” is difficult for the pupils.

As far as mathematical content in this environment is concerned, students expected pupils to enrich their ideas of geometrical figures in plane (2D). They also believed that the SLE would on an intuitive level develop pre-concepts of the concept of isometry (the figures if placed in a certain way overlap), of angle and symmetry. Differentiated records of the solution require from the pupils the ability to draw a figure with certain properties, e.g. the length of sides in a particular ratio, with particular interior angles.

Both SLEs however also confirmed our initial opinion that pre-service teachers are not very “fond” of geometry, they have very little idea of how to teach it at primary school level<sup>[2]</sup>. In consequence they highly appreciated the opportunity to work in SLE.

### **Seminar with teacher educators and researchers; new stimuli**

The described environments were also used for work with a group of researchers and teacher-educators within the frame of a working seminar on CME 2010 conference (Hošpesová, Roubíček, & Tichá, 2010). Our work with this group helped us considerably in our effort to deepen the theoretical background of the concepts of SLE and natural differentiation. All the participants welcomed the focus on geometry and emphasized the wide range of stimuli stemming from geometrical environments and the potential for their use in the system of propaedeutics. The discussion concentrated on the mathematical core and didactical possibilities as well as pre-service teacher training for primary school level. The participants emphasised the question: What knowledge do we develop? They asked for an example of the task which cannot be taken as SLE.

The conclusion of the discussion was that all the environments provide a wide range of opportunities for natural differentiation. The proposals of the participants included structured requests for: related mathematical knowledge, activities, and tasks.

The environment *Way* was seen as suitable for: (a) orientation in space and plane, (b) utilization of pictures as universal language, (c) development of coding and decoding (requiring from the pupils to erase unnecessary words, substitute words by signs in

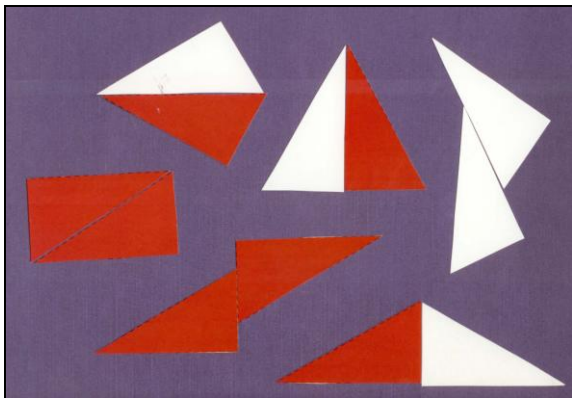
the description of the way), (d) development of algorithmic approach, optimization of the procedure, (e) adding/completing estimation and comparing distances (lengths), (f) challenging pupils to create their own ratio scale/measurement.

The environment *Room* was regarded as suitable for development of intuitive ideas, pre-concepts: (a) projection in two dimensions and in three-dimensional space and transition between the two, (b) covering the plane, (c) orientation in space, (d) measuring and using the scale, (e) solids, shapes, ..., (f) transformations – isometry, symmetry, congruency, similarity, (g) mathematical ideas, meaning, (h) language (describe your room, the room you have designed, create a room according to a description), (i) sense of representation (design furniture and cut it out from a piece of paper).

In their comments the participants also paid attention to the relation to reality and proposed questions such as: Why do we furnish a room? What kind of a room? Why are wardrobes cuboids? How high should wardrobes be? What happens if our room is not a cuboid? Why do you use double beds in a bedroom? What would be the price of having the room redecorated? [How does the position of objects in a room influence the social dimension of that room?](#)

The environment *Mosaics* in their opinion provides the opportunity:

- To assemble and identify geometric figures (e.g. they posed the following problem with triangles whose one side is red and the other white: create various shapes combining two triangles – see examples in Fig. 3 – and questions: propose criteria for classification of the created shapes, why are some of them monochromatic and others dichromatic?



**Fig. 3 Shapes from triangles**



**Fig. 4 Material for tessellations**

- To seek similarities and differences in geometric figures; on intuitive level to identify their properties such as length of their side, size of the interior angles
- To enrich notions of polygonal shapes
- To develop intuitive notions of congruence
- To gain experience with homothetic transformations on visual level
- To introduce pupils to measuring area – problem: cover the rectangle using geometrical figures; what figures did you use?

- To use this knowledge in modelling of practical situations

The participants also suggested other environments and activities: tessellations (proposed shapes designed by participants in fig. 4), constructions in Cabri.

## CONCLUDING REMARKS

As stated above, this paper presents information of ongoing research, on its background and its preliminary results. We are convinced that creation of SLE pertaining to the subject matter taught should be at the centre of attention of didactics of mathematics. It is not only research but also teacher training that should be related to it. The first experience of work targeted on this issue show that it is a long-term, systematic research focusing both on theoretical and experimental solution of the promising issues of teaching goals, contents and methods and pupils' learning of mathematics.

Work with pre-service teachers confirmed our belief that teacher training must involve activities of this type as students often fail to “see mathematics in the world that surrounds them”. It is even in-service teachers with long teaching experience who lack this ability. They often fail to realize that what they deal with is an open problem, open situation – they cannot make any use of it for they are not able to see, understand, and grasp mathematics inherent in it. It is becoming more and more apparent that it is mathematics that is the leading agent. It confirms again that the teacher must master mathematics – mere teaching experience, however long it may be, not anchored in knowledge of mathematics is not sufficient.

In our work with pre-service teachers we observed that the nature of the posed problems gradually changed: from simple problems with easy solutions of “textbook” nature, commonplace (both with respect to the context and to the topics) and often badly worded problems, to challenging uncommon problems with diversiform assignments (graphs, tables, diagrams, ...), representations (iconic, symbolic) facilitating various solving procedures and approaches, asking for justification, leading to further meditation (open problems). Referential contexts were enriched.

It is gratifying that the atmosphere in mathematics lessons changed: teacher students became more active, everybody participated in the lesson as their abilities were respected, not only their attitude but also their beliefs and self-confidence improved. Again joint reflection proved to be of utmost benefit as it shows that students penetrate mathematical concepts, procedures and structures in greater depth.

Working with teacher educators we realized their orientation towards mathematics and their ability to grasp the depth of the content. At the same time they discussed the pedagogical aspects in relation to the mathematical classroom and teacher training. We saw that realization of benefit of natural differentiation requires the change of approaches to coping with heterogeneity in mathematics classroom.

## NOTES

1. Similar ideas are in the background of our studies of the process of grasping a situation (Koman & Tichá, 1998). By grasping a situation we understand the thinking process which blends activities focusing especially on formulation of questions and problems springing from the situation, solution of the posed problems and interpretation of the results. To be able to grasp a situation with comprehension, it is necessary not only be able to construct a mathematical model, but also to communicate and reason. The social aspect is of great importance here. We want to make pupils and pre-service teachers see mathematics in the world that surrounds us, in different “mathematical” and “non-mathematical” situations. Our exploration of grasping became the starting point of our research of development and use of the ability to pose problems. We perceive it not only as a goal, but also the means of mathematics education and also as a diagnostic and motivational tool (Tichá, Hošpesová, 2010).
2. In the Czech Republic, geometry is a component of primary school curricula. The Framework Education Programme for Basic Education (2007) demands that: “In the thematic area of two- and three-dimensional Geometry, pupils identify and draw geometric figures and model practical situations, seek similarities and differences in common figures, study an object’s position on a plane (or in space), learn to compare, estimate and measure length, angle, circumference and area (surface area and volume), and to improve their graphic skills.”
3. This research was partially supported by the grant projects: GACR 406/08/0710; AS CR, Institutional Research Plan AV0Z 10190503; 142453-LLP-1-2008-1-PL-COMENIUS–CMP.

## REFERENCES

- Bell, A.W. (1984). Book review of Lesh, R. and Landau, M., *Acquisition of Mathematical Concepts and Processes*, New York 1983. *Educational Studies in Mathematics* 15, 103–110.
- Benke, G., Hošpesová, A., & Tichá, M. (2008). The Use of Action Research in Teacher Education. In K. Krainer, T. Wood (eds.) *The International Handbook of Mathematics Teacher Education (volume 3), Participants in Mathematics Teacher Education: Individuals, Teams, Communities and Networks*. Rotterdam: Sense Publishers, 283-301.
- Framework Education Programme for Basic Education, (2007). Retrieved from <http://www.msmt.cz/vzdelavani/framework-education-programme-for-basic-education>
- Hošpesová, A., Matějů, B., & Fantová Š. (2010). Mozaiky jako podnětná výuková prostředí. [Mosaics as substantial learning environment.]. In M. Uhlířová (ed.) *Matematika 4. Matematické vzdělávání v kontextu proměn primární školy. [Mathematical education in a context of changes in primary school]*. Olomouc : Univerzita Palackého v Olomouci, 119-123.



- Hošpesová, A., Roubíček, F., & Tichá, M. (2010). Substantial learning environments in geometry. In *Motivation via Natural Differentiation in Mathematics. The Conference Proceedings Abstracts*. Jarosław: Wydawnictwo Papirus. 24-26.
- Jonassen, D.H. & Grabowski, B.L. (1993). *Handbook of Individual Differences, Learning, and Instruction*. Hillsdale: Lawrence Erlbaum.
- Koman, M., & Tichá, M. (1998). On Travelling Together and Sharing Expenses. *Teaching Mathematics and its Applications*. Oxford University Press, vol.17, no. 3, 117 - 122.
- Krauthausen, G. & Scherer, P. (2010). Natürliche Differenzierung im Mathematikunterricht der Grundschule – Theoretische Analyse und Potenzial ausgewählter Lernumgebungen. In C. Böttinger et al. (Eds.) *Mathematik im Denken der Kinder. Anregungen zur mathematikdidaktischen Reflexion*. Seelze-Velber: Kallmeyer. 53-59
- Skalková, J. (1999). *Obecná didaktika*. [General Didactics]. Praha : ISV.
- Tichá, M. (2009). Problem posing in teacher training and as a research object in didactics: two complementary perspectives. In *Quaderni di Ricerca in Didattica (Scienze Matematiche), Suplemento n2. Proc. CIEAEM 61*. Palermo: G.R.I.M, 445-449. Retrieved from [http://math.unipa.it/~grim/cieaem/quaderno19\\_suppl\\_2.htm](http://math.unipa.it/~grim/cieaem/quaderno19_suppl_2.htm)
- Tichá, M. (2010). Podnětná výuková prostředí pro přirozenou diferenciaci (Substantial learning environments for natural differentiation). In M. Uhlířová (ed.) *Matematika 4. Matematické vzdělávání v kontextu proměn primární školy. [Mathematical education in a context of changes in primary school]*. Olomouc : Univerzita Palackého v Olomouci, 298-302.
- Tichá, M., & Hošpesová, A. (2006). Qualified Pedagogical Reflection as a Way to Improve Mathematics Education. *Journal for Mathematics Teachers Education. Special Issue: Inter-Relating Theory and Practice in Mathematics Teacher Education*. Springer Netherlands, 9, 2, 129-156.
- Tichá, M., & Hošpesová, A. (2010). Problem posing and development of pedagogical content knowledge in pre-service teacher training. In V. Durand-Guerrier, S. Soury-Lavergne & F. Arzarello (eds.) *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education, January 28<sup>th</sup> - February 1<sup>st</sup> 2009*. Lyon: Institut National de Recherche Pédagogique. 1941-1950. Retrieved from [www.inrp.fr/editions/cerme6](http://www.inrp.fr/editions/cerme6)
- Wittmann, E.Ch. (1995). Mathematics Education as a “Design Science”. *Educational Studies in Mathematics*, 29, 355–374.
- Wittmann, E.Ch. (2001). Developing mathematics education in a systemic process. *Educational Studies in Mathematics*, 48, 1–20.

# DIFFERENCES IN THE PROPOSITIONAL KNOWLEDGE AND THE KNOWLEDGE IN PRACTICE OF BEGINNING PRIMARY SCHOOL TEACHERS

Fay Turner

University of Cambridge

*In this paper I draw on a four year study into the development of mathematics teaching of beginning primary school teachers. I focus on the finding that in the early stages of teaching, it appeared to be difficult for the teachers to apply 'propositional' pedagogical content knowledge (PCK) that had been 'taught' in the university, even though they were able to demonstrate this PCK in discussion. As the teachers became more experienced, their supported reflection on practice seemed to enable them to draw on, and activate, their propositional knowledge. This perhaps suggests that mathematics educators should not 'despair' that their students do not appear to apply the content of their mathematics methods courses when they begin teaching. Rather, they should encourage beginning teachers to reflect on their practice with a focus on mathematical content knowledge.*

Key words: elementary teachers, content knowledge , professional development

## INTRODUCTION

From the start of their careers, beginning primary school teachers have a bank of knowledge about mathematics teaching on which they can draw in their practice. This knowledge originates variously from their own experience as a pupil, from their observations of mathematics teaching and from the mathematics methods courses they undertook as part of their teacher education. However, the knowledge of beginning teachers might be considered to be qualitatively different to that of experienced teachers because it is not 'embedded in practice'. There a number of theoretical perspectives which help us to understand the qualitative difference between the knowledge of beginning teachers and knowledge which is embedded in practice.

Shulman (1986) proposed three *forms* of knowledge: *propositional knowledge*, or academic knowledge; case study knowledge, i.e. knowing what worked from observation or from past experience and *strategic knowledge*, i.e. the ability to make appropriate strategic decisions while in the act of teaching by drawing on relevant propositional and case study knowledge. The knowledge of beginning teachers might be categorised as propositional knowledge, gained from their teacher education courses, and also case study knowledge gained from reflections on their own teaching and the teaching of others as a student teacher. Strategic knowledge can only be developed through the act of teaching and beginning teachers will therefore have little access to this.



Other mathematics educators have argued for a kind of *active* knowledge that can only develop in practice. Mason and Spence (1999) distinguished between *knowing-about* and *knowing-to*. They drew on the seminal work of Ryle (1949) in which he distinguished between *knowing that* (factual knowledge), *knowing how* (knowledge of how to perform acts), and *knowing why* (having stories to account for actions). Mason and Spence suggested that these three ways of knowing may all be considered as *knowing about*, but that knowing about a subject does not guarantee being able to apply this knowledge in practice. They proposed *knowing-to* as a fourth way of knowing that is active rather than static. They suggested that *knowing to* might arrive ‘suddenly like a bolt of lightning’ (p. 146), which resonates with the model of teacher knowledge suggested by Hegarty (2000). In this model the teacher is seen as having a number of incomplete sets of relevant insights, elements of which come together in instances of teaching to form a new insight specific to that situation. According to Hegarty’s model, teacher knowledge is firmly situated within the act of teaching, and can only be developed in practice.

These ideas are consistent with a social theory perspective of knowledge and knowledge development. Lave and Wenger (1991) suggested that knowledge does not exist in the consciousness of individuals but rather in their participation in social practices. Rather than bringing knowledge to situations and applying it, knowledge is constructed in social practice, and can therefore only be identified or developed in context (Greeno, 1998; Putman and Borko, 2000). Hodgen (2003) found mathematical content knowledge to be *situated* in the professional context of practitioners. Experienced practitioners in his study demonstrated more sophisticated knowledge of mathematics in the context of their practice than they were able to do in an interview. These practitioners appeared to *know to* when involved in professional action and when surrounded by the artefacts of their practice. This was in contrast to the findings of the study reported here where the mathematical content knowledge of the beginning teachers did not initially appear to be supported in the same way by the contexts of their teaching. In this paper I explore why this might be, and suggest how working with the Knowledge Quartet (KQ) framework (Rowland, Huckstep and Thwaites, 2005) appeared to accelerate the teachers’ ability to turn their *propositional knowledge* into *strategic knowledge*.

The KQ framework was developed from observation of mathematics teaching and the categorisation of situations in which mathematical content knowledge was revealed. The original 18 ‘situations’ or codes were later classified into four ‘superordinate’ categories based on associations between them. These categories make up the four dimensions of the Knowledge Quartet; *foundation*, *transformation*, *connection* and *contingency*. The *foundation* dimension encompasses situations in which *subject matter knowledge* (SMK) (Shulman, 1987) including *common content knowledge* (CCK) and *specialised content knowledge* (SCK) (Ball, Thames and Phelps, 2008), become apparent. It also encompasses situations in which

propositional *pedagogical content knowledge* (PCK) (Shulman, 1987) becomes apparent. The prefix ‘propositional’ suggests that this knowledge is in the form of *knowing-about* (Mason and Spence, 1999) and is knowledge which may be gained ‘in the institution’, from personal research or experience.

The three remaining dimensions categorise types of situations in which teachers’ ‘propositional knowledge’ is activated to make it accessible to learners in the process of teaching. Knowledge relating to these three dimensions may be conceptualized in terms of Mason and Spence’s (1999) notion of *knowing-to*. The *Transformation* dimension encompasses situations in which the *active* forms of PCK are revealed through demonstrations, representations and examples used by teachers. The *Connection* dimension encompasses situations in which a teacher’s knowledge of connections in mathematics is made visible in their teaching and involves drawing on their SMK. *Connection* also encompasses situations in which knowledge of how to sequence mathematics teaching and make connections for learners becomes visible and relates to PCK. Finally, the *Contingency* dimension relates to situations in which teachers respond to the unplanned-for and the unexpected in their teaching, when they are seen to draw on combinations of knowledge from the categories defined by Shulman (1986) and Ball Thames and Phelps (2008).

## THE STUDY

The KQ framework was used in this study to facilitate reflection both as a means to, and as a measure of, professional development. I wanted to investigate how effective the KQ would be as a tool for supporting beginning teachers in reflection on practice that would facilitate developments in their mathematical content knowledge. This paper focuses on whether the KQ supported the ability to activate in practice aspects of mathematical content knowledge learned in the institution. The study was conducted in four phases, each lasting one year, and the data collection methods were developed to suit the objectives for each phase. These included observations and videotaping of teaching, post-lesson interviews, group and individual interviews and participants’ written reflective accounts. The KQ was used by me to analyse observed teaching and was used as a framework for discussion with participants about their teaching. The participants used the KQ to structure their written reflective accounts of their teaching. The study began with twelve student teachers from the 2004-5 cohort of primary (5-11 years) postgraduate pre-service teacher education course at the University of Cambridge. The numbers reduced, as anticipated, to 4 in the fourth and last phase of the study. Case studies were completed for three of the four teachers who participated throughout all four years of the study. Two of the teachers worked with Reception classes (4-5 year olds) and it was decided to complete a full case study for just one of these. The three case study participants were given the pseudonyms Amy, Kate and Jess. Amy was an early years trainee (3-7 years) and her highest formal mathematics qualification was a grade B at

GCSE<sup>93</sup>. Kate was a general primary trainee (5-11 years) and her highest formal mathematics qualification was an A\* at GCSE. Jess was a general primary trainee and had gained a B in her mathematics GCSE.

## **FINDINGS IN RELATION TO ACTIVATION OF PROPOSITIONAL KNOWLEDGE**

Amy's lesson that I observed during her final student teaching placement contained two instances in which it appeared that she did not draw on her propositional knowledge. This was a lesson on counting with a Reception class (4-5 years) and during the introduction part of the lesson Amy wanted to assess whether the children could match spoken and written numerals. She asked the children to write various numbers on their individual white boards. One example was '19' and several children wrote '1P', at least one wrote '99' and many wrote '91'. Amy then asked two children to demonstrate how to write '19' on the class white board. One of the children wrote '1P' and Amy spent some time demonstrating how to write '9' correctly. She focused on the reversal of the nine but did not address the problem of incorrect digit order that was apparent for several children. During the post-lesson interview I observed that I had seen several children write '91' and asked Amy if she knew why this might be:

Because you say nine first, then you say the teen that's why often they write the nine first they often want to write nine first then write it from right to left instead of left to right. (Amy, Phase One, Post-lesson reflective interview)

Amy knew about the problems children encounter in writing teen numbers (Wigley, 1997; Anghileri, 2007), but did not apply this knowledge in her practice.

Later in the lesson, the children worked at a number of tasks, some of which involved counting objects. I asked Amy why she had chosen these activities and she explained that she had selected sets of interesting objects in order that the children might practice their counting. I then asked what she thought the children needed to know in order to be able to count these objects:

It's number names and order ... it's not anything to do with cardinality or anything yet, it's just the rote and the reciting numbers ... (Amy, Phase One, Post-lesson reflective interview)

I suggested that some of the activities were practising other pre-requisites of counting (Gelman and Gallistel, 1978), and asked Amy if she remembered what these were:

One to one correspondence ... moving an object, that's in one to one really, so if you've got a random pile of objects you might want to move them out of that into a line or a

---

<sup>93</sup> National examinations taken at 15 years

group ... and remembering the final count is the total. (Amy, Phase One, Post-lesson reflective interview)

Amy also knew that “remembering the final count is the total” was called ‘cardinality’. Amy was able to explain this aspect of her PCK even though she said she had not drawn on these ideas when planning her teaching.

I observed Amy teaching another lesson on counting during the second term of her first year of teaching. This lesson suggested that Amy had become more likely to apply the propositional PCK in her teaching that she had demonstrated in our earlier discussion. In the introduction to this lesson the children guessed how many unseen objects they thought were in various boxes, and then counted them. When the children were counting the objects, Amy praised them for, and drew attention to, the strategies they used, i.e. putting the objects in line, pointing to each in turn with their fingers, saying the numbers in order, and saying that the last number was the answer to ‘how many’. At the end of the introductory part of the lesson, Amy reinforced what had been good about their counting, saying “We said the numbers in the right order, we touched each thing once, and the last number we said was how many there are.” Amy made several comments in her reflective account of this lesson that suggested she had drawn on her propositional PCK when planning and teaching the lesson:

When I was planning this lesson I drew on my knowledge of the pre-requisites for counting: knowing the number names in order, one to one correspondence, the cardinal principle, being able to count objects that cannot be moved/touched and counting objects that cannot be seen e.g. sounds or beats. These developmental stages formed the progression and structure to the lesson. (Amy, Phase Two, Reflective account of observed lesson)

The principles of counting that Amy referred to and made explicit in her teaching had been taught during her teacher education course. She had not made explicit use of this knowledge in the lesson observed in her training year however her knowledge became more explicit in the lesson observed the following year. It seems likely that Amy’s supported reflection on her teaching led her to make greater use of her knowledge of the pre-requisites for counting in the later lesson:

I could have stressed moving the objects rather than putting them in a line ...when they were counting sounds it would have been helpful to match each sound to a held up finger ...When I asked are there more frogs or more snakes I could have asked a child to come up and show these on the number line ... (Amy, Phase Two, Reflective account of observed lesson)

Amy’s reflection on her practice appeared to draw on her propositional knowledge of the principles of counting and enabled her to make informed suggestions for improvements.

Kate also appeared to have early difficulties in drawing on the propositional knowledge that had been taught in her teacher education course. In a lesson with a Year 1 class (5-6 years), observed during her final student teaching placement, Kate used a representation of a witch's cauldron to record doubling. The number to be doubled was written as an addition sentence in the cauldron, e.g.  $3 + 3$ , and the answer in a bubble above the cauldron. After demonstrating doubling the numbers 1-9, Kate moved on to doubling two digit numbers. Her method for recording these involved writing the doubled number of tens and of units in separate bubbles, e.g. double 13 was recorded as 2 tens in one bubble and 6 units in another. This method of recording would seem to reflect *column value* rather than *quantity value* (Thompson, 2003). When discussing this in the post-lesson reflective interview, Kate recognised that her recording of this procedure might have been problematic:

Maybe it wasn't the most helpful thing to do at all because the easy way to do it is to think about it as a column method and if you are not doing it that way then it is probably quite hard. (Kate, Phase One, Post-lesson reflective interview)

In discussion, Kate demonstrated that she knew that by recording the tens as single units she had obscured the value of the tens number and suggested a column method for addition. She recognised that this was not the most appropriate calculation method for these problems. Propositional knowledge concerning the advantages of using quantity value over column value and mental over written methods for addition was available to her in discussion but had not been drawn on during the lesson.

On a number of occasions during the lesson, Kate used a number line to help children complete addition calculations such as ' $8 + 3$ ' by beginning at one of the numbers and then counting on the second number, e.g. starting at 8 on the number line, counting on 3 and giving the answer as the number at which she had arrived. This pre-supposed that children had reached the 'count on' stage in addition (Carpenter and Moser, 1984). However, observation of the children's independent use of the number line suggested that some were still at the 'count all' stage. I asked Kate if she remembered the stages children go through in learning addition:

At first not knowing that you can just start at numbers, that you have to count the one, two, three ... so you have to count three to get up to three before you can carry on. (Kate, Phase One, Post-lesson reflective interview)

Although she knew that some children would not be able to understand the addition strategy of starting with one number and then counting on the second number, Kate had not drawn on this knowledge in her teaching. Like Amy, Kate demonstrated that when probed, she could refer to PCK that was not apparent in her teaching. In phase two of the study I observed a lesson in which Kate modelled a number of methods for solving addition problems, involving both 'count all' and 'count on' strategies. Kate demonstrated that she was now drawing on her propositional knowledge of the stages in which children learn addition by counting, and applying this to her

teaching. Kate's reflection on her phase one lesson appeared to have enabled her to draw on and activate her propositional knowledge in the later lesson.

In a lesson observed during her third year of teaching, Kate demonstrated that she had become adept at drawing on her propositional knowledge in the act of teaching. The lesson had been planned by another teacher. This teacher had prepared a power point presentation to demonstrate addition strategies for two-digit numbers. Kate had only briefly looked at this before starting the lesson. When a slide was displayed showing  $23 + 12$  as  $(20 + 10) + (3 + 2)$ , Kate 'deviated from the agenda'<sup>94</sup> and made a new slide showing  $23 + 12$  as  $23 + 10 + 2$ . In the post lesson reflective interview, I asked Kate why she had changed her demonstration in this way. Kate explained that for two-digit addition, she normally used the strategy of keeping one number whole and partitioning the second number as children found this easier. When she realised that the slide was introducing a different addition strategy, she had changed it to represent the method her class were used to. Although she did not use the terms 1010 and N10 (Beishuizen, 2001) which had been referred to in her mathematics methods course, Kate demonstrated that she was aware of these two different methods of addition, and was able to make decisions about the appropriateness of their use in the act of teaching.

There was evidence from my study that both Amy and Kate appeared more able to draw on their propositional knowledge as they gained experience of, and reflected on, teaching. I observed instances at the beginning of their teaching in which they did not draw on knowledge taught during their teacher education course although they did refer to this knowledge in post-lesson interviews. In later lessons, there was evidence that they were drawing on the same propositional knowledge and activating it in their teaching. The findings were slightly different for the third case study participant, Jess. I observed an instance in Jess' early teaching in which she did not seem to draw on propositional knowledge taught in her teacher education course. However, unlike Amy and Kate, Jess was not able to recall this propositional knowledge during discussion. There was evidence however that she was able to develop this knowledge and to draw on it as she reflected on her teaching experience.

The observation happened in the second term of Jess' first year of teaching. This was a lesson involving the relationship between multiplication and division and Jess wrote ' $3 \times 4$ ', ' $12 \div 3$ ' and ' $12 \div 4$ ' on the class white board. She represented the relationship between these operations by drawing four circles, each containing three dots. For  $12 \div 3$ , this reflected a 'quotition' structure of the division problem and for  $12 \div 4$  a 'partition' structure. Teachers commonly refer to these two structures as 'grouping' and 'sharing' respectively. Jess consistently used the term 'share' when

---

<sup>94</sup> Deviate from the agenda (DA) is a code of the Knowledge Quartet framework from the Contingency dimension

talking about this division calculation. During the post-lesson interview, I probed Jess' understanding of the two structures of division, and she admitted to being unsure of the distinction:

Explaining dividing in terms of grouping and sharing still gets me mixed up. It is something I need to work on myself. The aim was to explain in terms of grouping. In future I am going to sort this out before the lesson so my physical representations don't get mixed up. (Jess, term two first year of teaching, reflective account)

Helping student teachers understand the two structures of division was a key objective of one of the sessions in Jess' mathematics methods course. At this stage, Jess appeared unable to draw on the knowledge taught in this session. However, a year later there was evidence that Jess was more secure in her knowledge of the two structures and was able to draw on this in her teaching. During a lesson on division, Jess told the children that they would be focusing on the word 'share' and using sharing to answer some division problems. She displayed an interactive whiteboard screen showing the problem '16 divided by 2 equals [ ]' also written as ' $16 \div 2 = [ ]$ ' and 16 small circles which she referred to as buttons.

Jess demonstrated a procedure for solving this problem involving 'sharing' the sixteen 'buttons' one at a time into two groups and counting how many in each group. A similar 'sharing' procedure was demonstrated for  $20 \div 2$ ,  $20 \div 5$  and  $24 \div 4$ . All the models used in this lesson reflected Jess' intended focus on the partition (sharing) structure of division. During the group interview Jess explained how she had become clearer about the structures of division:

In the [previous] division lesson you saw I got my sharing and my grouping mixed up ... now I have got like this clear idea so when I was planning I did think about 'is this sharing, is this grouping' and did actually draw a diagram out to make sure I had my ideas clear. (Jess, first term second year of teaching, group interview)

It seems that when she began teaching, Jess did not have sufficiently secure knowledge about the two structures of division to support her teaching. However, reflection on her teaching supported by the KQ framework appeared to enable her to develop this propositional knowledge and to activate it in her teaching

If you make a mistake in a lesson, like I got sharing and grouping wrong, every time I do division now I get my book out and go 'that's sharing, that's grouping' and draw some pictures so I know I have got it clear in my head before I start. (Jess, Phase Three, term three, Group interview)

## DISCUSSION

There were instances in the early teaching of all three teachers that suggested they did not draw on PCK that had been taught during their mathematics methods courses. There was evidence, particularly in the cases of Amy and Kate, that some of the ideas that they had been taught appeared to be held as propositional knowledge but

that they were unable to draw on this and activate it in their teaching. Amy was able to state the principles for counting, but did not use them explicitly in her planning and teaching. Kate recognised the advantages of using quantity value rather than column value when recording calculations, but did not take account of this in her teaching. She also failed to recognise the need of some children to use ‘count all’ strategies when doing addition. However, Amy and Kate were both able to answer questions about these aspects of PCK during post-lesson reflective interviews. This would seem to be inconsistent with the notion of knowledge as being *situated*, which suggests that teachers should exhibit greater knowledge in their professional context than they could demonstrate through questioning (Hodgen, 2003). However, the professionals in Hodgen’s study were experienced ‘experts’, whereas Kate and Amy were both at the very beginning of their careers. It may be that some experience is necessary before beginning teachers are able to draw on *situated* knowledge. In her second year of teaching, Kate was able to draw on her knowledge of the 1010 and N10 strategies for addition when using a power point presentation prepared by another teacher. Two years of experience enabled her to bring together the knowledge *situated* in the context of her teaching and the propositional knowledge taught in the university.

There was evidence that all three teachers drew on propositional knowledge in their later teaching that they had not seemed able to draw on in earlier lessons. Amy explicitly referred to the principles for counting, Kate used both ‘count-on’ and count-all’ strategies and Jess clearly focused on the ‘partition’ structure in her lesson on division. The beginning teachers in this study appeared to become more able to draw on their propositional knowledge as they had more experience of teaching and as they reflected on that experience. Reflection on their practice close to the teaching context, in terms of both time and physical proximity, seemed to enable the teachers to draw on knowledge that had been inaccessible to them when planning and when teaching. Propositional knowledge appeared to become more likely to become activated in future lessons following such reflection. My study suggested that reflection, supported by the KQ framework to focus on the mathematical content of teaching, helped the teachers to draw on their propositional knowledge and to activate it in their teaching. This suggests that in order to help student teachers apply mathematical pedagogical content knowledge taught during their teacher education courses we should perhaps take a longer view and support beginning teachers in consistently reflecting on their practice in a way that focuses on the mathematical content.

## REFERENCES

Anghileri, J. (2007). *Developing number sense: Progression in the middle years*. London: Continuum.



## Working Group 17

- Ball, D. L., Thames, M. H. and Phelps, G. (2008). 'Content knowledge for teaching: What makes it special?' *Journal of Teacher Education*, 59(5), pp. 389-407.
- Beishuizen, M. (2001). 'Different approaches to mastering mental calculation strategies', in J. Anghileri (Ed) *Principles and practices in arithmetic teaching*, pp. 119-130. Buckingham, Philadelphia: Open University Press.
- Carpenter, T.P. and Moser, J.M. (1984). 'The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education*, 15(3), pp. 179-202.
- Gelman, R. and Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, MA: Harvard University Press.
- Greeno, J.G. (1998) The situativity of knowing, learning and research. *American Psychologist* (January), pp. 5-26.
- Hegarty, S. (2000). 'Teaching as a knowledge-based activity', *Oxford Review of Education*, 26, (3-4) pp. 451-465.
- Hodgen, J. (2003). 'Teacher identity and professional development in primary school mathematics', *Unpublished PhD thesis*, King's College, University of London.
- Lave, J. and Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.
- Mason, J. and Spence, M. (1999). 'Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment'. *Educational Studies in Mathematics*, 38, pp. 135-161.
- Putman, R. and Borko, H. (2000). 'What do new views of knowledge and thinking have to say about research on teacher learning?' *Educational Researcher*, 29(1), 4-15.
- Rowland, T., Huckstep, P. and Thwaites, A. (2005). 'Elementary teachers' mathematics subject knowledge: The Knowledge Quartet and the case of Naomi'. *Journal of Mathematics Teacher Education*, 8, pp. 255-281.
- Ryle, G. (1949). *The concept of mind*, Hutchinson: London.
- Shulman, L. (1986). 'Those who understand: Knowledge growth in teaching'. *Educational Researcher*, 15 (2), pp. 4-14.
- Shulman, L. (1987). 'Knowledge and teaching: Foundations of the new reform'. *Harvard Educational Review*, 57 (1), pp. 1-22.
- Thompson, I. (2003). 'Place value: the English disease', in Ian Thompson (Ed.) *Enhancing Primary Mathematics Teaching*, pp. 181-190. Buckingham: Open University Press.
- Wigley, A. (1997). 'Approaching number through language', in Ian Thompson (Ed.) *Teaching and learning early number*, pp. 113-122. Maidenhead: Open University Press

.

