When theory should guide action, what kind of theorizing do we need?

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Abstract. Theory in mathematics education can take different roles, among them the role of guiding the action of different stakeholders (e.g., teachers, teacher educators, curriculum designers, researchers, policy makers). Researchers and practitioners alike have argued that the theories typically provided by researchers are often not sufficient for guiding action and have called for research-practice partnerships as in part, a means of generating theory to guide action. So far, we have only partially understood what theories that can guide action entail, and how researchers and practitioners can engage in research-practice partnerships in order to generate theories that can better guide practitioners' actions. In this chapter, we disentangle different kinds of theory elements and their interplay, in service of guiding practice, and discuss typical challenges in reaching the goal. We draw from existing literature and our own experiences to suggest how discourses of researchers and practitioners can be transformed so that theories are better positioned to guide practitioners' actions. We suggest acts of joint theorizing and supportive conditions that can increase the chance that the developed theory can guide stakeholders' actions. Finally, we reflect on the implications for the culture and policy of the academic field of mathematics education.

Keywords. Theory, Theorizing, Theory-practice gap, guide action, stakeholders

Introduction

Kurt Lewin's (1951) maxim, "There is nothing so practical as a good theory" (p. 169) has often been cited and questioned, at the same time. As appealing as it is, the maxim can be questioned as it does not reflect the diversity of roles that theories can play in research and practice. The important diversity of the roles of theory is impressively documented in two research overviews (Assude et al., 2008; Silver & Herbst, 2007) and by the ensemble of five other chapters of this handbook. In this chapter, we solely focus on one of these diverse roles, the one that Lewin ascribed to theory: Theories might guide action; in our case, in mathematics education.

Within this particular focus, Lewin's claim still has to be considered critically because even if theory *could* guide action, many authors have problematized a theory-practice gap saying that the image of two different cultures, one belonging to professional researchers whose work is to theorize, and another belonging to practitioners whose work is to teach based on researcher-produced theory, needs to be reimagined (Boaler et al., 2003; Matthews et al., 2021; Silver & Lunsford, 2017). In this chapter, we intend to push our ambitions to contribute to realizing Lewin's (1951) maxim and explore how mathematics education theories can guide the action of mathematics teachers, mathematics teacher educators, curriculum designers, policy makers and other stakeholders. For this, we assume that generating practical theories cannot

be conducted by researchers alone for practitioners, but requires particular forms of cooperation (Cai et al., 2017; Penuel et al., 2011). We identify typical challenges and ask in a constructive, future-oriented way:

When theory should guide action, what kind of theory elements and joint acts of theorizing with various stakeholders do we need?

We treat this question along the following four theses that will be elaborated in the succeeding sections:

Thesis 1. *Combination of various functions of theory elements.* For a theory to be practical in a well-grounded way, it needs to explicate and explicitly connect different theory elements, so that prescriptive theory elements (prescribing lines of action) can be substantiated with theory elements of categorial, descriptive, explanatory, and normative functions.

Thesis 2. *Different research objects for different stakeholders of practice.* The stakeholders of "the practice of mathematics education" are manifold, e.g., mathematics teachers, facilitators of mathematics teacher professional development, curriculum designers, syllabus designers, policy makers, etc. Each of these stakeholders require different parts of mathematics education theories, so the objects of theorizing are manifold and extend from the classroom level to other levels.

Thesis 3. *Transformations between discourses.* The discourse in the research discipline and the discourses of other stakeholders are structured differently, so theory elements emerging from research need to be transformed into the discourse of the stakeholders, in order to unfold their potential to guide stakeholders' actions. This also impacts research and theorizing.

Thesis 4. *Supportive conditions for joint acts of situated theorizing.* As active participation of stakeholders in many steps of research and theorizing is crucial to overcome unfaithful hierarchies and a theory-practice gap, we can derive from Thesis 1-3 that besides the engagement of people, there are additional supportive conditions for acts of joint theorizing that can increase the chance that the developed theory can really guide the stakeholders' action.

1. Metatheoretical background: Functions of theory elements and their combination

When we talk about *theories for guiding action* in this chapter, we refer not to background theories that implicitly or explicitly guide the research practices as a philosophical or methodological stance, but about foreground theories that are explicitly developed by researchers as an outcome of research (Mason & Waywood, 1996; Assude et al., 2008). The action to be guided by foreground theories is usually the action of mathematics teachers, teacher educators, curriculum designers, or policy makers; we will come back to these different stakeholders in Section 2. First, we present some meta-theoretical background about functions of theory elements.

Niss (2007) defines (foreground) theory as an "organized network of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain . . . consisting of objects, processes, situations, and phenomena. . . . In a theory, the concepts are linked in a connected hierarchy . . . [and] the claims are either basic hypotheses . . . or statements obtained from the fundamental claims" (p. 1308). In general philosophy of science, the concepts are called *categories* or *constructs* and the claims are called *propositions;* the propositions are further distinguished by pointing to *functions* of theories. For instance, Thiel (1996) asserted that "[a theory is a] language entity in propositional or categorial form that orders the phenomena of a domain and describes the relevant features of its objects and their relations to each other; explains by general laws and allows predictions for the occurrence of phenomena" (p. 262). Similarly, McKenney and Reeves (2012) list the following functions of theories: "describe, explain, predict, or even prescribe how to change or affect certain phenomena" (p. 32). In addition to these functions, Prediger (2019a) argued that there is an additional, critical function of theories that can guide practice: a *normative* function of setting and justifying goals that might stem from analyzing problems, including problems of practice (Arcavi, 2000; Cai et al., 2017).

Theory elements that serve these different functions can be characterized by different logical structures (Beck & Krapp, 2006, p. 39ff); these are summarized in Table 1 and explained and illustrated with examples throughout the chapter.

