CONTINUITIES AND DISCONTINUITIES FOR FRACTIONS

A PROPOSAL FOR ANALYSING IN DIFFERENT LEVELS

Susanne Prediger – Bremen University, Germany

Published in Novotna, J. et al. (eds.): Proceedings of the 30th PME, Prag 2006, 4-377-384

Students' difficulties with fractional numbers have been treated in many empirical studies with different theoretical frameworks for explaining them. Among them, the theory of conceptual change has met an increasing interest, focussing on necessary discontinuities in the learning process. This article proposes an integrating model with different levels in which continuities and discontinuities between natural and fractional numbers can be found, including the often neglected level of meaning. The model proves to be useful for explaining phenomena found in the presented empirical study and for structuring the current state of research.

1. DIFFICULTIES WITH FRACTIONS AS AN ISSUE OF RESEARCH

In many different countries, empirical studies on students' competencies and conceptions in the domain of fractions have shown enormous difficulties. Whereas *algorithmic competencies* are usually fairly developed, *understanding* is usually weaker, as well as the *competencies to solve* word or realistic problems including fractions (e.g. Hasemann 1981, Barash/Klein 1996, Aksu 1997).

One common aspect of several approaches for explaining the difficulties is the emphasis on *discontinuities between natural and fractional numbers;* Streefland (1984) for example spoke of "N-distractors", Hartnett / Gelman (1998) described early understandings of natural numbers as barriers to the construction of new understanding and pointed out that students see continuities where discontinuities in the dealing with numbers should appear. Brousseau (1980) classified these hidden discontinuities as epistemological obstacles. The discontinuities have been systematized by different authors, e.g. Stafylidou/Vosniadou (2004), their lists comprise for example the fact that the uniqueness in the symbolic representation of natural numbers does not hold for fractions (since several fractions can represent the same fractional number). Other famous discontinuities are the density of numbers and the order-property of multiplication: Whereas multiplication always makes bigger for natural numbers (apart from 0 and 1), this cannot be applied to fractions.

Among different theoretical approaches to explain students' difficulties with these discontinuities, the conceptual change approach (Posner et al. 1982) has gained a growing influence in mathematics education research (e.g. Lehtinen/ Merenluoto/ Kasanen 1997, Stafylidou/Vosniadou 2004, Lehtinen 2006). On the basis of a constructivist theory of learning and inspired by Piaget's notion of accommodation, the conceptual change approach has emphasized that learning is rarely cumulative in the sense that new knowledge is only added to the prior (as a process of enrichment). Instead, learning often necessitates the discontinuous *reconstruction* of prior knowledge when confronted with new experiences and challenges. Problems of conceptual change can appear, when the learners' prior knowledge is incompatible with the new necessary conceptualisations. The key point in the conceptual change approach adopted here is that discrepancies between the intended mathematical conceptions and the real individual conceptions are not seen as individual deficits but as necessary stages of transition in the process of reconstructing knowledge.

Other authors have emphasized the importance of underlying mental models (Fischbein et al. 1985, Greer 1994) or 'Grundvorstellungen' (GVs, see vom Hofe et al. 2005) for explaining students' difficulties. This paper goes beyond the current state of research by integrating the so far competing approaches for explaining students difficulties.

2. PROPOSAL FOR AN INTEGRATING LEVEL MODEL

The purpose of the here presented integrating model (see Fig. 1) is to provide a conceptual tool for describing the precise locations of students' difficulties with discontinuities, i.e. the quality of the obstacles hindering students to master the necessary changes in the process of conceptual change.

Following Fischbein et al. (1985), the model differentiates between algorithmic, intuitive and formal understanding. The *formal level* includes the definitions of concepts and of operations, structures, and theorems relevant to a specific content

