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RESEARCH FORUMS

WHAT IS MATHEMATICS-SPECIFIC WHEN WE BUILD KNOWLEDGE IN MATHEMATICS EDUCATION RESEARCH?

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Besides their role in practical problem solving, research fields aspire to advance human understanding by building knowledge. Mathematics education research often asserts its special contribution to human understanding on the grounds of its focus on mathematics in various education practices. This mathematical sensibility in the mathematics education research gaze can enable unique contributions to the larger field of educational research and the interdisciplinary table of educational policy-making. What exactly are these mathematical specificities in our contributions to knowledge building? This research forum explores various ways in which mathematics shapes the mathematics education research gaze, contributing to the clarification and development of knowledge building in our field.

A NEEDED REFLECTION ON THE NATURE OF OUR GAZE

Patricio Herbst, Felix Lensing, and Nathan Lombard

Our field's production of knowledge is aided by a mathematics education research (MER) gaze, with which we interrogate the practices of mathematics education. The term *gaze* stems from social research, where scholars have used it to refer to the specialized ways in which disciplines construct and interact with their objects of study (e.g., Foucault, 2000). Indeed, different disciplines may focus on the same happenings in the world but do so through different gazes and as a result aspire to produce different kinds of transposable knowledge. By transposable knowledge we mean knowledge claims about concepts or constructs which are instantiated by the observations about facts associated with the sample or locale involved in the research. This research forum proposes a reflection on the nature of our field's gaze and urges a commitment to research that further develops this gaze. We suggest that such reflection and commitment are essential for establishing the *autonomy* of our research field through the distinctness of our transposable knowledge, and for warranting a seat at the table in discussions of educational policy and research funding: Our field does not merely consist of the aggregate of scholars whose empirical work takes them to places where people teach or learn mathematics; our field aspires to look at practice in those places with a gaze that allows it to distinguish its claims to knowledge from those that other

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fields can make. With this research forum, we aspire to get a conversation started about the nature of our knowledge-building enterprise and about the need for research that engages in knowledge-building (in addition to research that engages in problem-solving).

An aspect that has been historically constitutive of our field's gaze is our attention to mathematics in the practices of mathematics education. And yet, in recent years some have decried that this attention is not present in some of the research our community has done in the last couple of decades (e.g., Heid, 2010). When the core of our field's research was on students' cognition as they solved particular problems (e.g., Carpenter & Moser, 1982/2020), it might have been obvious how mathematics shaped our gaze and was present in the transposable knowledge our field produced (e.g., the research on those students' solving those problems informed about different uses of addition and subtraction). Rather than assuming that mathematics is vanishing from our gaze or that the way for mathematics to be present in our gaze requires naming topics recognizable to mathematicians, we offer reflections on the question: *How does mathematics shape the MER gaze and how does mathematics show up in the transposable knowledge our field produces?*

WHY DISCIPLINARY SELF-REFLECTION MATTERS TO MATHEMATICS EDUCATION RESEARCH

Those questions continue a reflective tradition on the nature of research in our field (e.g., Biehler et al., 1994; Freudenthal, 1981; Sierpinska & Kilpatrick, 1998; Wittman, 1995). This questioning is still needed for at least two reasons. First, the historicity of both our objects of study and MER itself makes it necessary to repeatedly pose and revisit questions of disciplinary identity as the field continues to evolve. Second, throughout its history, the autonomy of MER—its recognized capacity to identify its objects of study, using its own theories and addressing them with its own methods—has been challenged on both epistemological and sociopolitical grounds. Like any other social science, our autonomy is constantly challenged by societal developments and public opinion, both of which impose upon us many of our research objects and questions (e.g., AI in the classroom?). The meaning of mathematics implied in a research practice affected by such endogenous and exogenous change vectors must be examined.

Two historical characteristics, however, have to some extent reduced the pressure to engage in such reflection. First, in its early history, MER emerged in connection with curriculum development and the study of students' thinking (Suppes, 1969). This led to a tendency to define our relationship to mathematics by drawing on concepts from the discipline and how mathematics is institutionalized in schools. And yet, identifying MER as focused on everything that happens in spaces institutionally designated for mathematics education or on practices that deal with topics sanctioned by the discipline of mathematics leaves outside our field studies of mathematics in cultures, the workplace, or the playground which have been important in our field, even to better understand the mathematics done in classrooms (e.g., Carraher et al., 1987). Depending too much on the presence of recognizable mathematical topics or on the institutional designation of spaces as mathematical may not be of sufficient assistance for us to recognize mathematical activity in the teaching or learning of other subjects, for example. These latter foci for research suggest the need for critical reflection on how mathematics shapes our research gaze beyond institutional labels or disciplinary references.

Now, if the development of a more reflective discourse of our relationship with mathematics is meant to support our autonomy as a research field, and if this autonomy involves taking stances about what practices we study, what theories we build and use, what benchmarks we use for the application of designs and methods, and what constitutes research results in our field, we ought to consider explicitly what goes into developing such stances. *Ontological* and *epistemological* considerations are surely needed, as we need to reflect on the nature of our objects of study and on how researchers interact with them. But because the possibility of studying those objects is realized by scholars in particular social and institutional positions which afford them differential access to those objects of study, onto-epistemological considerations can at best represent too ideal a type. For a research community populated by scholars coming from many prior disciplinary experiences and locations across the world, *sociopolitical* considerations are also needed. The reflection also needs to take stock of where the community presently is, how its scientific practice enacts or produces ways of knowing within its ways of being, as our scientific practices are bounded by political and economic conditions that include access to resources with which research is carried out—including grants, institutional positions, libraries, equipment, regulations, access to data, educational policies (e.g., access, curriculum, testing), and the very role and status of social research in each culture.

Our examination of how mathematics is present in research in our field thus needs to combine the ambition of improvement with an appreciation of what we are able to do with what we have. We need to thrust our "onto-epistemological foot" ahead, directed on a reasonable vision for progress that may afford us greater autonomy, while keeping a "sociopolitical foot" behind, grounded in the real conditions for the inquiry that we have to sustain and build upon.

SEARCHING FOR THE MATHEMATICAL SPECIFICITY OF THE MATHEMATICS EDUCATION RESEARCH GAZE

How are we to inquire into the mathematical specificity of the mathematics education research gaze? If the question is being raised in earnest, we ought to have no holds barred. We cannot, for example, presume that the practice of mathematicians or the disciplinary knowledge they organize necessarily provides the primary meaning to mathematics in our search for specificity. That may be one possibility—but it is not a necessity. Likewise, the expectation that the question "What is mathematics?" must be settled before we can investigate the mathematical specificity of mathematics education research should be recognized as a possibility rather than a necessity.