Table 1. Five theory elements and their functions and structures (Prediger, 2019a, p. 8)

	Function of the theory element	**Structure of the theory element**
Categorial theory elements	Categories providing a language and thinking tool for perceiving and distinguishing	Conceptual structure, i.e., categories, constructs, and relations
Descriptive theory elements	Describing a certain phenomenon qualitatively or quantitatively, focused by specific categories	Propositions stating existence, categorial hierarchies, or frequencies
Explanatory theory elements	Explaining, giving causes, or identifying backgrounds	Propositions with cause-effect structure or phenomenon-background structure
Prescriptive theory elements	Purposeful acting or predicting effects of actions	Propositions in "in order to" structure or propositions in "if-then" structure
Normative theory elements	Specifying and justifying aims and rationales (e.g., learning goals or process qualities)	Propositions with an aim-reason structure

Categorial theory elements are needed as a language and thinking tool for perceiving and distinguishing phenomena. Their logical structure is conceptual and they are used to describe phenomena and relations. They consist of constructs coined for these particular purposes. Many researchers in mathematics education research have emphasized the relevance of categories or constructs for a theory's descriptive and explanatory power (e.g., Niss, 2007, p. 1308; Mason & Johnston-Wilder, 2004). di Sessa and Cobb (2004) emphasize: "[Theoretical categories] enable us to discriminate between relations that are necessary and those that are contingent. They delineate classes of phenomena that are worthy of inquiry and specify how to look and what to see in order to understand them" (p. 79). They have additionally emphasized that it can be the invention of an important category that brings a phenomenon into a new quality of being, a so-

called ontological innovation. Empirical research that generates new categories must make sure to provide an added value for articulating phenomena. The methodologies for generating categories in processes of data-led successive refinement have been carefully reflected, for example, in grounded theory (Strauss & Corbin, 1990).

As the name says, *descriptive theory elements* serve to *describe*, i.e. portray a certain phenomenon qualitatively or quantitatively. They answer typical questions such as: What characterizes this area? Which phenomena and relations occur? With which frequencies? Descriptive theory elements consist of propositions of different logical structures, for example, propositional structures describing features ("M has characteristics C" or "M can be C_1, C_2, or C_3"), categorial hierarchies ("Every *x* is also *y*"), or frequencies of occurrences ("20% of students exhibit characteristics C_1 and 30% have C_2"). Empirical research that generates new descriptive findings must make sure that the phenomena and eventually frequencies are described in trustworthy ways, with adequate categories (Strauss & Corbin, 1990).

Explanatory theory elements serve to explain, give causes, or identify backgrounds of described phenomena. They answer to questions such as: Why does this phenomenon occur? What might be the background or underlying mechanisms? Their logical structure consists of propositions with cause-effect or phenomenon-background structure ("Phenomenon *x* occurs because of *y*" or "phenomenon *x* can be traced back to phenomenon *y*"), where each scientific community has some norms of what counts as acceptable explanation (Kuhn, 1962). Empirical research that generates new explanatory theory elements requires categorial and descriptive findings and empirical evidence that the phenomena are really related to this claimed background. In qualitative research approaches, the required categorial and descriptive components are required and sometimes generated at the same time; quantitative research approaches identify explaining factors statistically, e.g., in statistical path models or regression models for testing hypotheses on potential connections (Creswell, 2003).

Normative theory elements serve to specify and justify aims and rationales of prescriptive claims for guiding action, for example, by questions such as: Which aims shall be reached (e.g., by an instruction)? In which context are they justified according to which norms? Normative theory elements can refer, for example, to content learning goals, but also to process qualities (e.g., participation of all students) that should be reached in a teaching-learning arrangement. In its most explicated form, the logical structure of normative theory elements consists of propositions connecting the aims to explicit reasons why the aim should be reached ("Students should acquire learning goal *x* because this is required for literacy aspect *y*" or "the learning process should reach process quality *z* because this has been shown to enhance *w*"). Making normative elements explicit is important, given their role as explicit or implicit components of prescriptive theory elements. Whereas the aim itself in a normative theory element cannot be "proven" empirically, the justification of this aim can refer to explanatory theory elements and therefore have an empirical foundation.

It is the fourth function in Table 1, the *prescriptive function*, that is in the center of this chapter with its focus *on guiding action*, and as we will illustrate below, these prescriptive theory elements can be co-constructed by practitioners and researchers. Prescriptive claims serve to ground purposeful acting or to predict effects of a design element or structural element (such as specific access to the structure of the content). They answer questions such as: What can be done to reach a certain goal? What could happen if a decision is taken in a specific way?

Their logical structures can be an "in-order-to" structure ("in order to reach aim *x* you are advised to do *y*") or an "if-then" structure ("if you do *x*, you could reasonably expect *y*,") obviously not in a deterministic logical sense. However, as we will elaborate via empirical examples, prescriptive theory elements cannot be articulated without *categorial*, *descriptive* or *explanatory* and *normative* theory elements.

Let us briefly illustrate these elements by an example, to which we will resort in several places of this chapter. Pinto and Koichu (2021) presented a study in which 25 experienced Israeli mathematics teachers adapted practices of educational inquiry in a partnership with two mathematics education researchers. The researchers started the cooperation by asking the teachers: "Suppose your school hires a professional mathematics education researcher, what questions about your teaching or your students' learning would you like to explore with the researcher in your classroom?" (p. 1089). The partners negotiated the diversity of their initial ideas while attempting to reconcile the apparent gap between teachers' and researchers' conceptions of and expectations from research and theory and finally collectively selected the following learning goal to strive for: students should learn to ask good mathematical questions. Successively, the topic for the joint inquiry was shaped: to investigate the kinds of questions students ask in lessons, their relation to students' understanding and possible teaching approaches to promote good questions.