# Working Groups: Poster Communications

## DIDACTICAL ENVIRONMENTS “STEPPING” AND “STAIRCASE”

Nad'a Stehlíková, Milan Hejný, Darina Jirotková

Charles University in Prague, Faculty of Education

*A scheme-oriented teaching strategy based on developing mental schemas (Hejný et al., 2007) has been systematically studied at the Faculty of Education. Its main tools are didactical environments (substantial learning environments, see Wittmann, 2001). The focus of the poster are two arithmetic environments that we use for learning operations with natural numbers, negative numbers and as a preparatory vehicle for equations. (The whole poster is available from the first author.)*

Keywords: substantial learning environments, natural numbers, integers, operations

The learning environments “Stepping” and “Staircase” were developed in the Czech context by M. Hejný to support teaching at the elementary school: handling integers and understanding simple equations and their systems. The poster presents a brief description of the recent implementation of this environment in Grade 6 with students who had not met any stepping in their elementary grades (see Stehlíková, 2010).

In the “Stepping” environment, students **took steps** according to instructions in the school corridor with square tiles on the floor (see fig.). E.g., Adam and Boris were standing next to each other



on the blue tiles facing forward. Adam got an instruction “Two steps forward, then three steps backward, begin now!” and carried it out. The teacher asked the class: “What is the instruction for Boris so that he stands next to Adam?” The class: “One step backward.” (A step backward was done without turning.)

Note: In “Stepping”, numbers are used as *operators*. If we take steps on numbered stairs, we work with two types of numbers: *addresses of stairs* and *operators of the position change*. This environment is called “Staircase”.

To record the instructions, the pupils spontaneously used arrows or plus/minus signs. The teacher negotiated with them that they would use the **arrow notation**: an arrow to the right meant one step forward and an arrow to the left, one step backward:  
 $|\rightarrow\rightarrow|\rightarrow\rightarrow\rightarrow|$  or  $|\rightarrow\rightarrow\rightarrow\rightarrow|\leftarrow\leftarrow|$ .

Next, a new instruction was introduced – ‘**turn about**’, which meant that a pupil turned 180° to the opposite direction. It always came in pairs (so that the pupil faced the original direction). The pupils took actual steps and solved tasks which should start their journey towards understanding the effect of the minus sign before a parenthesis; i.e., subtracting a negative number is the same as adding a positive one. A new abbreviation was used:  $|\check{C}V|\rightarrow|\check{C}V| = \leftarrow$ ,  $\rightarrow\rightarrow\rightarrow|\check{C}V|\leftarrow|\check{C}V| = \rightarrow\rightarrow\rightarrow\rightarrow$ , etc. (ČV as ‘turn about’ in Czech.) When taking steps, the pupils experienced that “turning about changes the direction of the arrow”.

Later, the pupils were to **transcribe the arrow tasks in numbers** and the question was what to do with ČV. The pupils felt the need to separate a part of the calculation so it was natural to introduce a convention: the first ČV is written as the minus sign and left parenthesis and the second ČV is written as the right parenthesis.

Instead of continuing with actual stepping, each pupil was given a **stepping slip** (see fig.) and a little figure with a marked nose (so that it was clear where it faced). The pupils put the figure on the blue tile, facing to the right, and then moved it as if they took steps. An important rule was that the figure had to begin and end with its face in the ‘forward’ direction (facing to the right).



With the tasks, the pupils were to get experience with operations with integers and to realise the meaning of minus before a parenthesis. E.g.,  $-(-1) = 1$ ,  $-(1) = -1$ ,  $3 - (-1) = 4$ ,  $-3 + 1 = -2$ ,  $-1 + 3 = 2$ ,  $-1 - 3 = -4$ ,  $-(-1 - 3) = 4$ ,  $-(1 + 3) = -4$ , etc. Differences in the pupils’ performances were seen. Some quickly spotted rules. Some began to calculate without the use of the figure. The aim was to let everyone use the slip as long as they wanted. After writing numbers 0, 1, 2, 3, ... -1, -2, -3, ... on their slips they created a model of a number line. Negative numbers explicitly acquired double meaning: *procedural* (movements backwards) and *conceptual* (an address).

## CONCLUSIONS

We have observed that Stepping-Staircase environments: 1) reinforce the idea of addition and subtraction as a movement on a number line and that is why it preceded the use of the stepping strip; 2) help pupils to grasp the meaning of calculating with negative numbers by means of the arrows notation; 3) illuminate the difference between subtraction and negative numbers; 4) help pupils to understand the rule of ‘minus before a parenthesis’; 5) serve as a model which bridges the gap between the informal and formal understanding of operating with integers.

Acknowledgement: The presented work is supported by the project MSM 0021620862.

## REFERENCES

- Hejný, M., Jirotková, D., Slezáková, J. (2007). Procept – Schema – Structure. In J. Novotná, H. Moraová (Eds.), *SEMT '07 – International Symposium, Elementary Maths Teaching*. Proceedings. Praha: UK v Praze, PedF, 134–141.
- Stehlíková, N. (2010). „Schritte – Stufen“: Eine Lernumgebung für negative Zahlen. In C. Böttinger et al. (Eds.), *Mathematik im denken der Kinder. Anregungen zur mathematikdidaktischen Reflexion*. Berlin: Klett, 60–66.
- Wittmann, E. (2001). Developing mathematics education in a systemic process. *Educational Studies in Mathematics Education*. 48, 1–20.

## UNDERSTANDING THE INFINITE SETS OF NUMBERS

**Cristian Voica**

*University of Bucharest, Romania*

**Florence Mihaela Singer**

*University of Ploiești, Romania*

### INTRODUCTION

We present an overview of a long term research program focused on students' understanding of the concept of infinity. Many questions connected to the idea of infinity could be addressed to school learning. Among these: Does an intuition of infinity really exist in children? Does the school build an understanding of infinity? How is this understanding reflected on problem solving along the schooling period? What capacities do trigger one's choice of a strategy in solving problems that deal with infinite sets? We answered these questions in a series of papers, some already published and others in progress. The poster was a summary of this research, organized around the themes listed below.

### PERCEPTIONS IN UNDERSTANDING INFINITY

When discussing infinity, children seem to highlight three categories of primary perceptions: processual, topological, and spiritual (Singer&Voica, 2003, 2007). Around 11–13 years old, processual and topological perceptions interfere each other, while before and after this age they seem to coexist and collaborate, one or the other being specifically activated by the nature of different tasks.

### REASONING ABOUT THE INFINITY OF THE SETS OF NUMBERS

School mathematics is dealing with infinite sets of numbers: these are, actually, the only infinite objects that we “know” so far. Based on a processual perception, children see the set of natural numbers as being infinite, and endow  $\mathbf{Q}$  with a discrete structure by making transfers from  $\mathbf{N}$  to  $\mathbf{Q}$ . In addition, children are able to make a transfer of reasoning from  $\mathbf{N}$  to  $\mathbf{Q}$  (Singer&Voica, 2008). Contrary to other studies, we found that, for many children in the lower secondary school, the concept of infinity of  $\mathbf{N}$  is strong enough to permit reasoning. For example, some students found it necessary to apply negation for arguing that a given set is finite.

### STRUCTURES ACTIVATED IN THE PROBLEM SOLVING PROCESS

When reasoning about infinite sets, children seem to activate four categories of conceptual structures (Singer&Voica, 2010): geometric (*g*-structures), arithmetic (*a*-structures), fractal-type (*f*-structures), and density-type (*d*-structures). Students select different problem-solving strategies depending on the structure they recognize within the problem domain. They naturally search for structures in challenging learning contexts.

## FURTHER RESEARCH: RELATION MISCONCEPTIONS - STRUCTURES

Students activate various misconceptions when comparing infinite sets of numbers (e.g. Tsamir, 1999). Our research leads to the conclusion that the emergence of a certain structure is responsible for the misconceptions the students show when dealing with infinite sets. An overview of the relations between the categories presented above is given in Figure 1.

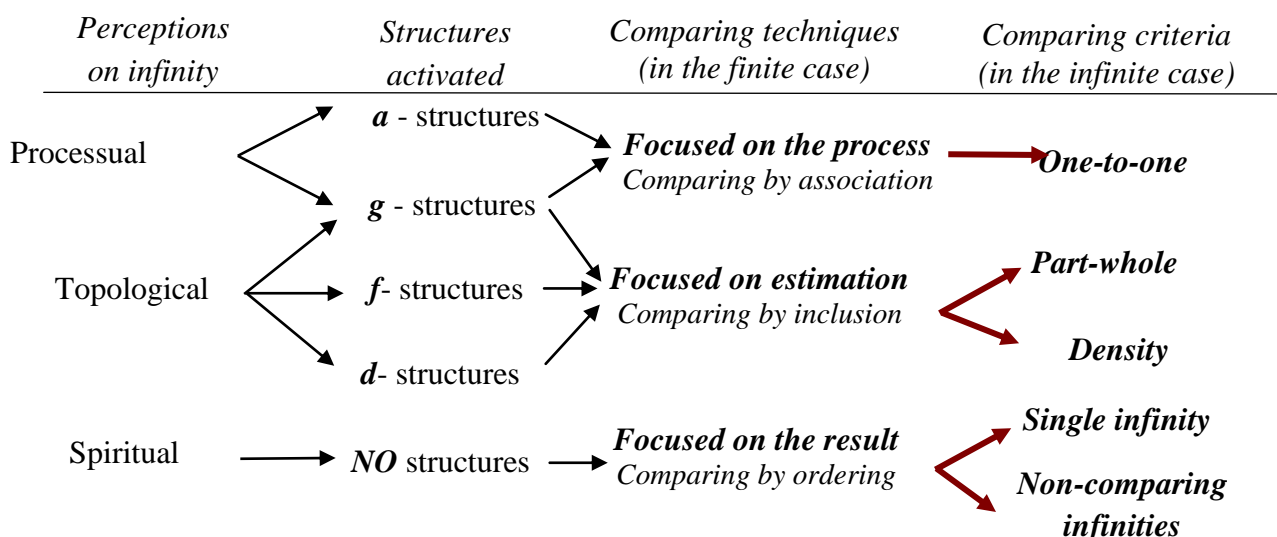


Fig. 1: Connections between perceptions, structures and comparing strategies

## BRIEF DESCRIPTION OF THE CONTENT OF THE POSTER

The poster contained the scheme presented in Figure 1, brief explanations, and some excerpts from relevant comments and drawings of the interviewed students. The pdf version of the poster is available under <http://gta.math.unibuc.ro/pages/cristi.html>.

**Keywords:** *infinity, structures, misconceptions.*

## REFERENCES

- Singer, M., Voica, C. (2003). Perception of infinity: does it really help in problem solving? In A. Rogerson (Ed.), *International Conference "The Decidable and the Undecidable in Mathematical Education"* (252-256).
- Singer, M., Voica, C. (2007). Children's perceptions on infinity: could they be structured? In D. Pitta – Pantazi and G. Philippou (Eds.), *CERME 5* (506-515).
- Singer, M., Voica, C. (2008). Between Perception and Intuition: Thinking about Infinity. *Journal of Mathematical Behavior*, 27, 188-205.
- Singer, M., Voica, C. (2010). In Search of Structures: How Does the Mind Explore Infinity? *Mind, Brain, and Education*, 4 (2), 81–93.
- Tsamir, P. (1999). The transition from the comparison of finite to the comparison of infinite sets: Teaching prospective teachers. *Educational Studies in Mathematics*, 38 (1-3), 209-234.

## EXPLORING PATTERNS IN ALGEBRAIC THINKING

António Borralho, Elsa Barbosa

Education and Psychology Research Centre, University of Évora, Portugal

*Nowadays algebraic thinking has become central to the mathematics curriculum. The development of algebraic thinking is seen as essential to the mastery of algebra. The transition between numbers and a higher level of abstraction is not trivial and in moving from arithmetic to algebra students experience genuine difficulties (Barbosa and Borralho, 2009 and Sinistsky, Ilany and Guberman, 2009). Teachers should diversify strategies, allowing their students to develop algebraic reasoning and symbol sense (Arcavi, 2006). Considering the classroom environment, the present study aims to understand the use of patterns in a context of research tasks as a way to improve the progress of algebraic thinking. We started out with two main research questions focused on: (1) algebraic reasoning and (2) mathematical communication.*

**Keywords:** mathematics education, investigations tasks, patterns, mathematic reasoning, Algebra, algebraic thinking.

### INTRODUCTION

The transition between numbers and a higher level of abstraction seems to be one of the thorniest stages in the mathematics teaching-learning process. Therefore, it is essential to choose adequate strategies that allow students to broaden and deepen their understanding of the algebraic language.

Algebraic equations may have multiple solutions, giving students the opportunity to explore different solution paths. This view acknowledges the crucial role played by teachers, in what concerns encouraging and exploring varied solutions; that is, helping students to improve algebraic thinking.

Orton and Orton (1999) claim that patterns are a possible way to approach algebra and, consequently, develop algebraic thinking. According to Bishop (1997), when a student identifies the relationship between quantities and patterns he/she acquires important mathematical knowledge, for example, the concept of function. This means he/she is learning to investigate and communicate algebraically. Solving research tasks related to patterns emphasizes, on the one hand, investigation, conjecture and proof. On the other hand, and no less important, these tasks need to be interesting and challenging to students (Vale and Pimentel, 2005). Finally, they need to promote the communication of mathematical ideas (Barbosa, 2007). In short, they need to promote a patterning approach to algebra furthers mathematical skills as it interconnects itself with exploration and investigation tasks.



## METHODS

A qualitative and interpretive method was chosen, where a class was considered the unit of analysis in what concerns research tasks involving patterns. The study was carried out in an eighth-grade class (13-14 years old) and between two of its students, considering definite criteria, and aimed a descriptive and analytical outcome. The methodological option in conducting this research led to a qualitative and analytical study case.

## CONCLUSIONS

Our study confirms Orton and Orton's results (1999) that the study of algebra may be initiated through inquiry and generalization of patterns. We found that exploring patterns in a context of investigative tasks enables the development of algebraic thinking and that such tasks are interesting and challenging to students. However, at the same time, teaching practices must be changed, leaving behind a "traditional" teaching that promotes routine and an "isolated" learning experience of contents, and move to teaching practices that give rise to meaningful and contextualized learning.

## REFERENCES

- Arcavi, A. (2006). El desarrollo y el uso del sentido de los símbolos. In I. Vale, T. Pimental, A. Barbosa, L. Fonseca, L. Santos and P. Canavarro (Org), *Números e Álgebra na aprendizagem da Matemática e na formação de professores* (pp. 29-48). Lisboa: Secção de Educação Matemática da Sociedade Portuguesa de Ciências da Educação.
- Barbosa, E. and Borralho, A. (2009). Exploring Patters and Algebraic Thinking. In M. Tzekaki, M. Kaldrimidou, and H. Sakonidis (Eds.). *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 344. Thessaloniki, Greece: PME.
- Bishop, J. W. (1997). *Middle School students' Understanding of Mathematical Patterns and Their Symbolic Representations*. Chicago: Annual Meeting of the American Educational Research Association.
- Orton, A. and Orton, J. (1999). Pattern and Approach to Algebra. In A. Orton (Ed.), *Pattern in the Teaching and Learning of Mathematics* (pp. 104-124). London: Cassel.
- Sinitsky, I., Ilany, B. and Guberman, R. (2009). From Arithmetic To Informal Algebraic Thinking Of Pre-Service Elementary School Mathematics Teachers. In M. Tzekaki, M., Kaldrimidou, and H. Sakonidis. (Eds.). *Proceedings of the 33rd Conference of the International Group r the Psychology of Mathematics Education*, Vol. 5, pp. 129-136. Thessaloniki, Greece: PME.
- Vale, I., Pimentel, T. (2005). Padrões: um tema transversal do currículo. *Educação e Matemática*, 85, 14-22.

## **PROSPECTIVE TEACHERS DOING MODELING ACTIVITIES AND INTERPREATING STUDENTS WORK**

Neusa Branco

Escola Superior de Educação do Instituto Politécnico de Santarém, Portugal  
Unidade de Investigação do Instituto de Educação, Universidade de Lisboa

*The poster presents strategies of prospective elementary school teachers in modeling situations and the contribution of the exploratory approach of their course of algebra and functions to their understanding of how to promote its teaching before the use of formal notation.*

**Keywords:** Prospective elementary school teacher, algebraic thinking, modelling.

The poster presents the understanding of prospective teachers about modeling situations and the contribution of the exploratory approach of their course of algebra and functions to the understanding of how to promote its teaching to elementary school children. Moreover, this approach intends to promote discussions with prospective teachers about the development of algebraic thinking in their future students and about some misunderstanding and errors starring by students.

### **MODELLING AS A STRAND OF ALGEBRAIC THINKING**

In Portugal algebra is taught throughout all grades. Thus, recent programs for elementary school try to articulate the algebra in all mathematical topics and to adapt the strands of algebra to each grade. Kaput (2008) presents two central aspects of algebra: (A) symbolic generalization of regularities, and (B) syntactically guided reasoning and actions on generalizations expressed in convectional symbol systems. These aspects are integrated into three strands: (1) the study of structures and systems abstracts, (2) the study of functions, relationships and covariation, and (3) application of a cluster of modeling languages. In elementary school, before the introduction of algebraic notation, students must be able to generalize situations using natural language and drawings in informal strategies. Students may guess and check solutions informed by the identification of linear relations and use their own notations (Sutherland, 2004). The focus of this poster is the third strand of algebra in two dimensions, the prospective elementary school teachers' understanding of algebraic activities and the expanding of these activities to elementary school students.

### **THE STUDY**

The research follows a qualitative approach, using case studies. The participants are a class of 20 prospective teachers that attended the algebra and functions course in the third year of a teacher education program. This course was taught by the researcher. The main instruments of data collection are two mathematical tasks where two quantities are unknown (Task A – carried out before the course and Task B – carried out after the course), participant observation and documents produced by

participants. Exploratory tasks proposed in the course contribute to develop participants' algebraic thinking and their knowledge about learning and teaching algebra in elementary school. One of the tasks proposes the analysis of strategies used by Grade 6 students to solve the chicken's problem and the reflection about the teaching approach (Reeves, 2000). Collecting and analyzing of data run, in part, in parallel (Merriam, 1988). The analysis of data takes into account the strategies used by the participants and discusses the contribution of the exploratory approach to the understanding of how to promote the learning of algebra.

## RESULTS

In task A, five prospective teachers gave the right solution but only four participants justified their solution with one guessing the values based on the difference of prices and three solving the system of equations. In task B, 15 participants present a correct solution and all explained their strategies, using pictorial representations, natural language or symbols: Twelve participants established their own strategies based on relationships and three solved the system of equations (not exactly the same three from task A). In the classes, the prospective teachers recognised the importance of discussion with students about different strategies to solve a problem with one or more unknown quantities and the initial uses of algebraic symbols:

The fact that the teacher valorised the reasoning more than the results also contributed to the development of algebraic thinking. (Student 1)

Some children used the letters and then explained their reasoning to colleagues and why they used them. From here the teacher can encourage the use of the letters, what begins to happen in a natural way. (Student 2)

## CONCLUSION

The study analyzed the participants' answers in modeling situations, before and after the course. After the study, a greater number of students present a solution based on the identifications of relations. The study shows the perspective of the prospective teachers about the exploration of modeling activities with their students, emphasizing the relevance of reflecting about pedagogical aspects, namely the work that can be used in the classroom to promote algebraic thinking.

## REFERENCES

- Kaput, J. (2008). What is Algebra? What is algebraic reasoning? In J. Kaput, D. Carraher, M. Blanton (Eds.), *Algebra in the Early Grades* (pp. 5-17). Mahwah, NJ: Lawrence Erlbaum.
- Merriam, S. (1988). *Case study research in education: A qualitative approach*. San Francisco, CA: Jossey-Bass.
- Reeves, C.A. (2000). The chicken problem. *Mathematics teaching in the middle school*, 5(6), 398-402.
- Sutherland, R. (2004). A toolkit for analysing approaches to algebra. In K. Stacey, H. Chick & M. Kendal (Eds.), *The future of the teaching and learning of algebra: The 12th ICMI Study* (pp. 73-96). Dordrecht: Kluwer.

## **DIFFERENTIATED LEARNING ROUTES FOR SCHOOL ALGEBRA USING ONLINE DATABASE SYSTEMS**

Julia PILET

*Paris-Diderot Paris 7 University, France  
Laboratoire de Didactique André Revuz (L.D.A.R.)*

*This poster presents the main lines of my current PhD work. It aims to design computer resources to help mathematics teachers to differentiate students' learning in school algebra in France in the final stage of compulsory education (15-16 years) and to study students' algebra activity.*

**Keywords:** School algebra differentiated learning routes, online database system, cognitive assessment

### **CONTEXT AND AIMS**

While algebra constitutes an essential element of the curriculum in order to have access to higher education, it has the reputation of being a difficult, disheartening subject, which poses many challenges for students (Kieran, 2007). Many teachers also have difficulties taking into account their students' cognitive diversity in algebra. Taking this into consideration, the *PépiMeP* project<sup>1</sup>, of which my PhD is a part, consists of implementing computer resources in the *LaboMeP* online database system to help teachers to differentiate students' learning in elementary algebra and to study mathematics teachers' practice and students' algebraic activity. *LaboMeP* is an online database of exercises developed by one of *PépiMeP*'s partners: *Sésamath*<sup>2</sup>, a French mathematics teachers association, which has had a central place in French online database systems for ten years. In the project, the *Pépité* software, a starting point of my research, based on a multidimensional analysis of algebraic skills (Grugéon, 1997), is integrated in *LaboMeP* and produces an automatic cognitive assessment of students' algebraic skills (Delozanne & al., 2008). My research consists of modelling differentiated learning routes for school algebra adapted to the cognitive assessment produced by *Pépité*, implementing them into *LaboMeP* and evaluating them as regards to students' algebraic activity. My research questions are: How to model differentiated learning routes? Which series of tasks to conceive in order to improve students' relationship with school algebra? How to conceive these tasks to integrate them into *LaboMeP*? Which feedbacks and tools to conceive in order to support students' activity? What are the effects of learning routes on students' activity?

### **FROM THE MODELLING OF DIFFERENTIATED LEARNING ROUTES ...**

The model of learning routes is based on a cognitive approach with Kieran's model for conceptualizing algebraic activity (Kieran, 2007), an epistemological approach and an anthropological approach (Chevallard, 1992) in order to construct the model. A differentiated learning route is defined by types of tasks and didactical variables

(expression complexity and task formulation) adapted to students' cognitive assessment established by *Pépité* and in order to develop transformational, generational and global/meta-level activities. Exercises are selected from *LaboMeP* or conceived with feedbacks and tools within the context of *PépiMeP* in order to enrich *LaboMeP* with new exercises. An example of learning routes is developed in the poster presented at CERME 7.

### ... TO THE EVALUATION OF STUDENTS' ALGEBRAIC ACTIVITY

Evaluation is linked to the use of technology because learning routes will be integrated and evaluated from the *LaboMeP* database. The following questions are considered: Are learning routes pertinent from a cognitive and epistemological point of view? How do students appropriate learning routes on the *LaboMeP* database? What are their effects on students' activity? Methodology combines qualitative and quantitative studies: a large-scale one with an ICT monitoring in *LaboMeP*, and some case studies with observations and interviews in classroom.

### NOTES

1. The three partners are: Laboratoire de didactique André Revuz (Paris 7 University), Laboratoire d'Informatique de Paris 6 (Paris 6 University) and the Sésamath association. Ile de France region supports the project. More information is given in "The PépiMeP Project: Online Database Systems and Differentiated Learning Routes for School Algebra" poster in working group 15 about technologies and resources in math education.

2. <http://www.sesamath.net/>

3. The actual poster presented at CERME 7 may be obtained from the author by emailing her at [pilet@math.jussieu.fr](mailto:pilet@math.jussieu.fr).

### REFERENCES

- Chevallard Y. (1992) Concepts fondamentaux de la didactique : perspectives apportées par une approche anthropologique. *Recherches en didactique des mathématiques*, 12(1), 73-111.
- Delozanne E., Previt D., Grugeon B. and Chenevotot, F. (2008), Automatic Multi-criteria Assessment of Open-Ended Questions: a case study in School Algebra, *Proceedings of ITS'2008, LNCS 5091* (pp.101-110). Dordrecht: Springer.
- Drijvers P., Kieran C. and Mariotti M-A. (2010), Integrating Technology into Mathematics Education: Theoretical Perspectives. In C. Hoyles and J.-B. Lagrange (Eds.), *Mathematics Education and Technology-Rethinking the Terrain*, 17th ICMI Study, (pp. 89-132). Dordrecht: Springer.
- Grugeon B. (1997), Conception et exploitation d'une structure d'analyse multidimensionnelle en algèbre élémentaire. *Recherches en didactique des mathématiques*, 17(2), 167-210.
- Kieran C. (2007) Learning and teaching algebra at the middle school through college levels. In F. K. Lester (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning* (pp.707-762). Greenwich, CT: Information Age Publishing .

## TEACHING AND LEARNING OF PARAMETERS IN FUNCTION FAMILIES <sup>[1]</sup>

Magda Nunes Pereira<sup>a</sup> and Manuel Joaquim Saraiva<sup>b</sup>

<sup>a</sup>Group of Schools Almeida, School EB 2,3/S of Vilar Formoso, Portugal

<sup>b</sup>University of Beira Interior and UIDEF, Portugal

**Abstract:** *In the study that supports this paper was created a model for teaching the concept of parameter in functions, which intended to answer to the students' difficulties in this mathematical subject. It results from the reflection on the semiotic representations and transformations associated with the development of algebraic thinking in the resolution of investigative and exploratory tasks. In this model were defined three main levels – First: Operational level of reference; Second: Operational informal level; and Third: Structural level. These three levels interact dynamically in semiotic-cognitive processes in the learning of the parameter concept in a functional relationship, and in a hierarchical way (from the first to the third).*

**Keywords:** *Function Families; Parameters; Semiotic Representations*

### THE PROBLEM OF THE STUDY

How do students represent, transform and convert the parameters in a functional relationship in achieving the structural level of knowledge?

### THE MODEL

In a functional relationship, that involves parameters, our model considers three levels of understanding:

First Level (Operational level of reference) – The student recognizes the correlation between dependent and independent variables, identifies and implements numerical parameters; the student uses informal representations (schemas, tables, graphs and statements in natural language) in his reasoning.

Second Level (Informal operational level) – The student recognizes the change of the variables (independent and dependent) and of the parameter; the student uses yet the informal representations (schemas, tables, graphs and statements in natural language) in his reasoning.

Third Level (Structural level) – The student turns and connects the representations that he used in the previous level; the student uses formal representations (symbolic, graphical, schematic, and statements in natural language) in his reasoning.

## THE RELEVANCE OF THIS MODEL TO THE LEARNING OF PARAMETER CONCEPT

In function families, a parameter is a generalized number that, when concretized, identifies functions of that family. For this understanding is necessary abstraction, but the progressive formalization of the concept causes difficulties for students. With the use of this model is intended to promote the learning of parameter concept by its three different levels – from the elementary to the structural one.

### NOTE

1. This poster is integrated in the TASK 1 (Estimation, Symbol sense, and functions) of the Research Project *Improving Mathematics Learning in Numbers and Algebra*, supported by FCT, MCTES, Portugal.

### REFERENCES

- Duval, R. (2006). Quelle sémiotique pour l'analyse de l'activité et des productions mathématiques? *Relime, Número Especial*, 2006, pp. 45-81.
- Filloy, E., Puig, L. & Rojano, T. (2008). *Educational Algebra. A Theoretical and Empirical Approach*. Mathematics Education.
- Gravemeijer, K. (2005). What makes mathematics so difficult, and what can we do about it? In L. Santos, A. P. Canavarro and J. Brocardo (Eds.), *Educação Matemática: Caminhos e Encruzilhadas* (pp. 83-101). Lisboa: APM.
- Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 11-49). Rotterdam: Sense Publishers.
- Pólya, G. (1957). *How to solve it*, second edition, Princeton.
- Radford, L. (2006). Elementos de una teoría cultural de la objectivación. *Relime, Número Especial*, 2006, pp. 103-129.
- Sfard, A. (1992). Operational origins of mathematical objects and quandary of reification: The case of functions. In E. Dubinsky and G. Harel (Orgs.), *The concept of function* (pp. 59-84). Washington, DC: Mathematical Association of America.
- Ursini, S. & Trigueros, M. (2004). How do high school students interpret parameters in algebra? In M. Johnsen Hoines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th international conference of the International Group for the Psychology of Mathematics Education*, PME 28 (pp. 361-368). Bergen, Norway, July 14– 18: Bergen University College.

## LEARNING RISK IN SOCIO-SCIENTIFIC CONTEXT

Hasan Akyuzlu

Institute of Education, University of London

*This poster gives you a glimpse at research on how students make sense of the concept of risk and how they create their own models when a technology-enhanced approach is taken.*

Keywords: probability, impact, risk.

### WHAT IS RISK?

Risk is an important socio-scientific concept in everyday life. There is a large body of literature on risk, covering a range of disciplines from mathematics to psychology. Each of these perspectives can contribute to a better understanding of how risk is constructed, perceived and managed by experts and the general public. However, neither mathematics nor psychology can explain risk by themselves; a combination of both sciences is needed, with risk at the socio-scientific intersection of the two. Adams (1996) proposed that risk is the product of the probability and utility of some future event. However, there is no agreed definition of risk in the literature; therefore this issue yields different epistemologies of risk. Sometimes, risk can be considered as possible harm, which is an issue relating to impact. Sometimes, it can be presented as a probabilistic idea, which is the quantification of likelihood. In some cases, risk can be presented as the coordination of likelihood and impact. This coordination makes risk a complex issue because, in situations which are not well defined, most people do not know how to coordinate likelihood and impact. In this theoretical account, we set out these issues in order to motivate the need for design research to explore students' thinking about risk.

### Methods:

We aim to design computer-based modelling tools which provide students with the opportunity to explore the context, to build their own model and to express their own understanding in that context. To do that, we have decided to use Deborah's Dilemma, which has been designed for teachers to explore and interrogate their knowledge of risk. We have conducted interviews with a pair of students (aged 17-18) and asked questions to trigger the discussion, to expose the idea behind their decision. In this sense, the students' activity provides us a window on their thinking about risk (Noss & Hoyles, 1996).

### Findings and Conclusion:

Pratt et al. (in press) found that teachers drew on personal experiences and values of their risk-based decision-making in a personal dilemma. The mapping tool was designed at the end of their study but it has not been systematically tested. I used



mapping tool in order to facilitate students' risk-based decision-making. In contrast to teachers, students in this study talked in a superficial way about their personal experiences and emphasising her condition in a social aspect.

Student: My mum was in pain and she had the operation, then she got rid of the pain. I knew Deborah is in pain and she has to get rid of it. So, she should have the operation... "Not having the operation" affects her relationship and her job. .. She will not be happy if she cannot do sports as she might likes.

Subsequently, the same study and methodology was tested with 15 years old students and they referred to impact separately from likelihood even though their discussions were more thoughtful and rational than 17-18 years old students.

## REFERENCES

Adams J.(1996). *Risk* . London: UCL Press.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: learning cultures and computers*. London: Kluwer Academic Publishers.

Pratt, D., Ainley, J., Kent, P., Levinson, R., Yogui, C., and Kapadia, R. (in press). Informal inferential reasoning as a contextualised risk-based enterprise. *Mathematical Thinking and Learning*.

## **FUTURE ELEMENTARY AND KINDERGARTEN TEACHERS' KNOWLEDGE OF STATISTICS AND OF ITS DIDACTICS**

Raquel Santos

Escola Superior de Educação de Santarém, Portugal

*This poster addresses prospective teachers and infant educators' knowledge of statistics and of its didactics. The poster presents the results of a pilot test made with a questionnaire given to two classes of students. Some results of the questionnaire are presented and some conclusions are made regarding the improvement of their training.*

*Keywords: statistical knowledge, professional development*

### **INTRODUCTION**

Most teachers do not feel able to teach statistics at the early years of schooling. Given their weaknesses in this field, they avoid teaching it or teach it in a superficial way. Nonetheless, in Portugal, the new mathematics curriculum indicates that statistics should be taught since the first years of school so that students can deal critically with the information around them. Therefore, there is a need to improve the professional development to prepare teachers to work accordingly to the curriculum. It is with this idea in mind that I propose to diagnose the prospective teachers' knowledge of statistics and of its didactics at the School of Education of Santarém.

The poster includes part of this investigation, since it regards only the results of a pilot test made with a questionnaire. 26 students in their 3<sup>rd</sup> and last year of the teachers' program answered the questionnaire. The questionnaire, composed by 14 questions, included both questions concerning statistical knowledge and concerning their knowledge of didactics of statistics, covering primarily the topics: organization of data, statistical measures and statistical investigations. A quantitative analysis of the answers to the questionnaire is presented and some conclusions are elaborated.

### **THEORETICAL FRAMEWORK**

Prospective teachers usually have weak or no training in statistics (Batanero et al., 2004), as well as in statistical pedagogy (Froelich et al., 2008). However, as Shulman (1986) underlines, to teach, it is essential to master the subject and the way of teaching it. In his perspective, the knowledge needed to teach a certain subject – “pedagogical content knowledge” (PCK) – includes not only knowledge of the subject, but also examples, applications, models and representations, connections between topics, etc.. This particular poster uses the Curcio's (1987) components of graphic comprehension (reading the data, reading between the data and reading beyond the data), Mokros and Russel's (1995) conceptions of average (as mode, as

algorithm, as reasonable, as midpoint and as balance), as well as Monteiro's (2009) different properties of average.