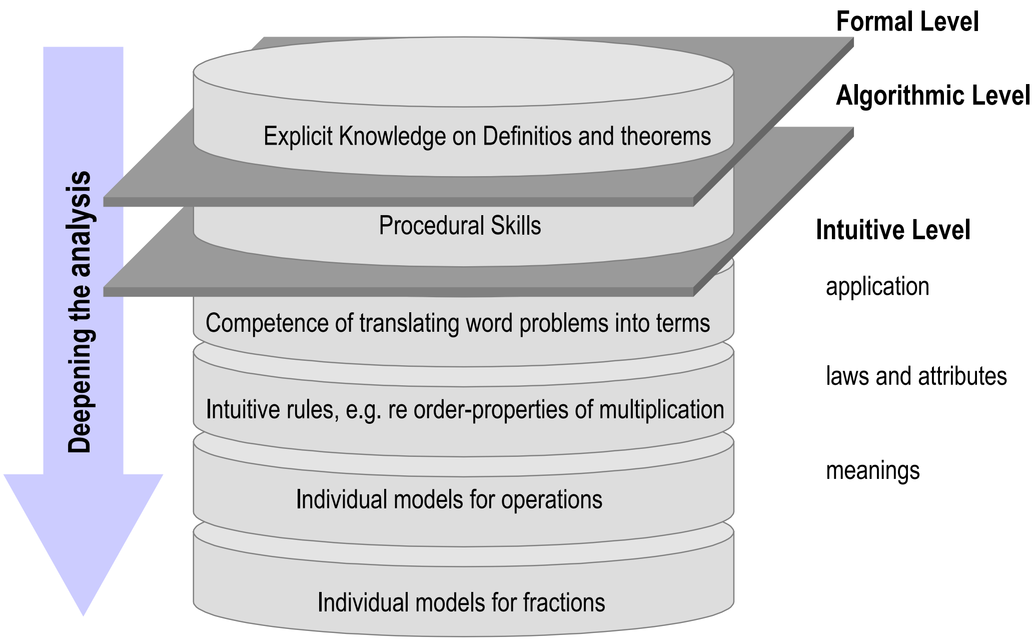

Figure 1: Obstacles can lie deeper – Different levels of students' difficulties

domain. This type of knowledge is formally represented by axioms, definitions, theorems and their proofs. It is not within the main scope of this paper. The *algorithmic level* of knowledge is basically procedural in nature and involves students' capability to explain the successive steps included in various, standard procedural operations. Although solving of word problems also has procedural aspects, it is assigned to the intuitive level since, as will be shown in the next sections, it is directly connected with other aspects of the intuitive level.

Intuitive understanding is characterized as the type of mostly implicit knowledge that we tend to accept directly and confidently as being obvious. On the *intuitive level,* we distinguish between conceptions about mathematical laws or properties called *intuitive rules* (like "multiplication makes bigger") from those about the *meanings* of concepts (like the interpretation "multiplication means repeated addition").

Nearly all studies dealing with conceptual change in the field of fractions have treated intuitive knowledge, but they have mainly focused on the level of intuitive rules. In contrast, they have neglected the level of meanings (modelled by the constructs of 'Grundvorstellungen' by vom Hofe et al. 2005 and mental models by Fischbein et al. 1985). The following sections will show why both levels must be considered integratively for understanding processes of conceptual change adequately. The next section sketches how this model can help to structure the current state of research. Furthermore, the presented empirical study about the multiplication of fractions gives evidence for the fact that the difficulties on different levels are highly connected, each level giving reasons for obstacles on the level above.

3. RESEARCH QUESTION FOR THE EMPIRICAL STUDY

This paper presents results of an empirical study dealing with students' competencies, content knowledge and conceptions of fractions and their operations as well as the connections between different conceptions (Prediger 2004). The report is here restricted to the specific part of the study which is related to multiplication.

This part of the study started from a phenomenon which has been shown by many empirical studies (cf. e.g. Brousseau 1980, Streefland 1984, Fischbein et al. 1985, Barash/Klein 1996): Although most students' show relatively good algorithmic skills in multiplying fractions, many of them work with the intuitive rule that 'multiplication makes bigger', which is mostly inherited from dealing with natural numbers. This phenomenon is also often cited within the framework of conceptual change and was hence an interesting case for being elaborated.

The survey of existing literature showed that the conception "multiplication makes bigger" and its generalization from natural to fractional numbers offers an obstacle for activating the multiplicative operation when mathematizing word problems from which they know that the result must be smaller than the factors (cf. Bell et al. 1981, vom Hofe et al. 2005). This is a first example for the fact that the problems on one level (*translating word problems*) can be influenced by a problem on the level underneath (the *intuitive rule* concerning the order property).

Fischbein et al (1985) gave empirical evidence for the thesis that the pertinacity of the *intuitive rule* "multiplication makes bigger" is often connected with the continuing maintenance of the *interpretation* of multiplication in the repeated addition model (which does not work for fractions). Whereas the influence of the repeated addition model is well studied, the great variety of other individual models for the multiplication of fractions and naturals must be explored more systematically.