What does seem ineluctable, however, is that the world in which we live includes mathematics in multiple forms: mathematics as a community of knowledge producers and a body of scholarly knowledge; mathematics as a library science classification; mathematics as the name of a school subject; mathematics as an acknowledged tool of professional practices of different sorts; mathematics as a component of public literacy; mathematics as a set of developmental skills; and perhaps more. The practices of teachers, students, teacher educators, curriculum developers, and policymakers often concern one or more of those real-world phenomena. And yet, how we ought to think of these realities also offers possibilities rather than necessities.

We could, for example, follow a structuralist impulse and interpret that set of practices as nestings around a core that provides a primary sense of what's mathematical. Thus, if the status of mathematics in library science is taken as evidence of mathematics as an organized body of knowledge, a first level of nestedness might be mapped onto the practices of humans engaged with that body of knowledge, as in writing mathematics textbooks. And the question could be posed as to how research on textbook writing (or on the textbooks written) maintains attention to mathematics in the claims it lays on transposable knowledge. And, of course, nestings of higher level could be recognized; for example, students studying those textbooks, teachers teaching from those textbooks, or teacher educators teaching prospective teachers to use those textbooks; in all of those nestings, the question could be asked of how research on such activities maintains attention to mathematics in the claims it lays on transposable knowledge. Here, each level of nestedness becomes a potential site for examining how mathematics remains present—and how it is transformed—within the MER gaze.

But we could also resist that structuralist impulse and instead concentrate on the mathematics as it is experienced by students, teachers, or teacher educators in institutions, and investigate how they arrive at meaningful accounts of what they experience as mathematics. We could find in such investigations some grounds for the legitimacy of associations that may be less available in the former approach–associations of mathematics with pleasure or pain, with social sortings or opportunities, for example. We should remain open to constructing a sense of mathematical specificity of our gaze that attends to how people's experiences of mathematics differ from other experiences in regard to what those experiences make people feel and become. And we should wonder whether a synthesis exists that could accommodate different possibilities to inquire into the mathematical specificity of our research under the common purpose of maintaining and deepening the autonomy of our field.

This is particularly important as we think of our field's capacity to sharpen its gaze: How do we put an onto-epistemological foot forward while holding strong to the sociopolitical foot back? Even the most basic orienting principles—what counts as an ontology, or the meanings and methods of epistemology—are heavily contested among scholars. The resulting diversity can produce a cacophony that might appeal to certain postmodern theorists, but that offers little guidance for flesh and blood researchers who must navigate real-world challenges, arguing for resources, positions, and the value of their work before both the public and other academic fields with greater stature. If we are to act constructively, we must ask whether we can move beyond entrenched fiefs and inherited traditions toward a vision of the field that is both open to multiple approaches and in control of the mechanisms necessary to safeguard its autonomy.

To start to tackle this self-reflective task of unfolding how mathematics can be present in diverse ways within the MER gaze, we now develop analyses of a series of examples from various sectors of MER.

MATHEMATICS AND EDUCATION GAZES MEET IN MATHEMATICS EDUCATION RESEARCH

Andrew Izsák

This forum on mathematics education research (MER) focuses on gaze, or the specialized ways in which disciplines construct and interact with the objects they study. Whether MER has a characteristic gaze raises as a question whether it is a distinct discipline. What is and what is not MER can be contested when researchers view the importance of mathematics and education in their work differently. More fundamentally, what is and what is not MER can be contested when researchers differ in what they recognize as mathematics or as education. How we think about mathematics, education, and basic research can inform how we think about a gaze that is characteristic of an MER discipline that is distinct yet, at the same time, in conversation with mathematics and education.

The adjective *basic* has multiple, related meanings. One meaning connotes something that is introductory, an entry point, and possibly perceived as straightforward. Examples would be a basic course in algebra or a basic course in theories of learning. A second meaning connotes research that does not have immediate practical applications. As an example, Schoenfeld et al. (1993) reported a case of one student's learning that was characterized by complex, multi-level coordination of knowledge and, therefore, revealed limitations of learning theories that described more straightforward accumulation of new knowledge on top of prior knowledge. A third meaning connotes research that (re)examines foundations upon which a field is built. An example from mathematics is work on the structures and logic that underpin the discipline. In what follows, I discuss two examples of "basic" MER in this third sense. In both examples, the objects of study, and therefore the gaze, make contributions that are both mathematical and educational.

The first example comes from Ball and colleagues' work on mathematical knowledge for teaching (e.g., Ball, Thames, & Phelps, 2008). These researchers have pursued a complex research program that includes elements directed at teacher preparation policies as well as ones that have uncovered mathematics with which mathematicians are less familiar. In particular, they have refined Shulman's (1986) knowledge categories in order to better understand the mathematical demands of teaching. One part of this multi-faceted research program has involved positing a category of mathematical knowledge employed in teaching but not in other professions, a category they have termed *specialized content knowledge* (SCK). One example of SCK is knowledge of different meanings for division that can be used to design tasks intended to support students' understandings of division with fractions. A related example is knowledge of ways that whole-number factor-product combinations can be leveraged to partition quantities when solving problems that can be modeled with fraction arithmetic (e.g., Izsák et al., 2018). Research that characterizes and maps this specialized mathematical knowledge has been a form of basic research uncovering mathematics with which mathematicians are less familiar, but that is critical for reform-oriented teaching.

The second example comes from research that Sybilla Beckmann and I have conducted to construct a coherent approach to multiplication topics across elementary, secondary, and tertiary grades (e.g., Izsák & Beckmann, 2019). Decades of research exists on topics related to multiplication, including multiplication and division with whole numbers and with fractions, ratios and proportions, rates of change, and so on. At the same time, we observed that oftentimes the field has seemed satisfied with multiple meanings for multiplication, each used in particular circumstances. For example, the oft criticized repeated addition meaning works with whole numbers, while the part-of-a-part meaning works with fractions. At the same time, the field has given less attention to ways in which these different meanings might be connected and seen as different manifestations of the same thing. From our point of view, this omission has consequences for how we might design opportunities for students to use prior knowledge of multiplication when studying new topics across the grades.

In order to develop a more coherent approach to a central swath of school mathematics, we have developed a meaning for multiplication based on coordinated measurement with two different units (e.g., Izsák & Beckmann, 2019). As an example, one can think of a journey measured simultaneously in kilometers and in hours. In the multiplication statement, $N \bullet M = P$, P is the measure of the journey in kilometers, M is the measure of the journey in hours, and N relates measures in the two units, kilometers and hours. On the face of it, this observation might seem innocuous, but it turns out that applying this perspective consistently across situations has led us to critically reexamine widely accepted MER results and to engage in new, at least for us, mathematical thinking.