## FINDINGS

A student answered, in average, to 70% of the questionnaire and left the other 30% blank. This can be related to the fact that it was the final of the year when they answered it or may be consequence of their statistical knowledge. Regarding the organization of different types of data, these students revealed more difficulty with the organization of quantitative continuous data. An interesting fact is that 7 students, when asked to make generalizations based on a graphic representation, used data from their own experience. Concerning the statistical investigation implementation in class, the majority of the students who answered this question stated a theme (recycling) and a survey or questions to be answered, which is probably related to their own experience in class as learners. Only one student was able to briefly discuss the class implementation of the statistical investigation. All the others connected the task to doing the investigation itself.

## CONCLUSION AND FURTHER RESEARCH

The most interesting conclusion of this pilot test was that students are stuck to their own experience as learners when thinking about teaching. They are not able to go further on the planning of a lesson, stating only tasks that they did during their training. This may be an indication that students need more diverse experiences as learners, where afterwards they discuss its usefulness for students and its implementation in a classroom. The actual poster presented at CERME7 may be obtained from the author by emailing her at raquelfms@gmail.com.

## REFERENCES

- Batanero, C., Godino, J. D., & Roa, R. (2004). Training teachers to teach probability. *Journal of Statistics Education*, 12(1).
- Curcio, F. (1987). Comprehension of mathematical relationships expressed in graphs. *Journal for Research in Mathematics Education*, 18(5), 382-393.
- Froelich, A. G., Kliemann, W., & Thompson, H. (2008). Changing the statistics curriculum for future and current high school mathematics teachers: A case study. In C. Batanero, G. Burrill, C. Reading & A. Rossman (Eds.), *Proceedings of the ICMI Study 18 and 2008 IASE Round Table Conference*. Monterrey, Mexico.
- Mokros, J., & Russell, S. J. (1995). Children's concepts of average and representativeness. *Journal for Research in Mathematics Education*, 26, 20-39.
- Monteiro, C. (2009). Que conhecimentos são necessários para se ensinar a média aritmética? Comunicação apresentada no *XIX Encontro de Investigação em Educação Matemática: Números e Estatística: Reflectindo no Presente, Perspectivando o Futuro*, Vila Real, Portugal.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Research*, 15(2), 4-14.

## COMPARING ATTITUDES TOWARDS MATHEMATICS AND STATISTICS OF K-10 STUDENTS: PRELIMINARY RESULTS

Assumpta Estrada and Ana Serradó

Universitat de Lleida, Spain, aestrada@matematica.udl.cat

La Salle-Buen Consejo, Spain, ana.serrado.bayes@gmail.com

*Attitudes towards mathematics and statistics of 48 grade-10 Spanish students are analysed. The descriptive analysis of students' response to ATMI and SATS as well as a qualitative analysis of students' justifications for their scores are used to comparing the value component of attitudes towards mathematics and statistics.*

Keywords: statistics, attitudes, secondary compulsory education

### INTRODUCTION

Attitudes are a key factor in improving the learning process (Estrada, 2009). They are manners of acting, feeling or thinking that show a person's disposition or opinion, and may involve negative or positive feelings, which result from positive or negative experiences over time in learning a topic (Phillips, 2007). A hypothesis suggested by Gal and Ginsburg (1994) is that students may transfer their feelings towards mathematics into statistics. In this paper we present results from an exploratory study that is oriented to provide empirical results that support this conjecture.

### METHOD

The ATMI (Attitudes Towards Mathematics Inventory) by Tapia & March (2004), and the Survey of Attitudes Towards Statistics (SATS) by Schau, Stevens, Dauphinee & Del Vecchio (1995) were given to the same sample students. The subjects were 48 undergraduate students, with an average age of 15 years and 60.4 % of girls in the sample. We carried out a descriptive analysis of scores in questionnaires as well as a content analysis of the students' written justifications for their scores in each item. We also compare our results with previous research (Carmona, 2004; Estrada & cols, 2005, 2009 and Tapia, 2004).

### RESULTS

The results of the descriptive analysis of the ATMI suggest a slightly positive attitude in mathematics, since the mean score is over 3 (neutral point) in all the items. An exception is item 33 (average score only 2.79). *I plan to take as much mathematics as I can during my education* that corresponds to the factor "motivation". As regards attitudes towards statistics (SATS), three items (22, 13, 26) resulted in scores lower than 3. In general, ATMI scores were higher than SATS scores, and the factor "value" had higher scores in both scales. The content analysis of written justifications suggested that: (a) Mathematics was seen as important and

useful in personal and professional life by these students; and (b) Statistics was not considered to be so important in the personal and professional life for these students. Moreover, the importance given to statistics by these students depends of the type of work and they were not aware of the full range of application of statistics.

## CONCLUSIONS

Attitudes towards statistics and towards mathematics were moderate or positive in both groups, but attitudes towards mathematics were better, in general. Contrary to the suggestion by Gal & Ginsburg (1994), this result suggests that not all the students transfer their feelings towards mathematics into statistics. Students have different view of the usefulness, relevance and value of statistics and mathematics in personal and professional life. All these results should be interpreted carefully, given the sample size, but they provide an interesting starting point to future research.

Acknowledgments: Research supported by the project: SEJ2010-14947/EDUC. MCYT-FEDER

## REFERENCES

- Carmona, J. (2004). Revising the reliability and validity evidence of attitudes and anxiety towards statistics questionnaires. *Statistics Education Research Journal*, 3(1), 5-28. On line: <http://www.stat.auckland.ac.nz/~iase/serj/>.
- Estrada, A. (2009). *Las actitudes hacia la estadística en la formación de los profesores*. (Attitudes towards statistics in training teachers) Milenio. Lleida
- Estrada, A., Batanero, C Fortuny, J. M. & Diaz, M. C. (2005). A structural study of future teachers' attitudes towards statistics. In M. Bosch (Ed.), *Proceedings of CERME IV. European Research in Mathematics Education*. Sant Feliu de Guisols: ERME. CD-ROM
- Gal, I. & Ginsburg, L. (1994). The role of beliefs and attitudes in learning statistics: towards an assessment framework. *Journal of Statistics Education*, 2(2). On line: <http://www.amstat.org/publications/jse/>. .
- Phillipp, R. A. (2007). Mathematics teachers' beliefs and affects. In F. Lester (Ed.). *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Charlotte, NC: Information Age Publishing and National Council of Teachers of Mathematics.
- Schau, C., Stevens, J., Dauphine, T., & del Vecchio, A. (1995). The development and validation of the survey of attitudes towards statistics. *Educational and Psychological Measurement*, 55 (5), 868-875.
- Tapia, M. & Marsh, G. E., II (2004). An instrument to measure mathematics attitudes. *Academic Exchange Quarterly*, 8(2), 16-21

## TEACHING DIFFERENTIAL EQUATIONS WITH MODELLING

Guerrero-Ortiz, C.

Camacho-Machín, M.

Cinvestav- IPN, México

University of La Laguna, Spain

*Some results obtained from the implementation of a teaching experiment are presented in this paper. Particular attention has been paid in the modelling process of a problem leading to an Ordinary Differential Equation (ODE). The research is framed within the methodology of Design Research and the problem used in the teaching experiment was designed taking into account certain elements of modelling.*

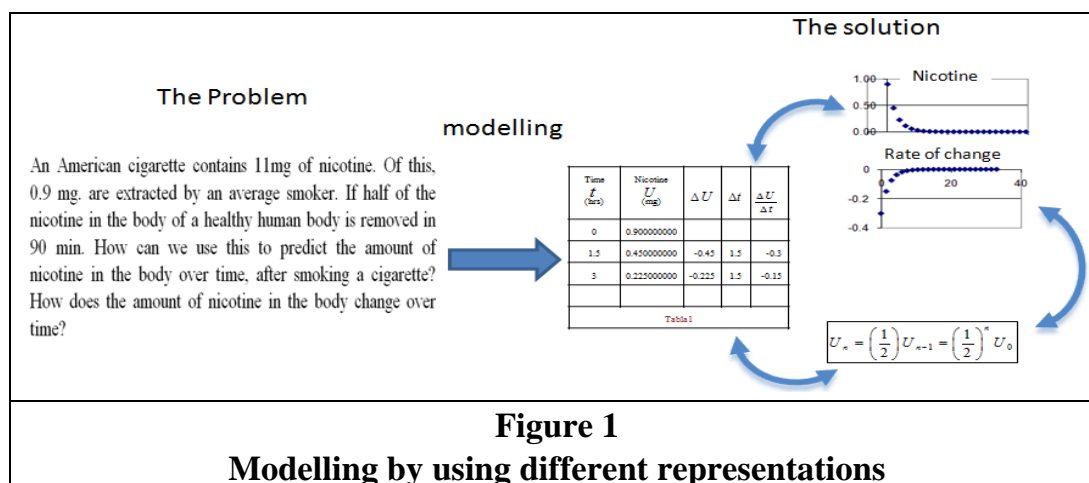
### Introduction and Conceptual Framework

The activity presented in this paper is part of a teaching sequence that has been developed with the aim of introducing the concept of Differential Equation. Our main purpose is to provide students with tools to interpret an ODE and its solutions when they model a phenomenon. The results shown here are part of the first cycle of investigation in accordance with the Design Research Methodology (Brown, 1992; Drijvers, 2003). Several representations of the solution appear in the modelling process, therefore, the focus is to show how one can promote the conversion between different representations of the solutions (Duval, 1988) and what the students' interpretations are regarding the rate of change by using different representations. As Doerr & Tripp (1999) said, modelling activities stimulate students to show different solution strategies as a result of the mental activities generated. Thus, they provoke the development of their conceptual comprehension by trying to represent their ideas.

### Methodology

As part of Design Research, there are two key aspects: the cyclic character of design research and the central position of the design of instructional activities. This research methodology consists of three phases: preliminary design, teaching experiments and retrospective analysis. Here we present the results of the first cycle of investigation. The instruction took place with activities involving mathematical skills and processes related to Problem Solving (Barrera & Santos, 2002).

The teaching activity was carried out with a group of seven engineering students, who chose to participate voluntarily in the experiment. One of the primary goals of the teaching was to begin modelling simple phenomenon by ODE such as  $\frac{dU(t)}{dt} = kU(t)$ , the activity in figure 1 was designed in this way. The following three representations are present in the process of modelling: numerical, graphical and algebraic. The students can obtain the algebraic representation of problem by looking for patterns of behavior and the visual representation can help them to elaborate conjectures about behavior of the substance in the body.



## Results and Comments

Among other results, we can observe, by implementing the teaching sequence, some of the student's knowledge of concepts such as: rate of change, continuity, and some representations of the derivative. A part of these results show that students think in terms of amounts when they interpret a rate of change, this difficulty is verified in the different representations of rate of change. However, later activities showed that the students built the instantaneous rate of change more easily. We support the idea that building the concept of the ODE through modelling can help students to develop interpreting skills between the real context and the mathematical context and establish relationships between different mathematical concepts. The change from the discrete function to the continuous function is not trivial. It is important to work on the connections between average rate of change and instantaneous rate of change. It is necessary to introduce activities that promote reflection on these topics in the future design of teaching sequences.

## REFERENCES

- Barrera, F. y Santos, M. (2002). *Cualidades y procesos matemáticos importantes en la resolución de problemas: un caso hipotético de suministro de medicamento*. Matemáticas aplicadas y su enseñanza. Ed. Iberoamérica.
- Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences*, Vol 2, pp. 141-178.
- Drijvers, P. (2003). *Learning Algebra in a Computer Algebra Environment. Design research on the understanding of the concept of parameter*. Utrecht: CD-β Press, Center for Science and Mathematics Education. – (CD-β wetenschappelijke bibliotheek; 2003)
- Doerr, H & Tripp, J (1999). Understanding How Students Develop Mathematical Models. *Mathematical Thinking and Learning*. Vol.1(3),pp.231-254.
- Duval, R. (1988). Graphiques et Equations: L'Articulation de deux registres. *Annales de Didactique et de Sciences Cognitives*. Vol.1.pp.235-253. IREM de Strasbourg.

## **BELIEFS ON THE USEFULNESS OF MATHEMATICS AND MATHEMATICS SELF-BELIEFS AS IMPORTANT FACTORS FOR MATHEMATICS ATTITUDES**

Peter Vankúš<sup>1</sup>

Comenius University in Bratislava, Slovakia<sup>1</sup>

*The poster deals with the results of the questionnaire surveys of pupils' attitudes towards mathematics. The aim of the poster is to stress results which show the importance of specific attitudes in areas for the whole of pupils' mathematics education. The questionnaire used was a modified version of the Mathematics Related Beliefs Questionnaire and the research sample were students of the 5<sup>th</sup> and 9<sup>th</sup> school year in Slovakia. The results of the questionnaire were processed by the statistical software CHIC. The pupils' beliefs on the usefulness of mathematics and their mathematics self-beliefs were identified as important factors for their liking of mathematics and their mathematics achievements.*

Keywords: affects towards mathematics, attitudes, MRBQ, questionnaire survey, improving of mathematics education

### **INTRODUCTION**

The importance of pupils' attitudes towards mathematics is supported by the opinion, believed to be true in scientific and teacher communities, which states that pupils learn more effectively and they are more interested in the mathematics lesson and are performing better if they have positive attitudes towards mathematics (Ma & Kishor, 1997).

In our poster, we deal with the results of questionnaire surveys on pupils' attitudes towards mathematics performed in Slovakia. In our study, that was part of the comparative research study led by Jose Diego-Mantecon and Paul Andrews from the University of Cambridge, the modified Mathematics Related Beliefs Questionnaire (De Corte & Op't Eynde, 2002) was used. The questionnaire was designed to find out the compatibility of its use in various European states, specifically in England, Spain and Slovakia (Andrews et al., 2007; Andrews et al., 2008). The aim of this poster is to report on part of the results from the surveys carried out in 2007 on a sample of 204 Slovak pupils, 76 in the 5<sup>th</sup> year (9–11 years old) and 128 pupils in the 9<sup>th</sup> year of the primary school (14–16 years old) and in 2008 on a sample of 241 pupils in the 9<sup>th</sup> year. The outcomes were analyzed by the statistical software CHIC (Gras et al., 2008) and showed significant impacts of the beliefs on the usefulness of mathematics and mathematics self-beliefs on other components of pupils' mathematics education as liking of mathematics, mathematics achievements and active learning strategies.



These results are very informative for mathematics education. They show that we can positively influence process of mathematics education by the improving of two elements: pupils' mathematics self-beliefs and their beliefs on the usefulness of mathematics.

## REFERENCES

- Andrews, P., Diego-Mantecón, J., Op 't Eynde, P., & Sayers, J. (2007). Evaluating the sensitivity of the refined mathematics-related beliefs questionnaire to nationality, gender and age. In. D. Pitta-Pantazi & G. Philippou (Eds.) *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education*. Larnaca: University of Cyprus, 209-218.
- Andrews, P., Diego-Mantecon, J., Vankúš, P., Op 't Eynde, P., & Conway, P. (2008). A tanulók matematikai meggyőződéseinek értékelése: Egy három országot érintő összehasonlító vizsgálat. [Construct consistency in the assessment of students' mathematics-related beliefs: A three-way cross-sectional pilot comparative study]. *Iskolakultúra Online*, 2008(2), 141-159.
- De Corte, E., & Op 't Eynde, P. (2002). Unraveling students' belief systems relating to mathematics learning and problem solving. In. Rogerson, A. (Ed.) *Proceedings of the International Conference "The Humanistic renaissance in mathematics education"*. Palermo: The Mathematics Education into the 21st Century Project, 96 -101.
- Gras, R., Suzuki, E., Guillet, F., & Spagnolo, F. (Eds.) (2008). *Statistical Implicative Analysis, Theory and Applications*. Springer.
- Ma X., & Kishor N. (1997). Assessing the Relationship Between Attitude Toward Mathematics and Achievement in Mathematics: A Meta-Analysis. *Journal for Research in Mathematics Education*, 28 (1), 26-47.

## ON THE MEANING OF MULTIPLICATION FOR DIFFERENT SETS OF NUMBERS IN A CONTEXT OF VISUALISATION

**Raquel BARRERA**

Ph.D. candidate in Didactics of Mathematics

Laboratoire André Revuz – Université Denis Diderot – Paris 7

**Abstract.** *This work is a study of definitions, meanings and representations of multiplication for different sets of numbers from middle school to high school in France. Specifically, it concentrates on the integer, rational and complex number domains in order to determine the role that visual and geometric representations could play in the learning process of multiplication.*

**Key-words.** Multiplication, semiotics, geometric representations, transformation.

**Description of the poster.** Could a geometric treatment of multiplication highlight the link between different meanings of multiplication related to different number domains? This question serves as the title of the poster, followed by different geometric representations of multiplication in the integer, rational, and complex number domains. Meanings of multiplication, even those corresponding to magnitudes, can be visually linked to transformations in the plane (*i.e.*, factors as operators). Considering that geometric figures help students' reasoning by giving meaning to an assigned problem (Kuzniak, 2006) and by leading students to describe mathematical situations "in order to master mathematical concepts" (Duval, 2003), we reflect on a historical text. Descartes' product (1637) is one way teachers could introduce complex number multiplication: what will we find if we describe what goes on between the points, angles and line segments composing Descartes' geometric construction? Possible responses could include relations of proportionality between line segments, co-linearity of segments, as well as relations based on transformation in the plane such as homothety and rotation.

**Summary of research and further questions to discuss.** Several studies show that the learning of multiplication in the rational, integer and complex number domains implies a break with the first meaning of multiplication. This is because the product can no longer be seen as the result of repeated addition or as the area of a surface. What are the other meanings? How should we teach them and why? Are they important in the learning process of multiplication? Is there a "geometrical common thread" between the different meanings of multiplication?

The theoretical framework that has motivated our research on the meanings of multiplication in geometry is a cognitive approach to semiotic representations. Our assumptions, following Raymond Duval, are that "mathematical objects are never accessible by perception or by instruments. The only way to have access to them and deal with them is using signs and semiotic representations" (Duval, 2006, p. 107)

and also that “There is not mathematical thinking without using semiotic representations” (Duval, 2008, p. 1). Thus, we propose that (1) establishing a link between a numeric or algebraic register and a geometric register may aid comprehension of a complex notion such as multiplication; (2) understanding different meanings for multiplication may facilitate the emergence of various strategies for solving problems linked to this mathematical Notion; (3) more specifically, linking multiplication and geometry may give students a deeper comprehension of factors as operators.

Through this research, we seek to relate our interest in **the meanings of multiplication to the cognitive process linked to a change of semiotic representations** that these meanings make possible. A few research questions are: can meanings of multiplication allow us to move between one semiotic representation and another? Can we suggest didactical treatments for multiplication and its meanings linking algebra, arithmetic and geometry?

An epistemological and didactical analysis of multiplication in three number domains—integers, rational and complex numbers (Glaeser, 1981; Flament, 2003)—and an ecological study (Chevallard, 1985) of middle school and high school curricula in France are the starting point of our methodology. Is there a historical link between the meanings of multiplication and geometric representations? If not, do epistemological or didactical elements exist which propose this link? Can we suggest some didactical treatments for multiplication and its meanings linking algebra, arithmetic and geometry?

Poster accessible online at: <http://tinyurl.com/5ubxboy>

## REFERENCES

- Chevallard, Y. (1985a), *La transposition didactique - Du savoir savant au savoir enseigné*, La Pensée sauvage, Grenoble. p.126.
- Davis, B. & Simm, E. (2006) *Mathematics-for-teaching: an ongoing investigation of the mathematics that teachers (need to) know*. Educational Studies in Mathemtaics.
- Duval, R. (2006). *The cognitive analysis of problems of comprehension in the learning of mathematics*, Springer Netherland, p. 103-131.
- Duval, R. (2008). *Eight problems for a semiotic approach in Semiotics in Mathematics Education* ( Eds Radford...) Sense publishers les pages p 40, et 48-49.
- Flament, D. (2003), *Histoire des nombres complexes, entre géométrie et algèbre*, CNRS histoire des sciences, CNRS Éditions, Paris.
- Glaeser, G. (1981), *Épistémologie des nombres relatifs*, Recherches en Didactique des Mathématiques, La pensée sauvage, Paris, Vol. 2, N° 3, pp. 303-336.

## **GESTURE AND VISUAL-SPATIAL THINKING**

Conceição Costa

*Escola Superior de Educação de Coimbra / UIED FCT UNL,*

José Manuel Matos

*Universidade Nova de Lisboa / UIED FCT UNL*

*This work attempts to grasp visual-spatial thinking in mathematics education. It analyses two classroom interaction episodes between two elementary students using a geometry software that stimulate the exploration of ideas of translation, reflection and rotation, through the motions slide, flip and turn. The excerpts are a follow-up of a study intended to create, explore and refine a theoretical model for visual-spatial thinking. This poster also presents part of conclusions of the research about the visual-spatial thinking, related to grasping “what is the role of the gesture for thinking?” and it intends to contribute to the reflection about the theme.*

Key-words: Visual-spatial thinking, gestures and objectification.

The interest in gestures for mathematics education is growing. Differences in perspective about the role of the gestures in cognition have, to an important extent, to do with theoretical components about thinking (Radford (2009a). Radford advocates that gestures are genuine constituents of thinking, and he presents a sensuous conception of thinking: "thinking cannot be reduced to impalpable mental ideas; it also made up of speech and our actual actions with objects and all types of signs. Thinking, hence, does not occur solely *in* the head, but *in* and *through* language, body and tools" (p.113). Within this perspective, gestures may be seen as part of the tactile mode that individuals use to conceptually grasp something; and thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporality of actions, gestures and artifacts (Radford, 2009b).

Our interest is in gestures as they appear in mathematics learning settings that promote the acquisition of visual-spatial thinking and this poster presets a follow-up of a previous work (Costa, 2005) that intended to create, explore and refine a theoretical model for visual-spatial thinking and through it to understand the development thereof by the identification of the visual-spatial thinking modes and the thinking processes associated to them. The theoretical framework for visual-spatial thinking model took into account research in the areas of cognitive processes in mathematics education, embodiment in mathematics, a socio-cultural perspective on learning with emphasis on the social construction of knowledge and on semiotic mediation, theoretical perspectives on the teaching and learning of geometric concepts.

In this poster, we will use the lens of *theory of knowledge objectification* (TO) (Radford, 2008) and the perspective about gesture of Radford to interpret two

classroom excerpts. In the TO, learning is conceptualized as the active and creative acquisition (thematized as a problem of objectification) of historically constituted forms of thinking through means and processes; these means are called, *semiotic means of objectification*; and the processes are called, *processes of objectification*.

The methodology was qualitative integrating video registrations of individual answers and tasks performed in classroom activity. The analysis of those episodes centres on the gestures and words of the students that will be interpreted through the socio-cultural perspective of Radford, where mathematics learning involves the social and semiotically mediated process of objectification.

## REFERENCES

- Costa, M. C. M. (2005). *Modelo do pensamento visual-espacial: transformações geométricas no início da escolaridade* [A model for visual-spatial thinking: geometric transformations in the beginning of the school]. (Doctoral dissertation). Universidade Nova de Lisboa, Portugal.
- Radford, L. (2009a). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111-126.
- Radford, L. (2009b). *Signs, gestures, meanings: algebraic thinking from a cultural semiotic perspective*. Paper presented at the meeting of CERME 6, Lyon, France.
- Radford, L. (2008). The ethics of being and knowing: Toward a cultural theory of learning. In Radford, L., Schubring, G. and Seeger, F. (eds), *Semiotics in mathematics education: epistemology, history, classroom and culture* (pp. 215-234). Rotterdam, NL: Sense Publishers.

## THE EVOLUTION OF SCHOOL MATHEMATICS DISCOURSE AS SEEN THROUGH THE LENS OF GCSE EXAMINATIONS

Candia Morgan\* and Anna Sfard\*\*

Institute of Education, University of London\* and University of Haifa\*\*

*The ways in which mathematics is communicated to students shape the ways in which they may engage in mathematical activity. A particularly powerful influence on the forms of tasks offered to students is the high-stakes examination. In a new project, we propose to analyse changes in the discourse of English national examinations at the end of compulsory schooling through a period of curriculum change and to investigate how these changes affect students' mathematical activity.*

*Keywords: examinations; discourse analysis*

Over the last decades there has been ongoing public and academic concern about the nature and standards of school mathematics. This concern has driven frequent revisions of curriculum and examinations, yet controversy continues and there are contradictory opinions about the effects of reforms. The overarching aim of our study is to investigate the evolution of school mathematics in England, as seen through the lens of examinations at 16+. Our main question asks: *What has changed since the introduction of the GCSE<sup>95</sup> examination in the mathematics that pupils are expected to learn and in the way they are expected to approach mathematics?* Rather than comparing syllabi or teaching methods, we seek to probe deeply into the nature of the mathematical activity expected of students. We do this by developing and applying a discourse analytic approach, drawing on Social Semiotics (Bezemer & Kress, 2009; Halliday, 1978; Hodge & Kress, 1988; Morgan, 2006) and Sfard's theory of mathematical thinking as communicating (Sfard, 2008). Studying discourse in this way, by focusing on the forms of language used in examination papers, allows a subtle characterisation of the ways they construct the nature of mathematics and of student mathematical activity. We argue that this type of analysis will provide insight into how changes in curriculum and assessment may affect students' mathematical learning.

There will be two principal parts to the study. First, discourse analytic methods will be applied to a sample of examination papers in mathematics for pupils aged 16+, taken from different points in time, including critical points in the development of mathematics examinations since the introduction of GCSE. This will allow us to:

---

<sup>95</sup> The GCSE examination is a national examination taken by almost all students at age 16+. It was introduced in 1988 at a time of major changes in the curriculum and has been revised several times since then.

## Working Group 9

- develop a means of analysis of the discourse of mathematics examinations at 16+ that will allow a characterisation of the nature of mathematical activity required of students and that will be sensitive to changes over time;
- identify and characterise differences over time in the nature of mathematical activity required of students at 16+.

Second, a group of current school pupils will answer a sample of questions from these examination papers. Analysis of their responses and detailed follow-up interviews will allow us to:

- investigate how differences in the discourse of mathematics examination questions may affect the ways students approach the mathematics;
- contribute to a fuller understanding of how changes in curriculum and assessment have affected the nature of students' mathematical learning.

These two parts, when taken together, are expected to help us in addressing the question of in what ways GCSE examinations of different periods can count as mathematically comparable, that is, (a) whether they are “mathematically equivalent” in the eyes of expert mathematics educators (the first part of the study); (b) whether they lead to “equivalent” mathematical activity (the second part of the study). The study will thus:

- contribute to debates about changes in curriculum, pedagogy and assessment methods;
- provide knowledge and analytic tools to inform the design of mathematics examinations and curricular materials.

In presenting the poster we will provide a brief overview of the project, our initial schema for analysing the discourse of examination papers, and examples of analysis that illustrate the differences in the nature of mathematical activity constructed in examination questions from different periods.

## REFERENCES

- Bezemer, J., & Kress, G. (2009). Visualizing English: a social semiotic history of a school subject. *Visual Communication*, 8(3), 247-262.
- Halliday, M. A. K. (1978). *Language as Social Semiotic: The Social Interpretation of Language and Meaning*. London: Edward Arnold.
- Hodge, R., & Kress, G. (1988). *Social Semiotics*. Cambridge: Polity Press.
- Morgan, C. (2006). What does social semiotics have to offer mathematics education research? *Educational Studies in Mathematics*, 61(1/2), 219-245.
- Sfard, A. (2008). *Thinking as Communicating: Human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University Press.

## SELF-REGULATION OF STUDENTS IN MATHEMATICS AND ORAL COMMUNICATION IN CLASSROOM

Sílvia Semana

Leonor Santos

PhD. Student of Institute of Education  
of University of Lisbon, AREA Project<sup>96</sup>

Institute of Education of University of  
Lisbon, CIE, AREA Project

*Through an interpretative case study, the poster focuses the teaching practices of a mathematics teacher, Joana, aimed at promoting students self-regulation, in a context of collaborative work between four teachers and one of the researchers. The results presented are based, mainly, on observation (with audio and video recorder) of two lessons of an 8th grade class of Joana. Special attention is given to oral communication in the classroom, as a central concern of Joana's reflection.*

**Keywords:** *self-regulation, oral communication in classroom, teaching practices, reflection, collaborative work*

### THEORETICAL FRAMEWORK

Research recognizes self-regulation as a process that can help improve students' learning and academic achievement (Boekaerts, Pintrich & Zeidner, 2000). It requires that students possess a concept of the standard level being aimed for, compare the current level of performance with that standard and engage in appropriate action which leads to the closure of the gap (Sadler, 1989). Oral communication in classroom could be an important element in the self-regulation, mainly if it is a thoughtful and informative communication that promotes reflection and argument, that is used to inform and adapt teaching practices and in which prevail discussion patterns (Voigt, 1995). Of course, this depends on how the teachers manage the discussions, question, listen and respond to their students. This requires the teachers to rethink what they do and how they do it, in other words, to reflect on their practice. But changes of practice are challenging. The collaborative work is a favourable setting for developing new forms of work in mathematics classroom (Sowder, 2007).

### JOANA'S CASE

In order to promote self-regulation of students, two lessons of Joana were planned by the collaborative group of teachers, including the proposal of a task in small groups, oral presentations with discussion in all-class and request of a written self-assessment of each working group. In the first lesson, the dominant role of the teacher and the few interventions of the students in reaction to peers interventions were noticed by Joana. An overview of the lesson shows that the number of

---

<sup>96</sup> Project financed by FCT, n° PTDC/CED/64970/2006



interventions of the teacher was about half of the total in group-class and that the voice of students tends to be “sandwiched” by Joana, that intervenes after each student. In the second lesson, Joana stress the need to students comment peers interventions and oral communication changed remarkably. The teacher passed much more unnoticed during oral presentations and the students asked questions to colleagues, commented and argued, communicating among themselves. Indeed, following this second episode we find 26 interventions by students, without any intervention by the teacher.

Eduardo: ... you are saying ... So, the number of the figure is one, isn't it? You have to do three times one plus one?

Duarte: No. You have to make the number of the figure, supposed that the figure is two (...) The number of the figure is two (...) three times, which are the points we add, and one more, which is the middle one.

Eduardo: Ah?

Filipe: Look, look ... Not counting this, two is what will be here [in the 2<sup>nd</sup> figure], three is what will be here [in the 3<sup>rd</sup> figure]...

Ana: Look. It's like this, I get it, we do the number of the figure, which is a ...

Eduardo: Yea...

## CONCLUSIONS

The collaborative work provided opportunities for Joana to plan, discuss and reflect about her classroom practices, especially concerning oral communication and the promotion of self-regulation of students. A remarkable development is visible in oral communication in the classroom, from the first lesson to the second one. Joana seems to become aware of her role in oral communication in the classroom, being more concerned with giving space and encouraging the students to discuss, argue and counter-argue their ideas. The reflection by the teacher, in a context of collaborative work (Sowder, 2007) gains especial importance, by contributing to changes in the teaching practices, which are expected gradual and challenging.

## REFERENCES

- Boekaerts, M., Pintrich, P. R., & Zeidner, M. (Eds). (2000). *Handbook of self-regulation*. San Diego: Academic Press.
- Sadler, D. R. (1989). Formative assessment and the design of instructional systems. *Instructional Science*, 18, 119–144.
- Sowder, J. T. (2007). The mathematical education and development of teachers. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the NCTM* (Vol. I, pp. 157-223). Charlotte: Information Age Publishing.
- Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb, & H. Bauersfeld (eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163-202). Hillsdale, NJ: Lawrence Erlbaum Associates.

## **TEACHERS' ENDORSED AND ENACTED NARRATIVES TO PROMOTE MATHEMATICAL COMMUNICATION**

Marie Bergholm

Linköpings University/ Gävle University Sweden

**Keywords:** mathematical discourse, teacher interviews, commognitive framework

*Ongoing study with the purpose of identifying teachers' strategies, tools and settings to give the students opportunities to participate in mathematical discourse. Several research methods are used to catch and compare the teachers' and students' discourse of mathematic. The study is guided by the commognitive framework and in this part teacher interviews are analyzed by a hermeneutic approach.*

### **INTRODUCTION**

The Swedish curriculum raises demands of individualization and an effect is that the students have been left to work individually at their own pace (Skolverket, 2009). When the mathematics communication decreases, the pupils cannot build up a useful language to master expressions and methods used in the textbook (Löwing, 2004). A conjecture is that teachers' awareness of the mathematical communication is fundamental for learning. The aim is to investigate the teachers' endorsed and enacted principles regarding mathematical communication and to identify teachers' discursive routines to encourage the participants' ability to take part in the classroom discourse.

### **THEORETICAL FRAMEWORK**

This empirical study sets out from the commognitive theoretical framework (Sfard, 2008) rooted in the participationist assumption that the discourse in mathematics plays a central role for learning. Learning is here seen as an activity in which the students modify and extend their discursive repertoire. The discourse analyses will be guided by the commognitive framework and critical hermeneutics is used to interpret the teachers' narratives about the role of mathematical communication.

### **METHODS AND RESEARCH QUESTIONS**

Several research methods are used to catch and compare the teachers' and students' discourse of mathematic; interviews, video recorded classroom observations and use of "smart pens". In this first part, with the purpose of investigating and identifying teachers' strategies, tools and settings to give the students opportunities to participate in mathematical discourse, 15 teachers in Upper Secondary School are interviewed in semi structured audio taped interviews. The transcribed interviews are analyzed according to a hermeneutic approach (Ödman, 2007). The research questions to deal with in the interviews of the teachers are: What narratives do teachers tell about the role of the discourse in learning mathematics and ways to foster the students'

mathematical conversation in different learning situations? What is the expected impact of the students' learning from the teachers' perspective?