That is why our study was guided by the following research questions: Which individual models for the multiplication do our students activate, and how do these models influence the intuitive rules about the order property and the use of multiplication? Where are the most crucial obstacles?

4. DESIGN OF THE STUDY

Our study was designed in a two step format, in which the written test of the first step was complemented by a qualitative clinical interview study. For the second step, 38 students in grade 7 to 10 (age 11 to 16) of different German schools have been asked in semi-structured pair interviews. 12 of the 19 interviews have been transcribed and analysed with respect to the interviewees' conceptions about multiplication of fractions and their connections on the different levels. The interviews have been videotaped or tape-recorded and transcribed. In a qualitative data analysis, the transcripts were interpreted on the basis of the individual conceptions derived from the written test and by careful comparison of cases (cf. Flick 1999).

The first step consisted of a 80 minutes paper and pencil test, written in all four Grade 7 classes of a German grammar school. 81 tests could be analysed, in total 44 boys and 37 girls (about 12 years old). The students' answers have been evaluated quantitatively in a points rationing scheme. Where appropriate, the answers have also been analysed qualitatively by categorizing the manifested conceptions about fractions and their operations in a data-driven, not theory-driven way (cf. Flick 1999).

Among the 11 test items, four concerned the multiplication on the different levels (see Fig. 1). Item 1 requested *algorithmic knowledge*, namely the skill to conduct the basic operations like $\frac{5}{6}\cdot\frac{2}{3}$. Item 3 posed a *word problem* that could be treated with multiplication when students knew the part-of-interpretation for the multiplication ($\frac{3}{4}$ *of* 60 as $\frac{3}{4}\cdot 60$). Item 2 operated on the *level of intuitive rules*, asking in a multiple choice format whether multiplication of fractions makes bigger or smaller or sometimes bigger, sometimes smaller. Item 6 ("Find a word problem that can be solved by means of the following equation: $\frac{3}{4}\cdot\frac{1}{3}=\frac{1}{4}$") operated exploratively on the level of meaning. It was given in an open item format in order not to impose a presupposed mental model but to gain a wide choice of impressions of the really existing *individual mental models*.

5. MOST IMPORTANT RESULTS

68 of 81 students, i.e. 84%, could calculate the multiplication item 1b correctly. The item's result 5/9 (which is bigger than both factors) could not prevent most of the students from approving the property "multiplication makes bigger" in Item 2. 29 of the 68 students with correct results in Item 1b chose an intuitive rule about the multiplication of fractions which is only true for natural numbers, hence, the known findings (see above) about this intuitive rule could be replicated in our sample.

Compared to the results given by Fischbein et al. (1985), the explorative item format for Item 6 facilitated a more detailed and multi-faceted impression of the students' individual models. The individual models for multiplication expressed by the probands were very heterogeneous and quite distant from the mathematically sustainable models. By coding and categorizing, the following individual models could be specified:

- No answers concerning meaning: 38 of 81 students could not show any individual interpretation of multiplication in Item 6. 12 students did not give any answer. 26 answers were only related to calculations (e.g. by explaining the way of calculation).
- Adequate individual models: Only 12 students formulated interpretations being coherent with the mathematical perspectives. 4 students formulated a story of a diminution lens and showed their *individual model of scaling up and down*. Two students used *multiplicative comparison*. Six students made explicit their *part-of-interpretation* for the multiplication (cf. Figure 3 for the different models).
- Traces of sustainable models: 14 students disposed of interesting traces of sustainable individual models. Two students translated the multiplication with $\frac{1}{3}$ by a *division by 3* and formulate a word problem of *sharing*. Twelve other students worked with the *part-of-interpretation* but formulated them in an incomplete way, e.g. "Peter has $\frac{3}{4}$ of a cake. He gives away $\frac{1}{3}$ of it. How much does he keep?"
- Non-sustainable models: 17 students expressed non-sustainable individual models of the multiplication of fractions, the most dominant being *additive* (e.g. "$\frac{3}{4}$ cake and then $\frac{1}{3}$.")

Although the sample size does not allow statistical significance for the dependencies between the order conceptions and the quality of manifested individual models, the results show a distinct tendency. Whereas 75% of those students who could not express a sustainable individual model have expressed an order conception which is only fruitful for natural numbers, there were only 50% among those with traces of a sustainable model and only around a third of those who expressed a sustainable individual model for the multiplication. That means that the formation of adequate individual models proves to be the major obstacle for overcoming the over-generalized intuitive rule "multiplication makes bigger". Not yet stable individual models like an incomplete part-of-interpretation can only partially suffice for the formation of adequate order conceptions.