One example is Vergnaud's (1983) well-known distinction between isomorphism-of-measures and product-of-measures. In his analysis, *speed • time = distance* is an example of the former and *length* • *width* = *rectangular area* is an example of the later. The coordinated measurement point of view supports a common analysis of the two categories, thereby dissolving the distinction, but this requires reinterpreting *length* • *width* as *number of square units in 1 column* • *the number of columns*.

A second example is the recognition that the two different meanings for division—partitive or sharing division versus quotitive or measurement division—leads to two different perspectives on proportional relations. Beckmann and Izsák (2015) referred to these as the multiple-batches and the variable-parts perspectives. Although the multiple-batches perspective has dominated past research on proportional relationships and rates of change, articulating the two perspectives and comparing them has helped us understand important limitations of developing rates of change using multiple batches as the foundational perspective on proportional relationships, especially when laying the ground for thinking about instantaneous rates of change and derivatives in calculus (e.g., Izsák & Beckmann, 2026).

A third example is the development of new ways to connect core calculus results to topics from secondary grades. As an example, if one uses coordinated measurement to span Vergnaud's (1983) categories of isomorphism-of-measures and product-of-measures, then one has an explanation for why rectangular areas can be used to approximate accumulation of a wide range of quantities (e.g., distance, mass, and work). Moreover, Izsák (2025) reported how integration-by-substitution can be justified by thinking about inversely proportional relationships.

These three examples illustrate ways in which we have simultaneously adopted a critical stance towards established MER results and engaged in what, for us, has been new mathematical discovery. Throughout, our work has been shaped by values and practices of the mathematics discipline that include reasoning from definitions, seeking common structure across situations, and reasoning logically.

TRANSPOSABLE KNOWLEDGE FOR EMBODIED GEOMETRY EDUCATION

Dor Abrahamson and Mitchell J. Nathan

We are mathematics-education researchers whose field of inquiry surfaces the other-than-symbolic modalities of mathematical practice, such as gesture, imagination, prosody, material construction, and intersubjectivity. This multimodal gaze on mathematical practice, which characterizes our work, resonates with a recent philosophical turn in the cognitive sciences loosely referred to as *embodiment*, or sometimes as *4E* (i.e., the notion that the mind is embodied, embedded, extended, and enactive; Newen et al., 2018). We believe that implicating the embodied and enactive nature of mathematical reasoning, while constituting a theoretical lens motivating empirical studies, bears fundamental implications for practice, too. For if these theories hold water—and expert mathematician testimonies support these claims theories (e.g., Hadamard, 1945; Tao, 2016)—then we might wonder how we could create conditions for students to partake in multimodal mathematical practices. This embodied mathematical gaze could inform what tasks educators engage students in, what guidance we offer them, what epistemic climate (Feucht, 2010) we nurture in the classrooms, and how we assess learning.

Didactic Transposition theory (Chevallard, 1989) recognizes that disciplinary knowledge is organized primarily to be put to use and must undergo considerable transformation to be in a form meant to be taught and learned, as it is for classroom lessons and educational technology. In the following, we briefly overview two independent lines of inquiry focused on the mathematical branch of geometry—one pertaining to two-column proof, one pertaining to model construction—that each foreground the function of other-than-textual sensory modalities, semiotic registers, and social behavior that, we claim, are inherent to and constitutive of expert mathematical practice which is embodied and enactive. Our aims, through these project case studies, are to: (1) embellish the mathematicians' gaze as more than visual and, as such, to offer authentic empirical mirrors to that multimodal gaze; and (2) describe unique insights the embodied–enactivist frame offers into the generation of transposable knowledge.

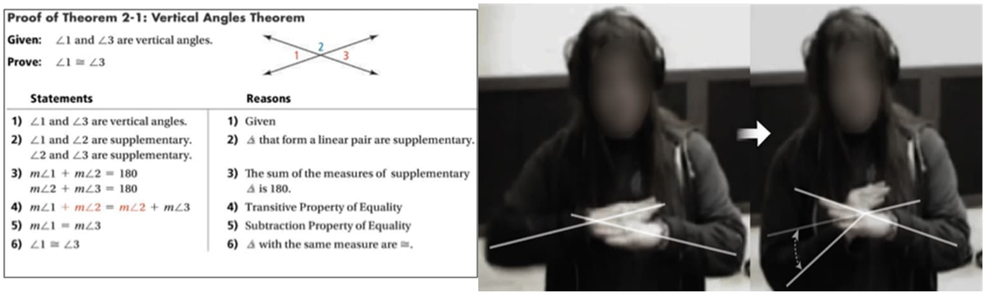
CASE1: GEOMETRY PROOF

Proof is a central disciplinary practice among mathematicians (Lakatos, 2015). When designing educational experiences for geometry proof practices, transposable knowledge starts with the mathematical gaze as it focuses on the nature of deductive proof (Figure 1). Geometry proof education is often out of reach for high school students and their teachers (e.g., Chazan, 1993; Herbst, 2002). Nathan and Walkington (2017) proposed the grounded and embodied mathematical cognition framework as an alternative embodied educators' gaze directed at students' verbal and nonverbal behaviors when producing transformational proofs. This includes surfacing students' embodied simulations of operations performed on imagined objects. This case of the embodied and multimodal nature of a high school student's transformational proof (Figure 1) demonstrates how gestures and operational speech enacting transformations on imagined geometric objects perform embodied simulation in support of insightful geometric reasoning. The student reasons:

> "The statement is true. Because the lines must always [Flat hands cross to make intersecting lines] be lines like they can. [Now crossed arms make lines] ((long pause)) Assuming the lines are straight, and not-, and we're operating on the plane, they the lines are [Now hands make lines that are uncrossed, then crossed] … To draw a straight line from any point to any point. [Switch hands] If one angle, one of the opposite angles was different than the other opposite angle, then the lines would not be straight, [Top (right) hand is now bent at palm] they would be, there would be an angle at the intersecting point in one of the lines."

The resulting embodied pedagogy has been used successfully in multiple laboratory and classroom interventions that lead to enhanced geometry proof performance (Nathan et al., 2022). Such findings provide a basis for didactic transposition from formal mathematics to teachable, learnable embodied geometry.