## RESULTS

The teachers emphasize that opportunities for students' learning in mathematics occur, when the students explain and discuss mathematics with others. Participating in the discourse the students can be aware of, and familiar with, the peers' strategies and methods to build up thinking structures in mathematics and identify their "error in thinking". The teachers stress that not only the students but also teachers themselves are learning in the classroom discourse. Their ability to understand the students' way of thinking increases and they can take more aspects of how students' struggle with mathematics into consideration in their planning. They put emphasis on the opportunities to observe the students when they are struggling with words and trying to enlarge the discourse in the current context. This enables them to help the students to modify and extend their use of words and fill the existing gaps in the used classroom language.

In order to promote the mathematical communication in the classroom, the most essential obstacle to clear away is the students' lack of self-confidence in the subject, to work against the feeling that it is possible to learn mathematics and defy negative attitudes from parents, teachers and society. Further to inspire the students' feeling that they have time to listen to each other, explain their thoughts with their own vocabulary and exchange ideas. The teachers have to work consciously to encourage the students to put their strategies and how they have figured things out into words. They underline the need to work towards a teaching learning agreement in the classroom where comprehensions is more important than getting a lot of exercises done. Most of the teachers try to avoid predetermined roles by randomly chosen working groups. To encourage the mathematical communication they prefer to arrange group discussions without right or wrong answers, and the students have to actively explore the explanations and interpretations from others and take sides for or against.

Author's email: mebm@soderkoping.se

## REFERENCES

- Ödman, P-J. (2007). *Tolkning, förståelse, vetande. Hermeneutik i teori och praktik*. Norstedts Akademiska Förlag.
- Löwing, M. (2004). *Matematikundervisningens konkreta gestaltning. En studie av kommunikationen lärare-elev och matematiklektionens didaktiska ramar*. Göteborg: Acta Universitatis Gothoburgensis.
- Sfard, A. (2008). *Thinking as Communicating. Human Development, the Growth of Discourses, and Mathematizing*. Cambridge, U.K.: Cambridge University Press.

## COMMUNICATION – A/THE KEY TO MATHEMATICS

Birgit Gustafsson

Department of Natural Sciences, Engineering and Mathematics,  
Mid Sweden University

*Abstract: This poster presents a part study of a larger project dealing with classroom communication and teacher-student interaction in mathematics education. The study focussed upon in the poster investigates communicative aspects and teachers' scaffolding strategies during lessons when algebraic concepts are being taught.*

Keywords: Communication, teacher-student interaction, algebra

### INTRODUCTION

The research project of which the presented study constitutes one part is dealing with teaching and learning mathematics in the latter part of compulsory education and the first year of upper secondary school. The focus of the project as such is upon classroom communication with respect to teacher-student interaction and the teachers' scaffolding strategies on the one hand and students' interpretation and understanding of the learning content on the other. The mathematical domain that is studied is algebra, and both mathematical concepts which are new to the students and concepts, which students are already familiar with, are of interest. It is frequently argued that algebra is an abstract and problematic area (e.g., Olteanu, 2007) which most students have not met in primary education in any formal sense, and which contains a body of new mathematical concepts. Shortcomings in the early teaching and learning of algebra and functions may, in turn, cause later problems in mathematics. There is a risk that difficulties occur if a large number of new concepts that the students have never met before are introduced within a short time and without ensuring an adequate understanding. The aim of the present study is to investigate the teacher-student interaction with focus on teachers' scaffolding strategies.

### THEORITICAL FRAMEWORK

To answer the research questions the Theory of Didactical Situations (TDS) has been adopted as a theoretical framework (Brousseau, 1997). In this theory, the didactical situation is regarded as three-dimensional, and as a relation between the teacher, the pupils and the learning content. Leaning on TDS the analysis has taken its points of departure in: the didactical situation vs. the adidactical situation; the milieu created; the didactical contract

There is a *didactical situation* if the teacher has an active role. If the teacher plays a modest role, we have an *adidactic situation*. The *milieu* is constituted not only by the physical surrounding but also by the design of the lesson. The didactical contract – is

an interdependent obligation between teacher and students. It has characteristic rules that each one has to fulfill.

## METHOD

In a first step, before their introductory lesson in algebra, semi structured interviews were carried out with each of the seven teachers participating in the project. These interviews focused on the learning content and the teachers' intentions with the lesson. Semi structured interviews were chosen because they allow both a certain flexibility and a relaxed conversation in which the teacher might relate to their experiences and understanding (Bryman, 2007). Next, the lesson in question was observed and both video and audio recorded in order to catch, as accurately as possible, the teacher-student interactions and scaffolding strategies used. Follow-up semi-structured interviews with the teacher were held after every classroom observation. Data have been collected through video-recorded observations of classroom situations as well as from interviews with the teachers in these situations. The analysis is inspired by the three dimensions suggested by Sensevy et al. (2005) and deals with (a) the teacher's organization of the milieu, (b) the teacher-student(s) interaction, and (c) the development of the knowledge.

## RESULTS AND DISCUSSION

The analyzed lessons are mainly didactic situations as defined by Brousseau (1997) because the teachers control the learning situation most of the time. There is certainly a didactic contract between the teacher and her/his students. The milieu is organization with materials for learning and tasks to be solved – these tasks are mostly of the same kind as examples they have been shown, which is in line with the teachers' intentions that the students "need to practice". Different kinds of classroom interaction are analyzed. The teachers' introductions differed with respect to scaffolding strategies and ways of communicating with their students. However, all the teachers used everyday language to a great extent and tended to avoid the adequate mathematical concepts. When the students' worked individually with textbook tasks or with problem solving in groups, two teacher strategies were found, namely piloting the students past difficulties and more elaborate guiding.

## REFERENCES

- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht, Netherlands: Kluwer.
- Bryman, A. (2007). *Social Research Method*. New York. Oxford University Press.
- Olteanu, C. (2007). *"What could  $x$  be?" Second degree equation and quadratic function as objects of learning*. Umeå, Sweden: Umeå University.
- Sensevy, G., Schubauer-Leoni, M-L., Mercier, A., Ligozat, F., & Perrot, G. (2005). An attempt to model the teacher's action in the mathematics class. *Educational Studies in Mathematics* 59, 153-181.

## **FAMILY MATHEMATICS INVOLVEMENT: DRAWING FROM A SOCIOLOGICAL POINT OF VIEW**

*Javier Díez-Palomar, Sandra Torras-Ortin*

*EMiCS – Universitat Autònoma de Barcelona*

*This poster presents the research project FAMA (Family Math for Adult Learners) which aim is to built a European network of good practices and resources to promote the family Mathematics education in schools over Europe. Using interviews, questionnaires and focus groups whose applied to students, teachers and families, we collected data. Our findings show the importance of the family involved in children's education, and the influence of family members' past experiences and background in mathematics.*

**Keywords:** Mathematics, family, involvement

FAMA (Family Math for Adult Learners)<sup>1</sup> is a research project funded by the European Commission (Grundtvig Program)<sup>2</sup>. International studies provide a plethora of evidences showing that family involvement in Mathematics Education is a key element to improve children performances in this topic (PISA, TIMSS, PIRLS). Drawing on this fact, FAMA intends to contribute to the development of quality lifelong learning practices among adult learners (parents) in order to promote high performance, innovation and a European dimension in systems and practices in this field (specific priority). FAMA work is framed by a theoretical and methodological perspective based on a social justice approach (making special consideration for vulnerable groups, such immigrant, and working class families). This project contributes to encourage more experiences of family involvement in mathematics education grounded on scientific criteria and successful prior experiences.

This project proposes the following objectives:

- To contribute to the development of quality lifelong learning and to promote high performances.
- To help provide adults with pathways to improving their knowledge and competences.
- To improve the quality and to increase the volume of cooperation between organizations in adult education.

In this project we collected data using interviews, questionnaires and focus groups in the five countries involved in this research study. These instruments have been applied to students, teachers and families. To analyze the data we use discourse analysis techniques (Gee, 1999) and critical communicative methodology (Gómez González & Díez Palomar, 2009; Gómez, 2006). According to our findings, we claim that when families get involved in children's education they are able to overcome

many of the barriers that make learning difficult for themselves as well as for their children. As result students improve their scores. In the same way, drawing on our data we have also observed that family members' past experiences and background also impact on their attitude towards mathematics, which mediates their involvement in their children's mathematics learning as well.

A lack of communication between families and teachers also arise from our findings.

### ACKNOWLEDGEMENTS

We would like to thank the European Commission for the support given to the authors of this study titled FAMA: Family Math for Adult Learners (504135-LLP-1-2009-1-ES-GRUNDTIG-GMP). This publication [communication] reflects the views only of the author, and the Commission cannot be held responsible for any use, which may be made of the information contained therein.

### NOTES

1. In this project six organizations are participating, which are linked in the field of education and formation: UAB and FACEPA (Spain), Kings' College of London (United Kingdom), University of Lisbon (Portugal), AFEC (France), GRIM (Italy). In addition, there are also three more participants that are involved in the proposal. This are: SVEB (Switzerland), VOX The University of Arizona (United States) and Monash University (Australia).

2. FAMA: Family Math for Adult Learners. Number of reference: 504135-LLP-1-2009-1-ES-GRUNDTIG-GMP.

### REFERENCES

European Comission (Grundtvig Program). Family Math for Adult Learners. [http://ec.europa.eu/dgs/education\\_culture/eve/alfresco/n/browse/workspace/SpaceStore/aa8d1e0b-d45f-11de-b89b-09f36ec598be](http://ec.europa.eu/dgs/education_culture/eve/alfresco/n/browse/workspace/SpaceStore/aa8d1e0b-d45f-11de-b89b-09f36ec598be) [Last query: September 2010]

Gee, J. P. (1999). *An introduction to discourse analysis: Theory and method* Psychology Press.

Gómez González, A., & Díez Palomar, F. J. (2009). Metodología comunicativa crítica. *Teoría de la Educación*, 10(3), 103.

Gómez, J. (2006). *Metodología comunicativa critica*. Esplugues de Llobregat, Barcelona: El Roure.

Mullis, I.V., Martin, M.O., Kennedy, A.M., and Foy, P. (2007). *IEA's Progress in International Reading Literacy Study in Primary School in 40 Countries*. Chesnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

National Center for Educational Statistics. (2010, December 1). *NAEP. Reading and Mathematics 2009 State Snapshot Reports*. Retrieved from <http://nces.ed.gov/pubsearch/pubsinfo.asp?pubid=2011456>

OECD (2010, December 1). *PISA 2009 Assessment Framework – Key Competencies in Reading, Mathematics and Science*. Retrieved from [http://www.oecd.org/document/24/0,3343,en\\_2649\\_35845621\\_45011928\\_1\\_1\\_1\\_1,00.html](http://www.oecd.org/document/24/0,3343,en_2649_35845621_45011928_1_1_1_1,00.html)

## LEARNING MATHEMATICS

### THOUGHTS AND INTERPRETATIONS OF STUDENTS WITH FOREIGN BACKGROUNDS

Petra Svensson

School of Education, Malmö University

*In this study, which is in the first phase, students' foregrounds and rationales for learning mathematics will be examined with the aim of producing a qualitative picture of thoughts and interpretations of students with foreign backgrounds on their possibilities to learn mathematics in the Swedish socio-political context.*

Keywords: Multicultural mathematics classrooms, students' foregrounds, rationales for learning.

#### BACKGROUND

My research interest originates from the complex of problems concerning the underachievement in school mathematics of students with foreign backgrounds; i.e. born abroad or born in Sweden with two parents born abroad (Siris databas 2009; SCB, 2002). In this text, I will use the term “student” instead of “students with foreign backgrounds”. In Malmö, the third largest city of Sweden, the academic achievement in mathematics is very low in some multicultural areas; at some schools the majority of the students finish compulsory school without a grade in mathematics (Siris databas, 2009). Hence, too many students do not reach the *goals to be attained*, which according to the National curriculum are the minimum level that students should have when they leave compulsory school. But even more important, this is one of the reasons why too many students do not get access to equal social opportunities in life (Wigerfelt, 2009). In these multicultural areas, the classrooms are multicultural and the majority of the students come from Iraq, Yugoslavia, Bosnia-Herzegovina and Lebanon.

Mellin-Olsen, (1987) describes two major rationales that can be identified as drivers for school learning, the I-rationale (instrumental) and the S-rationale (social). Simplified the I-rationale means that students decide to learn to achieve good grades and pass exams and the S-rationale that students decide to learn the particular knowledge because it has an importance beyond passing the exams (Mellin-Olsen, 1987). These rationales combined with the notion of foreground will constitute the theoretical framework of this forthcoming study. The notion of foreground was first introduced by Skovsmose in 1994 but later developed;

The notion of *foreground* refers to a person's interpretation of his or her learning possibilities and 'life' opportunities, in relation to what the socio-political context seems to make acceptable for and available to the person. (Alrø, Skovsmose, Valero, 2009, p. 7)



## PURPOSE AND RESEARCH QUESTIONS

The aim of my study is to learn more about students' thoughts and interpretations on their possibilities to learn – or not learn to mathematics – and what kind of rationales for learning they have, and to what extent the students' foregrounds have an impact on these rationales.

- How do students with foreign backgrounds experience their possibilities to learn mathematics and what are their reasons?
- To what extent do foregrounds of students with foreign backgrounds impact them when learning or not to learning mathematics?
- What do students with foreign backgrounds believe are their rationales for learning or not to learning mathematics?

## EMPIRICAL STUDIES AND METHODOLOGY

The study is intended to be carried out through classroom observations and interviews with students from 8<sup>th</sup> or 9<sup>th</sup> grade. The purpose of the observations is to get a picture of the learning situation and to get to know the students and how they act during the lessons, and also to help me to decide which students to interview. Since I am interested in learning more about the immigrant students' own thoughts and interpretations about learning mathematics I will use narratives where the data will be constructed by the informants and the researcher together (Goodson, 2001). I will interview students from 8<sup>th</sup> or 9<sup>th</sup> grade because they are about to make a decision about their future since they are about to leave compulsory school. Only the student interviews will be used in the data analysis and it will be carried out by using discourse analysis techniques (Winther Jørgensen, 2000).

## REFERENCES

- Alrø, H., Skovsmose, O., & Valero, P. (2009). Inter-viewing foregrounds. *Social Interactions in Multicultural Settings*, 13-37.
- Goodson, I. F. (2001). *Life history research in educational settings: Learning from lives*. Buckingham: Open University Press.
- Mellin-Olsen, S. (1987). *The politics of mathematics education*. Dordrecht: Kluwer.
- SCB, MIS. (2002). Statistics on persons with foreign background, Guidelines and recommendations, <http://www.scb.se/statistik/OV/AA9999/2003M00/X11OP0203.pdf>
- Siris databas, [http://siris.skolverket.se/portal/page?\\_pageid=33,90009&\\_dad=portal&\\_schema=PORTAL](http://siris.skolverket.se/portal/page?_pageid=33,90009&_dad=portal&_schema=PORTAL)
- Wigerfelt, B. (2009). En likvärdig skola? I: En gränsöverskridande skola: Om olika former av styrning och reglering i barndomen. *Educare vetenskapliga skrifter*, 2-3. Malmö: Lärarutbildningen, Malmö högskola.
- Winther Jørgensen, M. (2000). *Diskursanalys som teori och metod*. Lund: Studentlitteratur.

## THE EMERGENCE OF CULTURAL MATHEMATICS: AN ETHNOMATHEMATICAL APPROACH IN THE CONTEXT OF CLASSROOM

**Joana Latas \*, Darlinda Moreira \*\***

\*EBI/JI de Aljezur \*\*Universidade Aberta, UIDEF- U. Lisbon

*This poster presents an ongoing research whose main objective is to understand the role of cultural mathematics in the development of transversal mathematical processes. Using an ethnomathematical approach based on the cultural profile of students in a 7<sup>th</sup> grade class who participated in the research, a curricular project was designed, which was implemented in the classroom in five phases. An initial data analysis shows that the students: i) appropriated culturally distinct practices through relationships established with their prior knowledge ii) gradually reveal greater predisposition to establish mathematical connections. Throughout the research students improved their capacity of mathematical communication.*

Keywords: Cultural mathematics; mathematical connections; mathematical communication

The integration of cultural aspects in the mathematical curriculum contributes to the understanding of mathematics as part of everyday life, which, by its term, strengthens the possibility to establish meaningful connections between different contexts and, hence, is a factor of students' mathematical comprehension (Adam, Alangui & Barton, 2003 and Bishop, 2005; Boaler, 1993, Zaslavsky, 2002). However, when children come to school they bring a set of cultural experiences with them that might not be valued in the academic context (Gerdes, 2007; Moreira, 2002). The review of the literature highlights that students' foreground and background are important issues that are related to students' predisposition and involvement in their learning process. Thus, not only it is necessary to understand and take into consideration students' different *backgrounds*, but also it is important to combine this information with what students long for their future - *foreground* - and how they deal with their expectations in their social context, (Alrø, Skovsmose & Valero, 2009; Skovsmose & Vithal, 1997). Teachers' pedagogical decisions should, therefore, take into account the understanding of the background and foreground.

Inspired by the theoretical model for the implementation of an ethnomathematical curriculum (Adams, 2004), grounded in an integrated vision between the concepts and cultural practices of students and the predominantly formal mathematics, a project was developed and implemented in five phases: 1) search for each student's background and foreground, their relationships with local meanings, 2) the emergence of practices and connections among different cultural practices, 3) search for mathematical cultural experience, 4) mathematical formalization, and 5) deepening of cultural knowledge based on mathematics.

The methodology was qualitative in nature, following the interpretative paradigm. Data was collected using participant observation, interviews and documentary analysis. Participants were students in a class of the 7th grade in a seashore zone in the south of Portugal. The teacher played simultaneously the role of investigator.

A first analysis suggests that the ethnomathematical approach to explore cultural mathematics in the context of the classroom promotes the establishment of mathematical connections with students' daily lives and previous knowledge, contributing to the appropriation and assignment of meaning to mathematical concepts. Throughout the presentation of students' tasks, the development of oral and written communication and of arguments that used mathematical concepts related to the cultural contexts was further evidenced.

## REFERENCES

- Adam, S., Alangui, W., & Barton, B. (2003). A comment on Rowlands and Carson 'Where would formal academic mathematics stand in a curriculum informed by ethnomathematics? A critical review'. *Educational Studies in Mathematics*, 52(3), 327-335.
- Adam, S. (2004). Ethnomathematical Ideas in the Curriculum. *Mathematics Education Research Journal* 16(2), 49-68.
- Alrø H., Skovsmose, O. & Valero P. (2009). Inter-viewing foreground: students' motives for learning in a multicultural setting. In M. César & K. Kumpulainen (Eds.), *Social interactions in multicultural settings* (pp. 13-37). Rotterdam: Sense publishers.
- Bishop, A. (2005). Aproximación sociocultural a la educación matemática. Colombia: Universidad del Valle.
- Boaler, J. (1993, Junho). The role of contexts in the mathematics classroom: do they make mathematics more "real"? *For the Learning of Mathematics*, 13(2), 12-17
- Gerdes, P. (2007). *Etnomatemática - Reflexões sobre a diversidade cultural*. Ribeirão: Edições Húmus.
- Moreira, D. (2002). Educação Matemática, comunidades e mudança social. In D. Moreira, C. Lopes, I. Oliveira, J. M. Matos & L. Vicente, *Matemática e Comunidades – A diversidade social no ensino-aprendizagem da Matemática* (pp. 9-25). Lisboa: I.I.E. e S.P.C.E..
- Vithal, R. & Skovsmose, O. (1997). The end of innocence: a critique of ethnomathematics. *Educational Studies in Mathematics*, 34 (2), 131-157.
- Zaslavky, C. (2002, Fevereiro). Exploring world cultures in math class. [electronic version]. *Educational Leadership*, 66-69.

## **PARENTAL INVOLVEMENT IN CHILDREN'S ACHIEVEMENT: AN EXPLORATORY STUDY WITH FRENCH 2ND GRADERS IN MATHEMATICS**

**Ana Lobo de Mesquita**

Université d'Artois / Université Paris Diderot,

Laboratoire de Didactique André Revuz & IREM de Paris 7<sup>1</sup>

*Studies in literacy in mother tongue confirm the influence of the family behaviours in children's learning children in linguistic, cultural and urban backgrounds. With the idea of extending these studies, a group of teachers and researchers from two domains (mother tongue and mathematics) and from different countries, conducted studies about the influence of parental involvement in children achievement on these domains.*

*In our preliminary study, we analyse the case of parental involvement in pupils' achievement when mathematical activities are used. We focus on French 2<sup>nd</sup> graders (20 pupils from 7 to 8 years old), in a similar framework and setting than those used in literacy studies, issued from Villas-Boas (2001). We also use the notion of "communities of practice" developed by Wenger (1998). Our first results seem confirm the trend that parental involvement improves student achievement in mathematics<sup>2</sup>.*

Key words : Parental involvement, Achievement, Mathematics Education

Studies in literacy in mother tongue conducted in different contexts and countries have shown that the influence of the family behaviours appears to be strongly significant when those families help children to be aware of their cultural and linguistic backgrounds (Epstein, 1995; Henderson & Berla, 1994). Also, children's knowledge and achievement have been improved by the cooperation between generations (Villas-Boas, 2005).

In the study presented here, the notion of "communities of practice" - issued from a particular context (economics) - developed by Wenger (1998) and implemented in mathematics education by Díez-Palomar & Moratonas (2006), was used; two dimensions were specially important: the *mutual engagement* (of the child and the parent, engaged in their activities) and the *shared repertoire* (sharing texts, whose writing has a main role). (Note that the third dimension used by Wenger - the joint enterprise - was not considered here).

In our contribution we try to answer to the following questions: Is it possible to increase mathematics development by involving parents and children in activities about their common (urban, linguistic, mathematical) grounds? How do these activities influence pupils' achievement ?

With this purpose, we have planned an on-going qualitative and quantitative study, where the activities proposed to 2<sup>nd</sup> graders (7 to 8 years old) involved also mathematical concepts.

In the exploratory study, we work with one class, divided into two groups, the experimental and the control groups. Pupils were proposed tasks where mathematics concepts were progressively used. We ask pupils to develop some activities, in partnership with a parent (mother, father, grandmother, brother, ...) and to write a small text about the realization of this activity. Texts were compiled in a booklet untitled « *When my mother [or else] was a little child* ».

A questionnaire of home environmental processes related to school achievement was proposed to the parents, adapted from Villas-Boas (2001) and Davies (1996), aiming to scale the importance of different activities of pupils at home.

In the poster presentation, focus was done: i) on the activities presented to pupils and parents during the exploratory study, ii) a qualitative analysis of the progresses of pupils observed by the teacher, iii) on “profiles” of home environment issued from answers of parents, iv) on a discussion about the study, with suggestions for future interventions.

*The actual poster presented at CERME7 may be obtained from author by emailing to <mesquita@math.jussieu.fr>*

## REFERENCES

- Díez-Palomar, J. & Moratonas, M.P. (2010). Discussing a case study of family training in terms of communities of practice and adult education, in J. Trgalova (Ed.), *Proceedings of the sixth Congress of European Research in Mathematics Education* (CERME6), pp. 1462-1471. Lyon: INRP.
- Villas-Boas, M. A. (2001). *Escola e Família. Uma relação produtiva de aprendizagem em sociedades multiculturais*. Lisboa: Escola Superior de Educação Joao de Deus.
- Wenger, E. (1998). *Communities of Practice. Learning, Meaning, and Identity*. Cambridge: Cambridge University Press.

## NOTES

<sup>1</sup> This study had been conducted with the teacher of the class, Nathalie Pasquet-Fortuné, from IREM de Paris 7.

<sup>2</sup> The project, untitled *On Reducing the Intergenerational Gap through the Interaction with Common Cultural Grounds in Different European Countries*, has been presented at the 7th International Conference of the European Research Network About Parents in Education (ERNAPE), at Malmö University, Sweden, in 2009, in a Symposium organized by Maria Adelina Villas-Boas, from Lisbon University, Portugal.

## **STUDENTS' SELF-REGULATION, SELF-EFFICACY AND MATHEMATICAL COMPETENCE IN OECD'S PISA**

David Pepper

King's College London

*This paper is an iteration of a poster presented at CERME 2011. It describes the scope and methods of a planned international comparative study of the relationship between lower secondary students' self-regulation, self-efficacy and mathematical competence using the OECD's PISA survey and the ICCAMS study in England.*

Key words: Mathematics, comparative, self-regulation, self-efficacy, competence.

### **SCOPE**

This short paper is an iteration of a poster for CERME 2011 which presented the scope and methods of a proposed international comparative study. These have been revised as a result of feedback from CERME's Working Group 11- Comparative Studies in Mathematics Education and the author's subsequent reflections. The study's main research question is: to what extent is a relationship between lower secondary students' self-regulation, self-efficacy and mathematical competence generalisable across educational contexts and cultures in different countries?

Self-regulation is used with reference to students' monitoring, control or regulation of their cognition and behaviour (Pintrich, 1999). Self-efficacy is 'a judgment of one's capability to accomplish a certain level of performance' in a specific task (Bandura, 1986; Zimmerman, 2000). Self-efficacy is consistently highly correlated with academic performance, including mathematics (eg Pajares and Miller, 1994). Pintrich's (1999) general framework identified a strong and positive correlation between students' self-regulation, self-efficacy and academic performance. However, his call for future research to explore this beyond white, middle-class students in the USA requires attention. The study will therefore investigate the extent to which the correlation can be generalised across contexts and cultures in different countries.

The study will focus on students' monitoring of their own learning (as an aspect of self-regulation) and the calibration of their self-efficacy judgements to their attainment in mathematics. With reference to European and international policy discourses (Gordon, et al., 2009), it posits that this calibration should be viewed as an aspect of *mathematical competence*, enabling students' 'selection and use of the most appropriate solution strategy on a given mathematical item or problem, for a given individual, in a given sociocultural context' (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009, p. 343). Students' self-regulation and self-efficacy should therefore be predictive of their use of their solution strategies and their resultant solutions.

## METHODS

The study will use multiple methods. Firstly, there will be a correlation analysis of self-regulation, self-efficacy and attainment using the influential OECD PISA (Programme for International Student Assessment) 2003 data set, which focussed on mathematical competencies in 41 countries with varied educational contexts and cultures. The analysis will include student background, school and country variables. It will also explore whether a multi-level model provides the best fit for the data. Secondly, interpretation of the data for selected countries will be supported by the development and validation of country profiles (Hodgen, Pepper, Sturman, & Ruddock, 2010) detailing socio-mathematical norms (Yackel & Cobb, 1996). Thirdly, students will be interviewed through the Increasing Students' Competence and Confidence in Algebra and Mathematical Structures (ICCAMS) study in England (Brown, Hodgen, Kuchemann, Coe, & Pepper, 2010). The interviews will explore students' use of solution strategies in the context of their self-regulation and self-efficacy. These interviews will also help to assess the validity of the associated PISA questionnaire items in some contexts in England, perhaps with wider implications.

## REFERENCES

- Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs ; London: Prentice-Hall.
- Brown, M., Hodgen, J., Kuchemann, D., Coe, R., & Pepper, D. (2010). *Attitudes, gender and attainment: evidence from a large-scale survey in England*. Paper presented at the 34th Conference of the International Group for the PME, Belo Horizonte, Brazil.
- Gordon, J., Halasz, G., Krawczyk, M., Leney, T., Michel, A., Pepper, D., et al. (2009). *Key competences in Europe: Opening doors for lifelong learners across the school curriculum and teacher education*. Warsaw.
- Hodgen, J., Pepper, D., Sturman, L., & Ruddock, G. (2010). *Is the UK an Outlier? An international comparison of upper secondary mathematics*. London.
- Pajares, F., & Miller, M. D. (1994). *Role of self-efficacy and self-concept beliefs in mathematical problem solving: a path analysis* (Vol. 86). Washington DC: American Psychological Association.
- Pintrich, P. R. (1999). The role of motivation in promoting and sustaining self-regulated learning. *International Journal of Educational Research*, 31(6), 459-470.
- Verschaffel, L., Luwel, K., Torbeyns, J., & Van Dooren, W. (2009). Conceptualizing, investigating, and enhancing adaptive expertise in elementary mathematics education. *European Journal of Psychology of Education*, 24(3), 335-359.
- Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27(4), 458-477.
- Zimmerman, B. J. (2000). Self-Efficacy: An Essential Motive to Learn. *Contemporary Educational Psychology*, 25(1), 82-91.

## **CONTRASTING PROSPECTIVE TEACHER EDUCATION AND STUDENT TEACHING IN ENGLAND AND SLOVAKIA**

Ján Šunderlík, Soňa Čeretková

Constantine the Philosopher University in Nitra, Slovakia

*The poster presented a comparison of prospective teachers' preparation focused on student teaching in England and the Slovak Republic. Based on a small scale study, where we used observation of lesson structure and chosen lesson events, we try to understand some differences between English and Slovak approaches of prospective mathematics teachers' preparation.*

Key words: Prospective teachers, lesson structure, comparative studies

### **FOCUS OF THE POSTER**

Our poster presented some findings of a study in which we compared prospective mathematics teachers' preparation in England and Slovakia. To bring about a better understanding of mathematics teachers' preparation, we used a comparative approach based on the insight that "Being in an environment constantly, one usually takes things for granted and fails to see the characteristics of the environment as special or different from the others." (Runesson & Mok, 2004, p. 217)

### **THEORETICAL FRAMEWORK**

Based on large scale comparative studies as PISA and TIMSS many countries try to implement reform mathematics instruction and reform curricula. But these reforms of learning and teaching mathematics depart significantly from the school mathematics tradition of countries as presented by Cobb, Wood, Yackel and McNeal (as cited in Lloyd, 2005, p. 441). One of the countries that have recently implemented a reform curriculum is Slovakia (since 2008). From many critical areas we decided to focus on student teaching. For the comparison we chose England, a country with different cultural traditions in mathematics education. Our main research question was: What kinds of differences in prospective teachers' learning can we identify during their student teaching?

### **METHODS AND DATA ANALYSIS**

Our methodology is based on Clarke (2006). We used video based lesson observation, video simulated and narrative interviews with prospective teachers (Kaasila, 2007) teaching in the two specific cultural settings. All lessons and interviews were transcribed and coded. Based on the codes, we used cluster analysis by applying Ward's method to see the differences between observed lesson structures better. From all this information, we identified lesson events that were



critical for the prospective teacher and we considered them as an important part of prospective teacher learning.

## FINDINGS AND DISCUSSION

According to the classroom observations and interviews, we identified several evidences of Anna's learning, a Slovak prospective teacher. There was a good cooperation with the associate tutor (AT), who used to give the prospective teacher direct advice and, based on this intervention, she kept improving her teaching. She was familiar with the style of teaching, because, as she said, "I taught the way I had been taught". On the other hand, Anna expressed beliefs about more progressive and problem based teaching, even though she kept using traditional ways. She tried some activating strategies but those were implemented within the same pedagogy and questioning as in a traditional setting. Ben is a prospective math teacher from England who prefers problem based and innovative teaching and learning. In contrast to Anna, he could teach and develop his teaching in a more innovative environment. He was able to use developed materials that helped him to focus more on his teaching strategies. Based on the course organisation, he was encouraged to reflect on his practice based and set up action points for his further development. The comparative analysis gives us a wider perspective for possible future development of prospective teachers and also helps us to see some advantages and difficulties of more "open" teaching and learning in a cultural environment that is not aligned with this approach.

## NOTES

This work was supported in part by the EU, within the 7FP project, under the grant agreement 244380 "PRIMAS – Promoting Inquiry In Mathematics And Science Education."

The actual poster presented at CERME7 may be obtained from the authors by emailing them at <jsunderlik@ukf.sk; sceretkova@ukf.sk>

## REFERENCES

- Clarke, D.J. (2006). The LPS research design. In D.J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp.15-37). Rotterdam: Sense Publishers.
- Lloyd, M.G. (2005). Beliefs about the teacher's role in the mathematics classroom: one student teacher's exploration in fiction and in practice. *Journal of Mathematics Teacher Education*, 8, 441 – 467.
- Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM – The International Journal on Mathematics Education*, 39, 205-213.
- Runesson, U., & Mok, I.A. (2004). Lessons from a small-scale observational study: an example of the teaching of fractions. In M. J. Høines & A. B. Fugelstad (eds.) *Proceedings of the 28th International Conference for Psychology of Mathematics Education* (pp. 217-221). Bergen, Norway: Bergen University College.

## DEVELOPING A MODERN MATHEMATICS PEDAGOGICAL CONTENT KNOWLEDGE: THE CASE OF TELESOLA IN PORTUGAL IN THE MIDDLE 1960'

Mária Correia de Almeida and José Manuel Matos

UIED – FCT, New Lisbon University

*Modern mathematics reform spread through many countries during the 1960s and 1970s producing changes in the representations of learning, of mathematical content, of the social roles of mathematics, and in classroom teaching practices. Telescola – a national network of schools complemented by classes on television – implemented a program with the “modern” vision of mathematics. This poster focuses on changes in pedagogical content knowledge to appropriate the new ideas.*

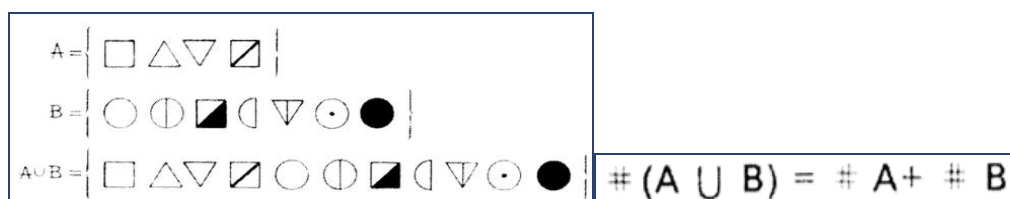
**Keywords:** Modern Mathematics, Pedagogical content knowledge, Telescola.