These quantitative results could be strengthened by the interview study in the second step. This can be illustrated by this prototypical passage:

Tim: That is clear, multiplication makes it bigger [...]

Interviewer: What does that mean when you multiply two numbers?

Tim: Well, this and this times plus itself!

Interviewer: Okay, but what does 5/6 times 2/3 plus itself mean, then?

Tim: How? [hesitates 3 sec] no idea!

Interviewer: Could you think about it in another way?

Tim: (draws a picture) 5/6 pizza and 2/3 pizza, how can I multiply them?

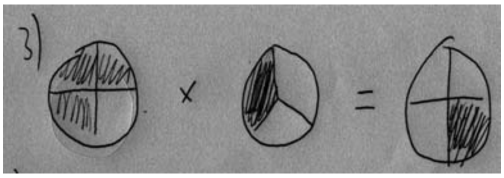

Figure 2: Individual representation of the multiplication

When in situations like this one, the interviewer headed for a part-of-interpretation by giving hints, an interesting new obstacle appeared. As Tim in this passage, many interviewees clang to the interpretation of a fraction as a part of a whole. This basic model for fractions is extensively taught in Germany. Tim's problem is represented in a pointed way by the individual representation in Figure 2, drawn similarly by several other interviewees. The inseparable link between fractions and their circle ("pizza")-representations makes it impossible for some interviewees to interpret the second factors in another way, for example like proportion or part of the first.

6. DISCUSSION: STRUCTURING EMPIRICAL FINDINGS

The findings of our and previous empirical studies about multiplication of fractions can be resumed to four connected findings that describe the learners' thinking in deeper and deeper levels in the model of Figure 1. Formal knowledge was not within the scope of the study, hence, it does not appear.

1. Finding: Algorithmic competencies for the multiplication of fractions alone do not qualify students to utilize their competencies in reality-oriented situations or word problems (Barash/Klein 1996, p. 35f.). In general, students' competencies to solve real problems or word problems are low (Hasemann 1981, Aksu 1997).
2. Finding: One important (but not the only) reason for the first finding is the intuitive rule "multiplication makes bigger". This intuitive rule incapacitates learners from choosing the multiplication for translating problems from which they know that the result must be smaller than the factors (cf. Bell et al. 1981, vom Hofe et al. 2005). This finding could be reproduced within the current study.
3. Finding: The pertinacity of the intuitive rule "multiplication makes bigger" (second finding) is linked to non-sustainable individual models for multiplication of fractions (the finding is supported by Greer 1994 and Fischbein et al. 1985).

Our written test and even more the interviews have shown the strong connection between both levels.

4. Finding: One possible reason for the incomplete formation of sustainable individual models of multiplication of fractions (third finding) could be found by the interviews in the limited conceptions of fractions, being only interpreted as parts of a whole.

In total, these findings give evidence for the thesis that the difficulties on the different levels are highly connected, each level giving reasons for obstacles in the upper level.

Additionally, the level model helps us to re-locate the exact place of the epistemological obstacles in the process of conceptual change from natural to fractional numbers. As sketched in the first section, most researchers in conceptual change locate the problem on the level of laws and rules. In this level, the transfer of rules from natural numbers to fractions simply appears to be a problem of hasty generalization. In contrast, our study could elaborate Fischbein et al.'s (1985) emphasis on the importance of the underlying level of meaning, namely the mental models. Whereas Fischbein et al. focused on the most important model 'repeated addition', our study could explore the factual variety of individual models for multiplication by using explorative data collection strategies (open item format and semi-structured interviews). By these means, we can enlarge Fischbein's findings considering all possible models of multiplication.

We can now complement the list of discontinuities on the level of laws about properties of fractions and their operations (given by Stafylidou/Vosniadou 2004) by another table: Figure 3 amends the list of discontinuities in the deeper level of mental models, i.e. in the *level of meaning* (cf. Greer 1994).