Figure 1: (Left) Mathematician's gaze of geometry proof based on axioms, ungrounded symbols, highly regulated diagrams, and a logical chain of deduction made through declarative steps of static states that hides processes and applies independently of any physical drawing or object. (*Right)* Educator's gaze of embodied simulations via body movements and operational speech.

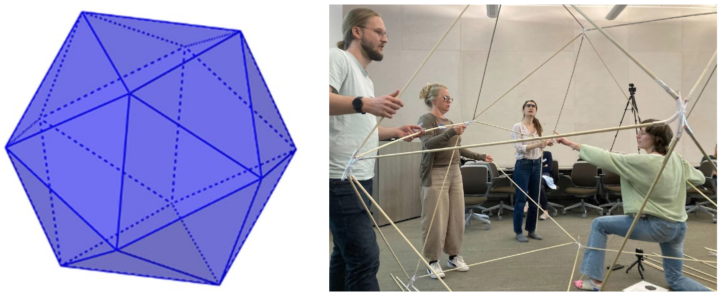


When researchers and teachers neglect the role of gestures and operational speech, they risk presenting only the symbolic, ritualized form of the proof (compatible with the mathematician's gaze), without teaching the dynamic, embodied meaning (the core transposable insight) that makes the proof logical, operational, and generalizable (Harel & Sowder, 2007) as well as convincing to the student. In this research, the mathematical specificity is found in the way bodily action is the transformed mathematical object. The specific challenge of transposing geometry proof is not about making it concrete, but about making the abstract transformations — which are the true mathematical specificities of the content—experiential through the body.

CASE2: OBJECT CONSTRUCTION

Geometry is traditionally taught through textbooks, in which standard shapes and forms are depicted along with formal definitions of their features. 3D geometric forms, in particular, such as a pyramid, present enduring challenges for students (Fujita et al., 2020). For example, students find it challenging to understand how three right pyramids compose a cube, a geometric fact supporting the formula that specifies the volume of a pyramid as one third of its containing cube. In a sense, students cannot perceive the voluminous forms as depicted in flat images let alone scrutinize angular relations between their constituent linear elements or animate complex transformations of these forms.

Figure 2: (Left) Mathematician's gaze of a geometric structure, the Platonic Solid icosahedron (= "twenty-faced"). In geometry, a notation of form {p, q}, named the Schläfli symbol, signifies and provides a uniform method to classify a regular polyhedron with p-gonal faces and q edges at each vertex (Coxeter, 1973). In particular, a regular icosahedron (20 planes, 12 vertices, and 30 edges) features five triangles around each vertex, thus, its Schläfli symbol is {3, 5}. (Right) Educator's gaze on a collaborative construction of a body-scale icosahedron. Participants receive a stack of dowels (edges), a pile of rubber connectors (vertices), and minimal information about the target structure. We monitor the emergence of understanding as situated discourse.



Educational responses to students' sense-making in geometry have varied with scholars' epistemological commitments and historical media. Catering to young students, Froebel (1895) introduced small wood models of 3D-geometric shapes as part of the "gifts" comprising his pedagogical regimen. For adult experts, Monge and Klein similarly sought to render 3D shapes more accessible through engaging with concrete models (Mueller, 1998). Currently, various interactive digital media enable students to inspect 3D shapes by constructing and/or transforming them (e.g., 3D GeoGebra). Mechatronic solutions for blind and visually impaired students offer accessible haptic-tactile inquiry matched with spoken explanation (e.g., the Quad; Lambert et al., 2022). Yet another approach involves the collaborative construction of body-scale Platonic solids, such as an icosahedron (see Figure 2).

Palatnik and Abrahamson (in press) combined theoretical constructs from ecological dynamics and multimodal interaction to interpret the process by which four participants built a body scale icosahedron. The authors thus looked to model individual development of geometric concepts as grounded in the enactive know-how they construct piecemeal through iteratively figuring out a common discourse for planning, executing, and evaluating the coordinated joint actions required for constructing the material structure. If you will, what you build together is what you know. Once the study participants had completed the giant icosahedron, a follow up standard quiz—such as, How many vertices does this shape have? How many edges? How many faces? How many parallel edges? and so on—revealed that the participants drew on their collective enactive know-how to articulate a disciplinary know-what. That is, the phenomenology of mathematical knowing is not about objects—it is itself a form of re-engaging the objects properly. The study thus offered a pedagogical gaze twin to the mathematical gaze. Future work will examine the durability and generativity of students' geometric know-how and know-that. Perhaps non-trivially, a decade of research on learning through building body-scale geometric models has persistently evidenced students and teachers' joy (Rosenbaum et al., 2024).

CONCLUSIONS

Embodied-enactive perspectives invite a broadening of the mathematical gaze by acknowledging the mathematical contributions of nonverbal behaviors. Students' embodied simulations of operations as exhibited by their dynamic depictive gestures and operative speech (Case 1) and enactive–discursive know-how developed through multimodally coordinating co-operative action (Case 2) reveal the invariant and variant properties of mathematical objects to support claims about universal properties of shape and space. We thus contribute a view of the mathematical gaze, along with its distinctive semiotic registers, as developmentally grounded in sensorimotor engagement with real and imaginary objects.

TRACING THE MATHEMATICAL GAZE IN TEXTBOOK STUDIES IN MATHEMATICS EDUCATION

Vilma Mesa and David Kollosche

Textbook studies are an interesting field to study the impact of the mathematical perspective: On the one hand, textbooks are mostly subject-specific, although there might be some subject-crossing textbooks on specific topics, for specific agendas (e.g., integrated STEM education for first-graders). A database search of "analysis of school textbooks" lists almost exclusively subject-specific studies. Indeed, the analysis of school textbooks might require subject-specific tools to analyze subject-specific knowledge structures and learning trajectories. On the other hand, the overall structure of textbooks is to some extent comparable across subjects, which suggests that at least some theoretical framings and methodological approaches are transferable between subjects and might constitute a part of a subject-independent field of textbook analysis.

We will show that textbook studies in mathematics education have resulted in knowledge building that is subject-independent as well as in knowledge building that is subject-specific, presenting first two studies that investigate the overall makeup of mathematics textbooks, and next a study that zooms into the analysis of specific aspects of mathematics textbooks using a conceptualization specific to mathematics education.