From 1965, a network of schools for 10 and 11 years-old supported by televised lessons was gradually put in place by the Portuguese Ministry of Education, in an effort to enlarge schooling after primary school, as it was demanded by economic development. Students that attended the Postos (the name of those schools) and finished a two-year course could enroll in the 7th grade of secondary schools. By 1968, this system that became known as Telescola covered the entire country, especially in remote areas. Modern Mathematics (MM) was gradually incorporated into these televised classes providing an experimental field for their later generalization to the entire population of 5th and 6th graders. Mathematics classes in Telescola were actually the first experience in the dissemination of the new ideas through an entire school sub-system in Portugal.

This study encompasses two dimensions aiming at understanding changes in pedagogical content knowledge (PCK) at this level: 1) a longitudinal study from 1965 to 1969 of the Mathematics “lessons” texts; 2) interviews with the teacher who prepared and enacted them on the screen, António Augusto Lopes (AAL). Qualitative procedures were performed (Bogdan and Biklen, 1994).

Shulman (1986) advanced thinking about teacher knowledge by introducing the idea of pedagogical content knowledge. This knowledge includes knowing what teaching approaches fit the content, and likewise, knowing how elements of the content can be arranged for better teaching. At the heart of PCK is the manner in which subject matter is transformed for teaching. This occurs when the teacher interprets the subject matter, finding different ways to represent it and make it accessible to learners. The gradual shaping (bricolage) of PCK was observed, so we could trace changes of school mathematics content put into practice and the difficulties that aroused for teaching. The distinctive characteristic of mathematical content is the adoption of set theory as an appropriate language to express mathematics.

*Change:* Now addition is the number of elements in the union of disjoint sets.



Now the teacher must draw the students' attention to the fact that addition may not be well defined. It depends upon the *universe*. In universe E,  $E = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  addition is not defined for all pairs of elements of E (3+4 is defined, but 5+8 is not). The possibility of addition was not in question at the primary school (its legitimacy was supported in the common sense of empirical, sensory experience).

The concept of *connected history* (Gruzinsky, 2003) developed by cultural historians, has been used to understand how communities elaborate their representations and practices in connection to other communities. The influence of one culture by another is seen as an act of hybridism (*métissage*) in which external appropriations are not seen as imitations from the original, but as producers of new originality. In this model, a key role is attributed to *mediators*, persons that travel among societies and cultures. The role of mediators was important in the expansion of the MM ideas in Portugal. A key mediator, AAL, author of the texts of the “lessons”, confirmed the influence of some authors and educators (Gattegno, Puig Adam, Servais) on the written materials he produced. The induction of change of classroom practices, due to external appropriation, is traceable in the perception that for students at this age it is important to see the real world applications of mathematic as well as to experience that mathematics embodied in objects that are real and accordingly in new materials used to support teaching. We could also trace the use of pedagogical materials, such as Geoboards and Cuisenaire Rods, on the written material produced for teaching.



The actual poster presented at CERME7 may be obtained from the authors by emailing them at <ajs.mcr.almeida@gmail.com>

## REFERENCES

- Bogdan, R. & Bliken, S. (1994). *Investigação qualitativa em educação* [Qualitative research for education]. Porto: Porto Ed.
- Gruzinsky, S. (2003). O historiador, o macaco e a centaura: a “história cultural” no novo milénio. *Estudos Avançados*, 17(49), 321-342.
- Shulman, L. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

## A STUDY ON THE FUNDAMENTAL CONCEPT OF “MEASURE” AND ITS HISTORY

Ana Amaral<sup>a</sup>; Alexandra Gomes<sup>b</sup>; Elfrida Ralha<sup>c</sup>;

<sup>a</sup>ESTM – IPLeiria, CIEC/IE, [anamaral@ipleiria.pt](mailto:anamaral@ipleiria.pt); <sup>b</sup>CIEC/IE, Universidade do Minho, [magomes@ie.uminho.pt](mailto:magomes@ie.uminho.pt); <sup>c</sup>CMAT/Dep.to de Matemática e Aplicações, Universidade do Minho [eralha@math.uminho.pt](mailto:eralha@math.uminho.pt)

*This paper exhibits a small part of a PhD degree project on the theme: “The fundamental concept of Measure: epistemological and pedagogical aspects related with the first six years of schooling”.*

*It aims at presenting the analyzis of the fundamental concept of measure at the so called elementary levels of schooling, in Portugal. We intend to: 1) distinguish elementary concepts and fundamental concepts; 2) explain why do we look at the mathematical concept of measurement as not only an elementary concept but, and above all, a fundamental concept in mathematics; 3) foresee implications of this distinction for teaching at the elementary level.*

**Keywords:** concept of measure, teaching of mathematics, history of mathematics.

### ELEMENTARY AND FUNDAMENTAL CONCEPTS

Concepts play a key role in the construction of mathematical knowledge with, certainly, different importance levels. At Elementary Mathematics’ schooling we are dealing with the so called elementary and fundamental concepts, deserving a special attention since their learning will influence the learning in higher levels.

In contrast to Ma (1999) (*Elementary Mathematics is fundamental mathematics*), we think that elementary and fundamental have different meanings, although a concept may be, at the same time, elementary and fundamental. On the one hand, a concept taught and learnt in elementary Mathematics created by Man to build a complex structure as a whole, or as a specific theme, is an elementary concept (Caraça, 2000). On the other hand, we say that a fundamental concept, emerges over time, its genesis is inherent to human activity and it is presented in many areas of mathematics as well as in several other areas of knowledge, both in the school context and in society, in general.

### THE FOCUS OF THIS POSTER

#### **Measure: Elementary and Fundamental concept**

We consider the specific mathematical concept of *Measure*! Through its history we understand that this is a concept emerging over time, its genesis is inherent to human activity (Astronomy, Agriculture, Economy) and it presents itself in many areas of mathematics (Trigonometry; Arithmetic; Probability) as well as in several other areas of knowledge, both in the school context and in society in general (Nanotechnology;

Medical issues; Biology). Therefore it is considered a fundamental concept. It also is an elementary concept since it is taught and learnt at an elementary level.

### **Possible implications for the Teaching of Mathematics**

We intend to answer the research question: *What is the approach of Measure in the Programs of Mathematics for Elementary Education, and in which way is History of Mathematics used to promote it as a fundamental concept?* To this end, we analyzed official portuguese mathematics programs (Mathematics Program for the 2<sup>nd</sup> Cycle of Basic Education, 1991; 1<sup>st</sup> Cycle Program for Mathematics, 1990 and the new Program of Mathematics for Basic Education, 2007), looking, in particular, at the role played by the history of *Measure*. Important works (Schubring (1998), Swetz (1995) and Siu (1996), among others) supported this analysis. We conclude that those programs present mainly the Geometric dimension of *Measure* and reduce the use of History of Mathematics related with this concept, to curiosities, examples, “historical aspects”. Seeing *Measure* as a seed of a future mathematical knowledge at a higher level, this approach may cause difficulties to the learning of mathematics. Namely, children may experience problems to integrate or to define units, develop misconceptions about measure/distance/length, fractions or when relating hours and angles in Trigonometry, misunderstanding measure statistics as measures, develop poor sense of money or work with temperature. This situation may aggravate if teachers aren’t aware of its importance to the development of mathematical knowledge and are unable to provide richer approaches.

Please recall that this is not a work developed directly with children, but meant to alert teachers, textbooks authors, programs and curriculum designers.

### **REFERENCES**

- Caraça, B. J. (2000). *Conceitos Fundamentais da Matemática*. Gradiva. Lisboa.
- Fischbein, E. (1993). The theory of figural concepts, *Educational Studies in Mathematics*, 24, 139-162.
- Kramer, S. N. (1985). *A História Começa na Suméria*. Publicações Europa-América. Lisboa.
- Man-Keung, Siu. (1996). *The ABCD of Using History of Mathematics in the (undergraduate) Classroom*. Department of Mathematics University of Hong Kong. 01/07/2010 < <http://hkumath.hku.hk/~mks/ABCD.pdf>>.
- Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers’ understanding of fundamental Mathematics in China and United States*. Lawrence Erlbaum Associates, Inc. Copyrighted Material.
- Neugebauer, O. (1969). “The Exact Sciences in Antiquity”. Dover Publications.
- Robson, E. (1996). From Uruk to Babylon: 4,500 Years of Mesopotamian Mathematics. *Historia e Educação Matemática*, I: pp. 35-44.

## **THE PROJECT OF MODERNIZATION OF THE MATHEMATICAL INITIATION IN PRIMARY SCHOOL AS CURRICULUM DEVELOPMENT (1965-1973)**

Rui Candeias

UIED – Faculdade de Ciências e Tecnologia, UNL, Lisboa, Portugal

Escola Básica 1/JI Quinta de Santo António, Amora, Portugal

*This poster refers to an ongoing research centered in a project of curriculum development in Mathematics, in Primary School (Project of Modernization of the Mathematical Initiation in Primary School, developed by the Educational Research Centre, of the Gulbenkian Foundation). This research intends to contribute to the knowledge of the evolution of mathematics teaching in primary schools, particularly the influence of “Modern Mathematics Movement” on curricular development and production of didactical material. This poster focuses on two aspects of the research: 1) a chronology of the project of Modernization of the Mathematical Initiation in Primary School; 2) a description of the documents produced for the 2nd grade according to the project’s scope, referring to mathematical subjects, the number of sessions dedicated to each subject, and a focus on the subject most worked (multiplication).*

**Keywords:** elementary school, history of mathematics education, modern mathematics movement.

### **THE “MODERNS MATHEMATICS MOVEMENT” REFORM IN ELEMENTARY SCHOOLS**

According to Moon (1986) Primary Schools only received the impact of Modern Mathematics Movement, from the mid 1960’s when several projects were developed. Some trends were detached in primary schools by introduction of Modern Mathematics Movement (Moon, 1986): (1) a structuralized trend, (2) an arithmetical trend, (3) an empirical trend.

In Portugal we can distinguish the work developed in some private schools and the work designed by the Gulbenkian Foundation (Candeias, 2008).

### **THE PROJECT OF MODERNIZATION OF MATHEMATICAL INITIATION IN PRIMARY SCHOOL - CHRONOLOGY**

In the 1960’s the Educational Research Centre initiated a project to introduce the teachers in the Didactics of Modern Mathematics (Kindergarten and Primary School level). In October 1967, a seminar was attended by 37 teachers, twenty-five of them belonging to five Primary Pilot - Schools, that would lead mathematics teaching to

the experimental groups. The teachers discussed scientific texts, exercises and had contact with educational material that would be used in the experimental classes. In 1969 this project was extended to two official schools from the Lisbon area.

## **PRELIMINARY RESULTS**

In this work we used two main sources: the bibliographic bulletin of the Gulbenkian Foundation (1965-1973) and the documents produced for the 2<sup>nd</sup> grade. From these sources it was made a qualitative analysis.

The project under study was an initiative of a private foundation, and in the beginning it was only implemented in private schools.

The mathematical themes and subjects for 2nd graders proposed in this project were similar to the Portuguese official programs at the time in study.

The most elaborated mathematical themes for 2nd graders were numbers and operations, especially the multiplication and division.

The approach to multiplication begins in the 2nd grade, and was based on the language of sets theory by means of “equipotent sets” which were used to introduce multiplication as repeated addition of equal terms. At this level, the multiplication was also addressed in rectangular arrangements, to introduce the commutative property of multiplication.

## **REFERENCES**

- Candeias, R. (2008). Contributo para a história das inovações no ensino da matemática no primário: João António Nabais e o ensino da matemática no Colégio Vasco da Gama. (Tese de Mestrado, Universidade de Lisboa)
- Centro de Investigação Pedagógica (1965-1973). Boletim Bibliográfico e Informativo nº 5-nº 17. Lisbon: Fundação Calouste Gulbenkian.
- Moon, B. (1986). The “New Maths” curriculum controversy. An International story. London: The Falmer Press.
- Moreira, D. & Matos, J. M. (Eds.). História do ensino da Matemática em Portugal: actas do XIII Encontro de Investigação em Educação Matemática Beja de 2 a 4 de Maio de 2004. Lisbon: Sociedade Portuguesa de Ciências da Educação.

# **WHO CAN UNDERSTAND THE GIFTED STUDENTS? A LESSON PLAN BASED ON HISTORY TO ENHANCE THE GIFTED STUDENTS' LEARNING**

Ersin İLHAN

Bayburt University

*The need of gifted students of a deeper understanding of mathematics has lead to the “enrichment programs”. Al-Khwarizmi’s work is used to develop a lesson plan aiming to create the appropriate atmosphere to help gifted students in a modern classroom understand the mathematics created by a gifted mathematician in the past. The objectives and the associated lesson plan developed by the author are suitable for implementation in Science and Art Centers which the gifted students in Türkiye attend after school.*

## **INTRODUCTION**

The National Ministry of Education of the Turkish Republic (NME, 2005) formed the 8<sup>th</sup> grade mathematics syllabus having algebraic expressions, intended to show algebraic equalities using models. What is missing there or the part that needs the enrichment is the modelling concept, which could benefit from Al-Khwarizmi’s work on the solution of quadratics equations (Desay and Akin, 1994). For the gifted students, this part of syllabus can be implemented in two teaching hours, during the enrichment program they follow after school.

For the Lesson Plan format, I choose The Integration Education Model, developed by Clark (2008). Using this model, all parts (cognitive, intuitive, physical and affective) of the needs of learning is activated, especially for the gifted students. I choose Al-Khwarizmi’s work on the solution of quadratics equations. For the curriculum, I choose the 8<sup>th</sup> grade mathematics where the algebraic equations exist. I choose the historical packages, suitable for two or three hours of enrichment programs in the Science and Art Centers.

The aim of the Lesson Plan is to make students gain the ability of concrete and abstract thinking and their relationship after the objectives one and objective two are realized. For this reason, after the second objective, the third one is developed for the gifted students, who are studying the extended mathematics after school in Science and Art Centers belonging to the National Ministry of Education.

## **The Lesson Plan**

**Learning Subject:** Algebra      **Sub-Learning Subject:** Algebraic Expressions

### **Objectives:**

1. To explain the differences between algebraic identities and equalities.
2. To explain identities via modelling (including equalities).



## Working Group 12

3. To explain the solution of equations via models. (*This objective which does not exist in the National Curriculum is developed by the author to integrate the history of mathematics and use it in the enrichment programs for the gifted students.*)

**Preparation before the class:** To let the students form pairs and ask them to prepare their presentation by studying and working on Al-Khwarizmi's life including writing poems, theatre, songs, drawings, etc.

### Procedure in the class:

1. To watch the documentary film about Al-Khwarizmi's life and works.
2. To teach in the light of objective 2 using modelling in the way described in the National Curriculum. To focus on the inductive thinking in the difference of squares ( $a^2-b^2$ ) and deductive thinking in the squares of parenthesis  $(a+b)^2$ .
3. To let the pairs give their Al-Khwarizmi presentations, with focus on the works and life of the mathematician.
4. To let the pairs do worksheet 1 (to solve  $x^2+10x=39$ ) for the objective 3 as a group.
5. To let the pairs continue their presentations with focus on sharing their feelings about the work of Al-Khwarizmi and his influence till today.
6. To let the pairs do worksheet 2 (to solve  $x^2+5x=56$ ) for the objective 3 as a group.
7. To ask the pairs the question "Do you think the work of Al-Khwarizmi is important? Explain and discuss as a class".

## REFLECTIONS

C. Tzanakis introduced a 2-D classification of the ICMI and Jankvist's *hows*; the lesson plan fits in the 2x2 cell; the cell history-as-a-tool (emphasis on inner-issues) & heritage ("learning mathematical topics" in the ICMI Study and Jankvist's "history based approaches"). Before CERME7 I implemented the lesson plan about history, but in the WG 12 sessions I realized that the above classification could help to further improve the teaching design and

## REFERENCES

- Clark, B. (2008). Growing up gifted: developing the potential of children at home and at school. United States: Pearson Prentice Hall.
- Desay, M. and Akin, Ö. (1994). Algebra's Five Important Mathematicians. (Beş Büyük Cebir Bilgini). Ankara: NME Publications.
- NME (2005). Mathematics Teaching Program: Grades 6, 7 and 8. (Matematik Dersi Öğretim Programı; 6-8. Sınıflar). Retrieved from <http://ttkb.meb.gov.tr/>

## **TEACHER TRAINING AT PEDRO NUNES NORMAL SECONDARY SCHOOL (1956-1969)**

**José Manuel Matos**

**Teresa Maria Monteiro**

*Universidade Nova de Lisboa, Instituto Politécnico de Beja*

*This poster<sup>1</sup> presents a longitudinal analysis of the papers prepared by mathematics teachers trainees in the Pedro Nunes Normal Secondary school between 1957 and 1969. Three periods can be considered: 1957-1962, subjects related to Modern Mathematics in general are presented and the papers are focused in conceptual approaches of the new ideas; 1962-1965, although the subjects related to Modern Mathematics still show general approaches, the papers present specific pedagogical proposals; 1965-1969, the works mainly discuss the pedagogical experiences related to the introduction of the Modern Mathematics in the final years of secondary school.*

Key-words: Mathematics education history, Mathematics teachers trainees, Portugal

The reform of Modern Mathematics in Portugal can be divided into three intertwined periods: the beginnings, from 1957 until 1963, in which the flow of new ideas can be detected; experimentation, from 1963 to 1968, during which the new ideas were implemented in classrooms; and dissemination, from 1968 onwards, that led to the gradual generalization of the reform to all students (Matos, 2009).

We have searched for articles printed in education publications authored by teachers of mathematics enrolled at a pedagogical training programme at Pedro Nunes Normal Secondary School (*Liceu*). The period of the study runs from 1957 (the restart of the teachers training program at the school) to 1969 (when changes were made to the teachers education programmes in Portugal). During this period, there were 36 mathematics trainees that published 12 papers, one in *Labor*, *Revista de Ensino Liceal* and the others in the *Palestra* (Matos & Monteiro, 2010).

New mathematical contents are mathematically studied in several papers. The axiomatic method-associated logic is probably the commonest mathematical topic. Although, in the analysed texts, logic is sometimes presented associated to axiomatics, it is usually linked to Set Theory. As an example, Fernanda Martins<sup>2</sup> describes it integrated in mathematical or symbolic logic. New contents mathematically studied include Modern Algebra, which itself includes the study of several structures with their corresponding composition laws, operations, unicity, neutral elements, inverse, commutativity, associativity and the distributive law. Fernanda Martins briefly discusses it, and Iolanda Lima<sup>3</sup> describes several examples of this kind of structures associated to the concepts of group, field and isomorphism.

New approaches to geometry are also analysed. Maria Bento<sup>4</sup>, considers that “Geometry taught in the euclidian way is out of date” (p. 136), and shortly discusses two axiomatic alternatives, one proposed by Choquet, and another by Papy. In the same year, Lourdes

Ruiz<sup>5</sup>, based in the concept of “Geometry as a group of drawings’ properties that do not change under a given set of transformations” (p. 141), briefly presents transformations’ geometry (also designated by this author as dynamic geometry) and the associated hierarchy. Few works discuss in detail the methodologies. However, some trainees study the most appropriate pedagogical approaches. Discussing regular secondary school education, Iolanda Lima rejects a program “leading to individuals that are mechanized in dealing only with formulas and problems similar to the ones that are usually presented in examinations” (p. 61); Dulce Nogueira<sup>6</sup> criticises Mathematics which is nothing more than a “mechanization and a theorem conglomerate” (p. 34). Several trainees declare to support an heuristic or active education, but only Iolanda Lima discusses in detail the meaning of the concept. Another two subjects, related with strategies to Mathematics classes, are discussed: workgroups and materials’ use. The constitution of these workgroups is proposed by some trainees. Dulce Nogueira discusses in detail their aims and working principles.

Many of these proposals were supported by valid studies at the time. There are many references to the books published by the CIEAEM: *L’enseignement mathématique* and *Le matériel pour l’enseignement des mathématiques*. Pedro Puig Adam is also referred, especially his book *La matemática y su enseñanza actual*.

From the school year 1964/65 the Pedagogical Lectures presented by the trainees discuss subjects concerning the practical aspects of the introduction of Modern Mathematics in high school. The aim was to stimulate reflections about the ongoing experiment in the last years of high school and how to expand it to the junior high school. Only two papers can be found in this period.

## NOTES

(1) This study is part of an historical comparative study of the school Mathematics Culture in Portugal and Brazil during the implementation of Modern Mathematics supported by the Brazilian CAPES and the Portuguese GRICES. (2) 1962, *Palestra*, 15, 48-71. (3) 1958, *Palestra*, 3, 58-74. (4) 1964, *Palestra*, 20, 126-140. (5) 1964, *Palestra*, 20, 141-148. (6) 1961, *Palestra*, 12, 32-53.

## REFERENCES

- Julia, D. (2001). A Cultura Escolar como Objecto Histórico. *Revista Brasileira de História da Educação*, Jan./Jun.(1), 9-43.
- Matos, J. M. (2009). Changing representations and practices in school mathematics: the case of Modern Math in Portugal. In K. Bjarnadóttir, F. Furinguetti & G. Schubring (Eds.), *"Dig where you stand" Proceedings of a Conference on On-going Research in the History of Mathematics Education, Gardabær, Iceland, June 20-24 2009*. Reykjavik: University of Iceland.
- Matos, J. M., & Monteiro, T. M. (2010). A Matemática Moderna nos Estágios do Liceu Normal de Pedro Nunes (1957-1969). Paper at XXI - SIEM, Aveiro.

# **THE PEDAGOGICAL CONSEQUENCES OF A LAISSEZ FAIRE INDIVIDUALISTIC SOCIETY**

**Judy Sayers**

*The University of Northampton*

*This poster reports on some of the early findings from an analysis of several case studies examining how English Early Years' teachers conceptualise the whole class interactive phases of mathematics lessons. The poster reported on a comparative examination of the very different approaches adopted by two of the case teachers to the development of young children's (ages 5-7 years) conceptual and procedural knowledge. The findings highlight the ways in which a laissez-faire, individualistic culture (Hofstede, 1986) may fail to construct equal opportunities to young children's mathematical thinking, learning and experiences.*

Key words: Pedagogical approaches, didactic foci, early years' mathematical thinking, measurable skills, early mathematical opportunities.

## **INTRODUCTION**

The United Kingdom in general and England in particular has been described an individualist rather than a collectivist culture (Hofstede 1986). Recent government interventions (DfEE, 2003) have highlighted the extent to which individualism permeates education systemically in its encouragement of teachers to personalise the learning of their students. Moreover, teacher education in England has traditionally eschewed notions of a commonly understood and accepted pedagogy, resulting in Simon's (1981) well known and repeated plea; "why no pedagogy in England?" and Alexander's (2004) more recent "still no pedagogy?"

## **METHODOLOGY**

A qualitative case study (Yin, 2009) approach was adopted to examine the practice of two teachers (Caz and Fiona). Initial and terminal interviews, videotaped lessons and stimulated recall interviews, and the method of constant comparison informed the analysis of the data.

## **FINDINGS**

The findings highlight how these two teachers' different approaches, drew on their experience as both learners and teachers of mathematics. Their experientially formed beliefs about mathematics and its teaching, informed the opportunities they provided for their children to engage with and learn mathematics. Three categories emerged from the data which have emphasised both differences and similarities of these two teachers' practices. The first *Mathematical Focus*, relates to conceptual and

procedural mathematical knowledge emphasised by the teacher through reasoning, problem solving or other mathematical processes, prior knowledge, explicit connections or other mathematical emphasis the teacher makes. The second category is the *Pedagogical approaches* used by the teacher, such as questioning, discussion, explaining and models used. And finally the *Classroom norms* created by the teacher which create specific classroom cultures towards learning. This will include the regular and routine attention the teacher makes towards children's behaviour, attitudes and responses to questions e.g. putting up their hand. The espoused practice discussed by the teachers expressed identical key pedagogical concepts; however the enacted emphasis and explained rationale by the teachers are quite different.

## DISCUSSION

The enacted practice and rationale provided by the teachers illustrate for Caz a deep understanding of mathematical structures which inform her practice in developing children's strategic competence, adaptive reasoning, productive disposition and procedural fluency (Kilpatrick et al., 2002). Whereas for Fiona, a deep focus on conceptual understanding through the development of a productive disposition and procedural fluency provide a very different experience and create very different classroom norms and approaches to mathematical learning.

The teachers work within an educational system that colludes and condones diversity in children's opportunities to engage with mathematics. We should then not be surprised that teachers do impose their own interpretation on the teaching and learning of mathematics. But can a society with a laissez-faire perspective on education ever offer consistent or equal opportunities?

## REFERENCES:

The actual poster presented at CERME7 may be obtained from the author by emailing her at [judy.sayers@northampton.ac.uk](mailto:judy.sayers@northampton.ac.uk).

Alexander. R. (2004) Still no pedagogy? Principle, pragmatism and compliance in primary education. *Cambridge Journal of Education*, 34, 7-33.

Department for Education and Employment (2003a) *Excellence and enjoyment: a strategy for primary schools* (London, Department for Education and Skills).

Greeno. J. (1994), 'Gibson's affordances', *Psychological Review*, 101, 336-342.

Hofstede. G. (1986). Cultural Differences in Teaching and Learning. *International Journal of Intercultural Relations*, 10, 301-320.

Kilpatrick J., Swafford. J., Findell. B. (2001). *Adding it up: helping children learn mathematics*. Washington, DC: National Academy Press.

Simon, B. (1981) Why no pedagogy in England? In B. Simon & W. Taylor (Eds.) *Education in the eighties: the central issues* (London, Batsford), 124-145.

Yin, R. K. (2009). *Case study research: Design and methods* (4th ed.). Thousand Oaks, CA: Sage.

## ICT SUPPORTED LEARNING OF MATHEMATICS IN KINDERGARTEN

**Martin Carlsen, Per Sigurd Hundeland and Ingvald Erfjord**

*University of Agder, Norway*

*Our project, ICT supported learning of mathematics in kindergarten<sup>97</sup>, aims to develop and disseminate experience from implementation and use of ICT for pre-school children's learning of mathematics in kindergarten environment. We will present aim, focus, research questions and preliminary analysis.*

**Keywords:** ICT, Kindergarten, Mathematics

### AIM AND FOCUS

Recently, there has been an emphasis on implementation of mathematics in kindergartens and on use of ICT in kindergartens, in Norway explicitly outlined in the national curriculum for kindergarten from 2006. In a review of research literature on ICT use in pre-school settings, mathematics is not mentioned (Plowman & Stephen, 2004). We see this as indicating a need for paying attention to this field. Lately, in a review of studies focusing on young children of age 3 to 6 learning with digital media, Lieberman, Bates and So (2009) give a list of important lines of research in this area. There they point to research focusing on how digital media support children's early "learning of mathematical concepts" (p. 275) and how collaborative learning and ability to interact can be facilitated in what they denote as well-designed digital technology. In our project, a number of ICT applications developed for children will be adapted and implemented in the kindergarten environment. We want to give special attention to 3 to 5 years old children's learning of mathematics supported by ICT, and to the communication between children and the kindergarten teachers in their learning processes. Our research question is: In what ways can use of ICT tools give new learning opportunities for mathematics in kindergarten? In particular, we are interested in investigating: (a) the content of conversations between adults and children; (b) the nature of the mathematics at stake, and (c) children's mathematical learning outcome.

### THEORETICAL PERSPECTIVE AND METHODOLOGY

In our study we take a sociocultural perspective on learning and development. Learning is viewed as a social and situated process where individuals, i.e. kindergarten teachers and children, appropriate concepts, tools, and actions through collaboration and communication (Rogoff, 1990).

---

<sup>97</sup> The project ICT mediated Learning of Mathematics in Kindergarten is funded by the LA2020 project at the University of Agder.

The methodology of this study is developmental research (Freudenthal, 1991). Researchers and kindergarten teachers collaborate in order to develop new forms of mathematical practice in kindergarten, i.e. we will establish a co-learning agreement (Wagner, 1997) with kindergarten teachers. Theoretically and methodologically the design of our project is a continuation of the TBM<sup>98</sup> project in Norway (Carlsen, Hundeland, & Erfjord, 2010). The participants are three kindergartens, six kindergarten teachers and children of age 3 to 5. Project activities are captured through use of video recordings, audio recordings and field notes.

## PRELIMINARY ANALYSIS

We have analysed children's use of ICT applications designed for Norwegian children of age 3 to 5. It seems as if the applications do not challenge the children appropriately. We suggest that the children are not met in their zone of proximal development (Vygotsky, 1978). However, we have experienced that applications designed for children age 6 to 7 are more appropriate.

## REFERENCES

- Carlsen, M., Erfjord, I., & Hundeland, P. S. (2010). Orchestration of mathematical activities in the kindergarten: The role of questions. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education. January 28th - February 1st 2009* (pp. 2567-2576). Lyon: Institut National De Recherche Pédagogique.
- Freudenthal, H. (1991). *Revisiting mathematics education: China lectures*. Dordrecht: Kluwer Academic Publishers.
- Lieberman, D. A., Bates, C. H., and So, J. (2009). Young Children's Learning With Digital Media. *Computers in the School*, 26, 271-283
- Plowman, L., & Stephen, C. (2003). A 'benign addition'? Research on ICT and pre-school children. *Journal of Computer Assisted Learning*, 19, 149-164
- Rogoff, B. (1990). *Apprenticeship in Thinking. Cognitive Development in Social Context*. New York: Oxford University Press.
- Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Wagner, J. (1997). The Unavoidable Intervention of Educational Research: A Framework for Reconsidering Research-Practitioner Cooperation. *Educational Researcher*, 26, 13-22.

---

<sup>98</sup> The Teaching Better Mathematics (TBM) project 2007-2010 was supported by the Research Council in Norway (NFR no. 176442/S20).

# **THE DESIGN AND IMPLEMENTATION OF MATHEMATICAL TASKS TO PROMOTE ADVANCED MATHEMATICAL THINKING**

Sinéad Breen, Ann O'Shea

CASTeL, St Patrick's College, Drumcondra, Dublin

National University of Ireland Maynooth

*This study concerns the design of mathematical tasks for first year large group mixed ability undergraduate classes. The tasks were designed by the two authors and assigned as homework in courses taught by them. The object was to foster deeper understanding through problems that facilitate engagement with concepts. We report on how the tasks were written and on preliminary student feedback.*

Keywords: task design; conceptual understanding; thinking processes.

## **INTRODUCTION**

Traditionally undergraduate mathematics courses have been defined in terms of mathematical content and the techniques students are expected to master or theorems they should be able to prove (Hillel, 2001). Although it is often assumed that students will develop an understanding of the higher level and often abstract mathematical concepts involved, such an understanding is seldom specifically fostered by the mathematical tasks and assessments students are required to complete (Pointon and Sangwin, 2004). Selden et al. (2000) found that more than half of their second year university calculus students could not solve any non-routine problems even though in a separate test they had demonstrated that they were familiar with the techniques required. They recommend that lecturers should 'scatter throughout a course a considerable number of problems for students to solve without first seeing very similar worked examples' (p.150).

## **TASK DESIGN**

Homework tasks on first year Calculus topics were written by both authors drawing on the frameworks of Swan (2008), Schoenfeld (1992), and Pointon and Sangwin (2004). The tasks required students to generalise and specialise, generate examples, make conjectures, reason, make decisions, explore, make connections, and reflect. For example, students were given an incorrect statement and proof and asked to critique the reasoning used; they were asked to give examples of functions with certain properties; they were asked to use graphs to solve problems that they would normally solve algebraically. They were also asked to investigate the truth of statements, for example: 'Suppose  $g(x)$  is an odd function, is  $1/g(x)$  odd? Justify your answer' (Task 1); 'Does every rational function have a vertical asymptote? Explain' (Task 2). The authors recognised that no one task could foster all of the desired thinking skills, but aimed to achieve a balance over a suite of assignment problems.



## IMPLEMENTATION AND FEEDBACK

During the Autumn 2010 semester, the homework tasks were assigned to large groups of undergraduate students at two institutions. Each assignment contained traditional tasks as well as the non-routine tasks designed (or selected from textbooks) using the frameworks mentioned above. At the end of the semester, student reaction to the tasks assigned was collected using questionnaires. In total 101 students gave feedback. In each institution, the questionnaire focused on pairs of tasks (one traditional and one non-routine) on the same mathematical topic. The majority of students reported that both task types contributed to their knowledge and understanding of the topic but felt that the non-routine tasks were more challenging or required more thinking or understanding than the procedural tasks. For example when discussing Task 1 and a related procedural task one student remarked:

There was more development into the knowledge of odd and even functions. The second task (*non-routine task*) was a lot more challenging.

For some students, Task 2 challenged their ideas about mathematics itself.

Problem 1 (*procedural task*) was mathematically based, and the other (*non-routine task*) was theory based.

## CONCLUSION

The use of the task frameworks helped the authors expand their range of assignment problems and the new problem types challenged their students in meaningful ways. Link to poster: [http://staff.spd.dcu.ie/breens/documents/BreenOSheaposter\\_A4.pdf/](http://staff.spd.dcu.ie/breens/documents/BreenOSheaposter_A4.pdf/)

## REFERENCES

- Hillel, J. (2001). Trends in curriculum: A working group report. In D. Holton, (Ed.), *The Teaching and Learning of Mathematics at University Level* (pp. 59-70). Kluwer Academic Publishers.
- Pointon, A. & Sangwin, C. J. (2004). An analysis of undergraduate core material in the light of hand-held computer algebra systems. *International Journal of Mathematical Education in Science and Technology*, 34 (5), 671-686.
- Selden, A, Selden, J., Hauk, S. & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems? In E. Dubinsky, A.H. Schoenfeld & J. Kaput (Eds), *Research in Collegiate Mathematics Education IV*, pp128-153. Providence: American Mathematical Society.
- Swan, M. (2008). Designing a multiple representation learning experience in secondary algebra. *Educational Designer*, 1, 1-17.
- Schoenfeld, A. H. (1992) *A Framework for Balance: Balanced Assessment Project*. Unpublished document, prepared for NSF-sponsored Calculus assessment workshop.