Natural numbers	→	**Fractions**
repeated addition (3x5 means 5+5+5, i.e. 3 wands of 5m length in a row)	→	???
area of a rectangle (3x5 is the area of a 3cmx5cm rectangle)	→	area of a rectangle (2/3 x5/4 is the area of a 2/3 cm x 5/4 cm rectangle)
????	→	part-of-interpretation (2/3 x 5/2 means 2/3 of 5/2)
multiplicative comparison (twice as much)	→	multiplicative comparison (half as much)
scaling up (3x5 means 5cm is stretched three times as much)	→	scaling up and down (2/3 x 5/2 means 5/2 cm compressed on 2/3 of it)
combinatorial interpretation (3x5 as number of combining 3 shirts + 5 trousers)	→	????

Figure 3: (Dis-)Continuities of mental models for multiplication in the transition from natural to fractional numbers

This compilation makes clear that not all mental models have to be changed, e.g. the interpretations as an area of a rectangle or as scaling up can be continued for fractions as well as the multiplicative comparison. In contrast, the basic model 'repeated addition' is not sustainable for fractions, neither the combinatorial interpretation. Vice versa, the basic model of multiplication, the part-of-interpretation, has no direct correspondence for the natural numbers. By this analysis of the mathematical structures behind, we can now specify the exact location of obstacles: Not the intuitive rules are the problem, but the necessary changes of mental models. Metaphorically speaking, the obstacles can be located in the flashes of Figure 3.

References

Aksu, M. (1997) 'Student performance in dealing with fractions', *Journal of Educational Research* 90(6), 375-380.

Barash, A. and Klein, R. (1996) 'Seventh Grades Students algorithmic, intuitive and formal knowledge of multiplication and division of non negative rational numbers', in Puig, L. and Gutiérrez, A. (eds.) *Proceedings 20th PME*, Vol. 2, 35-42.

Bell, A., Swan, M., and Taylor, G. M. (1981) 'Choice of operation in verbal problems with decimal numbers', *Educational Studies in Mathematics* 12, 399-420.

Brousseau, G. (1980) 'Problèmes de l'enseignement des décimaux', *Recherche en Didactiques des Mathématiques* 1, 11-59.

Fischbein, E. et al. (1985) 'The role of implicit models in solving problems in multiplication and division', *Journal of Research in Mathematics Education* 16 (1), 3-17.

Flick, U. (1999) *Qualitative Forschung. Theorie, Methoden, Anwendung in Psychologie und Sozialwissenschaften* [Qualitative research. Theory, methods, application in psychology and social sciences], Frankfurt, Rowohlt.

Greer, B. (1994) 'Extending the meaning of multiplication and division', in Harel, G. and Confrey, J. (eds.) *The development of multiplicative reasoning in the learning of mathematics,* Albany NY, SUNY Press, 61–85.

Hartnett, P. and Gelman, R. (1998) 'Early Understandings of Number: Paths or Barriers to the Construction of new Understandings?' *Learning and instruction* 8(4), 341-374.

Hasemann, K. (1981) 'On difficulties with fractions', *Educational studies in mathematics* 12(1), 71-87.

Lehtinen, E., Merenluoto, K. and Kasanen, E. (1997) 'Conceptual change in mathematics: From rational to (un)real numbers', *Europ. Journal of Psych. of Educ.* 12 (2), 131-145.

Lehtinen, E. (2006, in prep.): Mathematics education and learning sciences, to appear in: Proceedings of ICME-10, Kopenhagen 2004.

Posner, G., Strike, K., Hewson, P., & Gertzog, W. (1982) 'Accommodation of a scientific conception: Toward a theory of conceptual change', *Science Education* 66 (2), 211-227.

Prediger, S. (2004) 'Kompetenzen und Vorstellungen zu Brüchen von Gymnasiastinnen und Gymnasiasten. Bericht über eine empirische Untersuchung' [Students' competencies and conceptions of fractions. Report on an empirical study], *Preprint*, Bremen University.

Stafylidou, S. and Vosniadou, S. (2004) 'The development of students' understanding of the numerical value of fractions', *Learning and Instruction* 14 (5), 503-518.

Streefland, L. (1984) 'Unmasking N-distractors as a source of failures in learning fractions', in Southwell, B. et al. (eds.) *Proceedings of 8th PME*, Sydney, 142-152.

vom Hofe, R., Kleine, M., Blum, W., Pekrun, R. (2005) 'On the Role of 'Grundvorstellungen' for the Development of Mathematical Literacy - First Results of the Longitudinal Study PALMA', to appear in Bosch, M. (ed.) *Proceedings 4th Congress of ERME*, Spain 2005, here cited from the pre-conference paper, http://cerme4.crm.es/