The chosen learning goal "asking good questions" was successively developed into a *normative theory element* when the participants' first intuitions about what counts as "good question" were gradually examined, unpacked and explicated. In the due course of inquiry, a set of *categories* gradually emerged to distinguish different kinds of student utterances. After a while, researchers offered two categories from previous research to distinguish students' questions: elaboration questions (i.e., questions that lead the discussion into a new mathematical territory, e.g., "what happens if…") and clarification questions (i.e., requests such as "What do you mean by…?"). One teacher (here called Eva) took up these categories to *describe* her lesson observations concisely: "students asked many (13) clarification questions and only a few (4) elaboration questions" (ibid, p. 1093). Eva and most of her colleagues interpreted elaboration questions as "good questions", and articulated *predictive assumptions* in an implicit if-then structure such as "Students who ask elaboration questions will develop deeper understanding." This *predictive assumption* could be backed up by *explanative propositions* from instructional theories that the depth of student communication can promote the deepening of their learning. Other intuitive *predictive assumptions* articulated in the beginning did not hold when teachers analyzed their own lessons, so the observations led to new explanatory elements. For example, Eva found a counter-intuitive connection between her and her students' questions: "teacher questions, specifically high-order thinking questions, generally reduce student questions since in these situations the student thinking is led by the teacher." Finally, by the end of the inquiry Eva formulated for herself a set of *prescriptive propositions* for eliciting students' rich questions, "I should teach more such lessons [with a challenging task] and conduct whole classroom discussions only after the students exhausted all their questions." In this way, personal theories were increasingly aligned with the academic theories, and the research team profited from the teachers' inquiry by extending their theory elements, by turning insights such as Eva's insight on the relationship between teacher-generated and student-generated questions into hypotheses

to be tested in future studies. That is, the researchers attended to the emerging theory elements as those that can guide their further (research) action.

This brief example of joint theorizing in a research-practice partnership in Israel serves to illustrate the interplay of theory elements that can play a role in many contexts of basic and problem-driven research, with different degrees of logical and epistemological rigor. Moreover, it illustrates that the logical structures of theory elements need not always be very explicit, but their explication can be an added value of involving researchers who are trained in disentangling the theory elements. Whereas basic research can sometimes be restricted to single theory elements (e.g., a single hypothesis for an explanative theory element to be validated in an experiment), theory guiding practice nearly always combines different theory elements (Prediger, 2019a). Summing up, we re-articulate

Thesis 1. *Combination of various functions of theory elements.* For a theory to be practical in a well-grounded way, it needs to explicate and explicitly connect different theory elements, so that prescriptive theory elements (prescribing lines of action) can be substantiated with theory elements of categorial, descriptive, explanatory, and normative functions.

2. Diversification of research objects for different stakeholders of practice

The previous section discussed an example guiding action of *mathematics teachers*, the largest group of stakeholders in mathematics education, and of mathematics education researchers. But of course, teachers and researchers are not the only stakeholders in mathematics education. Other groups include curriculum designers and textbook authors, facilitators of mathematics teacher professional development, educational system leaders and policy makers. In this section, we suggest that various stakeholders require theories that focus on different objects, in relation to the focus of their particular practices.

For example, practice of curriculum designers relies on theory elements focused on task design. Design principles are their key predictive theory elements that combine typical design features (articulated with well-defined categories) and in-order-to-structures with normative elements (e.g., “In order to develop students’ conceptual understanding of fractions [normative element], we need to connect multiple representations, among them the fraction bar” [prescriptive element]).

As another example, we consider the practices of professional development (PD) facilitators in the German PD system. In Germany, PD facilitators had formerly been prepared for their jobs by learning about the classroom level (i.e., the pedagogical content knowledge that also the teachers should acquire) and about content-independent pedagogies for PD in general. When the DZLM project SiMa (Prediger, 2019b) started to cooperate with experienced PD facilitators, the content-independent pedagogies for PD were to be substantiated by reasoning about the PD content that turned out to be essential for developing and conducting targeted PDs, in the particular PD program, on language-responsive mathematics teaching. The researchers and expert PD facilitators jointly asked: What exactly is it, that mathematics teachers have to learn in the particular PD content, specific to language-responsive mathematics teaching? Developing categorial, descriptive and prescriptive theory elements was required but not only with respect to the mathematical contents for students, but with respect to PD content for

teachers. Those elements were needed for articulating normative goals for the PD in well-specified ways (Prediger, 2019a), and also for discussing PD activities with respect to their contributions for reaching these goals and for describing and explaining teachers' learning pathways.

Hence, a theory base for guiding PD facilitators actions was built upon research focused on other objects of research; theories guiding the action of PD facilitators do not only concern classroom phenomena, but have structurally analogical components in the PD tetrahedron as depicted in Figure 1.

The above example of the SiMa project illustrates that each group of stakeholders has practices specific to mathematics education, but these practices deal with different kinds of objects:

- Mathematics teachers and curriculum designers focus on the mathematical learning content, the students, the classroom resources. The theories guiding their action should theorize in particular about the relationships between these objects, i.e., the edges and faces in the lowest tetrahedron in Figure 1, e.g., the content-specific learning pathways (the face between student, content and classroom resources).
- Analogically, PD facilitators focus on teachers as learners and PD resources (e.g., videos, PD activities, etc.) and learning content from mathematics education. The theories guiding their action should theorize in particular about the relationships between these objects, i.e., the edges and faces in the middle tetrahedron in Figure 1.
- Analogically, when educational system leaders act as facilitator educators and designers for the facilitator PD level, they take into account the structurally analogical objects in the third tetrahedron of Figure 1, but also systemic aspects such as financial capacities.

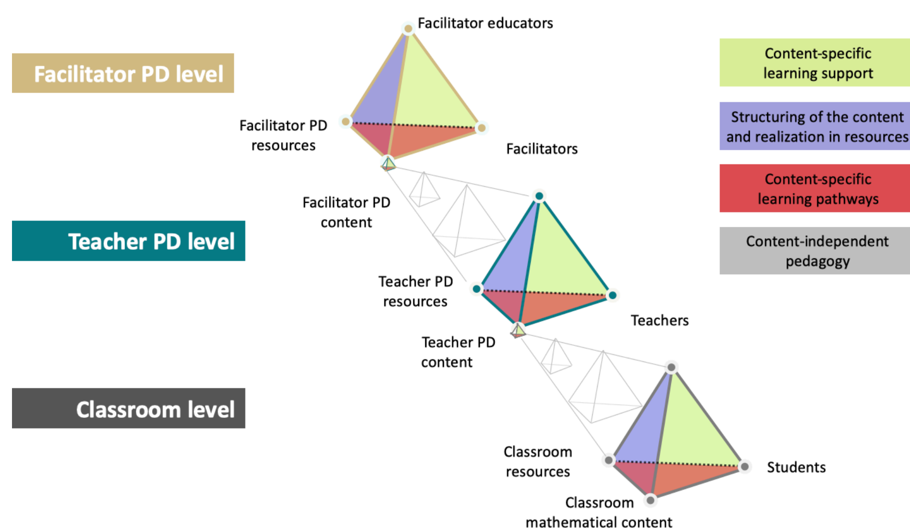

Figure 1. Widening the objects in theorizing for different stake holders – Examples from the Three-Tetrahedron-Model (Roesken-Winter et al., 2021)