## CRITICAL MULTICULTURAL INSTRUCTION FOR UNDERGRADUATE MATHEMATICAL THINKING COURSES

Irene M. Duranczyk

University of Minnesota

**Abstract.** *This poster reports the design and outcomes of a critical multicultural mathematics study on student performance and engagement in a required mathematics course for non-STEM (science, technology, engineering, or mathematics) majors.*

**Key Words:** postsecondary, math literacy, inclusion, multicultural

Students' sociocultural background and interests can enhance their understanding of academic mathematics, particularly statistics, if the instructor and the students are open to the idea and have beliefs/attitudes that allow the interplay of academic mathematics and the everyday mathematics used for negotiating information. We are bombarded daily with statistical information about our society and about us as members of multiple cultural identity groups. The Critical Multicultural framework interconnects these sources of information.

Critical multicultural instruction is an inclusive theoretical model, integrating multicultural education (Atweh, Forgasz, & Nebres, 2001; English, 2002; Joseph, 2000; Solomon, 2009), ethnomathematics (D'Ambrosio, 1985; Powell & Frankenstein, 1997), and universal instructional design (Higbee & Goff, 2008; Silver, Bourke, & Strehorn, 1998) for equity, access and success of tertiary learners.

A news story based on the writer/publisher's point of view, power, and privilege is often supported by numerical documentation. How does this information shape our ideas? How can this information be re-purposed to tell our story, a story that embraces diversity or be used to understand and address the dynamics of race, class, gender, sexual orientation, national identity, immigrant status, and other identity dimensions? What story would we tell given the same data and an empowering sociocultural lens? With statistical knowledge and skills; a critical lens (examining the hidden questions/assumptions) and purposive reflection, we can be empowered as individuals and members of society to reshape the story—and tell a powerful story of diversity and community strength.

### CONTEXT OF THE STUDY

The qualitative data was collected from an undergraduate course that fulfills a mathematical thinking, liberal education graduation requirement for a student in a tier one, doctoral research university in the US. This preliminary qualitative study measures the impact of using the critical multicultural instruction model on students' performance and engagement.

### POINTS OF EVIDENCE—ARTIFACTS AND OBSERVATION

Midterm grades, journal entries, analysis of evidence of mathematical thinking and

engagement within assignments and tests, within-semester and post-semester interviews, and comparisons between and among pre-enrollment characteristics were used to capture the model's impact.

## PRELIMINARY OUTCOMES

The preliminary outcomes: (a) Students in Service Learning (Fall 2010, 10 out of 38 students) were more thorough in their discussion, analysis, and summary of the research project and statistical concepts; (b) Many students (Fall 2010, 15 out of 38) thought that the use of real data important to non-profit would be valuable but they did not have time to commit to a "real" study; (c) Most students expressed (Fall 2010, 30 out of 38) that they learned more about statistics by doing the project than by engagement in in-class discussions, in-class and homework assignments, or preparing and taking tests; (d) Overall, Students in the course identified more with a Deep Approach to Learning (intrinsic interest, commitment to work, relating ideas, understanding) over a Surface Approach (fear of failure, aim for qualification, minimizing scope of study, memorization).

The actual poster presented at CERME7 may be obtained from the author by emailing [duran026@umn.edu](mailto:duran026@umn.edu)

## REFERENCES

- Atweh, B., Forgasz, H., & Nebres, B. (Eds.). (2001). *Sociocultural research on mathematics education: An international perspective*. Mahwah: Lawrence Erlbaum Associates.
- D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics*, 5(1), 41-48.
- English, L. D. (Ed.) (2002). *Handbook of international research in mathematics education*. Mahwah: Lawrence Erlbaum Associates.
- Higbee, J. L., & Goff, E. (2008). *Pedagogy and student services for institutional transformation: Implementing universal design in higher education..* Retrieved from University of Minnesota <http://www.cehd.umn.edu/passit/docs/PASS-IT-Book.pdf>
- Joseph, G. G. (2000). *The crest of the peacock: Non-European roots of mathematics*. Princeton, NJ: Princeton University Press.
- Powell, A. B., & Frankenstein, M. (Eds.). (1997). *Ethnomathematics: Challenging Eurocentrism in mathematics education*. Albany: State University of New York.
- Silver, P., Bourke, A., & Strehorn, K. C. (1998). Universal instructional design in higher education: An approach for inclusion. *Equity and Excellence in Education*, 31(2), 47-51.
- Solomon, Y. (2009). *Mathematical literacy: Developing identities of inclusion*. New York: Routledge.

# **PROBLEM-BASED LEARNING AS A METHODOLOGY OF STUDYING THE DIDACTIC KNOWLEDGE OF DERIVATIVES IN UNDERGRADUATE COURSES IN MATHEMATICS FOR ECONOMISTS**

Moreno, M.M.; García, L., and Azcárate, C.

University of Lerida (Spain); University of Andes (Venezuela) and University Autónoma of Barcelona (Spain)

*This poster reports a study focusing on the teaching of derivatives and their applications to an economic context at university level. It is an attempt to characterise the didactic content knowledge of teachers about the links between mathematical and economic concepts; and, about methodology of teaching and the use of problems for developing students' mathematical knowledge. We have organised four seminars, of two hours each, focused on the teaching of the derivative concept in an economic context.*

## **MOTIVATION AND OBJECTIVES**

Other studies suggest a lack of interest in teaching the derivative concept and its applications to economy. Teachers continue to be traditional in the sense that they seldom use derivatives to solve economic problems, and students use a large number of standardized procedures. There are no differences between the teaching of derivatives in chemistry, biology or economics. Our research aims: to characterise the didactic content knowledge of teachers about the teaching of derivatives and their links with economics; to evaluate the utility of seminars as a training tool for university mathematics teachers; and to evaluate a data collection instrument used to these purposes.

## **CONCEPTUAL FRAMEWORK AND DATA COLLECTING**

We define didactic content knowledge as the specific knowledge required for an effective teaching based on the following four components: mathematical content knowledge, curriculum knowledge, teaching knowledge and learning knowledge (Kahan *et al*, 2003). Problem-based learning (PBL) is both a teaching method and an approach to the curriculum. PBL consists of carefully designed problems that challenge students to use problem solving techniques, self-directed learning strategies, team participation skills and disciplinary knowledge (Lee & Bae, 2008).

The study was conducted with six teachers in a seminar consisting of four two-hour sessions. Data was collected through a questionnaire based on the seminar sessions and open, audio-recorded interviews. Thus, a teaching proposal was designed around four economic problems about derivatives and their economic meaning, the notion of domain of a function and extreme values, and the chain rule in an economic context.

### SOME RESULTS AND CONCLUSIONS: EXAMPLE PROBLEM 3

Problem 3 is a classical economic situation about the production of a product (pencils) in which we propose four different questions. We analyse the first one which concerns the domain of a function in a mathematical context ( $\mathbb{R}^+$ ) and in the economic context ( $[0,2]$ ). Referring to the content knowledge (mathematics and economics) teachers express their opinion as follows:

- Ramón: We should separate both contexts but we do not do so.
- Manuel: This point of view is very interesting and it would help students to understand the differences between contexts and the concept itself.
- Alexis: To teach this way, teachers need to make many cognitive and methodological changes. Thus, we should make connections between both contexts and show the analogies that are there.

Referring to the teaching knowledge, Kenya considers that this approach implies anticipation to the students' questions and doubts, and a student-centred teaching. This change is necessary, according to most of the teachers, but it takes time, especially the first time the teacher prepares the lessons and the corresponding learning activities.

Teachers evaluated our seminars in a positive way and said the seminars helped them to think about different approaches depending on the context. They also felt more conscious of the difficulties of the planning and the importance of their role as tutors:

- Ramón: I liked the seminars as they were useful for working with and thinking about economic problems. I feel I lack knowledge and ideas for teaching this way.
- Kenya: I had never thought about economic problems this way. In my opinion, those were only examples and applications of the mathematical concept. I feel better sharing my ideas and doubts with my colleagues.
- Elio: After the seminars I am really worried about how to approach the derivative with our students. Changes are necessary but I am not prepared for them.

Work partially supported by EDU2008-05254 of the Spanish Ministry of Science and Innovation. The poster is available from the authors.

### REFERENCES

- Kahan, J., Cooper, D. & Bethea, K. (2003). The role of mathematics teachers' content knowledge in their teaching: a framework for research applied to study of student teachers. *Journal of Mathematics Teacher Education*, 6, 223-252.
- Lee, H. & Bae, S. (2008). Issues in implementing a structured problem based learning strategy in a volcano unit: a case study. *International Journal of Science and Mathematics Education*, 6(4), 655-676.

## SOME MEANINGS OF THE DERIVATIVE OF A FUNCTION

Perdomo-Díaz<sup>a</sup>, J., Camacho-Machín<sup>a</sup>, M. and Santos-Trigo<sup>b</sup>, M.

<sup>a</sup>Universidad de La Laguna (Spain) ; <sup>b</sup>CINVESTAV-IPN (Mexico)

*The aim of this poster is to present the set of meanings that 15 first year university chemistry students associate with the concept of the derivative of a function. The research questions are as follows: Which meanings of the derivative of a function do students recognize and use? What kind of relationships do they establish between them? How do they use them in problem solving activities? The results obtained mark the starting point for the implementation of a teaching module designed to introduce Ordinary Differential Equations (ODE) using the link of this concept to the derivative of a function in a problem solving scenario.*

**Key words:** derivative, ordinary differential equations

### RESEARCH

A previous research project (Camacho-Machín *et al.*, submitted) showed there was some discontinuity in the learning of mathematics: several students do not relate the concepts of derivatives and ODE. This finding led to the design of a teaching sequence where the concept of ODE was introduced in a problem solving scenario, making the relationship between this concept and the derivative of a function explicit. As a prelude to the implementation of this teaching sequence we wanted to establish the set of meanings students associated with the concept of derivative, which is the aim of this poster<sup>99</sup>. Results will be used in the future to analyse how students' understanding of the derivative evolves during the learning of ODE.

### Conceptual framework

The following two main ideas shape the conceptual framework of this work: the importance of analyzing students' knowledge in order to explain how they solve problems (Schoenfeld, 1992) and that a mathematical concept is built by dealing with the different meanings associated with it that can be represented by different systems (Hiebert & Carpenter, 1992). In the case of derivatives, the concept can be thought of in different ways (Thurston, 1994) including the formal definition, an algebraic procedure, the slope of a line tangent to the graph, etc.

### Methodology

This research was carried out with 15 university students in the first year of a degree course in chemistry (the whole population). They had just studied calculus with one

---

<sup>99</sup> Some other research results related with the concept of derivative are included in the poster. This work was partially supported by Grant no. EDU2008-05254 of the National Research Plan I+D+I of the Spanish Ministry of Science and Innovation.

and several variables. Last sessions corresponded to the introduction of ODE. In order to answer the research questions formulated here, we designed a questionnaire with 11 items (included in the poster) that could be solved in different ways. Students answered it individually, only using paper and pencil.

**Analysis and results** (Evidence of the results are included in the poster)

All the students used the derivative of a function as a mathematical procedure (although 5 of them showed some difficulties). Another six different uses associated with the concept of derivatives were detected in the students' answers: formal definition, description of critical points, slope, monotonicity, variation and rate. Even though most of these uses are closely related, they are considered individually because students did not show how to link them. For example, some students described the derivative of a function by analysing the monotonicity of that function but they did not obtain the slope of the line tangent to the graph of a function at one point. The classification of the students (table) shows the diversity that occurs in the construction of the network of meanings associated with the derivative of a function.

Students who relate the derivative (in addition to consider it as an algebraic procedure)		
at most to 2 meanings	to 3 different meanings	to 4 or 5 meanings
Sonia (none) Nieves (monotonicity) Silvia (slope, monotonicity) Juan (slope, rate) Ginés (formal definition, rate)	Virginia- Carmen (formal definition, slope, monotonicity) Alberto (slope, variation, monotonicity) Zoraida – Naomi (slope, monotonicity, rate)	Mar (formal definition, slope, monotonicity, rate) Manuel (slope, monotonicity, rate, variation) Milagros - Nicanor (formal definition, slope, monotonicity, rate, variation) Alexis (formal definition, slope, critic points, rate, variation)

This classification will be useful for a future analysis of the evolution of the students' understanding of the concept of derivatives during the learning of ODE in a problem solving scenario. The conference poster is available at: [http://dl.dropbox.com/u/26255014/Poster\\_Perdomo-Camacho-Santos-CERME7.pdf](http://dl.dropbox.com/u/26255014/Poster_Perdomo-Camacho-Santos-CERME7.pdf)

## REFERENCES

- Camacho-Machín, M., Perdomo-Díaz, J. & Santos-Trigo, M. (submitted). An Exploration of Students' Conceptual Knowledge Built in a First Ordinary Differential Equation Course.
- Hiebert, J. & Carpenter, T. (1992). Learning and teaching with understanding. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning. A project of the NCTM*, pp. 65-97.
- Schoenfeld, A. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In Grouws, D. (Ed.) *Handbook of Research on Mathematics Teaching and Learning. A project of the National Council of Teachers of Mathematics*, pp. 334-370.
- Thurston, W. P. (1994). On Proof and Progress in Mathematics, *Bulletin of the American Mathematical Society*, 30 (3) 161–177.

## **TRANSITION SECONDARY-TERTIARY LEVEL EDUCATION VIA MATH-BRIDGE**

Julianna Zsidó, Viviane Durand-Guerrier

Université Montpellier II

*Abstract. With the poster presenting the Math-Bridge project we would like to draw attention to this multilingual resource for online mathematics courses and training exercises that aims to help the transition for students between secondary school and higher education. This resource is developed in an ongoing European project. The poster presents briefly the specifics of the project and possible usage modalities.*

Keywords: secondary-tertiary transition, bridging courses, e-learning, European project

### **THE AIM OF MATH-BRIDGE**

The goal is to provide an e-learning tool that contains relevant mathematical content for typically first and second year university students in need for mathematical bridging courses. The common European problem of high drop-out rates among science and engineering students because of gaps in mathematical knowledge is tackled by this project. A large pool of common remedial courses and training exercises is created for use and reuse across European universities.

### **FUNCTIONALITY**

Math-Bridge is based on an intelligent tutoring system called ActiveMath which adapts to the student users' field of study, competency level and progression profile. Moreover the multilingual and multicultural aspects (Melis et al. 2009) play an important role: the content is available in seven languages easing cross-cultural training and Europe-wide mobility.

Students can create their own personalized exercise books, exploratory books for new fields or even simulate tests from the pool of available learning objects and progress at their own speed. When doing exercises the system analyses errors in answers, gives feedback and adapts further reasoning guiding the student, instead of returning only “correct” or “wrong” feedback with the expected answer.

Tutors can create books for their classes containing learning material, exercises and tests. Student user statistics can be recorded in order to determine which content is more valuable.

Authoring of content is possible at any time, that is, adding learning material and exercises and translating into new languages. The pool of learning objects becomes richer and Math-Bridge evolves constantly.



## MATH-BRIDGE SUPPORTS SECONDARY-TERTIARY TRANSITION

Since Math-Bridge provides remedial course material and training exercises, it includes secondary level learning material that proved to be partially acquired together with higher educational level material. Acquiring knowledge is guided by different learning models and the exercises can be repeated as many times as needed. Moreover the fact that it is an e-learning tool favours students' personal initiative and skills for self-organization typical for higher education level; see also (Mercat, 2009).

## MATH-BRIDGE IN MONTPELLIER

A possibility for a (team of) teacher(s) at university is to elaborate a book with relevant training exercises for a given course; to use this book partly during their sessions and to invite students to go on working on the book as personal work. As a bridging course, it can be used by teacher teams encompassing both secondary and tertiary teachers, in order to smooth the transition.

In Montpellier, we are experimenting with these modalities at a rather limited scale in 2010-2011 in the first year Calculus course (and probably at a larger scale in 2011-2012) in order to investigate if the learning objectives are more likely to be reached by using Math-Bridge on the one hand, and the possibility of a smooth integration in the every-day work of students and teachers on the other.

It is possible for a student to switch between languages when learning or doing exercises, thus Math-Bridge claims to help to overcome linguistic difficulties, that arise when learning mathematics in a different language (Barton et al, 2005). Of course, further research in this matter needs to be developed. A first step has been made by an ongoing pre-experimentation with bilingual students.

The poster is available from the author at [julianna.zsido@univ-montp2.fr](mailto:julianna.zsido@univ-montp2.fr).

## REFERENCES

- Barton, B.; Chan, R.; King, C.; Neville-Barton, P.; Sneddon, J. (2005). EAL undergraduates learning mathematics, in *International Journal of Mathematical Education in Science and Technology*, 36, 7, pp. 721-729
- Math-Bridge project, funded by the eContentplus programme, ECP-2008-EDU-428046 Coordination: DFKI, Stuhlsatzenhausweg 3, D-66123 Saarbrücken, Germany, <http://www.math-bridge.org/>
- Melis, E.; Gougadze, G.; Libbrecht, P.; Ullrich, C. (2009). Culturally Adapted Mathematics Education with ActiveMath, in *Artificial Intelligence and Society, Special Issue on Enculturating HCI*, 24, pp. 251 - 265
- Mercat C. (2009). Math-Bridge: Remédiation entre lycée et université, *MathémaTICE*, n°16, <http://revue.sesamath.net/spip.php?article234>

## CHASE FOR A BULLET – USE OF ICT FOR DEVELOPING STUDENTS' FUNCTIONAL THINKING

Antonín Jančařík<sup>1)</sup>, Jarmila Novotná<sup>1)</sup>, Alena Pelantová<sup>2)</sup>

<sup>1)</sup> Charles University in Prague, Faculty of Education, <sup>2)</sup> ZŠ Na Slovance, Prague

*The poster presented an activity developed in the on-going Socrates Comenius project EdUmetrics, European Development for the Use of Mathematics Technology in Classrooms (503254-LLP-1-2009-1-UK-COMENIUS-CMP). The project focuses on integration of technology into secondary school mathematics. Its main goal is the creation of a research based course for pre- and in-service teachers to use new technologies in math classrooms with the goal of maximising students' learning.*

Keywords: Functional thinking, mathematical model, ICT, mathematical education

The outcome of the project will be teaching materials that illustrate the optimum use of ICT in mathematics teaching and learning. The activity presented in the poster, *Chase for a bullet*, has been designed for the module *Constructing functions and models*. It represents an example of a fruitful integration of ICT with paper-and-pencil activities. Two points A and B are given representing the current position of the Target (A) and the Bullet (B). A is moving on a trajectory at a given speed and B is trying to catch A. The main activity consists of choosing the trajectory of A and the following calculation of the approximation of the trajectory of B. The activity focuses on the potential of technology with respect to functions perceived as tools for modeling both the inner world of mathematics (internal modeling) and the outside world (external modeling). The proposed situation is based on the principles of the Theory of didactical situations (Brousseau, 1997). It possesses a high potential for dynamic access to families of functional objects depending on one or more parameters. It offers flexible access to a diversity of representations of functional objects and connections between them. The goal of *Chase for a bullet* is to enable students to learn indirectly about graphs of various functions, to make connections between parameters in formulas of functions and their properties. The situation is rich in the perspective of interdisciplinary relationships (physics, various environments for the same mathematical model). The foreseen age level is 14 years and up. The minimal mathematical knowledge is the Theorem of Pythagoras.

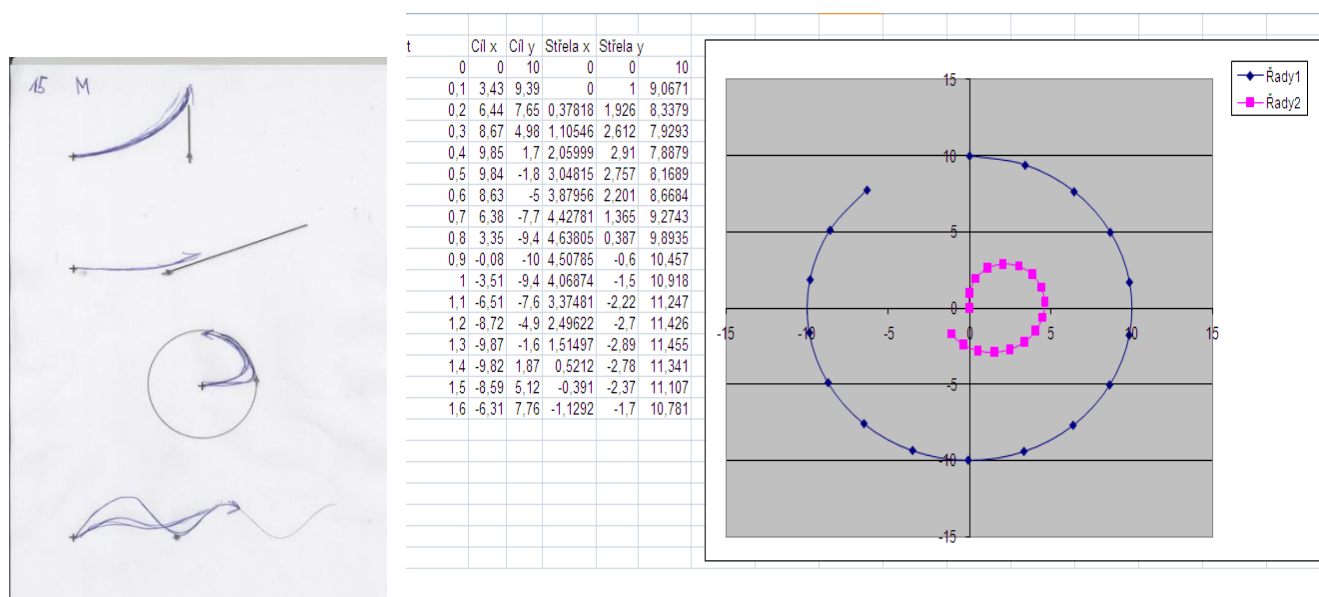
The design combines work without and with computers. ICT is mainly used for simulation of processes, as scaffolding for computations (e.g., solving quadratic equations), calculating new data using constructed formulas, verifying the correctness of the obtained results, and easy creation of examples and problems.

The added value of the use of computer technologies when solving bullet-target problems lies in the following two areas. Students learn how to model real situations with the help of a computer model. An analytic solution of the bullet-target problem is extremely difficult, even impossible. However a mathematical model constructed

by students for the solution of the problem enables a good enough approximation and offers a model solution applicable in other situations the student may come across in real life. In other words, the computer helps the student solve a problem that he/she would fail to solve without its support.

The benefit of the use of computer technology also lies in the fact that, after having developed a numerical model, the computer carries out the repetitive calculation of the position of the bullet using the formula that the student him/herself has developed within the frame of analysis. Thus the computer takes over the student's routine activity that he/she would be able to carry out on their own but due to time limitations only in a very limited number of problems. ICT allows modeling of the situation in real time and the student's attention shifts from routine calculations to analysis of the behavior of functions and to exploration of the dependence between the bullet and the target trajectory. Students may study the influence of the different parameters in an extent impossible without computer technology.

The poster presented the organization, course and analysis of the first pilot experiment of the situation in the Prague school Na Slovance (15-year old students). The following figures illustrate students' activities.



**Figure 1: Estimated trajectories**

**Figure 2: Trajectory of the target: circumference (MS Excel)**

## REFERENCES

- Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*. Kluwer Academic Publishers.
- European Development for the Use of Mathematics Technology in Classrooms, 503254-LLP-1-2009-1-UK-COMENIUS-CMP. Application form.

## THE CHALLENGE OF DEVELOPING A EUROPEAN COURSE FOR SUPPORTING TEACHERS' USE OF ICT

Michèle Artigue<sup>1)</sup>, Claire Cazes<sup>1)</sup>, Françoise Hérault<sup>2)</sup>, Gilles Marbeuf<sup>2)</sup>, Fabrice Vandebrouck<sup>1)</sup>

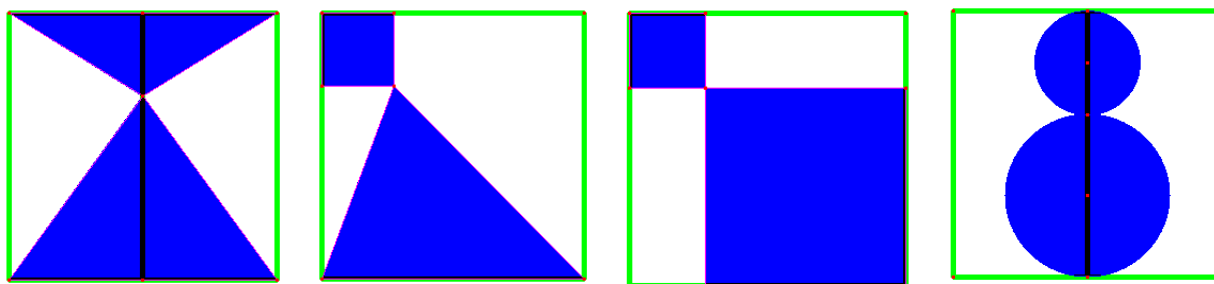
<sup>1)</sup> LDAR, Université Paris Diderot – Paris 7, <sup>2)</sup> Lycée Jacques Prévert, Taverny

*The poster concerns the on-going Socrates Comenius project EdUmaths (503254-LLP-1-2009-1-UK-COMENIUS-CMP), whose ambition is to develop a research based course which aims to educate in-service and pre-service teachers to use new technologies in their mathematics classroom to maximize students' learning. In the poster, we focus on issues raised by the development of one of the five modules for this course entitled "Constructing functions and models", and we take one family of situations used in this module: "The Sign Family" as an illustration [1] of the interaction between research and design.*

Key words: teacher education, ICT, functions and modeling

The teams involved in EdUmaths have all a research and training experience in ICT but they live in different educational contexts, with different curricular organizations, teacher education systems, and they face different institutional constraints. Even if the use of ICT in mathematics education is promoted by every country involved, school equipment and educational policy regarding ICT is also variable. The research that these teams have developed also situates in different traditions, relies on different theoretical frameworks, and as evidenced for instance by the European project ReMath [2], such differences substantially impact the design and use of ICT. Building a European course for teachers supposes a good awareness of this diversity and its possible consequences, a reached agreement on some core points for teacher education in that area in terms both of content and form, and a conception of design allowing easy adaptation to a diversity of contexts. In the poster, we illustrate one particular facet of our work in progress on these challenging issues: how we approach the necessary design flexibility through the idea of family of situations.

The Sign family that we use as an illustration denotes a family of mathematical situations at the interface between geometry, magnitudes and the functional world which can be described in the following way: a initial geometrical form is given (square, rectangle, circle ... or even a 3D form); and a point variable in this form (along a side, a bisector, a diameter...) allows its division into different parts; from this division a sign is created whose area depends on the position of the variable point (see examples in figure 1). Several questions naturally emerge regarding the variation of this area or of the area of its different components, its minimum and maximum values...



**Figure 1: Some Sign examples**

From this schema, according to the choices made in terms of *didactic variables* (Brousseau, 1997), one can generate a great diversity of didactic situations, varying the context of the task, the functional dependencies at stake, the autonomy given to students in the functional modeling, the technology used, and the didactic organization. The Sign family presents thus the adaptive character expected from the situations for this module. It also offers evident potential for addressing the issue of ICT affordances for the teaching and learning of functions: support to an inquiry based approach through the enrichment of the a-didactic *milieu*, support to the multimodal semiotic activity of students (Saenz-Ludlow & Presmeg, 2006).

In the poster, we will visually present how the research developed about the Sign family, combining an a priori analysis of the family of situations and analysis of screen captures and videos resulting from the experimentations carried out in the partner high schools, can be used in the design of the module for studying the affordances of ICT in the teaching and learning of functions, making teachers sensitive to instrumental issues (Guin, Ruthven & Trouche, 2004), discussing the specificities of the teacher's role in ICT sessions, and preparing the use of such situations in a diversity of educational contexts.

## NOTES

1. Another example is provided in the poster proposed by our colleagues from Prague.
2. The European STREP ReMath has especially addressed the impact of theoretical diversity on the design and use of digital technologies. Its results are accessible on the project website: <http://remath.cti.gr>

## REFERENCES

- Brousseau, G. (1997). *Theory of Didactical situations in mathematics 1970-1990*. Kluwer Academic Publishers.
- European Development for the Use of Mathematics Technology in Classrooms, 503254-LLP-1-2009-1-UK-COMENIUS-CMP. Application form.
- Guin, D. Ruthven, K., Trouche, L (Eds) (2004), *The didactic challenge of symbolic calculators*. Dordrecht : Kluwer Academic Publishers.
- Saenz-Ludlow, A., Presmeg, N. (Eds) (2006). Semiotic perspectives in Mathematics Education. A PME Special Issue. *Educational Studies in Mathematics*, 61(1-2).

## **REMEDIAL SCENARIOS FOR ONLINE AND BLENDED-LEARNING BRIDGING COURSES**

Rolf Biehler\*, Pascal R. Fischer\*\*, Reinhard Hochmuth\*\*, Thomas Wassong\*

\* University of Paderborn, \*\* University of Kassel

*The European Project Math-Bridge<sup>1</sup> aims at providing technical and pedagogical support for mathematical bridging courses. Based on the ITS ActiveMath<sup>2</sup> Math-Bridge (MB) extends it to the project's special needs. MB provides amongst others pedagogical remedial scenarios which enable teachers and learners to use MB for different course scenarios. This poster will present these remedial scenarios.*

Keywords: Bridging Courses, Remedial Scenarios, Self-Regulated Learning

### **THE LEARNING MATERIAL IN MATH-BRIDGE**

Most of the project European partners already have experience with the development and implementation of math bridging courses and provide a big amount of learning material for reuse within MB. This material needs to be enriched by pedagogical and mathematic-structural metadata. Our pedagogical structure (Biehler *et al.*, 2009) bases on existing competency models like PISA (OECD, 2003) and the German Bildungsstandards (KMK, 2003) and defines four competency clusters on the 1st dimension: technical, math problem solving, modelling, and communication and reasoning. Our 2nd dimension contains three achievement levels: reproduction, connection, and reflection. Our math structure is defined in an ontology which bases on the taxonomy for Mathematical Sciences Education<sup>3</sup>.

### **THE USE OF LEARNING OBJECTS IN MATH-BRIDGE**

In order to use the enriched content within the adaptive learning system, we extended the pedagogical scenarios of ActiveMath (Reiss *et al.*, 2005) to the purposes of math bridging courses. These scenarios aim at different learning goals (rehearse, workbook...) and select the most reasonable learning objects (LO) with regard to students' abilities and learning goals, bring them into a predefined order and hence give them an appropriate learning environment for their individual purposes.

An analysis of the learning material showed that some of the sequences of atomic LOs are not freely exchangeable and instead belong strictly together. These sequences should not be broken up since e.g. an "introduction" sometimes forms a holistic unit. Having the individual LOs and also keeping these units together, we introduced a new structure element called "complex learning objects" (CLOs). According to the learning material from the VEMA-project<sup>4</sup> (Biehler *et al.*, in press), we identified the following types of CLO: Introduction, Info/Interpretation/Explanation (IIE), Application, Misconception, Practice, and Supplement (Biehler *et al.*, 2010).

## FORMALIZED REMEDIAL SCENARIOS

For the remedial scenarios concerning the CLOs we developed a close order of units (overview, intro, info, IIE, application, typical mistakes, exercises, and supplement), where the learners can select relevant units for their learning process. To support the students in structuring their learning, MB provides scenarios with preselected units, e.g. Select Basic (overview, info, IIE, exercises) (Biehler *et al.*, 2010). Hence the students can create a book respecting their needs and pedagogical principles.

## THE LEARNING ADVICE COMPONENT

Most of the first-year students are not trained in self-regulated learning especially in ITS. We thus designed a self-assessment component: After having chosen a specific domain the learners get an overview of the topics and the relevant definitions and theorems. Then they estimate their knowledge and assess themselves using a diagnostic test providing feedback on their performance and abilities. Finally they compare their own solution to a model one and the assessment results with their initial self-estimation. With this feedback, the learners are able to select content and learning scenarios adequately. Besides the learning advice component trains the student's ability of self-estimation and self-regulation (Biehler *et al.*, 2010).

## NOTES

1. <http://www.math-bridge.org>
2. <http://www.activemath.org>
3. <http://people.uncw.edu/hermanr/mathtax/>
4. <http://www.mathematik.uni-kassel.de/vorkurs>

## REFERENCES

- Biehler, R., Hochmuth, R., Fischer, P. R., & Wassong, T. (2009). *D1.1: Target Competencies*. Retrieved from <http://mathbridge.math.upb.de/publ.html>
- Biehler, R., Hochmuth, R., Fischer, P. R., & Wassong, T. (2010). *D1.3: Pedagogical Remedial Scenarios*. Retrieved from <http://mathbridge.math.upb.de/publ.html>
- Biehler, R., Fischer, P.R., Hochmuth, R., Wassong, T. (in press). Self-regulated learning and self assessment in online mathematics bridging courses. In A.A. Juan, M.A. Huertas, S. Trenholm, C. Steegmann (Eds.), *Teaching Mathematics Online – Emergent Technologies and Methodologies*. Hershey, PA: IGI Global.
- KMK (Kultusministerkonferenz) (2003): *Bildungsstandards im Fach Mathematik für den mittleren Schulabschluss*. Bonn: KMK.
- Reiss, K., Moorman, M., Groß, Ch., & Ullrich, C. (2005). *D20: Formalized Pedagogical Strategies*. <http://www.leactivemath.org/publications1.html>
- OECD (2003). *The PISA 2003 Assessment Framework - Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Paris: OECD.