REZAT'S (2009) STUDY OF STUDENTS' TEXTBOOK USE

In his doctoral dissertation, parts of which have been published in English as well (e.g., Rezat, 2006), Rezat (2009) approached the question "how students use the mathematics textbook to learn mathematics" (p. 12, translation by DK). Rezat decided to use grounded theory to approach this question but found that its coding paradigm was too general for the question at hand. Therefore, he exchanged it by approaches that build on Vygotsky's concept of instrumental acts, which help to analyze how individuals interact with artefacts such as textbooks. In order to describe the affordances of mathematics textbooks, Rezat further distinguished the following structural levels: delimitation of textbooks by school year, mathematical field or otherwise; structure of textbook chapters; and structure of small coherent sections. For each of these structural levels, Rezat (2009) conducted a qualitative content analysis (Mayring, 2021), which resulted in the identification of types of structural element such as introductory exercises, introductions, explanations, and sample exercises within the structure of small coherent sections. These findings allowed him to compare the structures of different schoolbooks and to associate student use with textbook structure.

Thus prepared, for two to three weeks, Rezat (2009) had secondary school students mark everything they looked at in their textbook and take notes on what they did with the text in every instance. Additionally, he observed the mathematics lessons of the students participating in the study and interviewed the students about their textbook use. He found, among other things, that students use their textbook to do their homework, to exercise further, to learn contents out of sheer curiosity.

Tracing the mathematical gaze in Rezat's (2009) work, it can be argued that the theoretical framework employed, the research methods used, and even the concepts identified such as *types of structural elements* and *student use cases* are not specific to mathematics but could be applied to many other school subjects. Still, Rezat's insights are valuable for our understanding of the structure and use of mathematics textbooks. It is noteworthy that Rezat's references concerning textbook analysis nearly all originate in mathematics education research and that Rezat has not published his findings in a publication medium beyond mathematics education.

DOWLING'S (1998) SOCIOLOGICAL PERSPECTIVE ON TEXTBOOKS

Dowling's (1998) textbook analysis pursues knowledge-building that is specific to mathematics. Dowling aims at developing a sociological "language of description […] designed to enable the analysis of empirical data" (p. 1, without original emphasis). His socio-linguistic perspective supplies him with concepts, which he develops further with a mathematical gaze: For example, alienating common assumptions as socially shared myths, Dowling begins his book by discussing mathematical myths such as the myth that "mathematics justifies its existence on the school curriculum by virtue of its utility in optimizing the mundane activities of its students" (p. 9). To give another example, he distinguishes texts by the degree in which their expressions and contents are specialized (p. 135). This is especially important for academic mathematics, where both contents and expression are highly specialized, while learning arrangements might resort to using preliminary ideas about concepts and everyday language for their description. His theory building allows Dowling to show how English middle school and grammar school textbooks reinforce different myths about mathematics and provide text that allow for different degrees of understanding of academic mathematics.

Dowling’s (1998) work is an interesting example for a reflection on how a mathematical gaze can influence the development of mathematics-specific theory and research methods: Although the theoretical and methodological tools are taken from subject-independent disciplines, Dowling’s careful interrogation of the discourses in and about mathematics lead his choice of tools and allow him to develop mathematics-specific methods of analysis (such as the analysis of the specialization of mathematical texts) or to arrive at mathematics-specific insights that add to theory building in mathematics education (such as the myths he identified).

ANALYSIS OF MATHEMATICS TEXTBOOKS TO IDENTIFY POTENTIAL CONCEPTIONS

The following studies pertain to the application of a conceptualization with clear roots in mathematics education, Balacheff’s cK¢ model of conceptions (Balacheff & Gaudin, 2010), to understand how mathematical notions as presented in textbooks can lead to the multiple conceptions of those notions. Balacheff’s cK¢ model of conceptions attends to *knowings* (from the French *connaissance*, or, a learner’s personal constructs of mathematical notions) and differentiates it from *knowledge* (from the French *savoir*, or the intellectual constructs recognized by the community). In an instructional situation, the subject’s cognitive dimension interacts with aspects of the task that bring into play the knowledge at stake, in action-feedback loops with a *milieu* that end once the task has been completed. Balacheff models conception as a quadruplet that includes a problem (in a class of problems or “disequilibria” that call for mathematical notions needed to reach equilibrium, p. 190), the semiotic representations, or the “linguistic, graphical or symbolic means which support the interaction between the subject and the *milieu*” (p. 190), the operations or tools for “actions on the *milieu*” needed to simulate what the learner does to tackle the problems and “to transform and manipulate linguistic, symbolic or graphical representations” (p. 190), and the control structures, the “components supporting the monitoring of the equilibrium of the (student-*milieu*) system” (p. 190) or the strategies (e.g., metacognitive processes, use of definitions, checking existing answers) that allow learners to decide whether they had solved the problem and whether they had done so correctly. The cK¢ model gives a process for characterizing conceptions grounded in the analysis of students’ productions (what they write, what they say) and not on assumptions of what they are thinking.

This approach has been applied to textbook analyses to identify conceptions about functions, angles, span, linear independence in a wide range of textbooks (see Gerami et al., 2024; Mesa & Goldstein, 2016). The analysis uses the textbook content to identify the problem at stake, noting the expectations that the author has imposed on the content to identify the operations, the semiotic registers, and the control structures that are explicitly mentioned in the textbook. These constitute possible conceptions that could be assumed would emerge if students had read the textbook. Analysis of student responses has allowed for an expansion of the conception framework, to identify the criteria of application of the control structures and whether the inferences from the criteria are appropriate or not. This expanded system allows for an analysis of students' work that pinpoints the places in which a control structure use was warranted.

These fine-grained analyses of content in textbooks and student productions give better models of students' conceptions that do not depend on the elaboration of internal mental models (e.g., as in schemas), but that are directly tied to the mathematical knowledge at stake. In addition, it provides specificity to the control structures, as tied to mathematical notions, and, with the possibility of systematizing these analyses, it is possible to anticipate all the conceptions that textbooks can implicitly promote.

Concerning the subject-specificity of research approaches, we understand that Balacheff's model answers to analytical challenges that are paramount in mathematics education but not necessarily so in research on other school subjects, especially the concern for semiotic representations and operations—as these are objects specific to mathematics. Most noteworthy, however, is how the application of Balacheff's model to specific content areas of the mathematics curricula gives rise to local and content-specific theories as to what it means to understand and learn a certain content on levels that allow both for the analysis of written learning materials and of student productions.

DIVERSITY OF MATHEMATICAL GAZES IN RESEARCH ON MATHEMATICS CLASSROOM INTERACTION

Susanne Prediger

Research on mathematics classroom interaction is another interesting case in which knowledge building used two mechanisms, importing theories from other disciplines and establishing home-grown theories that refer more to mathematics-specific aspects. Interestingly, the kinds of mathematical gazes are highly diverse, as will be discussed.