Even if the three tetrahedrons in Figure 1 are only one particular model for organizing typical research objects in their structural analogies (others were suggested, e.g., by Luft & Hewson, 2014), they can help to identify gaps in the existing research to guide the stakeholders' actions. For a long time, PD research has focused on content-independent pedagogy (e.g.,

Timperley et al., 2007), whereas teachers' content-specific learning pathways have been rarely investigated (as identified in the large survey by Goldsmith et al., 2014). But this is insufficient for guiding the action of PD facilitators, as will be further exemplified in the DZLM SiMa project in the next section. Summing up, we re-state

Thesis 2. *Different research objects for different stakeholders of practice.* The stakeholders of "the practice of mathematics education" are manifold, e.g., mathematics teachers, facilitators of mathematics teacher professional development, curriculum designers, syllabus designers, policy makers, etc. Each of these stakeholders require different parts of mathematics education theories, so the objects of theorizing are manifold and extend from the classroom level to other levels.

3. Transformations between discourses

Even within one level (classroom level, teacher PD level, or facilitator PD level), mathematics education researchers and other mathematics education stakeholders are not necessarily involved in the *same* discourses. In this section, we briefly analyze major differences of the discourses and use this for suggestions about how these discourses can be transformed (or transposed, in Chevallard's (1985) terms), so that theory elements emerging from research would unfold their potential to guide different stakeholders' actions and vice versa. The starting point of our argument is that theories are as important and influential in the world of practice as they are in the world of research (Lewin, 1951), albeit subject to several notable distinctions. We outline three interrelated differences that unpack Chevallard's (1985) general idea of didactical transposition to other areas.

First difference: Highly structured explicit versus situated implicit theories

The first distinction relates to a paradigmatic difference between the highly-structured foreground theories that emerge from the effort of researchers and the often implicit theories that emerge based on practices of the other stakeholders. To recall, a foreground or explicit theory was referred to as an "organized network of concepts..." (Niss, 2007, p. 1308) or as "a language entity in propositional or categorial form" (Thiel, 1996, p. 262). In contrast, *implicit theories* "are intellectual constructions that reside in the minds of individuals … and are often revealed by behavior" (Sternberg & Zhang, 1995, p. 88). As such, implicit theories are not public and in most cases are shared only in communities of practice that produce them.

For many communities of practice of teachers, implicit theories frequently have a privileged status and "the most influence on actual life and practices" (Sternberg & Zhang, 1995, p. 89), and even when categorial or prescriptive elements of some explicit theories are offered, they are frequently rejected as irrelevant. However, explicit theory elements *can* enter teachers' discourse when these elements are offered in response to some genuine professional needs. In the example of the research-practice partnership presented in Section 1, the PD started with discussing teachers' implicit theories on what "a good question" can be. In further course, the need to understand others and be understood within the community of joint inquiry stipulated the search for an explicit vocabulary for characterizing students' questions. Then the researcher-proposed categorization of students' questions as "elaboration" vs. "clarification"

question was taken up by the teachers and refined to make sense of all students' questions that the teachers documented in their lessons. In turn, teachers' refinements of the initial categorization entered the researchers' discourse, so both discourses were further developed. This process is discussed in detail in Pinto and Koichu (2021) as crossing the boundary between practices of researcher inquiry and teacher inquiry.

In the case of the second example, the DZLM project SiMa on PD for language-responsive mathematics teaching, the research team started their specification of the PD content from the existing body of theory and research findings about the role of language in mathematics education. The focus on supporting teachers' learning immediately forced the research team to develop much more prescriptive knowledge on how to enhance language in mathematics classrooms. Pushed by the practical needs, design principles for language-responsive mathematics teaching were developed, investigated, and iteratively refined, and now constitute an empirically grounded body of research findings. In the first drafts of PDs, however, the first specifications of what teachers might learn consisted of unconnected bits of categorial and descriptive elements. The cooperation with very experienced PD facilitators was extremely fruitful to restructure the PD content. The PD facilitators engaged in actively restructuring the explicit, but inert and unconnected knowledge about language in mathematics into much more situated ways (Putnam & Borko, 2000), starting from teachers' typical classroom pedagogical demands and organizing the introduction of categories and explanatory findings in ways that really can guide teachers' actions. For the researchers, this collaboration was an important eye-opener that later led into the development of a content-related model of teacher expertise aiming at overcoming the gap between implicit, situated knowledge and explicit elements by conceptualizing the expertise starting from teachers' recurrent classroom practice and connecting all relevant normative, categorial, descriptive, explanative, and prescriptive elements directly to these teachers' work (Prediger, 2019b). This led to a substantial and theoretically important transformation of models for teacher expertise in order to better suit the practical needs in teacher PD. This theoretical shift would not have been possible without the strong support of the experienced PD facilitators who appreciated the relevance of implicit knowledge being explicated and organized around typical practices (Putnam & Borko, 2000).

Second difference: Interest in prescriptive or explanatory theory elements

The second difference in discourses relates to epistemic differences between researchers and non-researchers. For instance, both teachers and researchers are routinely engaged in various forms of inquiry about teaching and learning and produce theory-laden propositions based on the results of these inquiries. Both teachers and researchers try to assess what students know and how they think; they examine instructional materials and explore their pedagogical affordances; they design tasks and examine how different students engage with them; they try and compare different teaching approaches; they read, and think, and talk to their colleagues about education, and about how to improve it (e.g., Pinto & Koichu, 2021).

However, the inquiries teachers or PD facilitators engage with differ from inquiries conducted by researchers in many respects, and so the theories they produce. For example, in discussing tensions between the worlds of the teacher and the researcher, Labaree (2003) argues that teachers rarely engage in systematic analytical practices of educational scholarship because

they inquire in order to find practical solutions rather than explanations. Therefore, teacher-produced theories can sometimes be more embracing for prescriptive rather than explanatory theory elements, as the case of Eva illustrates (see Section 1 and Pinto & Koichu, 2021).