## **I2GEO.NET – A PLATFORM FOR SHARING DYNAMIC GEOMETRY RESOURCES ALL OVER EUROPE**

Jana Trgalova<sup>1</sup>, Ulrich Kortenkamp<sup>2</sup>, Ana Paula Jahn<sup>3</sup>, Paul Libbrecht<sup>4</sup>, Christian Mercat<sup>5</sup>, Tomas Recio<sup>6</sup>, Sophie Soury-Lavergne<sup>1</sup>

<sup>1</sup>INRP, Lyon, France - <sup>2</sup>CERMAT, University of Education Karlsruhe, Germany - <sup>3</sup>UNIFESP, São Paulo, Brasil - <sup>4</sup>DFKI, Saarbrücken, Germany - <sup>5</sup>University of Lyon, France - <sup>6</sup>Universidad de Cantabria, Santander, Spain

*The poster [1] presents the I2Geo platform developed within the Intergeo project [2] whose ambition was to develop a Pan-European math teacher community enabled to share resources and practices in using dynamic geometry (DG). We present the main tools and services i2geo offers to stimulate the use of DG in mathematics classes.*

**Keywords:** Intergeo, I2Geo platform, dynamic geometry

Dynamic geometry systems (DGS) are well known computer tools to support math teaching and learning by means of personal explorations and experience. Despite their availability and recommendations in European countries curricula to use them, their integration in schools is still unsatisfactory (Hendriks *et al.*, 2008). The Intergeo project (Kortenkamp *et al.*, 2009) tackles main obstacles to DGS integration: (1) difficulties to find suitable resources due to the lack of metadata describing accurately their content, (2) impossibility to exploit with a given DGS resources created with another one, (3) lack of quality guarantee of available resources. We briefly expose solutions proposed to overcome these obstacles.

### **SHARING DG RESOURCES**

An accurate resource content description is necessary for easy finding and sharing resources. The challenge in Intergeo consisted in defining metadata allowing a cross-curricular resource search regardless of language and cultural differences. This was achieved by defining a math topics and competencies ontology making it possible to find resources written in different languages. Sophisticated search tools (e.g., search associated to different national curricula key words) have also been developed.

### **INTEROPERABILITY OF DGS**

One of the obstacles in the use of DG is the issue of user lock-in with respect to a particular software product. Learning new software is time consuming and usually unrealistic for teachers. The project defines a common interoperable file format to describe constructions created with a DGS in a way to enable content exchange between DGSs. The standards are supervised by Intergeo A.s.b.l. association.



## QUALITY ASSESSMENT PROCESS

Quality assessment of DG resources (Trgalová *et al.*, 2009) aims at promoting access to best quality resources, as well as at ensuring their continuous improvement. The main tool supporting the assessment is a standardized questionnaire organized around nine dimensions of a resource related to its mathematical, didactical, pedagogical, technical and ergonomic aspects. Quantitative evaluation of a resource along these aspects in terms of a 4-level range of agreement can be complemented by qualitative comments, which are crucial for the resource improvement.



Figure 1: Main page of the platform

▶ ○○○○	I found easily the resource, the audience, competencies and themes are adequate
▶ ○○○○	The files are technically sound and easy to open
▶ ○○○○	The content is mathematically sound and usable in the classroom
▼ ○○○○	Translation of the mathematical activity into interactive geometry is coherent
○○○○	The math and the figures are related
○○○○	The figure is behaving consistently within the activity
○○○○	The figure shows no ill effect
○○○○	The numerical values (angles, lengths) are consistent
○○○○	Functionalities and specific behaviors like sliders, keyboard interaction or macros are well described
Comments:	
<input type="text"/>	
▶ ○○○○	In this resource, Interactive Geometry adds value to the learning experience
▶ ○○○○	This activity helps me teach mathematics
▶ ○○○○	I know how to set my class for this activity
▶ ○○○○	I found easily a way to use this activity in my curriculum progression
▶ ○○○○	The resource is user friendly and adaptable

Figure 2: I2Geo quality questionnaire

## I2GEO PLATFORM IN A FEW NUMBERS

By April 2011, around 3500 resources are available on the platform and more than 700 evaluations have been performed to this day. The platform has more than 1000 registered members.

## NOTES

1. The actual poster presented at CERME7 can be downloaded from <http://i2geo.net/xwiki/bin/download/Main/Proceedings/CERME7poster.pdf>
2. Intergeo was co-funded by the European Union within the eContentPlus programme, 2007-2010. See <http://i2geo.net>

## REFERENCES

- Hendriks, M., Ippersiel, M. *et al.* (2008), *Status quo report on DGS usage*. Intergeo deliverable D5.1, January 2008. Retrieved from [http://i2geo.net/files/i2g\\_status\\_quo\\_report\\_jan2008.pdf](http://i2geo.net/files/i2g_status_quo_report_jan2008.pdf)
- Kortenkamp, U., Blessing, A. *et al.* (2009), Interoperable interactive geometry for Europe – first technological and educational results and future challenges of the Intergeo project. In V. Durrand-Guerrier *et al.* (Eds.), *Proc. of CERM6* (pp. 1150-1160), Jan 28-Feb1 2009, Lyon, France.
- Trgalová, J., Jahn, A. P., Soury-Lavergne, S. (2009), Quality process for dynamic geometry resources: the Intergeo project. In V. Durrand-Guerrier *et al.* (Eds.), *Proceedings of the Sixth CERME conference* (pp. 1161-1170), Jan 28-Feb 1 2009, Lyon, France.

## MOBILE TECHNOLOGY IN MATHEMATICS COURSES FOR TEACHER STUDENTS

Iveta Kohanová

Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava,  
Slovakia

*In August 2008, Faculty of Mathematics, Physics and Informatics at the Comenius University in Bratislava received an HP Technology for Teaching Higher Education grant to innovate and transform learning and teaching at the Faculty. The poster focuses on the usage of the above-mentioned HP mobile technology in preparing future mathematics teachers in courses of Mathematical Analysis, Didactics of Mathematics and Didactical seminar in school mathematics. It demonstrates how we prepare our future teachers to the integration of modern mobile technology in teaching mathematics at primary and secondary schools; how these teachers improve their mathematical knowledge, attitudes towards mathematics, connecting theoretical knowledge with practical one and increase their motivation.*

Keywords: mathematics education, preparation of future teachers, technology in educational process

### IMPACT ON STUDENTS' LEARNING

Achievement of project objectives are measured in form of group comparison (pilot course vs. control course - offered without the support of HP mobile technologies), tests, and interviews related to the change in motivation and attitudes towards mathematics.

**Two Years Ago** – A typically successful completion in a Mathematical Analysis (*Didactical seminar*) course was approximately 55% (44%).

**One Year Ago** – A typically successful completion in Mathematical Analysis (*Didactical seminar*) course was approximately 74% (58%).

**Today** – To the date, our successful completion rate is approximately 75% (63%).

### IMPACT ON TEACHING

The teacher, as well as teacher students, work during the lessons with tablets. In this way, the students came from a passive role to the active role. They have all the materials in electronic form and therefore, they can edit them, share them, which leads to achieving better results.

During the course of the Didactics of Mathematics teacher students simulate the primary/secondary school classroom; one teacher student plays the role of the teacher, other teacher students represent pupils. By practicing of teaching and learning mathematics with the usage of the HP mobile technology, they prepare themselves for their future career and integration of technology into education.

## TECHNOLOGY IMPLEMENTATION

Each teacher student works during the lesson with a tablet (Fig. 1); the tablet capability of on screen inking enables him/her to work directly with a given problem or a representation of a mathematical concept, to write, calculate and draw, to model and observe the behavior of functions, parameters, etc.



**Figure 1: A photo of teacher students working with mobile technology**

## REFERENCES

- Gruska D., Kohanová I., Slavičková M. (2009) - *Application of mobile technology in preparation of future mathematics teachers*, Teaching, Learning, and Technology in Higher Education, Washington: ISTE, p. 21.
- Kohanová I., Slavičková M. (2009) - *Mobile technologies in mathematics teaching*, IMEM Congress 2009, Ružomberok: Catholic University, p. 112.

## **PEPIMEP PROJECT: ONLINE DATABASE SYSTEMS AND DIFFERENTIATED LEARNING ROUTES FOR SCHOOL ALGEBRA**

Françoise Chenevotot, Brigitte Grugeon-Allys

Laboratoire de Didactique André Revuz L.D.A.R., Paris Diderot University, France

*On this poster, we present PépiMep research project (running from 2010 to 2012). It articulates two research domains: educational research for school algebra (Kieran 2007) and Computer Learning Environments (C.L.E.). Our aim is to implement computer resources in online database systems to help teachers to differentiate students' learning in elementary algebra and to study the mathematic teachers practice and the algebra activity of students.*

Keywords: online database system, cognitive assessment, teaching and learning of school algebra, differentiated learning routes.

### **FROM PÉPITE-LINGOT PROJECT TO PÉPIMEP PROJECT**

Many teachers have difficulties taking into account their students' cognitive diversity. So maths teachers need computer resources to differentiate students' learning in elementary algebra. *Pépité-Lingot* project relies on a fifteen years collaboration between two partners: CLE researchers from LIP6 Paris and math education researchers from LDAR. *Pépité* software produces an automatic cognitive assessment for school algebra. This tool, based on a multidimensional analysis of algebraic skills (Grugeon 1997), generates automatic multi-criteria assessments of students' competence in school algebra (Delozanne and al. 2008). *PépiMEP* project is grounded on the collaboration of the two previous partners and a third one, Sésamath, a French maths teachers association which has had a central place in online database systems for ten years (Vanroyen 2008). The aim of *PépiMEP* project is to implement computer resources in the Sésamath online database *LaboMEP* to help teachers to differentiate students' learning in elementary algebra and to study the maths teachers' practises and the algebra activity of students.

### **AIMS**

*PépiMEP* project will design and implement assessment and differentiation tools in the *LaboMEP* database. For researchers, the objective is to have large-scale data. For Sésamath association, the objective is to develop a new version of their database, which includes assessment and differentiation. We present our research questions. In educational research: Do mathematic teachers use learning situations and differentiated learning routes, structured by students' cognitive learning, to organize work in groups in their classroom? Are these differentiated learning routes adapted to usual evaluation and differentiation practices of teachers? What are their effects

on students' activity? In ICT research: Is automatic analysis of pupils' answers developed in prototypes (*Pépité*) rather robust to pass on the scale (*LaboMEP*)? By combining numerical and didactical analyses, is it possible to develop a dynamic assessment tool?

## METHODS

Our method is based on both an analytical and a didactic engineering point of view. At the time of implementation in classrooms, we confront *a priori* analyses and models developed. Methodology is based on the combination of qualitative and quantitative studies: a large-scale study with an ICT monitoring in the *LaboMEP* database and some case studies with observations and interviews in classrooms (Abboud-Blanchard and al. 2007).

## FIRST RESULTS

Experimentations with assessment tool and adapted learning routes just began in 10<sup>th</sup> grade classrooms. For example, before the use of the assessment tool, a teacher has constituted 4 groups for school algebra learning in his class of 35 pupils: the very good pupils, the very good pupils with few difficulties, the average pupils, bad pupils. The teacher needs help mostly for the average group. The diagnosis agreed with the teacher's groups with only few differences and helped the teacher by lightning some pupils' difficulties that need special training.

The actual poster presented at CERME7 may be obtained from the authors by emailing them at [chenevotot.francoise@neuf.fr](mailto:chenevotot.francoise@neuf.fr).

## REFERENCES

- Abboud-Blanchard, M., Cazes, C., Vandebrouck, F. (2007). Teachers' activity in exercices-based lessons. Some case studies. *Proceedings of the Fifth Congress of the ESRME*, 1827-1836.
- Delozanne, E., Prévité, D., Grugeon, B., Chenevotot, F. (2008). Automatic Multi-criteria Assessment of Open-Ended Questions: a case study in School Algebra. *Proceedings of ITS'2008*, Montréal, juin 2008, LNCS 5091, Springer, 101-110.
- Grugeon, B. (1997). Conception et exploitation d'une structure d'analyse multidimensionnelle en algèbre élémentaire. *Recherches en didactique des mathématiques*, 17/2, Editions la Pensée Sauvage, 167-210.
- Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In Frank K. Lester (Eds.). *Second Handbook of Research on Mathematics Teaching and Learning*, 16, 707-762.
- Vanroyen, J.P. (2008). Un scénario pédagogique en Terminale S, *MathémaTICE*, 9, mars 2008. <http://revue.sesamath.net/spip.php?article133>.

## **EXTENDING [1] THE MATHEMATICS TEXTBOOKS ANALYSIS: QUESTIONS OF LANGUAGE AND ICT**

Carlos Alberto Batista Carvalho, José Manuel Leonardo de Freitas

Escola Secundária Lima de Freitas / UIED FCT UNL

*This poster refers to an ongoing research, which deals with the integration of ICT in Portuguese mathematics textbooks. However it is relatively unknown how this integration has been made. The main goal of this study is to develop a comparative analysis of how the syllabus for Mathematics A and for Professionals Courses (CP) integrates the technologies and how these technologies are discussed in school textbooks, and to characterize the different kinds of language used. For this purpose an analytical tool is being developed and tested.*

**Keywords:** ICT, Textbooks, language

In an attempt to characterize the language used in different types of textbooks and in the process of developing an analytical tool, crossed with an apparatus developed in a former work that analysed the levels of use of the graphing calculators in textbooks (Carvalho, 2006), we present here very briefly a part of literature review on the Theory of Social Activity (Dowling, 1998) directed to the analysis of textbooks.

A textbook should lead to the establishment of a pedagogical relationship between its authors and its content with its target audience to which it is intended. To be considered pedagogical, a text, according to Dowling (1998), must involve subjective relationship between two positions, on the one side, that dominates what should be taught, and on the other, someone with little or no knowledge about it.

A greater or lesser complexity of discourse adopted in school textbooks, called discursive saturation by Dowling (1998), is closely related to the pedagogical action being developed. The discursive saturation is directly related to metonymy and metaphor. When a math expression is looked at as a series of math symbols, ideally exemplified in a math equation or a demonstration, mathematics should be seen as a metonymy, presenting a high discursive saturation. If the school math often involves references to objects and not math relations, then they seem to have a metaphorical relationship, a low discursive saturation. According to Bernstein (1971, 2003), the discourse can be classified in two ways, one related to its specialization, the second related to the expression (language). For instance, a math expression has symbolic connotations in Portuguese, but the connotation with the non-mathematician is short. If the expression is translated into regular Portuguese, the content remains intact within the context of math, but the mode of expression is less specialized.

Dowling assigns three levels to the Theory of Social Activity: The Structural, the Textual and the Resources. The Structural – Activity level is further divided in other three levels: a) Practices – Domain, related to the forms of expression and content



relating to signifiers and signified, b) Practices – Discursive Saturation, related to the fact that the practices (DS+) present at the level of discourse, a highly complex organization and exhibit compared to a (DS-) practice a complete articulation, and c) Positions, related to the construction of hierarchical positions of transmission and acquisition. The textual level works on voices: who says what – the author, who hears – the reader, and messages, the content. The Resources level is essentially semiotic, assuming significant modes.

When testing the apparatus, we chose a topic: Properties of exponential functions, from the same textbooks authors, for two syllabuses: Mathematics A and Mathematics CP. Mathematics A is addressed to the students wishing to pursue studies, Mathematics CP is addressed to students who want to finish their studies. To the former, a more formal knowledge is needed, in the esoteric domain; to the later a more practical knowledge is expected, in the descriptive domain. However, in both textbooks, the presentation is equal, “copy and paste”. In the message, it is not possible to distinguish the intended audience. In both textbooks, the resource is iconic, no different strategies were used. A reference to the graphing calculator is made, in both textbooks, but nothing more than a reference. The pedagogic relationship, in the Mathematics CP is more of subject than dependent or even objectified, as it should be.

In conclusion, a student, who had problems with mathematics in a school route, will experience the same problems in the other route. The authors used the work of a manual for replicating the other, without regarding to whom it is addressed. The syllabus recommendations were not attended. Where we should find a practical approach, we find a formal approach. In the Mathematics CP textbooks where the references and examples with the use of ICT should prevail, the opposite is observed.

The actual poster presented at CERME7 may be obtained from the author by emailing them at [almo\\_mou@hotmail.com](mailto:almo_mou@hotmail.com)

## NOTES

1. Work within the Research Project QAMURT – Quality of learning in mathematics using technology resources, supported by FCT – Foundation for Science and Technology, under contract ner PTDC/CED/71744/2006.

## REFERENCES

- Bernstein, B. (1971/2003). *Class, Codes and Control: Theoretical studies towards a sociology of language*. London, UK: Routledge.
- Carvalho, C. (2006). *A calculadora gráfica na trigonometria do 11º ano, uma análise de manuais escolares*. Master Thesis in Portuguese, Universidade de Lisboa.
- Dowling, P. (1998). *The Sociology of Mathematics Education: Mathematical Myths/Pedagogic Texts*. London: Falmer Press.

## **INTENSIVE USE OF ICT IN PRE-SERVICE PRIMARY TEACHERS' PROFESSIONAL TRAINING IN MATHEMATICS: IMPACT ON TEACHING PRACTICES.**

Jean Baptiste Lagrange, Alexandre Becart

LDAR, Research Team in the Didactics of Mathematics, University Paris 7, France

*Up to this year, the training of primary teachers took place in France during two years in a Teacher Training College (IUFM). The first year was mainly theoretical and aimed at completing students' disciplinary knowledge. The second year focused on the teaching profession in primary schools, alternating training and practice. Research has found that in spite of a significant social development of digital technologies, teachers' use of ICT is very limited, especially in their classroom practice. The specific ICT courses taught at the IUFM did not really change this situation. My hypothesis is that an intensive use of ICT in professional training in mathematics can be more effective.*

### **CONTEXT AND PROBLEMATIC OF THIS RESEARCH**

On the one hand, the current development of ICT, particularly in mathematical applications, allows considering new teaching practices in teacher training in mathematics. On the other hand, few or no in-service primary teachers introduce ICT in classroom practice in contrast with institutional guidelines and most of them use ICT poorly or not at all to prepare their class lessons and collaborate with colleagues. These first observations appear paradoxical in light of the availability of ICT in both private (Lagrange, Lecas, & Parzysz, 2005) and professional sphere. They appear also paradoxical with regard to the institutional demands. Primary teacher training considered in my thesis is multidisciplinary and punctuated by periods of practices with full responsibility of a class. For both these reasons a tension exists between the different training periods and the different requirements of disciplinary subjects. In my assumption, ICT, especially the communication via networks, can offer a way to manage this tension positively.

### **THEORETICAL FRAMEWORK**

The theoretical framework should apply to the specific training of primary teachers. Pre-service primary teachers' ages and background are very diverse. The training is accessible to any graduate. Brought together in a permanent training group, students make a community of learners in which diversity can become an advantage to their learning if they collaborate. Many researchers believe that ICT can promote and guide this collaboration, especially Internet-oriented tools (Beatty & Geiger, 2010). The work of a school teacher in mathematics is to construct workable classroom sessions. The choice of the trainer is to support this construction and to promote reflexivity, especially by collective discussions. ICT can help the mutualisation and



the collective discussion relatively to classroom sessions. The framework of communities of practice developed by Wenger (2006) seems particularly well suited to the study of learning through exchange and collaboration within a permanent group and induced by the structure of training to share practices. Jonassen (2000) revisited the activity theory, in order to build learning environments based on the analysis of systems of activities in the professional sphere, which is consistent with my goal of preparing students to work as members of communities of teachers. That is why I choose these two approaches: Wenger's communities of practice and the activity theory by Jonassen.

## METHODOLOGY

As a trainer in mathematics, I taught courses for pre-service teachers and I choose to implement an intensive use of ICT. An agreement with the students allowed me to access to all digital exchanges in some groups. The experimental data collected during the courses was supplemented by questionnaires and group discussions with students in order to know more about their personal tools, skills, initial and final ICT use, how they work, their activities in training and how they use ICT in class.

## FIRST RESULTS

An initial analysis of data collected reinforces my choice of a dual theoretical approach. Indeed, we can see that students built a community after arriving at the Teacher Training College. This community first exchanged about work placements and then integrated teaching. In addition, the mathematics session production work was built around a division of the task and the activity theory revisited by Jonassen corresponds to this work. Students' personal equipment available was sufficient to consider any use of ICT in training. However while a few students were very good at using ICT tools, uses by most of the students were limited to a few tools and their knowledge was generally poor. Scaffolding or additional training was therefore often necessary in order to generalize in-training ICT practice to the whole group. Regarding classroom uses, the results are contrasted.

## REFERENCES

- Lagrange, J.-B., Lecas, J.-F., Parzysz, B. (2006). Les professeurs stagiaires d'IUFM et les technologies. Quelle instrumentation? *Recherche et Formation* n°52.
- Jonassen, D. H. (2000). Revisiting activity theory as a framework for designing student-centered learning environments. In D.H. Jonassen & S.M. Land (Eds.), *Theoretical foundations of learning environments* (pp. 89-121). Mahwah, NJ: Lawrence Erlbaum Associates.
- Beatty, R., & Geiger, V. (2010). Technology, Communication, and Collaboration: Re-thinking Communities of Inquiry, Learning and Practice. In C. Hoyles, J.-B. Lagrange (Eds.), *Mathematics Education and Technology – Rethinking the Terrain* (pp. 251-284). ICMI Study Series. Springer.
- Wenger, E (2006) *Communities of practice, a brief introduction*. Retrieved from <http://www.vpit.ualberta.ca/cop/doc/wenger.doc>.

# **THEORETICAL FRAMEWORK TO ANALYSE THE PROBLEM SOLVING PROCESS WHILE HANDLING COMPLEX MATHEMATICAL STORY PROBLEMS IN PRIMARY SCHOOL**

**Johannes Groß<sup>1</sup>, Katharina Hohn<sup>1</sup>, Sibel Telli<sup>1</sup>, Renate Rasch<sup>2</sup>,  
Wolfgang Schnotz<sup>3</sup>**

<sup>1</sup>DFG Graduate School “Teaching and Learning Processes”, University of Koblenz-Landau, Landau, Germany

<sup>2</sup>Institute of Mathematics, University of Koblenz-Landau, Landau, Germany

<sup>3</sup>Department of General and Educational Psychology, University of Koblenz-Landau, Landau, Germany

*One of the goals of mathematical education is the development of problem solving competences. This development is fostered by working on complex story problems. There are different models describing the sequences of operations and actions during the solving process for regular story problems. Yet, these models cannot be used when dealing with complex story problems. The idea of a new model is described in this paper. Two existing models were combined and adapted in order to get one that is suitable for our purposes.*

Keywords: problem solving, coordination, MLT-Model

## **THEORETICAL BACKGROUND**

The term complex story problems specifies a group of tasks that is different from regular story problems (Rasch, 2001). Unlike regular story problems, these tasks are based on very challenging mathematical structures and cannot be solved by arithmetic operation models that students are usually familiar with (Rasch, 2001). The development of problem solving competences is an important element of the curriculum for mathematics. Concerning this aim, such competences can be built when working on story problems (Staub & Reusser, 1995). An according study shows the positive influence of complex story problems on the development of problem solving competences of primary school students (Rasch, 2001). Steps of the problem solving process of regular story problems are defined differently in various models. Unfortunately, these models do not fit the problem solving process of complex story problems (Ruwisch, 1999). Since there are no models that describe the process of problem solving with regard to this special group of story problems, we aimed to develop a suitable one for such problems.

The first model (Reusser, 1993) is called “Student-Problem-Solver” (SPS). According to the SPS model the student recodes and gradually transforms the story problem into the solution. The second model is the „cognitive-metacognitive model of mathematical problem solving” (Montague & Applegate, 1993). This model defines seven cognitive processes as essential for effective and efficient problem

solving. Both models focus on different aspects of the problem solving process of story problems and complement each other. Through the “coordination” (Prediger, Bikner-Ahsbahr & Arzarello, 2008) of these two models we developed a new model (Radford, 2008). Our MLT-Model (Groß, Hohn, Telli, Rasch & Schnotz, 2010) describes the sequence of the problem solving process of complex story problems. Based on this model, we also developed a system of categories that can be used to analyse the problem solving process and the use of different external representations while working on complex story problems in primary school mathematics.

**The actual poster presented at CERME7 may be obtained from the author by emailing them at [gross@uni-landau.de](mailto:gross@uni-landau.de)**

## REFERENCES

- Groß, J., Hohn, K., Telli, S., Rasch, R. & Schnotz, W. (2010). Analyse des Problemlöseprozesses bei der Bearbeitung von problemhaltigen Textaufgaben durch Grundschul Kinder, 74. *Tagung der AEPF* in Jena, 13.-15.09.2010.
- Montague, M. & Applegate, B. (1993). Middle School Students` Mathematical Problem Solving. An Analysis of Thinking-Aloud Protocols. *Learning Disability Quarterly*, 16 (1), 19-32.
- Prediger, S., Bikner-Ahsbahr, A., Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches – First steps towards a conceptual framework, in: *ZDM - The International Journal on Mathematics Education*, 40 (2), 165-178.
- Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *Zentralblatt für Didaktik der Mathematik - The international Journal on Mathematics Education*, 40 (2), 317-327.
- Rasch, R. (2001). *Zur Arbeit mit problemhaltigen Textaufgaben im Mathematikunterricht der Grundschule*. Hildesheim: Franzbecker.
- Reusser, K. (1993). Tutoring systems and pedagogical Theory. Representational Tools for Understanding, Planning, and Reflection in Problem Solving. In S.P. Lajoie & S.J. Derry (Eds.), *Computers as Cognitive Tools*. Hillsdale: Lawrence Erlbaum Associates.
- Ruwisch, S. (1999). *Angewandte Multiplikation Klassenfest, Puppenhaus und Kinderbowle*. Frankfurt am Main: Peter Lang.
- Staub, F. & Reusser, K. (1995). The role of presentational structures in understanding and solving mathematical Word Problems. In C.A. Weaver, S. Mannes & C.R. Fletcher (Eds.), *Discourse Comprehension: Essays in honor of Walter Kintsch* (pp. 285-305). Hillsdale: Lawrence Erlbaum.

## THE COMPLEXITY OF ADVANCED MATHEMATICAL THINKING AT THE NON-UNIVERSITY LEVEL

Miguel Silva

Escola Secundária Jorge Peixinho - Montijo - FCT/UNL- (UIED)

António Domingos

Departamento de Matemática da FCT/UNL- (UIED)

*This poster intends to be a presentation of the PHD research project by the first author. Understanding the reasons to the (un)success in mathematics, has attracted the interest of researchers worldwide. Starting with some theories of knowledge construction, it aims to understand the different interactions occurring in the classroom. In addition we expect to further study this complexity, by understanding the links between the representations of various concepts, and knowing what non-higher education students make of it. The technology emergence, in all its forms, in schools as catalyst for learning, may enhance the connections between various representations of a mathematical object as a relevant factor contributing to the clarification of learning issues.*

**Key-words:** Advanced Mathematical Thinking, Mathematics Education, Significance, Connexions in curricula

In higher education, Domingos (2003) studied the understanding of mathematical concepts at the university. Internationally, several authors are references on this theme, namely Dubinsky(1991) work, on the APOS theory, inspired by Piaget. Equally important is the *proceptual* approach developed by Tall who bases his theory on three mathematical worlds of development of mathematical knowledge: conceptual-embodied world; *proceptual* symbolic-world, and formal-axiomatic world. (Tall, 2007). Dreyfus (1991), and his approach giving emphasis on representation and abstraction power, it will also be studied.

The main goal of this work is to understand how students acquire knowledge. Therefore, we want to: characterize the processes involved in understanding of mathematical concepts by students of the secondary school and characterize student actions in the appropriation process of mathematical concepts. Understand how they coordinate between different representations of the same mathematical object; understand how technology is able to influence the articulation and connection processes among the different concepts. The interrelation between the theories above is fundamental to understand the appropriation process of mathematical concepts. Dreyfus proposed the representation and abstraction as the main way to characterize this process. The *procept* notion and the *proceptual thinking* presented by Tall allow us to understand and explain the Dreyfus view. The Dubinsky approach became relevant in the way that allows us to describe more deeply the different steps of all

process and mainly the students' capacity of managing the encapsulation and de-encapsulation process. At this point the issues we bring to the investigation are: Does a better connection between different representations of a same object, makes a math student more skilled? Would these connection lines contribute significantly to an increased stage of abstraction leading students to a higher level of complexity? Is mathematics specific language used in the different representations of an object, too complex to promote peaceful interconnections among them? Would better links between different representations, allow a better understanding of upcoming knowledge? For studies of this nature makes more sense to use a qualitative interpretive methodology (Bogdan & Biklen, 1994). Field work consists on an 11<sup>th</sup> grade classroom observation on trigonometry topic. Later interviews will be conducted, and analyzed. Some documents analysis such as (notebooks, tests, worksheets) will be a complementary way of data collecting.

To analyze the data collected through the classroom observations and the teaching experiments based on proposed tasks, we intended to identify the representations that students have about the concepts studied and try to understand how the students abstract and reflect about them (Dreyfus position). To understand this process we use the notions of procedural and *proceptual lthinking* (Tall's theory) to categorize the student's answers to the proposed tasks, giving special attention to the *procept* notion. To analyze the transition from one to another kind of thinking we intended to take into account the interiorization of the student's actions on objects, the coordination and inversion of the processes developed and the encapsulation and de-encapsulation of these processes and objects are fundamental for our analyses of the mathematical objects studied (APOS)

## REFERENCES

- Bogdan, R., & Biklen, S. (2006). *Investigação Qualitativa em Educação: Uma Introdução à Teoria dos Métodos*. Porto, Portugal: Porto Editora.
- Domingos, A. M. (2003). *Compreensão de Conceitos Matemáticos Avançados - A Matemática no Ensino Superior*. Dissertação apresentada para a obtenção do grau de Doutor em Ciências de Educação .Lisboa.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In T. David, *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht, Netherlands: Kluwer Academics.
- Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall, *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht, Netherlands: Kluwer Academics.
- Tall, D. (2007). Embodiment, Symbolism and Formalism in Undergraduate Mathematics Education. Plenary at 10th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education, (pp. 22-27). San Diego, USA.

## **NETWORKING THEORIES**

### **THE ‘KOM PROJECT’ AND ‘ADDING IT UP’ THROUGH THE LENS OF A LEARNING SITUATION**

Yvonne Liljekvist and Jorrit van Bommel

Karlstad University

*Abstract: Mathematical skills and understanding have been described in different ways. Adding it Up and The Danish KOM project are two such frameworks and in this poster these are put side by side. Transcriptions of students working on a mathematical task were first categorized within each framework, where after comparing and contrasting revealed similarities and differences. We have classified these differences and similarities through four cases which are described in the poster.*

Keywords: competencies, proficiencies, comparing and contrasting

#### **INTRODUCTION**

In spring 2009 the authors attended a PhD course where one aim of the course was to understand two different reports; the Danish KOM project (KOM abbreviation in Danish: Competencies and the Learning of Mathematics) (Niss, Højgaard Jensen, et al, 2002) and the American report Adding It Up: Helping children learn mathematics (Kilpatrick, Swafford, & Findell, 2001). Kilpatrick et al state that their descriptions give “a framework for discussing the knowledge, skills, abilities, and beliefs that constitute mathematical proficiency” (p. 116). Niss et al describe their work as “an overarching conceptual framework which captures the perspectives of mathematics teaching and learning at whichever educational level.” (Niss, 2003, p. 1).

#### **USING NETWORKING STRATEGIES**

Our interest lies in *understanding the frameworks*. Since both frameworks are in their own way describing cognitive and behavioural activities for learning of mathematics, a logical question for us was to see whether the frameworks were disjoint or overlapping, and in case they would have common characteristics, whether they were all inclusive, only partly joined or whether one is a subset of the other. In order to understand the core concepts in the frameworks we needed a tool, and an arena, to be able to better grasp some characteristics. Using networking strategies when applying these frameworks on transcriptions from a videotaped learning situation made it possible to answer our research question: Do the two frameworks represent a situation of parallel development of theory in mathematics education?

In order to understand the frameworks we worked with a comparing and contrasting procedure (Bikner-Ahsbahr & Prediger, 2010). Each relevant sentence expressed by

the students was analyzed to define competencies or proficiencies exposed within the sentence, and when possible the sentence was categorized in terms of proficiencies and/or in terms of competencies. The focus on our analysis was to find examples of all the core components of the frameworks. Once each proficiency and competence had been identified by a cluster of sentences, we could start looking for patterns and reveal similarity and diversity. Since the sentences could be categorized with components from both frameworks, or from one or the other, a contrasting procedure pulling out typical (co) variation was conducted.

## FINDINGS AND IMPLICATIONS

The focus of the poster is the networking strategy method which is used to grasp core characteristics in two similar conceptual frameworks. When applying the two frameworks on the same data, patterns appeared that enlighten similarities and differences. Our results, presented through four cases, indicate that networking strategies can supply researchers (and teachers) with tools to understand the explanatory power of different conceptual frameworks. Hence the work to develop new theories by networking strategies goes side by side with the endeavor to more deeply understand existing frameworks.

Since we are participating in interdisciplinary projects where networking of theories is of importance the method using networking strategies to better understand (two) conceptual frameworks is useful in our further studies, even though we are at time not going to make a deeper study within these two frameworks.

The actual poster presented at CERME7 may be obtained from the authors by emailing them at [yvonne.liljekvist@kau.se](mailto:yvonne.liljekvist@kau.se), [jorryt.vanbommel@kau.se](mailto:jorryt.vanbommel@kau.se)

## REFERENCES

- Bikner-Ahsbals, A. & Prediger, S. (2010). Networking of theories – an approach for exploiting the diversity of theoretical approaches. In BharathSriraman, & Lyn English (Eds.), *Theories of mathematics education. Seeking new frontiers, Series: Advances in mathematics education* (pp. 483-506). New York, Berlin: Springer.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Mathematics learning study committee center for education division of behavioral and social sciences and education. Washington, D.C.: National Academy Press.
- Niss, M. (2003). *Mathematical competencies and the learning of mathematics: The Danish KOM project*. Paper presented at the Third Mediterranean conference on mathematics education, Athens.
- Niss, M., & Højgaard Jensen, T. (2002). *Kompetencer og matematiklæring : ideer og inspiration til udvikling af matematikundervisning i Danmark*. København: Undervisningsministeriets forlag.