Research on mathematics classroom interaction started in the 1970s with a focus on surface structures such as teachers' and students' talk time. Bauersfeld (1979) was one of the first who investigated deeper structures of classroom interactions by importing ethnomethodological and microsociological perspectives into mathematics. By his interactionist research approach, he and his students could identify subtle mechanisms for how typical interaction patterns between teacher and students shape the learning opportunities in mathematics classrooms, elaborating on well-known constructs such as funnel pattern and sociomathematical norms (Voigt, 1994). Subsequently, the interactionist research approach has influenced other research groups, such as, Yackel and Cobb (1996), who integrated it into their formerly constructivist approach. Other similar qualitative research approaches such as Ethnomethodology, Discourse Analysis, or Conversation Analysis were also imported from sociology to mathematics education research to study classroom interaction. Yet, they were often criticized for their missing mathematical gaze: When the main research focus was on interactive mechanisms on *how* norms or mathematical themes are established co-constructively in classrooms, this often neglected *what* was established mathematically (see Ingram, 2018, for a brief historical overview). In a survey on Discourse Analysis research in mathematics education research journals, Ryve (2011) counted that the majority of the 108 analyzed papers focused generic questions, while only a minority studied mathematical aspects of their discourses in view. To further grow the academic field with strong examples of mathematics-specific research, it is thereby interesting to make explicit what forms of mathematical gaze can be adopted: In which different ways can qualitative classroom interaction research be specific to mathematics?

Even research that does not address mathematical contents can still be specific to the role of mathematics in society, for example, the role of mathematics as an instrument reproducing inequity, being considered too hard for marginalized students. Jungwirth (1991), for example, studied the mechanisms by which teacher-girl interaction contributed to interactively "constructing girls' mathematical failures," while teacher-boy-interaction prioritized a pattern of "concealing the failure." Similarly, Gellert et al. (2018) determined patterns of interactive stratification for students with low socioeconomic and/or immigrant status. Based on these reconstructive findings, classroom design research started to identify productive practices which contribute to overcome inequities by strengthening marginalized students' agency and identities (DIME, 2007). It seems that the role of mathematics can raise particular challenges which are less virulent in other subjects and require subject-specific reflection. The authors themselves characterize their perspective as sociological (Gellert et al., 2018) without claiming to provide a unique contribution which could not be provided by sociologists. However, their work was still very important for theorizing on mathematics education, even beyond the practical purposes of strengthening equity.

For capturing learning opportunities specific to mathematical learning contents, Cobb et al. (2001) promoted the construct of mathematical practices and studied which practices of argumentation can and should be established. Selling (2016) took this further to identifying explicit teaching practices that make the expectations around argumentation explicit, once students have inquired possible argumentations individually. This research on mathematical practices might have contributed to establish them as important curricular content in educational standards of various countries.

Other content-specific forms of the mathematical gaze related to particular kinds of mathematical knowledge, such as, conceptual understanding of concepts: Case studies revealed that, although teachers often work with multiple representations, important learning opportunities are missed as long as surface translations (focusing only the numbers) are accepted and teacher-student-interaction does not make collectively explicit how underlying mathematical structures occur in the representations (Post & Prediger, 2024). Such kind of research contributes to content-specific theorizing on enhancing understanding, by identifying important conditions of productive teaching, which are of relevance across various mathematical topics.

Topic-specific forms of mathematical gaze can further unpack the research for topic-specific theorizing, for example, on important learning steps in a trajectory towards well-justified and understood volume calculation (Ademmer & Prediger, 2025). Although these steps (most importantly, explicating the multiplicative unit structures) were already specified in epistemological papers and design research papers focusing student learning, their constant neglect in classroom interaction is worth overcoming by topic-related reconstructions of classroom interaction. One major outcome is the typical pattern of productive topic-specific support and topic-specific pathways of classroom discourse, showing for example that even if one student quickly offers a formula, it is worth to relate it back to the meanings of counting in rows and layers. While the ideal learning trajectory is well known from design research studies, it is that topic-specific research on classroom interaction that can contribute to theorizing on productive teachers' steering pathways up and down the levels of the learning trajectory. Topic-specific classroom research is still rare and should be strengthened in the future in order to establish topic-specific theoretical accounts for each mathematical topic and their connections across topics.

Table 1: Different forms of mathematical gaze in classroom interaction research

	Descriptive research for understanding typical patterns	Research for identifying productive teaching practices
Specific to the role of **mathematics in society** as an instrument of inequity	Mechanisms for constructing girls' failure (Jungwirth, 1991) Stratification mechanisms (Gellert et al., 2018)	Equitable teaching practices strengthening (marginalized) students (DIME, 2007)
Content-specific to particular mathematical practices	What is established as a good argumentation? (Cobb et al., 2001)	Encouraging (Cobb et al., 2001) and explicit teaching practices on argumentation (Selling, 2016)

Content-specific to particular forms of mathematical knowledge	What opportunities for conceptual understanding are created/ missed? (Post & Prediger, 2024)	Teaching practices for explicitly connecting multiple representations (Post & Prediger, 2024)
Topic-specific to particular mathematical topics (e.g., area of prisms)	What opportunities for justifying volume calculation are provided along a learning trajectory? (Ademmer & Prediger, 2025)	Productive teaching practices for enhancing the justification of volume calculation (Ademmer & Prediger, 2025)

CONCLUSION

Summing up, the mathematical gaze can relate to different levels of classroom interaction research, even if for the moment, only a minority of studies hold a content-related or topic-specific form of mathematical gaze (Ryve, 2011). While a content-specific focus on mathematical practices is well established, expanding the topic-specific research could strengthen the mathematical gazes on classroom interaction and more consequently connect it to topic-specific epistemological and design research studies. This would allow, in the long run, to establish also theorizing activities on transposable knowledge across different topics of similar epistemological natures, such as in, classroom interaction for developing understanding for (different kinds of) formulas and the role of making explicit the underlying mathematical structures.

While there is no single criterion for judging the best way for mathematics-specific classroom interaction research, the maturing academic research field should elaborate on different forms of mathematical gazes in Table 1 and establish systematic connections to other research areas within mathematics education research to strengthen the knowledge building outcomes.