In the case of the DZLM project SiMa, the experienced PD facilitators drew on their rich experience to articulate prescriptive guidelines regarding how to situate the PD activities in essential classroom practices, without unpacking their backgrounds. It was the research team who then added empirical insights into teachers' learning pathways that helped to explain why some teachers could not yet profit from the PD activities. These additionally empirically grounded explanative insights (Prediger, 2019b) then helped to collectively sharpen the prescriptive consequences for the design of more targeted PD activities.

Labaree (2003) discusses another tension between the personal and particular on the one hand and the intellectual and universal on the other hand. Researchers' inquiries are conducted and communicated so that the data, argument and reasoning are intended to withstand scrutiny by other members of the scientific community (Shulman, 1981) and extend the scope beyond the individual case. Teacher inquiries are often intended to inform only the inquirer or small inquiry team, and most of the knowledge teachers develop through teacher inquiries is not documented or shared with a larger community (Kieran et al., 2012). Further, due to the high complexity of practical decision making taking, in which teachers need to consider many aspects of a classroom learning environment holistically, descriptive and normative theory elements are frequently intertwined in the teacher discourse, whereas researchers are trained to articulate them separately. This is not to say that teachers are incapable of clearly articulating their theories; this is just to suggest that they often do not encounter the need of so doing – and the need to connect many aspects holistically might even make a separate articulation nearly impossible. Accordingly, the structured theorizing consisting of systematic use of theory elements, as introduced above, is rather alien to the teachers' world with practically unnecessary separation.

Contrary to teachers' action research, a (good) researcher's inquiry is expected to be well-focused and thus is normally conducted in order to shed light on only a particular aspect of an immensely complex educational reality that mathematics teachers deal with daily in their classrooms. This strengthens the already mentioned tension, what has been described by different types of rationality: Altrichter et al. (2008) distinguished between technical rationality (in brief, general and generalizable solutions to practical problems can be developed in research) and reflective rationality (in brief, complex practical problems require particular solutions that can be developed only inside the context in which the problem arises). Stimulated by this distinction, Krainer (2021) suggested that mathematics education as a field of research-informed practice must accept societal rationality – in brief, "practical problems require an adequate link between general and particular solutions. The more complex the problem, the more important the particular" (p. 1180). In these terms, the personal and particular nature of teacher inquiry rooted in practical rationality "which allows the emergence of diversity within similarity" (Herbst & Chazan, 2003, p. 2) is an asset that researcher inquiries must capitalize upon. This can occur in cross-community learning, for example in our two examples of research-practice partnerships (and is further discussed In Pinto & Ko"chu,'2021).

The demarcation lines between the discourses of mathematics education researchers and the world of policy makers, in relation to theory-laden discourses and cultural orientations, are

even sharper than for the other stakeholder groups. In an overview on current research on implementation of mathematics education research, Koichu, Aguilar and Misfeldt (2021) observed that implementation projects initiated by researchers and policy makers tend to have different justifications. Namely, researcher-initiated projects tend to put forward theoretical justifications (e.g., theory X tells us that innovation Y is important, and it has been explored in context Z, so let's implement it in context W and see what happens), whereas policy-maker initiatives are usually justified by the need to resolve a systemic problem (e.g., the current standing of a country in an international test X should be improved, and this can be done by putting innovation Y in motion at large scale).

Furthermore, the between-stakeholder differences in the use of categorial, normative and explanatory theory elements inevitably affect the use of prescriptive theory elements and in turn, action. Krainer (2021) presented several (idealized) scenarios representing interactions of researchers and stakeholders, which begin differently but lead to similar ends. For example: (i) researchers propose an innovation—a small group of teachers successfully implement it – policy makers are informed of the success – further action is uncertain; (ii) policy makers identify a systemic problem—researchers advise – there are disagreements about focus, timescale and resources – further action is uncertain.

However, there is a growing body of evidence that researchers, practitioners, and policy makers can productively collaborate while being systematically engaged in attempts to align their discourses when they are sensitive to these differences in discourses and collectively strive for bridging the differences. An illustration for this point comes from implementation research reported by Wang et al. (2021). These scholars explore the process of launching, implementing and scaling up the Just Do Math program in Taiwan. In this program, stable channels of communication and mutual learning among the main stakeholders – teachers, educational leaders, mathematics education researchers and policy makers – were carefully designed as part of the overarching research-driven model of implementation from the very beginning. As a result, the program reached the national level during several years and was considered successful by various measures.

Third difference: Location of knowledge and role of fidelity vs. mutual adaptations

The third difference relates to paradigmatic variances of where (theoretical) knowledge is assumed to reside: Lewis (2015) describes experimental sciences as those presuming that knowledge is "in" a program relying on a particular explicit theory, whereas she calls improvement sciences as those assuming that knowledge is "in" people and institutions that use the program. The sharp distinction between experimental and improvement sciences have been dissolved in many approaches of design research with their dual aims of building basic knowledge and improving innovations (Cobb et al., 2017), and these design research approaches also demand to reconceptualize the location of the knowledge as dual, in programs *and* in people and institutions. This dual location also suggests the need to reconceptualize the perceived role of fidelity. According to Lewis (2015), experimental sciences typically value fidelity of implementation and assume that improvement occurs through "faithful" program implementation.

Alternatively, improvement sciences value "mutual adaptation" of programs (e.g., Russell et al., 2020). When researchers and practitioners carefully document and study the impact of adaptations, they are then in a good position to suggest what makes adaptations more and less productive. Of course, not all adaptations are productive; it is critical that researchers and practitioners alike are in a position to make principled judgments around which adaptations maintain the "integrity" of a program, and why other adaptations do not. In this way, both researchers and practitioners learn, by careful studies of implementations in new contexts, and lead also to revisions of the initial theory guiding implementation (and then the action of implementers in a new context.