## **A SUCCESSFUL COMBINATION OF COP AND CHAT TO UNDERSTAND PROSPECTIVE PRIMARY MATHEMATICS TEACHERS' LEARNING?**

Kicki Skog

Department of Mathematics and Science Education, Stockholm University, Sweden

*Can it be useful to combine two theoretical frameworks when attempting to understand what kind of challenges prospective teachers are facing and what makes sense in becoming a mathematics teacher? The poster suggests a combination of Lave and Wenger's theory of learning in Communities of Practice and Engeström's Cultural Historical Activity Theory. It also presents a discussion regarding commensurability within these frameworks, motives for using these theories in this study and what I find challenging.*

Keywords: Communities of practice, CHAT, mathematics, prospective teachers

### **INTRODUCTION**

There are times in our lives when learning is intensified: when situations shake our sense of familiarity, when we are challenged beyond our ability to respond, when we wish to engage in new practices and seek to join new communities. (Wenger, 1998, p. 8)

My primary aim for conducting this research is to understand what prospective primary mathematics teachers ascribe as meaningful and important in their process of learning to teach mathematics. From a first analysis issues regarding the role of mentors and peer-collaboration within the teacher education program are central, but also the individuals' background and school experiences. Thus I need to use theoretical frameworks that allow me to move from the present to history without changing research track. I believe this inside-view, i.e. social roles, agency etc., could be elaborated on as learning within different communities of practice, COP (Lave & Wenger, 1991; Wenger, 1998). When it comes to understanding how background, family and personal goals influence/affect the individual prospective teacher's learning, it could be fruitful to elaborate on these questions from a cultural and historical perspective. Therefore I consider using COP in combination<sup>100</sup> with Cultural Historical Activity Theory, CHAT (Engeström, 1999), as theoretical frameworks.

---

<sup>100</sup> For a further discussion on strategies and methods for connecting theoretical approaches, see Prediger et.al (2008)



## THE POSTER

Three excerpts from early interviews with two prospective primary mathematics teachers were viewed on the poster. Both immigrated to Sweden as adults. The excerpts indicate that an analysis through modes of belonging (Wenger, 1998) – here with a focus on imagination – could facilitate understanding of prospective mathematics teachers' identity and identification in communities of practice. From a CHAT perspective there is a tension between what mathematics was for them in their country of birth, and what mathematics is in the Swedish teacher education programme. It affects the prospective teachers and makes an analysis from a cultural historical activity theoretical perspective possible.

## CONCLUDING REMARKS

In this case, after having fruitful discussions with participants at the CERME 7 conference, I consider that the study can benefit from using both perspectives and create a broader picture of the process of becoming a mathematics teacher, but there are challenges to overcome. How, for example, do I distinguish between the different notions of community? Are the theories competing on the same level – and if they do: can that enrich the study? Can a combination of these two theoretical frameworks contribute with a deeper understanding of learning to teach mathematics as situated, but also as historically and culturally bound meaning?

## REFERENCES

- Engeström, Y. (1999). Activity theory and individual and social transformation. In Y. Engeström, R. Miettinen, & R.-L. Punamäki (Eds.), *Perspectives on activity theory* (pp. 19-38). Cambridge: Cambridge University Press.
- Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. New York: Cambridge University Press.
- Prediger, S., Bikner Ahsbals, A. & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM The international Journal on Mathematics Education* 40, 165–178.
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge: Cambridge University Press.

## **CONNECTING THEORIES TO THINK OF PLANE GEOMETRY TEACHING FROM ELEMENTARY TO MIDDLE SCHOOL**

Marie-Jeanne Perrin-Glorian

Laboratoire de Didactique André Revuz, Université d'Artois

*Abstract: To think of continuity of geometry teaching along compulsory school, we coordinate the theory of didactical situations with a theoretical framework concerning elementary geometry obtained by integrating Euclid's geometry with semiotic theory of Duval (analysis of processing in the register of figures).*

*Keywords: geometry, figures, instruments, figure restoration, connecting theories*

### **RESEARCH QUESTIONS AND NECESSITY TO CONNECT THEORIES**

*Motivation of the research:* There is a discontinuity between geometry in primary school, drawing figures with usual geometrical instruments and further geometry with abstract objects and demonstration. In primary schools, teachers usually regard geometry teaching as not problematic but find no relation between geometry they studied in their secondary courses and the geometry they teach. The use of usual instruments for drawing geometrical figures is generally considered as a technical problem. Nevertheless, these instruments are strongly related to mathematical objects and properties of these objects; this relation is not obvious for pupils.

*Our question:* Is there a way to think of continuity in teaching geometry along compulsory school (from 6 to 15) and help teachers?

*Necessity of connecting theories:* To address our problem, mathematical theories and theories on learning are not sufficient. It is necessary to construct intermediary theories integrating both. The theory of didactical situations (Brousseau, 1997) is itself such an intermediary theory but, to take into account the specificity of elementary geometry teaching, we can say, using the definitions of Bikner-Ahsbabs et al. (2010), that we needed to coordinate it with a specific theoretical frame we constructed, itself integrating two other theories: the axiomatic geometry of Euclide as a theory of space and the semiotic theory of Duval (2006).

### **CONNECTED THEORIES AND THE WAY TO CONNECT THEM**

First, our own theoretical frame about elementary geometry teaching leads to the following main hypotheses from an epistemological and cognitive point of view:

*In relation with Duval's theory:* There is a specific way to look at figures in geometry, different from the way to look at drawings in everyday life. The natural view of figures is to see a figure as composed of juxtaposed figural unities of dimension 2. However, to solve a geometrical problem as well as to construct figures with instruments, you have most of the time to see superposed figures composed of surfaces, lines, or points and relations between them and to articulate the register of

figures and the register of language. Many difficulties with geometry are rooted in problems to change the view on figures. An example is given in the poster.

*In relation with Euclid's theory:* The different axiomatic theories for geometry are not equivalent to define coherence between mathematical theory and a teaching progress compatible with students' development. By the important role given to the triangle, and its reference to magnitudes and ratios of magnitudes without numbers, the Euclidian axiomatic is adapted to help thinking of a progression from a view of figures as composed of two dimensional objects to a view in terms of relations between lines (straight lines and circles) and points.

We construct a progression, developing in primary school the identification of alignments, intersections... in 2D-figures, in relation with tracing with instruments and geometric objects and properties. Continuity in teaching geometry also demands extending the notion of instrument considering as well forms, templates, stencils, tracing paper, scissors and so on... as instruments. Some of these instruments are 2D-instrument, bearing 2D-information about the figure (e.g. templates) or not (e.g. tracing paper), some instruments are 1D-instruments, tracing lines (e.g. ruler), some are 1D or 2D-instruments according to how they are used (square, compass).

*Coordination with TDS:* This integrated theory of elementary geometry is then "coordinated" with the theory of didactical situations (TDS) seen as complementary. TDS is used to conceive a *milieu* (mainly figures and instruments) around production and reproduction of geometrical figures, possible modifications of this *milieu* and *situations* (rules to act on this *milieu*) helping students to progressively modify their view on figures and instruments and construct geometrical concepts. A situation, we called *figure restoration*, plays a particular role: a couple of figures is given such as one is the model, the other one is a "damaged" figure (partially deleted) that has to be restored and a "cost" is introduced for the use of instruments.

Examples and trends for discussion are proposed in the poster. The actual poster presented at CERME7 may be obtained from the author by emailing at [glorian@math.jussieu.fr](mailto:glorian@math.jussieu.fr)

## REFERENCES

- Bikner-Ahsbahs, A., et al. (2010). Networking of theories in mathematics education. In Pinto, M. M.F & Kawasaki, T. F. (Eds.). *Proceedings of the 34th Conference of the IGPME*, (Vol. 1, pp. 145-175). Belo Horizonte, Brazil: PME.
- Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics*. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Duval, R. (2008). Eight Problems for a Semiotic Approach in Mathematics Education. In L. Radford, G. Schubring, F. Seeger (Eds.) *Semiotics in Mathematics Education; Epistemology, History, Classroom and Culture*, 39-61. Sense Publishers.

## **DIDACTICAL ANALYSIS AND CITIZENSHIP WITH PROSPECTIVE MATHEMATICS TEACHERS**

Yuly M. Vanegas M., Joaquin Giménez and Vicenç Font

*Barcelona University*

*Our particular aim is to show some relationships between the development of didactical analysis and reflection about teaching of prospective mathematics teachers and developing citizenship, as a basic competence through the study of various professional math practices.*

**KEY WORDS:** didactical analysis; mathematical practices; citizenship

A persistent problem in mathematics education research is how to justify and design training programs giving professional problems that arise and influence the nature and quality of professional practices. Such practices should enable the future teacher for knowing describing, explaining, evaluating and improving situated and contextualized teaching and learning (Llinares and Krainer, 2006; Font, Planas and Godino, 2010). In particular to know how to assess general competencies as “training for citizenship through mathematical practices”. We want to describe the path of a future math teacher, characterizing its evolution in relation to training analysis expertise, identifying how he evolve being aware of considering citizenship when using mathematical practices.

Citizenship is interpreted here by using 4 axes: political appropriation of knowledge, responsible participation, critical mathematical perspective, responsibility for practices (Vanegas y Giménez, 2010). To perform didactical analysis, the future teacher use tools of description and explanation which have been learned in different subjects of the master. Above all, by applying the notion of criteria of suitability proposed by onto-semiotic approach (Godino, Batanero & Font, 2007) in which it's considered at least six criteria for valuing didactical suitability: Epistemic suitability, Cognitive suitability, Interactive suitability, Media/resources suitability, Emotional suitability, Ecological suitability.

Methodologically, we discuss two types of didactical analysis practices about instruction processes in two different moments: (m1) analyze experienced teacher practices by using videoclips, (m2) implement a practice discussing with a tutor being a teacher in a regular class of 12-13 years old students, and (m3) analyze their own practice two weeks after. With regard to m2 and m3, we analyze: (1) immediate written reflection about their school practice conducted, in which it was asked to explain from planning to analysis about what happened. (2) Delayed reflected report by which it was to proposed a replanning of the former practice after a certain time (part of the Final Master work). In these two tasks, the future teacher reflects on the design and implementation of a knowledge-driven sequence of extent of areas with flat figures.

In m1, the future teacher recognize the value of the context to interplay citizenship discussion when analyzing a presented task about comparison of a density population problem. In m2, we found that organizes his proposal for students based on a realistic mathematics perspective. He introduces the understanding of the formulas of the areas, with puzzles and discoveries. Sequence ends with a proposal to measure the oil slick from the Mexico Gulf through the NASA pictures, "to analyze the magnitude of the tragedy (using future teacher words)". After the experience in a real classroom (in a third moment), he think about his mistakes, and discuss about a need for a collaborative debate with the students, because "it was a monologue".

We acknowledge a greater depth of analysis of the practice m2 y m3 in relation to practice m1, based on the use of structured tools during training process. With regard to the didactical analysis levels achieved, we identify adequate descriptions of suitability levels in the development of mathematical practice. He recognizes that there are different ways to introduce the notion of area and identifies a potential cognitive conflict in the passage to the limit when considering the irrational measurement. Awareness of the interactional valuing can be seen as identifying rules and active positioning of students. With regard to developing critical reflective citizenship, we conclude that the future teacher (1) considers that mathematics enable the creation of powerful tools for interpretation, characterization and solution of problems (in this case, environmental), recognizing the complexity of the situation raised not only by the diversity of concepts, processes and mathematical procedures involved, but, for the opportunity to reflect on the social impact of such situations (2) allowing its students to value the measure as a conceptual tool for interpreting a phenomenon and social impact analysis and (3) fostering a reflective analysis of students through problem solving and the use of ICT as a working methodology.

## ACKNOWLEDGMENTS

This work was partially funded by Spanish Ministry of Education, EDU2009-01820/EDUC.

## REFERENCES

- Font, V., Planas, N. y Godino, J. D. (2010). Modelo para el análisis didáctico en matemática. *Infancia y Aprendizaje*, 33(1), 89-105.
- Godino, J. D., Batanero, C. & Font, V. (2007). The Onto-Semiotic Approach to Research in Mathematics Education, *ZDM-The International Journal on Mathematics Education*, 39(1-2), 127-135.
- Llinares, S. y K. Krainer (2006), "Mathematics (student) teachers and teachers educators as learners", en A. Gutierrez y P. Boero (eds.), *Handbook of Research on the psychology of Mathematics Education: Past, Present and Future*, Rotterdam/Taipe, Sense Publishers, 429-459.
- Vanegas, Y., Giménez, J. (2010). Aprender a enseñar matemáticas y educar en ciudadanía. En M.L.Callejo, J.M. Goñi, (Coords.) *Educación matemática y ciudadanía*. Barcelona. Graò, 147-166.

## HOW TO TEACH MATHEMATICAL KNOWLEDGE FOR TEACHING

Jorryt van Bommel

Karlstad University

*The paper presents the results of a learning study conducted at a teacher-training course in mathematics. The Object of Learning of the learning study was to develop Mathematical Knowledge for Teaching (MKT) for student teachers (ST). The data consists of three videotaped seminars and pre- and post tests (282) written by 47 ST. The data was analysed with help of variation theory and four critical features concerning MKT were found successively during the learning study cycle. Implementation of these critical features in the subsequent seminars increased ST results concerning their use of MKT.*

Keywords: Mathematical Knowledge for Teaching, Learning Study, Teacher Education.

### INTRODUCTION

Teachers in mathematics not only need a deep understanding of mathematics but also need knowledge on how to teach the mathematics (Ball & Bass, 2003; Grossman, 1990; Ma & Kessel, 2000). How to teach such MKT was the focus for the study described here. In what way would it be possible to work with MKT during a teacher-training course so that ST would consider aspects of MKT in their future teaching? Five aspects of MKT were chosen to work with: pupils' preconceptions, models for explanation, related (hands on) materials, suitable exercises and curricular knowledge.

### FRAMEWORKS

Two frameworks were considered in this learning study. The conceptual framework *MKT* describing the kind of mathematical reasoning, insight, understanding and skill teaching mathematics demands (Ball, Hill, & Bass, 2005) was used as a background when planning the lessons and also to describe the data. Furthermore the theoretical framework *Variation Theory* (VT) was worked with in planning the seminars as well as analysing the data. VT sees upon learning as to be aware of the world in a new way. To learn about an object one has to identify its parts, see connections within the object and discern it from other items. Through a focus on an Object of Learning one can identify differences in learning and describe conditions necessary for learning to take place. (Marton & Booth, 1997)

## ANALYSIS & RESULTS

The tests consisted of writing a lesson plan in which ST would reveal their use of the five aspects of MKT. VT was used to describe how the aspects were dealt with in relation to the way MKT was addressed during the seminar. What was made possible for ST to discern during the seminars and what did they actually discern? Four critical features were found and implemented in the subsequent seminars. There was a clear improvement in the way the 5 aspects of MKT were addressed by ST in the tests. The four critical features found were: 1) Curricular goals must be seen as something to choose from (not as a whole package to work with). 2) Contextual experiences have to be created and reflected upon from a teachers' perspective. 3) Components have to be described clearly in order to connect with other components. 4) The underlying mathematics should be understood when discussing an activity from a teachers' point of view.

The actual poster presented at CERME7 can be obtained by contacting the author: [jorrbomm@kau.se](mailto:jorrbomm@kau.se) or at [www.math.kau.se/jorrbomm](http://www.math.kau.se/jorrbomm)

## REFERENCES

- Ball, D., & Bass, H. (2003). *Towards a practice-based theory of mathematical knowledge for teaching*. Paper presented at 2002 AMCanadian Mathematics Education, Edmonton.
- Ball, D., Hill, H. C., & Bass, H. (2005). Knowing Mathematics for Teaching: Who Knows Mathematics Well Enough To Teach Third Grade, and How Can We Decide? *American Educator*.
- Grossman, P. L. (1990). *The making of a teacher : teacher knowledge and teacher education*. New York: Teachers College Columbia Univ.
- Ma, L., & Kessel, C. (2000). *Knowledge of fundamental mathematics for teaching*. Paper presented at the Knowing and Learning Mathematics for Teaching, Washington D.C.
- Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah, N.J.: Erlbaum.

## **ANALYSING EXAMS MATHEMATICAL QUESTIONS**

Mário José Miranda Ceia

Escola Superior de Educação, Instituto Politécnico de Portalegre

*The poster presents a model to analyse exams items. The model is inspired on the SOLO Taxonomy (Biggs & Collis, 1982), and in the cycles of learning they established. As in the SOLO taxonomy, three parameters were established to allow the items analysis: the capacity demanded to produced the answer (amount of knowledge required); the mental operations involved in the resolution (type of mental operation); and the type of answer required. The preliminary version of the model was designed to be used to any grade, and hypothetical answers were prepared to test the model. One example to exemplify the analysis made is provided.*

### **STUDY OBJECTIVES**

The present study pretends to develop a model that allows to establish different categories of exams questions complexity, and, in a second moment, to analyse exams quality.

### **THE MODEL (PRELIMINARY VERSION)**

The model was design using three parameters: Capacity Demanded, the amount of concepts involved in the answer, how the concepts are related among them, and if they are provided or the student has to search them; Involved Operations, the type of reasoning used, inductive or deductive, and if it was an original one or similar to others studied before; and Answers Requested, answers are unique or there are more than one possible answers, if there are more than one possible answer, are they coherent, and consistency between the conditions provided and the answer requested.

Five categories were considered, using the same designations that Biggs and Collis used in their taxonomy: Pre-structural, Uni-structural, Multi-structural, Relational and Abstract.

### **METHODOLOGY**

For the analysis of the examinations questions, it was considered hypothetical answers, produced by the authors according the level to which the examination was designed. With those optimal answers it was possible to realise the mathematical knowledge involved, the kind of reasoning, and possible coherence and consistency of the answers. To define the mathematical knowledge it was used the specific aims of the Curriculum of Mathematics for Basic Education (Ponte et al, 2007).



### AN EXAMPLE OF ANALYSIS (QUESTION 3, 4<sup>TH</sup> GRADE EXAM, 2009)

Question: A group of 47 children, from a holiday camp, are going to climb. Children are going by car. Each car carries 6 children. How many cars are needed to carry the 47 children? Show how you reach the answer.

Hypothetical Solution: The correct answer is 8 cars. Dividing the amount of children, 47, by 6 (the number of children by car) we realise that 7 cars became full and remain 5 children, which make necessary another car.

To reach the solution it is possible to use: 1) Additive structure; 2) Multiplicative structure. In both cases to answer this question it is necessary to understand the meaning of the operation involved, and realise what the results mean for the proposed context.

Knowledge Involved: To answer the question it is necessary to use arithmetical knowledge from the 3<sup>rd</sup> and the 4<sup>th</sup> grades. It is necessary to use several pieces of knowledge to reach the solution, and in the end it is required to establish the connection between the mathematical solution and the real context of the question.

Operations: To answer the question it is necessary to use deductive reasoning similar to others experienced in class.

Answer: In both solutions the answer is closed and unique. However, it is necessary to solve the inconsistency between the arithmetical result and the real world.

Category: The question is considered a *relational*.

### REFERENCES

- Biggs, J. B. (1992). Modes of Learning, Forms of Knowing, and Ways of Schooling. In A. Demetriou, M. Shayer, & A. Efklides (Ed.), *Neo-Piagetian Theories of Cognitive Development* (pp. 31-51). London: Routledge.
- Biggs, J. B., Collis, K. F. (1982). *Evaluating the quality of learning*. London: Academic Press, Inc.
- Kulm, G. (Ed.) (1990). *Assessing higher order thinking in mathematics*. American Association for the Advancement of Science Press.
- Ceia, M., Duarte, J. M. (2010). *A Taxonomia SOLO e os Exames (SOLO Taxonomy and Examinations)*. Aveiro: Associação de Professores de Matemática.
- Menezes, L., Santos, L., Gomes, H. & Rodrigues, C. (Orgs.) (2008). *Avaliação em Matemática: Problemas e Desafios*, Viseu: Secção de Educação Matemática da Sociedade Portuguesa de Ciências da Educação.
- Pegg, J., & Tall, D. (2010). The fundamental cycle of concept construction underlying various theoretical frameworks. In B. Sriraman & English, L. (Ed.), *Theories of Mathematics Education: Seeking New Frontiers* (pp. 173 – 192). NY: Springer-Verlag.
- Ponte, J. P., Serrazina, L., Guimarães, H. M., Breda, A., Guimarães, F., Sousa, H., Menezes, L., Martins, M. E. G. & Oliveira, P. A. (2007). *Programa de Matemática do Ensino Básico*. Lisboa: Ministério da Educação.

# **THE PRACTICES OF PROSPECTIVE TEACHERS IN SOUTH AFRICAN AND CANADIAN MATHEMATICAL LITERACY TEACHER EDUCATION PROGRAMS: WHAT WORKS AND WHAT DOES NOT?**

**\*Joany Fransman, \*\*Joyce Mgombelo & \*Marthie Van der Walt**

*\*North-West University & \*\*Brock University*

## **Abstract**

*The implementation of Mathematical Literacy (ML) as subject in South African schools in 2006 necessitated the re-training of teachers to teach the new subject. The teacher education program for ML was implemented at the North-West University (NWU) in 2005. This program is therefore quite “young” hence the need for an investigation into other similar programs at universities in developed countries. The purpose of the poster was to report on a qualitative study that aimed at exploring the experiences of in- and pre-service teachers enrolled in ML teacher education programs at the NWU in South Africa (SA) and Brock University in Canada. Findings indicated that SA can indeed learn from Canada: Put more emphasis on the mathematics processes (MP) in the mathematics module in the program.*

*Keywords: Mathematical Literacy, mathematics processes, mathematics teacher education programs.*

## **INTRODUCTION**

The central question in our study was: In what ways can developers and practitioners of Mathematical Literacy (ML) in teacher education programs in South Africa (SA), learn from programs for similar courses in Canada? The implementation of ML as subject in the SA school curriculum in 2006 necessitated the re-training of suitable teachers. This raises an empirical question: How does ML teacher education programs in SA relate to a wider context of other ML teacher education programs from developed countries like Canada?

## **THEORETICAL FRAMEWORK**

The definition of ML has been debated for decades, both nationally and internationally. A collage of the salient indicators from the various definitions of ML in extant literature (e.g. (Bowie & Frith, 2006) describe ML as the individual's abilities and competencies to make sense of, communicate and engage meaningfully in mathematical situations that are encountered in his/her daily life. The approach that needs to be adopted in developing ML is to engage with contexts rather than applying Mathematics already learned to the context (Department of Education, 2003). For Bowie & Frith (2006) ML is about seeing every context through a quantitative lens.

Prospective teachers in an effective ML teacher education program should pay attention to the development of all the important skills and the Mathematics Processes which are problem solving, communication, connections, reasoning and representation (Brown & Schäfer, 2006).

## **RESEARCH METHODOLOGY**

The study utilized a multiple case study approach due to its dual contexts (SA and Canada). Participants comprised of 61 out of 189 SA in-service teachers enrolled in the program and the 12 out of 30 Canadian pre-service teachers enrolled in the program. Data was collected from questionnaires, individual and focus group interviews, as well as lesson observation. The data was analysed using ATLAS.ti.5.0 – a computer-aided system used for the analysis of qualitative studies.

## **FINDINGS AND RECOMMENDATIONS**

The study indicates that ML as concept is not yet fully comprehended by the SA participants. A thorough understanding of the concept of ML will enhance teaching methods as well as the understanding of the learner.

Analysis of data indicates that sufficient emphasis is placed on the Mathematics Processes in the Canadian program as these processes are explicitly stated in the curriculum documents. In SA, it seems that the Mathematics Processes are not explicitly implemented in ML classrooms as teachers seem not to be aware of them. The MP's have a large role to play in any ML curriculum and should be explicitly mentioned in the National Curriculum Statements in SA and not only be implied as is currently the case.

## **CONCLUSION**

Developers of ML teacher education programs in SA can indeed learn from Canada by having a clear conceptualization of ML and placing more emphasis on the Mathematics Processes in the program. The actual poster presented at CERME7 may be obtained from the first author by emailing her at joany.fransman@nwu.ac.za.

## **REFERENCES**

- Bowie, L., & Frith, V. (2006). Concerns about the South African Mathematical Literacy curriculum arising from experience of materials development *Pythagoras*, 64, 62-69.
- Brown, B., & Schäfer, M. (2006). Teacher Education for Mathematical Literacy: A modelling approach. *Pythagoras*, 64, 45-51.
- South Africa. Department of Education. (2003). National Curriculum Statement Grades 10 – 12 (General). Mathematical Literacy. Pretoria: Government Printer.

# TEACHERS' USE OF GRAPHING CALCULATORS IN HIGH SCHOOL MATHEMATICS CLASSROOM - THE INFLUENCE OF TEACHERS' PROFESSIONAL KNOWLEDGE<sup>1</sup>

Helena Rocha

*Institute of Education, University of Lisbon, Portugal*

## ABSTRACT

The main focus of this poster is a new model of teacher's knowledge: Knowledge for Teaching Mathematics with Technology (KTMT). Central in this model is a set of inter-domains knowledge divided in two main areas: *Mathematics and technology*, and *Teaching & learning and technology*. This model is the basis for a study that intends to understand how teachers use graphing calculators in their practice and the influence of their professional knowledge on that use.

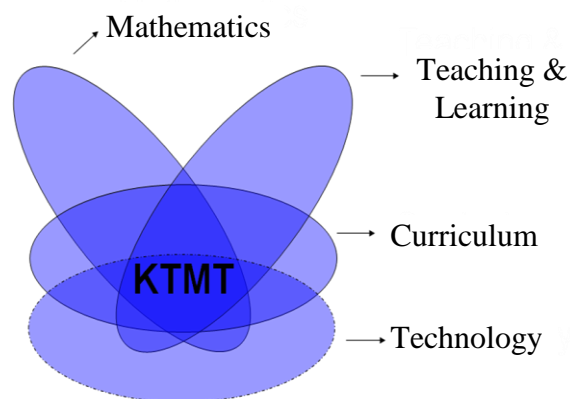
**KEYWORDS:** Knowledge for Teaching Mathematics with Technology (KTMT), professional knowledge, technology, graphing calculators

## RESEARCH GOALS AND QUESTIONS

This is an ongoing study that intends to examine the nature and extend of teachers use of graphing calculators. Among other questions, it aims to answer the following question: What is the influence of the teacher's professional knowledge?

## THEORETICAL FRAMEWORK

The evidence of how teachers' knowledge affects teachers' practice took to the development of frameworks such as *Didactical knowledge* (Ponte, 1999). Technology integration in schools raised new questions and prompted the need to a special focus on technology. *TPCK*, from Mishra & Koehler (2006), is one of the most



known characterizations that attends to technology issues, but problems still remain (Rocha, 2010). KTMT intends to articulate these two conceptualizations, overcoming the criticism and integrating important conclusions from technology research (Zbiek *et al.*, 2007). In KTMT, beyond the base knowledge domains (Mathematics, Teaching & learning, Curriculum, Technology), there are two important sets of inter-domains knowledge: *Mathematics and technology*, and *Teaching & learning and technology*. The first one involves knowledge of mathematical fidelity of the technology; knowledge of new emphasis on mathematical content; knowledge of new orders of mathematical content; and

representational fluency (i.e. knowledge of different representations and how to link them in meaningful ways). And the second one includes knowledge of effective ways of overcoming student difficulties in articulating technological and no technological information, promoting a critical attitude; knowledge of mathematical concordance; and knowledge of the diversity of classroom activities enhanced by technology, and of the new roles allowed. Curriculum is seen as a transversal domain that influences all the others. KTMT is then conceptualized as a unique body of knowledge constructed in an integrative way from the interaction of all the knowledges referred above.

## METHODOLOGY

This study adopts a qualitative methodological approach, performing three teacher case studies. Data gathering includes class observation, interviews, and documental gathering. All interviews and classes observed are audio taped and transcribed. Data analysis consists of data interpretation, considering the problem studied, and the theoretical framework.

## EXPECTED RESULTS

A better understanding of teacher's use of graphing calculator and how its effective integration can be achieved is the main outcome expected from this study.

The actual poster presented at CERME7 may be obtained from the author by emailing her at hcrocha@ie.ul.pt

## REFERENCES

- Mishra, P. & Koehler, M. (2006). TPACK: a framework for teacher knowledge. *Teachers College Record*, 108 (6), 1017-1054.
- Ponte, J. (1999). Didáticas específicas e construção do conhecimento profissional. In J. Tavares, A. Pereira, A. Pedro & H. Sá (Eds.), *Actas do IV Congresso da SPCE* (pp. 59-72). Porto: SPCE.
- Rocha, H. (2010). Ensinar Matemática com a tecnologia – uma nova conceptualização do conhecimento e das etapas subjacentes ao seu desenvolvimento. In *Actas do I Encontro Internacional TIC e Educação* (pp. 121-126). Lisboa: IE - UL.
- Zbiek, R.; Heid, M.; Blume, G. & Dick, T. (2007). Research on technology in mathematics education: a perspective of constructs. In F. Lester, Jr. (Ed.), *Second Handbook of research on mathematics teaching and learning* (pp. 1169-1207). Charlotte, NC: NCTM, IAP.

<sup>1</sup> work developed with the support of *Fundação para a Ciência e a Tecnologia* and *Ministério da Educação de Portugal*

## DEEPER MATHEMATICAL UNDERSTANDING THROUGH TEACHER AND TEACHING ASSISTANT COLLABORATION

Paul Spencer, Julie-Ann Edwards

University of Southampton, School of Education, UK

*This paper presents a summary of a poster presented at CERME7, the focus of which was to examine the way that teachers and teaching assistants (TAs) work together in mathematics classrooms in order to encourage mutual effective professional development. Within this paper the aims of the research project, the background to the research, the methodology employed and the significance of the research will be discussed.*

**Key Words:** Teachers, teaching assistants, collaboration, professional development.

The main aim of this study is to determine which characteristics of ways of working promote opportunities for developing deep understanding of mathematics in order to develop and trial an intervention strategy which will encourage mutual professional development. The study is designed to work from existing practice to address issues relating to the professional development of both mathematics teachers and their teaching assistants.

The expected outcomes of the study are:

- a) A characterisation of the ways in which mathematics teachers and TAs currently work together in secondary classrooms in the UK (11-16 years);
- b) Identification of models of teachers' and TAs' experiences which lead to effective professional development;
- c) Intervention strategies which promote effective professional development opportunities.

The UK standards framework for teachers (Teacher Development Agency [TDA], 2007) places an increasing emphasis on the effective working relationships between teachers and TAs in the classroom. Although it is generally acknowledged that the presence of TAs in the classroom has a positive effect on pupils' achievement, there is little research evidence in the UK (other than Muijs, 2003) about how this is achieved. However, recent findings of the Deployment and Impact of Support Staff (DISS) project (Blatchford et al. 2009) suggest that teaching assistants may have a negative impact on the mathematical progress of pupils.

Research conducted by Ma (1999) found that Chinese teachers' deeper understanding of mathematics is linked to the time and support they are given to work collaboratively on the content of their lessons. The importance of this deeper understanding of mathematics has been recognised previously by Ball (1989) who

concludes that deep understanding of mathematics and its interrelation with pedagogical knowledge is crucial to effective teaching.

This research project combines the work of Ma (1999) and Ball (1989) by utilizing the teacher and teaching assistant partnership in order to mutually develop deeper mathematical understanding, whilst meeting some of the recent recommendations made in the DISS project, thus mirroring the successes in China, but using established patterns of working in the UK. The design of this study utilizes embedded case study research methods (Scholz & Tietje, 2002) to characterise mathematics teachers' and their TAs' informal experiences and ways of working together, and systematic case studies (Yin, 2003) of particular teachers and TAs. Grounded theory methods provide the means for developing the models of current working practice through analysis of qualitative data. An intervention strategy will be designed, based on these findings, and will be implemented through a trial that involves both teachers and TAs. The development of an intervention strategy which encourages professional development is planned to have an impact on school and government policy which will, in turn affect the way mathematics teachers and teaching assistants work together in secondary school classrooms. Successful implementation of an intervention strategy is intended to develop teachers and TAs' mutual deeper understanding of mathematics which should impact on pupil attainment in the subject.

The actual poster presented at CERME7 may be obtained from the author by emailing them at [pcs1v07@soton.ac.uk](mailto:pcs1v07@soton.ac.uk)

## REFERENCES:

- Ball, D. (1989). Research on Teaching Mathematics: making subject knowledge part of the equation. In Brophy, J. (ed.) (1991), *Advances in Research on Teaching: teachers' knowledge of subject-matter as it relates to their teaching practice*. Greenwich, CT: JAI Press.
- Blatchford, P., Bassett, P., Brown, P., Koutsoubou, M., Martin, C., Russell, A., Webster, R., Rubie-Davies, C. (2009). *The Impact of Support Staff in Schools. Results from the Deployment and Impact of Support Staff (DISS) Project. (Strand 2 Wave 2)*. DCSF-RR148. London: Department for Children, Schools and Families.
- Ma, L. (1999). *Knowing and Teaching Elementary Mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Muijs, D. (2003). The Effectiveness of the Use of Learning Support Assistants in Improving the Mathematics Achievement of Low Achieving Pupils in Primary School. *Educational Research*, **45**(3) 219-230
- Scholz, R and Tietje, O. (2002). *Embedded Case Study Methods: Integrating Quantitative and Qualitative Knowledge*. Thousand Oaks, CA: Sage Publications
- Yin, R.K. (2003). *Case Study Research Design and Methods* (3<sup>rd</sup> ed.). Newbury Park: Sage Publications.