THE MATHEMATICAL GAZE, AND ITS NECESSITY, IN MATHEMATICS TEACHER EDUCATION

Gil Schwarts and Irene Biza

In this paper, we argue that research on mathematics teacher education and teaching practice should incorporate a mathematical gaze as an essential component. We briefly discuss the importance of this perspective, how it may be supported, and illustrate it through three research cases. In line with Prediger's (2024) call to foreground the content-related dimensions of research on mathematics teachers' collaboration, we seek to draw attention to the following issue: Even when situated within mathematics education, research on professional learning often focuses on teachers' experiences and reflections or on the design features of learning environments, while giving comparatively limited theoretical attention to how mathematics itself shapes the phenomena under investigation. We suggest that this issue is structural: In professional learning settings, unlike in classrooms, mathematical content is nested within discussions about teaching. In these contexts, attention often shifts toward generic teaching practices—such as discussion moves or classroom management—while the mathematical dimension receives limited attention. And while mathematics does not disappear in professional learning settings, its status changes: It becomes a layer within a complex configuration. When research on mathematics teaching phenomena attends only incidentally, or not at all, to the mathematics, the interpretation of those phenomena is compromised. In our view, mathematical and pedagogical discourses are entangled to form mathematics teaching discourses, and this entanglement should be reflected in the object of inquiry in mathematics teacher education. How, then, can research capture this entanglement? Herbst and Chazan (2012), who specify the didactical contract as subject-specific through constructs such as instructional situations that have specific norms, is an example of research capturing such entanglement. In another example from research, Weingarden and Heyd-Metzuyanim (2025) employ the diagrammatic visual of the Realization Tree, which represents how mathematical objects can be discussed in classrooms, to foreground an analysis of teaching discourse closely tied to the mathematical aspects of teaching. Prediger (2024) demonstrates how a general theoretical framework "gains its explanative power for content-related purposes when filled in content-related ways" (p. 281). We now briefly present three cases with more detail to illustrate the necessity (both conceptual and operational) of the mathematical gaze on how we inquire and analyze mathematics teaching and teacher learning.

MATHEMATICAL GAZE AND RESEARCH ON MATHEMATICS TEACHERS AND TEACHING: THREE ILLUSTRATIVE CASES

In the MathTASK program on research and development of mathematics teachers' pedagogical and mathematical discourses, Nardi, Biza and colleagues design fictional classroom situations that are informed by MER and teaching practice, and invite teachers to identify issues related to teaching and learning in those situations and to reflect on how they would respond. In Nardi et al. (2012) secondary mathematics teachers were invited to respond to one such situation in which two students debate whether a given line is tangent to the graph of a function, using algebraic and graphical arguments, respectively. Analysis of teachers' responses identified the complexity of the positions teachers take when they interpret students' responses and plan their actions (Nardi et al., 2012). To unpack such complexity, researchers analyzed the warrants (and their origins) in teachers' arguments. Initially, they drew on Toulmin's (1958) field-dependence account and Freeman's (2005) classification of warrants, but they found that such a classification was not enough to capture the mathematical specificity in teachers' multiple professional and institutional roles. To surface such specificity, they adapted Freeman's (2005) classification of warrants with attention to teachers' role as mathematicians and as teachers of mathematics. For example, they introduced the institutional warrant as "a justification of a pedagogical choice on the grounds of it being recommended or required in a textbook (institutional–curricular) or on the grounds that it reflects the standard practices of the mathematics community (institutional–epistemological)" (p. 161). A teacher may say a visual argument would not be acceptable as a full argument because within the mathematics community, visual arguments are not precise (institutional–epistemological) or because visual arguments are not acceptable in mathematics high-stake exams (institutional–curricular). This observation signals that interpretation, address, and uptake of teachers' argument need attention to the mathematical content as an epistemological object as well as a learning object.

In another study, Biza, Nardi, and colleagues unpacked the interrelationship between mathematical and pedagogical discourses and their critical role in teachers' competency in interpreting students' responses and responding to them (Biza et al. 2018). The study took place in the context of a post-graduate mathematics education program that incorporates MER with mathematical content. They used a similar classroom situation about a line tangent to a function graph, this time with teacher–student exchanges, to identify how MER and mathematical content are reified in participants' responses. Here, the term reification is defined as "the gradual turning of processes into objects" (Sfard, 2008, p. 118). Reification of pedagogical discourse addresses how MER theories and findings are reified into participants' responses "in order to describe the pedagogical and didactical issues of the classroom situations and the intended practice" (Biza et al. 2018, p. 64), while reification of mathematical discourse addresses how mathematical theories and practices are reified into participants' responses "in relation to the identification of the underpinning mathematical content of the classroom situations and the transformation of this mathematical content into the intended practice" (p. 64). The mathematical and pedagogical discourses are nested and shape what the mathematics teaching discourse is with attention to both mathematical and pedagogical routines and narratives.

Our third example draws on Schwarts' work on novice mathematics facilitators transitioning from classroom teaching to a teacher educator or facilitator role (Schwarts, 2025). In classrooms, mathematical authority is often anchored in the epistemology of the mathematics discipline, which provides relatively clear criteria for what counts as true or valid. When mathematics teachers move into facilitation, however, their practices as discussion leaders undergo substantial changes. From a generic gaze, these changes can be attributed to shifts in roles, norms, or authority relations. A mathematical gaze, however, reveals a more subtle shift: not only in authority, but also an epistemological shift. It concerns how mathematics teachers, as facilitators, draw on, suspend, or reconfigure mathematical criteria of correctness and justification as they navigate pedagogical dilemmas. In other words, the phenomenon is not only a change in social positioning but also a reorganization of the role of mathematics and its epistemology within professional discourse.

Through three cases, we illustrated the necessity of the mathematical gaze into research on mathematics teacher education and teaching practice towards three directions: (a) to increase the analytical power of existing theoretical instruments (e.g., the adaptation of Freeman's classification), (b) to highlight the nested role of mathematical and pedagogical discourses in teacher education and teaching (e.g., by addressing and evaluating both mathematical and pedagogical discourses), and (c) to situate social phenomena (e.g. shifts in authority or professional roles) into the disciplinary characteristics of mathematics teaching, thus supporting a deeper and subject-specific analysis of this phenomena.

HOW DOES MATHEMATICS SHAPE THE MER GAZE AND WHY MIGHT THIS MATTER?

Daniel Chazan

This Research Forum has provided examples of the gaze of researchers on mathematics education when it focuses on specific aspects of mathematics teaching and learning. As the examples bear out, the mathematical in MER's gaze is always more than just the conjectures, theorems, and axioms of the discipline. For example, in Abrahamson and Nathan's contribution, doing mathematics is conceptualized as an embodied, embedded, extended, and enactive activity, which allows them to see new aspects of student learning and to imagine other ways in which to teach mathematics. More generally, our gaze often combines attention to mathematics and educational matters.