Accordingly, researchers, teachers, facilitators and policy makers are likely to use normative elements of their theories in profoundly different ways, with the agency of also taking on decisions about the priorities of learning goals. To this end, we argue that research should accompany the adaptation processes as well as the process of accumulation of knowledge about typical adaptation processes (Brown, 2009). Again, the differences between theory and practice discourses can best be bridged when not only the stakeholders of classroom or PD practices learn about the research perspectives, but when researchers transform their discourse to better suit the practical needs and to build bridges to the classroom and PD complexities.

In sum, the above discussion leads us to

Thesis 3. Transformations between discourses. The discourse in the research discipline and the discourses of other stakeholders are structured differently, so theory elements emerging from research need to be transformed into the discourse of the stakeholders, in order to unfold their potential to guide stakeholders' actions. This also impacts research and theorizing.

As already illustrated in the examples, the transformation of discourse requires research-practice partnerships in which both discourses are considered as relevant and functional, not one as being better than the other. As transformations are always complex, they can best be conducted in strong collaborations. These collaborations involve joint acts of resolving problems, but in the ideal case they also include joint acts of situated theorizing, as will be discussed in the next section.

4. Supportive conditions for joints act of situated theorizing

The active participation of stakeholders in all steps of research, including theorizing, is sometimes claimed to be crucial to overcome unfaithful hierarchies and a theory-practice gap. We are *not* claiming that all stakeholders need to be involved in all parts of research for a resulting theory to guide practical action. However, we suggest the following:

Thesis 4. Supportive conditions for joint acts of situated theorizing. As active participation of stakeholders in many steps of research and theorizing is crucial to overcome unfaithful hierarchies and a theory-practice gap, we can derive from Thesis 1-3 that besides the engagement of people, there are additional supportive conditions for acts of joint theorizing that can increase the chance that the developed theory can really guide the stakeholders' action.

After having theorized on the meta-level about the theory-practice connection, this section is focused on the practicalities of theorization: Based on literature as well as our own experiences, we will outline a set of acts of theorizing that, in our view, would benefit from researchers' and practitioners' joint engagement. In addition, we elaborate an initial (but not exhaustive) set of

what we term “supportive conditions,” that is, conditions that enable researchers and practitioners to authentically co-participate in acts of theorizing.

Identifying and framing the phenomenon that is worthy of theorizing

One critical joint act of situated theorizing concerns *identifying and framing the phenomenon that is worthy of theorizing*. For theories to guide action, it is crucial that the focal issue that is being investigated resonates with the stakeholders who are “closest to” and are charged with acting on the problem (e.g., Cai et al., 2018; Penuel et al., 2011). In much of mathematics education research, this often means teachers or specialists who support teachers’ learning are critical partners in negotiating and framing the focal problem, as was shown in the first example (Pinto & Koichu, 2021) of specifying the collective focus of inquiry.

Specifying the nature and/or source of the problem requires generating descriptive and explanatory theory elements (see Section 1). It is unlikely that practitioners and researchers will share an understanding of the problem at the start, and for good reason: the various stakeholders bring insightful, albeit divergent, perspectives, knowledge, and histories. As Cai et al. (2018) suggest, practitioners bring knowledge of how persistent problems are rooted in and shaped by their local context, as well as of past efforts that may have failed to address the problem. Researchers draw on the research base, their theoretical frameworks, the collective norms and standards of the research community to which they belong as well as their knowledge of other contexts regarding the nature and source of the particular problem. In order to overcome the difference between holistic views versus a narrow focus, researchers often need to widen their focus and incorporate further aspects to adequately grasp the practitioners’ complexities. And, of course, all parties’ constructions of the problem are replete with assumptions and values about, for example, what is worth learning mathematically and what shapes students’ learning, for example, specific aspects of an instructional system (e.g., the role of teachers, tasks, discourse).

Articulating and justifying the normative elements of the theory

A second critical joint act of situated theorizing concerns *articulating and justifying the normative elements of the theory*. As described earlier, laying bare the aims of a particular intervention and/or theory, alongside the justifications, is an essential element of theories that are well-positioned to guide action. The aims are based, in part, on people’s values and commitments, and likely reflect the discourses with which people identify. In our experience, often, the normative elements of a theory remain implicit in research products and under-specified. The under-specification of normative elements makes it difficult to evaluate the fit between a given theory and a practical application, as in the above example of the underspecified PD learning goals in the early steps of the DZLM project SiMa (Prediger, 2019b). Building in genuine opportunities to name and justify normative elements of a theory with practitioners who are likely to act on the theory, and at various stages of theorizing, can support the refinement of the normative elements of a theory, and thus an understanding of the conditions in which the theory is conducive to generative action. In the example of the SiMa project, the shift towards learning goals tied to teachers’ classroom practices that were implicit in the PD

facilitators' agenda emerged in negotiations and raised the need to develop new categories to articulate the refined goals. For the facilitators, this act of situated theorizing led to a practical "learning landscape" in which the PD goals were restructured in relation to the teachers' instructional demands; for the researchers, it led to a more fundamental theorizing with a new model of teacher expertise (Prediger, 2019b) that was later used for other PD programs.

Iterative refinement of theory elements

We anticipate that the two acts of joint theorizing described above apply to a broad range of research methodologies that aim to generate theories that guide practice, where practitioners can be active stakeholders in the research. In discussing a third act of joint theorizing, we focus exclusively on research methodologies where practitioners are active partners throughout the process, specifically in educational design research (Cobb et al., 2017) or in improvement research (Peurach et al., 2022) that are characterized by successive cycles of inquiry. In these kinds of researcher-practitioner partnerships, the goal is often to generate a design for supporting learning, and to develop, test, and revise conjectures (i.e., generate theory) about the learning processes and the means of supporting that learning. In Cobb et al., 2017's terms, "The products of a design study therefore include both a practical artifact, program, technology, or system that serves as the design, and theory that constitutes the rationale for the design" (p. 208). The elements of a developing theory are actively revised and refined, as part of successive inquiry cycles (Prediger, 2019a). Thus, a third act of joint theorizing entails the substantial participation of practitioners in the refinement of elements of theory. As described earlier, acts of refining elements of theory may take on a number of forms, such as "refining categories in order to increase their explanatory power", "transforming an explanatory theory element into a conjecture for a predictive theory element" and "refining a predictive theory element" by adding a qualifiers as conditions of success (Prediger, 2019a, pp. 21-22).