Even when the forum shifts attention to textbook studies, where subject specificity seems prominent, the mathematics in the MER gaze combines ways of conceptualizing school mathematics, such as, topics, as well as more disciplinary perspectives that inform curricular decision making. The particular combination may change over time. Modern or New Math can be thought of as bringing early 20th century advances in the content of the discipline into the school curriculum across the world to provide students with new learning opportunities. By contrast, the NCTM Standards and the Common Core Standards in the US concentrated on ways in which the practice of doing of mathematics in the discipline might shape how students learn mathematics in school, and thus, again, their opportunities to learn. The careful studies of textbooks discussed by Mesa and Kollosche illustrate other ways in which the mathematical is present in MER, including a sensitivity to ways in which mathematical conceptions that originate in one curricular context may reappear in another to which they are less well suited.

Moving into the areas of MER where the mathematics in the activity is in some sense mediated and the objects of study do not necessarily carry mathematics as clearly with them, the mathematical in the MER gaze remains a combination of yet different elements. For example, as Prediger suggests, the mathematical in the MER gaze may be imbued with understandings of the societal roles that mathematics plays as gatekeeper from, and gateway to, opportunities for individual advancement. And, as outlined in the rows of Table 1 in Prediger's contribution, there are other ways in which the mathematical in the MER gaze can differ when classroom instruction is being studied. Similarly, Schwarts and Biza's contribution suggests that the mathematical in the MER gaze in studies of teacher development can include many elements of mathematics, from analyses of mathematical practices relevant to teachers' own mathematical activity, to understandings of the particular mathematics that is at the heart of the teaching teachers will do, to capturing the mathematical components of teachers' gaze as they work as teachers responsible for the education of children, not only instructors of particular mathematical content.

In the proposal for this Research Forum, we argued that Mathematics Education Research often asserts that it has the potential to make a special contribution to human understanding by focusing on the mathematical in education practices. What is it about a mathematical sensibility in the mathematics education research gaze that offers the potential to make unique contributions to the larger field of educational research and to being deserving of a seat at the interdisciplinary table of educational policy-making?

Based on the above summary of the initial offerings in the Research Forum, I would suggest that our field's claim is not based in a singular mathematical sensibility that researchers on MER hold in their gaze, but rather in the variety of mathematical sensibilities that MER makes available to the researcher of mathematics teaching and learning. This warrant fits with the contributions to the forum in emphasizing the importance of having a claim to a mathematical sensibility in our gaze, while at the same time challenging MER to grow and develop even more instantiations of mathematical sensibilities which will continue to enrich the insights that our field might develop. Rather than using one particular definition of the mathematical to police the boundaries of our field, we could continually consider the redefinition of the mathematical in ways that make our field relevant to the variety of ways in which mathematics is present in modern societies. If this line of argument is sensible, perhaps we should also be pushing ourselves to not only attend to the mathematical in how we examine matters of teaching and learning, and also consider ways in which we might be able to argue that the very claims generated by our research themselves might legitimately be described as mathematical as well. Of course, to do so would require us to explicate in what way such claims are mathematical, and especially how our descriptions of the mathematical may depart both from the discipline and from the school subject.

In closing, separate from attempting to answer this question about in what ways our mathematical sensibilities might warrant the importance of the insights of our field, I would like to underscore the importance of asking, and re-asking, this question. The potential for unique contributions to educational research and educational policy-making are what we—when we act as stewards (Herbst, 2023) of research in mathematics education—use to argue for investment in the field. For example, we as a field may need to argue periodically for funding for research, as well as support for journals.

Similarly, we as a field may want to argue for the importance of subject specificity in the training of doctoral students and in the focus of their dissertations, rather than having mathematics educators solely educated in generalist education courses or graduate level courses in mathematics. Instead, we might desire a doctoral education with attention to the history, philosophy, and sociology of mathematics, as well as to the results and methods of MER. Without a broad education of the mathematical sensibilities of future mathematics educators in the gaze they bring to educational matters, our field may be less likely to continue to respond to the mathematical throughout society.

CONCLUDING COMMENTS: ON GAZE AND BASIC RESEARCH

Patricio Herbst

How is mathematics present in the pluralistic research done in our field? Clearly, school mathematics is present in the practices we study—as the tasks, topics, or courses of study our research participants are occupied with. But some of the details of those particulars are to be forgotten when we make knowledge claims about those practices, while other particulars are likely to be taken as instances of abstractions that participate in the knowledge claims we seek to make. How do our knowledge claims about practice maintain a relationship with mathematics even if they rise above the particular mathematics of the practice being studied? Some studies make very apparent the mathematical entanglements of their knowledge claims, while other studies do not. But, in general, few studies rely on an explicit articulation of how they navigate the transition between the mathematical nature of the actual practices they study and the mathematical aspects of the knowledge of practice the researchers produce. In other words, mathematics education research studies vary widely in how they construct the relatedness of their object of study with mathematics. But our field values knowing more about this (see Yan, Mamolo, & Kontorovich, 2025) and our research forum has sought to promote that.

Because our field is pluralistic in what our research studies focus on, we cannot expect every research study to relate to mathematics in the same way. Furthermore, as the papers above demonstrate, the organization of knowledge in the discipline of mathematics does not provide sufficient resources for researchers in our field to identify how their knowledge claims are related to mathematics. We contend that it is important for our research field to elaborate on the various ways in which our objects of study relate to mathematics and to produce research that enhances the relatedness to mathematics of our knowledge claims. Because we often claim that our field provides knowledge which is distinct from that produced by other fields of education research, and we often rely on the special place of mathematics in our research to make that claim, it is important for our field to engage in work that clarifies the various ways in which our engagements with mathematics education practice result in knowledge claims that are related to mathematics. The unique contribution our research domain makes to scholarship, policymaking, and practice depends not only in our capacity for solving problems of mathematics education practice but in our capacity to provide fundamental understandings of mathematics education practice that distinguish our field from other fields that also sometimes draw their empirical data from places where mathematics is taught and learned (or that includes those).

Basic research has been defined as research which pursues fundamental understanding (see Herbst et al., 2022). Basic research is not antithetical to research that is use-inspired: Indeed, Stokes (1997) drew inspiration from the work of Louis Pasteur to characterize a kind of research that was both motivated by use and pursuant of fundamental understanding. The practices of mathematics teaching and learning are full of problems to solve to which many different fields are attracted to apply their gazes. Our special opportunity, similar to that of Pasteur's when he developed microbiology, is to demonstrate that our gaze affords us distinct understandings of such practices, fundamental understandings which are not mere applications of ideas from other disciplines. Research that hones our gaze, seeking to clarify how our knowledge claims are related to mathematics, can help us achieve such fundamental understanding and support our field's autonomy.

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