These acts can be illustrated in the research-practice partnership with PD facilitators in the SiMa project: the successive sharpening of categories for different components of teacher expertise was initiated by the facilitators who wanted to sharpen their PD goals. The researchers contributed to this process by connecting them to general situated theories of teacher expertise (Bromme, 1992; Putnam & Borko, 2000) which helped the research-practice team to *explain* why teachers were not able to activate learned categories in their practices for mastering particular classroom practices. From these explanative insights, the research-practice team derived conjectures of how the PD activities could be reshaped so that the use of learned categories was more successful. In the next design experiment, the research-practice team learned that they had to remediate also relevant aspects of teachers' pedagogical content knowledge before they could engage in the practices as intended, and thereby refined the predictive theory element by a condition of success. While the PD facilitators were mostly interested in the prescriptive knowledge needed to design productive PD activities, the collective analysis contributed to iteratively refine the descriptive and explanative elements needed for their empirical grounding. Within the process, the PD facilitators developed a strong expertise in analyzing teachers' learning pathways which started to guide their actions also in new contexts. Therefore, the developed and refined theoretical elements guided the actions of the PD designers as well as the practices of the PD facilitators.

Example to illustrate the interplay of joint acts of situated theorizing

To illustrate the interplay of the above acts of joint theorizing, we draw on work associated with the Middle School Mathematics and the Institutional Setting of Teaching (MIST) project (Cobb et al., 2018). MIST was an eight-year research-practice partnership between researchers in mathematics education, teachers, coaches, principals, and system leaders in several large, urban school districts in the United States. Conceived of as a design research project, the goal of MIST was to generate an empirically grounded theory of action for instructional improvement in middle-grades mathematics at the scale of a large district which included a set of instructional policies or strategies intended to support teachers' (and others') learning, and a rationale that justifies why it is reasonable to expect that these strategies will support instructional improvement (Argyris & Schön, 1978). The research team started by generating a set of initial broad prescriptive conjectures, drawing on literature about curriculum materials and associated resources, teacher PD, teacher collaboration, school instructional leadership, and district leadership. The partnership with the districts constituted a context in which to successively test, revise, and elaborate the conjectures, and, in the process, develop and refine an empirically grounded theory of action. Each year, data was collected and analyzed for 1) documenting the district's improvement strategies, 2) assessing how these strategies were playing out in schools and classrooms, and 3) reporting the findings back to the district together with recommendations about how the district might revise its strategies to make them more effective.

Throughout the partnership, substantial attention was given to the acts of joint theorizing discussed above. In terms of *articulating and justifying the normative elements of the theory*, the MIST project started by researchers approaching leaders in districts with explicit commitments to rigorous learning goals for students (e.g., conceptual understanding of key mathematical ideas, procedural fluency, problem-solving competencies); an associated inquiry-oriented vision of high-quality mathematics instruction; and who were investing in teachers' learning through the provision of high-quality curriculum materials and sustained professional development. In this sense, leaders "at the top" of the system and the researchers shared a set of normative elements of a theory of what it takes to improve mathematics teaching and learning.

However, these commitments and assumptions were not shared by all system and school leaders (or teachers) in a given district, and so the researchers and central office leaders designed and implemented activities to surface tensions in various role group's normative assumptions and to work toward a shared understanding. Interviews revealed that mathematics specialists and leaders (principals and their supervisors) often framed the issue of improving instruction in divergent ways. For example, mathematics specialists assumed there was a need for sustained teacher professional learning, whereas the supervisors of the principals prioritized a short-term goal of boosting student achievement, often at the expense of sustainable instructional improvement. The MIST research team discussed these differences with top central office leaders, and together, they co-designed and co-facilitated professional development sessions with the mathematics specialists and principal supervisors to develop coherent goals for teacher support.

Notably, these kinds of opportunities to surface and articulate differences in how the focal problem (improving mathematics teaching and learning at scale) was framed by different stakeholders, including the researchers, and to make visible the normative elements of the then-

current theory led to further evolution in the MIST team's empirically-grounded theory of action. For example, research revealed *explanatory elements* for making sense of crucial differences in what coaches were tasked to do in schools (e.g., focus on teacher or student learning; Kane & Rosenquist, 2019). Against this, researchers and central office leaders generated *conjectures* regarding how to negotiate these implicit differences in *normative elements* of educators' theories, and in so doing generated (and subsequently tested) predictions about how leaders could be supported to appreciate a more ambitious set of goals for students' learning, and the critical role of the coach in enabling teachers to, in turn, enable students to achieve those goals. And, given that these tensions emerged in a number of districts, it seems reasonable to suggest that engaging in these joint acts of theorizing made it more likely that the evolving theory of action would be of use to guiding the partner districts', as well as other districts' practices.

Supportive conditions for successful acts of theorizing

As discussed above, we are suggesting that engaging in joint acts of theorizing can increase the chance that the developed theory can really guide the stakeholders' action. However, engaging in joint theorizing is by no means a trivial task. It requires what we term "supportive conditions," that is, conditions that enable researchers and practitioners to authentically co-participate in acts of theorizing.

A primary condition for joint theorizing regards routines of and tools for communication between researchers and educators (Choy et al., 2017; Farrell et al., 2022) that helps to bridge the differences of discourses presented in Section 3. Drawing on studies of productive research-practice partnerships, Farrell et al. (2022) suggest that teams need to consider both *boundary practices* (routines) and *boundary objects* (tools) in order to collaborate while respecting the between-community differences and recognizing that full consensus cannot rarely be achieved. Boundary practices, they write, "are partnership activities that bring together multiple participants with varying roles, perspectives, experiences, and areas of expertise, and who are situated within organizations with different capacities and conditions" (p. 3), whereas boundary objects are "material and conceptual tools used in a partnership that support coordination between groups that work in different organizational settings and mediate activity within each organization" (p. 7).

As detailed in Farrell et al. (2022), the MIST project included intentional work to design both boundary practices and boundary objects that would support joint theorizing and sense-making. In particular, at the end of each academic year, the researchers sent a 15-page "feedback and recommendation report" to each district that detailed their findings about the implementation of the district's instructional improvement strategies and recommendations to improve the strategies for the upcoming academic year. The researchers and system leaders then met to discuss the report, to engage in genuine discussion about the findings and to consider the implications of the report. In two of the districts, system leaders and researchers also co-planned and co-facilitated follow-up 1.5 day "Instructional Improvement Institutes," in which system leaders continued to build from the report document and revised key instructional improvement strategies (e.g., coaching) for the upcoming year. The boundary practice of meeting across role groups to make sense of and refine instructional improvement strategies, organized,

in part, by the boundary object of the initial feedback and recommendation report, were invaluable in surfacing explanatory and normative elements in the ongoing theorizing of instructional improvement at scale, as well as for identifying revisions in conjectures that would then become the focus of inquiry the following year.

A second supportive condition for joint theorizing concerns the inclusion of people who are "fluent" in discourses of multiple role groups (e.g., researchers, teachers), sometimes referred to as "brokers" or "boundary spanners". Brokers support various stakeholders to both make visible and make sense of one another's perspectives – and not only in terms of "translations," but also in supporting the development of a new understanding, resulting in a "transformation" in discourse (Akkerman & Bakker, 2011). In MIST, system leaders who were also trained in research methods played critical role as brokers throughout the life of the project, but especially in "navigat[ing] turnover in district leadership and subsequent changes in district strategy" (Farrell et al., 2022, p. 6).

A third condition is the openness of the researchers for extending objects of theorizing beyond what they might initially express interest in. Joint situated theorizing must be responsive to the unfolding context; and when a diversity of actors are involved, undoubtedly, as described in Section 2, the objects of theorizing necessarily expand. As an example, at the start of the MIST project, the research team had not anticipated the prevalence across the partner districts of supplemental support mathematics classes for students identified as currently struggling. Interviews with teachers suggested that the learning goals and instructional methods of the supplemental support classes were often mis-aligned with the focus of the instructional improvement strategies in the various districts. The research team thus expanded its scope to include supplemental supports as an additional (related) object of theorizing instructional improvement at scale (Cobb et al., 2018). Of course, we are not suggesting that teams should *always* expand their focus to include new objects, that decision is shaped by available expertise, resources and preferences of the team.

A fourth condition is awareness by all involved stakeholders that the situated theorizing will still serve different rationalities which are tied to the different needs of the practice contexts and discourses. For example, as illustrated by the SiMa project, what counts as a relevant product of theorizing for a PD facilitator is distinct from a relevant product for a researcher of PD. In that project, PD facilitators were mainly interested in the prescriptive design principles for PD activities whereas the researchers also compiled the normative model of relevant PD content. It is not necessary that all products are of equal relevance to all stakeholders.

A *fifth condition* in designing and enacting supportive conditions for joint theorizing is explicit attention to the conceptions and distribution of expertise, and power relations (Farrell et al., 2022). In a more conventional researcher-participant relationship, it is common to position the researcher as holding expertise, and as the one constructing "knowledge" on a given topic or issue. In authentic acts of joint theorizing, it is assumed that multiple parties hold expertise that is valuable, and that diverse sources of expertise are valuable and necessary in improving the theory and generating knowledge. Needless to say, the perceived equity of parties in researcher-practitioner partnerships may be challenging to achieve, but the equity can grow when mutual trust is developed in joint work, as several of the examples used in this chapter suggest.

Although this preliminary list of supportive conditions is far from being complete, we hope it is a good starting point for continued reflection, with the goal of enhancing the prospect that

the theory being developed especially in research-practice partnerships can productively guide stakeholders' actions.

5. Outlook: Implications for the policies and practices of the field of mathematics education research

In earlier sections, we have outlined four theses regarding what generating theory that is intended to guide action entails. In brief, (1) theory that guides action needs prescriptive theory elements that are well-connected to elements that have categorial, descriptive, explanatory, and normative functions; (2) the objects of theorizing will necessarily be manifold, if they are to guide action of various stakeholders; (3) the discourses of various stakeholders vary, and therefore for theory to guide action there is a need for transformation of discourses; and (4) there are acts of joint theorizing and a corresponding set of conditions that can increase the chance that the developed theory can productively guide stakeholders' actions. In closing, we reflect on the implications of our four theses for the research policy of the academic field of mathematics education.

It is our position that engaging in joint theorizing requires changes primarily on the part of researchers, and that those changes require extended cultural practices and policies of the discipline. Developing a joint agenda that is grounded in issues of practice requires a substantial shift in the usual practices of researchers. It likely requires shifts in skills and knowledge, as well as in one's professional identity. Researchers typically investigate issues that they believe to be viewed as significant within their own research communities. Theorizing intended to guide action of stakeholders, in turn, requires identifying objects of theorizing that are significant from the point of view of practitioners. It requires an interest in and awareness of the "problems of practice" and rationalities of different stakeholders. It also requires desire and skill in participating in boundary practices, shaped by an active awareness of the imbalance in power relations that have historically characterized the relationship between researchers and practitioners.

The establishment of these kinds of extended cultural research practices should be supported by systematic capacity building for early career researchers. However, as observed by Choy et al. (2017), "the requirement to develop skills to be able to navigate between sites challenges programs designed to develop novice researchers preparing for global practices" (p. 278). The development of skills, knowledge and orientations underlying these extended research practices requires learning opportunities, e.g., by co-participating in research-practice partnerships with others who are already engaged in such partnerships and by training in methods that support joint inquiry and theorizing, and in how to communicate in a range of discourses. As such, we see promise in the provision of funded apprenticeships for doctoral students, post-doctoral fellows, and early career scholars with teams who have a record of having engaged in practice-forward work.

We anticipate at least one tension at a policy level. Our experiences suggest that engaging in joint theorizing requires an extensive time commitment, especially to build and sustain relationships and to engage in practice-engaged iterations of theorizing, like those described in the examples in the above sections. At least as communicated via promotion and tenure practices,

early career scholars are often under pressure to produce manuscripts quickly, and little value is ascribed to sustainable impact on practice. If engaging in joint theorizing is to take hold as a more normative cultural practice, there is a need to begin valuing the impact of research in practice as much as the rigor of research.

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